Relations (revisited)

Properties of Relations

- Definitions: Let R be a relation on a set A (i.e., $R \subseteq A \times A$).
  - R is said to be reflexive if for every $x \in A$, $(x, x) \in R$.
  - R is said to be irreflexive if for every $x \in A$, $(x, x) \notin R$.
  - R is said to be symmetric if for every $(x, y) \in R$, $(y, x) \in R$, i.e.,
    $$(\forall x)(\forall y)\,((x, y) \in R \rightarrow (y, x) \in R).$$
  - R is said to be antisymmetric if
    $$(\forall x)(\forall y)\,((x, y) \in R \wedge (y, x) \in R \rightarrow x = y).$$
  - R is said to be asymmetric if
    $$(\forall x)(\forall y)\,((x, y) \in R \rightarrow (y, x) \notin R).$$
  - R is said to be transitive if
    $$(\forall x)(\forall y)(\forall z)\,((x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R).$$

Example:

For $R = \{(a, b), (b, a)\}$ to be transitive, we need to add (a, a) and (b, b).

- Note:
  - If R is asymmetric then R is irreflexive.
  - If R is asymmetric then R is antisymmetric.

- Theorem: Let R be a relation on a set A. Then,

  (1) R is reflexive iff $E_A \subseteq R$.

  (2) R is irreflexive iff $R \cap E_A = \varnothing$.

  (3) R is symmetric iff $R = R^c$.

  E.g., $R = \{(a, b), (b, a)\}$ is symmetric.

  See that $R^c = \{(b, a), (a, b)\}$ and $R = R^c$.

  (4) R is antisymmetric iff $R \cap R^c \subseteq E_A$.

  E.g., $R = \{(a, a), (a, b)\}$ is antisymmetric.

  Then, $R^c = \{(a, a), (b, a)\}$ and $R \cap R^c = \{(a, a)\}$.

  (5) R is asymmetric iff $R \cap R^c = \varnothing$.

  (6) R is transitive iff $R \circ R \subseteq R$.

  E.g., $R = \{(a, b), (b, c) , (a, c)\}$ is transitive and $R \circ R = \{(a, c)\} \subseteq R$

Invalid proof of (6):

(if part) R is transitive if $R \circ R \subseteq R$

Assume $R \circ R \subseteq R$.

Let $(x, z) \in R \circ R$.

Then, there must exist a y such that $(x, y) \in R$ and $(y, z) \in R$.

Since $(x, z) \in R \circ R$ and $R \circ R \subseteq R$, we see that $(x, z) \in R$.

Therefore, R is transitive.

What is wrong with this proof?

Proof of (6): R is transitive iff $R \circ R \subseteq R$

(if part)

Assume $R \circ R \subseteq R$.

Let $(x, y) \in R$ and $(y, z) \in R$. Then, $(x, z) \in R \circ R$.

Since $(x, z) \in R \circ R$ and $R \circ R \subseteq R$, we see that $(x, z) \in R$.

Therefore, R is transitive.

(only if part)

Assume R is transitive.

Let $(x, z) \in R \circ R$.

Then, there must exist a y such that $(x, y) \in R$ and $(y, z) \in R$.

This implies $(x, z) \in R$ because R is transitive.

Therefore, $R \circ R \subseteq R$.

Relations and Graphs

- Definition:

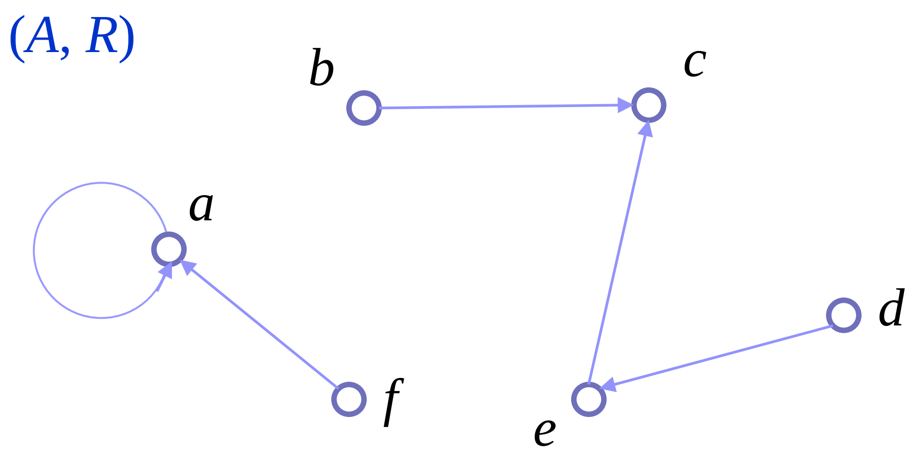
  Let R be a binary relation on a set A. Then (A, R) is called a directed graph, or digraph.

