Fundamentals of Logic
(Propositional Calculus)

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Formal Logic

- **Formal (or Symbolic) Logic**

  Mathematical model of deductive thought

  - ☐ Modeling: extract important features according to purposes

    (*Ex*) The law of gravity: $F = G\dfrac{m\,m'}{r^2}$

    masses, distance (○)　color (×)

  (Q1) What are the important features of deductive thought?

  - ☐ Syntax (structure), not semantics (meaning or interpretation)

    (*Ex*)　Borogoves are mimsy whenever it is brillig.

    It is now brillig and this thing is a borogove.

    Therefore, this thing is mimsy.　☐

(Q2) What does it mean for one sentence to "*follow logically*" from certain others?

☞ Truth table (valid consequence)

(Q3) If a sentence does follow logically from certain others, what methods of proof might be necessary to establish this fact?

☞ Formal proof

■ We need a different language since the natural language is ambiguous

Two models are considered:

They use different languages with different expressive power

■ Propositional Calculus

(Propositional Logic, Statement Calculus, Sentential Logic)

☐ Can express very simple, crude properties of deduction

■ Predicate Calculus

(Predicate Logic, First-Order Logic, First-Order Predicate Calculus)

☐ Suited for mathematical deduction

Propositional Calculus

- **Proposition**

  - An assertion (a declarative sentence) that can take a value *true* or *false*

    (*Ex*) $x + y = 4$.   This statement is false.   ???

         I work hard.   I am healthy.   !!!

- **Propositional Constants**

  - Stand for a particular proposition (such as W or H)

- **Propositional Variables**

  - Stand for some propositions (such as P, Q, R, S)

- **Compound Propositions**
  - More complicated propositions resulting from combining *primitive* propositions

    (*Ex*)   I work hard *and* I am healthy.

    If it rains *then* there is cloud in the sky.

- **Logical Connectives**

Symbol	Name	Read
¬	negation	not
∧	conjunction	and
∨	disjunction	or
→	implication (conditional)	if … then …
↔	bi-conditional	iff (if and only if)

- The meaning of logical connectives is defined by the truth table

P	$\neg P$
F	T
T	F

P	Q	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$Q \rightarrow P$	$P \leftrightarrow Q$
F	F	F	F	T	T	T
F	T	F	T	T	F	F
T	F	F	T	F	T	F
T	T	T	T	T	T	T

  - The blue-colored terms in the Borogove example are logical connectives ☞

(Example) Jane promises that on December 26 $P \rightarrow Q$.

P: I weigh more than 120 pounds.

Q: I shall enroll in an exercise class.

$P \rightarrow Q$: If P, then Q.

Case	P	Q	$P \rightarrow Q$
1	F	F	T
2	F	T	T
3	T	F	F
4	T	T	T

(Case 4) Jane enrolls just as she said. So $P \rightarrow Q$ is true.

(Case 3) Jane breaks her promise. So $P \rightarrow Q$ is false.

(Case 2) Jane enrolls even though her weight is less than or equal to 120 pounds. However, she does not violate her promise. So $P \rightarrow Q$ is true.

(Case 1) Jane does not violate her promise, either. So $P \rightarrow Q$ is true.
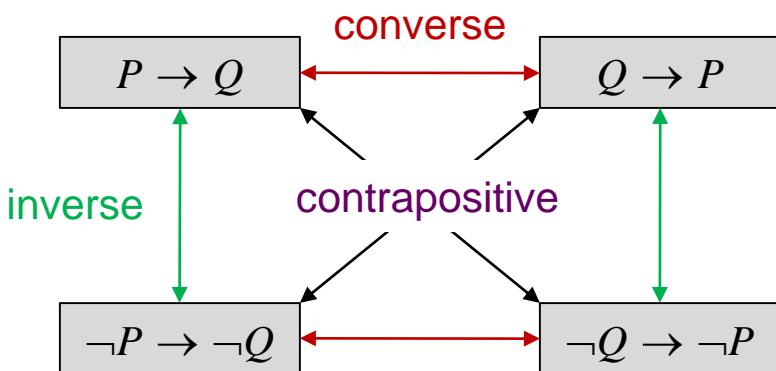
- Other expressions for implication

  P (antecedent) $\rightarrow Q$ (consequent)

