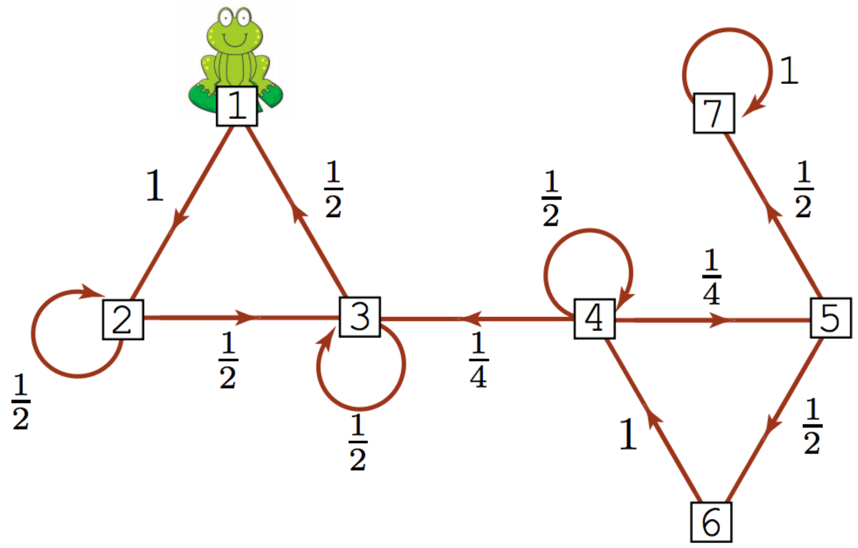
Markov Chains

Markov Chains – introduction

- Introduced in 1906 by Andrei Andreyevich Markov (1856–1922)
- (Example) A frog hops about on 7 lily pads. The numbers next to arrows show the probabilities with which, at the next jump, he jumps to a neighboring lily pad (and when out-going probabilities sum to less than 1 he stays where he is with the remaining probability).



Transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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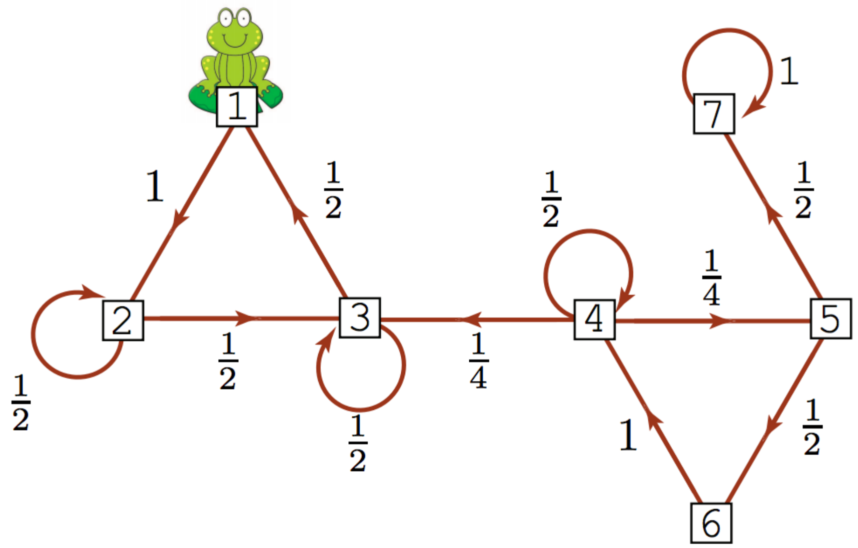

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- How long does it take to get there?
- What happens in the long run?
  - Starting in state 1, what is the probability that we are still state 1 after 3 steps? ($P_{1,1}{}^{(3)} = \frac{1}{4}$)
  - Starting in state 4, what is the probability that we ever reach state 7?

Markov Chains – definitions

- Let $I = \{i_0, i_1, i_2, \ldots, i_{n-1}, i, j\}$ where $i_k \in I$ (i_k is called a ***state*** *and* I the ***state-space***)

- Stochastic process
  - $X_n = i$: the process is said to be in state i at time n

- Markov Chain
  - Stochastic process where future state X_{n+1} is independent of the past states and depends only on present state X_n (discrete state space)
  - $P_{i,j} = P\{X_{n+1} = j | X_n = i, X_n = i_{n-1}, \ldots, X_0 = i_0\} = P\{X_{n+1} = j | X_n = i\}$
  - $P_{i,j} \geq 0,\ \sum_{j=0}^{\infty} P_{i,j} = 1, i = 0, 1, 2, \ldots$

Time homogeneous

Markov Chains – examples

- Example 4.1 (Forecasting the weather). Suppose that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, it will rain tomorrow with probability β
  - Transition probabilities with two states ? (denote “rain” as state 0 and “not rain” 1)
  - $P_{0,0} =$
  - $P_{0,1} =$
  - $P_{1,0} =$
  - $P_{1,1} =$
- Example 4.2 (A communication system). Two digits 0 and1. There is a probability *p* that the digit entered will be unchanged when it leaves.
  - $P_{0,0} = P_{1,1} =$
  - $P_{0,1} = P_{1,0} =$

Markov Chains – examples

- Example 4.3 State space = {Cheerful, So-so, Glum}
  - (C, C) = 0.5 = $P_{0,0}$, (C, S) = 0.4 = $P_{0,1}$, (C, G) = 0.1 = $P_{0,2}$
  - (S, C) = 0.3 = $P_{1,0}$, (S, S) = 0.4 = $P_{1,1}$, (S, G) = 0.3 = $P_{1,2}$
  - (G, C) = 0.2 = $P_{2,0}$, (G, S) = 0.3 = $P_{2,1}$, (G, G) = 0.5 = $P_{2,2}$
  - $P =$

Markov Chains – examples

- Example 4.4 (Transforming a process into a Markov Chain)
  - O: rain, X: not rain
  - 0: (O, O) → (O, O)
  - 1: (X, O) →
  - 2: (O, X) →
  - 3: (X, X) →
  - $P_{0,0} = P_{1,1} =$
  - $P_{0,1} = P_{1,0} =$

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- Example 4.5 (A random walk model)
  - State space = $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$
  - $X_i = \begin{cases} 1 \text{ with probability } p \\ -1 \text{ with } (1-p) \end{cases}$
  - $S_n = X_1 + X_2 + \cdots + X_n$
  - $P_{i,i+1} =$
  - $P_{i,i-1} =$
  - $P =$

Markov Chains – examples

- Example 4.6 (A gambling model)
  - State space = {0, 1, 2, 3, …, N}
  - $X_i = \begin{cases} 1 \text{ with probability } p \\ -1 \text{ with } (1-p) \end{cases}$
  - $P_{0,0} = 1$
  - $P_{N,N} = 1$
  - $P_{i,i+1} =$
  - $P_{i,i-1} =$
  - $P =$
  - {0, N} called absorbing states

Markov Chains – examples

- Example 4.7 (Bonus Malus system)

		Next state if			
State	Annual Premium	0 claims	1 claim	2 claims	$\geq$ 3 claims
1	200	1	2	3	4
2	250	1	3	4	4
3	400	2	4	4	4
4	600	3	4	4	4

- $a_k = e^{-\lambda}\frac{\lambda^k}{k!}, k \geq 0$
- $P =$