MTH240 Sequence Feb 7, 2018 Lei Yu

A sequence can be thought of as a list of number in a defined order.

$\{a_1, a_2, a_3, \dots a_n\}$ infinity sequence

It can also be denoted as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

For example $\{\sqrt{n-3}\}_{n=3}^{\infty} = \{0, 1, \sqrt{2}, \sqrt{3}, \dots\}$

If $\lim\limits_{n\to\infty} a_n = L$, then L is called the limit of the sequence.

So $\{a_n\}$ is convergent if $(\lim\limits_{n\to\infty} a_n)$ exists

Divergent if $(\lim\limits_{n\to\infty} a_n)$ doesn't exist

Theorem: If $\lim\limits_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for n being an interger, then $\lim\limits_{n\to\infty} a_n = L$.

Ex. $\lim\limits_{n\to\infty} \frac{1}{n^r} = \lim\limits_{x\to\infty} \frac{1}{x^r} = 0$

If $\{a_n\}$ & $\{b_n\}$ are convergent & C is a constant, then

$\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n$

$\lim a_n^p = (\lim\limits_{n\to\infty} a_n)^p \quad a_n > 0, \; P > 0$

$\lim (a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim b_n)$

$\lim\limits_{n\to\infty} C a_n = C \lim\limits_{n\to\infty} a_n$

$\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim a_n}{\lim b_n}$, if $\lim b_n \neq 0$

Squeeze Theorem. If $a_n \leq b_n \leq c_n$

& $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L \Rightarrow b_n = L$

✱ Theorem If $\lim\limits_{n\to\infty} |a_n| = 0 \Rightarrow \lim\limits_{n\to\infty} a_n = 0$

Ex. $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{x\to\infty} \frac{x}{x+1} = 1$

Ex. $\lim_{n\to\infty} \frac{n}{\sqrt{n+10}} = \lim_{x\to\infty} \frac{x}{\sqrt{x+10}} = \lim_{x\to\infty} \sqrt{x} = \infty$

Ex. $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{LH}{=} \lim_{x\to\infty} \frac{1}{x} = 0$

Ex. $\lim_{n\to\infty} \frac{(-1)^n}{n} = \lim_{x\to\infty} \frac{(-1)^x}{x} = 0$

we know $\lim_{x\to\infty} \frac{1^x}{x} = 0$

Theorem: If $\lim_{n\to\infty} a_n = L$ and F is continuous at L, then $\lim_{n\to\infty} F(a_n) = L$

Ex. $\lim_{n\to\infty} \sin(\frac{\pi}{n}) = \sin(\lim_{n\to\infty}(\frac{\pi}{n})) = \sin 0 = 0$

Ex. Investigate the convergence of $\{r^n\}$

We know that $\lim_{n\to\infty} r^x = \infty$, $r>1$

also $\lim_{x\to\infty} r^x = 0$, $0<r<1$

$\Rightarrow \lim_{n\to\infty} r^n \begin{cases} \infty & r>1 \\ 0 & 0<r<1 \end{cases}$

$r=0$ $\lim_{x\to\infty} 0^x = 0$

$r=1$ $\lim_{x\to\infty} 1^x = 1$

We also know that $-1<r<0 \Rightarrow 0<|r|<1$

$\lim_{n\to\infty} |r^n| = 0 \Rightarrow \lim_{n\to\infty} r^n = 0$, $-1<r<0$

$r=-1 \Rightarrow \lim_{n\to\infty} (-1)^n$ doesn't exist.

$r=-1 \Rightarrow \lim_{n\to\infty} r^n = \lim_{n\to\infty} (-1)^n (r)^n$, doesn't exis

Conclusion:

$\lim_{n\to\infty} r^n \begin{cases} 0 & -1<r<1 \\ 1 & r=1 \\ \text{Divergent for all other } r \end{cases}$

A sequence is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$

decreasing if $a_n > a_{n+1}$ for all $n \geq 1$

monotonic if it's either increasing or decreasing

Ex. Show that this sequence $\{\frac{n}{n^2+1}\}$ is decreasing

1) $a_n > a_{n+1} \Rightarrow \frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1}$

$\cancel{n^3} + \cancel{n} + 2n^2 + n > \cancel{n^3} + \cancel{n} + n^2 + 1$

$n^2 + n > 1$, $1<n<\infty$

2) equivalent function $f(x) = \frac{x}{x^2+1}$

$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$

$x>1 \Rightarrow f'(x) < 0$

$f(x)$ is decreasing for $x>1$

so $\{a_n\}$ decreasing for $n>1$

MTH240 Sequence Feb 7 2018 Lei Yu

Definition: A sequence $\{a_n\}$ is bounded above if there is number M such that $a_n \le M$ for $n \ge 1$
and bounded below if there is a number m such that $m \le a_n$ for $n \ge 1$.

If $\{a_n\}$ is bounded below and above. It's called a bounded sequence.

Theorem: Every bounded, monotonic sequence is convergent
Note: a sequence may be bounded but not convergent
e.g. $\{(-1)^n\}$

Ex. $\{a_n\} = \left\{\frac{\cos^2 n}{2^n}\right\}$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\cos^2 x}{2^x} = 0$$

Ex. $\{n \sin \frac{1}{n}\}$

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{n^{-1}} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{x^{-1}} \overset{L.H}{=} \lim_{x\to\infty} \frac{\cancel{\frac{-1}{x^2}} \cos\frac{1}{x}}{\cancel{\frac{-1}{x^2}}} = 1$$

Ex. $\int \frac{\sqrt{w}}{w-1}\,dw = \int \frac{2t^2 dt}{t^2-1} = 2\int \frac{t^2}{t^2-1}\,dt = 2\int \frac{t^2-1}{t^2-1}\,dt + \frac{1}{t^2-1}\,dt$

$\sqrt{w} = t$
$\frac{1}{2\sqrt{w}}dw = dt$
$dw = 2t\,dt$

$= 2\int \frac{t^2-1}{t^2-1}\,dt + 2\int \frac{1}{t^2-1}\,dt = 2t + 2\int \left(\frac{A}{t-1} + \frac{B}{t+1}\right) dt$ $A = 1$, $B = 1$

$= 2t + 2\int \frac{1}{t-1}\,dt \ 2\int \frac{1}{t+1}\,dt = 2t + 2\ln|t-1| - 2\ln|t+1| + C$

