CH12 3-Dimensional Coordinate System   Lei Yu   Mar 15, 2018

Function of two variable: (x,y) asigned to a set D, a unique real # F(x,y)

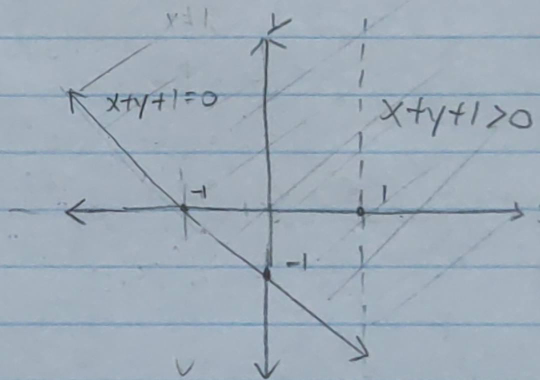
Set D is the Domain of F(x,y) and it's range is the set of values that F can take

We often write $z = f(x,y)$ to make explicit the value taken by f at a general point (x,y). x and y are called the independant variables, and z is called the dependant variable

Ex. For each of the following functions evaluate F(3,2) and sketch the domain.

a) $F(x,y) = \frac{\sqrt{x+y+1}}{x-1} \Rightarrow f(3,2) = \frac{\sqrt{6}}{2}$

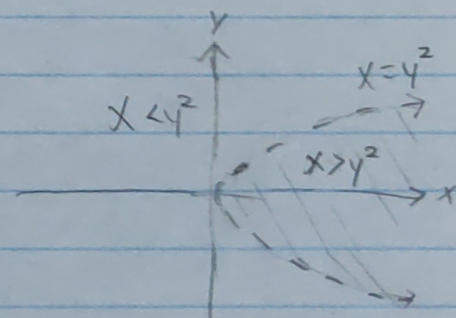
Domain: $(x+y+1 \geq 0,\ x \neq 1)$



b) $f(x,y) = x\ln(y^2 - x)$

$f(3,2) = 3\ln(1) = 0$

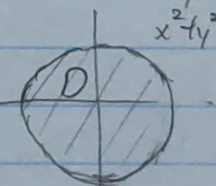
Domain: $y^2 - x > 0 \Rightarrow x < y^2$



Ex. Find the Domain and Range of $g(x,y) = \sqrt{9-x^2-y^2}$

Domain: $9 - x^2 - y^2 \geq 0$

$9 \geq x^2 + y^2$



Range: $z = \sqrt{9-(x^2+y^2)}$

$\max(x^2+y^2) = 9$
$\Rightarrow 0 \leq z \leq 3$
$\min(x^2+y^2) = 0$

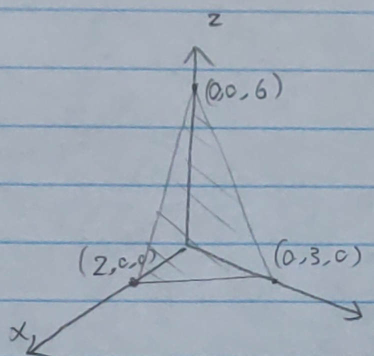
* Graph of a function of two variable with Domain D is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$ when $(x,y) \in D$

The graph of a function with two variable is a surface with equation $z = f(x,y)$

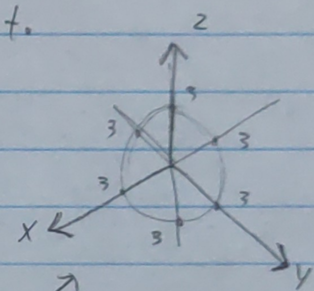
Ex. Sketch $F(x,y) = 6 - 3x - 2y$

$z = 6 - 3x - 2y \Rightarrow 3x + y + z = 6$, this represent a plane

$\begin{cases} x=0 \\ y=0 \end{cases}$ $z=6$ $\quad \begin{cases} x=0 \\ z=0 \end{cases}$ $y=3$ $\quad \begin{cases} y=0 \\ z=0 \end{cases} \Rightarrow x=2$



Portion of the surface in 1st octant.



only consider upper part.

Ex. Sketch $g(x,y) = \sqrt{9-(x^2+y^2)}$

$z = \sqrt{9-(x^2+y^2)}$

$z^2 = 9 - x^2 - y^2 \Rightarrow x^2 + y^2 + z^2 = 9$ equation of a sphere of radius 3

Note: the surface is the upper hemisphere b/c $z \geq 0$.

Limits & Continuity: Function of two variable   Mar 15, 2018   Lei Yu

Def: let f be a function of two variables: whose domain D includes points arbitrarily close to point (a, b), then we say

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ If for every $\varepsilon > 0$ there exist a corresponding $\delta > 0$

Such that if $(x,y) \in D$ and $0 < \sqrt{(x-a)^2+(y-a)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.

• The definition above indicates that the distance between $F(x,y)$ and L can be made arbitrarily small by making the distance between (x,y) and (a,b) sufficiently small but not zero.

∴ We can show the limit exists if $F(x,y)$ approaches the same limit no matter how (x,y) approaches (a,b). Thus, we can find two different path of approach along which $F(x,y)$ have different limits, the we can conclude that $\lim_{(x,y)\to(a,b)} F(x,y)$ DNE.

Ex. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = DNE$

approaching $(0,0)$ along x-axis. $y=0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1$ ∴ $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = DNE$

approaching $(0,0)$ along y-axis. $x=0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} = -1$

Ex. $\lim_{(x,y)\to(0,0)} F(x,y) = \frac{xy}{x^2+y^2}$

along x-axis. $y=0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0$ ∴ $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = DNE$

along y-axis. $x=0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{0}{y^2} = 0$

along $x=y$. $\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$

Function of Two Variables: Continuity Mar 21, 2018

A function $F(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Ex. $F(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ at $(1,2)$

This function is a polynomial so it is continuous at $(1,2)$

$\lim_{(x,y)\to(1,2)} = F(1,2) = 1(8) - 1(4) + 3 + 4 = 11$

- A Polynomial is continuous on all the points of its Domain.

Ex. $F(x,y) = \frac{x^2-y^2}{x^2+y^2}$ Domain: $\mathbb{R}^2 - \{(0,0)\}$

- This function is continuous over its domain, because it's a ratio of two polynomials.

Ex. $g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ Domain is $\mathbb{R}^2$.

