CHAPTER 9

MTH246 **Differential Equation DE** Jan 31, 2018 Lei Yu

A DE is an equation that contains an unknown function and one or more of its derivatives. The order of a DE is the order of the highest derivatives in the equation.

Separable DE

A separable DE is a first order DE in which the expression $\frac{dy}{dx}$ can be factored as a function of x times a function of y.

meaning $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ which can be solved like the following

$$\int h(y)\,dy = \int g(x)\,dx$$

Ex $\frac{dy}{dx} = \frac{x^2}{y^2} \Rightarrow \int y^2\,dy = \int x^2\,dx \Rightarrow \frac{y^3}{3} = \frac{x^3}{3} + 3 \Rightarrow y = \sqrt[3]{x^3 + 3C}$

Ex $\frac{dy}{dx} = \frac{y^2-1}{x^2-1}$, $y(2) = 2$

$$\int \frac{dy}{y^2-1} = \int \frac{dx}{x^2-1} \Rightarrow \int \frac{A}{y+1}\,dy + \int \frac{B}{y-1}\,dy = \int \frac{C}{x+1}\,dx + \int \frac{D}{x-1}\,dx$$

$1 = A(y-1) + B(y+1)$

$y=1$: $\frac{1}{2} = B$ $\quad C = \frac{-1}{2}$

$y=-1$: $\frac{-1}{2} = A$ $\quad D = \frac{1}{2}$

$$\frac{-1}{2}\int \frac{1}{y+1}\,dy + \frac{1}{2}\int \frac{1}{y-1}\,dy = \frac{-1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$\frac{-1}{2}\ln|y+1| + \frac{1}{2}\ln|y-1| = \frac{-1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

$$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

$$\frac{1}{2}\ln\left|\frac{1}{3}\right| = \frac{1}{2}\ln\left|\frac{1}{3}\right| + C$$

$$0 = C$$

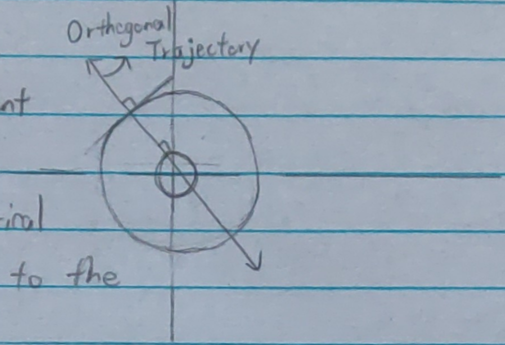
$$\therefore \ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right|$$

Orthogonal Trajectories

An orthogonal trajectories of a family of curves is a curve that intersects each curve of the family at right angles.

$x^2 + y^2 = k$, changing k will change curve, a family of curve.

if two curves are orthogonal, than the slope of the tangent line to one curve at one point is the negative reciprical of the slope of the tangent to the other curve at that point.



Ex. Find the orthogonal Trajectories of the curves $x = ky^2$ with k being an arbituary constant.

1. Find DE

$$x = ky^2$$

$$dx = 2ky\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2ky}$$

2. Make DE independant of k.

$$\frac{dy}{dx} = \frac{1}{2ky} \qquad x = ky^2$$

$$\frac{dy}{dx} = \frac{y}{2x} \qquad \frac{x}{y^2} = k$$

If we call the orthogonal trajectory y

Since it is orthogonal to the family of curves.

$$\left(\frac{dy}{dx}\right)_{traj} = -\frac{1}{\frac{y}{2x}} = \frac{-2x}{y}$$

$$\int y\,dy = \int -2x\,dx$$

$\frac{y^2}{2} = -x^2 + C$ → this is the orthogonal trajectory which it self is a curve

MTH240 First Order Linear DE Jan 31, 2018 Lei Yu

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Ex $x^2y' + y = 2x \Rightarrow y' = \frac{y}{x^2} = \frac{2}{x}$

Such equations can be solved by multiplying both sides by a suitable function $I(x)$. Which is called an intergrating factor.

Proof

$$y' + p(x)y = Q(x)$$

$$I(x)(y' + p(x)y) = I(x)Q(x) \quad ①$$

In this method $I(x)$ have to found such that the following equation is satisfied.

$$I(x)[y' + p(x)y] = [I(x)y]' \quad ②$$

$$I(x)Q(x) = [I(x)y]' \Rightarrow I(x)y = \int I(x)Q(x)\,dx$$

$$I(x)Q(x) = \frac{d(I(x)y)}{dx}$$

$$\Rightarrow y = \frac{1}{I(x)}\left[\int I(x)Q(x)\,dx + C\right]$$

$$② \; \cancel{I(x)y'} + I(x)p(x)y = \cancel{I(x)y'} + I(x)'y$$

$$I(x)P(x)\cancel{y} = \frac{dI(x)}{dx}\cancel{y}$$

$$I(x)P(x) = \frac{dI(x)}{dx} \Rightarrow \frac{dI(x)}{I(x)} = P(x)\,dx$$

$$\int \frac{dI}{I} = \ln|I| = \int P(x)\,dx + D$$

$$I(x) = e^{\int P(x)dx + D} = \pm e^{\int P(x)dx}e^{D} = Ae^{\int P(x)dx}$$

$$y = \frac{1}{Ae^{\int P(x)dx}}\left[Ae^{\int P(x)dx}Q(x)\,dx + C\right] = \frac{1}{e^{\int P(x)dx}}\left[\int e^{\int P(x)dx}Q(x)\,dx + \underbrace{\left(\frac{C}{A}\right)}_{k}\right]$$

