Fundamentals of Logic
- Propositional Calculus -

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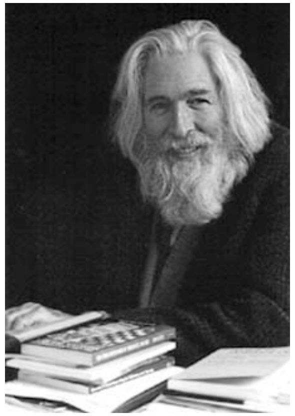
Problem: How to Reason With a Robot

- You might try the ploy successfully used by Mr. Spock on the malevolent android 'Norman' (*Star Trek*), wherein he posed the following to Norman: "Everything I say is a lie. I'm a liar".
  Norman self-fried.

- *Can you prove what Spock said is invalid?*

Knights and Knaves

- On a fictional island, all inhabitants are either knights, who always tell the truth, or knaves, who always lie. The puzzles involve a visitor to the island who meets small groups of inhabitants. Usually the aim is for the visitor to deduce the inhabitants' type from their statements, but some puzzles of this type ask for other facts to be deduced. The puzzle may also be to determine a yes/no question which the visitor can ask in order to discover what he needs to know.



Raymond Smullyan
(b1919)

Knights and Knaves : Question 1

- John and Bill are residents of the island of knights and knaves.
- John says: We are both knaves.
- *Who is who?*

J	**B**	
T	T	
T	F	
F	T	
F	F	

Knights and Knaves : Question 2

- John: We are the same kind.
- Bill: We are of different kinds.
- *Who is who?*

J	B	
T	T	
T	F	
F	T	
F	F	

Knights and Knaves : Question 3

- Logician: Are you both knights?

- John answers either Yes or No, but the Logician does not have enough information to solve the problem.

- Logician: Are you both knaves?

- John answers either Yes or No, and the Logician can now solve the problem.

- *Who is who?*

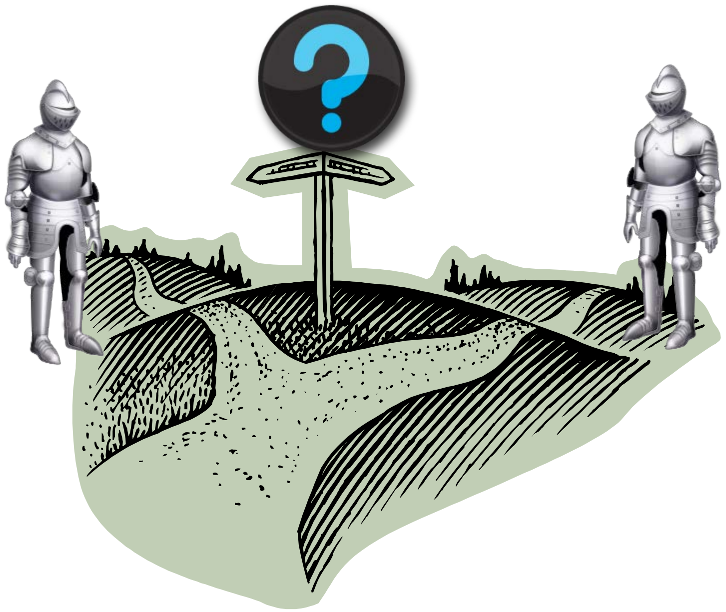
J	B	1st Question	2nd Question
T	T		
T	F		
F	T		
F	F		

Knights and Knaves : Question 4

- John and Bill are standing at a fork in the road.

  You know that one of them is a knight and the other is a knave, but you don't know which. You also know that one road leads to Freedom, and the other leads to Death.

- By asking one yes/no question, can you determine the road to Freedom?



- How many divisors for 6 ?
  - 6의 약수는 몇 개인가?
- How many divisors for 24?
  - 24의 약수는 몇 개인가?
- How many divisors for 210?
  - 210의 약수는 몇 개인가?
- How many divisors for 440?
  - 440의 약수는 몇 개인가?

- The answer of the problem. vs The solving process of the problem
  - 문제의 답? 문제의 풀이 체계?

Knights and Knaves : Question 1

- John and Bill are residents of the island of knights and knaves.
- John says: We are both knaves. Who is who?
- $\boldsymbol{P}$: John is a Knight; $\boldsymbol{Q}$: Bill is a Knight,
- $\boldsymbol{R}$: What John says is true; $\boldsymbol{R}: \neg\boldsymbol{P} \wedge \neg\boldsymbol{Q}$
  - $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ are variables
- We know that $\boldsymbol{P} \leftrightarrow \boldsymbol{R}$ is true. ; Premise
  - $\boldsymbol{P} \leftrightarrow \boldsymbol{R}$; if $\boldsymbol{P}$ is true then $\boldsymbol{R}$ is true and if $\boldsymbol{P}$ is false then $\boldsymbol{R}$ is false.

P	Q	$R\!: \neg P \wedge \neg Q$	$P \leftrightarrow R$	$\neg P \wedge Q$
T	T	F	F	F
T	F	F	F	F
F	T	F	T	T
F	F	T	F	F
			premise	conclusion

Knights and Knaves : Question 1

- Example : If we know that $\neg P \vee (P \wedge Q)$ is true.

$$\Leftrightarrow \big((\neg P \vee P) \wedge (\neg P \vee Q)\big) \Leftrightarrow \big(T \wedge (\neg P \vee Q)\big) \Leftrightarrow (\neg P \vee Q) \Leftrightarrow P \to Q$$

- We know that $\big(P \leftrightarrow (\neg P \wedge \neg Q)\big)$

$$\Leftrightarrow \big(P \wedge (\neg P \wedge \neg Q)\big) \vee \big(\neg P \wedge \neg(\neg P \wedge \neg Q)\big)$$

$$\Leftrightarrow F \vee \big(\neg P \wedge (P \vee Q)\big)$$

$$\Leftrightarrow \big((\neg P \wedge P) \vee (\neg P \wedge Q)\big)$$

$$\Leftrightarrow \neg P \wedge Q$$

Knights and Knaves : Question 2

- John: We are the same kind. Bill: We are of different kinds.
- *Who is who?*
- $\boldsymbol{P}$: John is a Knight; $\boldsymbol{Q}$: Bill is a Knight,
- $\boldsymbol{R}$: What John says is true; $\boldsymbol{P} \leftrightarrow \boldsymbol{Q}$
- $\boldsymbol{S}$: What Bill says is true; $\boldsymbol{P} \leftrightarrow \neg\boldsymbol{Q}$
- We know that $\boldsymbol{P} \leftrightarrow \boldsymbol{R}, \boldsymbol{Q} \leftrightarrow \boldsymbol{S}$; Premises

