HW6: Auctions, Network Economics and Clickmaniac

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1 Routing games with tolls

Notation declaration:

- $\mathbf{f}$: flow on the rout graph; f_e: flow on edge e;
- $Cost(\mathbf{f}) = \sum_{e\in E} c_e(f_e)f_e$: global cost;
- $p \subset E$: path of a player; $u(p) = \sum_{e\in p} c_e^*(f_e)$: cost for a user with path p under the new game.

Now assume that $\mathbf{f}^o$ is an globally optimal strategy, i.e.

$$\mathbf{f}^o = \arg\min_{\mathbf{f}} \ Cost(\mathbf{f}) = \sum_{e\in E} c_e(f_e)f_e.$$

We need to show that $\mathbf{f}^o$ is also a Nash equilibrium of the new game. Indeed, staring from state $\mathbf{f}^o$, if one of the user changes his path from p to any arbitrary p', and consequently the global flow changes from $\mathbf{f}^o$ to $\mathbf{f}'$, then the change in the global cost would be

$$\begin{aligned}
Cost(\mathbf{f}') - Cost(\mathbf{f}^o) =& \sum_{e\in E} c_e(f'_e)f'_e - \sum_{e\in E} c_e(f^o_e)f^o_e \\
=& \sum_{e\in p'\backslash p} \left[c_e(f'_e)f'_e - c_e(f^o_e)f^o_e\right] + \sum_{e\in p\backslash p'} \left[c_e(f'_e)f'_e - c_e(f^o_e)f^o_e\right] \\
=& \sum_{e\in p'\backslash p} \left[c_e(f^o_e+1)(f^o_e+1) - c_e(f^o_e)f^o_e\right] \\
&+ \sum_{e\in p\backslash p'} \left[c_e(f^o_e-1)(f^o_e-1) - c_e(f^o_e)f^o_e\right] \\
=& \sum_{e\in p'\backslash p} \left[c_e(f^o_e+1) + f^o_e(c_e(f^o_e+1) - c_e(f^o_e))\right] \\
&- \sum_{e\in p\backslash p'} \left[c_e(f^o_e) + (f^o_e-1)(c_e(f^o_e) - c_e(f^o_e-1))\right] \\
=& \sum_{e\in p'\backslash p} c^*_e(f^o_e+1) - \sum_{e\in p\backslash p'} c^*_e(f^o_e) \\
=& \sum_{e\in p'\backslash p} c^*_e(f'_e) - \sum_{e\in p\backslash p'} c^*_e(f^o_e).
\end{aligned}$$

On the other hand, the change in the cost for this user would be

$$u(p') - u(p) = \sum_{e\in p'} c_e^*(f_e') - \sum_{e\in p} c_e^*(f_e^o) = \sum_{e\in p'\setminus p} c_e^*(f_e') - \sum_{e\in p\setminus p'} c_e^*(f_e^o).$$

Therefore, we have

$$u(p') - u(p) = Cost(\mathbf{f}') - Cost(\mathbf{f}^o)$$

Since $\mathbf{f}^o$ is an optimal strategy for global cost by definition, we have

$$Cost(\mathbf{f}') - Cost(\mathbf{f}^o) \geq 0,$$

that is

$$u(p') - u(p) \geq 0,$$

which concludes that $\mathbf{f}^o$ is a Nash equilibrium of the new game with tolls.
We can also see this result in that the potential function of the new game $\Phi^*(\mathbf{f})$ is exactly the global cost function $Cost(\mathbf{f})$ given that $r_i = 1$ for $\forall i$, i.e.

$$\begin{aligned}
\Phi^*(\mathbf{f}) &= \sum_{e\in E}\sum_{i=1}^{f_e} c_e^*(i) \\
&= \sum_{e\in E}\sum_{i=1}^{f_e} \left[c_e(i) + (i-1)(c_e(i) - c_e(i-1))\right] \\
&= \sum_{e\in E}\sum_{i=1}^{f_e} \left[ic_e(i) - (i-1)c_e(i-1)\right] \\
&= \sum_{e\in E} c_e(f_e)f_e \\
&= Cost(\mathbf{f}).
\end{aligned}$$

2 Nonatomic routing games

Part 1 - Existence of equilibria

(a) An equilibrium for nonatomic routing game denotes a state where for each commodity $i \in I$, all paths $p \in \{p \in P_i : f_p > 0\}$ share the same cost that is no more than the cost of any path $p \in \{p \in P_i : f_p = 0\}$. Therefore at an equilibrium, no part of the flow volume of each commodity $i \in I$ is willing to change path because it won't make the cost of its new path less than the original cost, since the cost function $c_e(\cdot)$ is nondecreasing.
A similar idea applies to an atomic routing game, in which a Nash equilibrium also denotes a state where no user is willing to change path because they won't get a better cost for doing so.

