CHAPTER 7

MTH240 Integration Jan 10,2018 Lei Yu

Definite Intergral

If f is a function defined $a \le x \le b$, the interval $[a,b]$ is divided into n subinterval of equal width $\Delta x = \frac{b-a}{n}$ Such that $(x_0 = a)$.

$$\int_a^b f(x) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

- defines the area above the x and below the x-axis.

The Fundamental Theorem of Calculus

1. If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)dt$, $a \le x \le b$ is continuous on $[a,b]$ and differentiable on $[a,b]$ and $g(x) = f(x)$.
2. If $f(x)$ is continuous on $[a,b]$, then $\int_a^b f(x)dx = F(b) - F(a)$
   F is the antiderivative of $f(x)$.

Indefinite Intergration - finding the anti-derivative

$$\int f(x)\,dx = F(x) + C$$, general form of antiderivative

Intergration by Part - based on the product rule

$$(f(x)g(x))' = f'(x)g(x) + g'(x)f(x) = \frac{d(f(x)g(x))}{dx}$$

$f(x) = u \Rightarrow f'(x) = du$

$g(x) = v \Rightarrow g'(x) = dv$

$$= \int d f(x)g(x) = \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx$$

$$d f(x)g(x) = \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx$$

$\Downarrow$

$$\int u\,dv = uv - \int v\,du \qquad \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Ex. $\int \underbrace{x}_{u} \underbrace{\sin x\,dx}_{dv} = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$

① Get u & dv $\quad u = x \Rightarrow dx = du$

② Get the d $\quad dv = \sin x \Rightarrow \int \sin x\,dx = v = -\cos x$

③ plug eq. ④ Solve

Note: u = which one gets simpler when deferentiated

Ex $\int \ln x \, dx$

$\ln x = u \Rightarrow \frac{1}{x} = du$
$dx = dv \Rightarrow x = v$

$\int \ln x\, dx = x\ln x - \int x \frac{1}{x} dx$
$= x\ln x - \int dx + c$
$= x\ln x - x + c$

Ex $\int e^x x^2 dx$

Intergrate by part again

$\int e^x x^2 dx = x^2 e^x - \int e^x 2x\, dx$

$x^2 = u \Rightarrow 2x\,dx = du$
$e^x = dv \Rightarrow e^x = v$

$2x = u \Rightarrow 2dx = du$
$e^x = dv \Rightarrow e^x = v$

$\int e^x 2x\,dx = 2xe^x - \int e^x 2dx$
$=$

$\int e^x x^2 dx = x^2 e^x - 2xe^x + 2e^x + c$

Ex $\int e^x \sin x\, dx$

$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx$

$u = \sin x \Rightarrow du = \cos x\, dx$
$dv = e^x \quad v = e^x$

$u = \cos x \Rightarrow du = -\sin x\, dx$
$dv = e^x \quad v = e^x$

$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx$

$\int e^x \sin x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$
$\int e^x \sin x\, dx = \frac{e^x}{2}(\sin x - \cos x) + c$

Ex $\int_0^1 \tan^{-1} x\, dx$

$\tan^{-1} x = u \Rightarrow \frac{dx}{1+x^2} = du$
$dx = dv \Rightarrow x = v$

$\int_0^1 \tan^{-1} x\, dx = x\tan^{-1} x \Big|_0^1 - \left(\int_0^1 \frac{x}{1+x^2} dx\right)$

$\Downarrow$

$1+x^2 = z, \quad 2x\,dx = dz$
$x\,dx = \frac{dz}{2}$

$\leftarrow$ Substitution

$= x\tan^{-1} x \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{dz}{z}$

$= x\tan^{-1} x \Big|_0^1 - \ln\sqrt{1+x^2}\Big|_0^1$
$= \left(\frac{\pi}{4} - 0\right) - (\ln\sqrt{2} - 0)$
$= \frac{\pi}{4} - \ln\sqrt{2}$

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Ex. $\int (\sin^{-1}x)^2 dx$

① Inter by part

$u = (\sin^{-1}x)^2 \Rightarrow du = 2(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}dx$

$dv = dx \Rightarrow v = x$

$\int (\sin^{-1}x)^2 dx = x(\sin^{-1}x)^2 - \int x\, 2(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}dx$

$= x(\sin^{-1}x)^2 - \underbrace{\boxed{\int \frac{2x}{\sqrt{1-x^2}}(\sin^{-1}x)\,dx}}_{I_i}$

② Inter by part of I_i

$I_i = \int \frac{2x}{\sqrt{1-x^2}}(\sin^{-1}x)\,dx$

$u = \sin^{-1}x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}dx$

$dv = \frac{2x}{\sqrt{1-x^2}} \Rightarrow v = \int \frac{2x}{\sqrt{1-x^2}}dx$

$= -2\sqrt{1-x^2}$

③ Inter by Sub of $\int \frac{2x}{\sqrt{1-x^2}}dx$

$\int \frac{2x}{\sqrt{1-x^2}}dx = -\int \frac{dz}{\sqrt{z}} = -\int z^{-\frac{1}{2}}dz = -\frac{z^{\frac{1}{2}}}{\frac{1}{2}}$

$1-x^2 = z \Rightarrow -2x\,dx = dz$

$= -2\sqrt{z}$

$= -2\sqrt{1-x^2}$

④ Sub everything back, plus C.

$\int \frac{2x}{\sqrt{1-x^2}}(\sin^{-1}x)dx = -2\sqrt{1-x^2}\sin^{-1}x - \int -2\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}}dx$

$= -2\sqrt{1-x^2}\sin^{-1}x + \int 2\,dx$

$= -2\sqrt{1-x^2}\sin^{-1}x + 2x$

$\int (\sin^{-1}x)^2 dx = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + C$

Lecture 2

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Excercise:

$\int \frac{\sin^{-1}(\ln x)}{x}dx$ $u = \sin^{-1}(\ln x)$ $dv = \frac{1}{x}dx$

$du = \frac{1}{x\sqrt{1-\ln x^2}}dx$ $v = \ln x$

$I = \sin^{-1}(\ln x)\ln x - \int \frac{\ln x}{x\sqrt{1-(\ln x)^2}}dx$

$= \sin^{-1}(\ln x)\ln x - \int \frac{z}{\sqrt{1-z^2}}dz$

$= \sin^{-1}(\ln x)\ln x - \sqrt{1-z^2}$

$= \sin^{-1}(\ln x)\ln x - \sqrt{1-\ln x^2} + C$

$z = \ln x$

$dz = \frac{dx}{x}$

Trignometric Intergral

Recall:

$\int \sin x\,dx = -\cos x + C$ $\int \tan x\,dx = \ln|\sec x| + C$ $\frac{d}{dx}\tan x = \sec^2 x$

$\int \cos x\,dx = \sin x + C$ $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

Strategy for evaluating $\int \sin^m x \cos^n x\,dx$

1. if the power of cosin (n) is odd, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express remaining factor in term of sine.
2. if the power of sine (m) is odd, save 1 sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express remaining factor in term of cosine.