We need to check continuity at $(x,y) = (0,0)$.

from previous note ⇒ $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \text{DNE}$. $\therefore$ $g(x,y)$ is not continuous at $(0,0)$

Ex. Where is $h(x,y) = \arctan\left(\frac{y}{x}\right)$ continuous?

$f(x,y) = \frac{y}{x}$ is continuous everywhere on its domain.

$g(t) = \arctan(t)$ is continuous everywhere $t \in \mathbb{R}$

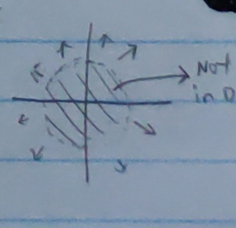
$g(f(x,y)) = \arctan\left(\frac{y}{x}\right)$ is continuous everywhere on its domain $\mathbb{R}^2 - (x=0)$

Ex. Determine the set of points at which $G(x,y) = \ln(x^2+y^2-4)$ is continuous.

$f(x,y) = x^2+y^2-4$ is continuous over $x,y \in \mathbb{R}^2$

$g(t) = \ln t$ is continuous over $t > 0$

$G(x,y)$ is continuous over $x^2+y^2-4 > 0$. $D = \{(x,y) \mid x^2+y^2 > 4\}$



Partial Derivatives Mar 21, 2018 Lei Yu

Recall For a function with one variable $F(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{dF}{dx}$$

For a function of two variables $f(x,y)$

$$f_x(x,y) = \frac{\delta f}{\delta x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$, The partial derivative of f with respect to x.

$$f_y(x,y) = \frac{\delta f}{\delta y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$, the partial derivative of f with respect to y.

The rule for partial derivative

To find $\begin{cases} f_x \\ f_y \end{cases}$ regard $\begin{cases} y \\ x \end{cases}$ as a constant and differentiate with respect $\begin{cases} x. \\ y. \end{cases}$

Ex. $f(x,y) = \sin\left(\frac{x}{1+y}\right)$

$$f_x = \frac{\delta f}{\delta x} = \frac{1}{1+y} \cos\left(\frac{x}{1+y}\right) \qquad f_y = \frac{df}{dy} = \frac{-x}{(1+y)^2} \cos\left(\frac{x}{1+y}\right)$$

Ex. Find $\frac{\partial z}{\partial y}, \frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x, y as follows

$$x^3 + y^3 + z^3 + 6xyz = 1$$

$$\left.\begin{aligned} 3x^2 + 0 + 3\frac{\delta z}{\delta x}z^2 + 6yz + 6yx\frac{\delta z}{\delta x} &= 0 \\ 3\frac{\delta z}{\delta x}z^2 + 6yx\frac{\delta z}{\delta x} &= -3x^2 - 6yz \\ \frac{\delta z}{\delta x}(3z^2 + 6yx) &= -3x^2 - 6yz \\ \frac{\delta z}{\delta x} &= \frac{-3x^2 - 6yz}{3z^2 + 6yz} \end{aligned}\right\}$$ derived with respect to x

$$\left.\begin{aligned} 3y^2 + 3\frac{\delta z}{\partial y}z^2 + 6xz + 6xy\frac{\delta z}{\delta y} &= 0 \\ 3\frac{\delta z}{\delta y}z^2 + 6xy\frac{\partial z}{\delta y} &= -3y^2 - 6xz \\ \frac{\partial z}{\delta y}(3z^2 + 6xy) &= -3y^2 - 6xz \\ \frac{\partial z}{\partial y} &= \frac{-3y^2 - 6xz}{3z^2 + 6xy} \end{aligned}\right\}$$ with respect to y.

Higher Derivaties

$$f_{xx} = (f_x)_x = f_{11} = \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x} = \frac{\partial^2 f}{\partial x}$$

$$f_{yy} = (f_y)_y = f_{22} = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial y} = \frac{\partial^2 f}{\partial y}$$

$$f_{yx} = (f_y)_x = f_{21} = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{xy} = (f_x)_y = f_{12} = \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Theorem: Suppose $f(x,y)$ is defined on D that contain (a,b) if f_{xy} & f_{yx} are both continuous on D then $f_{xy} = f_{yx}$

Ex. Find the second derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2y^3x \Rightarrow f_{xx} = 6x + 2y^3$

$f_y = 3x^2y^2 - 4y \Rightarrow f_{yy} = 6x^2y - 4$

$f_{xy} = 6xy^2$ $\qquad f_{xy} = f_{yx}$

$f_{yx} = 6y^2x$

Ex. $f(x,y,z) = \sin(3x + yz)$, Find f_{xxyz}

$f_x = 3\cos(3x + yz) \Rightarrow f_{xx} = -9\sin(3x + yz) \Rightarrow f_{xxy} = -9z\cos(3x + yz)$

$f_{xxyz} = -9\cos(3x + yz) + 9zy\sin(3x + yz)$

Partial Differential Equations Mar 21, 2018 Lei Yu

Partial derivatives occurs in Partial Differential Equations. The solution to a partial Differential Equation is factor of two or more variables. A well known PDE with application to heat conduction, fluid flow, electric potential is the Laplace Equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$$

Ex. Show that $u(x,y) = e^x \sin y$ is a solution to Laplace equation.

$U_x = e^x \sin y \Rightarrow U_{xx} = e^x \sin y$ $\qquad$ $u_{xx} + U_{yy} = e^x \sin y - e^x \sin y = 0$

$u_y = e^x \cos y \Rightarrow U_{yy} = -e^x \sin y$ $\qquad$ $\therefore$ $U(x,y) = e^x \sin y$ is a solution to Laplace eq.

Ex. Another famous PDE is the wave equation

$$\frac{d^2 u}{dt^2} = a^2 \frac{d^2 u}{dx^2}$$

Investigat if $u(x,t) = \sin(x+at)$ satisfies wave equation.

$U_x = \cos(x+at) \Rightarrow U_{xx} = -\sin(x+at)$ $\qquad$ $a^2 u_{xx} = u_{tt}$

$U_t = +a\cos(x+at) \Rightarrow U_{tt} = -a^2 \sin(x+at)$ $\qquad$ $\therefore$ Wave equation satisfied

Ex. If $U(x,y) = e^{-x}\cos y - e^{-y}\cos x$, then determine if u is a solution to Laplace equation.