$= 2\sqrt{w} + 2\ln|\sqrt{w}-1| - 2\ln\sqrt{\sqrt{w}+1} + C$

Ex. Investigate the convergence of $\{a_n\}$
Define the recurrence relations
$a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 6)$

$a_1 = 2 \Rightarrow a_2 = \frac{1}{2}(2+6) = 4$
$a_2 = 4 \Rightarrow a_3 = \frac{1}{2}(4+6) = 5$

If the $\{2, 4, 5, \ldots\}$ limit exists

$\lim_{n\to\infty} a_n = L$ $\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{a_n\to\infty} + 6\right)$

$\lim_{n\to\infty} a_{n+1} = L$ $L = \frac{1}{2}(L+6) \Rightarrow 2L = L + 6$
$L = 6$

Series Feb 8 2018

An infinit series $\sum_{n=1}^{\infty} a_n$ are the infinite addition of the Terms of an infinite sequence.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n$$

Partial sum S_n is the sum of the first n terms of a series, $S_n = a_1 + a_2 + a_3 + \dots + a_n$

$$= \sum_{i=1}^{n} a_i$$

Def: If the sequence is convergence, i.e. $\lim_{n\to\infty} S_n = S$, the the series Σa_n is also convergent. and $\sum_{n=1}^{\infty} a_n = S$.

If $\lim_{n\to\infty} S_n$ doesn't exist, the series Σa_n is divergent.

Geometric Series:

$\sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r|<1$ ($-1<r<1$), other wise it is divergent.

If $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $-1<r<1$

Ex. $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n} = \sum_{n=1}^{\infty} 4^n 3^{1-n} = \sum_{n=1}^{\infty} \frac{4^n}{3^n} 3 = 3\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n = \sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right) \times \left(\frac{4}{3}\right)^{n-1} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$

$r = \frac{4}{3}$, series is divergent.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$ telescopic series.

$n=1 \quad 1 - \frac{1}{2}$

$n=2 \quad \frac{1}{2} - \frac{1}{3}$

$n=\infty \quad \cancel{\frac{1}{n}} - \frac{1}{n+1}$

Theorem: if the series $\sum_{n=1}^{\infty} a_n$ is convergent, the $\lim_{n\to\infty} a_n = 0$.

☆ From this theorem we can conclude that if the limit of $a_n \neq 0$ ($n\to\infty$), the series Σa_n is Div

Note: A finite number of terms doesn't affect the convergence or divergence of a series.

e.g. If $\sum_{n=N+1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergence.

Ex. Find the sum of $\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$

$\lim_{n\to\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 0$ $\quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ $\quad \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

Can't say anything now.

$\therefore \sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3 + 1 = 4$

Ex. $\sum_{n=1}^{\infty} \arctan(n)$ $\quad \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \Rightarrow$ Divergent

Series pg.2 Feb 8, 2018

Ex. Write the number $2.3\overline{17}$ as a ratio of intergers.

$$2.3\overline{17} = 2.3 + \frac{17}{1000} + \frac{17}{10^5} + \frac{17}{10^7} + \dots$$

$$= 2.3 + \frac{17}{1000}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} \dots\right) =$$

$$= 2.3 + \frac{17}{1000}\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^{n-1} = 2.3 + \frac{\frac{17}{1000}}{1-\frac{1}{100}} = \frac{23}{10} + \frac{17/1000}{\frac{99}{100}} = \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}$$

Ex. Show that the harmonic Series $\sum \frac{1}{n}$ is divergent.

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$

$S_8 = 1 + \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$

$S_8 > 1 + \frac{3}{2}$ $\qquad \lim_{n\to\infty} S_{2^n} = \lim_{n\to\infty}\left(1 + \frac{n}{2}\right) = \infty$

$S_{2^n} > 1 + \frac{n}{2}$ $\qquad \{S_n\}$ is divergent $\Rightarrow \sum \frac{1}{n}$ is Divergent.

Ex. $$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \sum_{n=1}^{\infty} \frac{-12}{5}\left(\frac{-1}{5}\right)^{n-1} = \frac{\frac{-12}{5}}{1-\left(\frac{-1}{5}\right)} = \frac{\frac{-12}{5}}{\frac{6}{5}} = -2$$

Ex. $$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(\sum_{m=1}^{n} \frac{1}{m-1} - \frac{1}{m+1}\right) =$$

$$\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \dots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$+ \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{6}} - \frac{1}{9}\right)$$

$n \to \infty$

$$= \lim_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = \frac{3}{2}$$

aproximate Ex. $\sum \frac{1}{n^3}$ using sum of it's first 10 terms

$f(x) = \frac{1}{x^3}, n = 1$ $\qquad \int_{11}^{\infty} \frac{1}{x^3}dx < R_{10} < \int_{10}^{\infty} \frac{1}{x^3}dx, \quad R_{10} < \frac{1}{200} =$

Series Reminder

$\sum a_n$

Always Convergence test

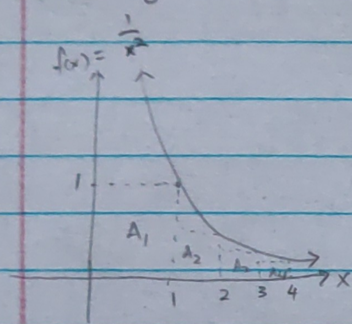
$\lim_{n\to\infty} a_n \neq 0$, div.

