1. Routing Games with tolls:

We can prove that the new potential function equals the cost function given that $r_i = 1$, for $\forall i$, i.e.

$$\Phi^*(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e^*(i)$$

$$= \sum_{e \in E} \sum_{x=1}^{f_e} \left[c_e(x) + (x-1)(c_e(x) - c_e(x-1)) \right]$$

$$= \sum_{e \in E} \sum_{x=1}^{f_e} (x c_e(x) - (x-1) \cdot c_e(x-1))$$

$$= \sum_{e \in E} c_e(f_e) f_e$$

$$= \mathrm{Cost}(f)$$

Therefore, any optimal strategy for our original game is a Nash Equilibrium for this new game.

2. Nonatomic Routing Games

2.1 The Model:

1. In ~~A~~ Nonatomic Selfish Routing Games, the equilibrium flows exist and are essentially unique.

We want to prove:

Let (G, r, c) be an nonatomic instance.

(a) The instance (G, r, c) admits at least one equilibrium flow.

(b) If f and $\tilde{f}$ are equilibrium flows for (G, r, c) then, $c_e(f_e)$ and $c_e(\tilde{f}_e)$ equals for each edge e.

2. How effecient are the equilibrium,

What's the price of anachy (PoA) and the price of stability?

2.2. Part 1 - Existance of Equilibrium:

(a) Equilibrium for nonatomic routing game means:

that equilibrium flows always exist;

and are essentially unique, which means all equilibrium flows have the same cost.

Connection with notion of Nash Equilibrium.

This is similar to Nash Equilibrium notion that when others remain unchanged, user will have no incentive to change its current paths as there is no better cost for doing so.

(b) Prove:

We prove (a) using the potential function method.

And we define potential function:

$$\Phi(f) = \sum_{e\in E} \int_0^{f_e} c_e(x)\,dx \qquad (2-1)$$

Let (G, r, c) be a nonatomic instance such that, for every edge e, the function $x \cdot c_e(x)$ is convex and continuously differentable. Let c_e^* denote the marginal cost function of the edge e. Then f^* is an optimal flow for (G, r, c) if and only if it is an equilibrium flow for (G, r, c^*).

Therefore, we characterize equilibrium flows as the global minimizers of the potential function Φ. (prove shown in other pages)

The set of feasible flows of (G, r, c) can be identified with a compact subset of $|\mathcal{P}|$-dimentional Euclidean Space. Since cost functions are continuous, by Weierstrass's Theorem, Φ achieves a minimum value on this set. And such minimum corresponds to ~~the~~ an equilibrium flow of (G, r, c).

(c) Prove that if f, f' are equilibrium flows, then $c_e(f_e) = c_e(f'_e)$ for all $e \in E$.

Prove:

As each cost function is nondecreasing, and hence each summand on the right-hand side of $(2-1)$ is convex, Hence Φ is also convex.

Suppose f, f' are both equilibrium flows, according to (a), they both minimize the potential function Φ. We consider all combination of f and f', that is:

all vectors of the form $\lambda f + (1-\lambda) f'$ for $\lambda \in [0, 1]$

all these vectors are feasible flows.

Since Φ is convex, a chord between two points on its graph cannot pass below its graph.

Prove: (2.2 (b) section)

$$f^o = \arg\min_{f\in S} \Phi(f), \quad f^o \in S$$ is an equilibrium flow for the non-atomic routing game.

Using Contradiction:

if f^o is not the equilibrium flow, then $\exists i \in I$, $p, p' \in P_i$, such that $f^o_p > 0$, and

$$c_p(f^o) - c_{p'}(f^o) > 0$$

Now we construct another new flow f' by moving a little of volumn $\delta r \in (0, f^o_p)$ of commodity i from path p to p',

And as:

$$f^*_p = f^*_p - \delta r, \quad f^*_{p'} = f^o_{p'} + \delta r$$

$$\sum_{p\in P_i} f^*_p = \sum_{p\in P_i} f^o_p = r_i$$

thus $f' \in S$ and

$$f'_e = \begin{cases} f^o_e - \delta r & e \in p \ \cancel{\text{not}}, e \notin p' \\ f^o_e + \delta r & e \in p', e \notin p \\ f^o_e & \text{others} \end{cases}$$