$U_x = -e^{-x}\cos y + e^{-y}\sin x$ $\qquad$ $U_y = -e^{-x}\sin y + e^{-y}\cos x$

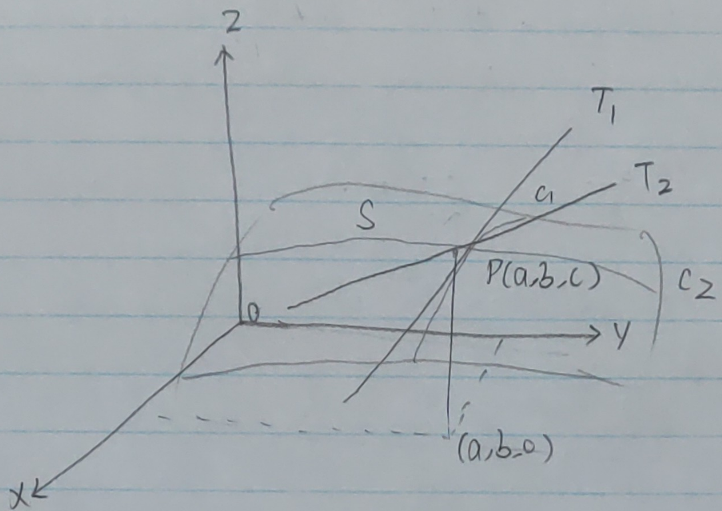
$U_{xx} = e^{-x}\cos y + e^{-y}\cos x$ $\qquad$ $U_{yy} = -e^{-x}\cos y - e^{-y}\cos x$

$$U_{xx} + U_{yy} = e^{-x}\cos y + e^{-y}\cos x - e^{-x}\cos y - e^{-y}\cos x$$
$$= 0$$

$\therefore$ $U(x,y) = e^{-x}\cos y - e^{-y}\cos x$ is solution to laplace equation.

Interpretation of Partial Derivative   Mar 22, 2018

We know that $f(x,y)$ represents a surface S.



So if $f(a,b)=c$, then the point $p(a,b,c)$ lies on S. by fixing $y=b$, we restrict our attention to the curve C_1 in which the vertical plane $(y=b)$ intersects S. Similarly the vertical plane $x=a$ intersects surface S in a curve C_2. We note that the intersect of C_1 and C_2 is the point $P(a,b,c)$. C_1 is the graph of $g(x)=f(x,b)$, so slop of its Tangent T_1 is $g'(x)=f_x(a,b)$ C_2 is the graph of $h(y)=f(a,y)$. So the slope of its tangent T_2 at P is $h'(y)=f_y(a,b)$. In other word, $f_x(a,b)$ and $f_y(a,b)$ represents the slopes of the tangent lines at $P(a,b,c)$ to the curves C_1, C_2 respectively.

Function with two Variable: Chain Rules Mar 22, 2018 Lei Yu

For functions of more than one variable, the chain rule has three versions.

$\boxed{V.1}$ Suppose $\underset{\text{dependant}}{z} = f(\underset{\text{Intermediate}}{x,y})$ is a differentiable function of x and y, where $x = g(\underset{\text{Independant}}{t})$, $y = h(\underset{\text{Independant}}{t})$ are both differentiable functions of t, then Z is a differentiable function of t and $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Ex. $Z = x^2y + 3xy^4$, $x = \sin 2t$, $y = \cos t$ $\qquad \frac{dz}{dt} = Z_x x' + Z_y y'$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \Rightarrow \frac{dz}{dt} = (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

$\boxed{V.2}$ $Z = F(x,y)$ $x = g(s,t)$, $y = h(s,t)$ $\qquad s,t$: independant variables, Z: dependant; x,y: intermediate variables

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} \quad \Big| \quad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Ex. $Z = e^x \sin y$, $x = st^2$, $y = s^2t$

$$Z_s = e^x \sin y(t^2) + e^x \cos y\, 2st \quad \Big| \quad Z_t = e^x \sin y(2st) + e^x \cos y(s^2)$$

<u>General Version</u> Suppos $\underline{u}$ is a function with n variables, $x_1, x_2, \dots, x_n$ and each x_j it self is a function of m variables, $t_1, t_2, t_3 \dots t_m$.

$\underset{\text{Dependant}}{u} = U(\underset{\text{intermediate}}{x_1, x_2, \dots, x_n})$ $\qquad x_j = x_j(\underset{\text{Independant variable}}{t_1, t_2, t_3, \dots, t_m})$

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_i} + \frac{du}{dx_2}\cdot\frac{dx_2}{dt_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

$i = 1, 2, 3, n$

Ex. Write out chain rule for $w=f(x,y,z,t)$, $x=x(u,v)$ $\quad z=z(u,v)$

$y=y(u,v) \quad t=t(u,v)$

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}+\frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}+\frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial v}$$

Ex. $u=x^4y+y^2z^3=u(x,y,z)$ $\quad x=rse^t,\ y=rse^{-t},\ z=r^2s\sin t,\ {x \atop {y \atop t}}(r,s,t)$

Find $\frac{\partial u}{\partial s}$ when $r=2, s=1, t=0$

$x=2(1)e^0=2$

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$y=2(1)e^0=2$

$z=0$

$$\left.\begin{aligned}\frac{\partial u}{\partial s}&=4x^3y(re^t)+(x^4+2yz^3)(re^{-t})+3z^2y^2(2r\sin t)\\&=4(2)^3(2)(2e^0)+(2^4+2(2)0^3)(2e^{-0})+3(0)^22^2(2\sin 0)\\&=128+32+\\&=160\end{aligned}\right\}=192$$

Ex. $g(s,t)$, $g=f(\underbrace{s^2-t^2}_{u},\underbrace{t^2-s^2}_{v})$ Show that g satisfies $t\frac{\partial g}{\partial s}+s\frac{\partial g}{\partial t}=0$.

$$\frac{\partial g}{\partial s}=\frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial s}+\frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial s} \qquad \frac{\partial g}{\partial t}=\frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial t}+\frac{\partial g}{\partial u}\cdot\frac{\partial v}{\partial t}$$

$$=\frac{\partial g}{\partial u}\cdot(2s)+\frac{\partial g}{\partial v}(-2s) \qquad =\frac{\partial g}{\partial u}\cdot(-2t)+\frac{\partial g}{\partial u}\cdot(2t)$$

$$t\left[\frac{\partial g}{\partial u}\cdot 2s+\frac{\partial g}{\partial v}(-2s)\right]+s\left[\frac{\partial g}{\partial u}(-2t)+\frac{\partial g}{\partial u}(2t)\right]=0$$

$$\frac{\partial g}{\partial u}\cdot 2st+\frac{\partial g}{\partial v}(-2st)+\frac{\partial g}{\partial u}(-2ts)+\frac{\partial g}{\partial u}(2ts)=0$$

$$t\frac{\partial g}{\partial s}+s\frac{\partial g}{\partial t}=0$$

Function w/ Two Variables: Chain Rule Pg. 2 Mar 28 Lei Yu

Ex. $z = f(x, y)$, $x = r^2 + s^2$, $y = 2rs$ find $\frac{\partial z}{\partial r}$, $\frac{\partial^2 z}{\partial^2 r}$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial h} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