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r}, \text{conv} & |r|<1 \\ \text{Div} & |r|>1 \end{cases}$$

11.3 Integral Test Feb 14, 2018 Lei Yu

Investigate the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2}$



$$\left.\begin{aligned} A_1 &= \frac{1}{1^2} \\ A_2 &= \frac{1}{2^2} \\ A_3 &= \frac{1}{3^2} \\ A_4 &= \frac{1}{4^2} \\ &\cdots \end{aligned}\right\} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

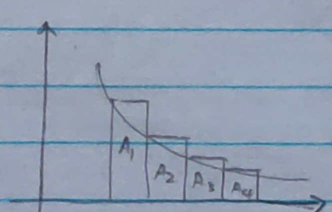
Sum of the rectangles except for the first rectangle is less than the area under the curve of $f(x) = \frac{1}{x^2}$ from 1 to ∞ which is $\int_1^{\infty} \frac{1}{x^2} dx$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < \left(1 + \int_1^{\infty} \frac{1}{x^2} dx\right)$$

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t\to\infty} \left.\frac{-1}{x}\right|_1^t = \lim_{t\to\infty} \left[\frac{-1}{t} + \frac{1}{1}\right] = 1$$

$\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$ $\therefore$ series is conv., series is a finite number

Investigate Convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{x}}$



$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}} dx$$

$\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ is Div

$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is Div

☆ Integral Test We can generalize last two example and state

Suppose $f(x)$ is continuous positive and decreasing, on $[1, \infty)$ and let $a_n = f(n)$ then:

1) if $\int_1^{\infty} f(x)dx$ is conv $\Rightarrow \sum a_n$ is conv

2) if $\int_1^{\infty} f(x)dx$ is Div $\Rightarrow \sum a_n$ is Div

Integral Test Examples Feb 14, 2018 Lei Yu

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$

$f(x) = \frac{1}{x^2+1}$ is continuous, positive and decreasing on $[1,\infty)$.

Integral test

$$\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{t\to\infty} \arctan x\Big|_1^t = \lim_{t\to\infty}(\arctan t - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ Conv}$$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is Conv.

NOTE : In using the integral test, it is not necessary to start the series or the integral from $n=1$.
Also, it is not necessary for f to be always decreasing. What is important is that f be ultimately decreasing for some $x \geq N$, then $\sum_{n=N}^{\infty} a_n$ will be convergent and so is $\sum_{n=1}^{\infty} a_n$

P series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

$$p<0 = \lim_{n\to\infty} \frac{1}{n^p} = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is Div}$$

$$P=0 = \lim_{n\to\infty} \frac{1}{n^0} = \lim_{n\to\infty} 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^0} \text{ is Div}$$

$$P>0 \Rightarrow \int_1^{\infty} \frac{1}{x^p}dx = \begin{cases} \text{Conv} & p>1 \\ \text{Div} & P\leq 1 \end{cases} \Rightarrow \boxed{\sum \frac{1}{n^p} = \begin{cases} \text{Con} & P>1 \\ \text{Div} & P\leq 1 \end{cases}}$$

Ex. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ $f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{1-\ln x}{x^2} < 0 \Rightarrow 1 < \ln x \Rightarrow x > e$

$f(x)$ is decreasing if $x>e \Rightarrow \int_e^{\infty} \frac{\ln x}{x}dx = \frac{\ln x^2}{x^2}\Big|_e^{\infty} = \lim_{t\to\infty} \frac{\ln t^2}{2} - \frac{1}{2} = \infty = \text{Div}$
positive
Continuous

$\sum_{n=e}^{\infty} \frac{\ln n}{n}$ is Div $\Rightarrow \sum_{n=1}^{\infty}$ is Div

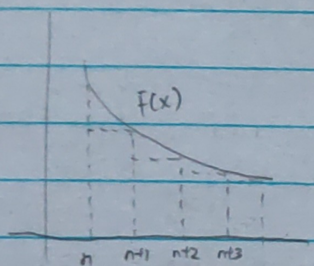
Estimating the sum of a series

Suppose that we have used the intergral test to show that $\sum a_n$ is Conv.

$$\sum_{n=1}^{\infty} a_n = S = a_1 + a_2 + a_3 + \dots$$

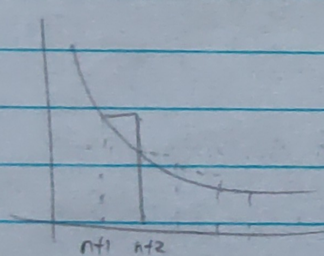
$S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots = R_n \rightarrow$ the remainder or error of approximation

To find bounds for R_n we use the same approach as the integral test



$f(x)$ such that $f(n) = a_n$

$$R_n = a_{n+1} + a_{n+2} + \dots < \int_n^{\infty} f(x)\,dx$$



$$R_n = a_{n+1} + a_{n+2} + \dots > \int_{n+1}^{\infty} f(x)\,dx$$

$$\therefore \int_{n+1}^{\infty} f(x)\,dx < R_n < \int_n^{\infty} f(x)\,dx$$

$\sum \frac{1}{n^3}$ the function correspond to this series is

$f(x) = \frac{1}{x^3}$, which is decreasing, positive, continuous.

So $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$

a) how many term are needed so that aquired sum is accurate within 0.0005?

This means that how many n so that $R_n \leq 0.0005$

$R_n \leq \int_n^{\infty} \frac{dx}{x^3} \leq 0.0005$

$$\int_n^{\infty} \frac{dx}{x^3} = \lim_{t \to \infty} \frac{-1}{2x^2}\Big|_n^t = \lim_{x \to \infty} \left(\frac{-1}{2t^2}\right) + \frac{1}{2n^2} \Rightarrow R_n \leq \frac{1}{2n^2} \leq 0.0005$$

$n^2 \geq \frac{1}{2(0.0005)}$

$n \geq \sqrt{1000}$

$n \geq 31.6$

$n \geq 32$

$\therefore$ We need at least 32 terms

MTH240 Other Test Feb 13 2018 Lei Yu

☒ Suppose $\sum a_n$ & $\sum b_n$ are series with positive terms and $\sum b_n$ is conv.

then if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, a_n is conv

☒ Suppose $\sum a_n$ & $\sum b_n$ are series with positive terms and $\sum b_n$ is Divergent

then if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \Rightarrow \sum a_n$ is Div.

Ex. $\sum \frac{\ln n}{n^3}$ $\qquad \sum b_n = \sum \frac{1}{n^2}$ conv

$\lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\ln n}{n} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

$\Rightarrow \sum \frac{\ln n}{n^3}$ is conv

Ex. $\sum \frac{\ln n}{\sqrt{n}\, e^n}$ at home

Ex. $\sum_{n=1}^{\infty} \frac{1}{\ln n} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{\ln n}$ $\qquad$ I choose $\sum b_n = \sum \frac{1}{n}$, Div

$\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty$ $\Rightarrow$ $\sum \frac{1}{\ln n}$ is Div

Estimating the Sum

Suppose we have used the comparison test to prove $\sum a_n$ is Conv. This means we could find a Conv series $\sum b_n$ such that $a_n \le b_n$ for $n \ge N$

$\sum a_n = S$ $\qquad$ then $R_n = S - \underset{\text{sum of first n term}}{S_n} = a_{n+1} + a_{n+2} + \dots$

$\sum b_n = T$ $\qquad$ $T_n = T - \underset{\text{sum of first n term}}{T_n} = b_{n+1} + b_{n+2} + \dots$

Since $a_n \le b_n$, we have $R_n \le T_n$

Ex. Estimate error resulting from considering only the first 100 terms to approximate $\sum \frac{1}{n^3+1}$

Since $\frac{1}{n^3+1} < \frac{1}{n^3}$

using intergral test for $\sum \frac{1}{n^3}$ we can conclude

$T_n \le \int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}$

$\Rightarrow n = 100 \Rightarrow R_n \le T_n \le \frac{1}{2(100)^2}$

Ex. Estimate the error from approximating $\sum \frac{\cos^2 n}{5^n}$ with only the 1st 10 term.