Therefore,

$$\Phi(f^o) - \Phi(f') = \sum_{e\in E} \int_0^{f^o_e} c_e(y)\,dy - \sum_{e\in E} \int_0^{f'_e} c_e(y)\,dy$$

$$= \underbrace{\sum_{e\in p\setminus p'} \int_{f^o_e - \delta r}^{f^o_e} c_e(y)\,dy}_{①} - \underbrace{\sum_{e\in p'\setminus p} \int_{f^o_e}^{f^o_e + \delta r} c_e(y)\,dy}_{②}$$

we have:

① : $= c_e(f^o_e)\,\delta r + o(\delta r)$

② : $= c_e(f^o_e)\,\delta r + o(\delta r)$

Therefore,

$$\Phi(f^{o}) - \Phi(f') = \sum_{e \in p \setminus p'} [c_e(f_e^{o})\delta r + o(\delta r)] - \sum_{e \in p' \setminus p} [c_e(f_e^{o})\delta r + o(\delta r)]$$

$$= \underbrace{[c_p(f^{o}) - c_{p'}(f^{o})]}_{>0} \delta r + o(\delta r)$$

and thus $\Phi(f^{o}) - \Phi(f') > 0$ contradicts ~~to~~ to the fact that f^{o} is the minimizer of $\Phi(f)$

Thus we prove f^{o} is the equilibrium ~~point~~ flow if it is the minimizer of $\Phi(f)$.

4 PoA analysis for GSP.

(a). For $n=2$:

We construct ~~an~~ a situation: $(X>1)$

$$\alpha_1 = X, \quad v_1 = X$$

$$\alpha_2 = 1, \quad v_2 = 1$$

and $(b_1, b_2) = (0, X)$, thus $\pi(b) = (2, 1)$

This biding is a Nash Equilibrium as:

$$U_1(b) = \alpha_2 v_1 = X, \quad U_2(b) = \alpha_1(v_2 - b_1) = X$$

Prove:

(1) fix $b_2 = X$, for any other $b_1' \geq 0$, $U_1(b_1', b_2)$ can only be $\begin{cases} \alpha_1(v_1 - b_2) = 0 \\ \alpha_2 v_1 = X \end{cases}$

no bigger than before

(2) fix $b_1 = 0$, for any other $b_2' \geq 0$, $U_2(b_1, b_2')$ can only be $\begin{cases} \alpha_1(v_2 - b_1) = X \\ \alpha_1 v_2 = \cancel{X}\, 1 \end{cases}$

no bigger than before.

Therefore, $b = (b_1, b_2) = (0, X)$ is a Nash Equilibrium.

$$W(b) = \sum_j \alpha_j v_{\pi_j(b)} = \alpha_1 v_2 + \alpha_2 v_1 = 2X = \underbrace{\min_{b \in NE} W(b)}_{\text{because it has only two values}}$$

because it has only two values $\begin{cases} X^2+1 \\ 2X \end{cases}$, as $X>1$, $2X < X^2+1$

Optimal Strategy:

$$W(b^*) = \alpha_1 v_1 + \alpha_2 v_2 = X^2+1$$

Therefore:

$$\text{(pure) PoA} = \frac{W(b^*)}{\min_{b \in NE} W(b)} = \frac{X^2+1}{2X}$$

Then for an arbitrary $r>1$, (pure) PoA $= \dfrac{X^2+1}{2X} = r$

(b). (i). Any $b_1 \neq X-1$ is a dominated strategy for advertiser 1.

①. if $b_1 < X-1$, then $\exists\, b_1'$ such that $X-1 > b_1' > b_1$, then

$$U_1(b_1', b_2) - U_1(b_1, b_2) = \begin{cases} 0 & b_2 < b_1 \\ X(X-1-b_2) > 0 & b_1 < b_2 < b_1' < X-1 \\ 0 & b_2 > b_1' \end{cases}$$

therefore b_1 is a ~~x~~ dominated strategy.

②. if $b_1 > X-1$, then $\exists\, b_1'$ such that $X-1 < b_1' < b_1$, then

$$U_1(b_1', b_2) - U_1(b_1, b_2) = \begin{cases} 0 & b_2 > b_1 \\ X(b_2 - X+1) > 0 & X-1 < b_1' < b_2 < b_1 \\ 0 & b_2 < b_1' \end{cases}$$

therefore b_1 is a dominated strategy.

