Hamilton Paths & Cycles

Section 11.5

부산대학교
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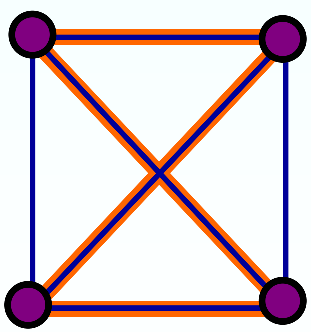
Hamilton Path & Cycle

Definition

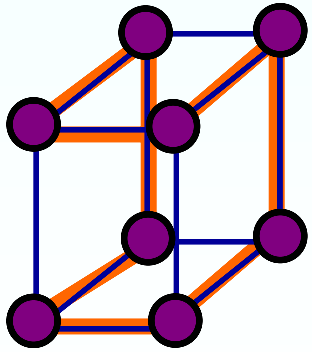
If $G=(V,E)$ is a graph or multigraph with $|V|\geq 3$, we say that G has a Hamilton cycle if there is a cycle in G that contains every vertex in V.

A Hamilton path is a path (and not a cycle) in G that contains each vertex.

K_4

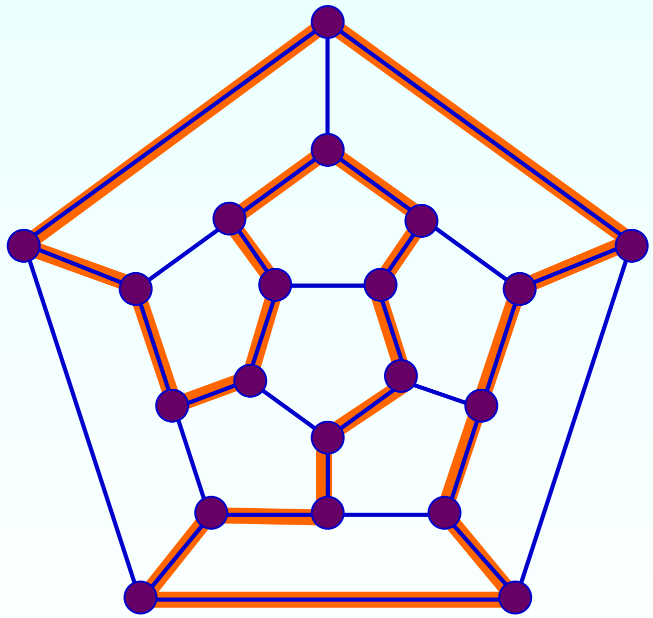


Q_3



Another Example

Hamilton Cycle



정12면체

Dodecahedron

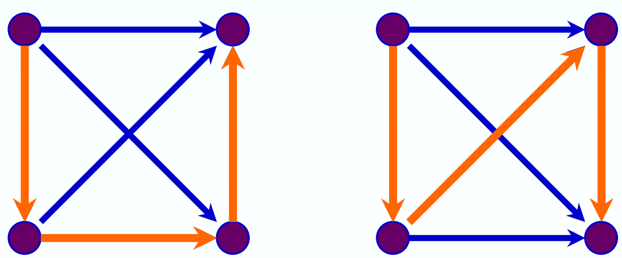
Theorem 11.7

- Existence of a Hamilton path in K_n^*

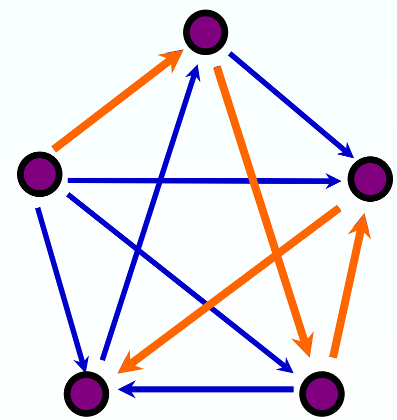
  Let K_n^* be a complete directed graph. That is, K_n^* has n vertices and for each distinct pair x,y of vertices, exactly one of the edges (x,y) or (y,x) is in K_n^*.

  Such a graph K_n^* always contains a (directed) Hamilton path.

K_4^*

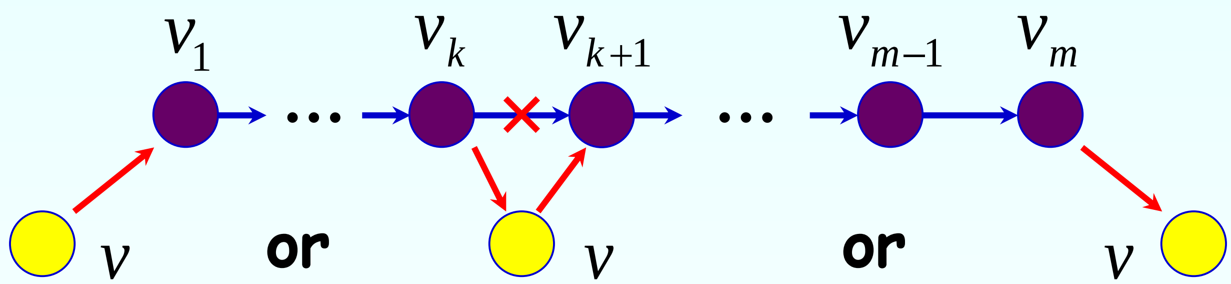


K_5^*



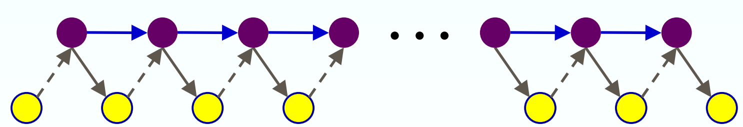
Proof of Th 11.7

Let P_m $(m \geq 2)$ a path containing the m-1 edges.