  - □ If P then Q

  - □ P only if Q

  - □ P is a sufficient condition for Q

  - □ Q is a necessary condition for P

- Converse: $Q \rightarrow P$
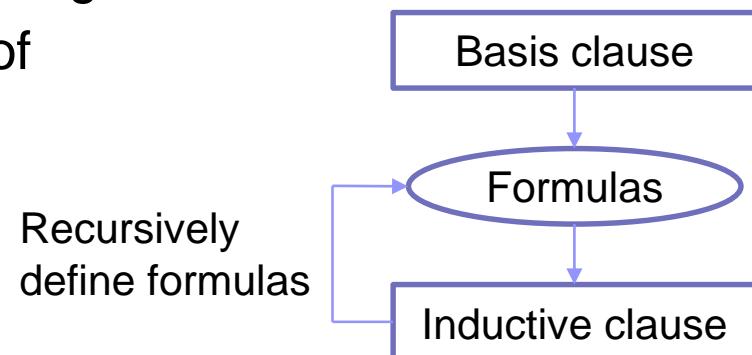- Inverse: $\neg P \rightarrow \neg Q$
- Contrapositive: $\neg Q \rightarrow \neg P$

- **Definition of Well-Formed Formula**

A propositional well-formed formula (wff) is a grammatically correct expression, which is defined *inductively* as follows.

1. (*Basis clause*) A truth symbol (T or F), a propositional variable, or a propositional constant is a wff.

2. (*Inductive clause*) If A and B are wffs, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$ are wffs.

3. (*Extremal clause*) Only the strings formed by *finite* applications of clauses 1 and 2 are wffs.

Recursively define formulas

Basis clause → Formulas → Inductive clause

- **Are the following wffs?**

  □ $((P \to Q) \to R)$

  □ $P \to Q$: technically not a wff

- **Precedence**

  To remove parentheses, we use the following conventions

  □ Hierarchy of evaluation: (highest) $\neg, \wedge, \vee, \to, \leftrightarrow$ (lowest)

  □ Each of the operations $\wedge, \vee$, and $\to$ is left associative

  (Ex) $F \wedge \neg P \vee \neg Q \to \neg Q \Rightarrow (((F \wedge (\neg P)) \vee (\neg Q)) \to (\neg Q))$

  $P \to Q \to R \Rightarrow ((P \to Q) \to R)$

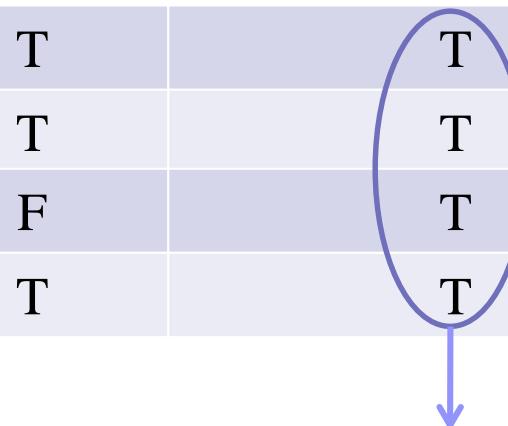- A wff has a truth value when interpreted
  - We draw a truth table

(*Ex*) $(P \to Q) \leftrightarrow (\neg P \vee Q)$? $P \to Q \leftrightarrow \neg P \vee Q$?

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$	$(P \to Q) \leftrightarrow (\neg P \vee Q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

Have different values depending on the interpretation

Tautology

(True regardless of the interpretation)

■ Definition

A tautology is a wff that takes the value true for every possible truth values assigned to the variables contained in the formula.

(*Ex*) $\models (P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

(Note) A tautology is true every time – not a definition!

■ Definition

A contradiction (or denial) is a wff that takes the value false for every possible truth values assigned to the variables contained in the formula.

■ Definition

A contingency is a wff that is neither a tautology nor a contradiction.

- **Definition**

  Let A and B be *two* wffs. The formulas A and B are said to be equivalent formulas (denoted by $A \Leftrightarrow B$), when A is true (false) iff B is true (false).

  (*cf.*) Let $P_1, P_2, \ldots, P_n$ be the propositional variables contained in A and/or B. The formulas A and B are said to be equivalent formulas (designated by $A \Leftrightarrow B$) if they have the same value for each of the 2^n sets of truth value assignment to the propositional variables.

- **Theorem 1**

  A and B are equivalent formulas iff $\models A \leftrightarrow B$.