Any $b_2 \neq 1-\frac{1}{X}$ is a dominated strategy for advertiser 2.

① if $b_2 < 1-\frac{1}{X}$, then $\exists\, b_2'$ such that $1-\frac{1}{X} > b_2' > b_2$ then

$U_1(b_1, b_2') - U_1($

therefore b_2 is a dominated strategy.

②. if $b_2 > 1-\frac{1}{X}$ ~~is~~, then $\exists\, b_2'$ such that $1-\frac{1}{X} < b_2' < b_2$, and

therefore b_2 is a dominated strategy.

(ii).

Assume that $b_i > v_i$, we will show that b_i is a dominated strategy for advertiser i, and we set $b_{\pi_{n+1}(b)} = 0$ for consistency.

since $b_i > v_i$, there exists some b_i' such that $b_i > b_i' > v_i$

Let $b = (b_i, b_{-i})$ and $b' = (b_i', b_{-i})$ then for arbitrary b_{-i}, we have:

(1) if $\pi(b) = \pi(b')$ assume that $\pi_j(b) = \pi_j(b') = i$, then

$$b_{\pi_{j+1}(b)} = b_{\pi_{j+1}(b')} = b'_{\pi_{j+1}(b')}$$

we have:

$U_i(b_i, b_{-i}) = \alpha_j(v_i - b_{\pi_{j+1}(b)}) = \alpha_j(v_i - b'_{\pi_{j+1}(b')}) = U_i(b_i', b_{-i})$

2). if $\pi(b) \neq \pi(b')$ assume that $\pi_j(b) = \pi_k(b') = i$, then $j < k$, since $b_i > b_i'$ and we must have

$$b_{\pi_{j+1}(b)} \geq b_i' > v_i , \quad b_{\pi_{j+1}(b)} \geq b_i' \geq b'_{\pi_{k+1}(b')}$$

since $\alpha_j > \alpha_i > 0$,

$$\alpha_j(v_i - b_{\pi_{j+1}(b)}) \leq \alpha_k(v_i - b_{\pi_{k+1}(b)}) \leq \alpha_k(v_i - b'_{\pi_{k+1}(b')})$$

that is:

$$U_i(b_i, b_{-i}) = \alpha_j(v_i - b_{\pi_{j+1}(b)}) \leq \alpha_k(v_i - b'_{\pi_{k+1}(b')}) = U_i(b_i', b_{-i})$$

Therefore, b_i is a dominated strategy for advertiser i.

(iii)

For a game of 2, $W(b)$ can only have 2 possible values,

$$\alpha_1 v_1 + \alpha_2 v_2 , \quad \alpha_1 v_2 + \alpha_2 v_1$$

if $\alpha_1 = \alpha_2$ or $v_1 = v_2$, then we always have

$$\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 v_2 + \alpha_2 v_1$$

Thus,

$$(\text{pure})\ PoA = \frac{W(b^*)}{\min_{b \in NE} W(b)} = 1$$

Then, we assume that $\alpha_1 > \alpha_2 > 0$ and $v_1 > v_2 \geq 0$,

then:

$$\alpha_1 v_1 + \alpha_2 v_2 > \alpha_1 v_2 + \alpha_2 v_1$$

and $\pi(b) = (2, 1)$, Then b is Nash Equilibrium implies that

$$U_1(b) = \alpha_2 v_1 \geq \alpha_1(v_1 - b_2)$$

$$U_2(b) = \alpha_1(v_2 - b_1) \geq \alpha_2 v_2$$

that is: (and both ader are conservative that $v_1 \geq b_1$, $v_2 \geq b_2$.