- Terminologies:

  For a graph (A, R),
  - A is called a set of nodes or a set of vertices.
  - R is called a set of arcs or a set of edges.

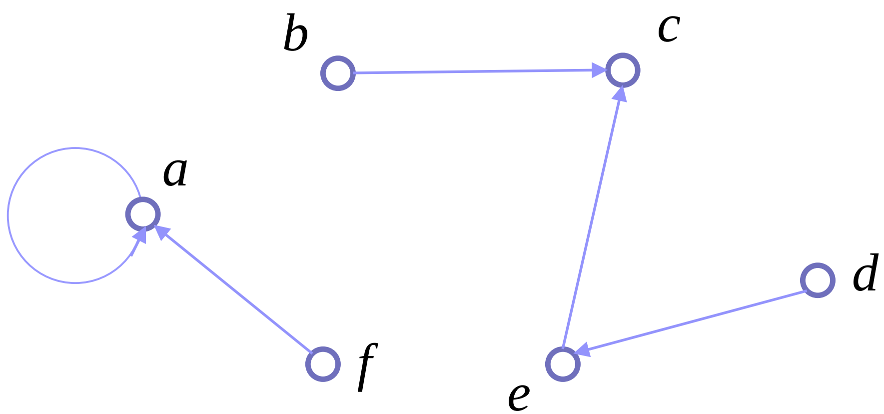
Example:

- $A = \{a, b, c, d, e, f\}$
- $R = \{(a, a), (b, c), (d, e), (e, c), (f, a)\}$



- Definitions: Let (A, R) be a digraph.
  - A sequence of nodes $x_0, x_1, \ldots, x_n$ is called a walk if $(x_i, x_{i+1}) \in R$ for all $0 \le i < n$, where n is the length of the walk.

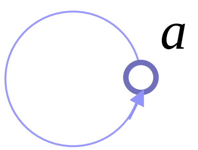
    Ex) $faaa$ is a walk. $aaaf$ is not a walk. $decb$ is not a walk.



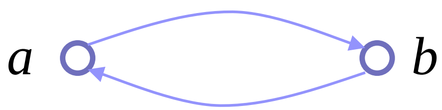
A node is a walk of length 0.

- Definitions: Let (A, R) be a digraph.
  - A walk $x_0, x_1, \ldots, x_n$ is called a path if $x_i \neq x_j$ for $i \neq j$, $0 \leq i, j \leq n$.
  - A walk $x_0, x_1, \ldots, x_n$ is called a cycle if $x_i \neq x_j$ for $i \neq j$, $0 \leq i, j \leq n$ except that $x_0 = x_n$.
  - A cycle of length 1 is called a loop.
  - A cycle of length 2 is called a sling.

loop aa



sling aba or bab



- Theorem: Let $G = (A, R)$ be a directed graph.
  - R is reflexive iff G contains a loop at every node.
  - R is irreflexive iff G has no loop.
  - R is symmetric iff for every edge $(x, y) \in R$ there is a sling between the nodes x and y.
  - R is antisymmetric iff G has no sling.
  - R is asymmetric iff G has no sling and no loop.
  - R is transitive iff there is a direct edge between two nodes x and y whenever there is a path of length two between x and y.

Closures

Example:

Let $A = \{a, b, c\}$ and $R = \{(a, b), (c, a)\}$. To make R reflexive, we need to add at least three tuples (a, a), (b, b), and (c, c).

- Definition:

  If R is a relation on a set A then the reflexive (symmetric, transitive) closure of R is a relation R' such that

  1. R' is reflexive (symmetric, transitive)
  2. $R \subseteq R'$
  3. If R'' is another reflexive (symmetric, transitive) relation such that $R \subseteq R''$, then $R' \subseteq R''$.

- Notations:

  Reflexive, symmetric, and transitive closure of R will be denoted by $r(R)$, $s(R)$, and $t(R)$, respectively.

- Theorem: Let R be a relation on a set A. Then,

  (a) $r(R) = R \cup E_A$.

  (b) $s(R) = R \cup R^c$.

  (c) $t(R) = \bigcup_{i=1}^{\infty} R^i$.

Proof of (a) $r(R) = R \cup E_A$

1. $R \cup E_A$ is obviously reflexive.
2. $R \subseteq R \cup E_A$
3. Let R'' be a reflexive relation such that $R \subseteq R''$.

   We need to show that $R \cup E_A \subseteq R''$.

   Since R'' is reflexive, $E_A \subseteq R''$.

   But $R \subseteq R''$, and thus $R \cup E_A \subseteq R''$.