3. If the power of both are even use half angle identity.

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\sin x \cos x = \frac{1}{2}\sin 2x$

4. If both power are odd use either case 1 or 2.

Ex. $\int \cos^3 x\, dx = \int \cos x(1-\sin^2 x)\,dx = \int \cos x\,dx - \int \cos x \sin^2 x\,dx$

$= \sin x - \int z^2 dz$

$\sin x = z$

$\cos x\, dx = dz$

$= \sin x - \frac{z^3}{3}$

$= \sin x - \frac{\sin^3 x}{3} + C$

Ex. $\int \sin^5 x \cos^2 x\, dx = \int \sin x(\sin^4 x)\cos^2 x\, dx = \int \sin x(1-\cos^2 x)^2 \cos^2 x\, dx$

$= \int \sin x(1-2\cos^2 x + \cos^4 x)\cos^2 x\, dx$

$\cos x = z$

$-\sin x\, dx = dz$

$= \int \sin x\cos^2 x - 2\sin x\cos^4 x + \sin x\cos^6 x\, dx$

$= \int \sin x(\cos^2 x - 2\cos^4 x + \cos^6 x)\,dx$

$= -\int (z^2 - 2z^4 + z^6)\,dz$

$= -\left(\frac{z^3}{3} - \frac{2z^5}{5} + \frac{z^7}{7}\right) = -\left(\frac{\cos^3 x}{3} - \frac{2\cos^5 x}{5} + \frac{\cos^7 x}{7}\right)$

$= \frac{-\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$

Ex. $\int \sin^2 x\, dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \int \frac{dx}{2} - \int \frac{\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

$\int \frac{\cos 2x}{2}\,dx = \int \frac{\cos z}{4}\,dz = \frac{\sin z}{4} = \frac{\sin 2x}{4}$

$2x = z \Rightarrow 2dx = dz \Rightarrow dx = \frac{dz}{2}$

Ex. $\int \sin^4 x\, dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx = \int \frac{1-2\cos 2x + \cos^2 2x}{4}\,dx$

$= \int \frac{1}{4}dx - \int \frac{\cos 2x}{2}dx + \int \frac{\cos^2 2x}{4}dx$

$\int \frac{\frac{1}{2}(1+\cos 4x)}{4}dx$

$= \frac{1}{4}x - \frac{\sin 2x}{4} + \int \frac{\frac{1}{2}(1+\cos 4x)}{4}dx$

$= \int \frac{1+\cos 4x}{8}dx = \int \frac{1}{8}dx + \int \frac{\cos 4x}{8}dx$

$= \frac{x}{8} + \frac{\sin 4x}{32}$

$= \frac{1}{4}x - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32}$

$= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$

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Intergrals of $\int \tan^m x \sec^n x \, dx$

1. If power of $\sec x$ (n) is even and $n \geq 2$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factor in term of $\tan x$.
2. If the power of tangent (m) is odd, save a factor of $\sec x \tan x$ and use $\tan^2 x = -1 + \sec^2 x$ to express the remaining factor in term of $\sec x$.

Ex. $\int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx = \int \tan^6 x \sec^2 x (1 + \tan^2 x) \, dx$

$z = \tan x$

$dz = \sec^2 x \, dx$

$= \int (\tan^6 x \sec^2 x + \tan^8 x \sec^2 x) dx$

$= \int \tan^6 x \sec^2 x \, dx + \int \tan^8 x \sec^2 x \, dx$

$= \int z^6 dz + \int z^8 dz$

$= \frac{z^7}{7} + \frac{z^9}{9} + C = \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$

Ex. $I = \int \tan^5 x \sec^7 x \, dx = \int \sec x \tan x \tan^4 x \sec^6 x \, dx = \int \sec x \tan x (-1 + \sec^2 x)^2 \sec^6 x \, dx$

$z = \sec x$

$dz = \sec x \tan x \, dx$

$I = \int \sec x \tan x (1 - 2\sec^2 x + \sec^4 x) \sec^6 x \, dx$

$= \int \sec x \tan x (\sec^6 x - 2\sec^8 x + \sec^{10} x) dx$

$= \int (z^6 - 2z^8 + z^{10}) dz$

$= \frac{z^7}{7} - \frac{2z^9}{9} + \frac{z^{11}}{11} + C = \frac{\sec^7 x}{7} - \frac{2\sec^9 x}{9} + \frac{\sec^{11} x}{11} + C$

Ex. $\int \tan^3 x \, dx = \int (\tan x \sec x)\left(\frac{\tan^2 x}{\sec x}\right) dx = \int \tan x \sec x \left(\frac{-1 + \sec^2 x}{\sec x}\right) dx$

$z = \sec x$

$dz = \sec x \tan x \, dx$

$= \int \frac{z^2 - 1}{z} dz$

$= \int (z - z^{-1}) dz = \int z \, dz - \int z^{-1} dz$

$= \frac{z^2}{2} - \ln|z| + C$

$= \frac{\sec^2 x}{2} - \ln|\sec x| + C$

Intergration by parts

Ex. $\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$

$\sec x = u \qquad \sec^2 x \, dx = dv$

$\sec x \tan x \, dx = du \qquad \tan x = v$

$\int \sec x \sec^2 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx$

$= \sec x \tan x - \int (-1 + \sec^2 x)(\sec x) \, dx$

$\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx - \int \sec^3 dx$

$2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$

$\int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$

Strategy for Intergrals of $\begin{cases} \int \sin mx \cos nx \, dx \\ \int \sin mx \sin nx \, dx \\ \int \cos mx \cos nx \, dx \end{cases}$

The following identities are used

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$
$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

Ex. $\int \sin 4x \cos 5x \, dx = \frac{1}{2}\int (\sin(4x-5x) + \sin(4x+5x)) dx$
$$= \frac{1}{2}\int [\sin(-x) + \sin(9x)] dx$$
$$= \int \frac{\sin(-x)}{2} dx + \int \frac{\sin 9x}{2} dx$$
$$= \int \frac{-\sin x}{2} dx + \int \frac{\sin 9x}{2} dx$$
$$= \frac{\cos x}{2} - \frac{\cos 9x}{18} + C$$