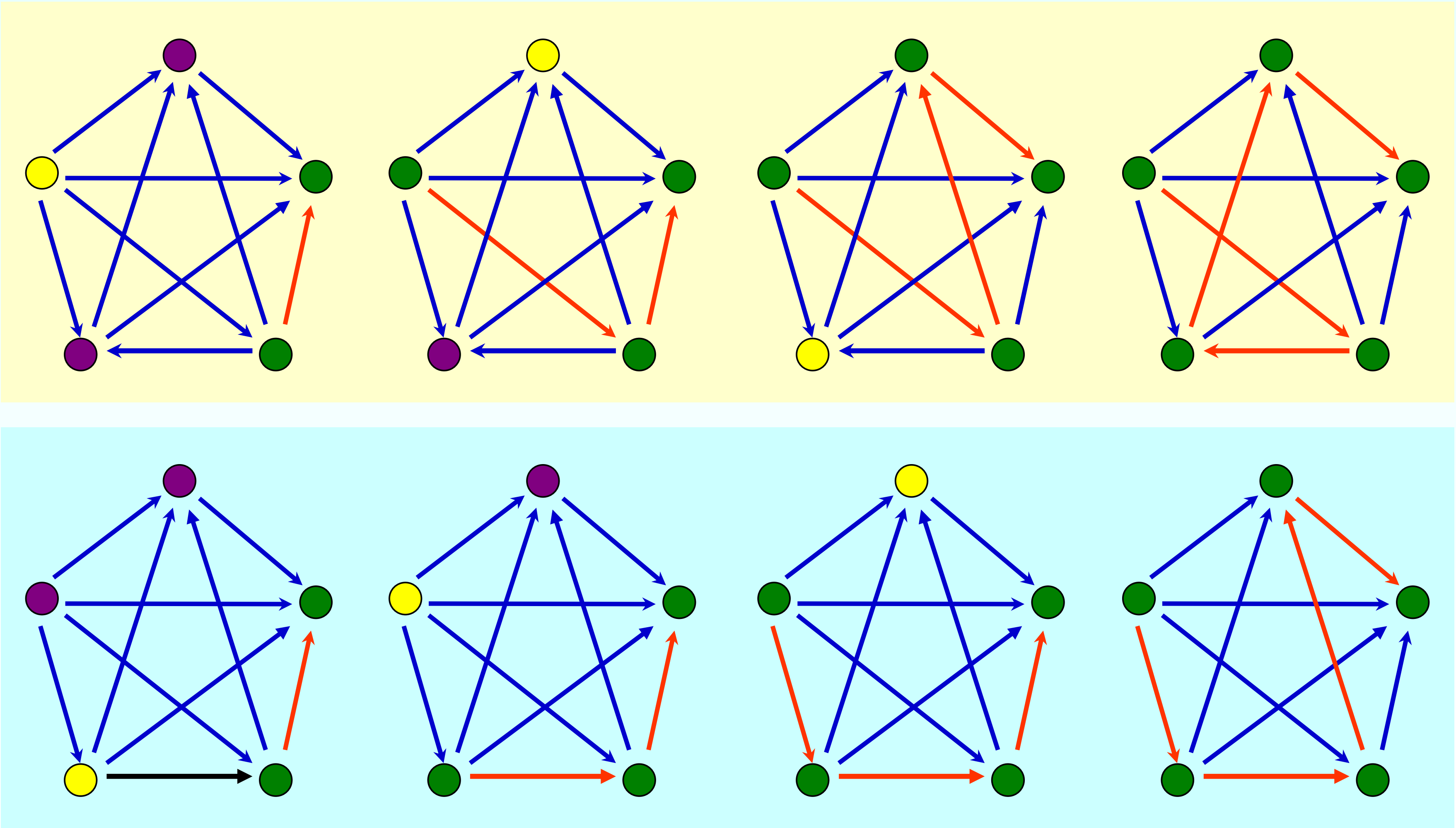
Extend

A new vertex v can be added into the path. Why?



This extension process can be repeated until a Hamilton path is found.

Examples



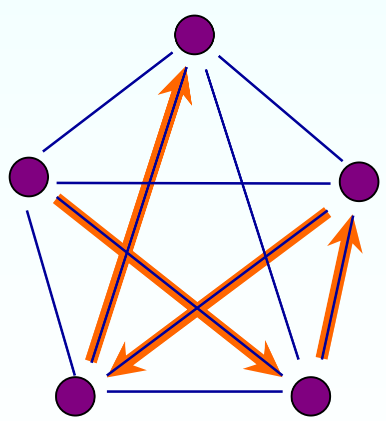
Theorem 11.8

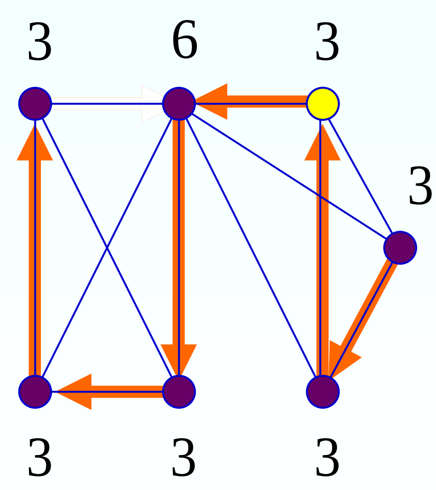
- Sufficient Condition for Existence of H.P.

  Let $G=(V,E)$ be a loop-free graph with $|V| = n \geq 2$.

  If $deg(x) + deg(y) \geq n-1$ for all $x, y \in V$, $x \neq y$,

  → G has a Hamilton path.



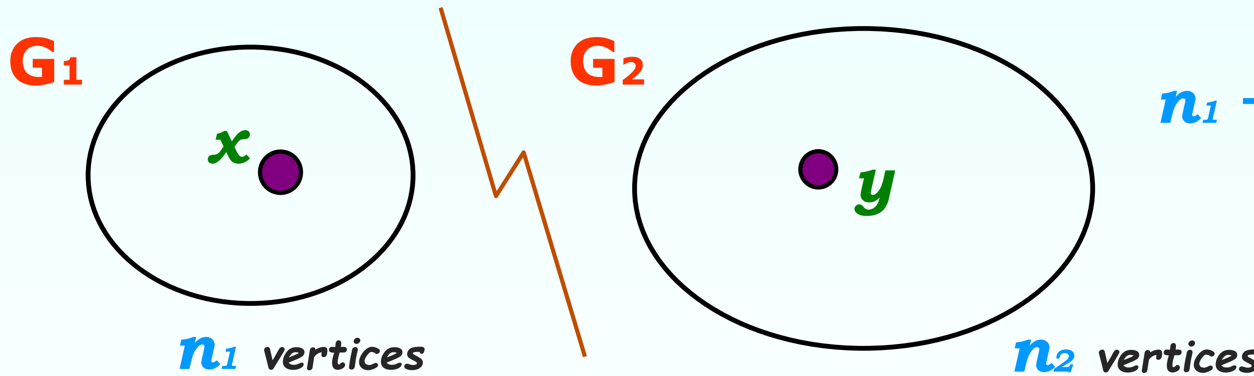


Proof of Th 11.8

(1) *G* is connected

Proof by contradiction. ($deg(x) + deg(y) \geq n-1$)

Suppose that *G* is disconnected



$n_1 + n_2 = n$

$deg(x) \leq n_1 - 1$ $\qquad$ $deg(y) \leq n_2 - 1$

$deg(x) + deg(y) \leq (n_1 + n_2) - 2 = n-2$

Contradiction

Proof of Th 11.8

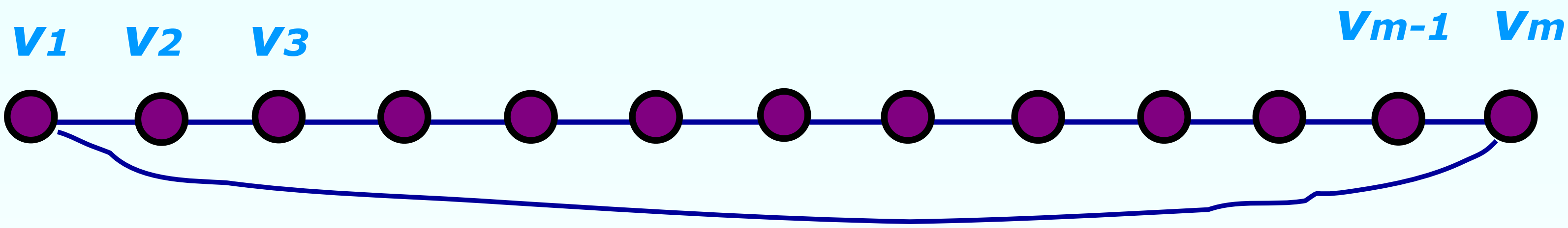
(2) Consider a path P_m of length m-1 (with m vertices)

