Equivalence Relations & Partitions

Equivalence Relations

- Definition:

  A binary relation is called an equivalence relation if it is reflexive, symmetric, and transitive.

Example:

W: set of all words in English dictionary

R: "has the same first letter as" relation

Claim: R is an equivalence relation on the set A.

$$(\forall w)\ \ w \in W \rightarrow (w, w) \in R$$

$$(\forall w_1)(\forall w_2)\ \ (w_1, w_2) \in R \rightarrow (w_2, w_1) \in R$$

$$(\forall w_1)(\forall w_2)(\forall w_3)\ \ (w_1, w_2) \in R \wedge (w_2, w_3) \in R \rightarrow (w_1, w_3) \in R$$

- $(w_1, w_2) \in R \Leftrightarrow w_1$ and w_2 are equivalent with respect to R

- More examples:
  - E_A for a set A is an equivalence relation.
  - $A \times A$ is an equivalence relation.

    (How many relations on A are equivalence relations?)
  - # of equivalence relations = # of partitions

    E.g., $A = \{a, b, c\}$ has five partitions

$\{a\}\ \{b\}\ \{c\}$	$R_1 = \{(a, a), (b, b), (c, c)\} = E_A$
$\{a\}\ \{b, c\}$	$R_2 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$
$\{b\}\ \{a, c\}$	$R_3 = \{(b, b), (a, a), (c, c), (a, c), (c, a)\}$
$\{c\}\ \{a, b\}$	$R_4 = \{(c, c), (a, a), (b, b), (a, b), (b, a)\}$
$\{a, b, c\}$	$R_5 = A \times A$

Who is equivalent with whom in each case?

- More examples:
  - $R_k = \{(x, y) \mid x, y \in \mathbf{Z},\ x - y = n \cdot k,\ k \in \mathbf{Z}^+,\ n \in \mathbf{Z}\}$

    We say "x and y are equivalent modulo k."

    E.g, when $k = 3$,

    $(7, 7) \in R_3$

    $(7, 4) \in R_3$ $(7 - 4 = 1 \cdot 3)$ $\rightarrow$ $(4, 7) \in R_3$ $(4 - 7 = (-1) \cdot 3)$

    $(4, 10) \in R_3 \wedge (10, 19) \in R_3 \rightarrow (4, 19) \in R_3$

- Theorem:

  If R_1 and R_2 are two equivalence relations on a set A then $R_1 \cap R_2$ is an equivalence relation.

Proof:

We need to show that $R_1 \cap R_2$ is reflexive, symmetric, and transitive.

(Reflexive)

Let $a \in A$. We must show that $(a, a) \in R_1 \cap R_2$.

Since R_1 and R_2 are equivalence relations, they must be reflexive and so $(a, a) \in R_1$ and $(a, a) \in R_2$.

Therefore, $(a, a) \in R_1 \cap R_2$ and $R_1 \cap R_2$ is reflexive.

Proof:

(Symmetric)

Let $(a, b) \in R_1 \cap R_2$. Then, $(a, b) \in R_1$ and $(a, b) \in R_2$.

Since R_1 and R_2 are symmetric $(b, a) \in R_1$ and $(b, a) \in R_2$.

Therefore, $(b, a) \in R_1 \cap R_2$ and $R_1 \cap R_2$ is symmetric.

(Transitive)

Left as an exercise. □

Example:

- Equivalence relations on $A = \{a, b, c\}$

$\{a\}\ \ \{b\}\ \ \{c\}$	$R_1 = \{(a, a), (b, b), (c, c)\}$
$\{a\}\ \ \{b, c\}$	$R_2 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$
$\{b\}\ \ \{a, c\}$	$R_3 = \{(b, b), (a, a), (c, c), (a, c), (c, a)\}$
$\{c\}\ \ \{a, b\}$	$R_4 = \{(c, c), (a, a), (b, b), (a, b), (b, a)\}$
$\{a, b, c\}$	$R_5 = A \times A$

$R_1 \cap R_2 = \{(a, a), (b, b), (c, c)\} = R_1$

$R_2 \cap R_3 = \{(a, a), (b, b), (c, c)\} = R_1$

$R_4 \cap R_5 = R_4$

- A counter example for $R_1 \cup R_2$:
  - $A = \{a, b, c\}$
  - $R_1 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ is an equivalence relation
  - $R_2 = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$ is an equivalence relation
  - ➡ $R_1 \cup R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)\}$
    is not transitive, and so is not an equivalence relation.