P	Q	$R: P \leftrightarrow Q$	$S: P \leftrightarrow \neg Q$	$P \leftrightarrow R$	$Q \leftrightarrow S$	$\neg P \wedge Q$
T	T	T	F	T	F	F
T	F	F	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	F	T	F

premises conclusion

Knights and Knaves : Question 2

- We know that $P \leftrightarrow (P \leftrightarrow Q), Q \leftrightarrow (P \leftrightarrow \neg Q)$

- We can infer that $P \leftrightarrow \neg Q$
  - $\neg(P \leftrightarrow Q) \Leftrightarrow (P \leftrightarrow \neg Q), \neg(P \leftrightarrow \neg Q) \Leftrightarrow (P \leftrightarrow Q)$
  - $P \leftrightarrow (P \leftrightarrow Q) \Leftrightarrow \neg(P \leftrightarrow \neg Q) \leftrightarrow \neg Q$

- $Q \leftrightarrow (P \leftrightarrow \neg Q) \leftrightarrow T$

Contents

Formal Logic

❖ Formal (or Symbolic) Logic (형식논리학: 形式論理學)

Mathematical model of deductive thought(연역적사고: 演繹的思考)

- Modeling: extract important features according to purposes

  (*Ex*) The law of gravity:

$$F = G \cdot \frac{m \cdot m'}{r^2}$$

masses, distance (○), color (×)

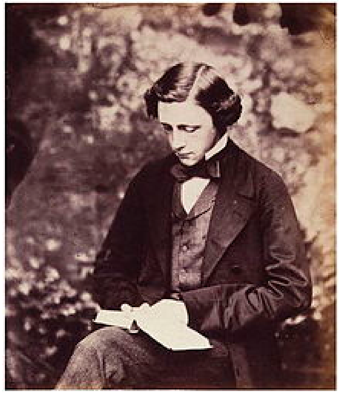
(Q1) What are the important features of deductive thought?

- syntax (structure), not semantics (meaning or interpretation)

  (*Ex*) Borogoves are mimsy whenever it is brillig.

  It is now brillig and this thing is a borogove. *Therefore*, this thing is mimsy.

- *http://en.wikipedia.org/wiki/Jabberwocky*
- *http://ko.wikipedia.org/wiki/재버워키* ,



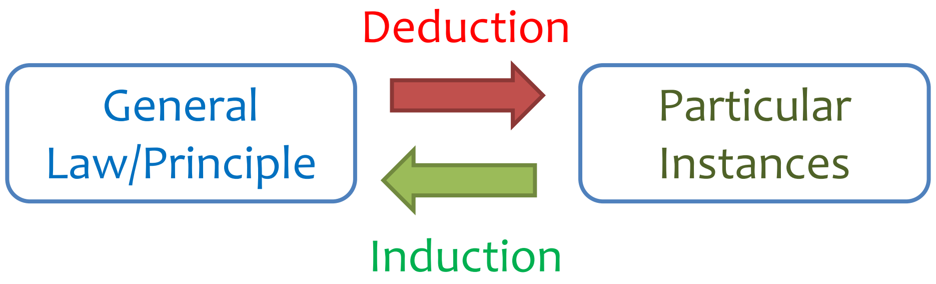
Lewis Carrol

(British, 1832~1898)

Deduction vs. Induction

❖ Deductive Reasoning (연역법 - 演繹法)

- Deduce : 추론하다. 추정하다. 유사어 : infer, derive, deduct
- The inference of particular instances by reference to a general law or principle
- Top-Down Logic
  - 합리론 (rationalism) – 프랑스의 데카르트
- Example)
  - 1. All men are mortal
  - 2. Aristotle is a man
  - 3. Therefore, Aristotle is mortal



❖ Inductive Reasoning (귀납법 – 歸納法)

- Induce : 유도하다. 유치하다. 유인하다. 유발하다.
- The inference of a general law from particular instances
- Bottom-Up Logic
  - 경험론 (empiricism) - 영국의 베이컨
- Example) Not deductively valid.
  - 1. All swans observed so far have been white
  - 2. Smoothy is a swan
  - 3. Therefore, Smoothy is white.
- In inductively valid arguments, the truth of the premises is very likely (but not necessarily) sufficient for the truth of the conclusion.
- Example) Mathematical Induction (수학적 귀납법)

(Q2) What does it mean for one sentence to *"follow logically"* from certain others?

(Q3) If a sentence does follow logically from certain others, what methods of proof might be necessary to establish this fact?

We need a different language since the natural language is ambiguous.

(Q2), (Q3) will be covered in "Ch3. Formal Reasoning"

Two models are considered:

(They use different languages with different expressive power.)

❖ Propositional Calculus (명제논리: 命題論理)

- Statement Calculus, Sentential Logic, Propositional Logic
- Can express very simple, crude properties of deduction

❖ First-Order Predicate Calculus (일차 술어논리: 一次 述語論理)

- First-Order Logic, Predicate Logic
- Suited for mathematical deduction

PROPOSITIONAL CALCULUS

Propositional Calculus

- ❖ Proposition
- ❖ Logical Connectives
- ❖ Well Formed Formula
- ❖ Equivalent Formula

- P, Q
- $P \vee Q, \neg P, P \rightarrow Q$
- $\neg P \vee Q$
- $(\neg P \vee Q) \Leftrightarrow (P \rightarrow Q)$

Propositional Calculus

❖ Proposition(명제: 命題)

- An assertion (a declarative sentence) that can take a value *true* or *false*

(*Ex*)	$x + y = 4.$	This statement is false.	???
	1 + 1 = 3.	I am a student.	
	I work hard.	I am healthy.	!!!

❖ Propositional Constants

- stand for a particular proposition (such as $\boldsymbol{W}$ or $\boldsymbol{H}$)

❖ Propositional Variables

- stand for some propositions (such as $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S}$)

❖ Compound Propositions (합성 명제: 合成 命題)

- More complicated propositions resulting from combining *primitive* propositions

(*Ex*) I work hard ***and*** I am healthy.

If it rains ***then*** there is cloud in the sky.