- Extend the P_m at the start or the end point by appending a new vertex that is not equal to any one of the m vertices of the P_m
- If $m = n$, then we get a Hamilton path
- If $m \neq n$ and we cannot extend the P_m any more,
- (2-1) The m vertices consists of a cycle
- (2-2) The P_m can be extended for a vertex that is not found on this cycle

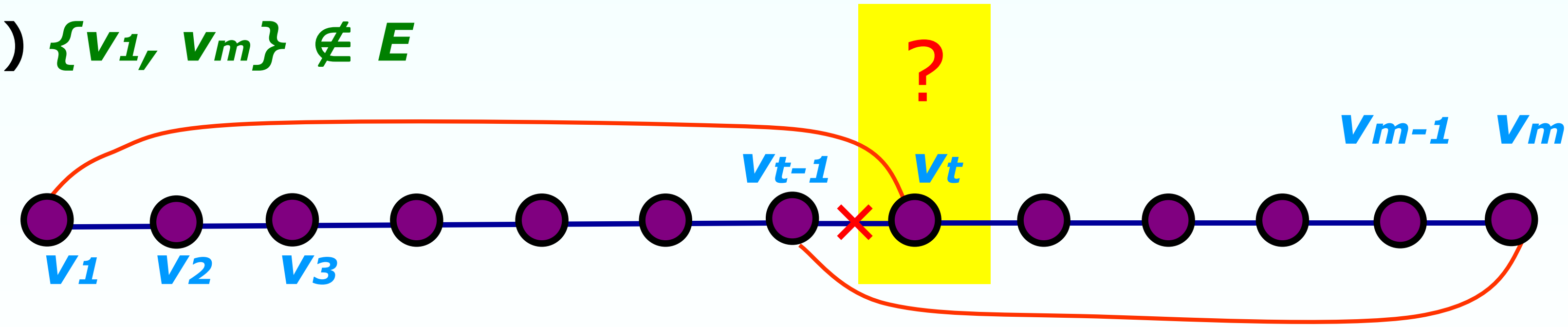
Proof of (2-1)

- ***G* contains a cycle on these *m* vertices**

i) $\{v_1, v_m\} \in E$



ii) $\{v_1, v_m\} \notin E$



Proof of (2-1)

- **Existence of v_t**

  Let $deg(v_1) = k$.

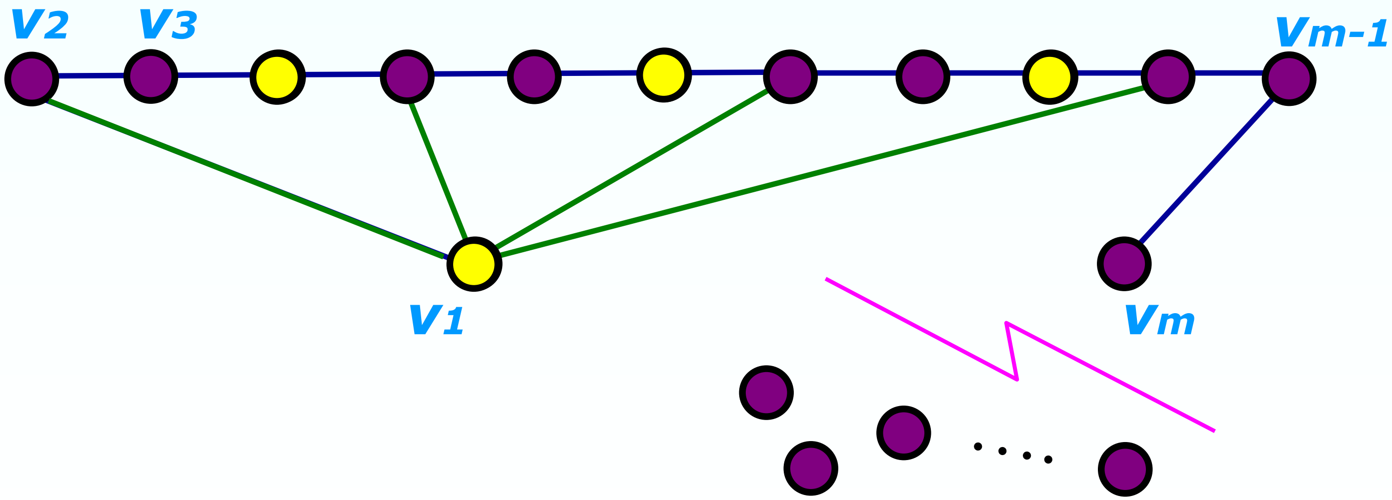
  If such v_t and v_{t-1} do not exist, then

