CS 144, HW1. LEIYA MA. 2018042.

1. Warmup

Have discuss with De Huang, Tao Liang

(i). Prove Markov's Inequality, $t > 0$

we have:

$$Pr(X \geq t) = \mathbb{E}(\mathbb{1}_{(X \geq t)})$$

where,

if $X \geq t$, $\mathbb{1}_{(X \geq t)} = 1 \leq X/t$

if $X < t$, $\mathbb{1}_{(X \geq t)} = 0 \leq X/t$

therefore,

$\mathbb{1}_{(X \geq t)} \leq \frac{X}{t}$ always exists.

then,

$$\mathbb{E}(\mathbb{1}_{(X \geq t)}) \leq \mathbb{E}\left(\frac{X}{t}\right) = \frac{\mathbb{E}(X)}{t}$$

finally,

$$Pr(X \geq t) \leq \frac{\mathbb{E}(X)}{t}$$

(ii) Prove Chebyshev's Inequality

we have:

$$Pr(|X - \mathbb{E}(X)| \geq t) = \mathbb{E}(\mathbb{1}_{(|X-\mathbb{E}(X)| \geq t)})$$

where:

if $|X - \mathbb{E}(X)| \geq t$, $\mathbb{1}_{(|X-\mathbb{E}(X) \geq t)} = 1 \leq \frac{X - \mathbb{E}(X)}{t}$

and thus:

$$\mathbb{1}_{(|X-\mathbb{E}(X)| \geq t)} \leq \frac{(X - \mathbb{E}(X))^2}{t^2}$$

if $|X - \mathbb{E}(X)| < t$, $\mathbb{1}_{(|X-\mathbb{E}(X)| \geq t)} = 0 \leq \frac{X - \mathbb{E}(X)}{t}$

and thus:

$$\mathbb{1}_{(|X-\mathbb{E}(X)| \geq t)} \leq \frac{(X - \mathbb{E}(X))^2}{t^2}$$

therefore:

$$\mathbb{E}(\mathbb{1}_{(|X-\mathbb{E}(X)| \geq t)}) \leq \frac{\mathbb{E}((X - \mathbb{E}(X))^2)}{t^2} = \frac{\sigma_X^2}{t^2}$$

finally:

$$Pr(|X - \mathbb{E}(X)| \geq t) \leq \frac{\sigma_X^2}{t^2}$$

(iii) Prove the weak law of large numbers.

we have:

$$Pr\left(\left|\frac{S_n}{n} - \mathbb{E}(X)\right| \geq \varepsilon\right) = Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}(X))\right| \geq \varepsilon\right)$$

$$= \mathbb{E}\left(\mathbb{1}_{(|\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}(X))| \geq \varepsilon)}\right)$$

from (i) and (ii), we can deduct.

$$\mathbb{E}\left(\mathbb{1}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mathbb{E}(X))\geq\varepsilon\right)\right)\leq\frac{\mathbb{E}\left[\left(\sum_{i=1}^{n}(X_i-\mathbb{E}(X))^2\right]}{n^2\varepsilon^2}$$

where the cross-term equals 0 when expand it,
that is:

$\mathbb{E}(X_i-\mathbb{E}(X_i))(X_j-\mathbb{E}(X_j))=0$ because X_i and X_j are indepedent,

therefore,

$$\mathbb{E}\left[\left(\sum_{i=1}^{n}(X_i-\mathbb{E}(X))\right)^2\right]=\sum_{i=1}^{n}\mathbb{E}(X_i-\mathbb{E}(X_i))^2$$

$$=\sum_{i=1}^{n}\sigma_{X_i}^2=n\sigma_X^2$$

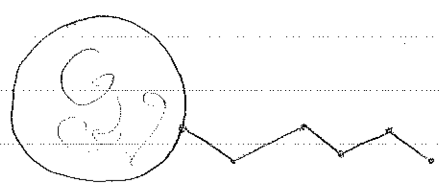
finally.

$$Pr\left(\left|\frac{S_n}{n}-\mathbb{E}(X)\right|\geq\varepsilon\right)\leq\frac{n\sigma_X^2}{n^2\varepsilon^2}=\frac{\sigma_X^2}{n\varepsilon^2}$$

$$\lim_{n\to\infty}Pr\left(\left|\frac{S_n}{n}-\mathbb{E}(X)\right|\geq\varepsilon\right)\leq\lim_{n\to\infty}\frac{\sigma_X^2}{n\varepsilon^2}=0\qquad\forall\varepsilon>0$$

(b).

Construct a graph where $\frac{\text{diameter}}{\text{average distance}}>3$.



m points n point

The graph is like this, it has 2 parts.