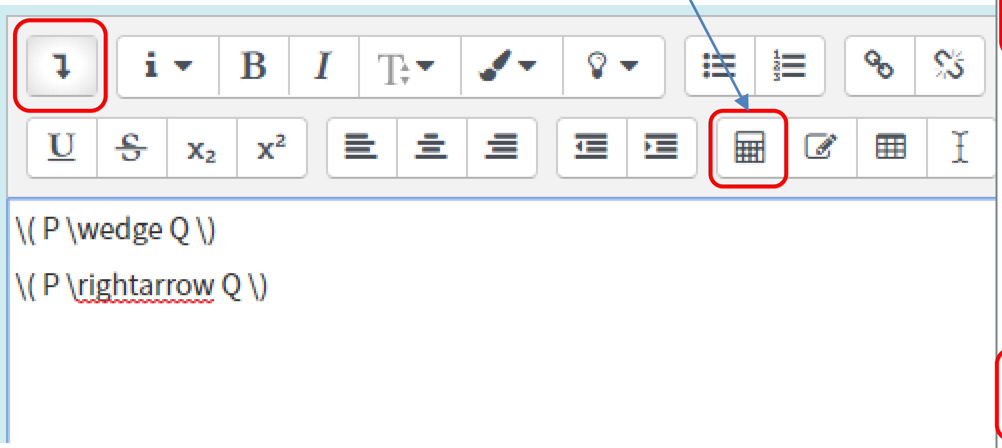
❖ Logical Connectives

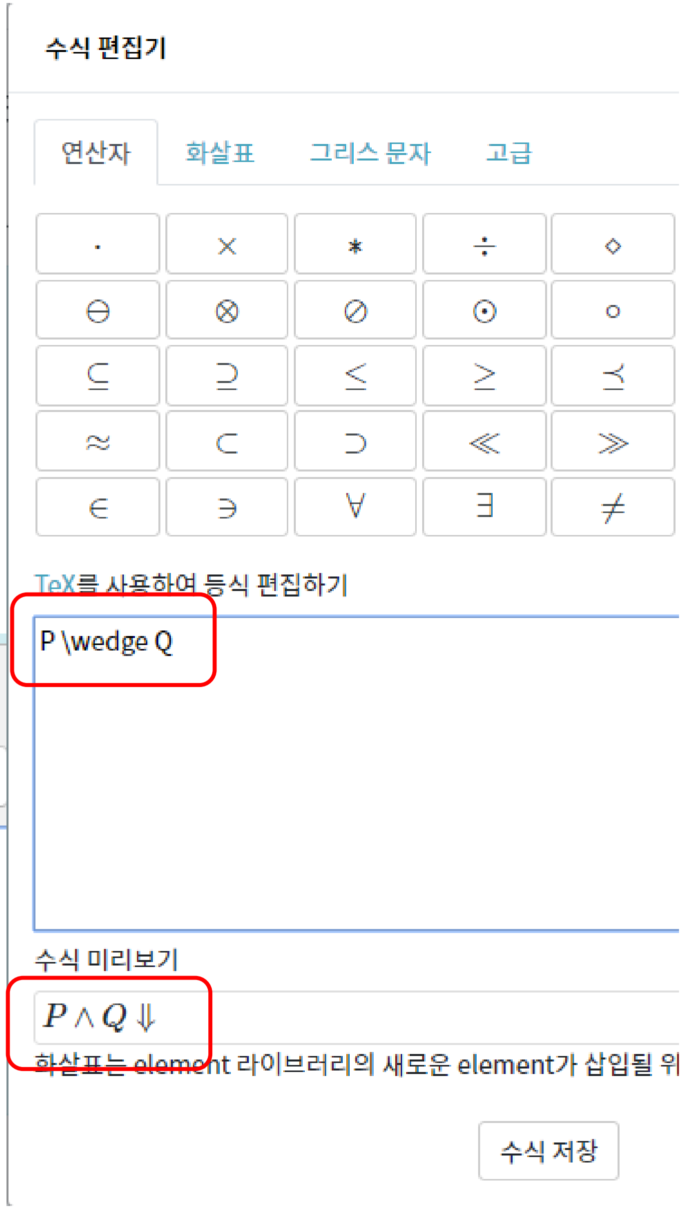
Symbol	Name	Read	MS Equation Editor	Web
$\neg$	negation	not	\neg	\neg
$\wedge$	conjunction	and	\wedge	\wedge
$\vee$	disjunction	or	\vee	\vee
$\rightarrow$	implication (conditional)	if … then …	\to, \rightarrow	\rightarrow
$\leftrightarrow$	bi-conditional	iff (if and only if)	\leftrightarrow	\leftrightarrow

Equation Editing

- LaTex Equation Editor
  - https://www.codecogs.com/latex/eqneditor.php
- Plato's HTML Editor
  - You can start an equation by "**\(**" and end it by "**\)**"
  - An WYSIWIG Editor is available

Launch the Equation Editor





❖ The meaning of logical connectives is defined by the truth table.

P	$\neg P$
F	T
T	F

P	Q	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
F	F	F	F	T	T
F	T	F	T	T	F
T	F	F	T	F	F
T	T	T	T	T	T

❖ (Example) Jane promises that on December 26 $\boldsymbol{P} \rightarrow \boldsymbol{Q}$.

- $\boldsymbol{P}$: I weigh more than 120 pounds.
- $\boldsymbol{Q}$: I shall enroll in an exercise class.
- $\boldsymbol{P} \rightarrow \boldsymbol{Q}$: If $\boldsymbol{P}$, then $\boldsymbol{Q}$.

Case	P	Q	$P \rightarrow Q$
1	F	F	T
2	F	T	T
3	T	F	F
4	T	T	T

✓ (Case 4) Jane enrolls just as she said. So $\boldsymbol{P} \rightarrow \boldsymbol{Q}$ is true.

✓ (Case 3) Jane breaks her promise. So $\boldsymbol{P} \rightarrow \boldsymbol{Q}$ is false.

✓ (Case 2) Jane enrolls even though her weight is less than or equal to 120 pounds. However, she does not violate her promise. So $\boldsymbol{P} \rightarrow \boldsymbol{Q}$ is true.

✓ (Case 1) Jane does not violate her promise, either. So $\boldsymbol{P} \rightarrow \boldsymbol{Q}$ is true.

❖ Let $\boldsymbol{p}, \boldsymbol{q}$ be primitive statements for which the implication $\boldsymbol{q} \rightarrow \boldsymbol{p}$ is false.

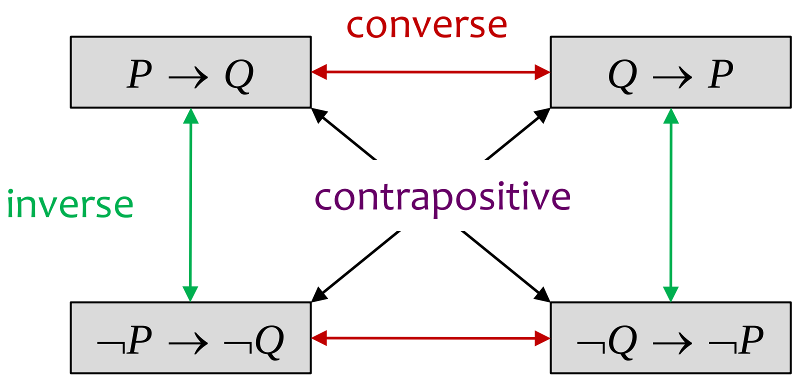
Determine the truth values for

- p
- q
- $p \wedge \neg q$
- $p \rightarrow q$

❖ Other expressions for implication

$$\boldsymbol{P} \text{ (antecedent) } \rightarrow \boldsymbol{Q} \text{ (consequent)}$$

- If $\boldsymbol{P}$ then $\boldsymbol{Q}$.
- $\boldsymbol{P}$ only if $\boldsymbol{Q}$.
- $\boldsymbol{P}$ is a sufficient condition (충분조건 : 充分條件) for $\boldsymbol{Q}$.
- $\boldsymbol{Q}$ is a necessary condition (필요조건: 必要條件) for $\boldsymbol{P}$.