Ex. Find the value of x for which $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$ is convergent.

$\sum_{n=1}^{\infty} \left(\frac{x}{3}\right)\left(\frac{x}{3}\right)^{n-1}$

$-1 < \frac{x}{3} < 1$

$-3 < x < 3$

MTH240 Comparison Test   Feb 15, 2018 Lei Yu

$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

We know that for $n \geq 1$, $\frac{1}{2^n+1} < \frac{1}{2^n}$, the term of given series are smaller than those of the geometric series $\sum (\frac{1}{2})^n$ $\therefore$ $\sum \frac{1}{2^n+1} < \sum \frac{1}{2^n}$. This means that its partial sum form a bounded increasing sequence which makes it convergent.

Now, the comparison test can be expressed as,

① If $\sum b_n$ is Con and $a_n \leq b_n$ for all n, then $\sum a_n$ is also conv.
② If $\sum b_n$ is Div and $a_n > b_n$ for all n, then $\sum a_n$ is also DIV

Note: in using the comparison test, we should already know some convergent or divergent series for comparison.
*Most of the time we use p series or geometric series.

Recall $\sum \frac{1}{n^p} = \begin{cases} \text{Conv}, & P > 1 \\ \text{Div}, & P \leq 1 \end{cases}$, $\sum ar^{n-1} \begin{cases} \text{Conv}, & |r| < 1 \\ \text{Div}, & |r| \geq 1 \end{cases}$

Ex. $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$, divergent test fail

we know that for $n \geq 1$ $\frac{5}{2n^2+4n+3} \leq \frac{5}{2n^2}$

we also know that $\sum \frac{1}{n^2}$ is Conv and so is $\sum \frac{5}{2}(\frac{1}{n^2})$

Based on comparison test, $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

Note: in the comparison test $a_n \leq b_n$ or $a_n \geq b_n$ should not necessarily be true for $n \geq 1$. In fact, even if the equality holds true for $n \geq N$ when N is some fixed Interger. we can still use the comparison test. Because the convergence is not affected by the finite # of term.

Ex. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

We know that $\ln n > 1$ when $n \geq 3 \Rightarrow \frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 3$

we know $\sum \frac{1}{n}$ is DIV $\Rightarrow \sum \frac{\ln n}{n}$ is Div

Ex. $\sum \frac{1}{2^n-1}$

$\sum \frac{1}{2^n-1} > \sum \frac{1}{2^n}$ $\Rightarrow$

No conclusion can be made

$\sum \frac{1}{2^n}$ is Conv $\nRightarrow$

Limit Comparison Test

Suppose that $\sum a_n$ & $\sum b_n$ are series with positive term

If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$

and C is a finite number and $C > 0 \Rightarrow$ Either both series are Conv or Div

Ex. $\sum \frac{1}{2^n-1}$, I choose $\sum b_n = \sum \frac{1}{2^n}$

$\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = 1 > 0$

$\sum \frac{1}{2^n}$ is Conv

$\sum \frac{1}{2^n-1}$ is Conv

Ex. $\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{n^5+5}}}{\frac{2n^2}{\sqrt{n^5}}} = 1 > 0$

$\Downarrow$

$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$

$\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ is Div

$b_n = \frac{2n^2}{\sqrt{n^5}} = \frac{2}{\sqrt{n}}$ $\Rightarrow$

NoTe : In testing many series we can find suitable comparison series by keeping the hisest power in numerator and denominator, similar to what we did in previous example

Alternating Series   Feb 28, 2018   Lei Yu

a series whose terms are alternately positive & negative

For example $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

In general an alternating series is represented by $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$

Alternating Series Test

If an alternating series satisfies the following conditions, it is Conv.

if $b_n' < 0$, decreasing → 1) $b_{n+1} \le b_n$ for $n \ge N$

2) $\lim_{n\to\infty} b_n = 0$

for e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

1) for $n \ge 1$ $\frac{1}{n+1} \le \frac{1}{n}$

2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

e.g. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3n}{4n-1}$, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4}$, 2nd not satisfied.

e.g. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1} = \sum_{n=1}^{\infty} (-1)^{n-1} (-1)^2 \frac{n^2}{n^3+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^3+1}$

2) $\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$, satisfied

To investigate condition 1.

$f(x) = \frac{x^2}{x^3+1}$, if $f(x)$ is decreasing for $x \ge N$ then 1) is satisfied.

$f'(x) = \frac{2x^4+2x-3x^4}{(x^3+1)^2} = \frac{-x^4+2x}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3-1)^2} \Rightarrow f'(x) < 0$ if $2-x^3 \le 0$

$x^3 \ge 2$

$\Leftarrow x \ge \sqrt[3]{2}$

$\therefore b_{n+1} \le b_n$ for $n \ge 2$

Alternating Series Estimation Theorem

If $S=\sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies $0\le b_{n+1}\le b_n$ and $\lim_{n\to\infty} b_n = 0$.

sum of 1st n term of series

$$\Rightarrow |R_n| = |S-S_n| \le b_{n+1}$$

e.g. Find the sum of the series $\sum(-1)^{n-1}\frac{1}{n!}$ correct to three decimal places.

0.5 0.167 0.041 0.0083 0.001383 0.00198

$b_n = \frac{1}{n!}$ $\quad S=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!} = 1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}-\frac{1}{6!}+\frac{1}{7!}-\frac{1}{8!}+\dots$

$b_7=\frac{1}{7!}=0.00198 \Rightarrow |S-S_6|\le b_7$ $\quad$ Note: Correct to m decimal places means $|S-S_n| < .\underbrace{00\dots}_{m-1}9$

Definition: a series $\sum a_n$ is called absolutely convergent if the series $\sum|a_n|$ is conv.