- **Definition**

A formula B is a substitution instance of a formula A if B is obtained from A by substituting formulas for propositional variables in A under the condition that the same formula is substituted for the same variable each time that variable appears in the formula A.

(*Ex*)

☐ A: $((P \rightarrow Q) \wedge (R \vee Q))$

☐ Substitute $(P \rightarrow Q)$ for P and $(R \vee S)$ for Q

☐ B: $(((P \rightarrow Q) \rightarrow (R \vee S)) \wedge (R \vee (R \vee S)))$

- **Theorem 2**

  A substitution instance of a tautology is a tautology.

- **Theorem 3**

  A formula B is equivalent to a formula A if B is obtained from A by replacing a subformula C of A with a formula D that is equivalent to C.

  (*Ex*) Note that $\vDash (P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$, that is, $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

  □ A: $((P \rightarrow Q) \wedge (R \vee Q))$

  □ B: $((\neg P \vee Q) \wedge (R \vee Q))$ by substituting $(P \rightarrow Q)$ with $(\neg P \vee Q)$

  □ Then, $A \Leftrightarrow B$

- **Definition**

  If a formula A contains the connectives $\neg$, $\wedge$, and $\vee$ only, then the dual of A, denoted by A^d, is the formula obtained from A by replacing each occurrence of $\wedge$ and $\vee$ with $\vee$ and $\wedge$, respectively.

  (*Ex*) $A \; : (\neg P \vee Q) \wedge (R \vee S) \wedge (\neg R \wedge \mathrm{T})$

  $\quad\;\; A^d : (\neg P \wedge Q) \vee (R \wedge S) \vee (\neg R \vee \mathrm{F})$

- **Theorem 4:** *De Morgan's Law*

  If $A(P_1, P_2, \ldots, P_n)$ is a formula and A^d is its dual, then

  $$\vDash \neg A(P_1, P_2, \ldots, P_n) \leftrightarrow A^d(\neg P_1, \neg P_2, \ldots, \neg P_n)$$

■ Theorem 5: *Principle of Duality*

Let A and B be two formulas and A^d and B^d be their respective duals, then

$$\vDash A \leftrightarrow B \ \text{ iff } \ \vDash A^d \leftrightarrow B^d$$

Formal Proof

- ## Valid Consequence

  Let $A_1, A_2, \ldots, A_m$, and B be formulas. Let $P_1, P_2, \ldots, P_n$ be the propositional variables occurring in $A_1, A_2, \ldots, A_m$, and B.

  The formula B is said to be a *valid consequence* of the formulas $A_1, A_2, \ldots, A_m$ if for each of the 2^n sets of truth value assignments to the variables $P_1, P_2, \ldots, P_n$ the formula B takes the value T each time the formulas $A_1, A_2, \ldots, A_m$ take the value T simultaneously.

  We write it

  $$A_1, A_2, \ldots, A_m \vDash B$$

(Example) $P \rightarrow Q, Q \rightarrow R \vDash P \rightarrow R$

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$P \rightarrow R$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	F	T
F	T	T	T	T	T
T	F	F	F	T	F
T	F	T	F	T	T
T	T	F	T	F	F
T	T	T	T	T	T

(Example) $P \lor Q, P \rightarrow R, Q \rightarrow R \vDash R$

P	Q	R	$P \lor Q$	$P \rightarrow R$	$Q \rightarrow R$
F	F	F	F	T	T
F	F	T	F	T	T
F	T	F	T	T	F
F	T	T	T	T	T
T	F	F	T	F	T
T	F	T	T	T	T
T	T	F	T	F	F
T	T	T	T	T	T

(Example) $\neg P \models P \rightarrow Q$

P	Q	$\neg P$	$P \rightarrow Q$	$\neg P \rightarrow (P \rightarrow Q)$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	T
T	T	F	T	T

$$\models \neg P \rightarrow (P \rightarrow Q)$$

- **Theorem 6**

    Let $A_1, A_2, \ldots, A_m$, and B be wffs. Then,

    $$A_1, A_2, \ldots, A_m \;\vDash\; B$$

    iff $\;A_1 \wedge A_2 \wedge \ldots \wedge A_m \;\vDash\; B$

    iff $\;\vDash (A_1 \wedge A_2 \wedge \ldots \wedge A_m \rightarrow B)$

- Theorem 7

$$A_1, A_2, \ldots, A_n \; \vDash \; A_i \; (1 \le i \le n)$$

- Theorem 8

If $A_1, A_2, \ldots, A_n \; \vDash \; B_j \, (1 \le j \le p)$ and $B_1, B_2, \ldots, B_p \; \vDash \; C,$