$$y = \frac{1}{e^{\int Pdx}}\left[\int e^{\int Pdx}Q(x)\,dx + k\right]$$

for

$$y' + P(x)y = Q(x)$$

$y = \frac{1}{e^{\int P dx}} \left[\int e^{\int P dx} Q(x) dx + k \right]$ $\qquad y' + P(x)y = Q(x)$

Ex. $\frac{dy}{dx} + 3x^2 y = 6x^2$

$P(x) = 3x^2$
$Q(x) = 6x^2$

$e^{\int P dx} = e^{\int 3x^2 dx} = e^{x^3}$

$y = e^{-x^3} \left[\int e^{+x^3} 6x^2 dx + k \right]$
$= e^{-x^3} \left[\int e^{+z} 2 dz + k \right]$
$= e^{-x^3} \left[+2e^{+z} + k \right]$
$= e^{-x^3} (+2e^{x^3} + k)$
$= +2 + ke^{-x^3}$

$x^3 = z$
$3x^2 dx = dz$

Ex. $x^2 y' + xy = 1$, $x > 0$, $y(1) = 2$
$y' + \frac{y}{x} = \frac{1}{x^2}$

$P(x) = \frac{1}{x}$
$Q(x) = \frac{1}{x^2}$
$y(1) = 2$

$\Rightarrow y = \frac{1}{e^{\int \frac{dx}{x}}} \left[\int e^{\int \frac{dx}{x}} \frac{1}{x^2} dx + c \right] = x^{-1} \left[\int \frac{x}{x^2} dx + c \right] = \frac{1}{x} \left[\ln x + c \right]$

$2 = \frac{1}{1} [\ln 1 + c)$ $\therefore y = \frac{1}{x} [\ln x + 2]$
$2 = c$

Ex. $y' + 2xy = 1$

$P(x) = 2x$
$Q(x) = 1$

$e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$

$y = \frac{1}{e^{x^2}} \left[\int e^{x^2} dx + C \right]$

$\int e^{x^2}$ have no closed form

MTH 240 Comparison Theorem Jan 31 2018 Lei Yu

Suppose f & g are continuous functions such that $f(x) \geq g(x) \geq 0$.

for $x \geq a$

① if $\int_a^\infty f(x)dx$ is Conv $\Rightarrow$ $\int_a^\infty g(x)dx$ is Conv

② if $\int_a^\infty g(x)dx$ is Div $\Rightarrow$ $\int_a^\infty f(x)dx$ is Div.

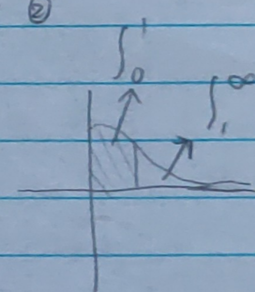
Ex. $\int_0^\infty e^{-x^2} dx$ have no closed formed solution

We know that e^x is increasing for $x>1$

$x < x^2 \Rightarrow -x > -x^2 \Rightarrow e^{-x} > e^{-x^2}$ ①

$\int_1^\infty e^{-x} dx = -e^{-x}\Big|_1^\infty = \lim_{t\to\infty} -e^{-t} + e^{-1} \Rightarrow \int_1^\infty e^{-x}dx$ is Conv ②

$\int_1^\infty -e^{x^2}dx$ is conv $\Rightarrow \int_0^\infty e^{-x^2}dx$ $\quad \int_0^\infty = \int_0^1 + \int_1^\infty$



$\int_1^\infty \frac{1}{x^p} \begin{cases} \text{Conv } p>1 \\ \text{Div } p<0 \end{cases}$

$\boxed{\int_a^\infty f(x)dx}$ If there exist that $b>a$ and $\int_b^\infty f(x)dx$ is Conv $\Rightarrow \int_a^\infty f(x)dx$ is Conv

Ex. Use comparison theorem to determine if the following intergral is Conv or Div.

$\int_0^\infty \frac{x}{x^3+1}dx$ we know $\frac{x}{x^3+1} < \frac{x}{x^3} \Rightarrow \frac{x}{x^3+1} < \frac{1}{x^2}$

$\int_1^\infty \frac{1}{x^2}$ is Conv $\Rightarrow \int_1^\infty \frac{x}{x^3+1}dx \Rightarrow \int_0^\infty \frac{x}{x^3+1}dx$ is convergence

Ex $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$ $\therefore$ is *divergent*

$-1 \leq \cos x \leq 1 \Rightarrow 0 \leq \cos^2 x \leq 1 \Rightarrow \frac{1}{\cos^2 x} \geq 1 \Rightarrow \sec^2 x \geq 1$

$\Rightarrow \frac{\sec^2 x}{x\sqrt{x}} \geq \frac{1}{x\sqrt{x}}$

$\int_0^1 \frac{1}{x\sqrt{x}}\,dx = \int_0^1 \frac{1}{x^{3/2}}\,dx = \frac{2x^{-1/2}}{-1}\Big|_0^1 = -\frac{2}{\sqrt{x}}\Big|_0^1 = \frac{-2}{1} + \lim_{x\to 0^+}\frac{1}{\sqrt{x}} = -2 + \infty$

This example shows that comparison theorem can be applied to proper intergral of type two. i.e when function has a discontinuity at either upper or lower bound.