② $\frac{dz}{dx} = h(x, y)$ $\frac{dz}{\partial y} = g(x, y)$

① $\boxed{\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y} \cdot (2s)}$ $\Rightarrow$ $= 2r \cdot h(x, y) + 2s \cdot g(x, y)$ ③

④ $\frac{\partial^2 z}{\partial x^2} = \frac{\partial h}{\partial x}$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial h}{\partial y}$

$$\frac{d^2 z}{dr^2} = 2h(x, y) + 2r\frac{\partial h}{\partial r} + 2s\frac{\partial g}{\partial r}$$

⑤ $\frac{\partial h}{\partial r} = \frac{\partial h}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial h}{\partial y} \cdot \frac{\partial y}{\partial r}$

$= 2r\frac{\partial h}{\partial x} + 2s\frac{\partial h}{\partial y}$

$= 2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y \partial x}$

⑥ $\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial r}$

$= 2r\frac{\partial g}{\partial x} + 2s\frac{\partial g}{\partial y}$

$= 2r\frac{\partial^2 z}{\partial x \partial y} + 2s\frac{\partial^2 z}{\partial y^2}$

$$\boxed{\frac{\partial^2 z}{\partial r^2} = 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 4s^2\frac{\partial^2 z}{\partial y^2} + 8rs\frac{\partial^2 z}{\partial x \partial y}}$$

before

$$\frac{d^2 z}{dr^2} = 2\frac{\partial z}{\partial x} + \left(4r^2\frac{\partial^2 z}{\partial x^2} + 4rs\frac{\partial^2 z}{\partial x \partial y}\right) + \left(4rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}\right)$$

Ex. $W = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$

$\frac{\partial w}{\partial r} = ?$, $\frac{\partial w}{\partial \theta} = ?$ when $\begin{cases} r = 2 \\ \theta = \frac{\pi}{2} \end{cases}$

$x = 2\cos\frac{\pi}{2} = 0$

$y = 2\sin(\frac{\pi}{2}) = 2$

$z = \pi$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

2π

$= (y + z)\cos\theta + (x + z)\sin\theta + (y + x)\theta$

$= (2 + \pi)\cos\frac{\pi}{2} + (0 + \pi)\sin(\frac{\pi}{2}) + (2 + 0)\pi$

$= \pi + 2\pi$

Ex. $z = \tan^{-1}(x^2+y^2) \quad x = s\ln t \,,\; y = te^s$

$$z_s = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{2x}{1+(x^2+y^2)^2}\cdot \ln t + \frac{2y}{1+(x^2+y^2)^2}\cdot te^s$$

$$z_t = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= \frac{2x}{1+(x^2+y^2)^2}\cdot\frac{s}{t} + \frac{2y}{1+(x^2+y^2)^2}\cdot e^s$$

Implicit Differentiation Mar 29, 2018

Suppose $F(x,y)=0$ defining y explicitly as a function of x that $y=f(x)$
$\Rightarrow F(x, f(x)) = 0$.

We can apply the chain rule to look both side of $F(x,y)=0$

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x} = 0$$

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

Ex. Find y' if $x^3+y^3-6xy=0$.

$$F(x,y) = x^3+y^3-6xy = 0$$

$$\frac{\partial y}{\partial x} = -\frac{3x^2-6y}{3y^2-6x}$$

z is given implicitly as a function of (x,y)

Now, let's suppose $z = f(x,y)$ by an equation $F(x,y,z) = z - f(x,y) = 0$
to find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ we can use the chain rule as follows

$$\frac{\partial F}{\partial x}\cdot\left(\frac{\partial x}{\partial x}\right)^{1} + \frac{\partial F}{\partial y}\cdot\left(\frac{\partial y}{\partial x}\right)^{0} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}}$$

Similarly, $\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_x}}$

Implicit Differentiation Pg.2   Mar 29, 2018   Lei Yu

Ex. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ when $x^3+y^3+z^3+6xyz=1$

$$F(x,y,z)=x^3+y^3+z^3-1+6xyz=0$$

$$\frac{\partial z}{\partial x}=-\frac{3x^2+6yz}{3z^2+6xy} \qquad \frac{\partial z}{\partial x}=-\frac{3y^2+6xz}{3z^2+6xy}$$