then $A_1, A_2, \ldots, A_n \; \vDash \; C.$

- The inference rules are fundamental valid consequences

$$A_1, A_2, \ldots, A_n \; \vDash \; B$$

$$A_1, A_2, \ldots, A_n \Rightarrow B$$

- **Derivability**

The fact that formula B is derivable from formulas $A_1, A_2, \ldots, A_n$ can be demonstrated by a sequence of formulas $F_1, F_2, \ldots, F_m$ where $F_m = B$ and the presence of any formula F_i $(1 \leq i \leq m)$ is *justified* by the following two rules (rules of inference):
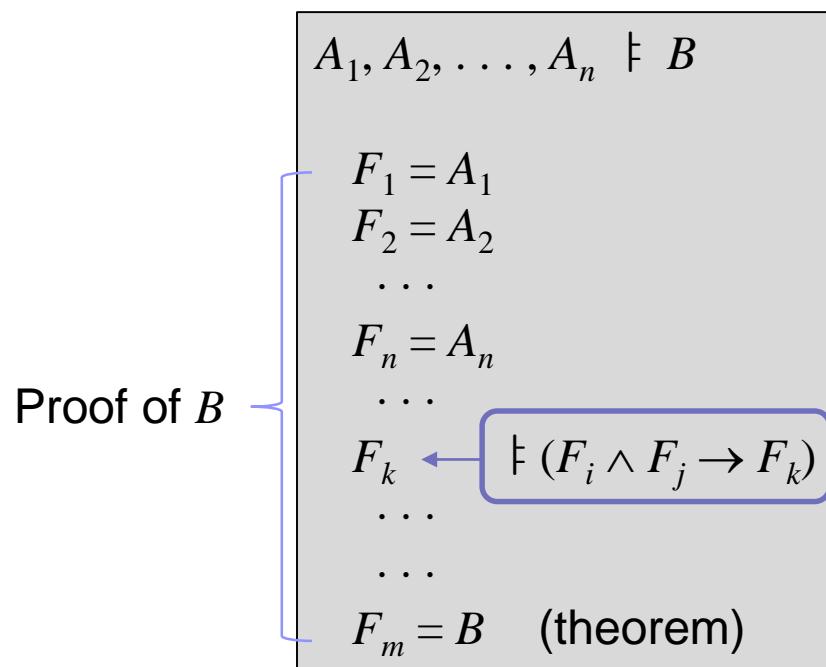
P rule (*premise* rule): $F_i = A_j$

T rule (*tautology* rule): F_i can occur in the sequence if there are formulas $F_{i1}, F_{i2}, \ldots, F_{ip}$ in the sequence before F_i and $\vDash (F_{i1} \wedge F_{i2} \wedge \ldots \wedge F_{ip} \rightarrow F_i)$.

We write it

$$A_1, A_2, \ldots, A_n \ \vdash \ B.$$

- **Proof**

A proof is a finite sequence of wffs with the property that each wff in the sequence either is an axiom or can be inferred from previous wffs in the sequence. The last wff in a proof is called a *theorem*.

$$A_1, A_2, \ldots, A_n \vDash B$$

Proof of B

$$F_1 = A_1$$
$$F_2 = A_2$$
$$\ldots$$
$$F_n = A_n$$
$$\ldots$$
$$F_k \quad \vDash (F_i \wedge F_j \rightarrow F_k)$$
$$\ldots$$
$$\ldots$$
$$F_m = B \quad \text{(theorem)}$$

(Note) Valid consequence vs. Derivability

Valid consequence	Derivable
$A_1, A_2, \ldots, A_n \vDash B$	$A_1, A_2, \ldots, A_n \vdash B$
Prove by truth table (Size 2^m grows rapidly)	Prove by rules (P rule, T rule)
Semantic (meaning)	Syntax (structure of the language)

- **Theorem 9**

$A_1, A_2, \ldots, A_n \vDash B$ iff $A_1, A_2, \ldots, A_n \vdash B$

- **Implications**

$I_1 \quad P \wedge Q \Rightarrow P$　　　　　　　⎤ (simplification)