Ex $\int_{-\pi/2}^{\pi/2} \frac{x\,dx}{\cos^2 x + 1} = 0$

$f(-x) = -f(x)$ point symmetry

odd function

Find the orthogonal trajectory. $y = \frac{2}{1+kx}$.

$\frac{dy}{dx} = \frac{1+k-kx}{(1+kx)^2} = \frac{1}{(1+kx)^2} = \frac{1}{(\frac{x}{y})^2} = \frac{y^2}{x^2}$

$\left(\frac{dy}{dx}\right)_{traj} = \frac{-x^2}{y^2} \Rightarrow \int y^2\,dy = -\int x^2\,dx$

$\frac{y^3}{3} = \frac{-x^3}{3} + C$ $\therefore \frac{y^3}{3} + \frac{x^3}{3} = C$

Bernoulli DE Feb 1, 2018 Lei Yu

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

If $n=0$ or 1, It is first order linear DE.

For other value of n. Make the following substitution. $u = y^{1-n}$

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y^n}{1-n}\frac{du}{dx}$$

$$① \Rightarrow \frac{y^n}{1-n}\frac{du}{dx} + P(x)y = Q(x)y^n$$

$$\frac{du}{dx} + (1-n)y^{1-n}p(x) = (1-n)y^{-n}y^n Q(x)$$

$$\boxed{\frac{du}{dx} + (1-n)u\,p(x) = (1-n)Q(x)}$$ Familiar Form.

Ex. $xy' + y = -xy^2 \Rightarrow y' + \frac{y}{x} = -y^2$

$n=2$

$P(x) = \frac{1}{x}$

$Q(x) = -1$

$$\frac{du}{dx} + (1-2)(u)\frac{1}{x} = (1-2)(-1)$$

$$\frac{du}{dx} - \frac{u}{x} = 1$$ first order

$$u = \frac{1}{e^{-\int\frac{dx}{x}}}\left[\int e^{-\int\frac{dx}{x}}dx + C\right]$$

$$e^{-\int\frac{dx}{x}} = e^{-\ln x} = e^{\ln\frac{1}{x}} = \frac{1}{x}$$

$P(x) = \frac{-1}{x}$

$Q(x) = 1$

$$u = x\left[\int\frac{dx}{x} + C\right] = x(\ln x + C)$$

$$y^{-1} = x(\ln x + C)$$

$$y = x^{-1}(\ln x + C)^{-1}$$

Ex. $xy' - y = x^2\cos x \Rightarrow y' - \frac{y}{x} = x\cos x$

$P(x) = \frac{-1}{x}$

$Q(x) = x\cos x$

$e^{-\int P dx} = e^{-\int\frac{dx}{x}} = e^{\ln\frac{1}{x}} = \frac{1}{x}$

$$y = x\left(\int \cos x + C\right)$$

$$y = x(\sin x + C)$$

Midterm I Review

Ex. $\int_0^1 (1+\sqrt{x})^8 dx = \int_0^1 u^8 \, 2(u-1) dx$

$u = 1+\sqrt{x}$ $\quad$ $u-1 = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$2\sqrt{x}\, du = dx$

$2(u-1)\, du = dx$

$= 2\int_0^1 (u^9 - u^8) dx$

$= 2\left(\frac{u^{10}}{10} - \frac{u^9}{9}\right)\Big|_0^1 = 2\left(\frac{(1+\sqrt{x})^{10}}{10} - \frac{(1+\sqrt{x})^9}{9}\right)\Big|_0^1$

$= 2\left[\left(\frac{2^{10}}{10} - \frac{2^9}{9}\right) - \frac{1}{10} + \frac{1}{9}\right] = 2\left(\frac{1023}{10} - \frac{513}{9}\right) =$

Ex. $\int \frac{xe^x}{\sqrt{1+e^x}} dx = \int \frac{\ln|u-1|\, du}{\sqrt{u}}$

D	I
$\ln\lvert u-1\rvert$	$u^{-\frac{1}{2}} du$
$\frac{1}{u-1} du$	$2u^{\frac{1}{2}}$

$u = 1+e^x$ $\quad$ $u-1 = e^x$

$du = e^x dx$ $\quad$ $\ln\lvert u-1\rvert = x$

$= 2u^{\frac{1}{2}} \ln|u-1| - \int \frac{2\sqrt{u}}{u-1} du$

$= 2\sqrt{u} \ln|u-1| - 2\int \frac{\sqrt{u}}{u-1} du$