Fundamental Principles of Counting

Rules of Sum and Product

- Useful for analyzing complicated problems through decomposing into parts and piecing together partial solutions

- The Rule of Sum

  If a first task can be performed in m ways (1),
  while a second task in n ways (2), and
  the two tasks cannot be performed simultaneously (3),
  then performing either task can be accomplished in any one of $m + n$ ways.

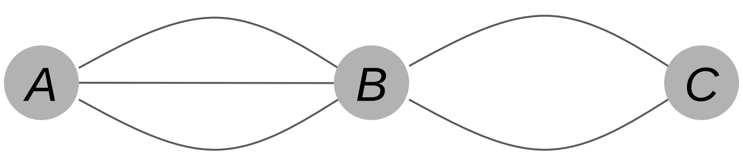
3 textbooks in English

2 textbooks in Korean

How many ways?

- The Rule of Product

  If a procedure can be broken down into first and second stages (1), and if there are m possible outcomes for the first stage (2) and if, for each of these outcomes, there are n possible outcomes for the second stage (3),
  then the total procedure can be carried out, in the designated order, in mn ways.



How many ways for $A \rightarrow C$?

Permutations

- Definition

  Given a collection of n distinct objects, any (linear) arrangement of these objects is called a permutation of the collection.

- In general, if there are n distinct objects and r is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size r for the n objects is

$$P(n,r) = n \times (n-1) \times (n-2) \times \cdots \times (n-(r-1))$$
$$= \frac{n!}{(n-r)!}$$

- Note that if repetitions are allowed, then by the rule of product there are n^r possible arrangements, with $r \geq 0$.
- In general, if there are n objects with n_1 indistinguishable objects of a first type, n_2 of a second type,…, n_r of an r-th type, where $n_1 + n_2 + \cdots + n_r = n$, then there are

$$\frac{n!}{n_1! n_2! \cdots n_r!}$$

(linear) arrangements of the given n objects. (Objects of the same type are indistinguishable.)

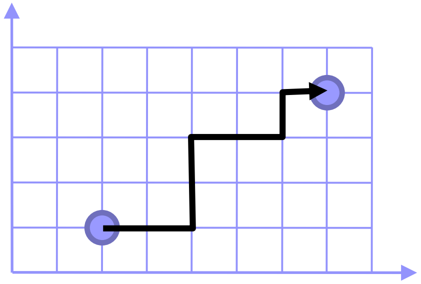
(*Ex*) PEPPER vs. $P_1E_1P_2P_3E_2R$

(Examples)

1. How many shortest paths from $(2, 1)$ to $(7, 4)$?

   Each path consists of 5 moves to the right and 3 ones upward. Thus the number of paths is

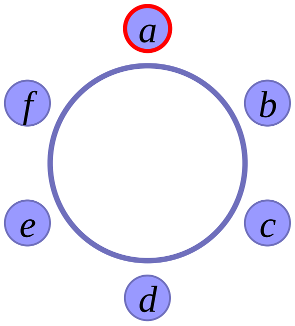
   $$8! \,/\, (5! \cdot 3!) = 56$$



2. How many circular arrangements of six people at a round table ?

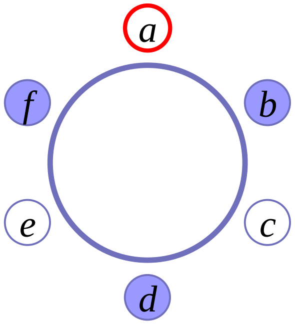
   $$6! \,/\, 6 = 120 \quad (5! = 120)$$

   (The rotation factor should be removed.)



3. How many sexually alternate arrangements of three males and three females at a round table ?

   $$3 \times 2 \times 2 \times 1 \times 1 = 12 \quad (2! \times 3! = 12)$$



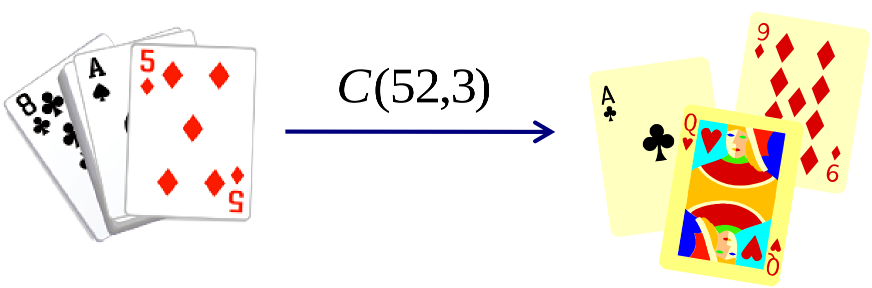
Combinations: The Binomial Theorem

- Combination: Selection with no reference to order
  - The number of combinations of size r from a collection of n distinct objects is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}, \quad 0 \le r \le n$$

☞ Selection of 4 from 10 distinct objects: X X X O X O O X O X

  - $C(n, r)$ is sometimes read as "n choose r".
  - Note that $C(n, 0) = 1$, for all $n \ge 0$, and $C(n, r) = C(n, n - r)$.



