# Random variables

# Objectives

1. To learn the definition of a random variable
2. To learn the definition of a probability mass function
3. Discrete random variables

# Random Variables

[Definition] **A Random Variable is a real-valued function of the elements of a sample space**, $S$.

- Given an experiment, $E$, with sample space, $S$, the random variable $X$ maps each possible outcome $\zeta \in S$, to a real number $f(\zeta)$ as specified by some rule

- We cannot describe a random variable $X$ by stating its value; rather, we must give it a probabilistic description by stating the probabilities that the variable $X$ takes on a specific value or values.

  → $\mathrm{P}(X = 3)$ , $\mathrm{P}(X > 8)$

- Example 2.1: Tossing two dice
  - X: the sum of two dice
  - $P(X = 7) = P\big((1,6), (2,5), \ldots, (6,1)\big) = \frac{6}{36}$

# Random Variables – example

- Example 2.3: Suppose that we toss a coin having a probability *p* of coming up heads until the first head appears. Let *N* denote the number of flips required.
  - $P(N = 1) = P\{H\} = p$
  - $P(N = 2) = P\{T, H\} = (1 - p)p$

    $\vdots$
  - $P(N = n) = P\{\underbrace{T, T, \dots, T}_{n\text{-}1}, H\} = (1 - p)^{n-1}p$

# Discrete random variables

- Discrete random variables: Random variables on either a finite or a countable number of possible values

- The **Probability Mass Function** (PMF), $\boldsymbol{p(a) = P(X = a)}$ of a random variable, *X*, is a function that assigns a probability to each possible value of the random variable, *X*.

- Example
  - Let $X$ be a random variable for the number of times tails occur in *n* coin-flipping trials. Then

$$P(X = k) = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

# Cumulative distribution function (cdf)

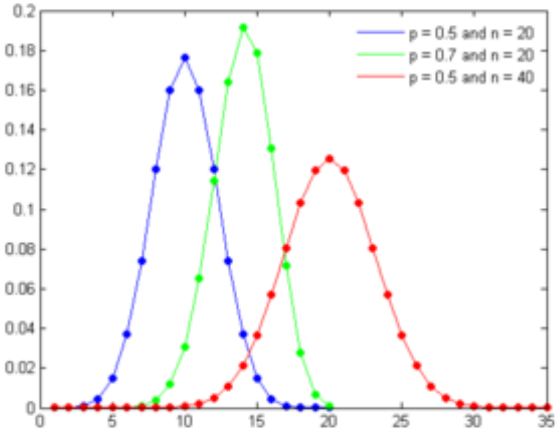
- $F(a) = P(X \leq a) = \sum_{all\ x_i \leq a} p(x_i)$
- Example:
  - Probability mass function for *X*: $p(1) = \frac{1}{2}, p(2) = \frac{1}{3}, p(3) = \frac{1}{6}$
  - The cumulative distribution function *F* of *X*

# Bernoulli Random Variable

- Experiments with 2 possible outcomes
- $P(0) = 1 - p, P(1) = p$
- Example ?

# Binomial Random Variable

- **Repeating a Bernoulli trial** *n* times
  Where the outcome of each trial is **independent**




- $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 0,1,2,\cdots,n$
- $\sum_{k=0}^{n} P_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$
  - Theorem of Binomial Coefficients

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- Example 2.7
  - For a binominal random variable *X* with $p = 0.1, n = 3$
  - $P(X \leq 1) =$

# Geometric Random Variable

- $X$ with parameter $p$
  - the number of trials required until the first success