Ex: $\sum\frac{(-1)^{n-1}}{n^2}$, convergent

Def: a series $\sum a_n$ is conditionally convergent if it is convergent but not absolutly.

i.e. $\sum\frac{(-1)^{n-1}}{n}$, Conditionally Convergent, b/c it is convergent but not absolutly convergent.

Theorem: If a series is absolutely convergent then it is convergent.

ie. $\sum|a_n|$ is convergent $\Rightarrow \sum a_n$ is convergent

The ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, if $L<1$, a_n absolutly conv

if $L>1$, a_n DIV

$L=1$, test inconclusive

e.g. $\sum(-1)^n\frac{n^3}{3^n}$

$$\lim_{n\to\infty}\frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty}\frac{(n+1)^3}{n^3}\times\frac{3^n}{3^{n+1}} = 1\times\frac{1}{3}=\frac{1}{3}, \quad \frac{1}{3}<1 \therefore \text{absolutly conv}$$

e.g. $\sum\frac{n^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\times\frac{\cancel{n!}}{(n+1)\cancel{n!}} = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\times\frac{1}{n+1}$$

$$=\lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$e>1$, $\therefore$ Absolutly div

Root test Feb 28 2018 Lei Yu

Root Test : $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \begin{cases} L<1 \text{ Conv} \\ L>1 \text{ div} \\ L=1 \text{ Inconclusive} \end{cases}$

Ex. $\sum \left(\frac{2n+3}{3n+2}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1 \Rightarrow \text{Conv}$$

Ex. $\sum \frac{2^n}{n!}$ ratio test: $\lim \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim \frac{2^{n+1}}{2^n} \times \frac{n!}{(n+1)!} = 2\lim \frac{n!}{(n+1)n!} = 2\lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$, Conv

Strategy for testing Series

1. P series $\sum \frac{1}{n^p} \begin{cases} \text{Con } P>1 \\ \text{DIV } P\le 1 \end{cases}$

2. Geometric Series $\sum ar^{n+1} \begin{cases} \text{Conv } |r|<1 \\ \text{DIV } |r|\ge 1 \end{cases}$

3. Divergent test works $\lim_{n\to\infty} a_n \neq 0 \Rightarrow a_n$ is DIV

4. Alternating Series test for $\sum (-1)^{n-1} b_n$

5. Series involving factorials or other products including a constant raised to the n^{th} power are conveniently tested using ration test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

6. If the series is of the form $(bn)^n$ then the root test may be helpful.

7. If a_n is equal f_n where $\int_1^\infty f(x)dx$ is easily evaluated and $f(x)$ is decreasing & continuous on $[1,\infty)$, the integral test may help.

8. If the series has a from similar to a p-series or a geometric series, then one of the comparison test should be used.

* Note that the comparison test only apply to series w/ positive term, but if $\sum a_n$ has some negative terms, we can apply the comparison test to $\sum |a_n|$ and test for absolute convergence.

divergent test — Ex. $\sum \frac{n-1}{2n+1}$, $\lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2} \neq 0 \Rightarrow$ DIV

limit comparison — Ex. $\sum \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

$\sum \frac{\sqrt{n^3}}{n^3} = \sum \frac{1}{\sqrt{n^3}} = \sum \frac{1}{n^{3/2}} \Rightarrow$ Conv

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^3+1}}{3n^3+4n^2+2}}{\frac{1}{\sqrt{n^3}}} = \lim_{n\to\infty} \frac{\sqrt{n^6+n^3}}{3n^3+4n^2+2} = \frac{1}{3} > 0$$

based on the limit comparison test we conclude that $\sum \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$ is conv

ratio test — Ex. $\sum n e^{-n^2}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)e^{-(n+1)^2}}{n e^{-n^2}} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{e^{n^2}}{e^{(n+1)^2}} = \lim_{n\to\infty} e^{n^2-n^2-2n-1} = \lim_{n\to\infty} e^{-2n-1} = 0 < 1$$

$\therefore$ Conv

Alternating test — Ex. $\sum (-1)^n \frac{n^3}{n^4+1}$ — Alternating test $\begin{cases} \lim_{n\to\infty} b_n = 0 \\ b_{n+1} \leq b_n \text{ for } n > N \end{cases}$

1) $\lim_{n\to\infty} \frac{n^3}{n^4+1} = 0$

2) $f(x) = \frac{x^3}{x^4+1} \Rightarrow f'(x) = \frac{3x^2(x^4+1) - 4x^3(x^3)}{(x^4+1)^2} = \frac{3x^6+3x^2-4x^6}{(x^4+1)^2} = \frac{-n^6+3n^2}{(n^4+1)^2} = \frac{n^2(3-n^4)}{(n^4+1)^2}$

$3 - x^4 \leq 0$ $\Rightarrow$ Convergent

$x^4 \geq 3$

$x \geq \sqrt[4]{3}$, $n \geq 2$ $b_{n+1} \leq b_n$

Comparison Test — Ex. $\sum \frac{1}{2+3^n}$ for $n \geq 1$ $\frac{1}{2+3^n} \leq \frac{1}{3^n}$

$\sum \left(\frac{1}{3}\right)^n = \sum \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$ is g-series wit $r < \frac{1}{3} < 1 \Rightarrow$ Conv

Power Series   March 1, 2018   Lei Yu

A power series is of the form

$\sum_{n=1}^{\infty} C_n x^n \rightarrow$ variable $\sum_{n=0}^{\infty} C_n (x-a)^n$

C_n: constant coefficient

Since x is a variable, a power series may converge for some value of x and diverge for other values.

$$f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots + C_n x^n$$

We can see that the sum of the series is a polynomial function whose domain is the set of all x for which the series is convergent

For example if $C_n = 1$ for all n's, the power series becomes

$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1}$ a geometric series with $a=1$ & $r=x$

which is conv for $|x|<1$

A more general form of power series is

$\sum_{n=0}^{\infty} C_n (x-a)^n$ which is called the power series centered at a.