❖ Converse (역: 逆): $\boldsymbol{Q} \rightarrow \boldsymbol{P}$

❖ Inverse (이: 裏): $\neg \boldsymbol{P} \rightarrow \neg \boldsymbol{Q}$

❖ Contrapositive (대우: 對偶): $\neg \boldsymbol{Q} \rightarrow \neg \boldsymbol{P}$

Translation

- s : Phyllis goes out for a walk.
- t : The moon is out
- u : It is snowing.
- For the above primitive statements s, t, u, translate each of the following into an English sentences.
  - $(t \wedge \neg u) \rightarrow s$
  - $t \rightarrow (\neg u \rightarrow s)$
  - $\neg(s \leftrightarrow (u \vee t))$
- Get the logical (symbolic) notation for the following sentences.
  - Phyllis will go out walking if and only if the moon is out.
  - If it is snowing and the moon is not out, then Phyllis will not go out for a walk.
  - It is snowing but Phyllis will still go out for a walk.

❖ Definition of Well-Formed Formula

A propositional well-formed formula (wff) is a grammatically correct expression, which is defined ***inductively*** as follows.

1. (*Basis clause*) A truth symbol (T or F), a propositional variable, or a propositional constant is a wff.
2. (*Inductive clause*) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are wffs, then $(\neg\boldsymbol{A}), (\boldsymbol{A} \wedge \boldsymbol{B}), (\boldsymbol{A} \vee \boldsymbol{B}), (\boldsymbol{A} \rightarrow \boldsymbol{B})$, and $(\boldsymbol{A} \leftrightarrow \boldsymbol{B})$ are wffs.
3. (*Extremal clause*) Only the strings formed by ***finite*** applications of clauses 1 and 2 are wffs.

❖ Precedence

- To remove parentheses, we use the following conventions.
- Hierarchy of evaluation: (highest) $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ (lowest)
- Operations $\wedge, \vee, \rightarrow$ are left associative.

(*Ex1*) $\boldsymbol{F} \wedge \neg \boldsymbol{P} \vee \neg \boldsymbol{Q} \rightarrow \neg \boldsymbol{Q} \Rightarrow \Big(\big((\boldsymbol{F} \wedge (\neg \boldsymbol{P})) \vee (\neg \boldsymbol{Q})\big) \rightarrow (\neg \boldsymbol{Q})\Big)$

(*Ex2*) $\boldsymbol{P} \rightarrow \boldsymbol{Q} \rightarrow \boldsymbol{R} \Rightarrow \big((\boldsymbol{P} \rightarrow \boldsymbol{Q}) \rightarrow \boldsymbol{R}\big)$

Meaning of a wff and Truth table.

- Meaning (or semantics) of a wff is defined by its truth table.
  - p : Combinatorics is a required course for sophomores.
  - q : Margaret Mitchell wrote *Gone with the Wind.*
  - r : $\mathbf{2+3=5}$
  - Margaret Mitchell wrote Gone with the Wind, and if $\mathbf{2+3\neq 5}$ then combinatorics is a required course for sophomores : $\boldsymbol{q \wedge (\neg r \rightarrow p)}$.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{r}$	$\boldsymbol{\neg r}$	$\boldsymbol{\neg r \rightarrow p}$	$\boldsymbol{q \wedge (\neg r \rightarrow p)}$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	F
F	T	T	F	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	T	F	F

❖ (*Ex*) $(\boldsymbol{P} \rightarrow \boldsymbol{Q}) \leftrightarrow (\neg \boldsymbol{P} \vee \boldsymbol{Q})$?

P	Q	$P \rightarrow Q$	$\neg P \vee Q$	$(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	T
T	T	T	T	T

Tautology

❖ Definition : A tautology (항진명제: 恒眞命題) is a wff which takes the value true for every possible truth values assigned to the variables contained in the formula.

- (Note) A tautology is true every time – not a definition!
- If a wff $\boldsymbol{P}$ is a tautology, it is denoted by $\vDash \boldsymbol{P}$.

❖ Definition : A contradiction (or denial, 모순명제: 矛盾命題) is a wff which takes the value false for every possible truth values assigned to the variables contained in the formula.

❖ Definition : A contingency (사건명제: 事件命題) is a wff which is neither a tautology nor a contradiction.

LOGICAL EQUIVALENCE & EQUIVALENCE RULES

Logical Equivalence

❖ Definition

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be *two* wffs. The formulas $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be equivalent formulas (denoted by $\boldsymbol{A} \Leftrightarrow \boldsymbol{B}$), when $\boldsymbol{A}$ is true (false) iff $\boldsymbol{B}$ is true (false).

- (*cf.*) Let $\boldsymbol{P_1}, \boldsymbol{P_2}, \cdots, \boldsymbol{P_n}$ be the propositional variables contained in $\boldsymbol{A}$ and/or $\boldsymbol{B}$. The formulas $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be equivalent formulas (designated by $\boldsymbol{A} \Leftrightarrow \boldsymbol{B}$) if they have the same value for each of the $\boldsymbol{2^n}$ sets of truth value assignment to the propositional variables.
- ⇔ : \Leftrightarrow, cf) ↔ : \leftrightarrow

❖ Theorem 1

$\boldsymbol{A}$ and $\boldsymbol{B}$ are equivalent formulas iff $\boldsymbol{A} \leftrightarrow \boldsymbol{B}$ is a tautology (denoted by $\models \boldsymbol{A} \leftrightarrow \boldsymbol{B}$).

- ⊨ : \models,

❖ Definition

A formula $\boldsymbol{B}$ is a substitution instance of a formula $\boldsymbol{A}$ if $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by substituting formulas for propositional variables in $\boldsymbol{A}$ under the condition that the same formula is substituted for the same variable each time that variable appears in the formula $\boldsymbol{A}$.