$$v_1 > v_2 \geq b_2 \geq \frac{\alpha_1 - \alpha_2}{\alpha_1} v_1 > \frac{\alpha_1 - \alpha_2}{\alpha_1} v_2 \geq b_1$$

let $\eta = \frac{v_2}{v_1}$, then,

$$1 > \eta > \frac{\alpha_1 - \alpha_2}{\alpha_1}$$

we have

$$(\text{pure})\ PoA = \frac{W(b^*)}{W(b)} = \frac{\alpha_1 v_1 + \alpha_2 v_2}{\alpha_1 v_2 + \alpha_2 v_1} = \frac{\alpha_1 + \alpha_2 \eta}{\alpha_2 + \alpha_1 \eta}$$

we have:

$$(\text{pure})\ PoA = \frac{\partial_1 + \partial_2 \eta}{\partial_2 + \partial_1 \eta} \le \left.\frac{\partial_1 + \partial_2 \eta}{\partial_2 + \partial_1 \eta}\right|_{\eta = \frac{\partial_1 - \partial_2}{\partial_1}} = 1 + \frac{\partial_2}{\partial_1} - \left(\frac{\partial_2}{\partial_1}\right)^2$$

Since $\cancel{\partial_2}\ \partial_1 > \partial_2 \ge 0$ then

$$\frac{\partial_2}{\partial_1} - \left(\frac{\partial_2}{\partial_1}\right)^2 = \frac{\partial_2}{\partial_1}\left(1 - \frac{\partial_2}{\partial_1}\right) \le \frac{1}{4}$$

Therefore

$$\boxed{(\text{pure})\ PoA \le 1 + \frac{1}{4} = 1.25}$$

that is:

$$\Phi(\lambda f + (1-\lambda) f') \le \lambda \Phi(f) + (1-\lambda)\Phi(f') \qquad (2-2)$$

for every $\lambda \in [0,1]$, since f and f' are both global minimal of Φ, ~~so~~ the inequality (2-2) must hold with equality for all of their convex combinations. Since every summand of Φ is convex, this can ~~imply~~ only occur if every summand $\int_0^x c_e(y)\,dy$ is linear between f_e and f'_e.

In turn, this implies that every cost function c is constant between f_e and f'_e. $c_e(f_e) = c_e(f'_e) = D_i$

$$C(f) = \sum_{p\in P} c_p(f_p) f_p = \sum_{i\in I}\sum_{p\in P_i,\, f_e>0} c_p(f_p) f_p$$
$$= \sum_{i\in I} D_i \sum_{p\in P_i,\, f_e>0} f_e$$
$$= \sum_{i\in I} D_i r_i$$
$$= C(f')$$

2.3 - Efficiency of equilibria

(a). For any $a, b \le 0$, let $c(x) = ax + b$, then:

$$\frac{r c(r)}{x c(x) + (r-x) c(r)} = \frac{ar^2 + br}{ar^2 + br + ax^2 - axr}$$
$$= \frac{ar^2 + br}{\frac{3}{4}ar^2 + br + a(x-\frac{r}{2})^2}$$
$$\le \frac{ar^2 + br}{\frac{3}{4}ar^2 + br}$$
$$\le \frac{ar^2 + br}{\frac{3}{4}ar^2 + \frac{3}{4}br}$$
$$= \frac{4}{3}$$

Therefore:

$$\alpha(C) = \sup_{c\in C} \sup_{x,r\ge 0} \frac{r c(r)}{x c(x) + (r-x) c(r)} \le \frac{4}{3}$$

In particular, when we take $a=1, b=0, r=1, x=\frac{1}{2}$, $\frac{r c(r)}{x c(x) + (r-x) c(r)} = \frac{4}{3}$

which imply: $PoA \le \alpha(C)$

$$\alpha(C) = \sup_{c\in C,}\ \sup_{x,r\ge 0} \frac{r(c(r))}{x c(x) + (r-x) c(r)} \ge \frac{4}{3}$$

Therefore $\alpha(C) = \frac{4}{3}$

(b).

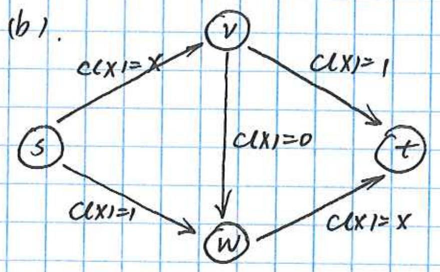


fig (2) – a

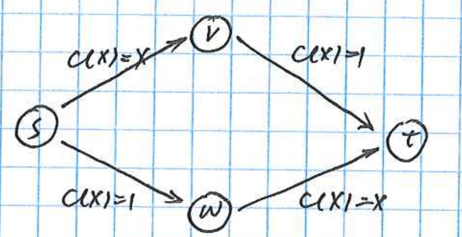
As edge $v \to w$ was added and had 0 cost, previous equilibrium ~~that~~ no longer exists,

And unique equilibrium flow routes all of the traffic on the new routes $r \to v \to w \to t$,

Because of the ensuing heavy congestion on edge (s, v), (w, t) all traffic now experience two units of costs,

But the optimal flow is remain the same as fig(2) – b.

the cost of optimal flows is $\frac{3}{2}$.