  $deg(v_m) \leq (m-1)-k$

  Contradiction

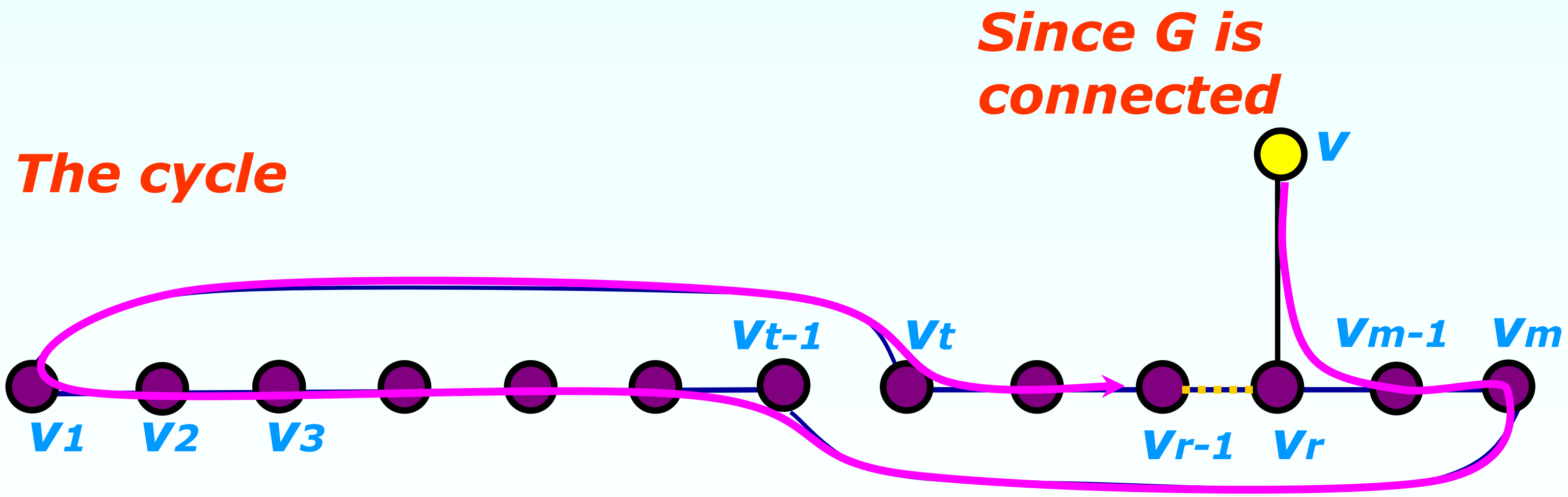
  $deg(v_1) + deg(v_m) \leq m-1 < n-1$

  $deg(x) + deg(y) \geq n-1$

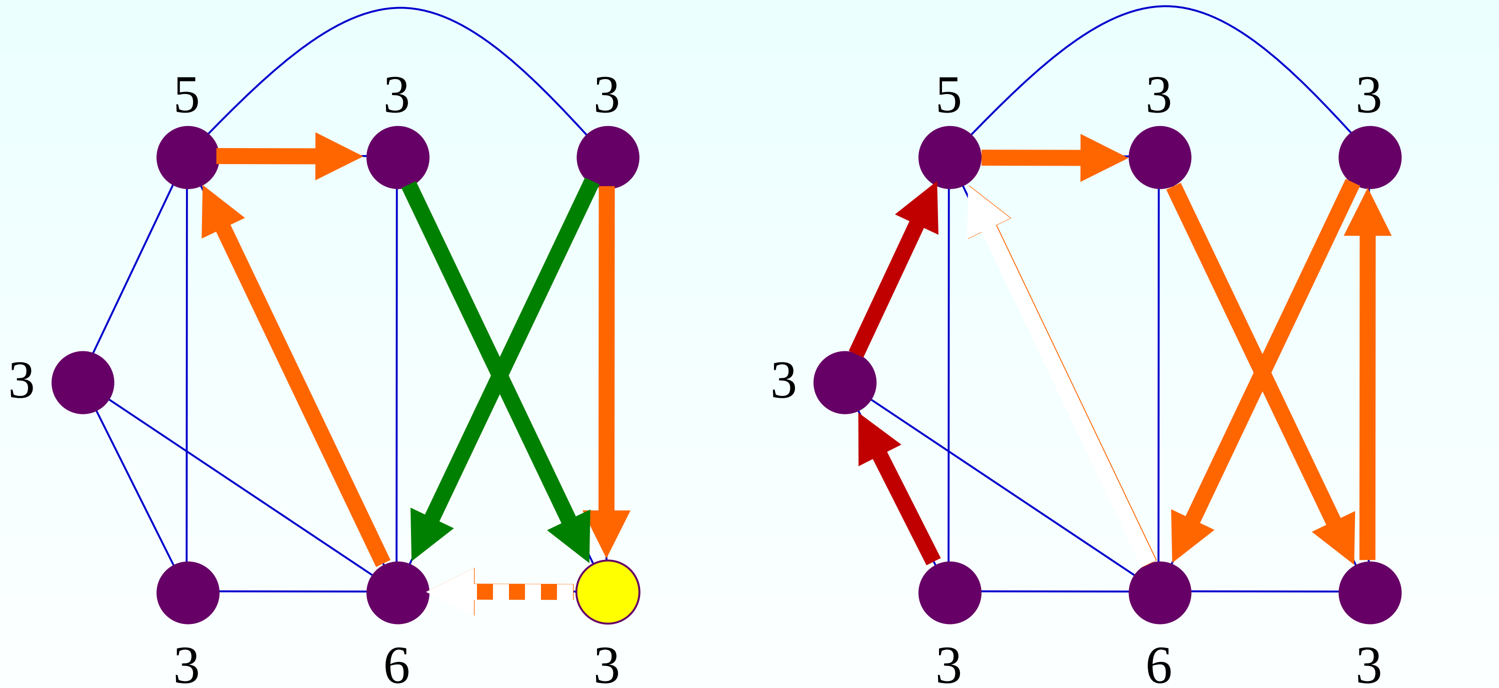


Proof of (2-2)

- P_m can be extended for a vertex that is not found on this cycle



Example



Corollary 11.4

- Another Sufficient Condition

  Let $G=(V,E)$ be a loop-free graph with $|V| = n \geq 2$.

  If $deg(v) \geq (n-1)/2$ for all $v \in V$,

  → G has a Hamilton path

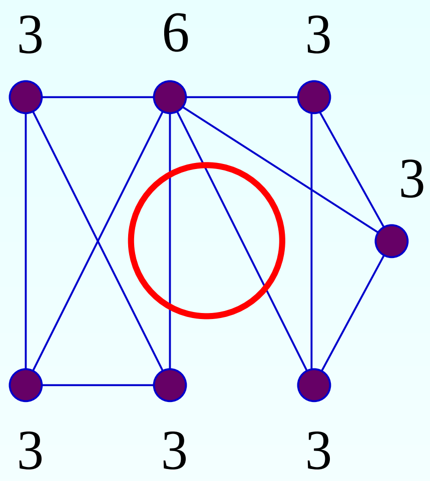
- **Proof**

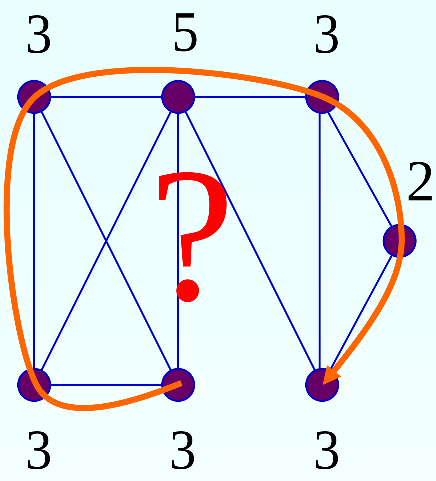
  For any distinct $u, v \in V$, $deg(u) + deg(v) \geq n-1$

  From Theorem 11.8, G has a Hamilton path

  (Note) Corollary 11.4 is more strict

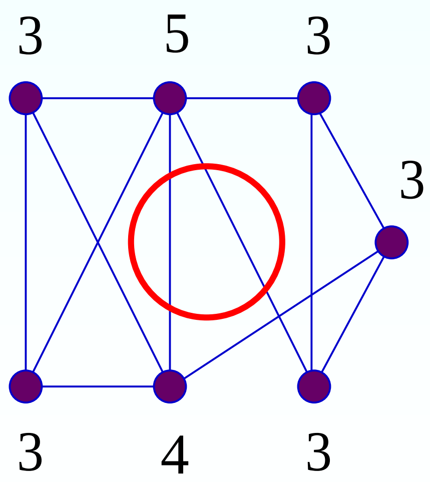
Some Examples for Hamilton Path

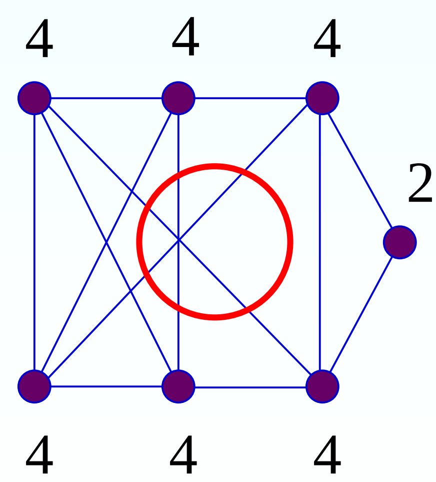




A → B

$\bar{A}$?