(b) We denote the feasible set as $S = \{f = (f_p, p \in P) \in \mathbb{R}_+^{|P|} : \sum_{p\in P_i} f_p = r_i, \forall i \in I\}$.
It is easy to see that the potential function

$$\Phi(f) = \sum_{e\in E} h_e(f_e) = \sum_{e\in E} \int_0^{f_e} c_e(y)dy$$

is continuous of each f_e, $e \in E$, and that each $f_e = \sum_{p:e\in p} f_p$, $e \in E$ is a continuous function of f, thus $\Phi(f)$ is a continuous function of f. Therefore, since S is a bounded set in a finite dimensional space, the minimum

$$\min_{f\in S} \Phi(f)$$

is achieved at some optimal point $f^o \in S$. Next, we will show that f^o is an equilibrium flow for the nonatomic routing game.

We prove this by contradiction. If f^o is not an equilibrium flow, then for this flow $\exists\ i \in I$ and $\exists\ p, p' \in P_i$ such that $f_p^o > 0$ and

$$\begin{aligned} c_p(f^o) - c_{p'}(f^o) &= \sum_{e\in p} c_e(f_e^o) - \sum_{e\in p'} c_e(f_e^o) \\ &= \sum_{e\in p\backslash p'} c_e(f_e^o) - \sum_{e\in p'\backslash p} c_e(f_e^o) \\ &> 0. \end{aligned}$$

Now we construct a new flow f' by moving a little amount of volume $\delta r \in (0, f_p^o)$ of commodity i from path p to path p'. We can do so because $f_p^o > 0$. Since $f'_p = f_p^o - \delta r$ and $f'_{p'} = f_{p'}^o + \delta r$, we still have

$$\sum_{p\in P_i} f'_p = \sum_{p\in P_i} f_p^o = r_i,$$

thus $f' \in S$. Further we have

$$f'_e = \begin{cases} f_e^o - \delta r, & e \in p\backslash p'; \\ f_e^o + \delta r, & e \in p'\backslash p; \\ f_e^o, & otherwise. \end{cases}$$

Therefore,

$$\begin{aligned} \Phi(f^o) - \Phi(f') =& \sum_{e\in E} \int_0^{f_e^o} c_e(y)dy - \sum_{e\in E} \int_0^{f'_e} c_e(y)dy \\ =& \sum_{e\in p\backslash p'} [\int_0^{f_e^o} c_e(y)dy - \int_0^{f'_e} c_e(y)dy] \\ &+ \sum_{e\in p'\backslash p} [\int_0^{f_e^o} c_e(y)dy - \int_0^{f'_e} c_e(y)dy] \\ =& \sum_{e\in p\backslash p'} \int_{f_e^o-\delta r}^{f_e^o} c_e(y)dy - \sum_{e\in p'\backslash p} \int_{f_e^o}^{f_e^o+\delta r} c_e(y)dy. \end{aligned}$$

Since $c_e(\cdot)$ is continuous, we have

$$\int_{f_e^o-\delta r}^{f_e^o} c_e(y)dy = c_e(f_e^o)\delta r + o(\delta r); \quad \int_{f_e^o}^{f_e^o+\delta r} c_e(y)dy = c_e(f_e^o)\delta r + o(\delta r).$$

Then we have

$$
\begin{aligned}
\Phi(f^o) - \Phi(f') &= \sum_{e\in p\backslash p'} \int_{f_e^o-\delta r}^{f_e^o} c_e(y)dy - \sum_{e\in p'\backslash p} \int_{f_e^o}^{f_e^o+\delta r} c_e(y)dy \\
&= \sum_{e\in p\backslash p'} \big[c_e(f_e^o)\delta r + o(\delta r)\big] - \sum_{e\in p'\backslash p} \big[c_e(f_e^o)\delta r + o(\delta r)\big] \\
&= \Big[\sum_{e\in p\backslash p'} c_e(f_e^o) - \sum_{e\in p'\backslash p} c_e(f_e^o)\Big]\delta r + o(\delta r) \\
&= \big[c_p(f^o) - c_{p'}(f^o)\big]\delta r + o(\delta r).
\end{aligned}
$$

Since $c_p(f^o) - c_{p'}(f^o) > 0$ by assumption, when we take δr sufficiently small, we will also have $\Phi(f^o) - \Phi(f') > 0$. Since $f' \in S$, this result contradicts the fact that f^o is the minimizer of $\Phi(f)$ in S.
Therefore, f^o must be an equilibrium flow for our nonatomic routing game, which means an equilibrium flow always exists.

(c) Since $c_e(\cdot)$ is continuous, we have $h_e'(x) = c_e(x)$. Then for any $x_2 \ge x_1 \ge 0$, and any $\lambda \in [0,1]$, we have

$$
\begin{aligned}
&\lambda h_e(x_1) + (1-\lambda)h_e(x_2) - h_e(\lambda x_1 + (1-\lambda)x_2) \\
=&(1-\lambda)\big[h_e(x_2) - h_e(\lambda x_1 + (1-\lambda)x_2)\big] - \lambda\big[h_e(\lambda x_1 + (1-\lambda)x_2) - h_e(x_1)\big] \\
=&(1-\lambda)\lambda(x_2 - x_1)h_e'(\xi) - (1-\lambda)\lambda(x_2 - x_1)h_e'(\eta) \\
=&(1-\lambda)\lambda(x_2 - x_1)(c_e(\xi) - c_e(\eta)) \\
\ge& 0,
\end{aligned}
$$