Ex. $\int \frac{\cos^5 x}{\sqrt{\sin x}} dx = \int \cos^5 x \sin^{-\frac{1}{2}} x \, dx = \int \cos x \cos^4 x \sin^{-\frac{1}{2}} x \, dx$
$$= \int \cos x (1-\sin^2 x)^2 \sin^{-\frac{1}{2}} x \, dx$$

$\sin x = z$
$\cos x \, dx = dz$

$$= \int \cos x (1 - 2\sin^2 x + \sin^4 x)(\sin^{-\frac{1}{2}} x) dx$$
$$= \int \cos x (\sin^{-\frac{1}{2}} x - 2\sin^{3/2} x + \sin^{7/2} x) dx$$
$$= \int (z^{-\frac{1}{2}} - 2z^{3/2} + z^{7/2}) dz$$
$$= 2\sqrt{z} - \frac{2(z)^{5/2}}{5/2} + \frac{z^{9/2}}{9/2} + C$$
$$= 2\sqrt{\sin x} - \frac{4\sin^{5/2} x}{5} + \frac{2\sin^{9/2} x}{9} + C$$

Excercise: $\int \cot^5 x \sin^4 x \, dx = \int \frac{\cos^5 x}{\sin^{5} x} \sin^4 x \, dx = \int \cos x (1-\sin^2 x)^2 \sin^{-1} x \, dx = \int \cos x \left(\frac{1}{\sin x} - \frac{2\sin^2 x}{\sin x} + \frac{\sin^4 x}{\sin x}\right) dx$

$u = \sin x$
$du = \cos x \, dx$

$$= \int \cos x \left(\frac{1}{\sin x} - 2\sin x + \sin^3 x\right) dx$$
$$= \int (u^{-1} - 2u + u^3) du = \ln u - \frac{2u^2}{2} + \frac{u^4}{4} + C = \ln|\sin x| - \sin^2 x + \frac{\sin^4 x}{4} + C$$

$\int \cot^{2/3} x \csc^3 x \, dx = \int \frac{\cos^{2/3} x}{\sin^{2/3} x} \cdot \frac{1}{\sin^3 x} dx = \int$

$\int \cot x \csc x \left(\frac{\csc^2 x}{\cot^{1/3} x}\right) dx = \int \cot x \csc x \left(\frac{1+\cot^2 x}{\cot^{1/3} x}\right) dx$

$u = \cot x$
$du = -\csc x \cot x \, dx$

$$= -\int \frac{1+u^2}{u^{1/3}} du = -\int \left(\frac{1}{u^{1/3}} + u^{5/6}\right) du$$
$$= -\frac{3u^{2/3}}{2} - \frac{6u^{11/6}}{11} + C$$
$$= -\frac{3(\cot^{2/3} x)}{2} - \frac{6\cot^{11/6} x}{11} + C$$

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$\int f(x)\sqrt{a^2-y^2}\,dx$

if $f(x)=x \Rightarrow I=\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{2}\int \sqrt{z}\,dz = -\frac{z^{\frac{3}{2}}}{3}+C$

$= \frac{-(a^2-x^2)^{\frac{3}{2}}}{3}+C$

$a^2-x^2=z \Rightarrow -2x\,dx=dz$

$x\,dx=\frac{-dz}{2}$

This works when it defines a one to one function (for every x there is only one y)

Expression	Substitution	Identity	
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$	$\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$1+\tan^2\theta=\sec^2\theta$	$\frac{-\pi}{2}<\theta<\frac{\pi}{2}$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\sec^2\theta-1=\tan^2\theta$	$0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta\le\frac{3\pi}{2}$

Ex. $\int\sqrt{a^2-x^2}\,dx = \int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,d\theta = \int\sqrt{a^2(1-\sin^2\theta)}\,a\cos\theta\,d\theta$

$= \int\sqrt{a^2\cos^2\theta}\,a\cos\theta\,d\theta$

$x=a\sin\theta$

$dx=a\cos\theta\,d\theta$

$= \int a^2\cos^2\theta\,d\theta$

$= a^2\int\cos^2\theta\,d\theta$

$= a^2\int\frac{1}{2}(1+\cos 2x)\,d\theta$

$= \frac{a^2}{2}\int 1+\cos 2x\,dx$

$= \frac{a^2}{2}(\theta - \sin 2x + C)$

Ex. $\int\frac{\sqrt{9-x^2}}{x^2}\,dx = \int\frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}\,3\cos\theta\,d\theta = \int\frac{\sqrt{\cos^2\theta}\,\cos\theta}{\sin^2\theta}\,d\theta = \int\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$

$x=3\sin\theta$ $\quad \frac{x}{3}=\sin\theta$

$dx=3\cos\theta\,d\theta$ $\quad \sin^{-1}(\frac{x}{3})=\theta$

$= \int\cot^2\theta\,d\theta = \int(\underline{\cot^2\theta+1}-1)\,d\theta = \int\csc^2\theta\,d\theta - \int d\theta$

$= -\cot^2\theta - \theta + C$

$1+\cot^2\theta = \frac{1}{\sin^2\theta} = \frac{1}{(\frac{x}{3})^2}$

$= \frac{9}{x^2}$

$= \frac{-9}{x^2} - \sin^{-1}(\frac{x}{3}) + C$

$1+\tan^2\theta=\sec^2\theta$ $\qquad \int\sec^2\theta\,d\theta=\tan\theta$

$1+\cot^2\theta=\csc^2\theta$ $\qquad \int\csc^2\theta\,d\theta=-\cot\theta$

$\cos^2\theta+\sin^2\theta=1$

Ex. $\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} = \int \frac{\sec^2\theta\, d\theta}{2\tan^2\theta\sqrt{4(\tan^2\theta+1)}}$

$= \int \frac{\sec^2\theta\, d\theta}{4\tan^2\theta\sqrt{\sec^2\theta}} = \int \frac{\sec\theta\, d\theta}{4\tan^2\theta}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$= \int \left(\frac{1}{\cos\theta} \times \frac{\cos^2\theta}{4\sin^2\theta}\right) d\theta = \int \frac{\cos\theta}{4\sin^2\theta}\, d\theta$

$z = \sin\theta$

$dz = \cos\theta$

$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}\int \frac{1}{z^2}\, dz = \frac{1}{4}\frac{z^{-1}}{-1} = \frac{-1}{4z} + C$

$= \frac{-1}{4\sin\theta} + C$

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta$

$=$

$1+\tan^2\theta = \frac{1}{\cos^2\theta} \Rightarrow \cos^2\theta = \frac{1}{1+\tan^2\theta}$