- Theorem:

  Let R be a non-empty relation on a set A. Then,
  - $tsr(R)$ is an equivalence relation.
  - If R' is any equivalence relation such that $R \subseteq R'$, then $tsr(R) \subseteq R'$.

Equivalence Classes

- Definition:

  Let R be an equivalence relation on a set A. For each $x \in A$, the equivalence class of x with respect to R, denoted by $[x]_R$, is defined as

$$[x]_R = \{\, y \in A \mid (x, y) \in R \}.$$

Example:

W: set of all words in English dictionary

R: "has the same first letter as" relation

$[\text{dog}]_R$: set of all the words that start with the letter 'd'

- Theorem:

  Let R be an equivalence relation on a set A. Then,

  $$[a]_R = [b]_R \text{ iff } (a, b) \in R$$

  (E.g., $[\text{dog}]_R = [\text{dummy}]_R$)

Proof:

(if part)

Assume $(a, b) \in R$.

Let $x \in [a]_R$.

Then, $(a, x) \in R$ and $(x, a) \in R$ because R is an equivalence relation and thus symmetric.

From $(x, a) \in R$ and $(a, b) \in R$, we have $(x, b) \in R$ because R is an equivalence relation and thus transitive.

Proof:

Then, $(b, x) \in R$ because R is an equivalence relation and thus symmetric. So, we have $x \in [b]_R$.

Therefore, $[a]_R \subseteq [b]_R$.

We can similarly show that $[b]_R \subseteq [a]_R$.

Therefore, $[a]_R = [b]_R$.

(only if part)

Assume $[a]_R = [b]_R$.

There must be an $x \in [a]_R = [b]_R$ as long as $[a]_R = [b]_R \neq \emptyset$.

Then, $(a, x) \in R$ and $(b, x) \in R$.

Since $(b, x) \in R$ and R is symmetric, we have $(x, b) \in R$.

From $(a, x) \in R$ and $(x, b) \in R$, $(a, b) \in R$ because R is transitive. □

- Theorem:

  Let R be an equivalence relation on a set A. Then,

  (1) For any $a \in A$, $[a]_R \neq \varnothing$

  (2) If $[a]_R \neq [b]_R$ then $[a]_R \cap [b]_R = \varnothing$

  (3) $\bigcup_{a \in A} [a]_R = A$

<u>Example</u>: English dictionary

- $\text{dog} \in [\text{dog}]_R$
- $[\text{dog}]_R = [\text{dummy}]_R$ $[\text{dog}]_R \cap [\text{cat}]_R = \varnothing$
- $\bigcup_{a \in W} [a]_R = W$ (set of all words)

Proof *of* (2):

Suppose $[a]_R \neq [b]_R$ and $[a]_R \cap [b]_R \neq \varnothing$.

Let x be an element of the nonempty set $[a]_R \cap [b]_R$.

Then, $x \in [a]_R$ and $x \in [b]_R$ and so $(a, x) \in R$ and $(b, x) \in R$.

But, $(x, b) \in R$ because R is symmetric.

From $(a, x) \in R$ and $(x, b) \in R$ we get $(a, b) \in R$ because R is transitive.

Then, $[a]_R = [b]_R$, which is a contradiction. □

- Definition:

  Let R be an equivalence relation on a set A. The quotient set of A modulo R, denoted by A/R, is defined as

$$A/R = \{[x]_R \mid x \in A\}$$

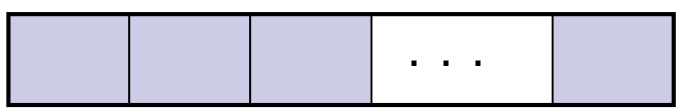
Example:

- W: English dictionary R: has the same first letter as

  $W/R = \{\{$all words starting with $a\}$, $\{$all words starting with $b\}$, . . .

  . . . , $\{$all words starting with $z\}\}$

W/R:



Example:

- $A = \{a, b, c\}$
- $R = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$
- $A/R = \{\{a, c\}, \{b\}\}$

Partitions

- Definition:

  Let π be a collection of nonempty subsets of A such that

  1. For any $X, Y \in \pi$, if $X \neq Y$ then $X \cap Y = \varnothing$
  2. $\bigcup_{X \in \pi} X = A$

  then π is called a partition of A.