$I_2 \quad P \wedge Q \Rightarrow Q$　　　　　　　⎦

$I_3 \quad P \Rightarrow P \vee Q$　　　　　　　⎤ (addition)

$I_4 \quad Q \Rightarrow P \vee Q$　　　　　　　⎦

$I_5 \quad \neg P \Rightarrow P \rightarrow Q$

$I_6 \quad Q \Rightarrow P \rightarrow Q$

$I_7 \quad \neg(P \rightarrow Q) \Rightarrow P$

$I_8 \quad \neg(P \rightarrow Q) \Rightarrow \neg Q$

$I_9 \quad P, Q \Rightarrow P \wedge Q$

$I_{10} \quad \neg P, P \vee Q \Rightarrow Q$　　　　　(disjunctive syllogism)

$^* P \Rightarrow Q$ is another way of writing $P \vDash Q$ or $\vDash P \rightarrow Q$

- Implications

I_{11} $P, P \rightarrow Q \Rightarrow Q$ (modus ponens)

I_{12} $\neg Q, P \rightarrow Q \Rightarrow \neg P$ (modus tollens)

I_{13} $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (hypothetical syllogism)

I_{14} $P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ (dilemma)

- Equivalences

E_1 $\neg\neg P \Leftrightarrow P$ (double negation)

E_2 $P \wedge Q \Leftrightarrow Q \wedge P$ (commutative laws)

E_3 $P \vee Q \Leftrightarrow Q \vee P$

*$P \Leftrightarrow Q$ is another way of writing $\vDash P \leftrightarrow Q$

- Equivalences

$E_4 \quad (P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ ⎤ (associative laws)

$E_5 \quad (P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ ⎦

$E_6 \quad P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ ⎤ (distributive laws)

$E_7 \quad P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ ⎦

$E_8 \quad \neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ ⎤ (De Morgan's laws)

$E_9 \quad \neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ ⎦

$E_{10} \quad P \vee P \Leftrightarrow P$

$E_{11} \quad P \wedge P \Leftrightarrow P$

$E_{12} \quad R \vee (P \wedge \neg P) \Leftrightarrow R$

$E_{13} \quad R \wedge (P \vee \neg P) \Leftrightarrow R$

- Equivalences

E_{14} $R \vee (P \vee \neg P) \Leftrightarrow \text{T}$

E_{15} $R \wedge (P \wedge \neg P) \Leftrightarrow \text{F}$

E_{16} $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

E_{17} $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$

E_{18} $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$

E_{19} $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

E_{20} $\neg(P \leftrightarrow Q) \Leftrightarrow P \leftrightarrow \neg Q$

E_{21} $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

E_{22} $P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

(Example 1) If I get the job and work hard, then I will be promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get the job or I will not work hard.

J : I get the job. W : I work hard.

P : I get promoted. H : I will be happy.

$$J \wedge W \to P, \ P \to H, \ \neg H \ \vDash \ \neg J \vee \neg W$$

No.	Formula	Rule	Just.	Taut.
1	$J \wedge W \to P$	P		
2	$P \to H$	P		
3	$\neg H$	P		
4	$J \wedge W \to H$	T	1, 2	$I_{13}{}^{*}$
5	$\neg(J \wedge W)$	T	3, 4	I_{12}
6	$\neg J \vee \neg W$	T	5	E_8

∗ A substitution instance of a tautology is a tautology

(Example 2) If I study law, then I will make a lot of money. If I study archeology, then I will travel a lot. If I make a lot of money or travel a lot, then I will not be disappointed. Therefore, if I am disappointed then I did not study law and I did not study archeology.

L : I study law. M : I make a lot of money.

A : I study archeology. T : I travel a lot.

D : I am disappointed.

$$L \rightarrow M, \ A \rightarrow T, \ M \vee T \rightarrow \neg D \ \models \ D \rightarrow (\neg L \wedge \neg A \,)$$

$$L \rightarrow M, \ A \rightarrow T, \ M \vee T \rightarrow \neg D \ \vDash \ D \rightarrow (\neg L \wedge \neg A)$$

No.	Formula	Rule	Just.	Taut.
1	$L \rightarrow M$	P		
2	$A \rightarrow T$	P		
3	$M \vee T \rightarrow \neg D$	P		
4	$\neg\neg D \rightarrow \neg(M \vee T)$	T	3	E_{18}
5	$D \rightarrow \neg(M \vee T)$	T	4	E_1
6	$D \rightarrow (\neg M \wedge \neg T)$	T	5	E_9
7	$\neg M \rightarrow \neg L$	T	1	E_{18}
8	$\neg T \rightarrow \neg A$	T	2	E_{18}
9	$D \rightarrow (\neg L \wedge \neg T)$	T	6, 7	I_{11}
10	$D \rightarrow (\neg L \wedge \neg A)$	T	8, 9	I_{11}

☞ Implication rules cannot be applied to a subformula.