for some $\xi \in (\lambda x_1 + (1-\lambda)x_2, x_2)$ and $\eta \in (x_1, \lambda x_1 + (1-\lambda)x_2)$, because $c_e(\cdot)$ is nondecreasing. Therefore $h_e(f_e)$ is convex of f_e. Moveover, $h_e(f_e)$ is convex of f since each f_e is a linear combination of $f_p,\ p \in P$. Then $\Phi(f)$ is convex of f because $\Phi(f)$ is a linear combination of all $h_e(f_e),\ e \in E$.
Another fact we will use is that a feasible flow is an equilibrium flow if and only if it minimizes $\Phi(f)$ over all feasible flows.
Still, let S denote the set of all feasible flows, and let

$$\Phi^* = \min_{f\in S} \Phi(f).$$

Now assume that $f, f' \in S$ are two equilibrium flows, then

$$\Phi(f) = \Phi(f') = \Phi^*.$$

Since $\Phi(\cdot)$ is convex, for any $\lambda \in [0,1]$, we have

$$\Phi(\lambda f + (1-\lambda)f') \le \lambda\Phi(f) + (1-\lambda)\Phi(f') = \Phi^*.$$

It is easy to check that S is a bounded, convex set, thus $\lambda f + (1-\lambda)f' \in S$, and

$$\Phi(\lambda f + (1-\lambda)f') \ge \Phi^*.$$

Therefore we indeed have

$$\Phi(\lambda f + (1-\lambda)f') = \Phi^*, \quad \lambda \in [0,1].$$

Then using the fact that $c_e(\cdot),\ e \in E$ is continuous, for $\lambda \in [0,1]$ we have

$$\begin{aligned}
0 &= \frac{d}{d\lambda}\Phi(\lambda f + (1-\lambda)f') \\
&= \frac{d}{d\lambda}\sum_{e\in E} h_e(\lambda f_e + (1-\lambda)f'_e) \\
&= \sum_{e\in E}\frac{d}{d\lambda}\int_0^{\lambda f_e+(1-\lambda)f'_e} c_e(y)dy \\
&= \sum_{e\in E}(f_e - f'_e)c_e(\lambda f_e + (1-\lambda)f'_e).
\end{aligned}$$

In particular, for $\lambda = 0, 1$ we have

$$\sum_{e\in E}(f_e - f'_e)c_e(f'_e) = 0,$$

$$\sum_{e\in E}(f_e - f'_e)c_e(f_e) = 0.$$

By subtracting we get

$$\sum_{e\in E}(f_e - f'_e)\big[c_e(f_e) - c_e(f'_e)\big] = 0.$$

Since $c_e(\cdot)$ is nondecreasing, we have $(f_e - f'_e)\big[c_e(f_e) - c_e(f'_e)\big] \geq 0,\ \forall e \in E$. Therefore the equation above actually implies

$$(f_e - f'_e)\big[c_e(f_e) - c_e(f'_e)\big] = 0, \quad \forall e \in E,$$

which is apparently equivalent to

$$c_e(f_e) - c_e(f'_e) = 0, \quad \forall e \in E.$$

Moreover we have

$$c_p(f) = \sum_{e\in p} c_e(f_e) = \sum_{e\in p} c_e(f'_e) = c_p(f').$$

Then by the definition of equilibrium flow, for each $i \in I$, we have

$$c_p(f) = c_p(f') = D_i,\ \forall p \in P_i \ s.t.\ f_p > 0 \ or \ f'_p > 0,$$

for some constant D_i. Therefore we have

$$C(f) = \sum_{p\in P} f_p c_p(f) = \sum_{i\in I}\sum_{p\in P_i, f_p>0} f_p c_p(f) = \sum_{i\in I} D_i \sum_{p\in P_i, f_p>0} f_p = \sum_{i\in I} D_i r_i,$$

$$C(f') = \sum_{p\in P} f'_p c_p(f') = \sum_{i\in I}\sum_{p\in P_i, f'_p>0} f'_p c_p(f') = \sum_{i\in I} D_i \sum_{p\in P_i, f'_p>0} f'_p = \sum_{i\in I} D_i r_i.$$

That is $C(f) = C(f')$.

Part 2 - Efficiency of equilibria

(a) For any $a, b \leq 0$, let $c(x) = ax + b, x \geq 0$, then

$$\begin{aligned}
\frac{rc(r)}{xc(x) + (r-x)c(r)} &= \frac{ar^2 + br}{ar^2 + br + ax^2 - axr} \\
&= \frac{ar^2 + br}{\frac{3}{4}ar^2 + br + a(x - \frac{1}{2}r)^2} \\
&\leq \frac{ar^2 + br}{\frac{3}{4}ar^2 + br} \\
&\leq \frac{ar^2 + br}{\frac{3}{4}ar^2 + \frac{3}{4}br} \\
&= \frac{4}{3}.
\end{aligned}$$

Therefore

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{x, r \geq 0} \frac{rc(r)}{xc(x) + (r-x)c(r)} \leq \frac{4}{3}.$$

In particular, when we take $a = 1, b = 0, r = 1, x = \frac{1}{2}$, we have

$$\frac{rc(r)}{xc(x) + (r-x)c(r)} = \frac{r^2}{r^2 + x^2 - xr} = \frac{4}{3},$$

which implies

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{x, r \geq 0} \frac{rc(r)}{xc(x) + (r-x)c(r)} \geq \frac{4}{3}.$$

Thus in all we have $\alpha(\mathcal{C}) = \frac{4}{3}$.