$\cos^2\theta = \frac{1}{1+(\frac{x}{2})^2} = \frac{1}{1+\frac{x^2}{4}} = \frac{4}{4+x^2}$

$= \frac{-1}{4\sqrt{\frac{x^2}{4+x^2}}} + C$

$= \frac{-\sqrt{4+x^2}}{4x} + C$

$1-\sin^2\theta = \frac{4}{4+x^2}$

$-\sin^2\theta = \frac{4}{4+x^2} - 1$

$-\sin^2\theta = \frac{\cancel{4-4}-x^2}{4+x^2}$

$\sin^2\theta = \frac{x^2}{4+x^2}$

$\sin\theta = \sqrt{\frac{x^2}{4+x^2}}$

Ex. $\int \frac{x}{\sqrt{x^2+4}}\, dx = \int \frac{1}{\sqrt{z}}\frac{dz}{2} = \int \frac{dz}{\sqrt{z}\,2}$

$= \frac{1}{2}\int z^{-\frac{1}{2}}\, dz = \frac{1}{\cancel{2}}\frac{\cancel{2}z^{\frac{1}{2}}}{1} + C = z^{\frac{1}{2}} + C$

$z = x^2+4$

$dz = 2x\, dx$

$\frac{dz}{2} = x\, dx$

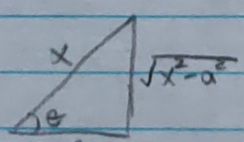
Ex. $I = \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}} = \int \frac{\cancel{a}\sec\theta\tan\theta\, d\theta}{\cancel{a}\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\cancel{\tan\theta}\, d\theta}{\cancel{\tan\theta}}$

$= \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$



$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C$

$\frac{x}{a} = \sec\theta$

$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$

$\cos\theta = \frac{a}{x}$

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Ex.

$$I = \int \frac{x^3}{\sqrt{(4x^2+9)^3}}\,dx = \int \frac{\frac{27}{8}\tan^3\theta\left(\frac{3}{2}\sec^2\theta\right)d\theta}{\sqrt{(9\tan^2\theta+9)^3}} = \int \frac{\frac{81}{16}\tan^3\theta\sec^2\theta\,d\theta}{9^{\frac{3}{2}}\sqrt{(\tan^2\theta+1)^3}}$$

$2x = 3\tan\theta$

$2dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{\frac{81}{16}\tan^3\theta\sec^2\theta\,d\theta}{27\sqrt{\sec^6 x}} = \frac{\frac{81}{16}}{27}\int \frac{\tan^3\theta\sec^2\theta\,d\theta}{\sec^3 x}$$

$$= \frac{3}{16}\int \frac{\tan^3\theta}{\sec x}\,d\theta = \frac{3}{16}\int \frac{\frac{\sin^3\theta}{\cos^3\theta}}{\frac{1}{\cos\theta}}\,d\theta = \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$I = \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta = \frac{3}{16}\int \frac{\sin\theta(1-\cos^2\theta)}{\cos^2\theta}\,d\theta$$

$z = \cos\theta$

$-dz = \sin\theta\,d\theta$

$$= \frac{3}{16}\int \frac{\sin\theta - \sin\theta\cos^2\theta}{\cos^2\theta}\,d\theta =$$

$$= \frac{3}{16}\int \frac{\sin\theta}{\cos^2\theta}\,d\theta - \frac{3}{16}\int \sin\theta\,d\theta$$

$$= \frac{-3}{16}\int \frac{1}{z^2}\,dz + \frac{3}{16}\cos\theta + C$$

$$= \frac{+3}{16z} + \frac{3}{16}\cos\theta + C$$

$$= \frac{3}{16\cos\theta} + \frac{3}{16}\cos\theta + C$$

$$\frac{3}{16}\left(\frac{1}{\cos\theta} + \cos\theta\right)\Bigg|_{x=0}^{x=\frac{3\sqrt{3}}{2}}$$

$x = \frac{3}{2}\tan\theta \Rightarrow x = \frac{3\sqrt{3}}{2} \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

$x = \frac{3}{2}\tan\theta \Rightarrow x = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$

$$\frac{3}{16}\left(\frac{1}{\cos\theta} + \cos\theta\right)\Bigg|_0^{\frac{\pi}{3}} = \frac{3}{16}\left(\frac{1}{\cos\frac{\pi}{3}} + \cos\frac{\pi}{3} - \frac{1}{\cos\theta} - \cos\theta\right)$$

$$= \frac{3}{16}\left(2 + \frac{1}{2} - 1 - 1\right) = \frac{3}{32}$$

$\frac{3\sqrt{3}}{2} = \frac{3}{2}\tan\theta$

$\sqrt{3} = \tan\theta$

$\frac{\pi}{3} = \theta$

$0 = \frac{3}{2}\tan\theta$

$0 = \tan\theta$

$0 = \theta$

Ex $x^2 + ax = x^2 + ax + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$

$= \left(x + \left(\frac{a}{2}\right)\right)^2 - \frac{a^2}{4}$

$$\int \sqrt{5 + 4x - x^2}\, dx$$

$$= \int \sqrt{-(x^2 - 4x - 5)}\, dx = \int \sqrt{-(x^2 - 4x + 2^2 - 2^2 - 5)}\, dx = \int \sqrt{-((x-2)^2 - 2^2 - 5)}\, dx$$

$$= \int \sqrt{-((x-2)^2 - 9)}\, dx = \int \sqrt{-(x-2)^2 + 9}\, dx = \int \sqrt{9 - (x-2)^2}\, dx$$

$x - 2 = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$x - 2 = 3\sin\theta$

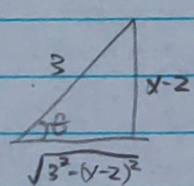
$\frac{x-2}{3} = \sin\theta$

$\sin^{-1}\left(\frac{x-2}{3}\right) = \theta$

$\cos\theta = \frac{\sqrt{9 - (x^2 - 4x + 4)}}{3}$

$= \frac{\sqrt{-x^2 + 4x + 5}}{3}$

$\sin 2\theta = 2\sin\theta\cos\theta$



$\cos\theta = \frac{\sqrt{3^2 - x^2 + 4x - 4}}{3}$

$= \frac{\sqrt{-x^2 + 4x + 5}}{3}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = \cos^2\theta - (1 - \cos^2\theta)$