Ex. Find $\frac{dy}{dx}$, $\tan^{-1}(x^2y)=x+xy^2$

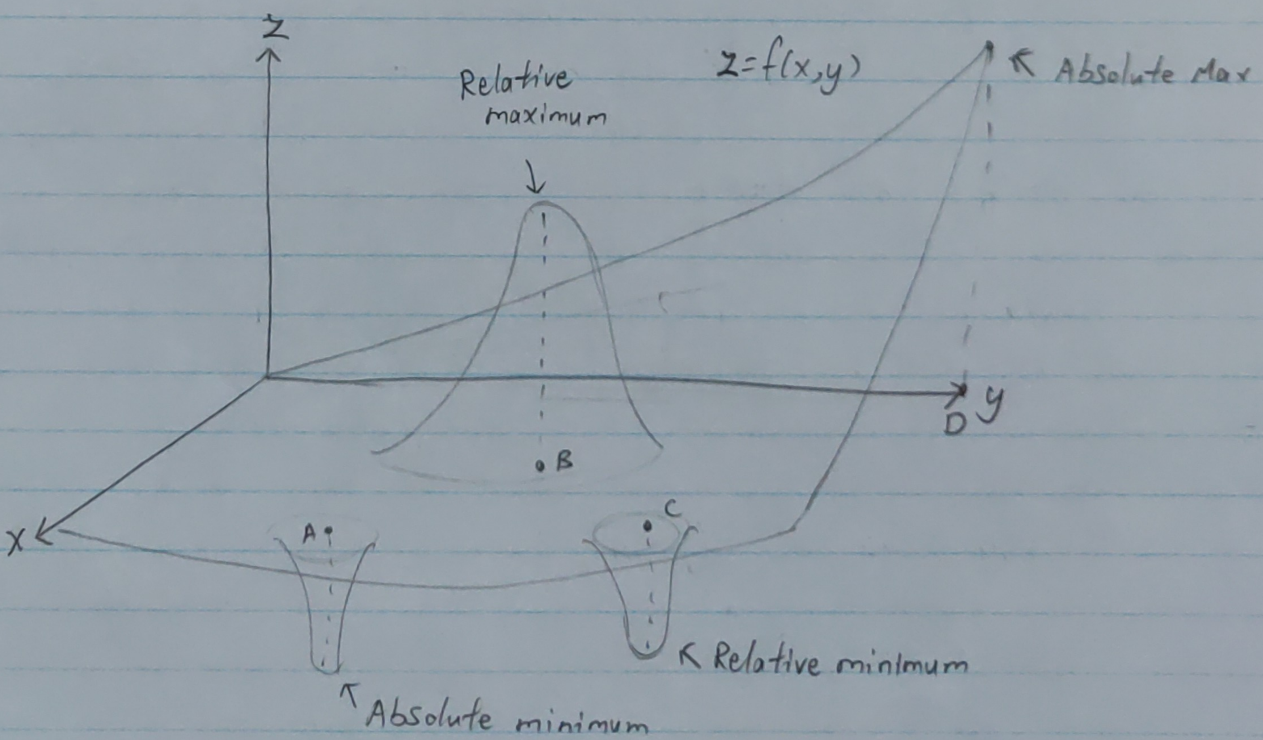
$$F(x,y)=x+xy^2-\tan^{-1}(x^2y)=0$$

$$\frac{\partial F}{\partial x}=-\frac{1+y^2-\frac{2xy}{1+(x^2y)^2}}{2yx-\frac{x^2}{1+(x^2y)^2}}=-\frac{[1+(x^2y)^2](1+y^2)-2xy}{[1+(x^2y)^2]2yx-x^2}$$

Ex. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ when $e^z=xyz$

$$F(x,y,z)=xyz-e^z=0$$

$$\frac{\partial z}{\partial x}=-\frac{yz}{xy-e^z}=\frac{yz}{e^z-xy} \qquad \frac{\partial z}{\partial y}=-\frac{xz}{xy-e^z}=\frac{xz}{e^z-xy}$$



Function of two Variable (Maximum and Minimum) Mar 29, 2012 Lei Yu

DEF: a function of two variable $F(x,y)$ has a local (relative) maximum at point (a,b) if $F(x,y) < F(a,b)$ when (x,y) is near (a,b) [This means that $F(x,y) \leq F(a,b)$ For all the points (x,y) in some disk centered at (a,b). The # $F(a,b)$ is called the local minimum value if $F(x,y) \geq F(a,b)$ when (x,y) is near (a,b), then $\underline{\underline{F}}$ has a local minimum value.

If the inequality in the above given definition hold for all points (x,y) in the whole domain of $\underline{\underline{F}}$, then F has an absolute maximum or minimum.

Theorem: If $f(x,y)$ has a local max/min at (a,b) and if the $f_x(a,b)$, $f_y(a,b)$ exist then $f_x(a,b) = f_y(a,b) = 0$.

proof: let $g(x) = f(x,b)$. If f has a local max or a local min at (a,b) then g has a local max or min at $x = a$, therefor $g'(a) = 0$. but $g'(x) = f_x(x,b)$

$\Rightarrow g'(a) = f(a,b) = 0$

Similarly: $h(y) = F(a,y) \Rightarrow h'(b) = 0 \Rightarrow h'(b) = f_y(a,b) = 0$

DEF: A point (a,b) is a critical / stationary pt. if $F_x(a,b) = F_y(a,b) = 0$ or if one of these partial derivaties does not exist.

NOTE: as in function of one variable not all CP are maxima or minima.

Ex. $F(x,y) = x^2 + y^2 - 2x - 6y + 14$

$F_x(x,y) = 2x - 2$
$0 = 2x - 2$
$1 = x$

$F_y = 2y - 6$
$6 = 2y$
$3 = y$

$\therefore$ $F(1,3)$ is a CP

$\therefore$ $F(1,3)$ is an absolute minimum.

$F(x,y) = (x^2 - 2x + 1) + (y^2 - 6y + 9) + 4$

$= (x-1)^2 + (y-3)^2 + 4$ ← minimum value will be 4 at $F(1,3)$

MTH240 OPtimization with multivariable function Lei Yu Apr 4, 2018

Shortes distance from plane to pt.

Ex. Find the shorest distance from the point (1, 0, -2) to the plane $x + 2y + z = 4$.

Sol: we assume that (x, y, z) is a point on the plane.

$\because$ The distance between (x, y, z) and $(1, 0, -2)$ is $D = \sqrt{(x-1)^2 + (y)^2 + (z+2)^2}$

we know $z = 4 - x - 2y$

$\Rightarrow d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2} \Rightarrow d^2 = (x-1)^2 + y^2 + (6 - x - 2y)^2$.

whatever minimizes d minimizes d^2 aswell. So we minimize d^2.

$F_x = 2(x-1) - 2(6-x-2y) = 2x - 2 - 12 + 2x + 4y = 4x + 4y - 14 = 0$

$F_y = 2y + 2(6-x-2y)(-2) = 2y - 4(6 - x - 2y) = 2y - 24 + 4x + 8y = 4x + 10y - 24 = 0$

$4x + 4y - 14 = 0$

$4x + 10y - 24 = 0$

$-6y + 10 = 0$

$y = \frac{10}{6} \Rightarrow x = \frac{11}{6}$

$\therefore (\frac{11}{6}, \frac{10}{6})$ is a CP.

$D = \begin{vmatrix} 4 & 4 \\ 4 & 10 \end{vmatrix} = 40 - 16 = 24 > 0$

$F_{xx} = 4 > 0$ $\therefore (\frac{11}{6}, \frac{10}{6})$ is ~~local~~ min.

the nature of the problem depicts that $(\frac{11}{6}, \frac{10}{6})$ is absolute min aswell.

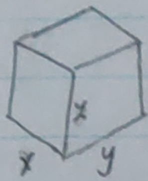
$F_{xx} = 4$ $F_{xy} = 4$

$f_{xy} = 10$ $F_{yx} = 4$

$d = \sqrt{(\frac{11}{6} - 1)^2 + (\frac{10}{6})^2 + (6 - \frac{11}{6} - \frac{10}{3})^2} = \frac{5}{\sqrt{6}}$

Rectangular Box with no lid.