(b) In this game, there are three paths

$$p_1 : s \to u \to d, \qquad p_2 : s \to v \to d, \qquad p_3 : s \to u \to v \to d.$$

Let $f_i \in [0, 1]$ be the volume allocation on path p_i, $i = 1, 2, 3$. Then it's easy to check that

$$C(f) = (f_1 + f_3)^2 + f_1 + f_2 + (f_2 + f_3)^2.$$

Given the condition that $f_1 + f_2 + f_3 = 1$, we have

$$\begin{aligned}
C(f) &= (f_1 + f_3)^2 + f_1 + f_2 + (f_2 + f_3)^2 \\
&= (1 - f_2)^2 + (1 - f_1)^2 + f_1 + f_2 \\
&= (f_1 - \frac{1}{2})^2 + (f_2 - \frac{1}{2})^2 + \frac{3}{2} \\
&\geq \frac{3}{2}.
\end{aligned}$$

The last inequality becomes equality when $f_1 = f_2 = \frac{1}{2}$, thus

$$C(f^*) = \frac{3}{2}.$$

As claimed before, $f^E = (f_1, f_2, f_3) = (0, 0, 1)$ is the unique equilibrium flow for this game, thus

$$C(f^E) = 2.$$

Then we have

$$\text{PoA} = \frac{C(f^E)}{C(f^*)} = \frac{4}{3}.$$

(c) Since by definition

$$\alpha(\mathcal{C}) = \sup_{c\in\mathcal{C}} \sup_{x,r\geq 0} \frac{rc(r)}{xc(x) + (r-x)c(r)},$$

for any $\epsilon > 0$, there exists some $c(\cdot) \in \mathcal{C}$ and some $x, r \geq 0$ such that

$$\alpha(\mathcal{C}) \geq \frac{rc(r)}{xc(x) + (r-x)c(r)} > \alpha(\mathcal{C}) - \epsilon.$$

Now fix $c(\cdot), x, r$. For this game, let $c_1(t) = c(t) \in \mathcal{C}$, $c_2(t) = c(r) \in \mathcal{C}$(here $c(r)$ is a constant), $r_1 = r$, and let $f = (t, r-t)$, $t \in [0, r]$ be a feasible flow on the graph. Then since $c(\cdot)$ is nondecreasing, we have

$$c_1(t) = c(t) \leq c(r) = c_2(t), \quad \forall t \in [0, r].$$

That is, path 2 is dominated by path 1. Thus $f^E = (r, 0)$ is a equilibrium flow, and

$$C(f^E) = rc_1(r) = rc(r).$$

Further, let $f^* = (t^*, r - t^*)$ be an optimal flow. If $x \in [0, r]$, then $f = (x, r-x)$ is a feasible flow, and we have

$$C(f) = xc_1(x) + (r-x)c_2(r-x) = xc(x) + (r-x)c(r) \geq C(f^*),$$

because f^* is optimal by definition. If $x > r$, then since $c(\cdot)$ is nondecreasing, we still have

$$xc(x) + (r-x)c(r) = x(c(x) - c(r)) + rc(r) \geq rc(r) = C(f^E) \geq C(f^*).$$

Therefore, we must have

$$C(f^*) \leq xc(x) + (r-x)c(r).$$

Then we have

$$\alpha(\mathcal{C}) \geq PoA = \frac{C(f^E)}{C(f^*)} \geq \frac{rc(r)}{xc(x) + (r-x)c(r)} > \alpha(\mathcal{C}) - \epsilon.$$

Since ϵ is arbitrary, PoA can be arbitrary close to $\alpha(\mathcal{C})$.

(d) We can also come the same result of (c) with a relaxed assumption that $\mathcal{C}$ includes all rational constant cost functions.
Again, for any $\epsilon > 0$, there exists some $c(\cdot) \in \mathcal{C}$ and some $x, r \geq 0$ such that

$$\alpha(\mathcal{C}) \geq \frac{rc(r)}{xc(x) + (r-x)c(r)} > \alpha(\mathcal{C}) - \epsilon.$$

Since rational numbers are dense in $\mathbb{R}$, there exists some small enough $\eta \geq 0$ and some rational number $s \in [c(r), c(r) + \eta]$ such that

$$\frac{rc(r)}{xc(x) + (r - x)s + \eta x} > \alpha(\mathcal{C}) - \epsilon,$$

Now fix $c(\cdot), x, r, s, \eta$. For this game, let $c_1(t) = c(t) \in \mathcal{C}$, $c_2(t) = s \in \mathcal{C}$, $r_1 = r$, and let $f = (t, r - t)$, $t \in [0, r]$ be a feasible flow on the graph. Then since $c(\cdot)$ is nondecreasing, we have

$$c_1(t) = c(t) \leq c(r) \leq s = c_2(t), \quad \forall t \in [0, r].$$

That is, path 2 is dominated by path 1. Thus $f^E = (r, 0)$ is a equilibrium flow, and

