(1)

(a). Degree distribution.

$$\lim_{n\to\infty} P(D=k) = \lim_{n\to\infty} \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$= \lim_{n\to\infty} \frac{(n-1)!}{k!(n-1-k)!} p^k (1-p)^{n-1-k}$$

we keep $(n-1)p$ constant, and denote $\lambda = (n-1)p$,
therefore,

$$\lim_{n\to\infty} P(D=k) = \lim_{n\to\infty} \frac{(n-1)^k p^k}{k!} \cdot \frac{(n-1)!}{(n-1-k)!(n-1)^k} \cdot \left(1-\frac{\lambda}{n-1}\right)^{n-1-k}$$

$$= \lim_{n\to\infty} \frac{\lambda^k}{k!} \cdot \frac{(n-1)(n-2)(n-3)\cdots(n-k)}{(n-1)^k} \left(1-\frac{\lambda}{n-1}\right)^{n-1-k}$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

(b) The expected number of triangles. $E[T]$
we first denote a random combination of vertices in graph,
define, $Y_m = 1$ when these 3 points form a triangle

$$P(Y_m = 1) = p^3$$

$Y_m = 0$ when not;
There are $\binom{n}{3}$ such combinations. $T = \sum_{m=1}^{\binom{n}{3}} Y_m$
Therefore,

$$E[T] = \sum_{m=1}^{\binom{n}{3}} E[Y_m]$$

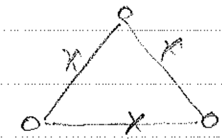
$$= \binom{n}{3} \cdot [p^3 \times 1 + (1-p^3) \times 0] = \frac{n(n-1)(n-2)}{6} p^3$$

(c) It is obvious that
$P(\text{diameter}(G(n,p)) = 1) + P(\text{diameter}(G(n,p)) = 2)$
$+ P(\text{diameter}(G(n,p)) > 2) = 1$
we need to prove when $n\to\infty$, the first & third part $\to 0$.

(i) $P(\text{diameter}(G(n,p)) = 1) = p^{\binom{n}{2}} = p^{\frac{n(n-1)}{2}}$
$\because$ every vertice connect to each other, $\lim_{n\to\infty} p^{\frac{n(n-1)}{2}} = 0$ $(p \in (0,1))$

(iii) For computing $P(\text{diameter}(G(n,p)) \geq 2)$,

We denote event A_i: all $d_i \geq 2$ in graphs G
$\wedge$ the collection of G.



$P(A_i) = P(d_i \geq 2)$

which mean no connection between two vertices, and no connections to common neighbor

$P(d_i \geq 2) = (1-p)(1-p^2)^{n-2}$

therefore:

$$P(A = \{G : D \geq 2\}) = P(A_1) \cup P(A_2) \cup \cdots \cup P(A_{\binom{n}{2}})$$
$$\leq P(A_1) + P(A_2) + \cdots + P(A_{\binom{n}{2}})$$
$$= \frac{n(n-1)}{2}(1-p)(1-p^2)^{n-2}$$

$$\lim_{n\to\infty} P(\text{diameter}(G(n,p)) \geq 2)$$
$$= \lim_{n\to\infty} \frac{n(n-1)}{2}(1-p)(1-p^2)^{n-2}$$
$$= 0$$

Therefore, $\lim_{n\to\infty} P(\text{diameter}(G(n,p)) = 2) = 1$