Functions

Functions

- Definition:

  A function f from a set A to a set B, denoted by $f : A \to B$, is a relation between A and B, in which for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$.

- Uniqueness check: $(\forall x)(\forall y)(\forall z)\ [(x, y) \in f \wedge (x, z) \in f \to y = z]$

Example:

$A = \{a, b, c\}$, $B = \{\alpha, \beta, \gamma, \delta\}$

$R_1 = \{(a, \alpha), (c, \gamma)\}$ is not a function.

$R_2 = \{(a, \alpha), (a, \delta), (b, \gamma), (c, \beta)\}$ is not a function.

$R_3 = \{(a, \alpha), (b, \alpha), (c, \delta)\}$ is a function.

- Note:
  - For a function $f : A \to B$, $(c, \delta) \in f$ is also written as $f(c) = \delta$ where c is called the argument and δ is called the value.
  - As a relation, $(c, \delta) \in f$ is sometimes written as $c\, f\, \delta$.
  - Domain $\mathcal{D}(f) = A$ and range $\mathcal{R}(f) \subseteq B$ (B: codomain of f).
  - The writing order of function composition is the reverse of relations.

    Let $A = \{a, b, c\}$, $B = \{\alpha, \beta\}$, and $C = \{x, y, z\}$.

    Let $f = \{(a, \beta), (b, \beta), (c, \alpha)\}$ and $g = \{(\alpha, x), (\beta, z)\}$.

    Then, $\cancel{f \circ g} = \{(a, z), (b, z), (c, x)\} = g \circ f$.

    $g \circ f(a) = g(f(a)) = g(\beta) = z$.

- Commutative Diagram

$$
\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C
\end{tikzcd}
$$

- Theorem:

  If $f : A \to B$ and $g : B \to C$ then $g \circ f : A \to C$.

Proof:

Obviously, $\mathcal{D}(g \circ f) = A$.

Let $(a, c_1) \in g \circ f$ and $(a, c_2) \in g \circ f$.

Then, there must be a $b_1 \in B$ such that $(a, b_1) \in f$ and $(b_1, c_1) \in g$.

Also, there must be a $b_2 \in B$ such that $(a, b_2) \in f$ and $(b_2, c_2) \in g$.

Since $(a, b_1) \in f$, $(a, b_2) \in f$, and f is a function, $b_1 = b_2$.

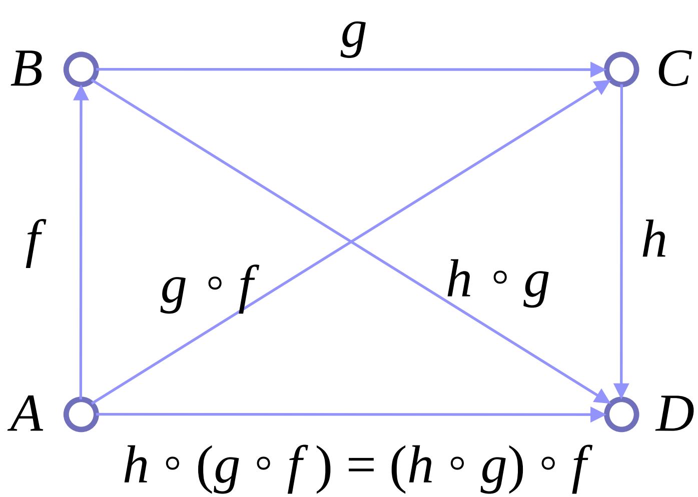
Then, from the fact that $(b_1, c_1) \in g$, $(b_2, c_2) \in g$, $b_1 = b_2$, and g is a function, we get $c_1 = c_2$.

Therefore, $g \circ f : A \to C$. $\square$

- Theorem:

  Let $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then $(h \circ g) \circ f = h \circ (g \circ f)$.

  (i.e., function composition is associative.)



- Notation:

  B^A denote the set of all functions from A to B.

  Suppose $|A| = n$ and $|B| = m$. Then, $|B^A| = m^n$.