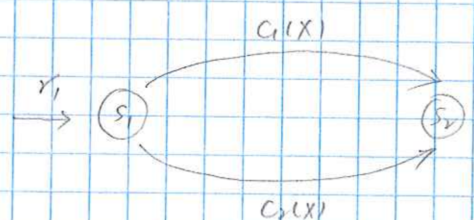
fig(2) – b

(flow volumn is split equaly ~~in~~ between the two paths)

then: $PoA = \dfrac{C(f^E)}{C(f^*)} = \dfrac{2}{\frac{3}{2}} = \dfrac{4}{3}$

(C) We want to prove that:

$$\partial(C) \geq \frac{r c(r)}{x c(x) + (r-x) c(r)} > \partial(C) - \varepsilon \qquad 2\text{-}4$$



we assume $r c(r)$ is the equilibrium flow cost

then we assume $c_1(x) = \underline{c(x)}$

fit $c(x)$ which 满足 2-4

$c_2(x)$ is a constant $c(r)$

then equilibrium flow is that all flow goes c_1

Optimal:

if $x < r$, we assume $[x, r-x]$ a feasible flow that

cost: $x c_1(x) + (r-x) c_2(r-x) = x c(x) + (r-x) c(r) \geq C(f^*)$

if $x > r$, we have $c(\cdot)$ is nondecreasing, we still have

$$x c(x) + (r-x) c(r) = x(c(x) - c(r)) + r c(r) \geq r c(r) = C(f^E) \geq C(f^*)$$

we have:

$$\partial(C) \geq PoA = \frac{r c(r)}{x c(x) + (r-x) c(r)} \geq \partial(C) - \varepsilon$$

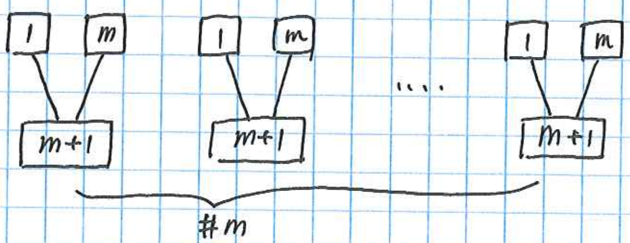
3. Tightness of PoA bound in loading balance games.

Now we look at a load balancing games with $2m$ jobs and m servers. we define that in these $2m$ jobs, m of them have demand 1
m of then have demand m

For optimal action,

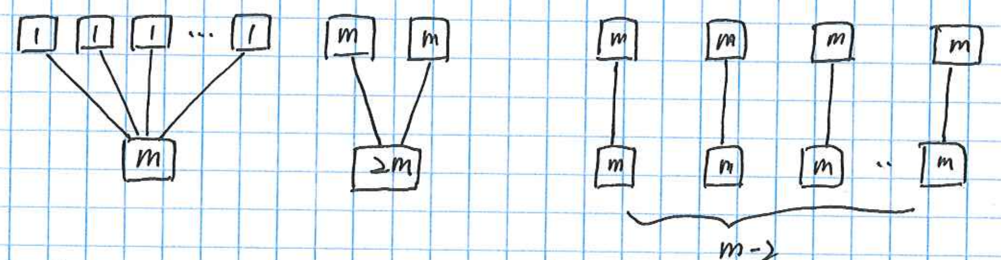
$$C(A^*) = \max_j L_j = m+1$$

we combine a 1 and a m together to be a pair:



For a Nash Equilibrium action

$$C(A^o) = 2m$$



Then we have

$$2 - \frac{2}{m+1} \geq PoA \geq \frac{C(A^o)}{C(A^*)} = \frac{2m}{m+1} = 2 - \frac{2}{m+1}$$

Therefore:

$$\boxed{PoA = 2 - \frac{2}{m+1}}$$