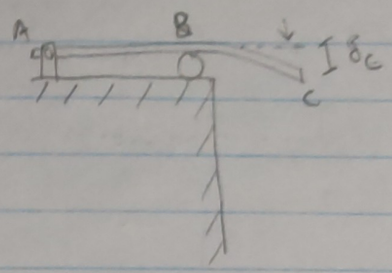
81029

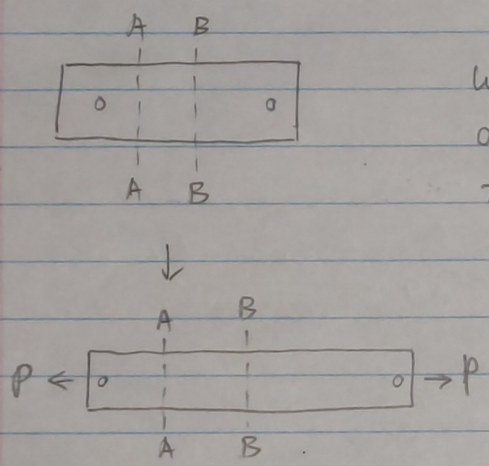
Lec 12: Intro to Mechanics of Material (MM 1.1)

Statics - study of bodies in equilibrium
Dynamics - " " " " motion
Mechanics - " " behaviors of bodies and materials under loading.



In mechanics, we ask questions like,
1. What weight W would cause the board to break?
2. What is the relationship b/w deflection δ_c and weight W?

Normal Stress

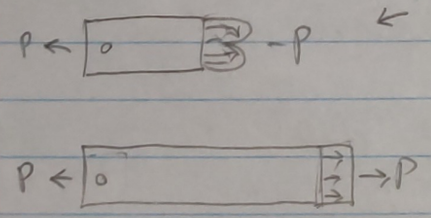


A straight bar undergoing tension (loading) will elongate. While the force distribution near the point of application of force is non-linear, far away from the point of loading, the internal force distribution is approximately uniform.

$$\text{Stress} = \frac{\text{force}}{\text{area}} \approx \text{Pressure}$$

unit: N/m^2 or Pascal S.I.

psi or ksi (kips per square in) English



Internal force not uniform

we assume force uniform.

Internal force uniform

① $$\boxed{\text{normal stress} = \frac{\text{force normal to area}}{\text{area}}}$$

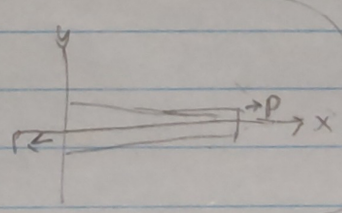
② Average normal stress at cross section $\Big\}$ $\sigma_{av} = \frac{F}{A}$ (F → total force, A → area)

$\sigma \begin{cases} +ve = \text{tension} \\ -ve = \text{compression} \end{cases}$

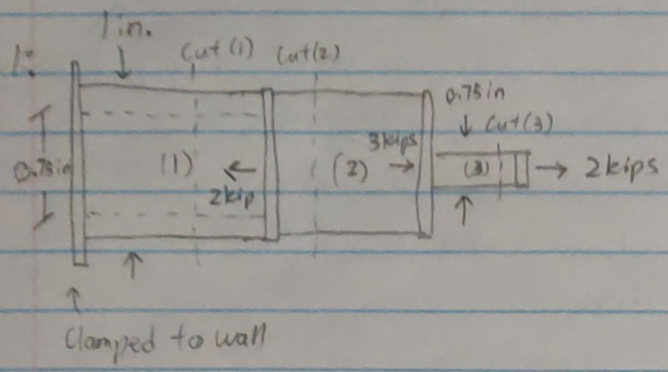
③ More generally:

$$\sigma(x) = \frac{F(x)}{A(x)}$$

function of location



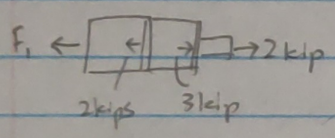
Example 1:



The three-part axially loaded member consists of a tubular segment (1) with outer diameter $(d_o)_1 = 1$ in, and inner diameter $(d_i)_1 = 0.75$ in, on solid circular rod segment (2) with diameter $d_2 = 1$ in, and another solid rod segment (3) with diameter $d_3 = 0.75$ in. Find axial stress, σ_1, σ_2, σ_3.

loading in axial direction

Cut (1), right side



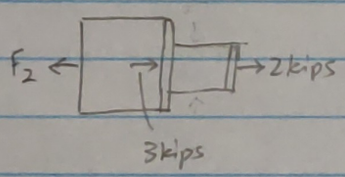
$\Sigma F_x = 0$; $2 - 2 + 3 - F_1 = 0$, $F_1 = 3$ kips

$\sigma_1 = \frac{F_1}{A_1} \doteq 8.73\ \text{kip/in}^2$ $\quad A_1 = \pi r_o^2 - \pi r_i^2$

$= \pi(0.5^2 - 0.375^2) =$

+ve = tension

Cut (2)

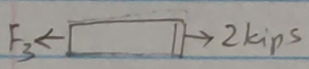


$\Sigma F_x = 0$ $\quad -F_2 + 2\text{ kips} + 3\text{ kips} = 0$, $F_2 = 5$ kips

$\sigma_2 = \frac{F_2}{A_2} = \frac{5\text{ kips}}{\pi(0.5)^2} \doteq 6.37$ ksi

Tension

Cut (3)



$\Sigma F_x = 0$; $2\text{ kips} - F_3 = 0$, $F_3 = 2$ kips

$\sigma_3 = \frac{F_3}{A_3} = \frac{2\text{ kips}}{(0.375)^2\pi} \doteq 4.53$ ksi [tension]

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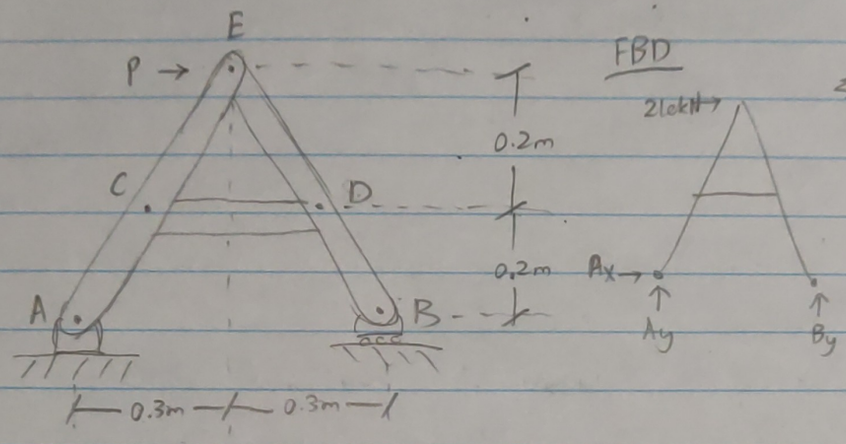
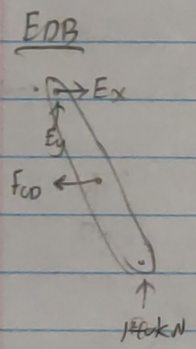
Example 2: Cross-brace member CD has a rectangular cross-section measuring 30mm × 50mm. If p = 210kN, what is the axial stress in member CD?



$\Sigma M_A = 0$; $B_y(0.6) - 210(0.4) = 0$

$B_y = 140kN$

no force P. ∴ P is not applied to 1 single member. Ex, Ey will account for P



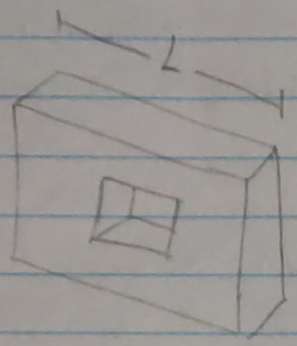
$\Sigma M_E = 0$; $140kN(0.3m) - F_{CD}(0.2m) = 0$

$F_{CD} = 210kN$

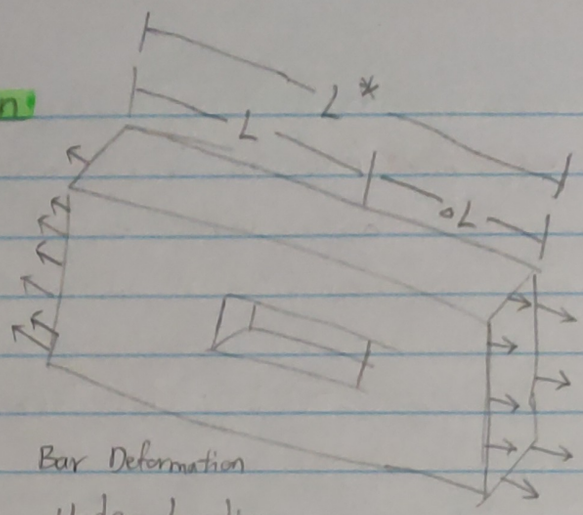
$$\sigma_{CD} = \frac{F_{CD}}{A_{CD}} = \frac{210kN}{30mm \times 50mm} = \frac{210 \times 10^3 N}{} = 140 \times 10^6 \, N/m$$

Convert units.

Extensional / Thermal Strain:



undeformed bar



Bar Deformation Under Loading

When a solid body is subjected to external and/or temperature change, it deforms.

Axial strain, is defined as $\boxed{\varepsilon = \frac{\Delta L}{L}}$, when $\varepsilon > 0$, tensile strain

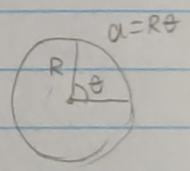
$\varepsilon < 0$, compressive strain

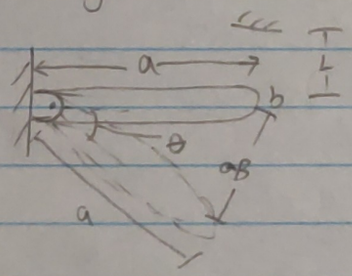
Note: strain is dimensionless, but it is common to report in term of $\frac{\text{length}}{\text{length}}$. eg. in/in, μin/in, or $\frac{\mu m}{mm}$

$$\boxed{\varepsilon = \frac{L^* - L}{L}}$$

Small-displacement / small-strain behavior:

We can generalize the small-strain behavior to the following,





When $\theta \ll 1$, we assume $\overset{\text{arc length}}{\overset{\frown}{AB}} = \underset{\text{radius}}{a}\ \theta \rightarrow$ angle (rad)

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Lec 13

External/Thermal Strain Cont'd (MM2.1)

Thermal Strain: Most engineering materials respond to a uniform increase in temperature, ΔT, by expanding in all directions by uniform amount.