- Definitions: Let $f : A \to B$.
  - f is said to be surjective if $\mathcal{R}(f) = B$.
  - f is said to be injective if for every $(a, b) \in f$ and $(a', b) \in f$, $a = a'$.
  - If f is both surjective and injective then it is called bijective.

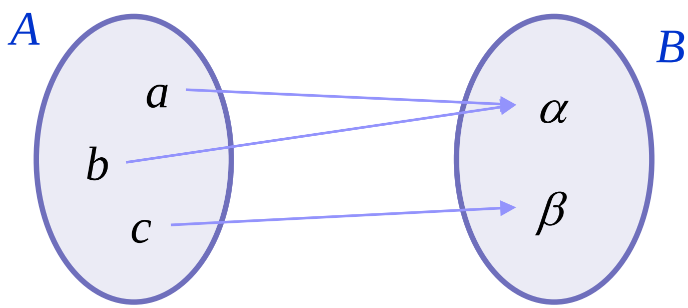
- Note: f is injective $\Leftrightarrow$ $(\forall x)(\forall y)(\forall z)\,[(x, z) \in f \wedge (y, z) \in f \to x = y]$

- Terminologies:

  is a surjection $\cdots$ is surjective $\cdots$ is onto

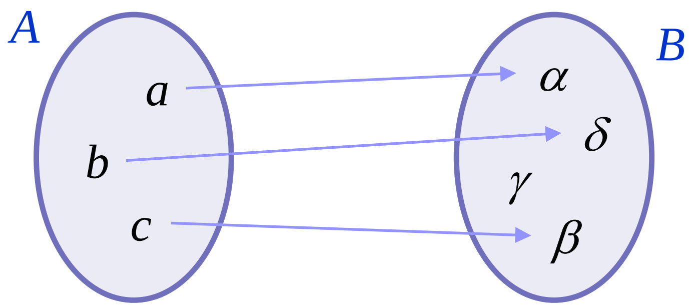
  is an injection $\cdots$ is injective $\cdots$ is one-to-one

  is a bijection $\cdots$ is bijective $\cdots$ is one-to-one correspondence



surjective
not injective

- Terminologies:



injective
not surjective

- Note that for a function $f: A \rightarrow B$ to be bijective, A and B should have the same cardinality.

- Theorem: Let $f : A \to B$ and $g : B \to C$.
  - (1) If f and g are surjective then $g \circ f$ is surjective.
  - (2) If f and g are injective then $g \circ f$ is injective.
  - (3) If f and g are bijective then $g \circ f$ is bijective.

Proof of (2)

Note that $g \circ f : A \to C$.

Let (a, c) and (a', c) be elements of $g \circ f$.

Since $(a, c) \in g \circ f$, there must be a $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$.

Similarly for $(a', c) \in g \circ f$, there must be a $b' \in B$ such that $(a', b') \in f$ and $(b', c) \in g$.

Proof of (2)

But $b = b'$ because $(b, c) \in g$, $(b', c) \in g$, and g is injective.

Since $(a, b) \in f$, $(a', b') \in f$, $b = b'$, and f is injective, we get $a = a'$.

$\square$

- Theorem: Let $f : A \to B$ and $g : B \to C$.
  - (1) If $g \circ f$ is surjective then g is surjective.
  - (2) If $g \circ f$ is injective then f is injective.
  - (3) If $g \circ f$ is bijective then g is surjective and f is injective.

Proof of (1)

Suppose that g is not surjective.

Then there exists a $c \in C$ such that $g(b) \neq c$ for any $b \in B$.

But there is an $a \in A$ such that $(a, c) \in g \circ f$ (i.e., $g \circ f(a) = c$) because $g \circ f$ is surjective.

From $(a, c) \in g \circ f$, we know that there must exist a $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$, which contradicts our assumption.