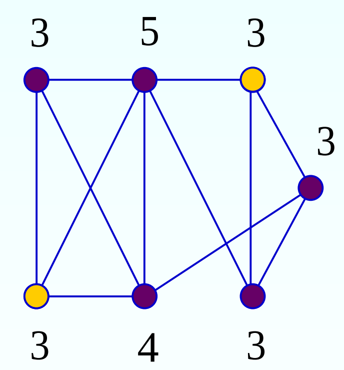


Theorem 11.9

- **Hamilton Cycle**

  Let $G=(V,E)$ be a loop-free undirected graph with $|V| = n \geq 3$. If $deg(x) + deg(y) \geq n$ for all nonadjacent $x, y \in V$, $x \neq y$,

  → G contains a Hamilton cycle



- **Corollary 11.5**

  If $G=(V,E)$ be a loop-free undirected graph with $|V| = n \geq 3$, and if $deg(v) \geq n/2$ for all $v \in V$,

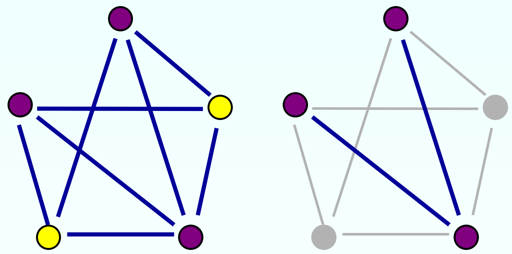
  → G has a Hamilton cycle

Corollary 11.6

- Another Condition

  If $G=(V,E)$ be a loop-free undirected graph with $|V| = n \geq 3$, and if $|E| \geq {}_{n-1}C_2 + 2$,

  ➔ G has a Hamilton cycle



- **Proof**

  Let $a,b \in V$, where $\{a,b\} \notin E$. Let $H=(V', E')$ be the subgraph of G by removing both the a,b vertices and their edges

  Then, $|E| = |E'| + deg(a) + deg(b)$

(Proof)

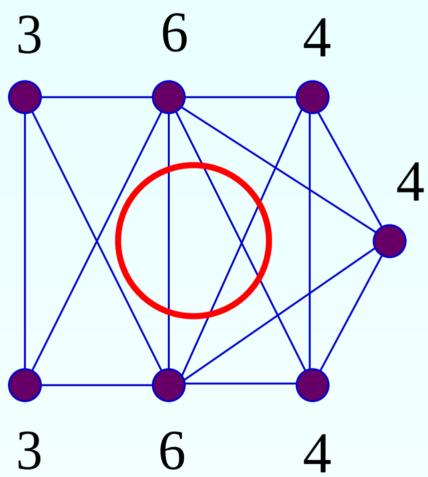
Since $|V'| = n-2$ and H is a subgraph of the complete graph K_{n-2}, $|E'| \leq {}_{n-2}C_2$

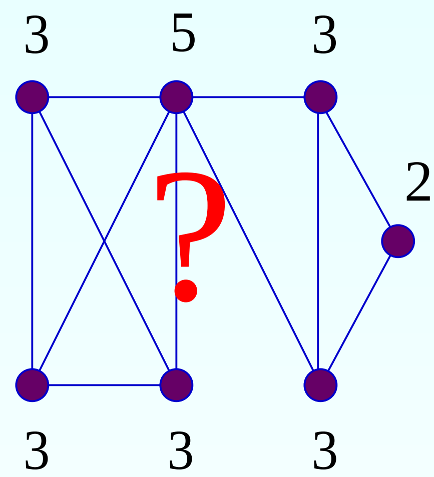
$$ {}_{n-1}C_2 + 2 \leq |E| = |E'| + \deg(a) + \deg(b) $$
$$ \leq {}_{n-2}C_2 + \deg(a) + \deg(b) $$