$$C(f^E) = rc_1(r) = rc(r).$$

Further, let $f^* = (t^*, r - t^*)$ be an optimal flow. If $x \in [0, r]$, then $f = (x, r - x)$ is a feasible flow, and we have

$$C(f) = xc_1(x) + (r - x)c_2(r - x) = xc(x) + (r - x)s \geq C(f^*),$$

because f^* is optimal by definition. If $x > r$, then since $c(\cdot)$ is nondecreasing, we have $s - c(x) \leq s - c(r) \leq \eta$, and thus

$$\begin{aligned}
C(f^*) - \big(xc(x) + (r - x)s\big) &\leq C(f^E) - \big(xc(x) + (r - x)s\big) \\
&= rc(r) - \big(xc(x) + (r - x)s\big) \\
&= r(c(r) - s) + x(s - c(x)) \\
&\leq x(s - c(x)) \\
&\leq \eta x.
\end{aligned}$$

Therefore, we must have

$$C(f^*) \leq xc(x) + (r - x)s + \eta x.$$

Then we have

$$\alpha(\mathcal{C}) \geq PoA = \frac{C(f^E)}{C(f^*)} \geq \frac{rc(r)}{xc(x) + (r - x)s + \eta x} > \alpha(\mathcal{C}) - \epsilon.$$

Since ϵ is arbitrary, PoA can be arbitrary close to $\alpha(\mathcal{C})$.

(e) Let f be an equilibrium flow, then by definition, for each $i \in I$, there exits a constant D_i such that for $p \in P_i$

$$c_p(f) = D_i, \text{ if } f_p > 0; \qquad c_p(f) \geq D_i, \text{ if } f_p = 0.$$

Then for any feasible flow f', we have

$$\begin{aligned}
\sum_{e\in E} f'_e c_e(f_e) &= \sum_{e\in E} c_e(f_e) \sum_{p:e\in p} f'_p \\
&= \sum_{p\in P} f'_p \sum_{e\in p} c_e(f_e) \\
&= \sum_{p\in P} f'_p c_p(f) \\
&= \sum_{i\in I} \sum_{p\in P_i} f'_p c_p(f) \\
&\geq \sum_{i\in I} \sum_{p\in P_i} f'_p D_i \\
&= \sum_{i\in I} r_i D_i.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\sum_{e\in E} f_e c_e(f_e) &= \sum_{p\in P} f_p c_p(f) \\
&= \sum_{i\in I} \sum_{p\in P_i, f_p>0} f_p c_p(f) \\
&= \sum_{i\in I} \sum_{p\in P_i, f_p>0} f_p D_i \\
&= \sum_{i\in I} r_i D_i.
\end{aligned}$$

Therefore

$$\sum_{e\in E}(f_e - f'_e)c_e(f_e) = \sum_{e\in E} f_e c_e(f_e) - \sum_{e\in E} f'_e c_e(f_e) \leq 0.$$

Now let f^* be an optimal flow, then we have

$$\sum_{e\in E}(f_e - f^*_e)c_e(f_e) \leq 0.$$

and

$$\begin{aligned}
\sum_{e\in E} f^*_e c_e(f^*_e) &\geq \sum_{e\in E} f^*_e c_e(f^*_e) + \sum_{e\in E}(f_e - f^*_e)c_e(f_e) \\
&= \sum_{e\in E} \left[f_e c_e(f^*_e) + (f_e - f^*_e)(c_e(f_e) - c_e(f^*_e)) \right] \\
&\geq 0.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\text{PoA} &= \frac{C(f)}{C(f^*)} \\
&= \frac{\sum_{e\in E} f_e c_e(f_e)}{\sum_{e\in E} f_e^* c_e(f_e^*)} \\
&\le \frac{\sum_{e\in E} f_e c_e(f_e)}{\sum_{e\in E} f_e^* c_e(f_e^*) + \sum_{e\in E}(f_e - f_e^*)c_e(f_e)} \\
&= \frac{\sum_{e\in E} f_e c_e(f_e)}{\sum_{e\in E} \left[f_e^* c_e(f_e^*) + (f_e - f_e^*)c_e(f_e)\right]} \\
&\le \max_{e\in E} \frac{f_e c_e(f_e)}{f_e^* c_e(f_e^*) + (f_e - f_e^*)c_e(f_e)} \\
&= \frac{f_{\tilde{e}} c_{\tilde{e}}(f_{\tilde{e}})}{f_{\tilde{e}}^* c_{\tilde{e}}(f_{\tilde{e}}^*) + (f_{\tilde{e}} - f_{\tilde{e}}^*)c_{\tilde{e}}(f_{\tilde{e}})},
\end{aligned}$$

where

$$\tilde{e} = \arg\max_{e\in E} \frac{f_e c_e(f_e)}{f_e^* c_e(f_e^*) + (f_e - f_e^*)c_e(f_e)}.$$