- Definition:

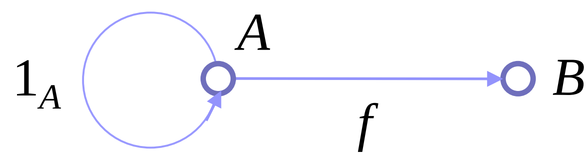
  The function $1_A : A \to A$, denoted by $1_A(a) = a$ for all $a \in A$, is called the identity function for A.

Example:

If $A = \{a, b, c\}$ then the identity function 1_A for A is

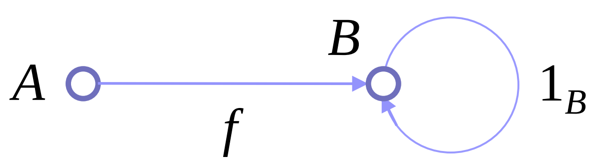
$$\{(a, a), (b, b), (c, c)\}.$$

- Right and left identities: Let $f : A \to B$.
  - $f \circ 1_A = f$, where 1_A is called the right identity function of f.



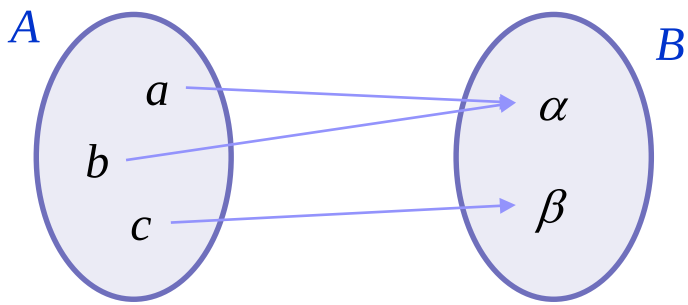
($1_A \circ f = f$? $\cdots$ meaningless if $A \neq B$)

  - $1_B \circ f = f$, where 1_B is called the left identity function of f.



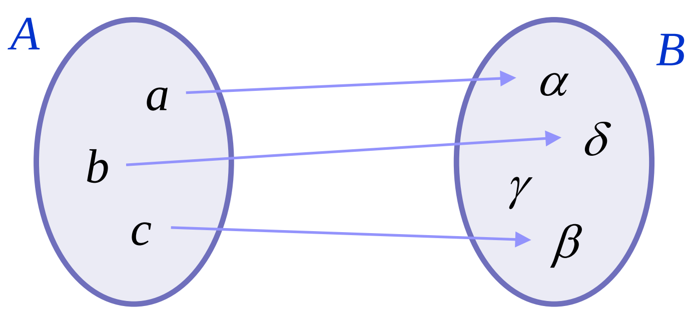
- Inverse function:

  Let $f: A \to B$. Then the converse of f is $f^{c} = \{(y, x) \mid (x, y) \in f\}$.



$f_1 : A \to B$

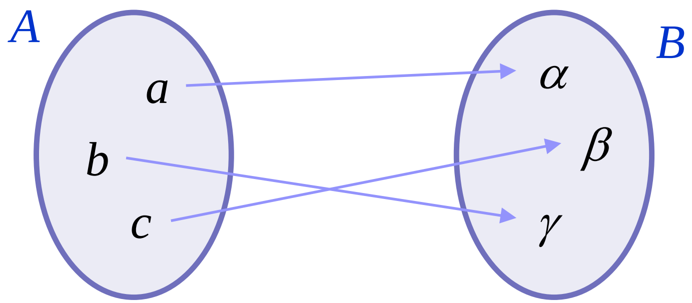
$f_1^{\,c}$ is not a function



$f_2 : A \to B$

$f_2^{\,c}$ is not a function

- Inverse function:

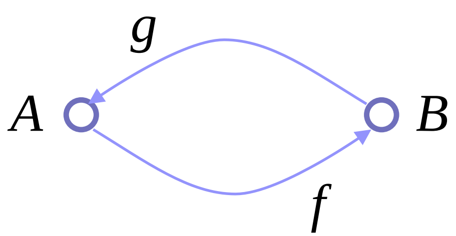


$f_3 : A \to B$

$f_3^{\,c}$ is a function

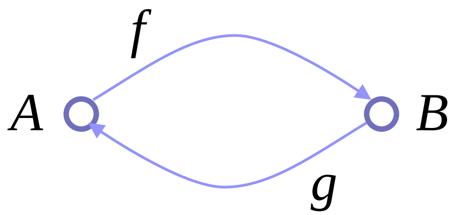
- The converse of a function is not necessarily a function.
- If there exists a converse that is a function, it is called the inverse of f, and is denoted by f^{-1}.
- Note that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