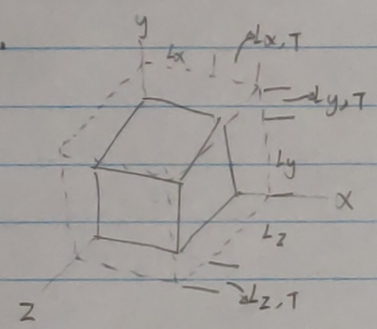
$$\boxed{\varepsilon_T = \alpha \Delta T} \text{ (Thermal Strain)}$$

ε_T, thermal strain

α, coefficient of thermal expansion [Temperature]

ΔT, temperature change. [Temperature]

e.g.



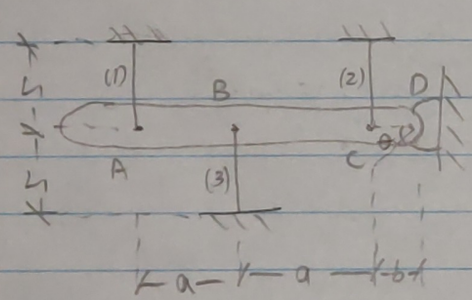
If the block is free to expand in all direction, then $\varepsilon_x(T) = \varepsilon_y(T) = \varepsilon_z(T) = \alpha \Delta T$

So, $\Delta L_{x,T} = (\alpha \Delta T) L_x$

$\Delta L_{y,T} = (\alpha \Delta T) L_y$

$\Delta L_{z,T} = (\alpha \Delta T) L_z$

Example 1:



A rigid beam AD is supported by a smooth pin at D and by vertical rods attached at A, B, C. Rods are all strain free when the beam is horizontal. Then rod (2) is heated until its extensional strain $\varepsilon_2 = 80 \times 10^{-6}$ in/in

$L_1 = L_3 = 30$ in $\quad a = 20$ in

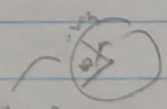
$L_2 = 40$ in $\quad b = 60$ in

a) find resulting θ

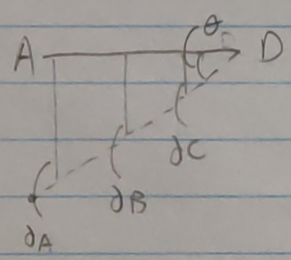
b) Determine, strains $\varepsilon_1, \varepsilon_3$.

a) geometry of deformation

Assume $\theta \ll 1$ rad.



arclength $= R\theta$



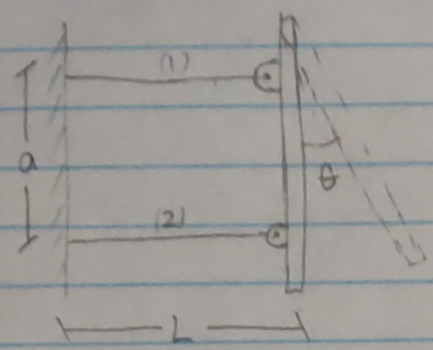
$\delta C = b\theta$

$\varepsilon_2 = \frac{\delta C}{L_2} \rightarrow 80 \times 10^{-6} = \frac{60\text{in}(\theta)}{40\text{in}}, \therefore \theta = 5.3 \times 10^{-5}$ rad

b) $\varepsilon_1 = \frac{\delta A}{L_1} = \frac{(a+a+b)(\theta)}{L_1} = \frac{(100\text{in})(5.3 \times 10^{-5}\text{rad})}{30\text{in}} \doteq 1.77 \times 10^{-4}$ in/in

other direction $\leftarrow \varepsilon_3 = \frac{-\delta B}{L_3} = \frac{-(a+b)\theta}{L_3} = \frac{-80\text{in}(5.3 \times 10^{-5}\text{rad})}{30\text{in}} = -1.41 \times 10^{-4}$ in/in

Example 2:



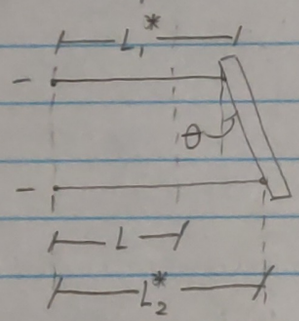
The angular orientation θ of a rigid mirror is controlled by length of rods (1) and (2). At the reference temperature, the rod length are $L_1 = L_2 = 2m$. The distance between the rods, $a = 1.2m$.

If the thermal coefficient of expansion of the rods is $\alpha = 14 \times 10^{-6}\ /^{\circ}C$, determine an expression for θ (in radians) as a function of ΔT_1, and ΔT_2.

$\varepsilon_T = \alpha \Delta T$

$\frac{\Delta L}{L} = \alpha \Delta T \rightarrow \Delta L = L \alpha \Delta T$

Geometry of Deformation



$L_1^* = L + \alpha \Delta T_1 L$

$L_2^* = L + \alpha \Delta T_2 L$

$$\tan\theta = \frac{L_2^* - L_1^*}{a}$$

$$\theta = \tan^{-1}\left(\frac{\cancel{L} + \alpha \Delta T_2 L - (\cancel{L} + \alpha \Delta T_1 L)}{a}\right)$$

$$= \tan^{-1}\left(\frac{\alpha \Delta T_2 L - \alpha \Delta T_1 L}{a}\right)$$

$$\theta = \tan^{-1}\left(\frac{(\Delta T_2 - \Delta T_1)\alpha L}{a}\right) \text{ rad}$$

$$\theta = \tan^{-1}\left[\frac{70}{3} \times 10^{-6} (\Delta T_2 - \Delta T_1)\right] \text{ rad}$$

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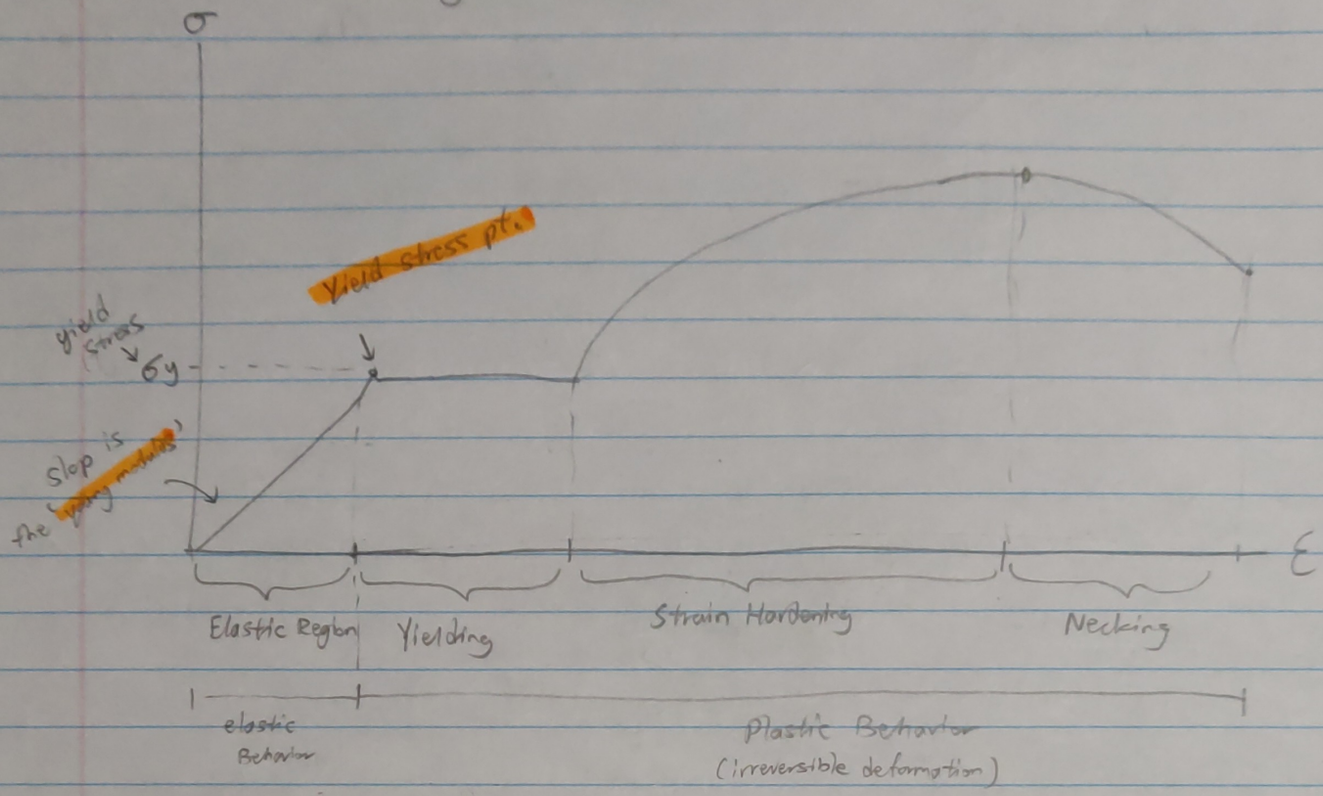
Stress-Strain Diagram/Material Property of material [MM 2.4]



The slope of the elastic region of the stress-strain curve is called the Modulus of Elasticity (or Young's Modulus) of the Material.

$$\boxed{\text{Young's Modulus} \rightarrow E = \frac{\partial \sigma}{\partial \varepsilon}}$$

σ_y - the yield stress, corresponds to the pt at which the material is no longer elastic.

Robert Hooke:
- discoverd Cell
- optic
- gravity

Linear Elasticity

Hooke's Law: For the case of uni axial stress applied to a homogeneous (same materials everywhere), isotropic (same properties in all directions) member oriented along the x-axis, the linear relationship between stress and strain. $(0 \leq \underbrace{\sigma \leq \sigma_y}_{\text{before yield stress.}})$ is,

$$\boxed{\text{Hooke's Law} \quad \sigma_x = E\varepsilon_x}$$

σ_x: Stress in x-direction $\left(\frac{\text{Force}}{\text{area}}\right)$

E: Young's Modulus $\left[\frac{\text{Force}}{\text{area}}\right]$

ε_x: Strain in x-direction (dimensionless)

Analogus to $F = kx$

Typical Values:

$E = 200\,GPa$ (steel)

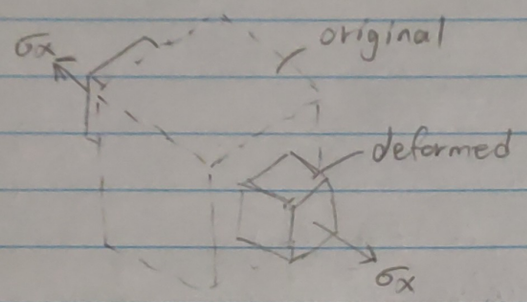
$E = 2\,GPa$ (Plastics)

Poisson's Ratio

When there is elongation of a member in the direction of axial tension, there is also a transverse contraction, described by Poisson's Ratio.

$$\boxed{\varepsilon_{transverse} = -\nu\,\varepsilon_{elongation}}$$

where, ν "nu" is the poisson's ratio.