Since $R \cup E_A$ satisfies all the three conditions in the definition of the reflexive closure of R, $R \cup E_A$ is the reflexive closure of R, i.e., $r(R) = R \cup E_A$. □

Proof of (b) $s(R) = R \cup R^c$

1. $R \cup R^c$ is symmetric because for every $(x, y) \in R \cup R^c$, $(y, x) \in R \cup R^c$.
2. $R \subseteq R \cup R^c$
3. Let R'' be a symmetric relation on A such that $R \subseteq R''$.

   We must show that $R \cup R^c \subseteq R''$.

   $R \subseteq R''$ is given.

   Since R'' is symmetric, $R^c \subseteq R''$.

   - Let $(x, y) \in R^c$. Then $(y, x) \in R$.

     Since $R \subseteq R''$, $(y, x) \in R''$.

     But, R'' is symmetric and so $(x, y) \in R''$.

     Therefore, $R^c \subseteq R''$. $\square$

- Lemma: Let R be a relation on a set A. Then,

$$R^n \subseteq t(R), \text{ for all } n \geq 1.$$

Proof of lemma:

(Basis Step) For $n = 1$,

$R \subseteq t(R)$ by the definition of $t(R)$.

(Inductive step)

Assume $R^n \subseteq t(R)$.

We want to prove that $R^{n+1} \subseteq t(R)$.

Note that $R^{n+1} = R \circ R^n$.

Proof of lemma:

Since $R \subseteq t(R)$, $R^n \subseteq t(R)$, and $t(R)$ is transitive, $R \circ R^n \subseteq t(R)$.

- Let $(x, z) \in R \circ R^n$.

  There must exist a y such that $(x, y) \in R$ and $(y, z) \in R^n$.

  But $R \subseteq t(R)$ and $R^n \subseteq t(R)$.

  Hence $(x, y) \in t(R)$ and $(y, z) \in t(R)$.

  Since $t(R)$ is transitive, $(x, z) \in t(R)$.

  Therefore, $R \circ R^n \subseteq t(R)$.

Therefore, $R^n \subseteq t(R)$ for all $n \geq 1$. □

Proof of (c) $t(R) = \bigcup_{i=1}^{\infty} R^i$

By the previous lemma $R^n \subseteq t(R)$ for all $n \geq 1$.

Thus, $\bigcup_{i=1}^{\infty} R^i \subseteq t(R)$.

Now we must show that $t(R) \subseteq \bigcup_{i=1}^{\infty} R^i$.

Obviously, $R \subseteq \bigcup_{i=1}^{\infty} R^i$.

All that remains to be shown now is that $\bigcup_{i=1}^{\infty} R^i$ is transitive.

Let $(x, y) \in \bigcup_{i=1}^{\infty} R^i$ and $(y, z) \in \bigcup_{i=1}^{\infty} R^i$.

Since $(x, y) \in \bigcup_{i=1}^{\infty} R^i$, there must exist an s such that $(x, y) \in R^s$.

Similarly, there must exist a t such that $(y, z) \in R^t$.

Then, $(x, z) \in R^s \circ R^t = R^{s+t}$ and $R^{s+t} \subseteq \bigcup_{i=1}^{\infty} R^i$.

Thus, $(x, z) \in \bigcup_{i=1}^{\infty} R^i$.

Therefore, $\bigcup_{i=1}^{\infty} R^i$ is transitive. □

- Theorem: Let R be a binary relation. Then,
  - (a) R is reflexive iff $r(R) = R$.
  - (b) R is symmetric iff $s(R) = R$.
  - (c) R is transitive iff $t(R) = R$.

Proof of (a)

(if part): R is reflexive if $r(R) = R$.

Assume $r(R) = R$.

Since the reflexive closure is reflexive, R is obviously reflexive.

Proof of (a)

(only if part): R is reflexive only if $r(R) = R$.

Assume R is reflexive.

Since $R \subseteq R$ and R is reflexive, $r(R) \subseteq R$ by the definition of the reflexive closure.

But $R \subseteq r(R)$ also by the definition of the reflexive closure.

Therefore, $R = r(R)$. $\square$

- Theorem: Let R be a binary relation.
  - (a) If R is reflexive then so are $s(R)$ and $t(R)$.
  - (b) If R is symmetric then so are $r(R)$ and $t(R)$.
  - (c) If R is transitive then so is $r(R)$.

Example:

$R = \{(a, b)\}$ is transitive.

$s(R) = R \cup R^c = \{(a, b), (b, a)\}$ is not transitive.

Proof of (a)

Assume R is a reflexive relation.

We prove that $s(R)$ is reflexive.

Since R is reflexive, $E \subseteq R$.

We know by the definition of $s(R)$ that $R \subseteq s(R)$.