$$ \therefore \ \deg(a) + \deg(b) \geq {}_{n-1}C_2 + 2 - {}_{n-2}C_2 = n $$

According to theorem 11.9, there is a Hamilton cycle in the given G

Some Examples for Hamilton Cycle

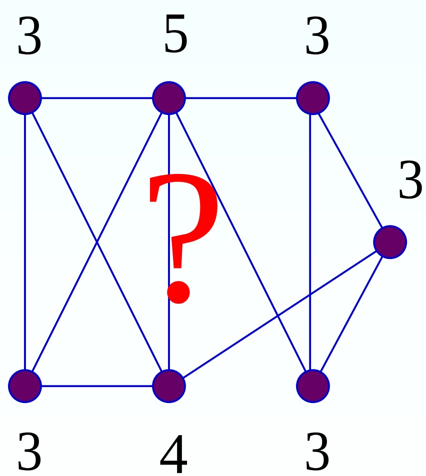


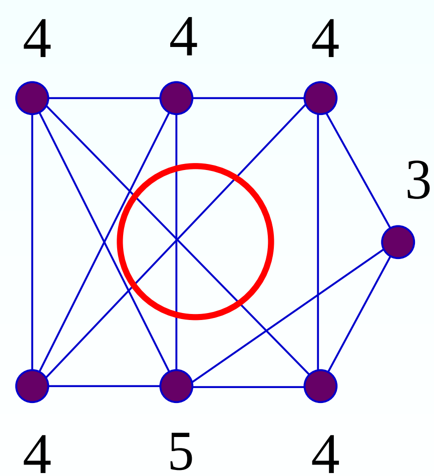


$\Sigma\deg(v) = 22$

${}_{n\text{-}1}C_2 + 2 = 17$

$\Sigma\deg(v) = 24$



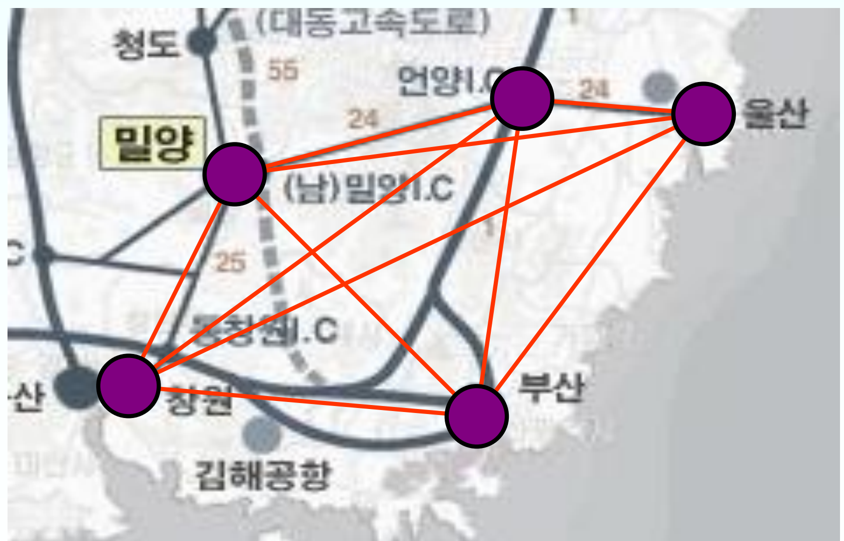


Traveling Salesman Problem

- Weighted graph $G = (V, E, W)$

  최소의 비용으로 모든 도시를 각 한번씩 방문하고 집으로 돌아오는 cycle 찾는 문제

  No optimal solution within a reasonable amount of time



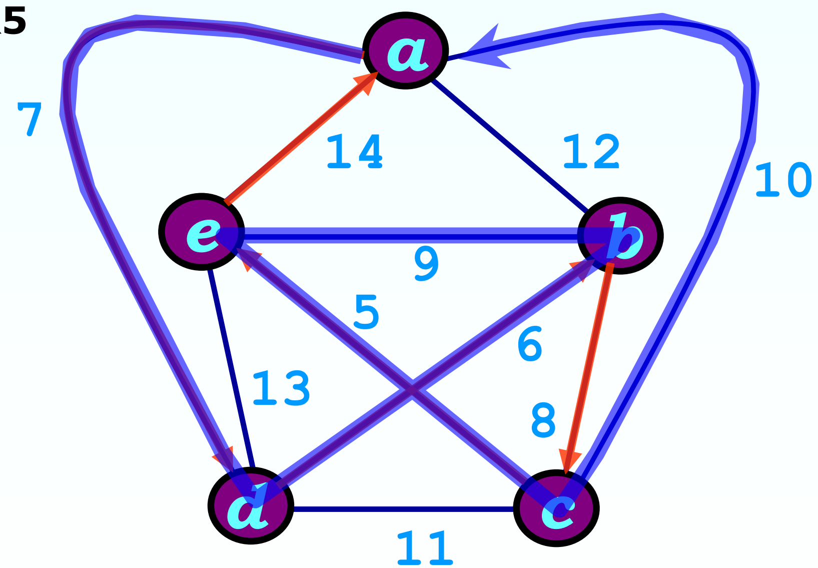
Traveling Salesman Problem

- Greedy Algorithm

  각 vertex에서 아직 방문하지 않은 vertex와 edge 중 가장 비용이 작은 것을 선택하는 heuristic 적용

  Cannot provide us the optimal solution

K5



Greedy solution (40)

; a → d → b → c → e → a

7 6 8 5 14

Another solution (37)

; a → d → b → e → c → a

7 6 9 5 10

Summary of Hamilton Path & Cycle

- Path (cycle) that contains every vertices
- H.P. in Complete Directed Graph : K_n^*

❶ If $deg(x)+deg(y) \geq n-1$ for all $x,y \in V$, $x \neq y$ or

❷ If $deg(v) \geq (n-1)/2$ for all $v \in V$

➔ G has a Hamilton Path

❶ If $deg(x) + deg(y) \geq n$ for all nonadjacent $x, y \in V$, $x \neq y$ or

❷ If $deg(v) \geq n/2$ for for all $v \in V$ or

❸ If $|E| \geq {}_{n-1}C_2 + 2$,

➔ G contains a Hamilton Cycle

Graph Coloring

Section 11.6

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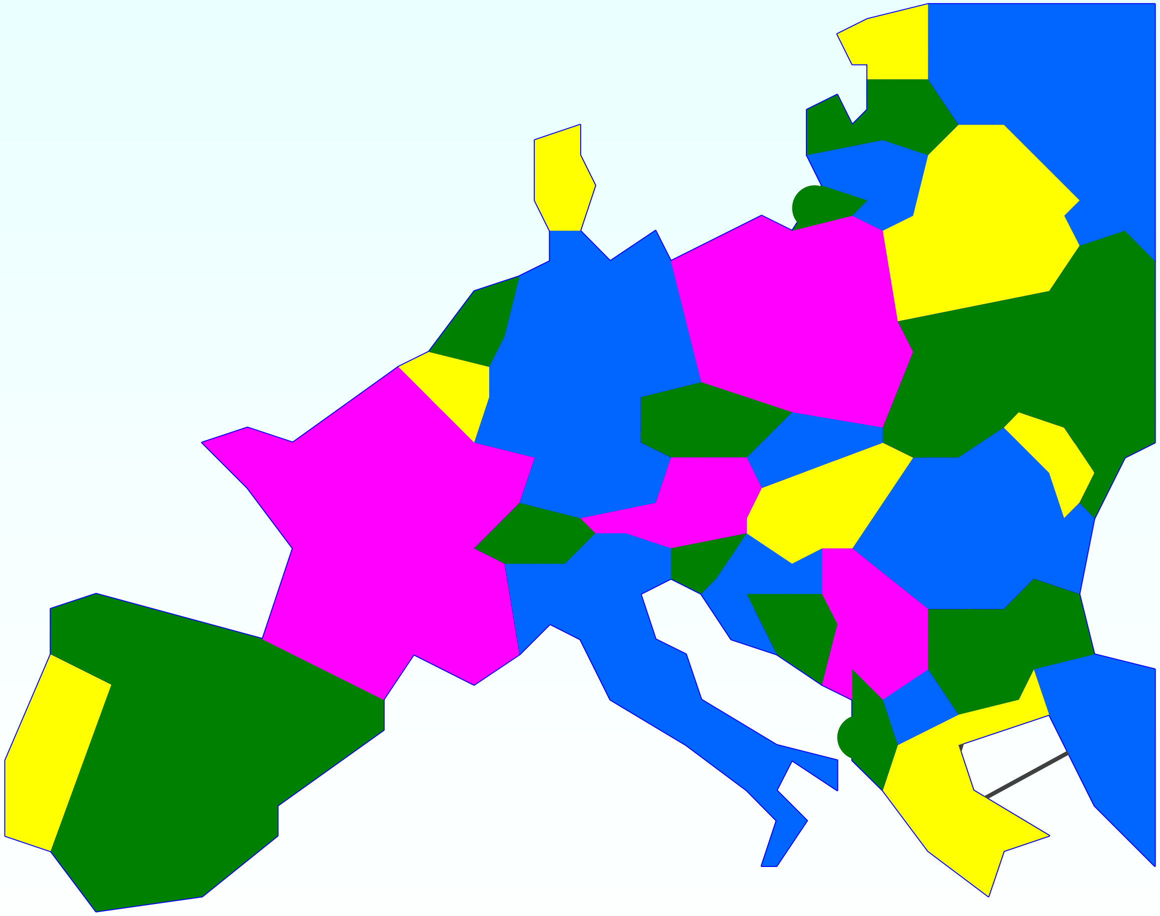
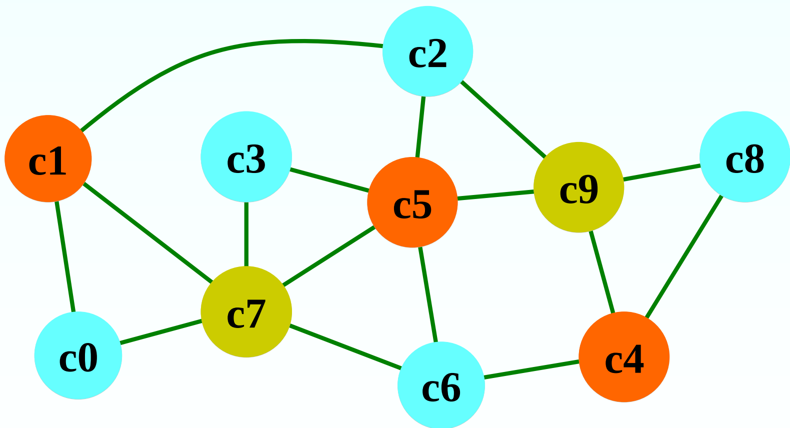
Map Maker's Problem