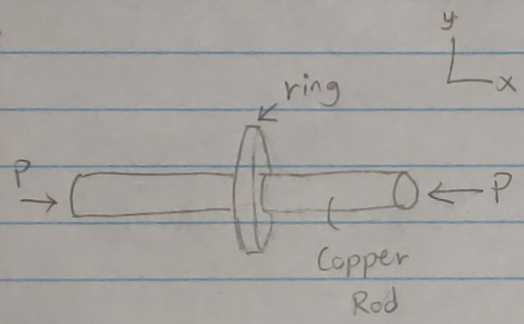
in this diagram,

$$\varepsilon_y = \varepsilon_z = -\nu\,\varepsilon_x$$

$$\sigma_x = E\,\varepsilon_x$$

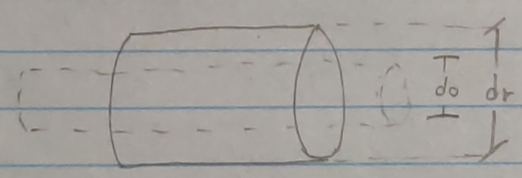
The same expression holds in compression.

Example 1:



The copper rod has $E = 17\times10^3$ ksi, $\nu = 0.33$, and initial diameter, $d_o = 1.9998$ in. The ring has inner diameter, $d_r = 2.000$ in. What is the critical load P_{cr}, below which the rod is able to stick freely through the ring?

Deformation Diagram



y, x

① $\varepsilon_y = -\nu\,\varepsilon_x$

$$\frac{-\varepsilon_y}{\nu} = \varepsilon_x$$

③ $$\varepsilon_x = \frac{-1.0001\times10^{-4}}{0.33}$$

$$\doteq -3.03\times10^{-4}$$

→ negative means compression

② $$\varepsilon_y = \frac{d_r - d_o}{d_o}$$

$$= \frac{2 - 1.9998}{1.9998}$$

$$= 1.0001\times10^{-4}\ \text{in/in}$$

→ + means increase in size

④ $$\sigma_x = E\,\varepsilon_x$$

$A = \pi r^2$

$r = \frac{1}{2}d$

$A = \pi\frac{1}{4}d^2$

$$\sigma_x = \frac{P_{cr}}{\frac{1}{4}\pi d_o^2} = E\,\varepsilon_x$$

$$\frac{P_{cr}}{\frac{1}{4}\pi(1.9998)^2} = (17\times10^3\,\text{ksi})(-3.03\times10^{-4})$$

$$P_{cr} \doteq +16.2\ \text{kips}$$

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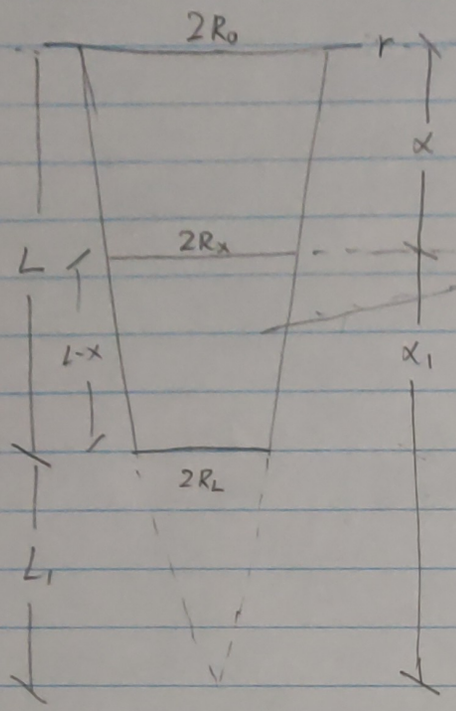
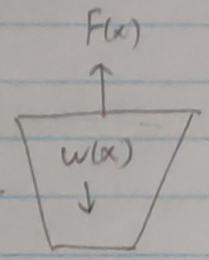
Lab 8

Prob 2.2-4

- Candidate for final Exam



FBD



$V_{cone} = \frac{\pi}{3} r^2 h$

Proportionality:

$\frac{R_L}{L_1} = \frac{R_x}{x_1}$; $\boxed{x_1 = L_1 + (L - x)}$

$\frac{L_1}{R_L} = \frac{L_1 + L}{R_0}$; $L_1 R_0 = R_L(L_1 + L)$

$L_1 = \left(\frac{R_L}{R_0 - L}\right) L$

$x_1 = \frac{R_0 L}{R_0 - R_L} - x$

$\sigma(x) = \frac{F(x)}{A(x)}$

$w(x) = \rho V(x) g$

$V(x) = V_{cone}(x_1) - V_{cone}(L_1)$

$= \frac{\pi}{3}(R_x^2 x_1 - R_L^2 L_1)$

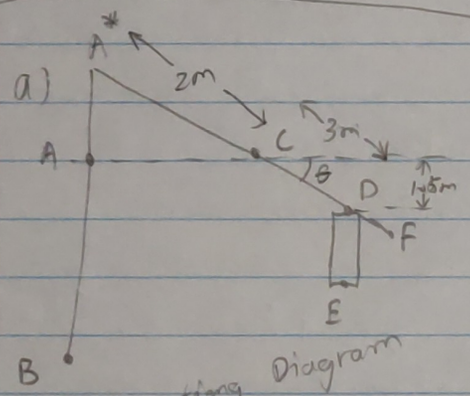
$\sigma(x) = \frac{\rho g v(x)}{A(x)} = \frac{\rho g \pi}{3}\left(\frac{R_x^2 x_1 - R_L^2 L_1}{\pi R_x^2}\right)$

$= \frac{\rho g \pi}{3}\left(x_1 - \left(\frac{R_L}{R_x}\right)^2 L_1\right)$

$= \frac{\rho g \pi}{3}\left[\frac{R_0 L}{R_0 - R_L} - x - \left(\frac{R_L}{R_x}\right)^2 L_1\right]$

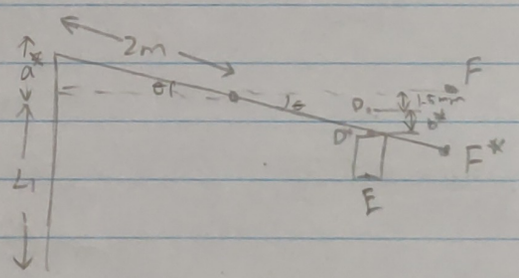
Prob 2.3-7

a)



Deformation Diagram

b)



a) $\sin\theta = \frac{1.5}{3} = 5\times10^{-4}$

$a^* = 2\sin\theta = 0.001m$

$L_1^* = L_1 + a^* = 4.001m$

$\varepsilon_1 = \frac{L_1^* - L_1}{L_1} = 2.5\times10^{-4} m/m$

$\varepsilon_1 = 0.01 mm/mm$

$\varepsilon_2 = ?\ (DE)$

① $\sin\theta = \frac{a^*}{2}$

$a^* = \varepsilon_1 L_1 = (0.001)(4) = 0.004m$

$\sin\theta = 0.002$

② $\sin\theta = \frac{1.5 + b^*}{3}$

$b^* = 0.0045m$

$L_2^* = L_2 - b^* = 2.9955m$

$\varepsilon_2 = \frac{L_2^* - L_2}{L_2} = -0.0015 m/m$

(compresive strain)

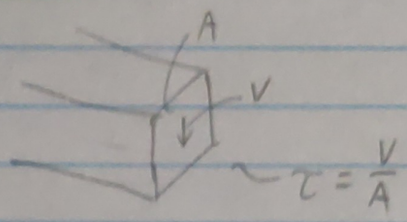
negative compresive strain

Shear Stress and Shear Strain

Definition of shear stress

$$\tau = \frac{V}{A}$$

- τ – 'tau' shear stress
- V – shear force
- A – area which shear force acts



F.2.30

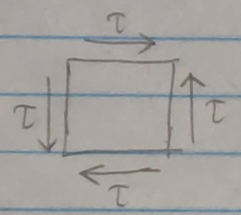
Direct shear: caused by forces that act parallel to a surface, that results in shearings of the material at the surface.

Figs. 2.28 and 2.30 illustrates examples of shear stress and equilibrium req.

Remarks:

- shear stress on parallel surfaces have equal magnitude and opposite direction.
- On adjacent faces, at right angles, shear stress are equal in magnitude, and must point in the directions towards (or away from) the intersection of 2 faces. [$\Sigma M = 0$]

e.g.

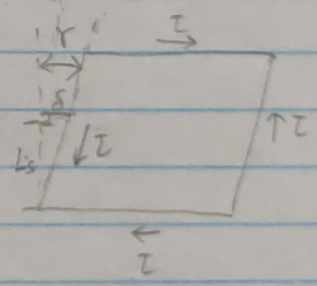


Hooke's Law for Shear

$$\boxed{\tau = G\gamma}$$

- τ – 'tau' shear stress
- γ – 'gamma' shear strain
- G – shear modulus (proportionality constant)

G is typically described in ksi, GPa, MPa etc



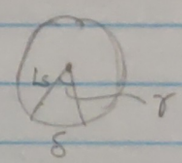
$$\sin\gamma = \frac{\delta}{L_s}$$

Generally, G, E, and ν (Poisson's ratio) are related by

$$G = \frac{E}{2(1+\nu)}$$

for small small γ; $\gamma = \frac{\delta}{L_s}$

$$\delta = L_s\gamma$$



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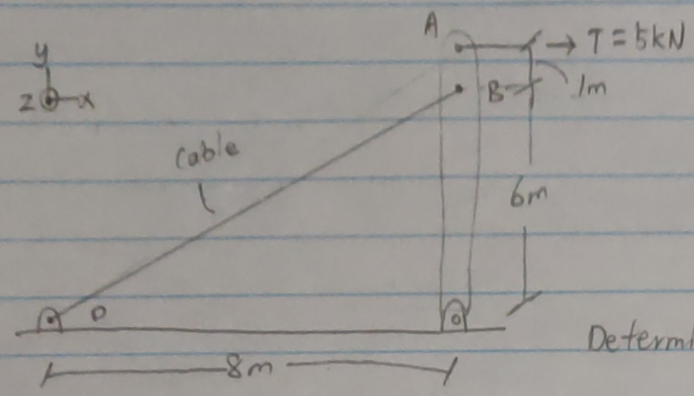
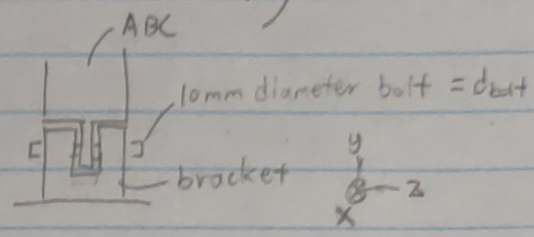
It would break in 2 cross section

Example 1:



Side View:

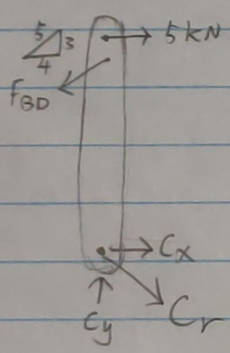


Determine Shear Stress at C.

No deformation → Draw FBD

- find reaction forces at pt C.