(Caution)

$$(A \wedge B) \to C,$$

$$A \wedge B \Longrightarrow B$$

$$\vDash B \to C$$

$$(A \wedge B) \to C,$$

$$B \Longleftrightarrow D$$

$$\vDash (A \wedge D) \to C$$

$$A \to C, A \to B \ \vDash B \to C \ \ ???$$

☞ Only equivalence rules can be applied to a subformula.

$$L \rightarrow M, \ A \rightarrow T, \ M \lor T \rightarrow \neg D \ \vdash \ D \rightarrow (\neg L \land \neg A)$$

No.	Formula	Rule	Just.	Taut.
1	$L \rightarrow M$	P		
2	$A \rightarrow T$	P		
3	$M \lor T \rightarrow \neg D$	P		
4	$\neg\neg D \rightarrow \neg(M \lor T)$	T	3	E_{18}
5	$D \rightarrow \neg(M \lor T)$	T	4	E_1
6	$D \rightarrow (\neg M \land \neg T)$	T	5	E_9
7	$\neg D \lor (\neg M \land \neg T)$	T	6	E_{16}
8	$(\neg D \lor \neg M) \land (\neg D \lor \neg T)$	T	7	E_7
9	$(\neg D \lor \neg M)$	T	8	I_1
10	$(\neg D \lor \neg T)$	T	8	I_2
11	$D \rightarrow \neg M$	T	9	E_{16}
12	$D \rightarrow \neg T$	T	10	E_{16}
13	$\neg M \rightarrow \neg L$	T	1	E_{18}
14	$\neg T \rightarrow \neg A$	T	2	E_{18}

$$L \to M, \; A \to T, \; M \lor T \to \neg D \;\; \vdash \;\; D \to (\neg L \land \neg A)$$

No.	Formula	Rule	Just.	Taut.
11	$D \to \neg M$	T	9	E_{16}
12	$D \to \neg T$	T	10	E_{16}
13	$\neg M \to \neg L$	T	1	E_{18}
14	$\neg T \to \neg A$	T	2	E_{18}
15	$D \to \neg L$	T	11, 13	I_{13}
16	$D \to \neg A$	T	12, 14	I_{13}
17	$\neg D \lor \neg L$	T	15	E_{16}
18	$\neg D \lor \neg A$	T	16	E_{16}
19	$(\neg D \lor \neg L) \land (\neg D \lor \neg A)$	T	17, 18	I_9
20	$\neg D \lor (\neg L \land \neg A)$	T	19	E_7
21	$D \to \neg L \land \neg A$	T	20	E_{16}

(Exercise)

$$C \land B \to R, \; \neg B \to \neg P \;\; \vdash \;\; \neg R \land P \to \neg C$$

Conditional and Indirect Proof

- Theorem 10 Let $A_1, A_2, \ldots, A_n$ and B be wffs.

$$A_1, A_2, \ldots, A_n \;\vDash\; B$$

$$\text{iff} \quad A_1, A_2, \ldots, A_{n-1} \;\vDash\; (A_n \to B)$$

$$\text{iff} \quad \vDash\; (A_1 \to (A_2 \to (\ldots (A_{n-1} \to (A_n \to B)))\ldots))$$

- **Rule of Conditional Proof (CP rule)**

The formula $(F_i \to F_j)$ is justified in a derivation from the premises A_1, $A_2, \ldots, A_n$ if F_j can be derived from $A_1, A_2, \ldots, A_n, F_i$.