Ex. For what value is the series $\sum_{n=0}^{\infty} n!\, x^n$ is convergent

Using ratio test, $x \neq 0$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)n!\, x^n x}{n!\, x^n}\right| = \lim_{n\to\infty} |(n+1)x| = \infty \text{ if } x \neq 0 \quad \underline{DIV}$$

If $x=0 \Rightarrow \lim_{n\to\infty} |n+1||x| = 0 < 1$, Conv

This series is convergent only when $x=0$, $\therefore \sum_{n=0}^{\infty} n!\, x^n$ is DIV

Ratio test, $x \neq 3$

Ex. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

$$\lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right| = \lim_{n\to\infty} |x-3| <$$

$x-3<1$ $-x+3<1$ $2<x<4$, $x\neq 3$
$x<4$ $-x<-2$ $\Rightarrow$ is convergent
$x>2$

if $x=3 \Rightarrow \sum_{n=1}^{\infty} \frac{0^n}{n} = 0$, the series is conv ratio test inconclusive for $|x-3|=1$. Investigate separetly

$$|x-3|=1 \begin{cases} x=2 \\ x=4 \end{cases}$$

$x=2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, Conv by alternating test

$x=4 \Rightarrow \sum \frac{1}{n} = 0$, Div.

$\therefore \Rightarrow$ the series is Conv for $\boxed{2 \leq x < 4}$

Ex. $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$. the Bessel function of order 0.

We apply ratio test for $x \neq 0$
if $x=0$, Conv

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{2^{2n+2}((n+1)!)^2} \times \frac{2^{2n}(n!)^2}{x^{2n}} \right| = \lim_{n\to\infty} \frac{x^2}{2^2(n+1)^2} = 0 < 1, \text{ regardless of } x.$$

Conv for $x \in \mathbb{R}$

Theorem: For the power series $\sum_{n=0}^{\infty} C_n (x-a)^n$
there are only 3 posibilities
1. The series converges only when $x=a$
2. The series conveges for all values of x.
3. There is a positive number R that the series is convergent if $|x-a|<R$ and divergengent when $|x-a|>R$.
R is called the radius of convergence
The interval of convergence is the interval consisting of all the values of x for which the series is convergent.

Power Series Pt.2 March 1, 2018 Lei Yu

Case 1: interval of convergence is $\{0\}$. The R of convergence is 0.

Case 2: interval of convergence is $(-\infty, \infty)$ and $R=\infty$

Case 3: when $|x-a|<R$, This indicates that we can use the ration test to find R. But to determine interval of convergence the series needs to be investigated at $x=a+R$ & $x=a-R$.

$a-R<x<a+R$

$a-R\leq x<a+R$

$a-R<x\leq a+R$

$a-R\leq x\leq a+R$

Ex. $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$

ratio $\lim_{x\to\infty}\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}=3|x|<1$

$|x|<\frac{1}{3}$

$x<3 = R=3$

we have to check $|x|=\frac{1}{3}\begin{cases}-\frac{1}{3}\\ \frac{1}{3}\end{cases}$

$y=\frac{1}{3}\Rightarrow\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}\Rightarrow$ Conv by alternating test

$y=\frac{-1}{3}\Rightarrow\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$ DIV

$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}}=1>0$

Conv: $\{\frac{-1}{3}<x\leq\frac{1}{3}\}$

Ex. $\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}}\Rightarrow a=-2,\ C_n=\frac{n}{3^{n+1}}$

at $x=-2$, the series is conv

Using the ratio test $\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}\times\frac{3^{n+1}}{n(x+2)^n}=\lim_{n\to\infty}\left|\frac{(x+2)(n+1)}{3n}\right|$

$=|x+2|\lim_{n\to\infty}\left|\frac{n+1}{3n}\right|=\frac{|x+2|}{3}$ if $\frac{|x+2|}{3}<1$, Conv

$|x+2|<3$

Considering $|x+2|=3$ separatly

$|x+2|=3$

$x=1$

$x=-5$

if $x=1\Rightarrow\sum_{n=0}^{\infty}\frac{n(3)^n}{3^{n+1}}=\sum_{n=0}^{\infty}\frac{n\cancel{3^n}}{\cancel{3^n}3}=\sum_{n=0}^{\infty}\frac{n}{3}$, DIV

if $x=-5\Rightarrow\sum_{n=0}^{\infty}\frac{n(-3)^n}{3^{n+1}}=\sum_{n=1}^{\infty}\frac{n(-1)^n}{3}$, DIV

$\therefore\begin{cases}|x+2|<3, \text{ Conv}\\ |x+2|=3, \text{ DIV}\end{cases}$, $R=3$ $\quad -5<x<1$

Representation of functions as Power Series. Mar 7, 2018 Lei Yu

• allows us to intergrate functions that doesn't have anti derivative or solving defferential equations.

Ex. $F(x) = \frac{1}{1-x} = 1 + x^2 + x^3 + x^4 \ldots$

$F(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^{n-1}$ is a geometric series which convergent for $|x| < 1$

Ex. $f(x) = \frac{1}{1+x^2}$ as a power series and find interval of convergence

$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=1}^{\infty} (-1)^{n-1} (x^2)^{n-1}$ conv whe $|x|^2 < 1$

$-1 < x < 1$ interval of conv

Ex. $f(x) = \frac{1}{x+2} = \frac{1}{2} \frac{1}{\frac{x}{2}+1} = \frac{1}{2} \frac{1}{1-\left(\frac{-x}{2}\right)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{-x}{2}\right)^n = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{-x}{2}\right)^{n-1}$

$\left|\frac{-x}{2}\right| < 1$, $\frac{-x}{2} < 1$ $\quad \frac{-x}{2} < 1$ $\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} (x)^n$

$-x < 2 \quad x < 2$

$x > -2$

Ex. Power Series representation of $f(x) = \frac{x^3}{x+2}$, $R = ?$

$f(x) = \frac{x^3}{x+2} = \frac{x^3}{2} \left(\frac{1}{\frac{x}{2}+1} \right) = \frac{x^3}{2} \left(\frac{1}{1-\left(\frac{-x}{2}\right)} \right) = \frac{x^3}{2} \sum_{n=1}^{\infty} \left(\frac{-x}{2}\right)^{n-1}$, $|x| < 2$

Differentiation & Intergration of power series.