(1) "body" a complete graph with m nodes

(2) "leg" a chain with n nodes

Then we calculate its average distance.

① the amount of shortest distance in "body".

$$N_1=\frac{m(m-1)}{2}\times 1$$

② the amount of shortest distance in "leg"

$$N_2=\sum_{i=1}^{n}(n+1-i)\,i$$

$$=\frac{(n+1)(n-1)n}{2}-\frac{n(n+1)(2n+1)}{6}$$

③ the amount of shortest distance between "body" and "leg"

$$N_3=\frac{n(n+3)}{2}(m-1)$$

in total, the average distance of this graph is.

$$\text{average distance} = \frac{N_1 + N_2 + N_3}{\frac{(m+n)(m+n-1)}{2}}$$

$$= \frac{\frac{m(m-1)}{2} + \frac{n(n+1)(n-1)}{2} + \frac{n(n+1)(2n+1)}{6} + (m-1)\frac{n(n+1)}{2}}{\frac{(m+n)(m+n-1)}{2}}$$

when $m \to \infty$, average distance $\to 1$

while diameter $= n+1$;

therefore, $\frac{\text{diameter}}{\text{average distance}} \xrightarrow{m\to\infty} n+1$

So, our graph have 3 points in "leg" and after calculation, our main part "body" has 40 points to make.

$$\frac{\text{diameter}}{\text{average distance}} > 3$$

Theoratically, if we have enough points in "body", we can have any number of c.

2. Job Interview Problems:

(a) In order to get an unbiased random bit 0/1.

Using the unknowned biased p coin, (p top, (1-p) bottom)

we flip twice independently in one row.

(a). 1 top, 2 top: $Pr = p^2$

(b) 1 top, 2 bottom: $Pr = p(1-p)$

(c) 1 bottom, 2 top: $Pr = p(1-p)$

(d) 1 bottom, 2 bottom: $Pr = (1-p)^2$

only when (b) or (c) happenes we counts.

then, if is (b), I win

if is (c), my friends win,

in this case, $Pr(b \mid b \text{ or } c) = Pr(c \mid b \text{ or } c) = \frac{1}{2}$

The expected number of flips:

we flip the coin twice in each row under it shows 1 top, another bottom,

$Pr(1 \text{ top and another bottom}) = 2p(1-p)$

using X to represent the number of rows we play,

$Pr(X=j) = [2p(1-p)][1-2p(1-p)]^{j-1}, \quad j=1,2,3,\cdots$

$$\mathbb{E}(X) = \sum_{j=1}^{\infty} j(2p(1-p))(1-2p(1-p))^{j-1} = \frac{1}{2p(1-p)}$$

then,

the expected number of flips is:

$$2\mathbb{E}(X) = \frac{1}{p(1-p)}$$

(b) a Peer-to-peer system:

1 round: t_1

2 round. $\sum_{k=0}^{\infty} (\frac{1}{n})^k (1-\frac{1}{n})(kt_2+t_1)$

$\vdots$

m round. $\sum_{k=0}^{\infty} (\frac{m}{n})^k (1-\frac{m}{n})(kt_2+t_1)$

(round) (expected value of time used)

therefore, the expected value of time to get all n chunks.

$$\mathbb{E}(t) = \sum_{m=0}^{n-1} \sum_{k=0}^{\infty} \left(\frac{m}{n}\right)^k \left(\frac{n-m}{n}\right) (kt_2 + t_1)$$

$$= nt_1 + \sum_{m=0}^{n-1} \sum_{k=0}^{\infty} \left(\frac{m}{n}\right)^k \left(\frac{n-m}{n}\right) kt_2$$

$$= nt_1 + \sum_{m=0}^{n-1} \left(\frac{n-m}{n}\right) t_2 \underbrace{\sum_{k=0}^{\infty} \left(\frac{m}{n}\right)^k \cdot k}$$

$$\Downarrow$$

$$\partial \sum_{k=0}^{\infty} p \int p^{k-1} \cdot k \, dp \quad \left(p = \frac{m}{n}\right.$$

$$= \partial \left(\sum_{k=0}^{\infty} p^k\right) p$$

$$= \frac{p}{(1-p)^2}$$

therefore,

$$\mathbb{E}(t) = nt + \sum_{m=0}^{n-1} \left(\frac{n-m}{n}\right) t_2 \cdot \frac{\frac{m}{n}}{\left(1 - \frac{m}{n}\right)^2}$$

$$= nt + \sum_{m=0}^{n-1} \left(\frac{m}{n-m}\right) t_2$$

(c)

Dealer's Offer:

When the first record occurs at time step $n > 1$,
it means: X_1 is the max among $X_1, X_2, \cdots, X_{n-1}$,
X_n is the max among $X_1, X_2, \cdots, X_n$,
therefore, the probability:

$$p_n = \frac{1}{n-1} \cdot \frac{1}{n}$$

where, $\frac{1}{n-1}$ represents that X_1 is the 2nd biggest and $\frac{1}{n}$ represents that X_n is the biggest. It doesn't matter what random variable I choose.
therefore, the expected value of N is:

$$\mathbb{E}[N] = \sum_{i=2}^{\infty} i \frac{1}{i-1} \cdot \frac{1}{i} = \infty$$

which indicates this game is not in my favor.

Techer's Offer:

In this model, the probability of getting a record at time step n is:

$$p_n = \frac{1}{n}$$

then,
the expected number of records in M steps is:

$$\mathbb{E}[M] = \sum_{i=1}^{M} \mathbb{1}_{(X_i \text{ is a record})}$$

$$= \sum_{i=1}^{M} \frac{1}{i} \approx \ln M$$

to determine which M to choose, we have

$$\ln M = x$$

$$M = e^x$$

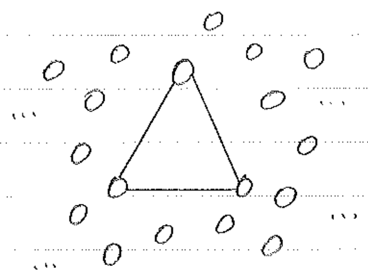
3. Understanding Clustering

Based on the definition of average clustering coefficient and overall clustering coefficient,

$$Cl^{avg}(G) = \frac{\sum_{i=1}^{n} Cl_i(G)}{n} \quad , \quad Cl(G) = \frac{3 \times \#\text{triangles}}{\#\text{connected tuples}}$$

we can construct examples:

(i) when # number of nodes $\to \infty$, $Gl^{avg}(G) \to 0$, $Gl(G) = 1$



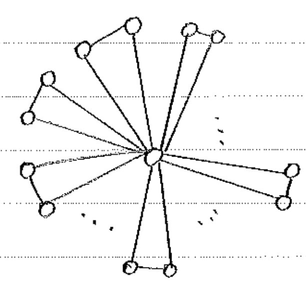
in this example, when $n \to \infty$,
it is only counted in $Cl^{avg}(G)$ and is neglected in $Cl(G)$, therefore, $Gl^{avg}(G) \to 0$

$$Gl(G) = \frac{3 \times 1}{3} = 1$$

(ii) another example where:

$Gl(G) \to 0$,

$Gl^{avg}(G) = 1$



we prove it:

in this graph, we can divide the node in two classes,

A: the only one node in the center, $Cl_{i \sim A}(G) = \frac{\#\text{triangles}}{\#\text{triples}} < 1$

B: the rest of the other nodes $Cl_{i \sim B}(G) = 1$

when $n \to \infty$,

even $Gl_{i \sim A}(G) < 1$, all the other nodes $Cl_i(G) = 1$

then,
$$Gl^{avg}(G) = \frac{\sum_{i=1}^{n} (Cl_{i \sim A}(G) + Cl_{i \sim B}(G))}{n} \to 1$$

But for the overall clustering coefficient:

tuples for node in class A equals ∞ when $n \to \infty$ which will be counted directly in denominator in $Cl(G)$

when $n \to \infty$, $Cl(G) \to 0$

4. Six degree of separation.

(a)

(i) first path: (3 hops)

Network congestion → fiber-optic → electromagnetic interference → capatitor

(ii) shortest path: the same as (i) 3 hops

*: The main thought for this problem is to turn this concept to a more specific electronic entity. (Network congestion)

(capacitor)

So I first look for entities (fiber-optic) and then electronic related entity (electromagnetic interference → capacitor).

(i) first path: (6 hops)

Paul Erdős → Princeton University → Association of American Universities → California Institute of Technology → California Institute of Technology people → California Institute of Technology faculty → K. Mani Chandy

(ii) shortest path: (4 hops)

Paul Erdős → Mathematics → Computer Science → distributed computing → Chandy, Mani

(b)

(i) first path: (5 hops)

Adam Wierman → Nikhil Bansal → Joel H. Spencer → Ernst Gabor Straus → Albert Einstein → Erwin Schrödinger

(ii) shortest path: same as (i)

(i) first path: (4 hops)

Paul Erdős → Zoltán Füredi → Robert P. Kurshan → James C. Browne → K. Mani Chandy

(ii) Same as (i) for shortest way.