$\cos 2\theta = \cos^2\theta - 1 + \cos^2\theta$

$\frac{1}{2}(\cos 2\theta + 1) = \cos^2\theta$

$= \int \sqrt{9 - 9\sin^2\theta}\; 3\cos\theta\, d\theta$

$= \int \sqrt{9(1 - \sin^2\theta)}\; 3\cos\theta\, d\theta$

$= \int 9\sqrt{\cos^2\theta}\, \cos\theta\, d\theta$

$= \int 9\cos^2\theta\, d\theta$

$= 9\int \cos^2\theta\, d\theta$

$= 9\int \frac{1}{2}(1 + \cos 2\theta)\, d\theta$

$= \frac{9}{2}\int (1 + \cos 2\theta)\, d\theta$

$= \frac{9\theta}{2} + \frac{9}{2}\int \cos 2\theta\, d\theta$

$= \frac{9\theta}{2} + \frac{9}{4}\sin 2\theta + C$

$= \frac{9\sin^{-1}\left(\frac{x-2}{3}\right)}{2} + \frac{9}{4}\sin\left(2\sin^{-1}\left(\frac{x-2}{3}\right)\right) + C$

$= \frac{9\sin^{-1}\left(\frac{x-2}{3}\right)}{2} + \frac{9}{2}\sin\theta\cos\theta + C$

$= \frac{9\sin^{-1}\left(\frac{x-2}{3}\right)}{2} + \frac{9}{2}\left(\frac{x-2}{3}\right)\left(\frac{\sqrt{-x^2+4x+5}}{\cancel{3}}\right) + C$

$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{1}{2}(x-2)\left(\sqrt{-x^2+4x+5}\right) + C$

MTH240 7.4 Intergration of Rational Function by Partial Function. Jan 18, 2018 Lei Yu

- Based on decomposing fraction into simpler fraction whose intergral can be easily evaluated.

$$f(x) = \frac{P(x)}{Q(x)}$$

Preliminary Step: If $\deg(P) \geq \deg(Q)$, P has to be divided by Q (long division) until a remainder $R(x)$ is obtained such that $\deg(R) < \deg(Q)$

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

Ex. $\int \frac{x^3+x}{x-1}\,dx$

$$\begin{array}{l} x^3+x \;\underline{|\, x-1} \\ -\underline{x^3-x^2}\;(x^2+x+2) \\ \quad x+x^2 \\ \;-\underline{x^2-x} \\ \quad 2x \\ \;-\underline{2x-2} \\ \quad 2 \end{array}$$

$$\Rightarrow \frac{x^3+x}{x-1} = \underbrace{(x^2+x+2)}_{S(x)} + \frac{\overbrace{2}^{(R(x))}}{x-1}$$ ← $Q(x)$

$$= \int\left((x^2+x+2) + \frac{2}{x-1}\right)dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln(x-1) + C$$

After the preliminary step (if needed) in order to $\int \frac{R(x)}{Q(x)}\,dx$ $\deg(R) \leq \deg Q$

4 Cases:

Case 1. $Q(x)$ is a product of distinct linear factors. In this case (according to partial fraction theorem.

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_2} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_K}{a_K+b_K}$$

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\ldots(a_Kx+b_K)$$

Ex. $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$Q(x) = 2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$

$$\Rightarrow \int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \frac{A}{x}\,dx + \int \frac{B}{2x-1}\,dx + \int \frac{C}{x+2}\,dx$$

$x^2+2x-1 = A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1)$

$x^2+2x-1 = A(2x^2+3x-2) + B(x^2+2x) + C(2x^2-x)$

$= A2x^2+3Ax-2A + Bx^2+2Bx + C2x^2 - Cx$

$= x^2(2A+B+2C) + x(3A+2B-C) - 2A$

$$\left.\begin{array}{l} 2A+B+2C = 1 \\ 3A+2B-C = 2 \\ -2A = -1 \end{array}\right\} \quad \begin{array}{l} A = \frac{1}{2} \\ B = \frac{1}{5} \\ C = \frac{-1}{10} \end{array}$$

Simpler way to find the constants (A, B, C)

$(x^2+2x-1) = A(2x-1)(x+2) + B(x+2)x + C(2x-1)x$

Since the expression above would be valid for any x

root

$x=\frac{1}{2} \Rightarrow \frac{1}{4}+1-1 = 0 + B(\frac{1}{2}+2)\frac{1}{2} + 0$

$\frac{1}{4} = B\frac{5}{4}$

$\frac{1}{5} = B$

$x=0 \Rightarrow -1 = A(-1)(2) + 0 + 0$

$-1 = -2A$

$\frac{1}{2} = A$

$x=-2 \Rightarrow 4-4-1 = 0+0+C((-5)(-2))$

$\frac{-1}{10} = C$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \frac{\frac{1}{2}}{x}\,dx + \int \frac{\frac{1}{5}}{2x-1}\,dx + \int \frac{\frac{1}{10}}{x+2}\,dx$$

$$= \frac{1}{2}\ln x + \frac{1}{10}\ln|2x-1| + \frac{1}{10}\ln|x+2| + C$$

Case II | $Q(x)$ is a product of linear factors, some of which are repeated. For example if the k^{th} linear factor is repeated r times. Then instead of a single term $\frac{A_k}{a_kx+b_k}$, we will have $\frac{A_{1k}}{a_kx+b_k} + \frac{A_{2k}}{(a_kx+b_k)^2} + \dots + \frac{A_{rk}}{(a_kx+b_k)^r}$

Ex $\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$

$Q(x) = x^2(x-1)^3$

Ex $I = \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int (x+1)\,dx + \int \frac{4x}{x^3-x^2-x+1}\,dx$ I_1

$x^3-x^2-x+1 = x^2(x-1)-x-1 = x-1(x^2-1) = (x-1)^2(x+1)$

$I_1 = \frac{4x}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$x=1 \Rightarrow 4 = 0 + B(2) + 0$

$2 = B$

$x=0 \Rightarrow 0 = A(-1) + 2 - 1$

$1 = A$

$x=-1 \quad -4 = 0 + 0 + C(4)$

$-1 = C$

MTH240 7.4 pg.2 Jan 18, 2018 Lei Yu

Ex. from end of pg.1

$$I_1 = \int \frac{4x}{x^3-x^2-x+1}\,dx = \int \frac{dx}{x-1} + \int \frac{2}{(x-1)^2}\,dx + \int \frac{-1}{x+1}\,dx$$

$$= \ln|x-1| + \frac{2}{x-1} - \ln|x+1| + C$$

$$\int (x+1)\,dx + I_1 = \frac{x^2}{2} + x + \ln|x-1| + \frac{2}{x-1} - \ln|x+1| + C$$

discriminant ↑

Case 3: $Q(x)$ contains irreducible quadratic factor ax^2+bc+c, where $b^2-4ac<0$.
In this case the fraction corresponding to this factor would be a

$$\frac{Ax+B}{ax^2+bx+c}$$

For example, $\frac{x}{x\underbrace{(x^2+1)(x^2+4)}_{\text{irreducible}}} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+4)}$