FBD ABC

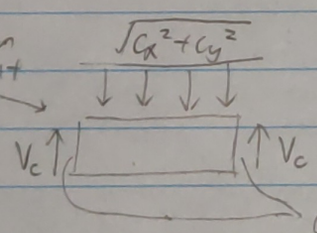


$\Sigma M_c = 0;\ -5kN(7m) + F_{BD}(\frac{4}{5})(6m) = 0,\ F_{BD} \doteq 7.29\,kN$

$\Sigma F_x = 0;\ 5kN - (7.29kN)\frac{4}{5} + C_x = 0,\ C_x \doteq 0.83\,kN$

$\Sigma F_y = 0;\ C_y - F_{BD}(\frac{3}{5}) = 0,\ C_y \doteq 4.38\,kN$

force on the bolt

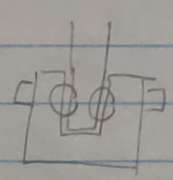


$\Sigma F = 0;\ 2V_c - \sqrt{C_x^2 + C_y^2} = 0$

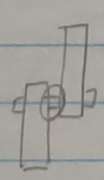
$V_c \doteq 2.23\,kN$

$a = \frac{1}{4}\pi d_{bolt}^2$

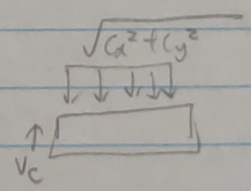
then, $\tau = \frac{V_c}{a} = \frac{2.23kN}{\frac{1}{4}\pi(10\times10^{-3}m)^2} \doteq 28.4\,MPa$



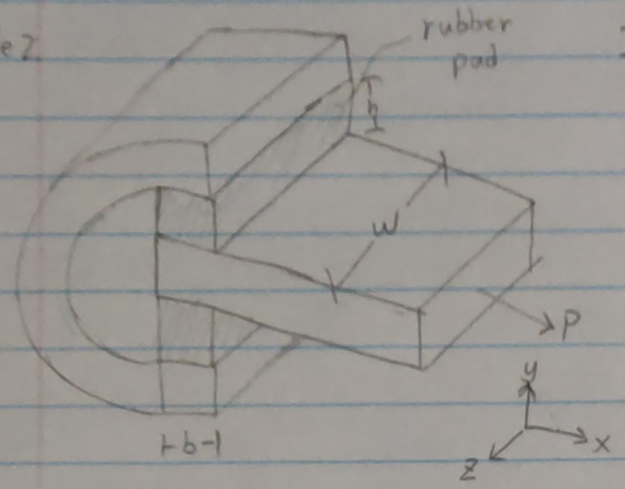
2 areas being sheared



1 area being sheared



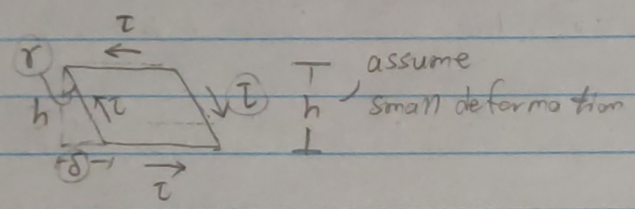
Example 2



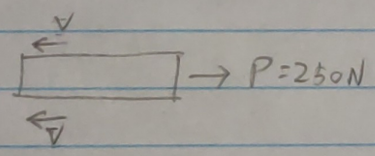
If $P = 250N$, $b = 50mm$, $w = 80mm$, $h = 25mm$, $G = 0.6MPa$

Determine the shear stress τ in the rubber pads, and find the shear strain γ, and the displacement δ of the rubber pads.

Geometry of Deformation of rubber pad (symmetrical)



FBD of bar



$\Sigma F_x = 0$; $250N - 2V = 0$, $V = 125N$

$\tau = \frac{V}{A} = \frac{125N}{(50 \cdot 80)mm^2} \doteq 31.250\ Pa$

$b \cdot w$

$\tau = G\gamma$

$31.250\ Pa = (0.6MPa)\gamma \rightarrow \gamma \doteq 0.052$, $\gamma = \frac{\delta}{h} \Rightarrow 0.052 = \frac{\delta}{25mm}$, $\delta \doteq 1.3mm$

181108

Introduction to Design (MM2.8)

Modes of failure:

1) Design for strength (eg. fail by yielding). material plastic — why stress strain curve
2) Stiffness (Buckling)
3) Ductility (fracture)
4) Displacement (elongation), non-exhasive

Factor of Safety

Uncertainty in material size and materials (manufacturer tolerance, approx comp. sim).

Ensure: **failure load** > **allowable load**

↳ when structure fail ↳ according to requirement of design

Factor of safety: $FS = \frac{\text{Failure load}}{\text{Allowable load}}$ $(FS > 1)$ always

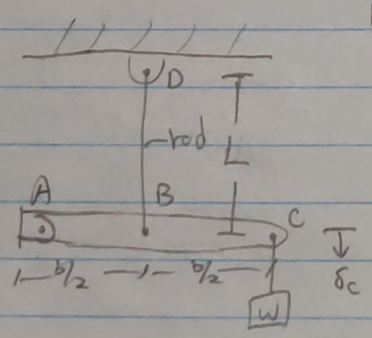
Allowable (load / stress):

$\boxed{\sigma_{allow} = \frac{\sigma_y}{FS}}$ → failure by axial deformation (yeild stress)

or

$\boxed{\tau_{allow} = \frac{\tau_y}{FS}}$ failure by shear strain (yield stress due to shear)

Example 1:



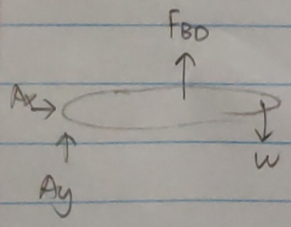
Dimensions:
$b = 3m$
$L = 2.5m$

Rod BD:
- Steel
- $\sigma_y = 250MPa$
- $E = 200GPa$
- diameter $d = 25mm$

Design
- $\delta_{c_{max}} = 10mm$
- yielding $FS_\sigma = 3.3$
- point C displacement $FS_\delta = 3.0$

What is W_{allow}?

Assum failure due to yielding



$\Sigma M_A = 0$:

$F_{BD}(1.5m) - W(3m) = 0$

$F_{BD} = 2W$

$\sigma_{BD} = \frac{F_{BD}}{\frac{1}{4}\pi d^2}$

$\sigma_{BD} = \frac{8W}{\pi d^2}$

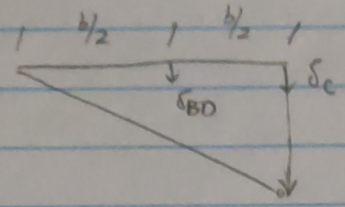
$\frac{8W_\sigma}{\pi d^2} = \frac{\sigma_y}{FS_\sigma}$ (yielding stress)

this will find the W_{allow}.

$\boxed{W_\sigma \doteq 18.6kN}$ allowed W if yielding

Now assume failure by displacement $\qquad \delta_{c\,max} = 10\,mm$

Geometry of Deformation

① 

In this geometry

② $2\delta_{BD} = \delta_c$

(similar triangle)

③ $\dfrac{\delta_{c\,max}}{FS_\delta} = \delta_{c_{allow}}$

$3.33\,mm = \delta_{c_{allow}}$

$$\delta_{c_{allow}} = \frac{\delta_{c\,max}}{FS_\delta}$$

$$\frac{16}{\pi}\frac{\overset{2.5m}{L}}{\underset{25\times10^{-3}m}{d^2}}\frac{W_\delta}{\underset{200\times10^9 Pa}{E}} = \frac{\overset{10\times10^{-3}m}{\delta_{c\,max}}}{FS_\delta} \rightarrow 3.3$$

$$\sigma_{BD} = E\,\varepsilon_{BD}$$

$$\sigma_{BD_{avg}} = E\frac{\delta_{BD}}{L}$$

$$\sigma_{BD} = E\frac{\delta_c}{2L}$$

$$\frac{\overset{2W}{F_{BD}}}{\frac{1}{4}\pi d^2} = E\frac{\delta_c}{2L}$$

$$\frac{8W}{\pi d^2} = E\frac{\delta_c}{2L}$$

$$\delta_c = \frac{16LW}{\pi d^2}$$

$$\boxed{\therefore W_\delta \doteq 32.7\,kN}$$

$W_\delta > W_\sigma$

$$\boxed{\therefore W_{allow} = W_\sigma = 18.6\,kN}$$

181108

Saint Venant's Principal (MM 2.10)

if cross section is close to applied force normal force distribution is equally distributed.

Avg stress, is same on all cross section, $\sigma = \frac{P}{A}$

At $x \geq b$, the force distribution is uniform

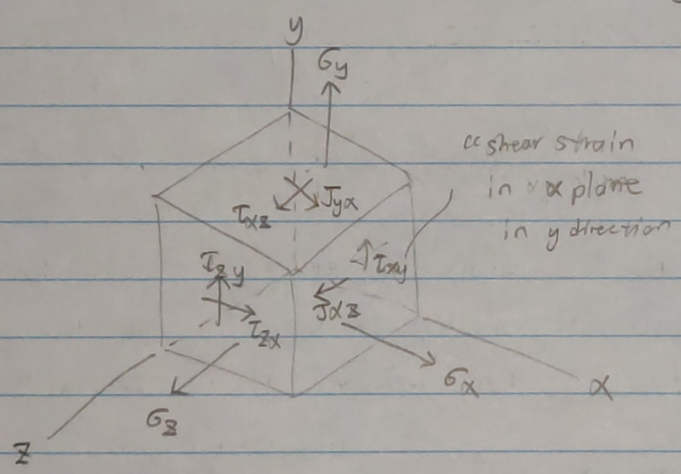
width of beam

Generalized Hooke's Law

Stress/strain effects - Axial compression / tension, thermal strain, poisson's ratio.

↳ can be combined. (Superpositioned

Total strain = thermal strain + σ_x + σ_y + σ_z , etc



"shear strain in x plane in y direction

x-plane, normal to x axis

y-plane, normal to y axis -

Generalized hooke's law for strains

Extensional strains:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \alpha \circ T$$

young's modulus — poisson — coefficient of thermal extension — temperature change.

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] + \alpha \circ T$$

$$\varepsilon_z = \underbrace{\frac{1}{E}\left[\sigma_z\right.}_{\text{Contribution from axial stress}} \underbrace{- \nu(\sigma_y + \sigma_x)}_{\text{effect of poisson's ratio}}\left.\right] + \underbrace{\alpha \circ T}_{\text{thermo expansion}}$$

Shear Strains (just shear stress)

$$\gamma_{xy} = \frac{1}{4}\tau_{xy}, \quad \gamma_{yz} = \frac{1}{4}\tau_{yz}, \quad \gamma_{xz} = \frac{1}{4}\tau_{xz}$$

where, $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ are shear strain.

