Introduction to Graph Theory

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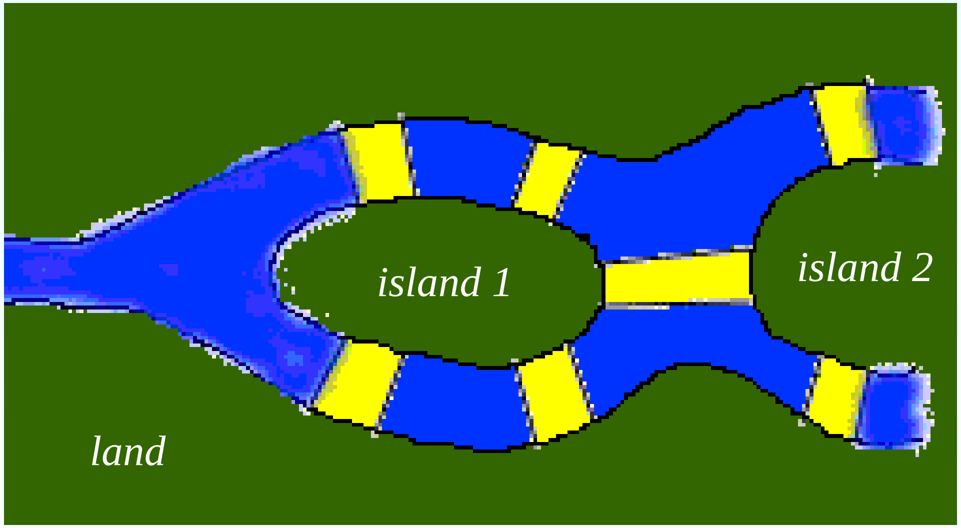
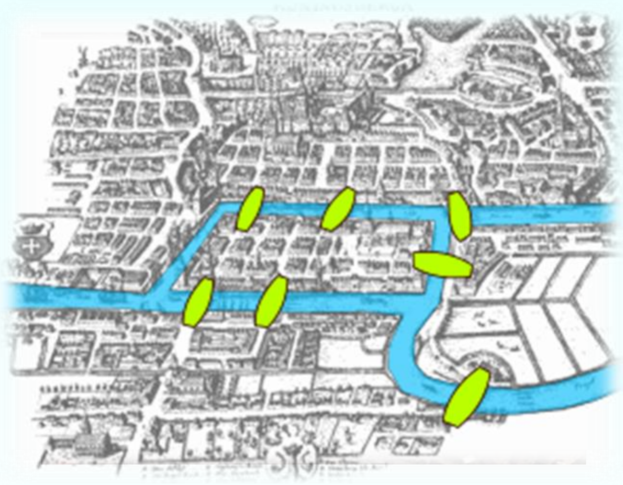
Graph Application (1)