- Right and left inverses: Let $f : A \to B$ and $g : B \to A$.
  - If $f \circ g = 1_B$, then g is called the right inverse of f.



(*f* is the left inverse of g)

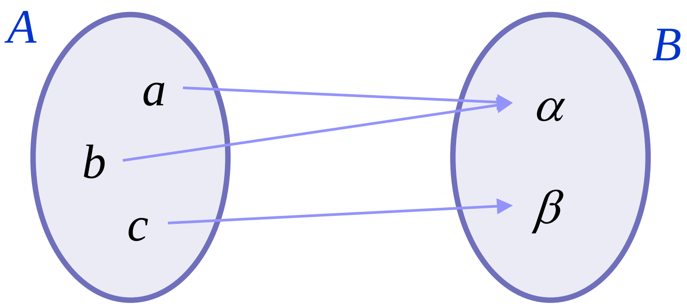
  - If $g \circ f = 1_A$, then g is called the left inverse of f.



(*f* is the right inverse of g)

- Right and left inverses:

When a function (e.g., f_1) is surjective but not injective:



$f_1 : A \to B$

f_1^c is not a function

If we compose the two relations f_1 and f_1^c (beware of the order):

$$f_1 \circ f_1^c = \{(a, a), (a, b), (b, a), (b, b), (c, c)\} \neq E_A$$

$$f_1^c \circ f_1 = \{(\alpha, \alpha), (\beta, \beta)\} = E_B \ (= 1_B)$$

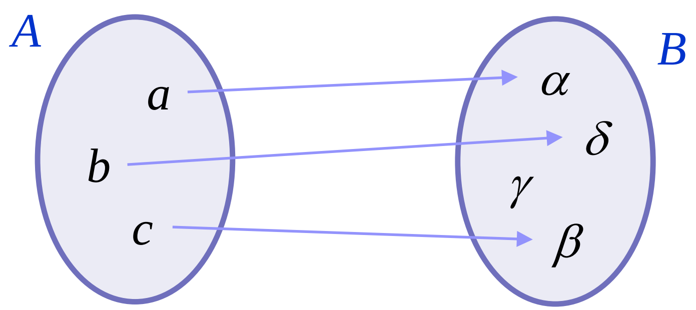
If f_1^c is modified to make a function, e.g., $g_1 = \{(\alpha, b), (\beta, c)\}$,

$$g_1 \circ f_1 = \{(a, b), (b, b), (c, c)\}$$

$$f_1 \circ g_1 = \{(\alpha, \alpha), (\beta, \beta)\} = 1_B$$ (g_1 is the right inverse of f_1)

- Right and left inverses:

When a function (e.g., f_2) is injective but not surjective:



$f_2 : A \to B$

f_2^c is not a function

If we compose the two relations f_2 and f_2^c (beware of the order):

$f_2 \circ f_2^c = \{(a, a), (b, b), (c, c)\} = E_A \ (= 1_A)$

$f_2^c \circ f_2 = \{(\alpha, \alpha), (\beta, \beta), (\delta, \delta)\} \neq E_B$

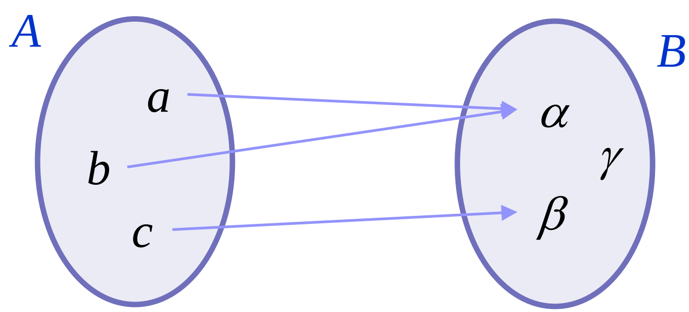
If f_2^c is modified to a function, e.g., $g_2 = \{(\alpha, a), (\beta, c), (\gamma, a), (\delta, b)\}$,