Generalized Hooke's Law for Stress

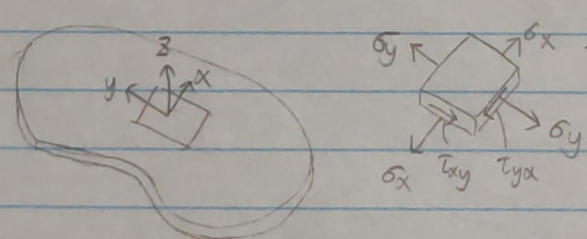
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y+\varepsilon_z) - (1+\nu)(\alpha\Delta T)\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y - \nu(\varepsilon_x+\varepsilon_z) - (1+\nu)(\alpha\Delta T)\right]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z - \nu(\varepsilon_y+\varepsilon_x) - (1+\nu)(\alpha\Delta T)\right]$$

$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx} \qquad G = \frac{E}{2(1+\nu)}$$

Example 1. A small portion of a thin aluminum-alloy is under plane stress ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$). At a particular point on the plate, $\varepsilon_x = 600\times10^{-6}$, $\varepsilon_y = -200\times10^{-6}$, $\gamma_{xy} = 200\times10^{-6}$. $E = 10\times10^{3}$ ksi, $\nu = 0.33$. Find σ_x, σ_y, and τ_{xy} at this point. $\Delta T = 0$



stress is only on plane no z.

do not know ε_z and $\varepsilon_z \neq 0$

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y+\varepsilon_z)\ldots\right]$$

2eqn, 2unknow

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \cancel{\sigma_z}^{\,0}) + \cancel{\alpha\Delta T}^{\,0}\right] \rightarrow \sigma_x, \sigma_y \text{ 2 unknown 1eqn}$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \cancel{\sigma_z}) + \cancel{\alpha\Delta T}\right]$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

Solve for $\boxed{\sigma_x \doteq 6\text{ksi}, \quad \sigma_y \doteq -22\text{ksi}}$

and $\tau_{xy} \doteq G\gamma_{xy}$ $\quad G = \frac{E}{2(1+\nu)} \doteq 3737\text{ksi}$

$\doteq 0.75\text{ksi}$

181112

Axial Deformation

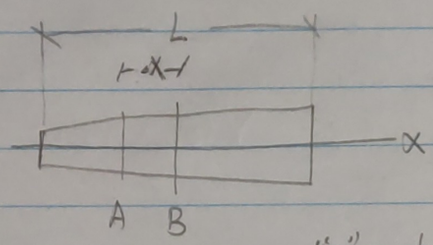
Axial Deformation (longest axial):

1. Axial of member remain straight
2. Cross section remain plane and perpendicular to the axis after deformation. Cross section do not rotate about the long axis.

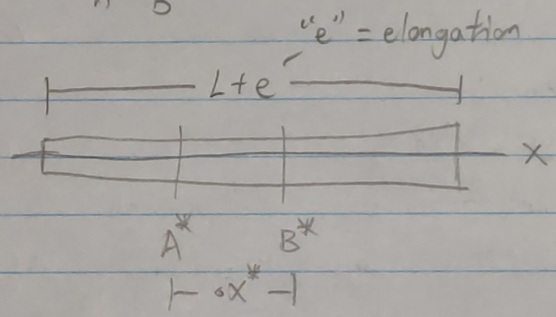
Sometime, axial deformation can have vary cross section. In which ε become function of $\varepsilon(x, y, z)$.

e.g.

before deformation



after deformation



Total elongation of the member: $e = \int_0^L \varepsilon(x)\,dx$

Then Hooke's Law for uniaxial: stress in x-direction.

$$\sigma(x) = E\,\varepsilon(x)$$

Stress distribution $\sigma(x)$ and strain distribution $\varepsilon(x)$ are constant at each cross-section, but vary at diff. cross section.

For linearly elastic members,

Axial stress $\sigma_x(x) = \dfrac{F(x)}{A(x)}$

Axial Strain $\varepsilon(x) = \dfrac{1}{E}\dfrac{F(x)}{A(x)}$

Axial Deformation $e = \dfrac{1}{E}\int_0^L \dfrac{F(x)}{A(x)}\,dx$

When a member is an uniform axial-deformation member, it has:

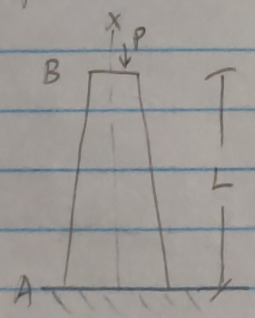
1) Constant cross-sectional area
2) " Young's modulus,
3) Axial forces only applied at member ends.

The above eqns can be simplified to.

$$e = \frac{1}{E}\frac{F}{A}\int_0^L dx = \frac{FL}{AE} = fF \quad , \quad f = \frac{L}{AE} \text{ 'flexibility constant'}$$

or $F = ke$, where $k = \frac{AE}{L}$ 'stiffness constant'

Example1. The tapered column is subjected to a downward force P acting through the centroid of the top cross section at B. The column has circular cross-section, with a diameter that varies <u>linearly</u> from d_A at to d_B at B. Determine an expression for the elongation of the column, as a result of P.



$$e = \frac{1}{E}\int_0^L \frac{F(x)}{A(x)}$$

② $A = \frac{1}{4}\pi d(x)^2$

Linear Interpolation

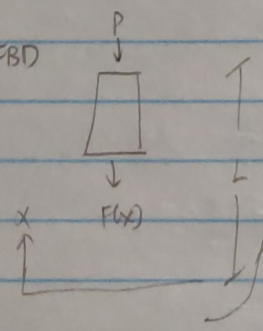
$$\frac{d_B - d_A}{L-0} = \frac{d_B - d(x)}{L-x}$$

$$d_B - (L-x)\frac{d_B - d_A}{L} = d(x)$$

$$d_A - \frac{x}{L}(d_A - d_B) = d(x)$$

$$A = \frac{1}{4}\pi\left[d_A - \frac{x}{L}(d_A - d_B)\right]^2$$

① FBD



$\Sigma f_x = 0$; $-P - F(x) = 0$

$F(x) = -P$

③

$$e = \frac{-P}{E}\int_0^L \frac{1}{\frac{1}{4}\pi\left[d_A - \frac{x}{L}(d_A - d_B)\right]^2}\,dx$$

$$= \frac{-4P}{\pi E}\int_0^L \left[d_A - \frac{x}{L}(d_A - d_B)\right]^{-2} dx$$

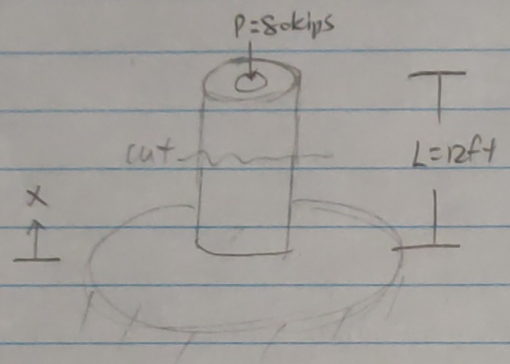
$$= \frac{-4P}{\pi E}\,\frac{\left[d_A - \frac{x}{L}(d_A - d_B)\right]^{-1}}{+1}\left(+\frac{L}{d_A - d_B}\right)\Bigg|_0^L$$

$$= \frac{-4PL}{\pi E d_A d_B}$$

Example 2: A steel pipe is filled with concrete and the resulting column is subjected to a compressive load, $P = 80$ kips. The pipe has an outer diameter $d_o = 12.75$ in, and inner diameter, $d_i = 12$ in. $E_s = 30 \times 10^3$ ksi (steel), and $E_c = 3.6 \times 10^3$ ksi (concrete).

a) Stress in steel and concrete

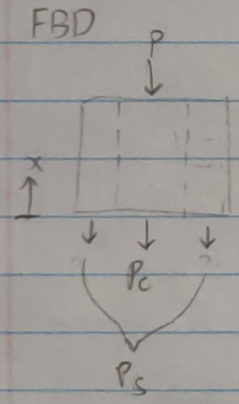
b) How much does column shorten when loaded by P?



Assume there's no sliding b/w the steel & concrete.

$$A_s = \frac{1}{4}\pi d^2 - \frac{1}{4}\pi d^2$$
$$= \frac{1}{4}\pi(12.75\,in^2 - 12\,in^2)$$
$$= 14.58\,in^2$$
$$A_c = \frac{1}{4}\pi(12\,in)^2$$
$$= 113.097\,in^2$$

FBD



$F_x = 0$; $-P - P_c - P_s = 0$

$$\boxed{-P_c - P_s = P} \quad ①$$

$\epsilon_s = \epsilon_c = \epsilon$

$$\sigma_s = \frac{P_s}{A_s} \rightarrow P_s = \sigma_s A_s = E_s \epsilon_s A_s = \boxed{E_s \epsilon A_s} \quad ②$$

$$\sigma_c = \frac{P_c}{A_c} \rightarrow P_c = \sigma_c A_c = E_c \epsilon_c A_c = \boxed{E_c \epsilon A_c} \quad ③$$

substitute ②,③ into ①

$$-E_c \epsilon A_c - E_s \epsilon A_s = P$$
$$\epsilon(+E_c A_c + E_s A_s) = -P$$
$$\epsilon = \frac{-P}{E_c A_c + E_s A_s}$$

$$\epsilon = \frac{-4}{\pi} \frac{P}{(d_i^2(E_c - E_s) + d_o^2 E_s)}$$
$$= \frac{-4}{\pi} \frac{80\,kips}{(12^2(E_c - E_s) + 12.75^2 E_s)}$$
$$= \frac{-101.86\,kips}{1.075 \times 10^6\,kips}$$
$$= -94.729$$

$\therefore \sigma_c = E_s \epsilon \doteq -0.34$ ksi

$\sigma_s = E_c \epsilon \doteq -2.84$ ksi

b) $\epsilon = \frac{\delta}{L}$

$e = L\epsilon \rightarrow$

$= 12ft($

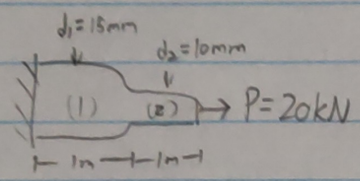
$= -0.014in$

Example of Axial Deforestation for Statically Determinant System

Basic Steps

1) Geometry of Deformation (optional)
2) Force - equilibrium eqns (i.e. $\Sigma F = 0$)
3) Force - deformation eqn (ie. $e = \frac{1}{E}\int_0^L \frac{F(x)}{A(x)}dx$

Example 1

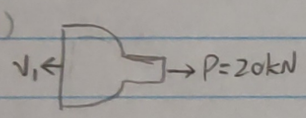


$E = 200 GPa$

What is the total elongation of member?