Additional Premise (AP)

(Ex. of CP)

$$L \rightarrow M, \ A \rightarrow T, \ M \vee T \rightarrow \neg D \ \ \vDash \ \ D \rightarrow (\neg L \wedge \neg A)$$

$$L \rightarrow M, \ A \rightarrow T, \ M \vee T \rightarrow \neg D, \ D \ \ \vDash \ \ (\neg L \wedge \neg A)$$
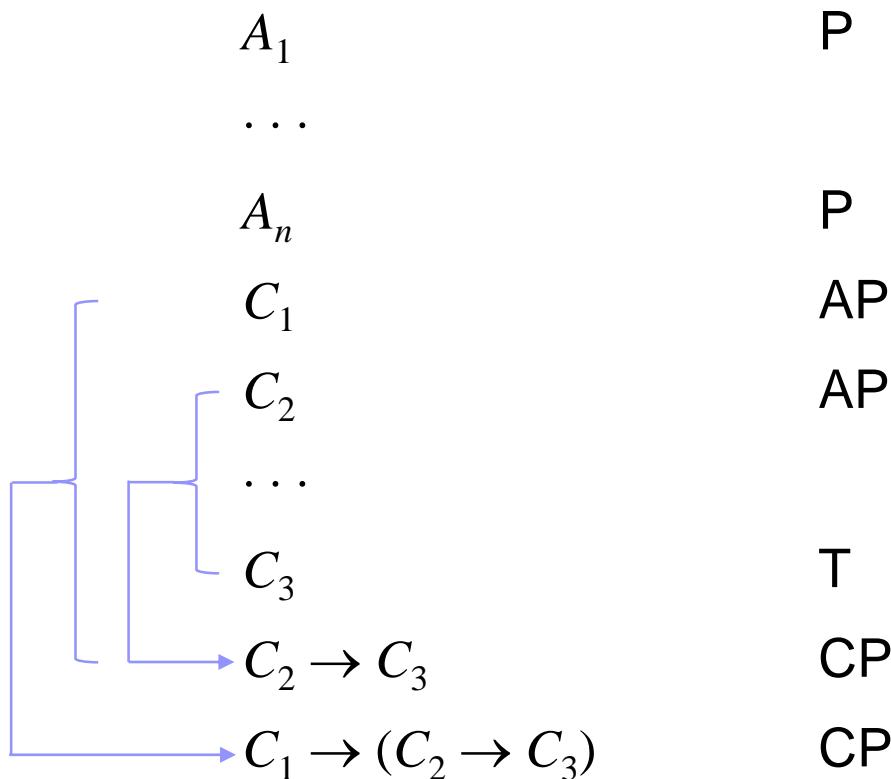
No.	Formula	Rule	Just.	Taut.
1	$L \rightarrow M$	P		
2	$A \rightarrow T$	P		
3	$M \vee T \rightarrow \neg D$	P		
4	D	**AP**		
5	$\neg\neg D$	T	4	E_1
6	$\neg(M \vee T)$	T	3, 5	I_{12}
7	$\neg M \wedge \neg T$	T	6	E_9
8	$\neg M$	T	7	I_1
9	$\neg T$	T	7	I_2
10	$\neg L$	T	1, 8	I_{12}

$$L \rightarrow M, \ A \rightarrow T, \ M \vee T \rightarrow \neg D \ \vDash \ D \rightarrow (\neg L \wedge \neg A)$$

No.	Formula	Rule	Just.	Taut.
1	$L \rightarrow M$	P		
2	$A \rightarrow T$	P		
3	$M \vee T \rightarrow \neg D$	P		
4	D	AP		
5	$\neg\neg D$	T	4	E_1
6	$\neg(M \vee T)$	T	3, 5	I_{12}
7	$\neg M \wedge \neg T$	T	6	E_9
8	$\neg M$	T	7	I_1
9	$\neg T$	T	7	I_2
10	$\neg L$	T	1, 8	I_{12}
11	$\neg A$	T	2, 9	I_{12}
12	$\neg L \wedge \neg A$	T	10, 11	I_9
13	$D \rightarrow (\neg L \wedge \neg A)$	CP	4, 12	

(Nested Application of CP rules)

$$A_1, A_2, \ldots, A_n \models C_1 \rightarrow (C_2 \rightarrow C_3)$$

A_1	P
. . .	
A_n	P
C_1	AP
C_2	AP
. . .	
C_3	T
$C_2 \rightarrow C_3$	CP
$C_1 \rightarrow (C_2 \rightarrow C_3)$	CP

- **Consistent Set**

  Let $\{A_1, A_2, \ldots, A_n\}$ be a set of wffs. Let $P_1, P_2, \ldots, P_m$ be the propositional variables occurring in $A_1, A_2, \ldots, A_n$. The set is said to be *consistent* if there exists at least one set of truth assignment to $P_1, P_2, \ldots, P_m$ for which $A_1, A_2, \ldots, A_n$ are simultaneously true.