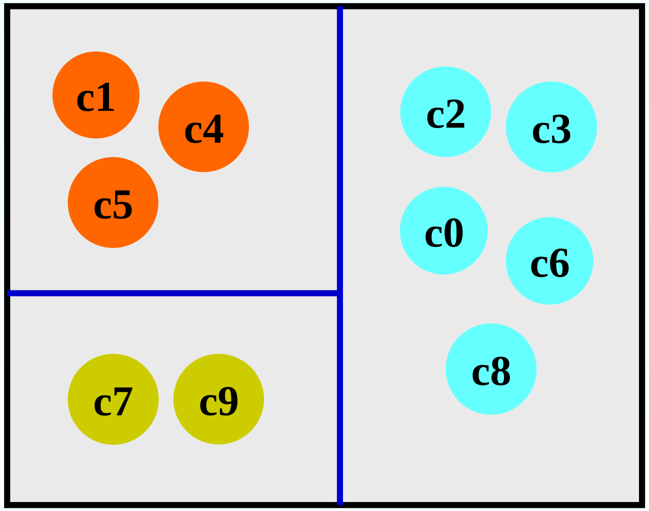


(Another Ex.) Compartment Building

Problem

함께 보관할 수 없는 화학약품들을 분리하여 보관할 수 있도록 창고에 분리 칸막이 공사를 하라.





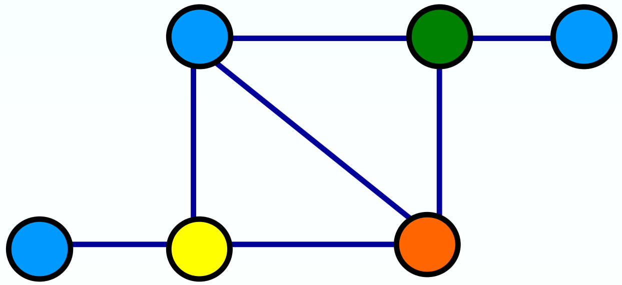
Graph Coloring

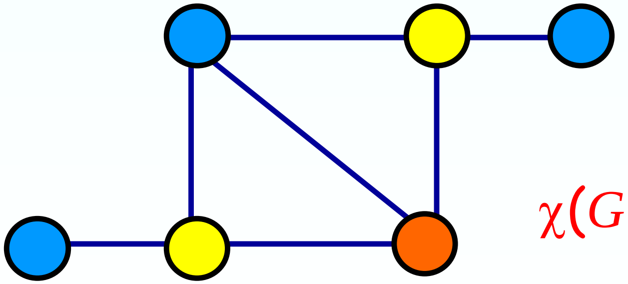
- Definition

  If $G = (V,E)$ is an undirected graph, a proper coloring of G occurs when adjacent vertices have different colors.

- Chromatic number

  The minimum number of colors needed to properly color G, which is written $\chi(G)$.





$\chi(G) = 3$

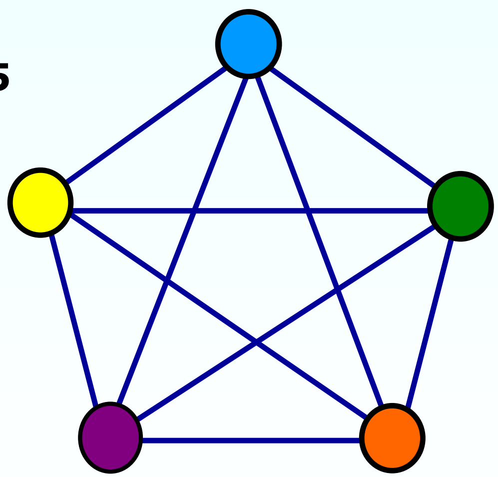
Coloring Complete Graphs

- The chromatic number of a complete graph K_n is n

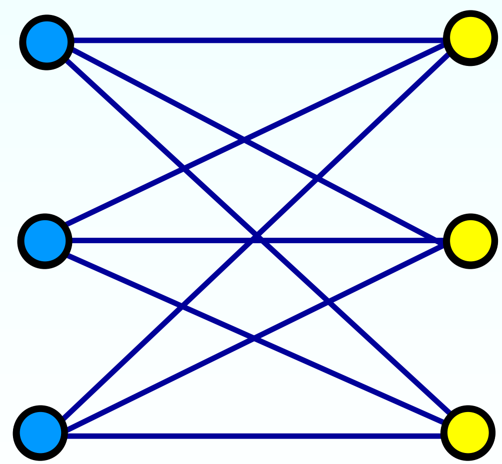
$$\chi(K_n) = n$$

$K_{3,3}$?