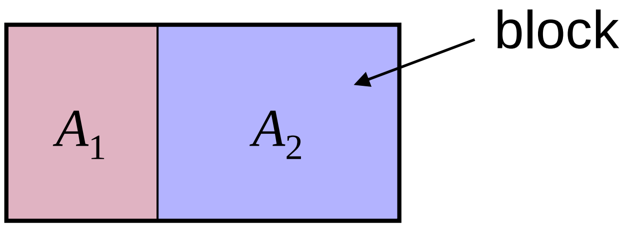
Example: Given a set $A = \{a, b, c\}$,

$\{\{a\}, \{b,c\}\}$ is a partition, but $\{\{a,b\}, \{b,c\}\}$ is not

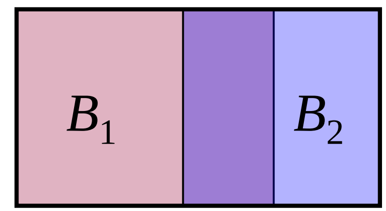
- Note:
  - $\varnothing \notin \pi$ and $\pi \subset \wp(A)$
  - If only 2 holds, then it is called a cover.

Example:

A



B



$\{A_1, A_2\}$: partition

$\{B_1, B_2\}$: cover, not a partition

- Definition:

  Let π be a partition on a set A. If π is finite then $|\pi|$ is called the rank of the partition A.

- Theorem:

  If R is an equivalence relation on a set A, then A/R is a partition of A.

Proof:

Let $X, Y \in A/R$.

Then, X and Y are equivalence classes, each of which is a nonempty subset of A.

We know that (1) if $X \neq Y$ then $X \cap Y = \varnothing$ and (2) $\bigcup_{X \in A/R} X = A$.

This implies that A/R is a partition of A.

□

- Definition:

  Let π be a partition on a set A. The relation induced by the partition π, denoted by R_π, is defined as

  $$R_\pi = \{(x, y) \mid \text{there exists an } S \in \pi \text{ such that } x \in S \text{ and } y \in S)\}$$

<u>Example</u>:

- $A = \{a, b, c, d, e, f, g\}$
- $\pi = \{\{a, b, c\}, \{d, e\}, \{f, g\}\}$
- $R_\pi = \{a, b, c\} \times \{a, b, c\} \cup \{d, e\} \times \{d, e\} \cup \{f, g\} \times \{f, g\}$

  $= \{(a, a), (b, b), \ldots, (g, g), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b),$
  $(d, e), (e, d), (f, g), (g, f)\}$

- Theorem:

  Let R_π be the relation induced by a partition π on a nonempty set A.

  Then,

  (1) R_π is an equivalence relation.

  (2) $A/R_\pi = \pi$.

Example:

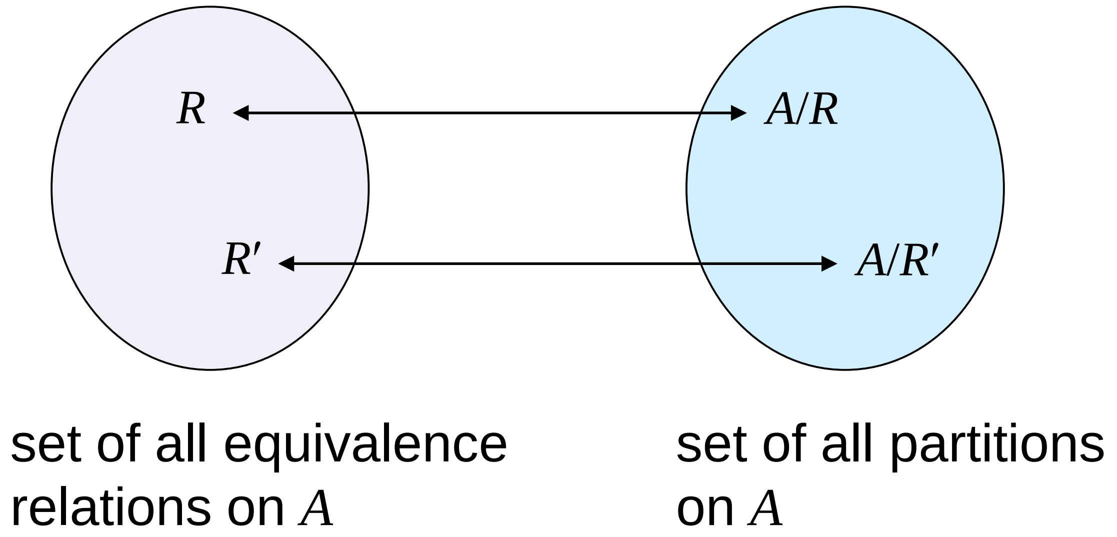
- $A = \{a, b, c\}$
- $\pi = \{\{a, c\}, \{b\}\}$
- $R_\pi = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$ is an equivalence relation
- $A/R_\pi = \{\{a, c\}, \{b\}\} = \pi$