(*Ex*)

- $\boldsymbol{A}: \left((\boldsymbol{P} \rightarrow \boldsymbol{Q}) \wedge (\boldsymbol{R} \vee \boldsymbol{Q})\right)$
- Substitute $(\boldsymbol{P} \rightarrow \boldsymbol{Q})$ for $\boldsymbol{P}$ and $(\boldsymbol{R} \vee \boldsymbol{S})$ for $\boldsymbol{Q}$
- $\boldsymbol{B}: \left(\left((\boldsymbol{P} \rightarrow \boldsymbol{Q}) \rightarrow (\boldsymbol{R} \vee \boldsymbol{S})\right) \wedge \left(\boldsymbol{R} \vee (\boldsymbol{R} \vee \boldsymbol{S})\right)\right)$

❖ Theorem 2

A substitution instance of a tautology is a tautology.

❖ Theorem 3

A formula $\boldsymbol{B}$ is equivalent to a formula $\boldsymbol{A}$

if $\boldsymbol{B}$ is obtained from $\boldsymbol{A}$ by replacing a subformula $\boldsymbol{C}$ of $\boldsymbol{A}$

with a formula $\boldsymbol{D}$ which is equivalent to $\boldsymbol{C}$.

(Ex) For $\vDash (\boldsymbol{P} \to \boldsymbol{Q}) \leftrightarrow (\neg \boldsymbol{P} \lor \boldsymbol{Q})$, that is, $(\boldsymbol{P} \to \boldsymbol{Q}) \Leftrightarrow (\neg \boldsymbol{P} \lor \boldsymbol{Q})$

- $\boldsymbol{A}$: $\left((\boldsymbol{P} \to \boldsymbol{Q}) \land (\boldsymbol{R} \lor \boldsymbol{Q})\right)$
- $\boldsymbol{B}$: $\left((\neg \boldsymbol{P} \lor \boldsymbol{Q}) \land (\boldsymbol{R} \lor \boldsymbol{Q})\right)$ by substituting $(\boldsymbol{P} \to \boldsymbol{Q})$ with $(\neg \boldsymbol{P} \lor \boldsymbol{Q})$
- Then, $\boldsymbol{A} \Leftrightarrow \boldsymbol{B}$

❖ Definition

If a formula $\boldsymbol{A}$ contains the connectives ¬,∧, and ∨ only, then the dual of $\boldsymbol{A}$, denoted by $\boldsymbol{A^d}$, is the formula obtained from $\boldsymbol{A}$ by replacing each occurrence of ∧ and ∨ with ∨ and ∧, respectively, and each occurrence of T and F by F and T, respectively.

- Ex) $\boldsymbol{A} : (\neg \boldsymbol{P} \vee \boldsymbol{Q}) \wedge (\boldsymbol{R} \vee \boldsymbol{S}) \wedge (\neg \boldsymbol{R} \wedge \boldsymbol{T}) \Rightarrow \boldsymbol{A^d} : (\neg \boldsymbol{P} \wedge \boldsymbol{Q}) \vee (\boldsymbol{R} \wedge \boldsymbol{S}) \vee (\neg \boldsymbol{R} \vee \boldsymbol{F})$

❖ Theorem 4: *De Morgan's Law*

If $\boldsymbol{A(P_1, P_2, \cdots, P_n)}$ is a formula and $\boldsymbol{A^d}$ is its dual, then

$$\neg \boldsymbol{A(P_1, P_2, \cdots, P_n)} \Leftrightarrow \boldsymbol{A^d(\neg P_1, \neg P_2, \cdots, \neg P_n)}$$

❖ Theorem 5: *Principle of Duality*

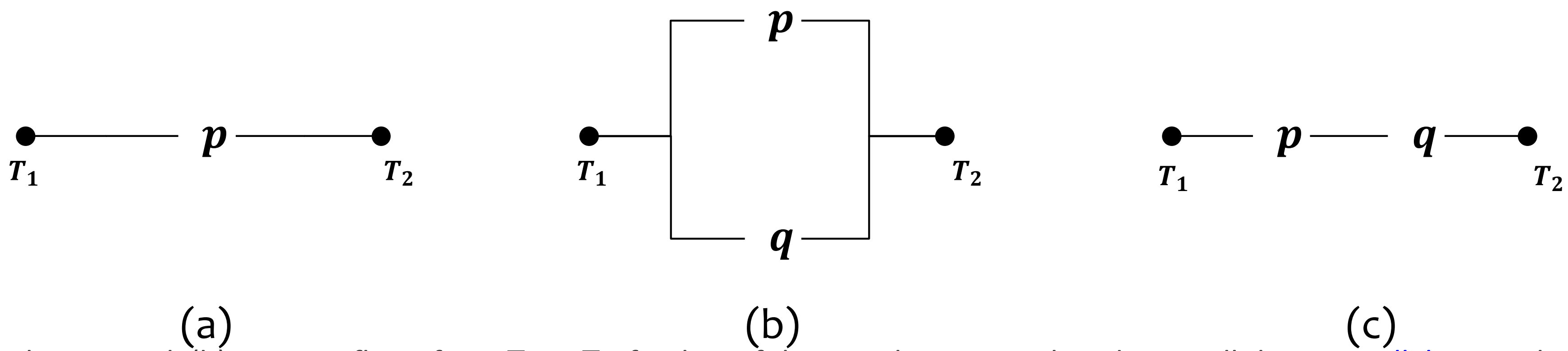
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two formulas and $\boldsymbol{A^d}$ and $\boldsymbol{B^d}$ be their respective duals, then $\boldsymbol{A} \Leftrightarrow \boldsymbol{B}$ iff $\boldsymbol{A^d} \Leftrightarrow \boldsymbol{B^d}$