- The Binomial Theorem

  If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \cdots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k} = \sum_{k=0}^{n}\binom{n}{n-k}x^ky^{n-k}$$

$(x+y)$	$(x+y)$	$(x+y)$
x	x	x
x	x	o
x	o	x
x	o	o
o	x	x
o	x	o
o	o	x
o	o	o

- Corollary: For each integer $n > 0$,

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

$$(-1+1)^n = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n} = 0$$

- The Multinomial Theorem

  For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$ in the expansions of $(x_1 + x_2 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1! n_2! n_3! \cdots n_t!}$$

  where $0 \le n_i \le n$ for all $1 \le i \le t$ and $n_1 + n_2 + \cdots + n_t = n$.

- The coefficient of $x^k y^{n-k}$ in the expansion of $(x + y)^n$ is

$$\frac{n!}{k!(n-k)!}$$

  (Special case of multinomial theorem)

- Coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$

$$\overbrace{(x_1 + x_2 + \cdots + x_t)\ (x_1 + x_2 + \cdots + x_t)\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ (x_1 + x_2 + \cdots + x_t)}^{n \text{ times}}$$

Select n_1 places for x_1 from all the n places $\rightarrow C(n, n_1)$

Select n_2 places for x_2 from the remaining $n - n_1$ places

$\rightarrow C(n - n_1, n_2)$

Select n_3 places for x_3 from the remaining $n - n_1 - n_2$ places

$\rightarrow C(n - n_1 - n_2, n_3)$

. . . .

Select n_t places for x_t from the remaining $n - n_1 - n_2 - \cdots - n_{t-1}$ places

$\rightarrow C(n - n_1 - n_2 - \cdots - n_{t-1}, n_t) = C(n_t, n_t) = 1$

Proof:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{t-1}}{n_t}$$

$$=\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{(n-n_1-\cdots-n_{t-1})!}{n_t!0!}$$

$$=\frac{n!}{n_1!n_2!n_3!\cdots n_t!}$$

which is also written as $\binom{n}{n_1,n_2,n_3,\cdots,n_t}$

and is called a multinomial coefficient.

Combinations with Repetition

- An Example

On their way home from track practice, seven high school freshmen stop at a restaurant, where each of them has one of the following: a cheese-burger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible (from the viewpoint of the restaurant)?

Answer : The number of ways for selecting 7(r) of 4(n) different objects, *with repetition*, is $C(n + r - 1, r) = C(4 + 7 - 1, 7) = C(10, 7)$.

c, c, c, h, h, t, f $\rightarrow$ x x x | x x | x | x

c, c, h, h, h, t, f $\rightarrow$ x x | x x x | x | x

t, t, t, t, t, t, f $\rightarrow$ | | x x x x x x | x

- When we wish to select, with repetition, r of n distinct objects, we find that we are considering all arrangements of r x's and $(n-1)$ | 's and the total number of ways is

$$\frac{(n+r-1)!}{r!(n-1)!}=\binom{n+r-1}{r}$$

- From the viewpoint of the students, there are 4^7 different purchases (The problem becomes counting permutations with repetitions)

- It is crucial that we recognize the equivalence of the following:

1. The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \quad x_i \geq 0,\ 1 \leq i \leq n$$

2. The number of selections, with repetition, of size r from a collection of size n.
3. The number of ways r identical objects can be distributed among n distinct containers.

(*Ex*. 1) Consider the following program segment, where i, j, and k are integer variables. How many times is the **print** statement executed?

for $i = 1$ **to** 20 **do**
  for $j = 1$ **to** i **do**
    for $k = 1$ **to** j **do**
      print $(i * j + k)$;

1	2	3	12	13	20
	xx		x		
		x	x	x	

Answer : The selection of i, j, and k where the **print** statement is executed satisfies the condition $1 \le k \le j \le i \le 20$. In fact, any selection a, b, c $(a \le b \le c)$ of size 3, with repetitions allowed, from the list $1, 2, \ldots, 20$ results in one of the correct selections, here, $k = a$, $j = b$, $i = c$. Consequently the **print** statement is executed

$$\binom{20+3-1}{3} = \binom{22}{3} = 1540 \text{ times.}$$

Note that the answer is $C(n + 3 - 1, 3)$ in general.

for $i = 1$ **to** n **do**
  for $j = 1$ **to** i **do**
    for $k = 1$ **to** j **do**
      print$(i * j + k)$;

$$\binom{n+2}{3} = \frac{(n+2)!}{3!(n-1)!} = \frac{1}{6}n(n+1)(n+2)$$

Another Approach : The **print** statement is executed T times that can be represented as follows;

$$T = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j = \sum_{i=1}^{n} i(i+1)/2 = \frac{1}{2}\sum_{i=1}^{n}(i^2+i)$$

$$= \frac{1}{2}\left\{n(n+1)(2n+1)/6 + n(n+1)/2\right\}$$

$$= \frac{1}{6}n(n+1)(n+2)$$

(*Ex*. 2) In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?

Answer

(1) ways of banana distribution $(n = 4, r = 3)$: $C(4 + 3 - 1, 3)$

(2) ways of orange distribution $(n = 4, r = 6)$: $C(4 + 6 - 1, 6)$

$\therefore$ By the rule of product,

the total # of ways is $C(6,3) \times C(9,6) = 1680$.

(*Ex*. 3) A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

Answer

$$12! \cdot C(11 + 12 - 1, 12).$$