(1) & (2) have different rate of elongation

(1)



$\Sigma F_x = 0$; $P - V_1 = 0$

$V_1 = 20kN$

$\sigma = \frac{V}{A}$ $\quad A = \frac{1}{4}\pi d^2$

$e = \frac{FL}{EA}$

(2) $V_2 \leftarrow \square \rightarrow P$ $\quad V_2 = 20kN$

$$e = e_1 + e_2 = \frac{V_1(L)}{E(A_1)} + \frac{V_2(L)}{E(A_2)} = \frac{L}{E}\left(\frac{4V_1}{\pi d_1^2} + \frac{4V_2}{\pi d_2^2}\right) = \frac{4L}{\pi E}\left(\frac{V_1}{d_1^2} + \frac{V_2}{d_2^2}\right)$$

$$= \frac{4(1m)}{\pi(200\times10^9 \frac{N}{m^2})}\left(\frac{20\times10^3 N}{(15\times10^{-3}m)^2} + \frac{20\times10^3 N}{(10\times10^{-3})^2}\right)$$

$$= 6.37\times10^{-10}\frac{m^2}{N}\left(288.89\times10^6 \frac{N}{m^2}\right) = 1.839\times10^{-3} m$$

Example of Axial Deformation for Statically Indeterminant System (MM 3.5)

Steps

1) Geometry of Deformation (optional)
2) Force - equilibrium eqn
3) Force - deformation eqn
4) Compatibility eqn

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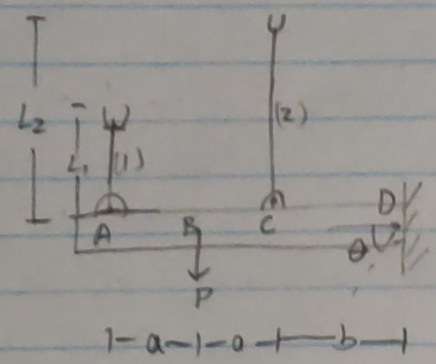
Example



Steel rod (1) & (2) stress free when $P=0$

a) Find F_1 & F_2 in rod when P is applied.

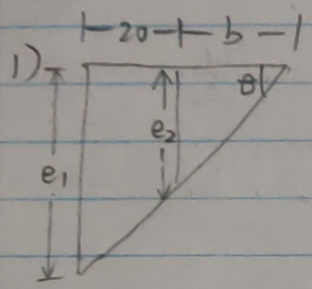
b) Find e_1, corresponding to load P.

$A_1 = A_2 = 500\,mm^2$, $L_1 = 1m$, $L_2 = 2m$

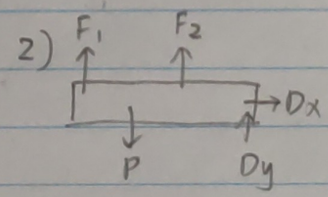
$E_1 = E_2 = 210\,GPa$, $P = 50\,kN$

$a = 0.5m$, $b = 1.5m$

1)



2)



$\Sigma M_D = 0$; $\boxed{-F_1(2a+b) + P(a+b) - F_2(b) = 0}$ ①

2 unknowns (F_1, F_2)

Eqn

3) $e_1 = \dfrac{F_1 L_1}{A_1 E}$, $e_2 = \dfrac{F_2 L_2}{A_2 E}$

$\dfrac{F_1 L_1}{\cancel{A_1 E}}\left(\dfrac{b}{2a+b}\right) = \dfrac{F_2 L_2}{\cancel{A_2 E}}$

$\boxed{\left(\dfrac{b}{2a+b}\right)\dfrac{F_1 L_1}{L_2} = F_2}$ ②

Compat. eqn using linear Interpolation

4) $\dfrac{e_1 - 0}{2a+b-0} = \dfrac{e_2 - 0}{b}$

$b e_1 = e_2(2a+b)$

$e_2 = \dfrac{b e_1}{2a+b}$

$\sigma = E\epsilon$

$\dfrac{F}{A} = E\left(\dfrac{e}{L}\right)$

b) $e_1 = \dfrac{F_1 L_1}{A_1 E}$

$= 0.32mm$

a) Sub ② into ①

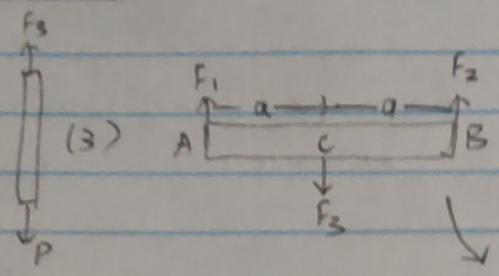
$-F_1(2a+b) + P(a+b) - \dfrac{F_1 b^2}{4a+2b} = 0$

$F_1 = 33.9kN$

$F_2 = 10.1kN$

Lab 10

Prob 3.4-12



$\hookrightarrow F_3 = P$

$e = \frac{FL}{AE}$ $\quad E_1 = E_2 = E_3 = E$

$A_1 = A_2 = A_3 = A$

$L_3 = 2L$

$L_1 = L_2 = L$

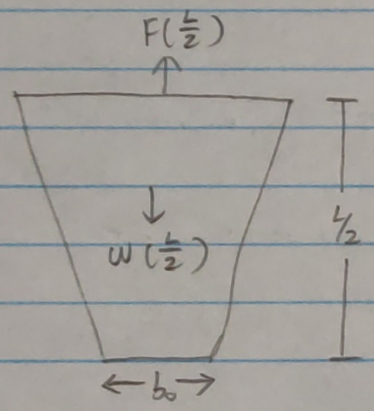
$\hookrightarrow \Sigma M_C = 0, \rightarrow F_1 = F_2$

$F_1 = F_2 = F_3/2 = P/2$

$e_2 = e_1 = \frac{F_1 L_1}{A_1 E_1} = \frac{PL}{2AE}$, $e_3 = \frac{F_3 L_3}{A_3 E_3} = \frac{2PL}{AE}$

$$\left.\begin{aligned} \Delta_D &= e_3 + e_1 \\ &= \frac{PL}{2AE} + \frac{2PL}{AE} \\ &= P\left(\frac{L+4L}{2AE}\right) \end{aligned}\right\} = \frac{5}{2}\frac{PL}{AE}$$

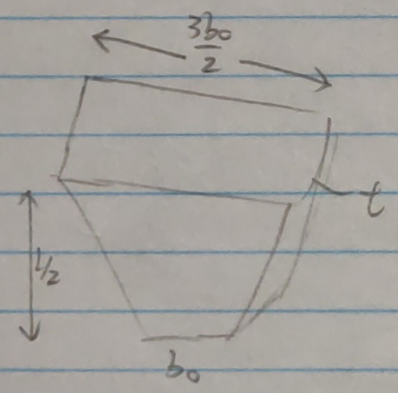
Prob 3.3-16



width: $w(\frac{L}{2}) = \frac{1}{2}(b_0 + 2b_0) = \frac{3b_0}{2}$

$A(\frac{L}{2}) = (\frac{3b_0}{2})(t)$

Volume $(\frac{L}{2}) = \underbrace{\frac{1}{2}(b_0 + \frac{3b_0}{2})}_{\text{Trapozoid}}(\frac{L}{2})(t) = \frac{5b_0 tL}{8}$



$\Sigma F_y = 0$; $F(L/2) = w(L/2)$

$= \gamma V(\frac{L}{2})$

$= \frac{5b_0 tL\gamma}{8}$

$\sigma = \frac{F(L/2)}{A(L/2)}$

$= \frac{\left(\frac{5b_0 tL\gamma}{8}\right)}{\left(\frac{3b_0}{2}t\right)} = 28.4$ psi

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Thermal Effect on Axial Deformation (MM 3.6)

$$e = \underbrace{\frac{1}{E}\int_0^L \frac{F(x)}{A(x)}dx}_{\text{axial stress}} + \underbrace{\alpha \int_0^L \Delta T(x)dx}_{\text{thermal stress}}$$

α - coefficient of thermal expansion

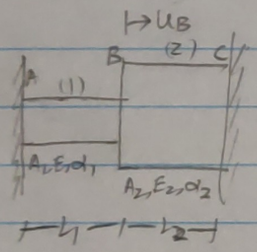
ΔT - change in temperature

→ in this course α & ΔT is constant

General steps:

1) Geometry of deformation (optional)
2) Force Equilibrium eqn
3) Force - Temperature deformation eqn.
4) Compatibility Eqn

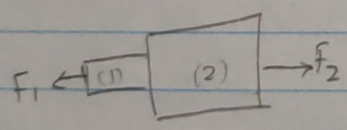
Example 1.



Two rod elements are stress free when they are assembled together and attached to the rigid walls at A and C. Determine expressions of stresses $\sigma_1, (\Delta T)$ and $\sigma_2(\Delta T)$ that would result from an uniformed temperature ↑. $\Delta T_1 = \Delta T_2 = \Delta T$. Express answer in term of $E_1, E_2, A_1, A_2, \alpha_1, \alpha_2$ and ΔT.

2) Force Equilibrium Eqn

FBD



$\Sigma F_x = 0$

$-F_1 + F_2 = 0, \quad F_1 = F_2$

statically indeterminant

Ans:

$$\boxed{\sigma_1 = \frac{F_1}{A_1} \qquad \sigma_2 = \frac{F_2}{A_2}}$$

3) Force Temp. deformation eqn.

$$e_1 = \frac{1}{E}\int_0^{L_1} \frac{F_1}{A_1}dx + \alpha_1 \int_0^{L_1} \Delta T dt = \frac{F_1 L_1}{E_1 A_1} + \alpha_1 \Delta T L_1$$

$$e_2 = \frac{F_2 L_2}{E_2 A_2} + \alpha_2 \Delta T L_2$$

(2 eqn, 2 unknowns)

4) Compatibility eqn

$e_1 + e_2 = 0$

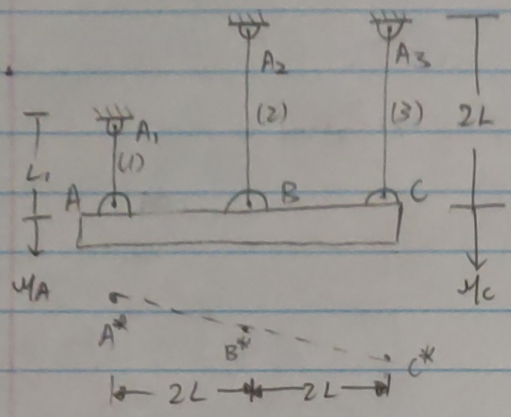
(Structure Constrained by 2 walls)

$$\frac{F_1 L_1}{E_1 A_1} + \alpha_1 \Delta T L_1 + \frac{F_2 L_2}{E_2 A_2} + \alpha_2 \Delta T L_2 = 0$$

→ isolate for F_1 & F_2

$$\boxed{F_1 = F_2 = \frac{-(\alpha_1 L_1 + \alpha_2 L_2)\Delta T}{\left(\frac{L_1}{A_1 E_1}\right) + \left(\frac{L_2}{A_2 E_2}\right)}}$$

Example 2.