- **Theorem 11**

  If $\{A_1, A_2, \ldots, A_n\}$ is an *inconsistent* set of wffs, then $A_1 \wedge A_2 \wedge \ldots \wedge A_n$ is a contradiction.

- **Theorem 12**

  If $A_1, A_2, \ldots, A_n, B$, and C are wffs, then
  $$A_1, A_2, \ldots, A_n \ \vDash \ B \ \text{ iff } A_1, A_2, \ldots, A_n, \neg B \ \vDash \ \boxed{C \wedge \neg C}.$$

  Contradiction

- **Rule of Indirect Proof (IP rule)**

  The formula B is justified in a derivation from the premises $A_1, A_2, \ldots, A_n$ if $C \wedge \neg C$ can be derived from $A_1, A_2, \ldots, A_n, \neg B$.

  Additional Premise (AP)

  "Proof by Contradiction"

(Ex. of IP) $\quad J \wedge W \rightarrow P, \ P \rightarrow H, \ \neg H \ \models \ \neg J \vee \neg W$

$$J \wedge W \rightarrow P, \ P \rightarrow H, \ \neg H, \ \neg(\neg J \vee \neg W) \ \models \ C \wedge \neg C$$

No.	Formula	Rule	Just.	Taut.
1	$J \wedge W \rightarrow P$	P		
2	$P \rightarrow H$	P		
3	$\neg H$	P		
4	$\neg(\neg J \vee \neg W)$	**AP**		
5	$\neg\neg J \wedge \neg\neg W$	T	4	E_9
6	$J \wedge W$	T	5	E_1
7	P	T	1, 6	I_{11}
8	H	T	2, 7	I_{11}
9	$H \wedge \neg H$	T	3, 8	I_9
10	$\neg J \vee \neg W$	**IP**	4, 9	

- **Conjunctive Normal Form (CNF)**

A CNF is either a fundamental disjunction or a conjunction of two or more fundamental disjunctions

- □ A *literal* is a propositional variable or its negation
- □ A *fundamental disjunction* is either a literal or the disjunction of two or more literals

(*Ex*)

- □ P
- □ $\neg Q$
- □ $P \vee \neg Q \vee R$
- □ $(P \vee Q) \wedge (R \vee \neg Q)$

- **Conversion of wff to CNF**

Every wff can be transformed to a CNF as follows.

1. Remove all occurrences of the connective $\rightarrow$ by using the equivalent formulas: $(A \rightarrow B) \Leftrightarrow \neg A \vee B$

2. Move all negations inside to create literals by using the De Morgan's law.

3. Apply distributive laws to obtain a CNF.

(*Ex*) $(P \vee Q \rightarrow R) \vee S$

$\Leftrightarrow (\neg(P \vee Q) \vee R) \vee S$; $(A \rightarrow B) \Leftrightarrow \neg A \vee B$

$\Leftrightarrow (\neg P \wedge \neg Q) \vee (R \vee S)$; De Morgan, Associative

$\Leftrightarrow (\neg P \vee R \vee S) \wedge (\neg Q \vee R \vee S)$; Distributive

- **Disjunctive Normal Form (DNF)**

  A DNF is either a fundamental conjunction or a disjunction of two or more fundamental conjunctions

  - A *fundamental conjunction* is either a literal or the conjunction of *two or more literals*

  (*Ex*)

  - P

  - $P \wedge \neg Q \wedge R$

  - $(P \wedge Q) \vee (R \wedge \neg Q)$

- **Conversion of wff to DNF**

  Every wff can be transformed to a DNF as follows.

  1. Remove all occurrences of the connective $\rightarrow$.

  2. Move all negations inside.

  3. Apply distributive laws.

  (*Ex*) $(P \wedge Q \rightarrow R) \wedge S$

  $\Leftrightarrow (\neg(P \wedge Q) \vee R) \wedge S$; $(A \rightarrow B) \Leftrightarrow \neg A \vee B$

  $\Leftrightarrow (\neg P \vee \neg Q \vee R) \wedge S$; De Morgan, Associative

  $\Leftrightarrow (\neg P \wedge S) \vee (\neg Q \wedge S) \vee (R \wedge S)$; Distributive