Since $c_{\tilde{e}} \in \mathcal{C}$, we thus have

$$\begin{aligned}
\text{PoA} &\le \frac{f_{\tilde{e}} c_{\tilde{e}}(f_{\tilde{e}})}{f_{\tilde{e}}^* c_{\tilde{e}}(f_{\tilde{e}}^*) + (f_{\tilde{e}} - f_{\tilde{e}}^*)c_{\tilde{e}}(f_{\tilde{e}})} \\
&\le \sup_{c\in\mathcal{C}} \sup_{x,r\ge 0} \frac{rc(r)}{xc(x) + (r-x)c(r)} \\
&= \alpha(\mathcal{C})
\end{aligned}$$

3 Tightness of PoA bound in load balancing games

For each $m \ge 1$, consider a load balancing game with m servers and $n = 2m$ jobs. Among the $2m$ jobs, m of them have demand 1 and the other m of them have demand m, that is $p_j = 1, j = 1, 2, \cdots, m$, and $p_j = m, j = m+1, m+2, \cdots, 2m$.
For any optimal action A^*, we must have

$$C(A^*) \ge \frac{\sum_{j=1}^{2m} p_j}{m} = \frac{m + m^2}{m} = m + 1.$$

Figure 1 gives an action A such that $C(A) = m + 1$. Therefore we must have

$$C(A^*) = m + 1.$$

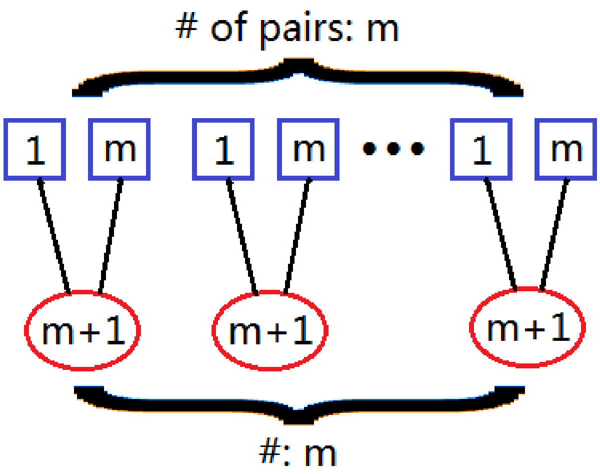

Figure 1: $C(A^*) = m + 1$.

Figure 2 gives an Nash equilibrium action A^{NE} because no job is willing to move to another server. For this Nash equilibrium, we have

$$C(A^{NE}) = 2m.$$

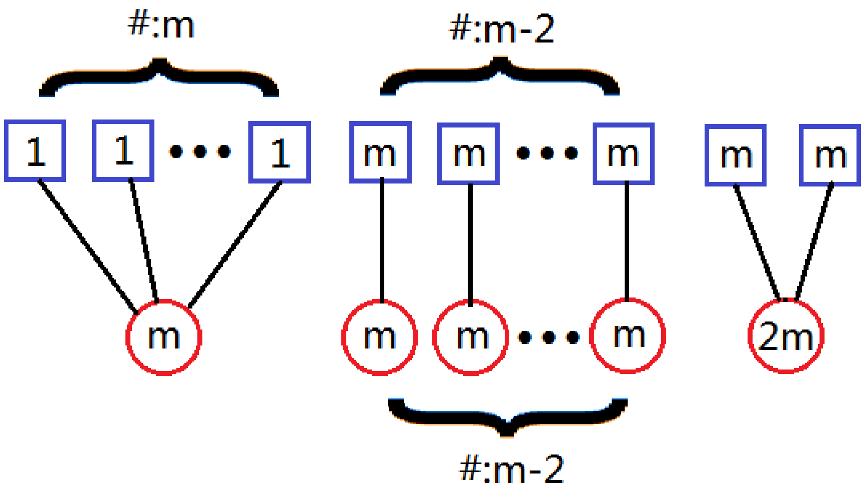

Figure 2: $C(A^{NE}) = 2m$.

Then we have

$$2 - \frac{2}{m+1} \geq \text{PoA} \geq \frac{C(A^{NE})}{C(A^*)} = \frac{2m}{m+1} = 2 - \frac{2}{m+1}.$$

That is to say, for this game we exactly have

$$\text{PoA} = 2 - \frac{2}{m+1}.$$

4 PoA analysis for GSP

In what follows, we only consider pure strategies, and NE only denotes pure Nash equilibria.

(a) For n=2, consider an auction with $\alpha_1 = X, \alpha_2 = 1$ and $v_1 = X, v_2 = 1$, where $X > 1$. Let $b = (b_1, b_2) = (0, X)$ be a biding action for advertiser 1 and 2. We will prove that this biding is a Nash equilibrium for this game.
Indeed, under this biding $b = (b_1, b_2) = (0, X)$, we have $\pi(b) = (2, 1)$, and

$$U_1(b) = \alpha_2 v_1 = X, \qquad U_2(b) = \alpha_1(v_2 - b_1) = X.$$

- If we fix $b_2 = X$, then for any other $b_1' \geq 0$, $U_1(b_1', b_2)$ only has two possible values, $\alpha_1(v_1 - b_2) = 0$ or $\alpha_2 v_1 = X$, either of which is no more that $U_1(b_1, b_2) = X$.
- If we fix $b_1 = 0$, then for any other $b_2' \geq 0$, $U_2(b_1, b_2')$ only has two possible values, $\alpha_1(v_2 - b_1) = X$ or $\alpha_2 v_2 = 1$, either of which is no more that $U_2(b_1, b_2) = X$.