* $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Ex. $\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx = \int dx + \underbrace{\int \frac{x-1}{4x^2-4x+3}\,dx}_{= I_1}$

$$4x^2-3x+2 \,\big)\, 4x^2-4x+3 \quad 1$$
$$-\;4x^2-4x+3$$
$$x-1$$

$$\frac{x-1}{4x^2-4x+8} = \frac{Ax+B}{4x^2-4x+8}$$

$I_i = \int \frac{x-1}{4x^2-4x+3}\,dx$ case III

$x-1 = Ax+B$

$A=1$

$B=-1$

$\Delta = b^2-4ac = 16-48 = -32 < 0$

$$I_i = \int \frac{x-1}{4x^2-4x+3}\,dx = \int \frac{x-1-\frac{1}{2}+\frac{1}{2}}{4x^2-4x+3}\,dx = \underbrace{\int \frac{x-\frac{1}{2}}{4x^2-4x+3}\,dx}_{I_2} - \underbrace{\int \frac{\frac{1}{2}}{4x^2-4x+3}\,dx}_{I_3}$$

$8x-4 = 8(x-\frac{1}{2})$

$I_2 = \int \frac{x-\frac{1}{2}}{4x^2-4x+3}\,dx = \frac{1}{8}\int \frac{du}{u}$

$u = 4x^2-4x+3 \qquad = \frac{1}{8}\ln|u| + C$

$du = (8x-4)\,dx \qquad = \frac{1}{8}\ln|4x^2-4x+3| + C$

$\frac{du}{8} = (x-\frac{1}{2})\,dx$

$I_3 = \frac{1}{2}\int \frac{1}{4x^2-4x+3}\,dx$

$= \int \frac{1/8}{4x^2-4x+3}\,dx = \int \frac{1/8}{(x-\frac{1}{2})^2+\frac{1}{2}}\,dx$

$\therefore I = x + I_1 = x + I_2 + I_3$

$= x + \frac{1}{8}\ln|4x^2-4x+3| - \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{\sqrt{2}}}\right) + C$

Case 4 Ex. $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$1-x+2x^2-x^3 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)(x)$

$= A(x^4+2x^2+1) + (Bx^2+Cx)(x^2+1) + Dx^2 + Ex$

$= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$

$1-x+2x^2-x^3 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$

$1 = A$

$\cancel{1}-x+2x^2-x^3 = (1+B)x^4 + Cx^3 + (2+B+D)x^2 + (C+E)x\ \cancel{+1}$

$-1 = B$

$-x+2x^2-x^3 = Cx^3 + (1+D)x^2 + (C+E)x$ $\quad C=-1 \quad E=0$

$=$ $\quad D=1$

$\int \frac{1}{x}\,dx + \int \frac{-x-1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$

$= \ln|x| - \int \frac{x\,dx}{x^2+1} - \int \frac{1}{x^2+1}\,dx + \int \frac{x\,dx}{(x^2+1)^2}\,dx$ $\quad u = x^2+1$

$= \ln|x| - \tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + \int \frac{1}{2}\frac{1}{z^2}\,dz$ $\quad du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$= \ln|x| - \tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| - \frac{1}{2(x^2+1)} + C$

$x^2+1 = z$

$2x\,dx = z\,dz$

$x\,dx = \frac{z}{2}\,dz$

if $\Rightarrow$ $\int \frac{dx}{(x^2+1)^2}$ use $x = \tan\theta$

Rationalizing Substitution Jan 24, 2018 Lei Yu

Non-rational functions can be change to rational functions.

·When intergrand contains $\sqrt[n]{g(x)}$, sub $u=\sqrt[n]{g(x)}$

$\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx \qquad x^2+1=u \Rightarrow 2x\,dx=du \Rightarrow x\,dx=\frac{du}{2}$

$x^2=u-1$

Ex. $=\int \frac{x^2 x}{\sqrt[3]{x^2+1}}\,dx = \frac{1}{2}\int \frac{u-1}{u^{1/3}}\,du = \frac{1}{2}\int \frac{u}{u^{1/3}}\,du - \frac{1}{2}\int \frac{1}{u^{1/3}}\,du = \frac{1}{2}\cdot\frac{3u^{5/3}}{5} - \frac{1}{2}\cdot\frac{3u^{2/3}}{2}$

$= \frac{3u^{5/3}}{10} - \frac{3u^{2/3}}{4} + C$

$= \frac{3(x^2+1)^{5/3}}{10} - \frac{3(x^2+1)^{2/3}}{4} + C$

Ex. $\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2u\,du}{2u+u^2-3} = \int \frac{2u\,du}{u^2+2u-3} = \int \frac{2u\,du}{(u+3)(u-1)}$

$\sqrt{x+3}=u \Rightarrow \frac{dx}{2\sqrt{x+3}}=du \Rightarrow dx=2\sqrt{x+3}\,du=2u\,du$

$x+3=u^2$

$x=u^2-3$

$\int\left(\frac{A}{u+3}+\frac{B}{u-1}\right)du = \frac{3}{2}\int \frac{1}{u+3}\,du + \frac{1}{2}\int \frac{1}{u-1}\,du = \frac{3}{2}\ln|\sqrt{x+3}+3| + \frac{1}{2}\ln|\sqrt{x+3}-1| + C$

u=1 $2u=A(u-1)+B(u+3)$ u=-3 $-6=-4A$

$2=4B$ $\frac{3}{2}=A$

$\frac{1}{2}=B$

$$\begin{array}{r} u^2+u+1 \\ u-1\,\overline{)\,u^3+0^2+0^1+0} \\ u^3-u^2 \\ \hline u^2 \\ u-u \\ \hline u \\ u-1 \\ \hline 1 \end{array}$$

LCD

Ex. $\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3-u^2} = 6\int \frac{u^3\,du}{u-1} = 6\int\left(u^2+u+1+\frac{1}{u-1}\right)du$

$u=\sqrt[6]{x} \Rightarrow u^6=x \Rightarrow 6u^5\,du=dx \qquad = 6\left(\frac{u^3}{3}+\frac{u^2}{2}+u+\ln|u-1|\right)+C$

$u^2=\sqrt[3]{x},\ u^3=\sqrt{x} \qquad = 2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+\ln|\sqrt[6]{x}-1|+C$