Theorem: If the power series $\sum_{n=0}^{\infty} C_n (x-a)^n$ has a radius of conv R, then the function $f(x) = C_n (x-a)^n$ is differentiable over $|x-a| < R$ and

$$F'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$$

$$\int F(x)\,dx = \int \sum_{n=0}^{\infty} C_n (x-a)^n\,dx = \sum_{n=0}^{\infty} \int C_n (x-a)^n\,dx = \sum_{n=0}^{\infty} \frac{C_n (x-a)^{n+1}}{n+1} + C$$

Note: the Radius of convergence of power series remains unchanged after differentiation and intergration. However, the interval of convergence may change.

Representation of function as Power Series PT.2 Mar 7, 2018 Lei Yu

ex. $f(x) = \frac{1}{(1-x)^2}$, Sol $f(x) = \frac{1}{1-x}$, $f'(x) = \frac{1}{(1-x)^2}$

$f(x) = \sum_{n=0}^{\infty} x^n \Rightarrow f'(x) = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

Ex. $\ln(1-x)$

$f(x) = \frac{1}{1-x} \Rightarrow \int f(x)\,dx = -\ln(1-x) \Rightarrow \ln(1-x) = -\int f(x)\,dx$

$f(x) = \sum_{n=0}^{\infty} x^n \Rightarrow \ln(1-x) = -\sum_{n=0}^{\infty} \int x^n dx = -\left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\right) + C$ $x=0 \Rightarrow \ln(1) = 0 + C$

$0 = C$

Hmwk: $\tan^{-1}(x)$, $\Rightarrow$ power sepries

Taylor Series

Suppose $f(x)$ is a function that can be represented by a power series.

$f(x) = \sum_{n=0} C_n(x-a)^n = C_0 + C_1(x-a)^1 + C_2(x-a)^2$

We are going to find the relationship between C_n and $f(x)$

$f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 \dots$

$f'(a) = C_1$

$f''(x) = 2C_2 + 6C_3(x-a) + 12C_4(x-a)^2 \dots$

$f''(a) = 2C_2$, $C_2 = \frac{f''(a)}{2}$

Simillary $f'''(a) = 6C_3 \Rightarrow C_3 = \frac{f'''(a)}{6}$

$C_0 = f(a)$

$C_1 = \frac{f'(a)}{1}$

$C_2 = \frac{f''(a)}{2}$

$C_3 = \frac{f'''(a)}{6}$

$\vdots$

$C_n = \frac{f^{(n)}(a)}{n!}$

$\Rightarrow f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

☆ Can proof $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

This is called Taylor series centered at a. If $a=0$, then Maclauren series.

Ex. Find the Maclauren Series of $f(x) = e^x$ and it's Radius of Conv

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \cdot x^n$

$f(x) = e^x \Rightarrow f(0) = 1$ $\Rightarrow f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 < 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$\therefore R = \infty$

Continuation of Taylor Series   Mar 8, 2018 Lei Yu

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n}(x-a)^n$$

$T(n)$ $R(n)$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \frac{f^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!} + \dots$$

$f(x) = T(n) + R(n) \longrightarrow$ The remainder

$\downarrow$

the n^{th} degree taylor polynomial of $f(x)$ at $x=a$

Theorem: If $F(x) = T_n(x) + R_n(x)$ and $\lim_{x\to\infty} R_n(x) = 0$ for $x-a<R$, then $f(x)$ is equal to the sum of it's taylor series on $|x-a|<R$

Taylor's Inequality

if $|f^{n+1}(x)| \leq M$, for $|x-a| < d$ then remainder $R_n(x)$ of the Taylor Series satisfies the following inequality.

$$|R_n(-1)| < \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

Ex. Prove that $f(x) = e^x$ is equal to it's Mac lauren Series.

$F(x) = T_n(x) + R_n(x)$

$'f(x) = e^x \Rightarrow f^{(n+1)}a = e^x$

If d is any positive int, then for $|x| \leq d$

we have $|f^{(n+1)}(n)| = e^x \leq e^d \rightarrow M$

$$\lim_{x\to\infty} R_n(x) = 0$$

$$\text{So } |R_n(a)| \leq \frac{e^d}{(n+1)!}|x|^{n+1}$$

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$$

$$\lim_{n\to\infty} \frac{e^d (x)^{n+1}}{(n+1)!} = 0$$

$\Uparrow$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$, ratio test

$= 0 < 1$, Conv $R = \infty$

Taylor PT.2 Mar 8, 2018 Lei Yu

Ex Find taylor series of $f(x)=e^n$ at $a=2$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(2)(x-2)^n}{n!}, \quad f^{(n+1)}(2) = e^2$$

$$= \sum_{n=1}^{\infty} \frac{e^2(x-2)^n}{n!}$$

Ex Find Maclauren Series for $f(x)=\sin x$ and proof it represents $\sin x$ for all x.

$$f(x) = \sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

$$\left.\begin{array}{l} f_n = \sin x \Rightarrow f(0) = 0 \\ f'(n) = \cos x \Rightarrow f'(0) = 1 \\ f''(n) = -\sin x \Rightarrow f''(0) = 0 \\ f'''(n) = -\cos x \Rightarrow f'''(0) = -1 \end{array}\right\}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$f''''(n) = \sin x \Rightarrow f''''(0) = 0$

repeated

Now to show that $f(x)$ is equal to its maclauren series for all values of x.

$\lim_{n\to\infty} R_n(x) = 0$, So we use Taylor's inequality

$\boxed{M=1}$

$|F^{(n+1)}(x)| \leq 1$, for all x

$$\Rightarrow R_n(x) \leq \frac{x^{n+1}}{n+1}, \quad \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \quad \Rightarrow \lim_{n\to\infty} R_n(x) = 0$$

Ex Find the Maclauren Series for $\cos x$.

$$(\sin x)' = \cos x = \frac{d(\sin x)}{dx}$$

$$\frac{d\left(\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}\right)}{dx} = \sum_{n=0}^{\infty} (-1)^n \frac{\frac{d(x)^{2n+1}}{(2n+1)!}}{dx} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x$$

Ex. Find maclauren for $x\cos x$.

$$x\cos x = x\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n)!}$$

Ex. Find the Maclauren series for $f(x)=(1+x)^k$ $k\in\mathbb{R}$

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} \quad \text{Binomial Series}$$