Therefore, $b = (b_1, b_2) = (0, X)$ is a Nash equilibrium for this game, and we have

$$W(b) = \alpha_1 v_2 + \alpha_2 v_1 = 2X.$$

Since for any b', we have either $\pi(b') = (1, 2)$ or $\pi(b') = (2, 1)$, the social welfare can only have two possible values

$$W(b') = \begin{cases} \alpha_1 v_1 + \alpha_2 v_2 = X^2 + 1, & \pi(b') = (1, 2); \\ \alpha_1 v_2 + \alpha_2 v_1 = 2X, & \pi(b') = (2, 1). \end{cases}$$

Given $X > 1$, we have $X^2 + 1 > 2X$, therefore

$$\min_{b' \in NE} W(b') = W(b) = 2X.$$

It is easy to check that

$$W(b^*) = \alpha_1 v_1 + \alpha_2 v_2 = X^2 + 1.$$

Therefore

$$\text{PPoA} = \frac{W(b^*)}{\min_{b' \in NE} W(b')} = \frac{W(b^*)}{W(b)} = \frac{X^2 + 1}{2X}.$$

Then for arbitrary $r > 1$, we can always find a $X > 1$ so that

$$\text{PPoA} = \frac{X^2 + 1}{2X} = r.$$

(b) (i) Any $b_1 \neq X - 1$ is a dominated strategy for advertiser 1.

- If $b_1 < X - 1$, then $\exists b_1'$ such that $X - 1 > b_1' > b_1$, and

$$U_1(b_1', b_2) - U_1(b_1, b_2) = \begin{cases} 0, & b_2 < b_1; \\ X(X - 1 - b_2) > 0, & b_1 < b_2 < b_1' < X - 1; \\ 0, & b_2 > b_1'. \end{cases}$$

Therefore b_1 is a dominated strategy.

- If $b_1 > X-1$, then $\exists b_1'$ such that $X-1 < b_1' < b_1$, and

$$U_1(b_1', b_2) - U_1(b_1, b_2) = \begin{cases} 0, & b_2 > b_1; \\ X(b_2 - X + 1) > 0, & X-1 < b_1' < b_2 < b_1; \\ 0, & b_2 < b_1'. \end{cases}$$

  Therefore b_1 is a dominated strategy.

Any $b_2 \neq 1 - 1/X$ is a dominated strategy for advertiser 2.

- If $b_2 < 1 - 1/X$, then $\exists b_2'$ such that $1 - 1/X > b_2' > b_2$, and

$$U_1(b_1, b_2') - U_1(b_1, b_2) = \begin{cases} 0, & b_1 < b_2; \\ X - Xb_1 - 1 > 0, & b_2 < b_1 < b_2' < 1 - 1/X; \\ 0, & b_1 > b_2'. \end{cases}$$

  Therefore b_2 is a dominated strategy.

- If $b_2 > 1 - 1/X$, then $\exists b_2'$ such that $X-1 < b_2' < b_2$, and

$$U_1(b_1, b_2') - U_1(b_1, b_2) = \begin{cases} 0, & b_1 > b_2; \\ 1 + Xb_1 - X > 0, & X-1 < b_2' < b_1 < b_2; \\ 0, & b_1 < b_2'. \end{cases}$$

  Therefore b_2 is a dominated strategy.

(ii) Assume that $b_i > v_i$, we will show that b_i is a dominated strategy for advertiser i. In what follows, we set $b_{\pi_{n+1}(b)} = 0$ for consistency.
Indeed, since $b_i > v_i$, there exists some b_i' such that $b_i > b_i' > v_i$. Let $b = (b_i, b_{-i})$ and $b' = (b_i', b_{-i})$. Then for arbitrary b_{-i}, we have

- If $\pi(b) = \pi(b')$, assume that $\pi_j(b) = \pi_j(b') = i$, then

$$b_{\pi_{j+1}(b)} = b_{\pi_{j+1}(b')} = b'_{\pi_{j+1}(b')},$$

  and we have

$$\begin{aligned} U_i(b_i, b_{-i}) &= \alpha_j(v_i - b_{\pi_{j+1}(b)}) \\ &= \alpha_j(v_i - b'_{\pi_{j+1}(b')}) \\ &= U_i(b_i', b_{-i}). \end{aligned}$$

- If $\pi(b) \neq \pi(b')$, assume that $\pi_j(b) = \pi_k(b') = i$, then $j < k$ since $b_i > b_i'$, and we must have

$$b_{\pi_{j+1}(b)} \geq b_i' > v_i, \qquad b_{\pi_{j+1}(b)} \geq b_i' \geq b'_{\pi_{k+1}(b')}$$