Ex. $\int \frac{dx}{x\sqrt{4x+1}} = \int \frac{dx}{x\sqrt{(2\sqrt{x})^2+1}}$

$2\sqrt{x} = \tan\theta$

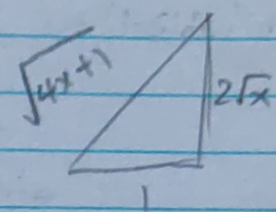
$4x = \tan^2\theta$

$x = \frac{\tan^2\theta}{4}$

$dx = \frac{1}{4}\, 2\tan\theta \sec^2\theta\, d\theta$

$= \int \frac{2\tan\theta\sec^2\theta}{\tan^2\theta\sec\theta}\, d\theta$

$= 2\int \frac{\sec\theta}{\tan\theta}\, d\theta = 2\int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\, d\theta$



$= 2\int \csc\theta\, d\theta = 2\ln|\csc\theta - \cot\theta| + C$

$= 2\ln\left|\frac{\sqrt{4x+1}}{2\sqrt{x}} - \frac{1}{2\sqrt{x}}\right| + C$

$= 2\ln\left|\frac{\sqrt{4x+1}-1}{2\sqrt{x}}\right| + C$

Ex. $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\, dx = \int \sin\theta\sqrt{2-\cos\theta}\,\cos\theta\, d\theta$

$x = \sin\theta$

$dx = \cos\theta$

$= \int \sqrt{u}\,(2-u)\, du = 2\int \sqrt{u}\, du - \int u^{3/2}\, du$

$= 2\,\frac{2u^{3/2}}{3} - \frac{2u^{5/2}}{5} + C$

$2-\cos\theta = u \Rightarrow 2-u = \cos\theta$

$\sin\theta = du$

$= \frac{4}{3}(2-\cos\theta)^{3/2}\Big|_0^{\frac{\pi}{2}} - \frac{2}{5}(2-\cos\theta)^{5/2}\Big|_0^{\frac{\pi}{2}} + C$

$0 \le x \le 1$

$0 \le \sin\theta \le \frac{\pi}{2}$

$= \frac{4}{3}\left((2-0)^{3/2} - 1\right) - \frac{2}{5}\left((2)^{5/2} - 1\right)$

$= \frac{4}{3}(2^{3/2}-1) - \frac{2}{5}(2^{5/2}-1)$

Ex. $\int \frac{dx}{1+2e^x - e^{-x}} = \int \frac{dx}{1+2e^x-\frac{1}{e^x}} = \int \frac{dx}{\frac{e^x+2e^{2x}-1}{e^x}} = \int \frac{e^x\, dx}{e^x+2e^{2x}-1}$

$e^x = u$

$e^x dx = du$

$= \int \frac{du}{u+2u^2-1} = \int \frac{du}{2u^2+u-1} = \int \frac{du}{(2u-1)(u+1)} = \int \left(\frac{A}{2u-1} + \frac{B}{u+1}\right) du$

$1 = A(u+1) + B(2u-1)$

$= \frac{2}{3}\int \frac{1}{2u-1}\, du - \frac{1}{3}\int \frac{1}{u+1}\, du = \frac{2}{6}\ln|2u-1| - \frac{1}{3}\ln|u+1| + C$

$u=-1$ $\quad \frac{-1}{3} = B$

$u=\frac{1}{2}$ $\quad 1 = A\left(\frac{3}{2}\right)$

$\frac{2}{3} = A$

$= \frac{1}{3}\left(\ln|2e^x-1| - \ln|e^x+1|\right) + C$

$= \frac{1}{3}\ln\left|\frac{2e^x-1}{e^x+1}\right| + C$

MTH240 Improper Intergrals Type 2 Jan 25, 2018 Lei Yu

Type II: when f(x) has an infinite discontinuity on the interval of intergration.

Definition:

a) if f(x) is continuous on [a, b) and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx = \begin{cases}\text{Convergent if limit exist}\\ \text{divergent if limit doesn't exist}\end{cases}$$

b) If f is continuous on (a, b] and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_a^b f(x)dx = \begin{cases}\text{Convergent}\\ \text{Divergent}\end{cases}$$

c) If f(x) has a discontinuity at c where $a<c<b$ and both $\int_a^c f(x)dx$, $\int_c^b f(x)dx$ are convergent, then we defined

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Ex. $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}} = \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}dx = \lim_{t\to 2^+} 2(\sqrt{x-2})\Big|_t^5$

$= \lim_{t\to 2^+}(2\sqrt{3}-2\sqrt{0})$

f(x) is discontinuous at $x=2$ $= 2\sqrt{3}$

Ex. $\displaystyle\int_0^{\pi/2} \sec x\,dx = \lim_{t\to\frac{\pi}{2}^-}\int_0^t \sec x\,dx = \lim_{t\to\frac{\pi}{2}^-}(\ln|\sec x+\tan x|)\Big|_0^t = \lim_{t\to\frac{\pi}{2}^-}(\ln|\sec t+\tan t|) = \infty$, divergent

Ex. $\displaystyle\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1} = \lim_{t\to 1^-}\int_0^t \frac{dx}{x-1} + \lim_{t\to 1^+}\int_t^3 \frac{dx}{x-1}$

$= \lim_{t\to 1^-}\ln(x-1)\Big|_0^t + \lim_{t\to 1^+}\ln|x-1|\Big|_t^3$

$= \lim_{t\to 1^-}[\ln(t-1) - \cancel{\ln|-1|}] + \lim_{t\to 1^+}[\ln|2| - \ln|t-1|]$

$= -\infty - \infty$

$\therefore \displaystyle\int_0^3 \frac{dx}{x-1}$ is divergent

$u = \ln x$ $\qquad$ $du = dx$

$du = \frac{1}{x} dx$ $\qquad$ $u = x$

Ex $\int_0^1 \ln x \, dx = \lim_{t\to 0^+} \int_t^1 \ln x \, dx = \lim_{t \to 0^+} \left(x \ln x \Big|_t^1 - \int_t^1 dx \right)$

$= \lim_{t\to 0^+} \left(\cancel{\ln 1} - t\ln t - 1 + \cancel{t} \right) = - \lim_{t \to 0^+} \left(1 + t \ln t \right) = -1$

L.H $= \lim_{t\to 0^+} \frac{\ln t}{t^{-1}} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{-t^{-2}} = \lim_{t\to 0^+} \frac{-t^2}{t} = \lim_{t\to 0^+} (-t) = 0$

Ex $\int \frac{\sqrt{1+x^2}}{x} dx = \int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta} \sec^2\theta \, d\theta = = \int \frac{\sec^3\theta}{\tan\theta} d\theta = \int \frac{\frac{1}{\cos^3\theta}}{\frac{\sin\theta}{\cos\theta}} d\theta = \int \frac{1}{\cos^2\theta \sin\theta} d\theta$