Rod (1) is maintained at the original stress-free temperature, and rod (2) and (3) are heated so that $\Delta T_2 = \Delta T_3 = \Delta T$. Determined stress induced in the 3 supports.

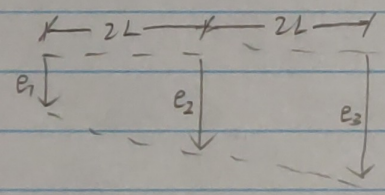
$A_1 = A$, $A_2 = A_3 = 2A$

$E_1 = E_2 = E_3 = E$

$\alpha_2 = \alpha_3 = \alpha$

Rod(1) experience axial stress through expansion of rod (2) & (3)

1) Geometry of Deformation



(4) Compatibility Eqn

-beam has to be in a straight line

use linear interpolation

$$\frac{e_3 - e_1}{4L} = \frac{e_2 - e_1}{2L}$$

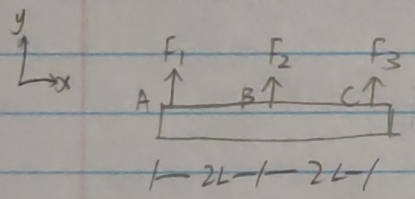
$$e_2 = \frac{1}{2}(e_1 + e_3)$$

→ e_2 in b/w e_1 & e_3

∴ e_2 the avg of $e_1 + e_3$

(2) Force Equilibrium Eqn

FBD



$\Sigma M_C = 0;\ -F_1(4L) - F_2(2L) = 0$ (2 eqns 3 unknowns)

$-2F_1 = F_2$

$\Sigma F_y = 0;\ F_1 + F_2 + F_3 = 0$

$F_3 = F_1$

(3) Force Deformation Eqn

$$e_1 = \frac{1}{E}\int_0^L \frac{F(y)}{A(y)}dy + \cancel{\alpha\int_0^L \Delta T\,dt}^{\,0} = \frac{F_1 L}{EA}$$

$$e_2 = \frac{1}{E}\int_0^{2L} \frac{F(y)}{A(y)}dy + \alpha\int_0^{2L}\Delta T\,dt = \frac{2F_2 L}{2EA} + \alpha\Delta T(2L) = \frac{F_2 L}{EA} + 2\alpha\Delta T L$$

$$e_3 = \frac{F_3 L}{EA} + 2\alpha\Delta T L$$

Substitute eqn (3) into (4) and solve w/ (1)

$F_2 = -2F_1$, $F_3 = F_1$

$$\frac{F_2 L}{EA} + 2\alpha\Delta T L = \frac{1}{2}\frac{F_1 L}{EA} + \frac{1}{2}\frac{F_3 L}{EA} + \alpha L \Delta T$$

$F_1 = \frac{1}{3}\alpha E A \Delta T$ $\qquad \sigma_1 = \frac{F_1}{A} = \frac{1}{3}\alpha E \Delta T$ (tension)

$F_2 = \frac{-2}{3}\alpha E A \Delta T$ $\qquad \Rightarrow \quad \sigma_2 = \frac{F_2}{2A} = \frac{-1}{3}\alpha E \Delta T$ (compression)

$F_3 = \frac{1}{3}\alpha E A \Delta T$ $\qquad \sigma_3 = \frac{F_3}{2A} = \frac{1}{6}\alpha E \Delta T$ (tension)

18/1/19

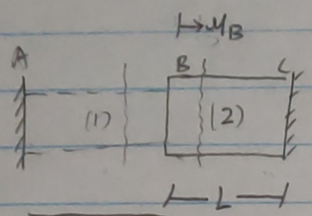
Geometry Misfit (MM3.7)

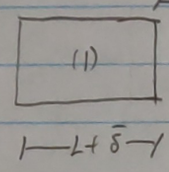
Geometric misfits (geometric incompatability) may be unintended.

In this course we'll modify the compatibility eqn.

☆ Note: when drawing FBD & incompatibility equations, we ignore small gaps $\bar{\delta}$.

Example 1.





$A_1 = A$

$A_2 = 1.5A$

$E_1 = E_2 = E$

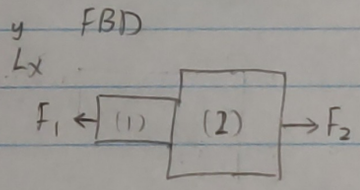
(1) and (2) are supposed to fit exactly b/w walls at A and C. (1) was mismanufactured by a small amount $\bar{\delta}$, too long. Someone cooled (1) just enough to fit it b/w A and B.

a) determine expression for σ_1 & σ_2 when T of (1) return to ambient T.

b) determine the resulting displacement u_B.

(2) Force Equilibrium

FBD



$\Sigma F_x = 0: \; -F_1 + F_2 = 0, \quad \boxed{F_1 = F_2}$

(3) Force Deformation

$$e_1 = \frac{1}{E}\int_0^L \frac{F(x)}{A(x)}\,dx = \frac{F_1 L}{EA} \qquad e_2 = \frac{1}{E}\int_0^L \frac{F(x)}{A(x)}\,dx = \frac{F_2 L}{1.5EA}$$

(4) Compatibility eqn

Total $e = -\bar{\delta} \Rightarrow e_1 + e_2 = -\bar{\delta}$, condition has to be met, for (1,2) to fit.

Sub (3) to (4)

$F_1 = F_2$

$$\frac{F_1 L}{AE} + \frac{F_2 L}{1.5AE} = -\bar{\delta} \Rightarrow \therefore \; F_1 = \frac{-3}{5}\frac{EA\bar{\delta}}{L}, \quad F_2 = \frac{-3}{5}\frac{EA\bar{\delta}}{L}$$

a) $\therefore \; \sigma_1 = \frac{F_1}{A_1} = \frac{-3}{5}\frac{E\bar{\delta}}{L}$

$\sigma_2 = \frac{F_2}{A_2} = \frac{-2}{5}\frac{E\bar{\delta}}{L}$

b) $u_B = -e_2 = \frac{-2}{3}\frac{F_2 L}{AE} = \frac{2}{5}\bar{\delta}$

55% No Torsion

- 3D eq
- From & Machine
- Axial deform
  - Gen hooke's & Poisson
  - Geometric Misfits

Displacement Method Solution (MM 3.8)

DMS procedure:

1) Equilibrium Equation
2) Force - temperature - deformation equation
3) Compatibility Equatn

} fundamental eqns

4) Sub ③ into ②

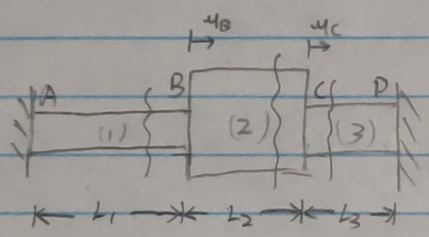
5) Sub result of (4) into (1)
6) Obtain sys: of equation for u_i and solve.

} Solve for u_i

7) Sub result of ⑥ to ② to find F_i

} Solve Forces, F_i

Example 1



Rods are stress free when attached to wall.
(2) is suddenly heated, $\Delta T_2 = \Delta T$, while $\Delta T_1 = \Delta T_3 = 0$.

a) Use displacement method to obtain an expression of u_B and u_C

b) Obtain expressions for $\sigma_1, \sigma_2, \sigma_3$

$A_1 = A_3 = A$, $A_2 = 2A$

$L_2 = L_3 = L$, $L_1 = 2L$

$E_1 = E_2 = E_3 = E$

$\alpha_1 = \alpha_2 = \alpha_3 = \alpha$

A, B, C, D are 'nodes' in FEA
Analysis takes place at these nodes in FEA software

1) Equilibrium

FBD

$F_1 \leftarrow \boxed{(1)}\boxed{(2)} \rightarrow F_2$ $\qquad$ $F_2 \leftarrow \boxed{(2)}\boxed{(3)} \rightarrow F_3$

$\Sigma F_x = 0;\ F_2 - F_1 = 0$ $\qquad$ $\Sigma F_x = 0;\ F_3 - F_2 = 0$

2) Force - Temperature Deformation

Normally... $e = \frac{1}{E}\int_0^L \frac{F(x)}{A(x)}\,dx \rightarrow e = \frac{FL}{AE}$

$\therefore F_1 = \frac{A_1E_1}{L_1}e_1 = \left(\frac{A_1E_1}{L_1}\right)e_1$ - or $\boxed{F_1 = k_1 e_1}$ where $k_1 = \frac{AE}{2L}$. stiffness Coefficient

$F_3 = \frac{A_3E_3}{L_3}e_3$, so $\boxed{F_3 = k_3 e_3}$, $k_3 = \frac{AE}{L}$

$\Delta T \rightarrow e_2 = \frac{1}{E}\int_0^{L_2}\frac{F_2(x)}{A_2(x)}dx + \alpha_2\int_0^{L_2}\Delta T_2(x)\,dx = \frac{LF_2}{2AE} + \alpha\Delta T \Rightarrow F_2 = 2\frac{AE}{L}(e_2 - \alpha L\Delta T)$

$= k_2(e_2 - \alpha L\Delta T)$, $\qquad k_2 = \frac{2AE}{L}$

181122

3) Compatibility Eqn

$e_1 + e_2 + e_3 = 0$, $e_1 = u_B$, $e_3 = -u_C$

$u_B + e_2 - u_C = 0$, $\rightarrow e_2 = u_C - u_B$

4) Sub ③s into ②

$\boxed{F_1 = k_1(u_B)}$ $\boxed{F_2 = k_2[(u_C - u_B) - \alpha L_0 T]}$ $\boxed{F_3 = -k_3 u_C}$

5) Sub ④ into ①

$$\left.\begin{aligned} F_2 - F_1 = 0 &\longrightarrow k_2[(u_C - u_B) - \alpha L_0 T)] - k_1(u_B) = 0 \\ F_3 - F_2 = 0 &\longrightarrow -k_3 u_C - k_2[(u_C - u_B) - \alpha L_0 T)] = 0 \end{aligned}\right\} \text{System of eqn for } u_i$$

6) Solve

$\boxed{u_B = \frac{-4}{7}\alpha L_0 T \qquad u_C = \frac{2}{7}\alpha L_0 T}$ solved by computer program

7) Sub ⑥ into ④

$$F_1 = k_1 e_1 = \frac{AE}{2L}\left(\frac{-4}{7}\alpha L_0 T\right) = \frac{-2}{7} AE\alpha_0 T$$

$$F_2 = k_1(e_2 - \alpha L_0 T) = 2\frac{AE}{L}\left[\left(\frac{2}{7}\alpha L_0 T + \frac{4}{7}\alpha L_0 T\right) - \alpha L_0 T\right] = \frac{-2}{7} AE\alpha_0 T$$

$$F_3 = k_3 e_3 = \frac{AE}{L}\left(\frac{2}{7}\alpha L_0 T\right) = \frac{-2}{7} AE\alpha_0 T$$

b) $\sigma_1 = \frac{F_1}{A_1} = \frac{-2}{7} E\alpha_0 T$

$\sigma_2 = \frac{F_2}{A_2} = \frac{-1}{7}\alpha E_0 T$

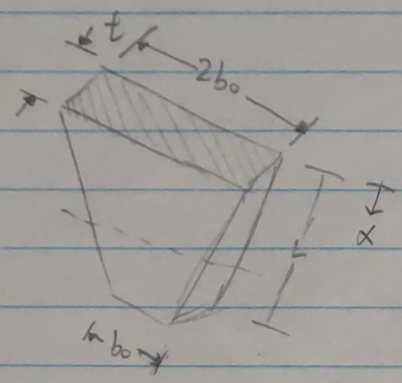
$\sigma_3 = \frac{F_3}{A_2} = \frac{-2}{7}\alpha E_0 T$