Ex. a rectangular box without a lid is to be made from $12\ m^2$ of cardboard. Find the maximum volume of such a box.



$$V(x,y,z) = xyz \qquad \frac{12-yx}{2y+2x} = z$$

$$12 = 2zy + 2xz + yx \Rightarrow \frac{12-yx}{2(y+x)} = z$$

$$V(x,y,z) = \frac{xy(12-yx)}{2(y+x)} = \frac{12xy - y^2x^2}{2(y+x)}$$

$$V_x = \frac{(12y-2xy^2)(2y+2x) - 2(12xy - y^2x^2)}{4(y+x)^2} = \frac{24y^2 \cancel{+24yx} - 4xy^3 - 4x^2y^2 \cancel{-24xy} + 2y^2x^2}{4(y+x)^2}$$

$$V_x = \frac{-4xy^3 + 24y^2 - 2x^2y^2}{4(y+x)^2} = \frac{2y^2(-2xy + 12 - x^2)}{4(y+x)^2} = \frac{y^2(-2xy+12-x^2)}{2(y+x)^2}$$

$$V_y = \frac{x^2(-2xy+12-y^2)}{2(y+x)^2}$$

$x \& y \neq 0$

$$\begin{cases} x^2(12-2xy-y^2) = 0 \\ y^2(12-2xy-x^2) = 0 \end{cases} \Rightarrow \begin{matrix} 12-2xy-y^2 = 0 \\ 12-2xy-x^2 = 0 \end{matrix}$$

$x^2 = y^2$

$x = \pm y$

$x = -y$ not possible

$\therefore x = y$

$x = y \Leftarrow$

$12 - 2x^2 - x^2 = 0$

$3x^2 = 12$

$x = 2$

$\therefore (2,2)$ is a critical pt.

$\therefore$ We can show that $(2,2)$ is a local max.

(Max, Min)

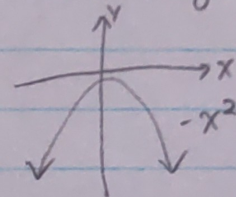
MTH 240 Apr 4, 2018 Week 11 — Lei Yu

Ex. Find the local extream values of $f(x,y) = y^2 - x^2$

$f_x = -2x = 0$
$f_y = 2y = 0$ $\Rightarrow (0,0)$ is a CP

NOTE: that for the points on the x-axis, when $y=0$

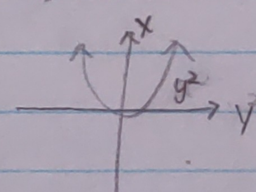
$F(x,y) \Rightarrow F(x,0) \Rightarrow F(x) = -x^2$



when $y=0$
maximum along x-axis

For the points on the y-axis, when $x=0$

$F(x,y) \Rightarrow F(0,y) \Rightarrow f(x) = y^2$



when $x=0$
minimum along y-axis

$\therefore$ this point $(0,0)$ is a saddle point

Recall function of one variable $f(x) = x^3$

$f'(x) = 3x^2 \Rightarrow x=0$ $f'(x)=0$, $(0,0)$ critical but it's neither max nor min.

It is a inflection point.

☆ Second derivative test

Suppose the second derivative of $f(x,y)$ are continuous on a disk centered at (a,b) and suppose $F_x(a,b) \approx f_y(a,b) = 0$. ← CP

① Find CP

let $D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

② Use 2nd to test local max or min.

a) If $D > 0$ and $F_{xx}(a,b) > 0$ then (a,b) is local min.
b) If $D > 0$ and $F_{xx}(a,b) < 0$ then (a,b) is local max.
c) If $D < 0 \Rightarrow (a,b)$ is a saddle point.
d) If $D = 0 \Rightarrow$ inconclusive

Ex. $f(x,y) = x^4 + y^4 - 4xy + 1$. Find local max, min & saddle pt.

$f_x = 4x^3 - 4y = 0 \Rightarrow y = x^3 \Rightarrow y = 0$

$f_y = 4y^3 - 4x = 0 \Rightarrow 4x^9 - 4x = 0 \Rightarrow x^9 - x = 0 \Rightarrow x(x^8 - 1) = 0$

$x(x^4+1)(x^4-1) = 0 \Rightarrow x(x^4+1)(x^2+1)(x^2-1) = 0$

$x(x^4+1)(x^2+1)(x-1)(x+1) = 0$

$0, -1, 1 = x$

$0, 1, -1 = y$

$\Rightarrow (0,0)$, $(1,1)$, $(-1,-1)$

$f_{xx} = 12x^2 \quad f_{xy} = -4$

$f_{yy} = 12y^2 \quad f_{yx} = -4$

$(0,0)$

$$D = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0, \text{ Saddle point}$$

$(1,1)$ $\quad f_{xx}(1,1) = 12$

$$D = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 144 - 16 = 128 > 0, \therefore \text{Local min}$$

$(-1,-1)$

$$D = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 144 - 16 = 128 > 0 \therefore \text{Local min}$$

Ex. Find and classify CP of $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$

find all derivatives

$f_x = 20yx - 10x - 4x^3 = 0 \Rightarrow f_{xx} = 20y - 10 - 12x^2 \quad f_{xy} = 20x$

$f_y = 10x^2 - 8y - 8y^3 = 0 \Rightarrow f_{yy} = -8 - 24y^2 \quad f_{yx} = 20x$

$(0,0)$ is a CP

finding $f_x = f_y = 0$ CP

$2x(10y - 5 - 2x^2) = 0 \quad \begin{cases} x = 0 \\ 10y = 5 + 2x^2 \end{cases}$

$5x^2 - 4y - 4y^3 = 0$

Case 1: if $x = 0$ ② $\Rightarrow -4y - 4y^3 = 0 \Rightarrow y(1+y^2) = 0$

Case 2: $x^2 = 5y - \frac{5}{2} \Rightarrow$ then $5(5y - \frac{5}{2}) - 4y - 4y^3 = 0$

$25y - \frac{25}{2} - 4y - 4y^3 = 0$

$4y^3 - 21y + \frac{25}{2} = 0$

$\begin{cases} y = -2.54,\ x^2 < 0 \text{ not possible} \\ y = 0.65,\ x = \pm 0.86 \\ y = 1.9,\ x = \pm 2.64 \end{cases}$

CP	f	f_{xx}	D	Conclusion
0,0				
(0.86, 0.65)				
(-0.86, 0.65)				
(2.64, 1.9)				
(-2.64, 1.9)				

Absolute Min and Max for Function of two Variable Apr 5, 2018 Lei Yu

Reall: that for function of one variable $F(x)$ the extream value theorem states that if F is continuous on a closed interval $[a,b]$, then f has an absolute max and an absolute min $[a,b]$.