  Since $\alpha_j \geq \alpha_k > 0$, we have

$$\alpha_j(v_i - b_{\pi_{j+1}(b)}) \leq \alpha_k(v_i - b_{\pi_{j+1}(b)}) \leq \alpha_k(v_i - b'_{\pi_{k+1}(b')}),$$

  that is

$$U_i(b_i, b_{-i}) = \alpha_j(v_i - b_{\pi_{j+1}(b)}) \leq \alpha_k(v_i - b'_{\pi_{k+1}(b')}) = U_i(b_i', b_{-i}).$$

  In particular, if $b_{\pi_{j+1}(b)} > b'_{\pi_{k+1}(b')}$, which can happen, then

$$\alpha_j(v_i - b_{\pi_{j+1}(b)}) \leq \alpha_k(v_i - b_{\pi_{j+1}(b)}) < \alpha_k(v_i - b'_{\pi_{k+1}(b')}),$$

  and we have

$$U_i(b_i, b_{-i}) = \alpha_j(v_i - b_{\pi_{j+1}(b)}) < \alpha_k(v_i - b'_{\pi_{k+1}(b')}) = U_i(b_i', b_{-i}).$$

Therefore, b_i is a dominated strategy for advertiser i.

(iii) For a game with $n = 2$, the social welfare with under pure strategy can only have two possible values,

$$\alpha_1 v_1 + \alpha_2 v_2 \quad \text{or} \quad \alpha_1 v_2 + \alpha_2 v_1.$$

If $\alpha_1 = \alpha_2$ or $v_1 = v_2$, then we always have

$$\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 v_2 + \alpha_2 v_1.$$

Thus

$$\text{PPoA} = \frac{W(b^*)}{\min_{b \in NE} W(b)} = 1.$$

Next, without loss of generality, we may assume that $\alpha_1 > \alpha_2 \geq 0$ and $v_1 > v_2 \geq 0$. Let b be worst pure Nash equilibrium. In order for PPoA to be greater than 1, we need

$$W(b) = \alpha_1 v_2 + \alpha_2 v_1 < \alpha_1 v_1 + \alpha_2 v_2 = W(b^*),$$

that is $\pi(b) = (2, 1)$. Then b is a Nash equilibrium implies that

$$U_1(b) = \alpha_2 v_1 \geq \alpha_1 (v_1 - b_2),$$

$$U_2(b) = \alpha_1 (v_2 - b_1) \geq \alpha_2 v_2.$$

That is

$$b_2 \geq \frac{\alpha_1 - \alpha_2}{\alpha_1} v_1 > \frac{\alpha_1 - \alpha_2}{\alpha_1} v_2 \geq b_1.$$

Since both advertisers are conservative, we have $v_1 \geq b_1$ and $v_2 \geq b_2$. In all we have

$$v_1 > v_2 \geq b_2 \geq \frac{\alpha_1 - \alpha_2}{\alpha_1} v_1 > \frac{\alpha_1 - \alpha_2}{\alpha_1} v_2 \geq b_1,$$

Let $\eta = v_2 / v_1$, then

$$1 > \eta \geq \frac{\alpha_1 - \alpha_2}{\alpha_1}.$$

Then we have

$$\text{PPoA} = \frac{W(b^*)}{W(b)} = \frac{\alpha_1 v_1 + \alpha_2 v_2}{\alpha_1 v_2 + \alpha_2 v_1} = \frac{\alpha_1 + \alpha_2 \eta}{\alpha_2 + \alpha_1 \eta}.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \frac{\alpha_1 + \alpha_2 \eta}{\alpha_2 + \alpha_1 \eta} = \frac{\alpha_2}{\alpha_2 + \alpha_1 \eta} - \frac{\alpha_1^2 + \alpha_1 \alpha_2 \eta}{(\alpha_2 + \alpha_1 \eta)^2} = \frac{\alpha_2^2 - \alpha_1^2}{(\alpha_2 + \alpha_1 \eta)^2} \leq 0,$$

we have

$$\text{PPoA} = \frac{\alpha_1 + \alpha_2 \eta}{\alpha_2 + \alpha_1 \eta} \leq \frac{\alpha_1 + \alpha_2 \eta}{\alpha_2 + \alpha_1 \eta}\bigg|_{\eta = \frac{\alpha_1 - \alpha_2}{\alpha_1}} = 1 + \frac{\alpha_2}{\alpha_1} - \left(\frac{\alpha_2}{\alpha_1}\right)^2.$$

Since $\alpha_1 > \alpha_2 \geq 0$, we have

$$\frac{\alpha_2}{\alpha_1} - \left(\frac{\alpha_2}{\alpha_1}\right)^2 = \frac{\alpha_2}{\alpha_1}\left(1 - \frac{\alpha_2}{\alpha_1}\right) \leq \frac{1}{4}.$$

Therefore

$$\text{PPoA} \leq 1 + \frac{1}{4} = 1.25.$$

In particular, when $\alpha_1 = 2, \alpha_2 = 1$ and $v_1 = 2, v_2 = 1$, one of the worst Nash equilibria is $b = (0, 1)$, and we have

$$W(b) = \alpha_1 v_2 + \alpha_2 v_1 = 4, \qquad W(b^*) = \alpha_1 v_1 + \alpha_2 v_2 = 5.$$

Therefore we exactly have

$$\text{PPoA} = \frac{W(b^*)}{W(b)} = \frac{5}{4} = 1.25.$$