$g_2 \circ f_2 = \{(a, a), (b, b), (c, c)\} = 1_A$ (g_2 is the left inverse of f_2)

$f_2 \circ g_2 = \{(\alpha, \alpha), (\beta, \beta), (\gamma, \alpha), (\delta, \delta)\}$

- Right and left inverses:

When a function (e.g., f_3) is neither injective nor surjective:



$f_3 : A \rightarrow B$

f_3^c is not a function

If we compose the two relations f_3 and f_3^c (beware of the order):

$f_3 \circ f_3^c = \{(a, a), (a, b), (b, a), (b, b), (c, c)\} \neq E_A$

$f_3^c \circ f_3 = \{(\alpha, \alpha), (\beta, \beta)\} \neq E_B$

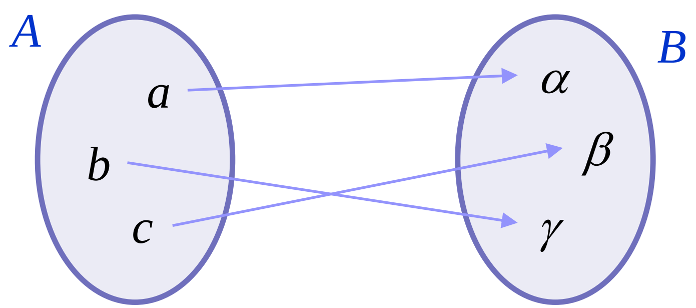
If f_3^c is modified to a function, e.g., $g_3 = \{(\alpha, a), (\beta, c), (\gamma, b)\}$,

$g_3 \circ f_3 = \{(a, a), (b, a), (c, c)\}$

$f_3 \circ g_3 = \{(\alpha, \alpha), (\beta, \beta), (\gamma, \alpha)\}$

- Right and left inverses:

When a function (e.g., f_4) is bijective:



$f_4 : A \rightarrow B$

f_4^c is a function ($f_4^c = f_4^{-1}$)

If we compose the two functions f_4 and f_4^c:

$$f_4^c \circ f_4 = f_4^{-1} \circ f_4 = \{(a, a), (b, b), (c, c)\} = 1_A$$

(f_4^{-1} is the left inverse of f_4)

$$f_4 \circ f_4^c = f_4 \circ f_4^{-1} = \{(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma)\} = 1_B$$

(f_4^{-1} is the right inverse of f_4)

- Theorem: Let $f : A \to B$.

  (1) f has a left inverse if and only if f is injective.

  (2) f has a right inverse if and only if f is surjective.

  (3) f has a left and right inverse if and only if f is bijective.

  (4) If f is bijective then the left inverse of f is equal to the right inverse of f.

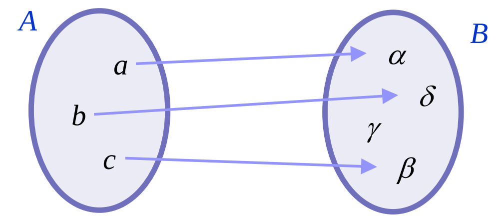
Proof of (1)

(if part): f has a left inverse if f is injective.

Assume that f is injective.

Let $g : B \to A$ be defined as follows.

Proof of (1)



For $b \in B$,

$$g(b) = \begin{cases} a & \text{if } b \in \mathcal{R}(f) \text{ and } f(a) = b \\ c & \text{otherwise, where } c \text{ is a unique element of } A \end{cases}$$

Obviously g is a relation and $\mathcal{D}(g) = B$.

Let $(x, y) \in g$ and $(x, z) \in g$.