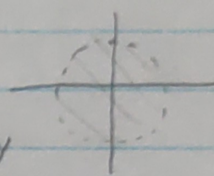
Closed Set

There's a similar situation for function of two variable. Similar to a closed interval that contains its end points. a close set in $\mathbb{R}^2$ is the one that contains all its boundary points.

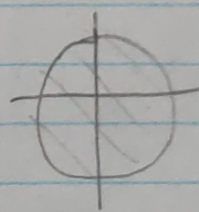
For example Disk $D = \{(x,y) \mid x^2+y^2<1)$

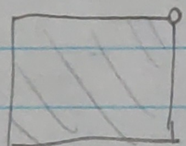
Only point inside the disk not point on boundary

· does not contain its boundary. NOT CLOSED



$D = \{(x,y) \mid x^2+y^2 \leq 1\}$ is a closed set.



Ex. 

↘ This point is not included

· The set is not closed.

A Bounded set in $\mathbb{R}^2$ is one that is contained in some disk. in other word, it is finite in extent.

NOTE: A set can be closed but not bounded. I.e (A line is closed but not bounded)

Extreme Value Theorem for Functions of two variable

If $f(x,y)$ is continuous on a closed, bounded set D in $\mathbb{R}^2$, then F has an absolute max and min in D.

· Finding absolute max, min, of a continuous function $f(x,y)$ on a closed, bounded set D.

1. Find CP of f in D ($F'(x)=0$, or DNE
2. Find extream value of f on boundary of D.
3. The largest and smallest values found in step 1 and 2 are absolut max and min.

Eg. Find the absolute max and min of $f(x,y) = x^2-2xy+2y$ on rectangle $D = \{(x,y) \mid 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$.

① CP on D

$F_x = 2x-2y = 0 \Rightarrow CP: (1,1)$

$F_y = 2-2x = 0$

② max min on Boundary

L_1 $y=0$, $\boxed{0 \leq x \leq 3}$ max $h_1(3)=9$ min $h_1(0)=0$

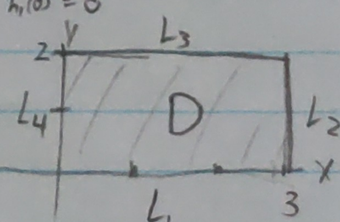
$\boxed{f(x,0) = x^2 \Rightarrow h_1(x)}$

L_2 $x=3$ $0 \leq y \leq 2$ max $h_2=9$ min $h_2=1$

$f(3,y) = 9-6y+2y = 9-4y = h_2(y)$

L_3

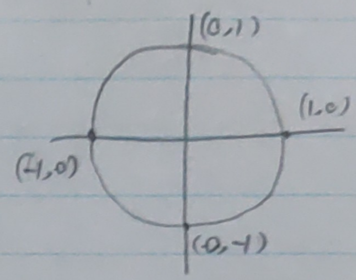
③ Compare values to find max min.



4 line Segment

Ex. Find the absolute max and min of $F(x,y)=2x^3+y^4$ on the closed set $D=\{(x,y) \mid x^2+y^2\le 1\}$

First, the CPs



$F_x=6x=0$
$F_y=4y^3=0$ $\Rightarrow$ CP: $(0,0)$ $\quad F(0,0)=0$

On the boundary $x^2+y^2=1 \Rightarrow y^2=1-x^2$

$\Rightarrow F(x,y)=2x^3+(1-x^2)^2=2x^3+1-2x^2+x^4=\overset{h(x)}{x^4+2x^3-2x^2+1} \quad -1\le x\le 1$

extrema should be found

$h'(x)=4x^3+6x^2-4x=2x(2x^2+3x-2)=0 \quad \begin{cases} x=0 \quad y^2=1-x^2 \Rightarrow y=\pm 1 & (0,1) \; (0,-1) \\ 2x^2+3x-2=0 & (\frac{1}{2},\frac{\sqrt{3}}{2}),(\frac{1}{2},\frac{-\sqrt{3}}{2}) \end{cases}$

$2x^2+3x-2=0$

$x=\frac{-3\pm\sqrt{25}}{4}=\frac{-3+5}{4}$ or $\frac{-3-5}{4}=\frac{1}{2}$ or -2 $\quad \begin{cases} x=\frac{1}{2}, \; y^2=1-\frac{1}{4}=\pm\frac{\sqrt{3}}{2} \\ \cancel{x=-2, \; y^2=1-4<0} \text{ Not possible} \end{cases}$

CP found: $(0,0),(0,1),(0,-1),(\frac{1}{2},\frac{\sqrt{3}}{2}),(\frac{1}{2},\frac{-\sqrt{3}}{2})$

$F(0,0)=0,\quad F(\frac{1}{2},\frac{\sqrt{3}}{2})=2(\frac{1}{8})+(\frac{\sqrt{3}}{2})^4=\frac{1}{4}+\frac{9}{16}=\frac{4}{16}+\frac{9}{16}=\frac{13}{16}$

$F(0,1)=1 \quad F(\frac{1}{2},\frac{-\sqrt{3}}{2})=\frac{13}{16}$

$F(0,-1)=1$

$\therefore$ Absolute max is 2 at $(1,0)$
min is -2 at $(-1,0)$

End point: $(-1,0),(1,0)$

$F(-1,0)=-2$

$F(1,0)=2$

MTH240 Final Review Apr 10, 2018 Lei Yu

Ex Evaluate the intergral as power series and find its radius of conv.

$$\int x^2 \ln|1+x|\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+2}}{n}\,dx$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad R=1$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \int \frac{x^{n+2}}{n}\,dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+3}}{n(n+3)} + C = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x^{n+3})}{n^2+3n} + C$$

R still gets 1.