K_5

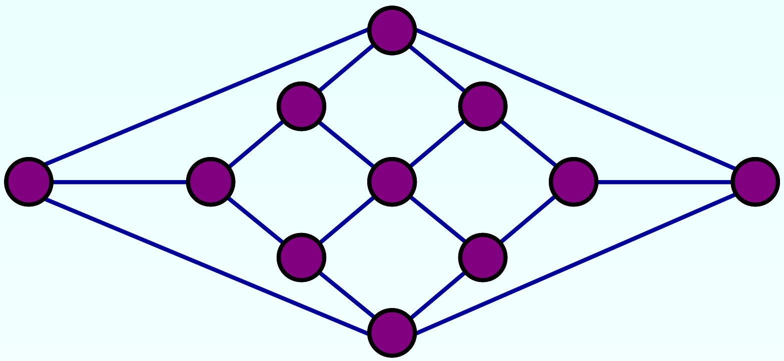


$\chi(K_5) = 5$



Coloring Planar Graph

- Any planar graph is 5-colorable



Herschel Graph

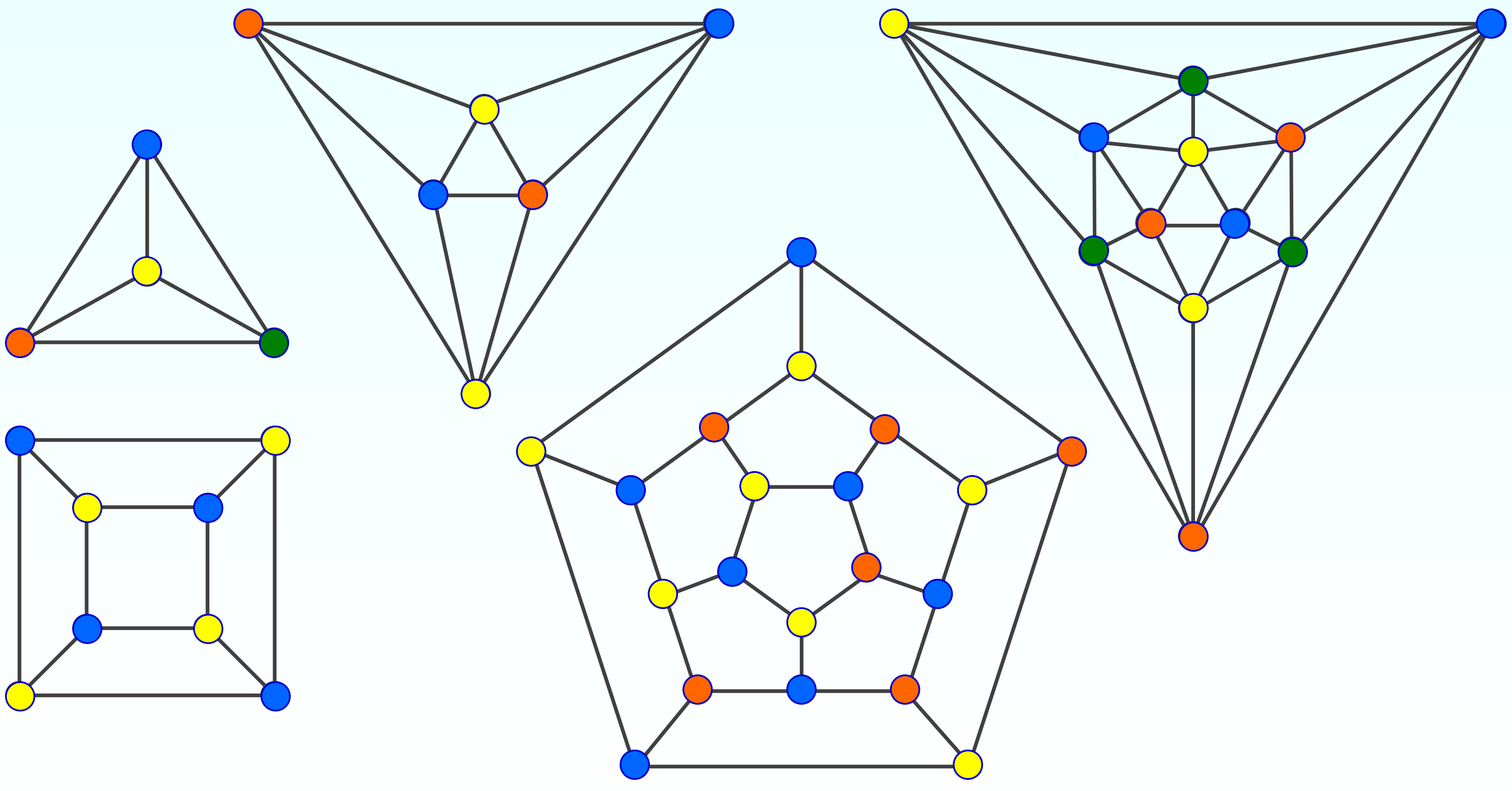
- Four Color Conjecture

  Any planar graph is 4-colorable.

  Posed in 1850s, proved by K.Appel & W.Haken in 1976 through analyzing 2,000 different cases

Some Questions

(Q1) $\chi(G)$ for the five platonic graphs ?



Some Questions

(Q2) If a graph G_1 is a subgraph of G_2, what can you say about $\chi(G_1)$ and $\chi(G_2)$?

(Q3) What is the only graph with n vertices and chromatic number 1 ?

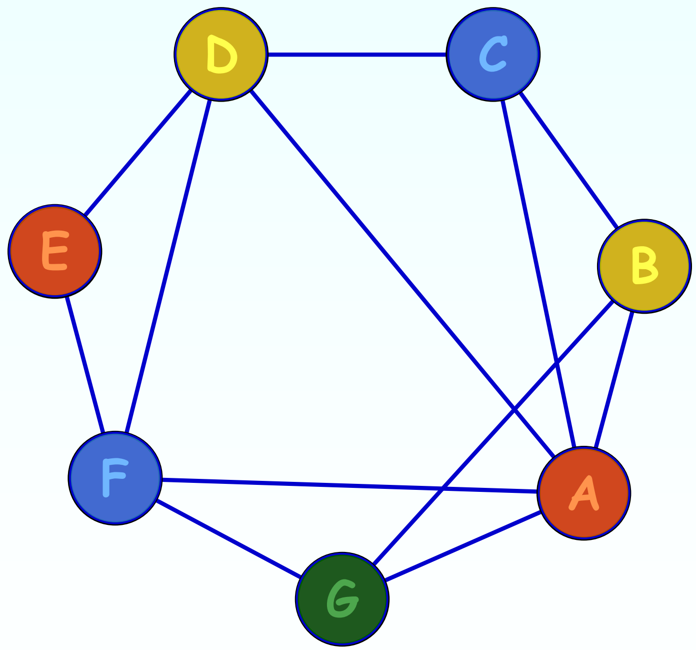
(Q4) If a graph is a tree, what is its chromatic number ? (A tree is a connected acyclic graph.)

Some Questions

(Q5) 각 학생이 하루에 한 과목만 시험칠 수 있도록 시험계획표를 작성함. 최소로 필요한 시험일수는?

	A	B	C	D	E	F	G
A		v	v	v		v	v
B	v		v				v
C	v	v		v			
D	v		v		v	v	
E				v		v	
F	v			v	v		v
G	v	v				v	

한 학생 이상이 동시에
수강하고 있는 과목 표기

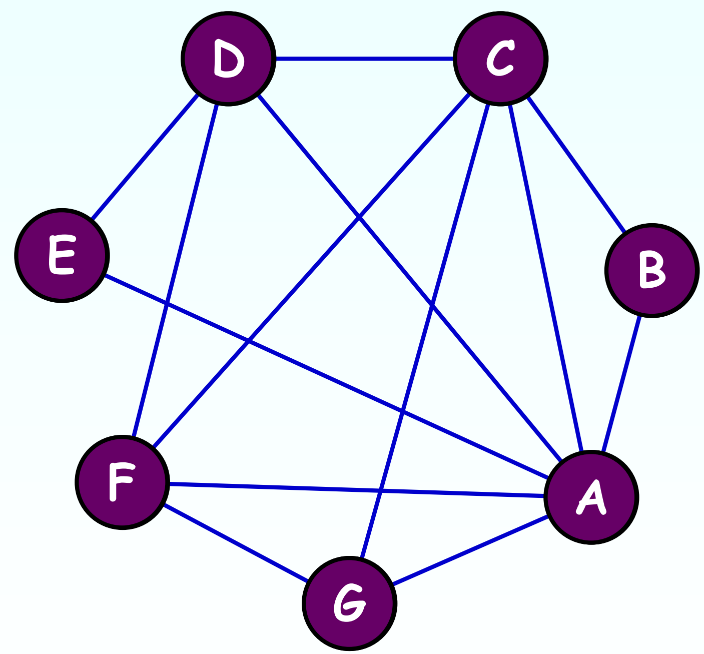


Some Questions

(Q6) 유아교육원에서 최소의 사물함 마련 문제

	A	B	C	D	E	F	G
7:00	v			v	v		
8:00	v	v	v				
9:00	v		v	v		v	
10:00	v		v			v	v
11:00	v					v	v
12:00	v				v		

	A	B	C	D	E	F	G
A		v	v	v	v	v	v
B	v		v				
C	v	v		v		v	v
D	v		v		v	v	
E	v			v			
F	v		v	v			v
G	v		v			v	



Summary of Graph Coloring

- Chromatic Number $\chi(G)$
- $\chi(K_n)$ = n
- Four Color Conjecture
  - Any planar graph is 4-colorable
- Application Examples
  - Map Making
  - 화학약품 창고 칸막이
  - 1과목/일 시험 스케줄링
  - 유치원 사물함