Thus, $E \subseteq s(R)$.

Therefore, $s(R)$ is reflexive.

We can similarly show that $t(R)$ is reflexive.

$\square$

- Theorem: Let R be a binary relation.
  - (a) $rs(R) = sr(R)$.
  - (b) $rt(R) = tr(R)$.
  - (c) $st(R) \subseteq ts(R)$.

Proof of (a)

$$
\begin{aligned}
rs(R) &= r(R \cup R^c) \\
&= R \cup R^c \cup E = R \cup R^c \cup E \cup E = R \cup R^c \cup E \cup E^c \\
&= (R \cup E) \cup (R^c \cup E^c) \\
&= (R \cup E) \cup (R \cup E)^c \\
&= s(R \cup E) \\
&= sr(R) \quad \square
\end{aligned}
$$

- Lemma: Let R_1 and R_2 be two relations.

  If $R_1 \subseteq R_2$, then $s(R_1) \subseteq s(R_2)$ and $t(R_1) \subseteq t(R_2)$.

Proof of (c) $st(R) \subseteq ts(R)$

$R \subseteq s(R)$ by the definition of the closure.

$t(R) \subseteq ts(R)$ by the above lemma.

$st(R) \subseteq sts(R)$ again by the above lemma.

Since $s(R)$ is symmetric, $ts(R)$ is symmetric by the previous theorem.

Proof of (c) $st(R) \subseteq ts(R)$

Since $ts(R)$ is symmetric, it must be equal to its symmetric closure, by one of the previous theorems.

Hence, $sts(R) = ts(R)$.

Therefore $st(R) \subseteq ts(R)$. □

- A counter example for $ts(R) \subseteq st(R)$.

  Let $R = \{(a, b)\}$.

  Then, $t(R) = \{(a, b)\}$ and $st(R) = \{(a, b), (b, a)\}$.

  Also, $s(R) = \{(a, b), (b, a)\}$ and $ts(R) = \{(a, b), (b, a), (a, a), (b, b)\}$.

  We can see that $ts(R) \not\subseteq st(R)$.

Cardinality

- Definitions:
  - A set A has a cardinality n, denoted by $|A| = n$, if there exists a bijection from the set of the first n positive integers to A.
  - A set is said to be finite if it has a cardinality n, where n is a positive integer.
  - A set is said to be infinite if it is not finite.
  - A set A is said to be denumerable or countably infinite if there exists a bijection from the set of all positive integers to the set A.
  - A set is said to be countable if it is finite or countably infinite.

Example:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$.

Let $f : \mathbf{Z}^+ \to \mathbf{N}$, where $f(a) = a - 1$ for $a \in \mathbf{Z}^+$.

Since f is a bijection, $\mathbf{N}$ is countably infinite.

Example:

$\mathbf{E} = \{0, 2, 4, 6, \ldots\}$.

Let $f : \mathbf{Z}^+ \to \mathbf{E}$, where $f(a) = 2(a - 1)$ for $a \in \mathbf{Z}^+$.

Since f is a bijection, $\mathbf{E}$ is countably infinite.

- Theorem: The set $[0, 1]$ is not denumerable.

Proof (by a diagonalization argument)

Suppose the set $[0, 1]$ is denumerable.

Then, there exists a bijection $f : \mathbf{Z}^+ \to [0, 1]$.

Suppose we enumerate f as shown in the following table.

$\mathbf{Z}^+$	$[0, 1]$
1	$0\ .\ x_{11}\ x_{12}\ x_{13} \cdots$
2	$0\ .\ x_{21}\ x_{22}\ x_{23} \cdots$
3	$0\ .\ x_{31}\ x_{32}\ x_{33} \cdots$
$\vdots$	$\vdots$

Proof (by a diagonalization argument)

Now, take a number $0.y_1y_2y_3\cdots$, where $y_i \neq x_{ii}$.

This number is not the same as any number in the table.

So f is not surjective, which is a contradiction.

Therefore, there does not exist a bijection from $\mathbf{Z}^+$ to $[0, 1]$, and so $[0, 1]$ is not denumerable.

$\square$

- Implications:
  - We can see that $|\mathbf{Z}^+| \neq |\mathbf{R}|$.

    In fact, $|\mathbf{Z}^+| < |\mathbf{R}|$.

    Is there anything in-between? That is still an open question.
  - $|\mathbf{R}| = |\wp(\mathbf{Z}^+)| = 2^{|\mathbf{Z}^+|}$
  - $2^{|\mathbf{R}|}$ is yet another infinity.

- Theorem:

  Let A be an infinite set and let $\wp(A)$ be the power set of A. Then,

$$|A| < |\wp(A)| = 2^{|A|}.$$