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$= \int \frac{\sin\theta}{\cos^2\theta \sin^2\theta} d\theta = \int \frac{\sin\theta}{\cos^2\theta(1-\cos^2\theta)} d\theta = \int \frac{\sin\theta}{\cos^2\theta - \cos^4\theta} d\theta$ $\qquad$ $u = \cos\theta$

$du = -\sin\theta$

$= -\int \frac{du}{u^2 - u^4} = \int \frac{du}{u^2(u^2-1)} = \int \frac{du}{u^2(u+1)(u-1)} = \int \left(\frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} + \frac{d}{u-1} \right) du$

$1 = A(u)(u+1)(u-1) + B(u+1)(u-1) + C(u-1)(u^2) + D(u^2)(u+1)$

$u=0$ $\qquad$ $-1 = +B$

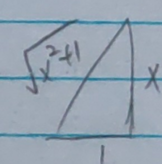
$1 = -2C$ $\qquad$ $I = -\int \frac{1}{u^2} du - \frac{1}{2} \int \frac{1}{u+1} du + \frac{1}{2} \int \frac{1}{u-1} du$ $\qquad$ $x = \tan\theta$

$u=-1$ $\qquad$ $\frac{-1}{2} = C$ $\qquad$ $= + \frac{1}{u} - \frac{1}{2} \ln|u+1| + \frac{1}{2} \ln|u-1| + C$

$1 = 2D$ $\qquad$ $= \frac{1}{\cos\theta} - \frac{1}{2} \ln|\cos\theta + 1| + \frac{1}{2} \ln|\cos\theta - 1| + C$

$u=1$ $\qquad$ $\frac{1}{2} = D$ $\qquad$ $= \sqrt{x^2+1} - \frac{1}{2} \ln\left|\frac{1}{\sqrt{x^2+1}} + 1\right| + \frac{1}{2} \ln\left|\frac{1}{\sqrt{x^2+1}} + 1\right| + C$

$0 = A$



Comparison Theerem for Improper Intergral Type 2

Suppose f and g are continuous functions such that

$F(x) \geq g(x) \geq 0$ for $x \geq a$

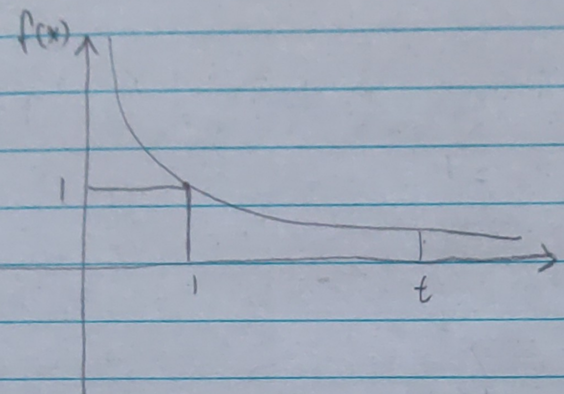
a) If $\int_a^\infty f(x) dx$ is conv $\Rightarrow \int_a^\infty g(x) dx$ is conv.

b) If $\int_a^\infty g(x) dx$ is Div $\Rightarrow \int_a^\infty f(x) dx$ is Div.

MTH240 Improper Intergrals Jan 25, 2018 Lei Yu

Type 1: when interval of integration is infinite.

$f(x) = \frac{1}{x^2}$



$$A(t) = \int_1^t f(x)\,dx = \int_1^t \frac{dx}{x^2} = \left.\frac{-1}{x}\right]_1^t = \frac{-1}{t} + \frac{1}{1} = 1 - \frac{1}{t}$$

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1 = \int_1^{\infty} \frac{1}{x^2}\,dx$$

Definition of an improper intergrals of Type 1:

a) If $\int_a^t f(x)\,dx$ exist for every t, $t \geq a$, then $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$.

1. If limit exist, the intergral is convergent
2. If limit doesn't exist, then intergral is divergent

b) If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$ $\begin{cases} \text{Convergent} \\ \text{Divergent} \end{cases}$

c) If both intergral $\int_a^{\infty} f(x)\,dx$ & $\int_{-\infty}^a f(x)\,dx$ (a can be any real number)

are convergent, then we define $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^{\infty} f(x)\,dx$

Ex. $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t \Big|_1^{t} = \lim_{t\to\infty} (\ln t - \ln 1) = \infty$, divergent, limit doesn't exist

Ex. $\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty} \int_t^0 xe^x\,dx = \lim_{t\to-\infty} xe^x - e^x \Big|_t^0$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$= \lim_{t\to\infty} (-1-(te^t - e^t)) = \lim_{t\to\infty} (-1 - te^t) = -1$

$\overset{L.H}{=} \lim_{t\to\infty} te^t = \lim_{x\to\infty} \frac{t}{e^{-t}} = \lim_{x\to\infty} \frac{1}{-e^{-t}} = \lim_{x\to\infty} -e^t = 0$

Ex. $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)}\,dx = \int_{-\infty}^{0} \frac{dx}{(1+x^2)} + \int_0^{\infty} \frac{dx}{(1+x^2)} = \arctan(x)\Big|_{-\infty}^{0} + \arctan(x)\Big|_0^{\infty}$

$= \lim_{t\to\infty} (0 - \arctan t) + \lim_{t\to\infty} (\arctan t - 0) = (0 - (\frac{-\pi}{2})) + (\frac{\pi}{2} - 0) = \pi$

Ex. $\int_1^{\infty} \frac{dx}{x^p} = \lim_{t\to\infty} \int_1^{t} \frac{dx}{x^p}$

If $P = 1 \Rightarrow \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$ is divergent

If $P \neq 1 \Rightarrow \int_1^t \frac{1}{x^p}\,dx = \frac{x^{1-P}}{1-P}\Big]_1^t = \frac{t^{1-P}}{1-P} - \frac{1}{1-P}$

$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{t\to\infty} \left(\frac{t^{1-P}}{1-P} - \frac{1}{1-P}\right) = \lim_{t\to\infty} \frac{1}{1-P}(t^{1-P} - 1)$

$1-P<0$

if $P>1 \Rightarrow \lim_{t\to\infty} \frac{1}{1-P}\left[\cancel{\frac{1}{t^{P-1}}}^{\,0} - 1\right]$

$= \frac{-1}{1-P}$

if $P<1 \Rightarrow \lim_{t\to\infty} \frac{1}{1-P}\left[t^{1-P} - 1\right] = \infty$

$\Rightarrow \int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{Convergent if } P>1 \\ \text{divergent if } P\leq 1 \end{cases}$