The Laws of Logic - Equivalence Rules

#	Rule of Equivalence	Name of Rule
E_1	$\neg\neg \boldsymbol{P} \Leftrightarrow \boldsymbol{P}$	*Law of Double Negation*
E_2	$\neg(\boldsymbol{P} \wedge \boldsymbol{Q}) \Leftrightarrow \neg \boldsymbol{P} \vee \neg \boldsymbol{Q}$ $\neg(\boldsymbol{P} \vee \boldsymbol{Q}) \Leftrightarrow \neg \boldsymbol{P} \wedge \neg \boldsymbol{Q}$	*De Morgan's Law*
E_3	$\boldsymbol{P} \wedge \boldsymbol{Q} \Leftrightarrow \boldsymbol{Q} \wedge \boldsymbol{P}$ $\boldsymbol{P} \vee \boldsymbol{Q} \Leftrightarrow \boldsymbol{Q} \vee \boldsymbol{P}$	*Commutative Law*
E_4	$(\boldsymbol{P} \wedge \boldsymbol{Q}) \wedge \boldsymbol{R} \Leftrightarrow \boldsymbol{P} \wedge (\boldsymbol{Q} \wedge \boldsymbol{R})$ $(\boldsymbol{P} \vee \boldsymbol{Q}) \vee \boldsymbol{R} \Leftrightarrow \boldsymbol{P} \vee (\boldsymbol{Q} \vee \boldsymbol{R})$	*Associative Law*
E_5	$\boldsymbol{P} \wedge (\boldsymbol{Q} \vee \boldsymbol{R}) \Leftrightarrow (\boldsymbol{P} \wedge \boldsymbol{Q}) \vee (\boldsymbol{P} \wedge \boldsymbol{R})$ $\boldsymbol{P} \vee (\boldsymbol{Q} \wedge \boldsymbol{R}) \Leftrightarrow (\boldsymbol{P} \vee \boldsymbol{Q}) \wedge (\boldsymbol{P} \vee \boldsymbol{R})$	*Distributive Law*
E_6	$\boldsymbol{P} \wedge \boldsymbol{P} \Leftrightarrow \boldsymbol{P}; \boldsymbol{P} \vee \boldsymbol{P} \Leftrightarrow \boldsymbol{P}$	*Idempotent Law*
E_7	$\boldsymbol{R} \vee \boldsymbol{F_0} \Leftrightarrow \boldsymbol{R}\,; \boldsymbol{R} \wedge \boldsymbol{T_0} \Leftrightarrow \boldsymbol{R}$	*Identity Law;* $\boldsymbol{F_0}$ *: False;* $\boldsymbol{T_0}$ *: True*
E_8	$\boldsymbol{P} \vee \neg \boldsymbol{P} \Leftrightarrow \boldsymbol{T_0}\,; \boldsymbol{P} \wedge \neg \boldsymbol{P} \Leftrightarrow \boldsymbol{F_0}$	*Inverse Law*
E_9	$\boldsymbol{R} \vee \boldsymbol{T_0} \Leftrightarrow \boldsymbol{T_0}\,; \boldsymbol{R} \wedge \boldsymbol{F_0} \Leftrightarrow \boldsymbol{F_0}$	*Domination Law*
E_{10}	$\boldsymbol{P} \vee (\boldsymbol{P} \wedge \boldsymbol{Q}) \Leftrightarrow \boldsymbol{P}\,; \boldsymbol{P} \wedge (\boldsymbol{P} \vee \boldsymbol{Q}) \Leftrightarrow \boldsymbol{P}$	*Absorption Law*
E_{11}	$\boldsymbol{P} \rightarrow \boldsymbol{Q} \Leftrightarrow \neg \boldsymbol{Q} \rightarrow \neg \boldsymbol{P}$	*Contrapositive Law*
E_{12}	$\boldsymbol{P} \rightarrow \boldsymbol{Q} \Leftrightarrow \neg \boldsymbol{P} \vee \boldsymbol{Q}$	
E_{13}	$\boldsymbol{P} \leftrightarrow \boldsymbol{Q} \Leftrightarrow (\boldsymbol{P} \wedge \mathrm{Q}) \vee (\neg \boldsymbol{P} \wedge \neg \boldsymbol{Q})$	

❖ A Switching Network

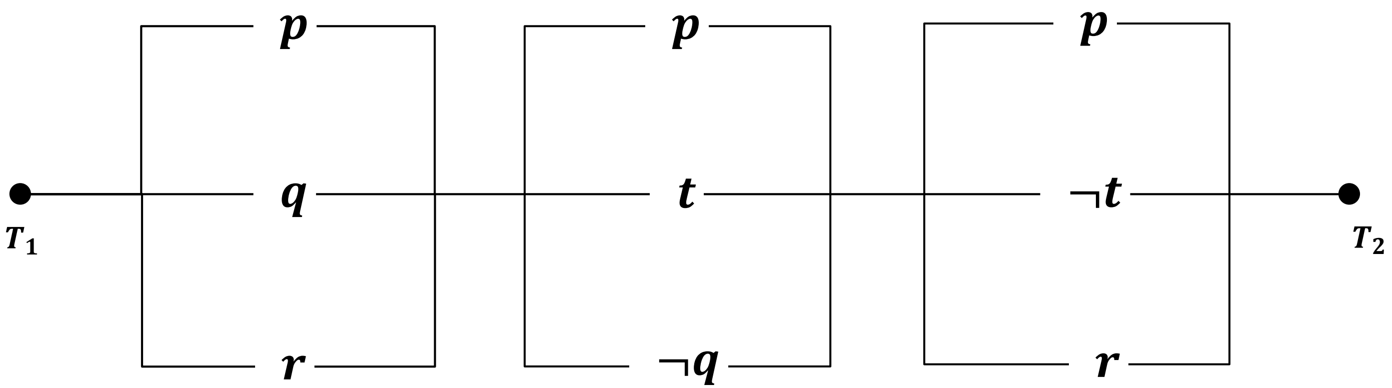
- is made up of wires and switches connecting two terminals T_1 and T_2.
  - In such a network, each switch is either open (0 / False), so that no current flows through it, or close (1 / True), so that current does flow through it.
  - In the figure (a) we have a network with one switch p. In the figure (b) and (c), we have two independent switches p and q.



  - For the network (b), current flows from T_1 to T_2 if either of the switches p, q is closed. We call this a *parallel* network and represented by $\boldsymbol{p} \vee \boldsymbol{q}$.
  - For the network (c), it requires that each of the switches p, q be closed in order for current to flow from T_1 to T_2. Here the switches are in series; this network is represented by $\boldsymbol{p} \wedge \boldsymbol{q}$.

❖ An Example

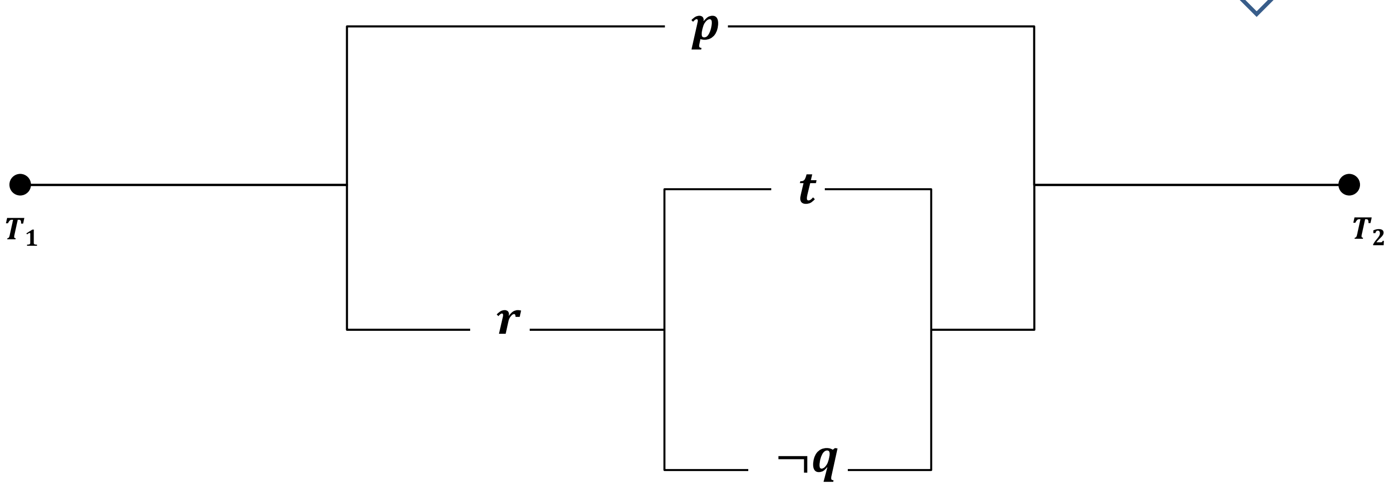
- (a)