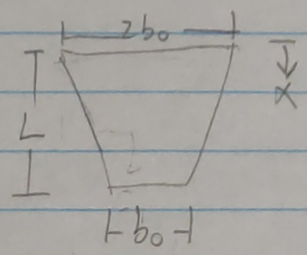
Axial Deformation of an Object under loading by its own weight (MM 3.3 & 3.7)

Example 1. A flat bar is suspended from above and loaded by its own weight. Bar thickness is constant. The specific (desity × gravity) weight of the material is ρg.

a) What is the normal stress midway from the top of the bar?

b) How does the normal stress change if bar thickness is doubled?





width: $b(x) : \frac{b_0 - 2b_0}{L} = \frac{b(x) - 2b_0}{x - 0}$

$b(x) = b_0(2 - \frac{x}{L})$

Check: $b(0) = 2b_0$, $b(L) = b_0$

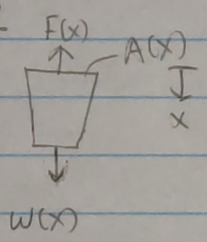
not looking for total V

$V(x) = \int A(x)\,dx$ $\quad A(x) = t \cdot b(x) = t \cdot b_0(2 - \frac{x}{L})$

$= 2tb_0x - \frac{1}{2}\frac{tb_0x^2}{L}$

Force equilibrium Eqn:

FBD



$\Sigma F_x = 0;\ -F(x) + W(x) = 0$

$W(x) = F(x)$

(density × gravity) $\rho g\, V(x) = F(x)$

$F(x) = \rho g\left(2tb_0x - \frac{1}{2}\frac{tb_0x^2}{L}\right)$

$= \rho g\, tb_0\left(2x - \frac{1}{2}\frac{x^2}{L}\right)$

$=$

a) $\sigma_1 = \frac{F(x)}{A(x)}$

$= \frac{\rho g\, \cancel{tb_0}\left(2x - \frac{1}{2}\frac{x^2}{L}\right)}{\cancel{t \cdot b_0}\left(2 - \frac{x}{L}\right)}$

$= \frac{\rho g\left(2x - \frac{1}{2}\frac{x^2}{L}\right)}{\left(2 - \frac{x}{L}\right)}$

a) $x = \frac{L}{2}$

$\sigma_1 = \frac{\rho g\left(2\frac{L}{2} - \frac{1}{2}\frac{(\frac{L}{2})^2}{L}\right)}{\left(2 - \frac{L/2}{L}\right)}$

$= \frac{\rho g\left(L - \frac{1}{8}L\right)}{2 - \frac{1}{2}}$

$= \frac{7}{12}\rho g L$

b) Stress does not chang when t changes. $\sigma(x)$ is independant of t.

$\underbrace{F(x) \propto W(x) \propto V(x) \propto A(x)}$

$\sigma(x) = \frac{F(x)}{A(x)} \Rightarrow \therefore t$ does not affect

8/1/26

Introduction to Torsion (MM 4.1)

Torsional Loading: couple or moments that causes the twisting of the member along the long axis.

(Fig. 4.1) assume slender member in this course

- max at outer edge
- 0 at center

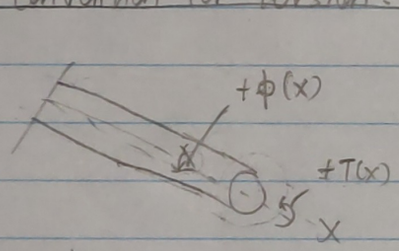
Torsional Deformation of Linear Elastic Circular Bars

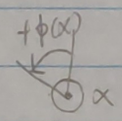
- Circular member, cross section stays plane
- Square member, cross section does not remain plane.

Three fundametal torsion-deformation assumption:

1. Long axis remain straight & inextensible
2. Cross section remain plane and perpendicular to long axis.
3. Radial line remain straight and radial as the cross section rotates about the long axis.

Sign Convention for torsion:



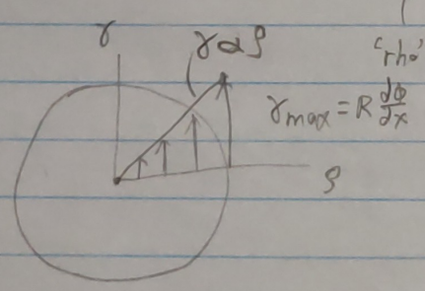


$\phi(x)$, angle of rotation 'phi'

Strain-displacement eqn:

Shear strain, $\gamma(x, \rho) = \rho \frac{d\phi}{dx}$

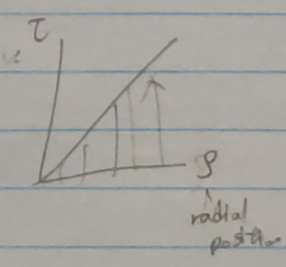
'rho' radial position in cross section



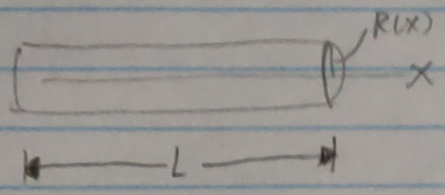
Linear Elastic Stress-Strain Behavior

Hooke's Law: $\tau = G\gamma$

- τ: Shear stress
- G: Shear modulus
- γ: Shear strain



Torsional equation for linearly elastic bar, where G is constant everywhere



Shear Stress $\tau(x, \rho) = \frac{T(x)\rho}{I_p(x)}$

where $I_p(x)$ is the polar moment of inertia, $I_p = \int_A \rho^2 \, dA$

Circle : $I_p = \frac{1}{2}\pi r^4$

stiffness size

$\boxed{G \cdot I_p}$, ↑ resist twisting

Maximum shear stress : $\tau_{max}(x) = \frac{T(x)\,R(x)}{I_p(x)}$

Twisting angle (angle of deformation) $\phi = \frac{1}{G}\int_0^L \frac{T(x)}{I_p(x)}dx$

Total twisting angle, $\phi_t = \phi(L) - \phi(0)$

If cross section area is same through out member, remove (x).

181129

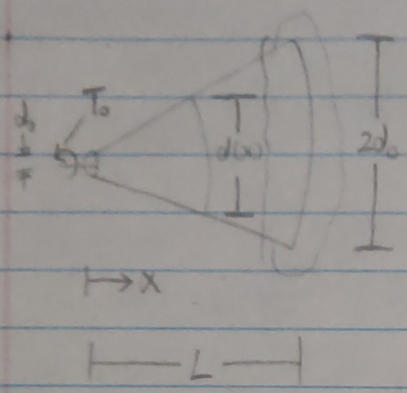
Torsional Deformation of Linearly Elastic-Circular bars (MM 4.2, 4.3)

Equations of Torsion:

$$I_p(x) = \int_A \rho^2 \, dA \quad , \quad \tau_{max}(x) = \frac{T_{max}(x)\, r(x)}{I_p(x)} \quad , \quad \phi_{total} = \phi(L) - \phi(0)$$

maximum twist ↗ ↗ min twist

Example 1.



Material shear modulus, G

At $x=0$, $d = d_0$

At $x=L$, $d = 2d_0$

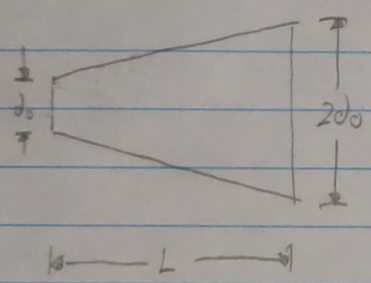
Torque T_0 is applied at $x=0$

Shaft is attached to a wall at $x=L$.

a) Determine expression for $\tau_{max}(x)$

b) Determine total angle of twist.

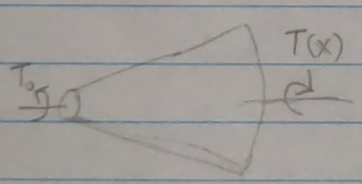
deformation geometry



$$\frac{2d_0 - d_0}{L} = \frac{d(x) - d_0}{x - 0}$$

$$d(x) = d_0\left(1 + \frac{x}{L}\right)$$

force equilibrium



$\Sigma M_x = 0$; $\quad T_0 - T(x) = 0$

about x-axis

$\therefore T(x) = T_0$

a) $\tau_{max}(x)$

$$\tau_{max}(x) = \frac{T_{max}(x)\, r(x)}{I_p(x)}$$

$$= \frac{T_0\left(\frac{1}{2}d(x)\right)}{\frac{1}{32}\pi\, d(x)^4}$$

$$= \frac{16}{\pi}\frac{T_0}{d(x)^3}$$

$$\therefore \quad = \frac{16}{\pi}\frac{T_0}{d_0^3\left(1+\frac{x}{L}\right)^3}$$

rdr, $\rho d\rho$ ↗ $2\pi r$ → integrating the circumference

$$I_p(x) = \int_A \rho^2 dA = \int_0^r \rho^2 (2\pi\rho)\, d\rho$$

$$= 2\pi \int_0^r \rho^3 d\rho = 2\pi \left.\frac{\rho^4}{4}\right|_0^r = 2\pi\left(\frac{r^4}{4}\right) = \frac{1}{2}\pi r^4 = \frac{1}{2}\pi\left(\frac{d}{2}\right)^4$$

$$= \frac{1}{32}\pi\, d(x)^4$$

$$\phi_t = \phi(0) - \phi(L)$$

$$= \frac{1}{G}\int_0^L \frac{T(x)}{I_p(x)}\, dx$$

$$= \frac{T_0}{G}\int_0^L \frac{32}{\pi d_0^4 \left(1+\frac{x}{L}\right)^4}\, dx = \frac{32}{\pi}\frac{T_0}{G d_0^4}\int_0^L \left(1+\frac{x}{L}\right)^{-4} dx$$

$$= \frac{32}{\pi}\frac{T_0}{G d_0^4}\left[\left(1+\frac{x}{L}\right)^{-3} \cdot \frac{1}{3}L\right]_0^L$$

$$\phi_t = \frac{28}{3\pi}\frac{T_0 L}{G d_0^4}$$

$\uparrow T_0, L, \uparrow \phi$

$\uparrow G, \boxed{d_0^4}, \downarrow \phi$

greates impact