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  - $P(N = 1) = P\{H\} = p$
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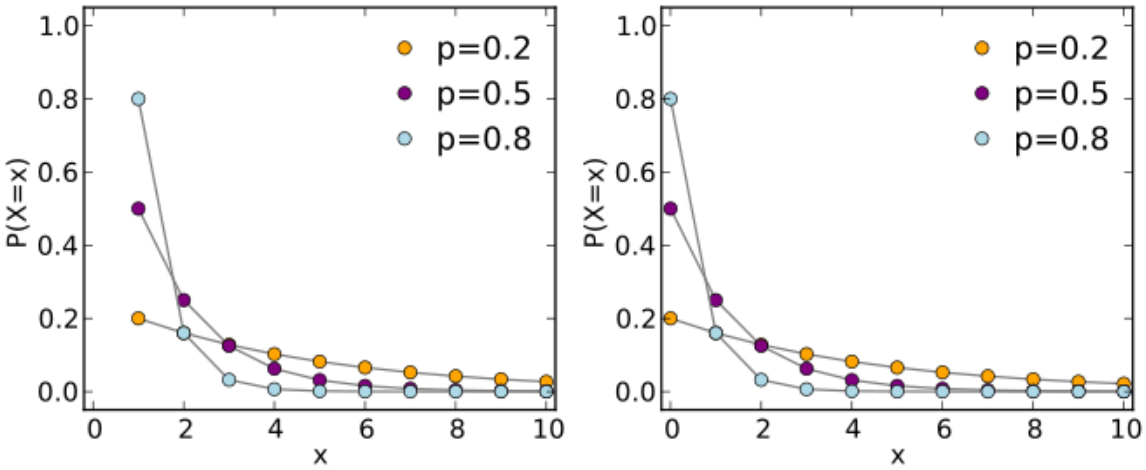
$$\vdots$$

  - $P(N = n) = P\{\underbrace{T, T, \ldots, T}_{n-1}, H\} = (1 - p)^{n-1}p$

# Geometric Random Variable - cdf

- Cumulative distribution function of a geometric random variable $X$
  - $F(x) = P(X \leq x) = 1 - (1-p)^x$

- Probability mass functions



# Negative binomial

- $X$ with parameter $(p, r)$
  - the number of trials required until the $r^{\text{th}}$ success
  - $P(X = n) = \binom{n-1}{r-1} p^{r-1}(1-p)^{n-r}p = \binom{n-1}{r-1} p^{r}(1-p)^{n-r}$

# Poisson Random Variable

- $X$ with parameter $\lambda$
  - $P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!},\ i = 0,1,\ldots$
- May be used to approximate a binomial random variable when *n* is large and *p* is small.

# Approximation of a binomial random variable

$$\lim_{n\to\infty} np = \lambda$$

$$\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

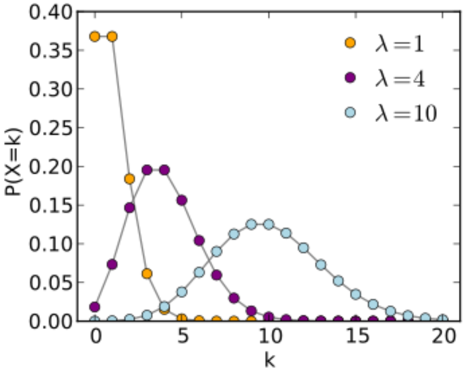
$$\lim_{n\to\infty} P(X = i) = \lim_{n\to\infty} \binom{n}{i} p^i (1-p)^{n-i} = \lim_{n\to\infty} \frac{n!}{(n-i)!\, i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-i+1}{n}\right)}_{\approx 1} \left(\frac{\lambda^i}{i!}\right)\left(1 - \frac{\lambda}{n}\right)^n \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-i}}_{\approx 1}$$

$$\boldsymbol{P(X = i) = \frac{\lambda^i}{i!} e^{-\lambda}, \qquad i = 0, 1, 2, \cdots}$$

# Poisson Random Variable - applications

- $P(k) = \frac{\lambda^k}{k!} e^{-\lambda},\ k = 0, 1, 2, \cdots$
  - $\sum_{k=0}^{\infty} P(k) = 1$ ; From Taylor Series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$




- Describes the behavior of many physical phenomena.
  - The number of cars that pass through a certain point on a road (sufficiently distant from traffic lights) during a given period of time.
  - The number of spelling mistakes one makes while typing a single page.
  - The number of phone calls at a call center per minute.
  - The number of times a web server is accessed per minute.
  - The number of mutations in a given stretch of DNA after a certain amount of radiation.

# Poisson Random Variable - examples

- Example 2.12: average 3.2 $\alpha$-particles are given off per second ($\lambda$).
  - $P(X \leq 2) =$