$$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$$

⇕ Equivalent

- (b)



$$p \vee [r \wedge (t \vee \neg q)]$$

Digital Logic and Boolean Algebra

❖ Digital Logic Circuits

- is electronic circuits that handle information encoded in binary form (deal with signals that have only two values, 0 and 1)
- Logic Functions can be described by Binary Logic/Truth Table.
- Boolean algebra is used as a formal / mathematical tool to describe and design complex binary logic circuits.

❖ Example – Half Adder : Adding two bits

				Input
0	0	1	1	A
+ 0	+ 1	+ 0	+ 1	B
---	---	---	---	
0	1	1	10	
Sum	Sum	Sum	Carry Sum	

Inputs		Outputs	
A	B	Carry	Sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

- Carry : $A \wedge B$, Sum : $(A \wedge \neg B) \vee (\neg A \wedge B)$

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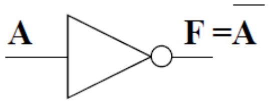
Digital Logic and Gates

- Quite complex digital logic circuits (e.g. entire computers) can be built using a few types of basic circuits called gates, each performing a single elementary logic operation

- Gates : NOT, AND, OR, NAND, NOR, XOR, XNOR, …

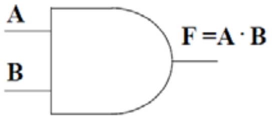
NOT

A	$\overline{A}$
0	1
1	0



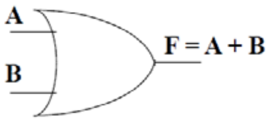
AND

A	B	$A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1



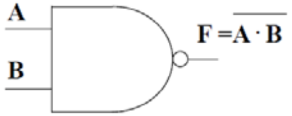
OR

A	B	A + B
0	0	0
0	1	1
1	0	1
1	1	1



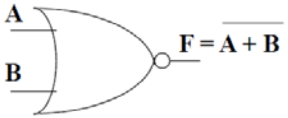
NAND

A	B	$\overline{A \cdot B}$
0	0	1
0	1	1
1	0	1
1	1	0



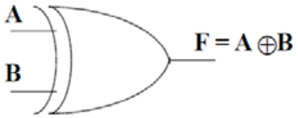
NOR

A	B	$\overline{A + B}$
0	0	1
0	1	0
1	0	0
1	1	0



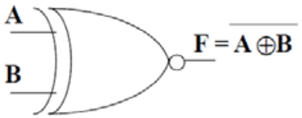
XOR

A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0



EQU **or** *XNOR*

A	B	$\overline{A \oplus B}$
0	0	1
0	1	0
1	0	0
1	1	1



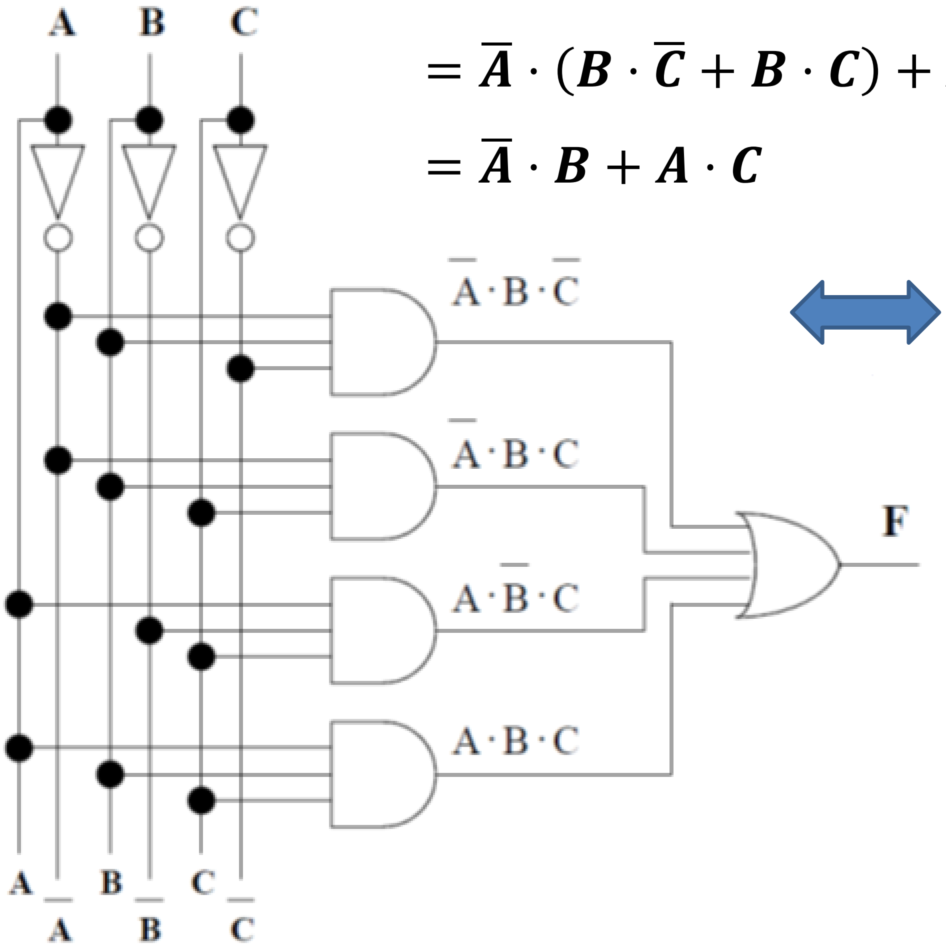
Simplifying logic functions using Boolean Algebra Rules