Ex Find the Maclaurin Series expansion for $f(x) = xe^x$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(x) = xe^x$

$f'(x) = e^x + xe^x$

$f''(x) = 2e^x + xe^x$

Ex Find if limit exist or show it doesn't

$$\lim_{(x,y)\to(0,0)} \frac{xy^2\cos y}{x^2+y^4}$$

$$y=0 \quad \lim_{x\to 0} \frac{x \times 0 \times \cos 0}{x^2} = 0$$

$$x=y^2 \quad \lim_{x\to 0} \frac{y^4 \cos y^2}{y^4+y^4} = \lim_{x\to 0} \frac{\cos y}{2} = \frac{1}{2}$$

For every $\varepsilon > 0$, there exist $\delta > 0$ such that

if $|F(x,y) - L| < \varepsilon \Rightarrow (x,y) - (0,0) < \delta$

Ex. $\lim_{(x,y)\to(0,0)} \sin(x^2+y^2)$ $1 \le \sin(x^2+y^2) \le 1$ limit exist by

$1 \le \lim_{(x,y)\to(0,0)} \sin(x^2+y^2) \le 1$ squeeze theorem

Ex. $f(x,y,z) = z^{(\sin z)(\cos x + \tan y)}$

Find $f_z = \frac{\partial f}{\partial z}$ $f = z^{\sin z(\cos x + \tan y)}$

$\ln f = \sin z(\cos x + \tan y)\ln z$

$\frac{f_z}{f} = (\cos x + \tan y)\cos z \ln z + \frac{\sin z(\cos x + \tan y)}{z}$

$f_z = z^{\sin z(\cos x + \tan y)}\left[(\cos x + \tan y)\cos z \ln z + \frac{\sin z(\cos x + \tan y)}{z}\right]$

Ex. Find $\frac{\partial z}{\partial x}$ if z is implicitly defined as a function of other variables.

$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ $x^{5z} + \sin y^x + \cos(wy) = 0 = F(x,y,w,z)$ $z(x,y,w)$

with respect to x

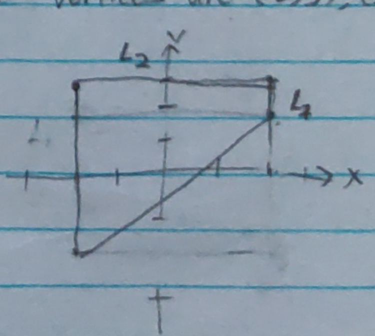
$\frac{\partial F}{\partial x} = 5z\, x^{5z-1} + (y^x)'\cos(y^x) = 5z\, x^{5z-1} + (\ln y)\, y^x \cos y^x$

$\frac{\partial F}{\partial z} = 5(\ln x)\, x^{5z}$

Ex. Find Max and min of $f(x) = x^3 - 3x - y^3 + 12y$ on a trapezoid! vertices are $(-2,3), (2,3), (2,-2), (-2,2)$.

CP $\begin{cases} f_x = 3x^2 - 3 \Rightarrow 3(x^2-1) = 0 \Rightarrow (x-1)(x+1) = 0 & x = \pm 1 \\ f_y = -3y^2 + 12 \Rightarrow 3(4-y^2) = 0 \Rightarrow (2-y)(2+y) = 0 & y = \pm 2 \end{cases}$

(1,2), 1,-



$\boxed{\begin{array}{l} f(1,2) = 1 - 3 - 8 + 24 = 14 \\ f(-1,2) = -1 + 3 - 8 + 24 = 18 \end{array}}$

on L_1: $x = 2$ $F(2,y) = 8 - 6 - y^3 + 12y$ Find CP and End point

$h(y) = -y^3 + 12y + 2$

$h'(y) = -3y^2 + 12 = 0 \Rightarrow y = 2, x = 2$ | $F(2,2) = 18$

$\cancel{y = -2, x = 2}$ not on domain

$y = 3, x = 2$ | $F(2,3) = 11$

MTH240 Review Pg.2 Apr 12, 2018 Lei Yu

On L_2: $y=3$, $-2 \le x \le 2$

$F(x,3) = x^3 - 3x - 27 + 36 = x^3 - 3x + 9 = h_2(x)$

$h'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0 \Rightarrow x = \pm 1$

$(1,3), (-1,3), (2,3), (-2,3)$

$$\boxed{\begin{array}{ll} F(1,3)=7 & F(2,3)=11 \\ F(-1,3)=11 & F(-2,3)=7 \end{array}}$$

On L_3 $x=-2$, $-2 \le y \le 3$

$F(-2,y) = -y^3 + 12y - 2 = h_3(y)$

$h_3'(y) = -3y^2 + 12 = 0 \begin{cases} y=2, x=-2 \Rightarrow \boxed{F(-2,2) = 14} \\ y=-2, x=-2 \Rightarrow \boxed{F(-2,-2) = -18} \end{cases}$

end pt. $\begin{cases} \boxed{F(-2,3) = 7} \\ F(\end{cases}$

On L_4: $y=x$

$F(x,x) = x^3 - 3x - x^3 + 12x = 9x = h_4'(x)$

$h'(x) = 9$

$\therefore$ Absolute min is $(-2,-2)$, absolute max $(2,2)$, $(-1,2)$, Ex. Approximate $F(x) = x^{\frac{2}{3}}$ by a taylor polynomial of degree 3 at $a=1$. Use taylor's inequality to estimate the accuracy of this approximation when $.8 \le x \le 1.2$

$F(x) = x^{\frac{2}{3}}$ $\qquad F(1) = 1$

$F'(x) = \frac{2}{3} x^{-\frac{1}{3}}$ $\qquad F'(1) = \frac{2}{3}$ $\qquad F(1) = F(a) + F'(1)(x-a) + \frac{F''(1)(x-a)^2}{2!} + \frac{F'''(1)(x-a)^3}{3!}$

$F''(x) = \frac{-2}{3^2} x^{-\frac{4}{3}}$ $\qquad F''(1) = \frac{-2}{3^2}$

$F'''(x) = \frac{8}{3^3} x^{-\frac{7}{3}}$ $\qquad F'''(1) = \frac{8}{3^3}$ $\qquad = 1 + \frac{2}{3}(x-1) - \frac{2(x-1)^2}{3^2(2!)} + \frac{8(x-1)^3}{3^3(3!)} + \frac{-56(x-1)^4}{3^4\, 4!}$

$F''''(x) = \frac{-56}{3^4} x^{-\frac{10}{3}}$

$R_3 \le \frac{M}{(n+1)!}|x-a|^{n+1}$ $\qquad M = F^{(n+1)}(x)$ max

$\le \frac{1.45457}{4!}|0.2|^4$ $\qquad (x-1)^{n+1}$ sub 1.2 for min

$R_3 \le 9.697 \times 10^{-5}$

$0.8 \le x \le x$ $\quad -0.2 \le |x-1| \le 0.2$

$|x-1| \le 0.2$