If $x \notin \mathcal{R}(f)$, then $y = z$ $(= c)$ by the lower part of g's definition.

If $x \in \mathcal{R}(f)$, then $f(y) = x$ and $f(z) = x$.

But since f is injective, $y = z$.

Hence, when $(x, y) \in g$ and $(x, z) \in g$, $y = z$.

Thus, g is a function.

Proof of (1)

Note that $g \circ f : A \to A$.

Let $a \in A$.

Then $g \circ f(a) = g(f(a)) = a$ by the top part of g's definition.

Hence, $g \circ f = 1_A$.

Therefore, g is a left inverse of f.

(only if part): f has a left inverse only if f is injective.

Assume that f has a left inverse.

Let $g : B \to A$ be a left inverse of f, i.e., $g \circ f = 1_A$.

We want to prove that f is injective.

Proof of (1)

Assuming $(x, y) \in f$ and $(z, y) \in f$, we have to show $x = z$.

$$x = 1_A(x) = g \circ f(x) = g(f(x)) = g(y) = g(f(z)) = g \circ f(z) = 1_A(z) = z$$

Hence, f is injective. □

Another proof for $x = z$:

Note that $(x, x) \in g \circ f$ and $(z, z) \in g \circ f$.

From $(x, y) \in f$ and $(x, x) \in g \circ f$, we get $(y, x) \in g$.

Similarly, from $(z, y) \in f$ and $(z, z) \in g \circ f$, we get $(y, z) \in g$.

Since $(y, x) \in g$, $(y, z) \in g$, and g is a function,

we get $x = z$.

Proof of (4): f is bijective $\Rightarrow f_L = f_R$

Since f is surjective, it has a right inverse.

Let that inverse be $f_R : B \to A$.

Since f is injective, it has a left inverse.

Let that inverse be $f_L : B \to A$.

We have to prove that $f_L = f_R$.

Since $f_L \circ f = 1_A$ and $f \circ f_R = 1_B$,

$$f_L = f_L \circ 1_B = f_L \circ (f \circ f_R) = (f_L \circ f) \circ f_R = 1_A \circ f_R = f_R.$$

□

Proof of (4)

Another proof for $f_L = f_R$:

Let $(b, a) \in f_L$.

Note that $(a, a) \in 1_A = f_L \circ f$ and $(b, b) \in 1_B = f \circ f_R$.

From $(b, a) \in f_L$ and $(a, a) \in f_L \circ f$, we get $(a, b) \in f$.

Then, from $(a, b) \in f$ and $(b, b) \in f \circ f_R$, we get $(b, a) \in f_R$.

Therefore, $f_L \subseteq f_R$.

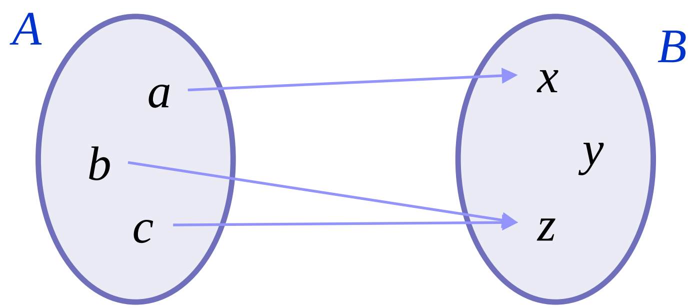
We can similarly show that $f_R \subseteq f_L$.

From $f_L \subseteq f_R$ and $f_R \subseteq f_L$, we conclude $f_L = f_R$.

- Image and inverse image:

  Let $f : A \to B$, where $A = \{a, b, c\}$, $B = \{x, y, z\}$, and

  $f = \{(a, x), (b, z), (c, z)\}$.



  - The image of the set $\{a, b\}$ under f:

    $$f(\{a, b\}) = \{f(a), f(b)\} = \{x, z\}$$

  - The inverse image of $\{z\}$ under f is $\{b, c\}$.
  - The inverse image of $\{x, z\}$ under f is $\{a, b, c\}$.