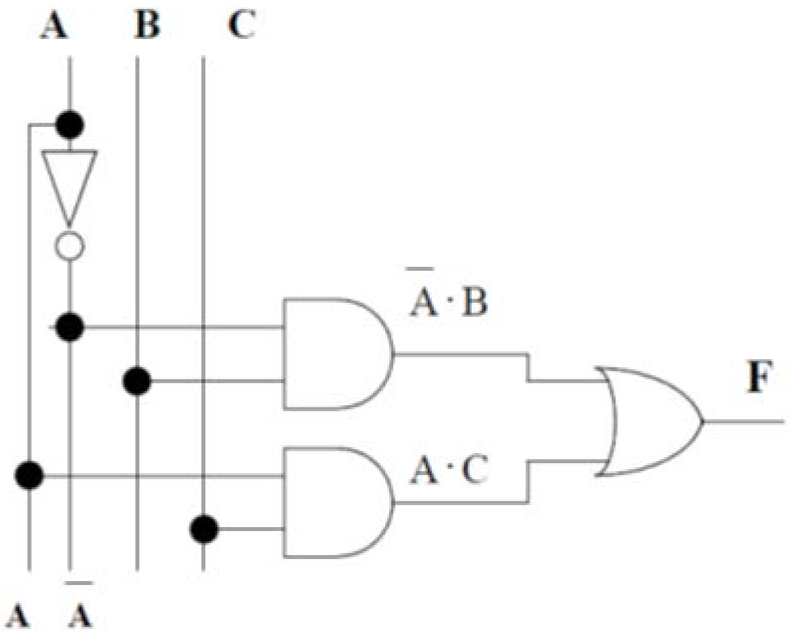
❖ Example) Sum of Products form logic function :

$$F = \overline{A}\cdot B\cdot\overline{C} + \overline{A}\cdot B\cdot C + A\cdot\overline{B}\cdot C + A\cdot B\cdot C$$

$$= \overline{A}\cdot(B\cdot\overline{C} + B\cdot C) + A$$

$$= \overline{A}\cdot B + A\cdot C$$





❖ Conjunctive Normal Form (CNF) : 논리곱 정규형(표준형)

A CNF is either a fundamental disjunction ($\vee$) or a conjunction ($\wedge$) of two or more fundamental disjunctions ($\vee$)

- A *fundamental disjunction* is either a literal or the disjunction of two or more literals
- A *literal* is a propositional variable or its negation

(*Ex*)

- P
- $P \vee \neg Q \vee R$
- $(P \vee Q) \wedge (R \vee \neg Q)$

❖ Conversion of wff to CNF

Every wff can be transformed to a CNF as follows.

1. Remove all occurrences of the connective $\rightarrow$ by using the equivalent formulas. $(A \rightarrow B) \Leftrightarrow \neg A \vee B$
2. Move all negations inside to create literals by using the De Morgan's law.
3. Apply distributive laws to obtain a CNF.

(*Ex*) $(P \vee Q \rightarrow R) \vee S$

$\Leftrightarrow (\neg(P \vee Q) \vee R) \vee S$; $(A \rightarrow B) \Leftrightarrow \neg A \vee B$

$\Leftrightarrow (\neg P \wedge \neg Q) \vee (R \vee S)$; De Morgan, Associative

$\Leftrightarrow (\neg P \vee R \vee S) \wedge (\neg Q \vee R \vee S)$; Distributive

❖ Disjunctive Normal Form (DNF) : 논리합 정규형 (표준형)

A DNF is either a fundamental conjunction or a disjunction of two or more fundamental conjunctions

- A *fundamental conjunction* is either a literal or the conjunction of two or more literals

(*Ex*)

- P
- $P \wedge \neg Q \wedge R$
- $(P \wedge Q) \vee (R \wedge \neg Q)$

❖ Conversion of wff to DNF

Every wff can be transformed to a DNF as follows.

1. Remove all occurrences of the connective $\rightarrow$.
2. Move all negations inside.
3. Apply distributive laws.

(*Ex*) $(P \wedge Q \rightarrow R) \wedge S$

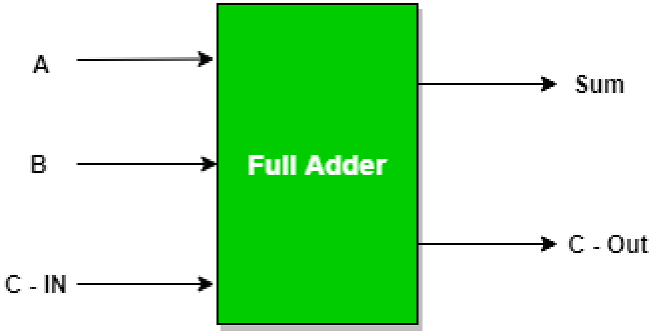
$\Leftrightarrow (\neg(P \wedge Q) \vee R) \wedge S$; $(A \rightarrow B) \Leftrightarrow \neg A \vee B$

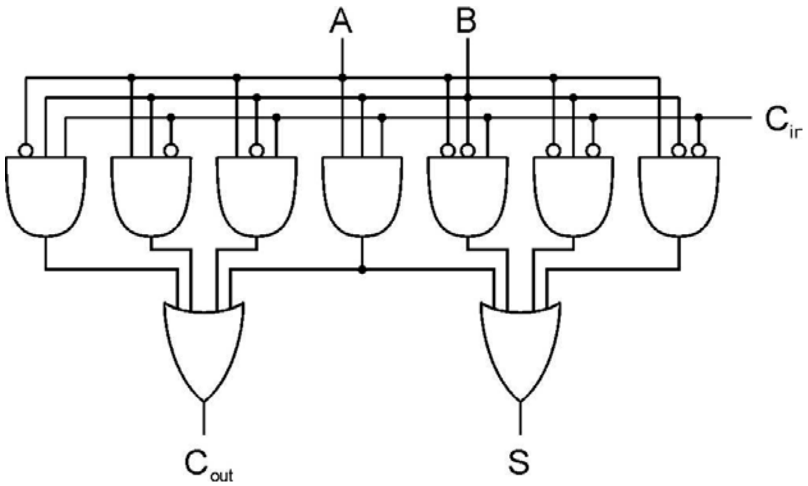
$\Leftrightarrow (\neg P \vee \neg Q \vee R) \wedge S$; De Morgan, Associative

$\Leftrightarrow (\neg P \wedge S) \vee (\neg Q \wedge S) \vee (R \wedge S)$; Distributive

Full Adder

❖ Full Adder is the adder which adds three inputs and produces two outputs. The first two inputs are A and B and the third input is an input carry as C-IN. The output carry is designated as C-OUT and the normal output is designated as S which is SUM.





A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Four-bit Adder

보조 자료