$f(x)=(1+x)^k \Rightarrow f(0)=1$

$f'(x)=k(1+x)^{k-1} \Rightarrow f'(0)=k$

$f''(x)=k(k-1)(1+x)^{k-2} \Rightarrow f''(0)=k(k-1)$

$f'''(x)=k(k-1)(k-2)(1+x)^{k-3} \Rightarrow f'''(x)=k(k-1)(k-2)$

$\vdots$

$f^{(n)}(x)=k(k-1)(k-2)\ldots(k-(n-1))(1+k)^{k-n} \Rightarrow f^{(n)}(0)=k(k-1)(k-2)\ldots(k-(n-1))$

$$\Rightarrow f(x)=(1+x)^k=\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\ldots x^n}{n!} = \sum_{n=0}^{\infty}\frac{k!}{n!(k-n)}x^n = \sum_{n=0}^{\infty}\binom{k}{n}$$

Let's use ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{k(k-1)\ldots(k-n+1)}(k-n)x^{n+1}}{(n+1)!}\times\frac{n!}{x^n\cancel{k(k-1)\ldots(k-n+1)}}\right| = \lim_{n\to\infty}\frac{|k-n||x|}{n+1} = |x|<1$$

- the series is conv when $|x|<1$, when k is not an int.
- b/c when k is an int non negative the the terms are eventually 0. So conv regardless of x.

To Summarize

$f(x)=(1+x)^k=\sum_{n=1}^{\infty}\binom{k}{n}x^n$, Conv for all all value of x if x is an positive integer.

, Con for $|x|<1$, if k is not an Integer

, at $x=\pm1$ Conv depends on the value of k.

Important Maclauren Series

$\frac{1}{1-n}=\sum_{n=0}^{\infty}x^n$, $R_n=1$, $|x|<1$

$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n(x)^{2n}}{(2n)!}$, $R=\infty$

$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n$, R depends on k

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$, $R_n=\infty$

$\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$, $R=1$

$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n(x)^{2n+1}}{(2n+1)!}$, $R=\infty$

$\ln(1+x)=\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, $R=1$

Application of Taylor Series   Mar 12, 2018   Lei Yu

1. Integration

Ex. Evaluate $\int_0^1 e^{-x^2} dx$ Correct to an error of 0.001.

Sol: knowing that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-x^2} = \sum_{n=1}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{n!} \Rightarrow \int_0^1 e^{-x^2} dx = \int_0^1 \left(\sum_{n!}^{\infty} (-1)^n \frac{x^{2n}}{n!} \right) dx$$

$$\sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(n!)} \Big|_0^1$$

Recall: alternating Series estimation theorem

If $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is the sum of an alternating series satisfying $0 \leq b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, $\Rightarrow |R_n| = |S - S_n| \leq b_{n+1}$

$$\int_0^1 e^{-2x} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(n!)} \Big|_0^1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)(n-1)!} \Rightarrow b_n = \frac{x^{2n-1}}{(2n-1)(n-1)!}$$

$$0 \leq b_n \leq b_{n+1} \text{ and } \lim_{n\to\infty} b_n = 0$$

$$\int_0^1 e^{-2x} dx = \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} \cdots \right]_0^1$$

$$\int_0^1 e^{-2x} dx = \left[1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{11 \cdot 5!} + \ldots \right]$$

we stop when $b_{n+1} \leq 0.001$

2. Limits

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x} = \lim_{x\to\infty} \frac{\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 - x}{x} = \lim_{x\to\infty} \frac{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right] - 1 - x}{x}$$

$$\lim_{x\to\infty} \frac{\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)}{x} = \lim_{x\to\infty} \frac{\cancel{x}\left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots\right)}{\cancel{x}} = \infty$$

3. Approximating functions by polynomial.

When using taylor polynomial T_n to approximate a function.
We should ask the following questions.
1. How accurate we want our approximation to be
2. How large n should be to achieve desired accuracy.

To answer these questions we need to look at the remainder $|R_n| = |f(x) - T_n|$
there are two methods to estimate the size of the error.

1. Use the alternating series estimating theorem if the series is alternating.
2. Use taylor inequality, which says if $|f^{(n+1)}(x)| \le M \Rightarrow |f(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$

Ex. Approximate the function $f(x) = \sqrt[3]{x}$ by a taylor polynomial of degree 2 at $a = 8$.
b) How accurate is the approximated value?

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(8)}{n!}(x-8)^n \approx \frac{f(8)}{0!} + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$f(x) = \sqrt[3]{x} \Rightarrow f(8) = 2$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \Rightarrow f'(8) = \frac{+1}{12}$

$f''(x) = \frac{-2}{9}x^{-\frac{5}{3}} \Rightarrow f''(8) = \frac{-2}{288}$

$$f(x) \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{2}{288}(x-8)^2$$

when $7 \le x \le 9$

b) The taylor series is not an alternating series when $x < 8$. Taylor Inequality.

If $f'''(x) \le M \Rightarrow |R_2(x)| \le \frac{M}{3!}|x-8|^3$

$f^{(3)}(7) = \frac{10}{27} 7^{-\frac{8}{3}} \Rightarrow f^{(3)}(x) \le \frac{10}{27}(7)^{-\frac{8}{3}} \rightarrow M$

$7 < x < 9 \qquad f^{3}(x) = \frac{10}{27}x^{-\frac{8}{3}} \Rightarrow f^{(3)}(9) = \frac{10}{27}9^{-\frac{8}{3}}$

$$|R_2(x)| \le \frac{\frac{10}{27}7^{-\frac{8}{3}}}{3!}|x-8|^3 < \frac{10}{164}7^{-\frac{8}{3}} = 3.44e-4$$

$7 \le x \le 9, \quad -1 \le |x-8| \le 1, \quad |x-8| < 1$

Ex. What is the maximum error possible in using this approximation?: $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}, \ -.3 \le x \le .3$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Alternating Series Estimation Theorem

$$\sum_{n=1}^{\infty}(-1)^{n-1} b_n \Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} \qquad |R_n| \le b_{n+1} \xrightarrow{n=3} |R_n| \le b_4$$

$b_4 = \frac{x^7}{7!} \Rightarrow |R_n| \le \left|\frac{x^7}{7!}\right|$, if $-.3 \le x \le .3 \Rightarrow |x| \le |.3| \Rightarrow |x|^7 \le |.3|^7$

$\therefore \quad |R_n| \le \frac{x^7}{7!} \le \frac{(.3)^7}{7!} \le 4.3e-8$