$$\begin{array}{ccc} & R \longrightarrow A/R & \\ A & & R = R_\pi \text{ iff } \pi = A/R \\ & \pi \longrightarrow R_\pi & \end{array}$$

- **Theorem**:

  Let R be an equivalence relation on a nonempty set A and let π be a partition on the set. Then,

  $$R = R_\pi \text{ iff } \pi = A/R$$



- Theorem:

  There exists a one-to-one correspondence between the set of all equivalence relations on a nonempty set A and the set of all partitions on A.

- Definition:

  Let π and π' be two partitions on a nonempty set A. Then π' is said to refine π (π' is a refinement of π) if every block of π' is a subset of some block of π.

Example:

- $A = \{a, b, c, d, e, f, g\}$
- $\pi = \{\{a, b, c\}, \{d, e\}, \{f, g\}\}$
- $\pi' = \{\{a, b\}, \{c\}, \{d, e\}, \{f, g\}\}$ is a refinement of π.
- $\pi'' = \{\{a, b\}, \{c, d\}, \{e\}, \{f, g\}\}$ is not a refinement of π.

- Note:
  - $\pi_0 = \{\{a, b, c, d, e, f, g\}\}$

    Every partition is a refinement of π_0.
  - $\pi_\infty = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}\}$

    π_∞ is a refinement of every partition.
  - A partition refines itself.

- Definition:

  If a partition π' refines a partition π and if $\pi' \neq \pi$, then π' is called a proper refinement of π.

- Definitions:
  - Let π_1 and π_2 be two partitions on a nonempty set A. $\pi_1 \cdot \pi_2$ is a partition on A that refines both π_1 and π_2, and if π' is another partition that refines π_1 and π_2 then π' refines $\pi_1 \cdot \pi_2$.
  - Let π_1 and π_2 be two partitions on a nonempty set A. $\pi_1 + \pi_2$ is a partition on A that is refined by both π_1 and π_2, and if π' is another partition that is refined by π_1 and π_2 then π' is refined by $\pi_1 + \pi_2$.

Example:

- $\pi_1 = \{\{a, b, c\}, \{d, e\}, \{f, g\}\}$
- $\pi_2 = \{\{a, b, c, d\}, \{e\}, \{f, g\}\}$
- $\pi_1 \cdot \pi_2 = \{\{a, b, c\}, \{d\}, \{e\}, \{f, g\}\}$
- $\pi_3 = \{\{a, b\}, \{c\}, \{d\}, \{e\}, \{f, g\}\}$

  π_3 refines both π_1 and π_2, but $\pi_3 \neq \pi_1 \cdot \pi_2$.

  π_3 is a proper refinement of $\pi_1 \cdot \pi_2$.
- $\pi_1 + \pi_2 = \{\{a, b, c, d, e\}, \{f, g\}\}$

- Theorem:

  The relation “refines” on the set of all the partitions on a nonempty set is reflexive, antisymmetric, and transitive.

Example:

$$
\begin{tikzcd}
 & \pi_0 = \{\{a, b, c\}\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\pi_1 = \{\{a\}, \{b, c\}\} \arrow[dr, dash] & \pi_2 = \{\{b\}, \{a, c\}\} \arrow[d, dash] & \pi_3 = \{\{c\}, \{a, b\}\} \arrow[dl, dash] \\
 & \pi_\infty = \{\{a\}, \{b\}, \{c\}\} &
\end{tikzcd}
$$

- **Theorem**:

  Let π_1 and π_2 be two partitions on a nonempty set A. Then

  (1) $\pi_1 \cdot \pi_2 = A/(R_{\pi_1} \cap R_{\pi_2})$.

  (2) $\pi_1 + \pi_2 = A/t(R_{\pi_1} \cup R_{\pi_2})$.

- **Corollary**:

  Given two partitions π_1 and π_2 on a nonempty set A, there is a unique $\pi_1 \cdot \pi_2$ and a unique $\pi_1 + \pi_2$.