Seven bridges of Königsberg





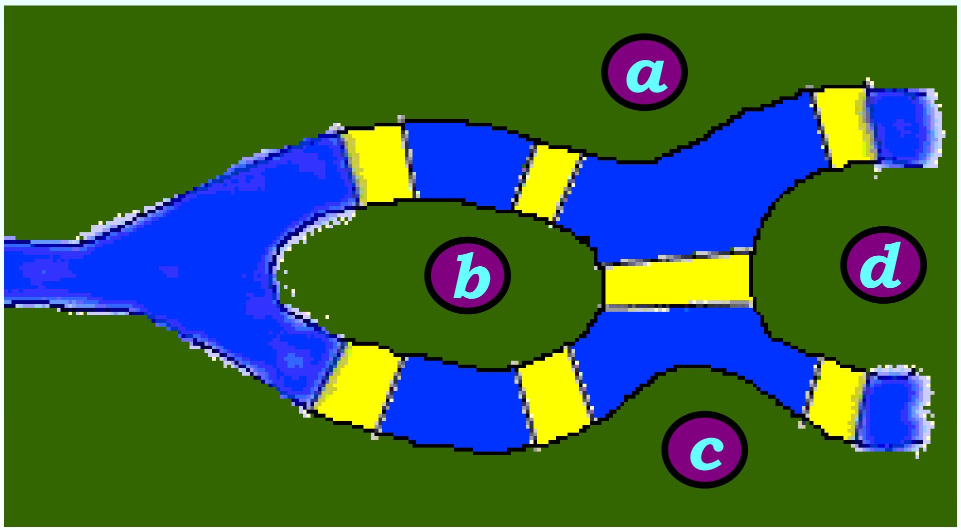
Königsberg in Euler’s time

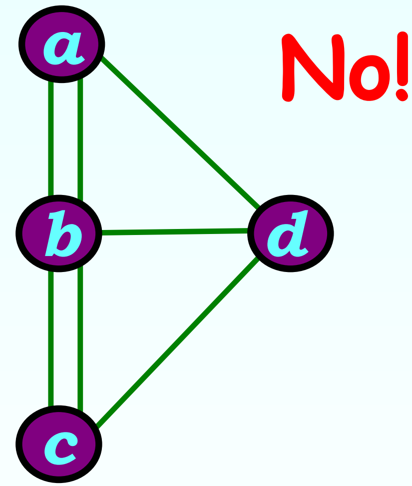
Problem (1)

Can we cross each bridge exactly once ?

Graph Application (1)

Seven bridges of Königsberg

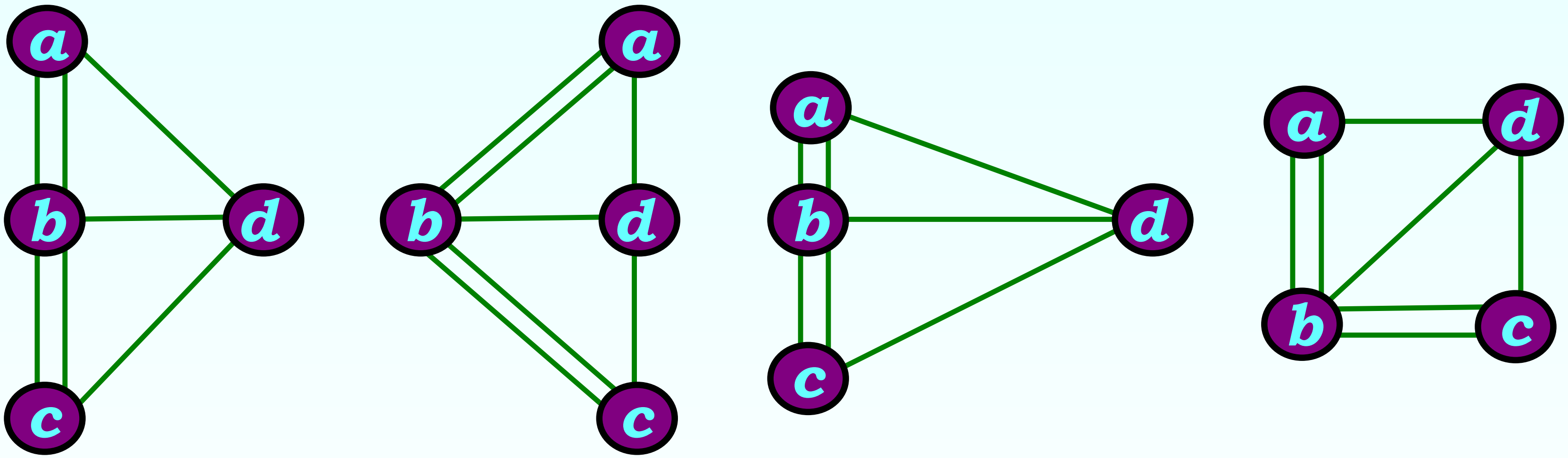


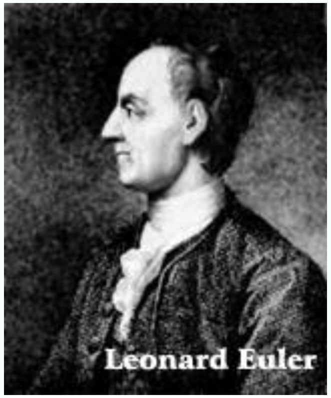


Topology: Rubber Sheet Geometry

Study the properties of shapes that remain the same when the shapes are stretched or compressed

Graph Application (1)



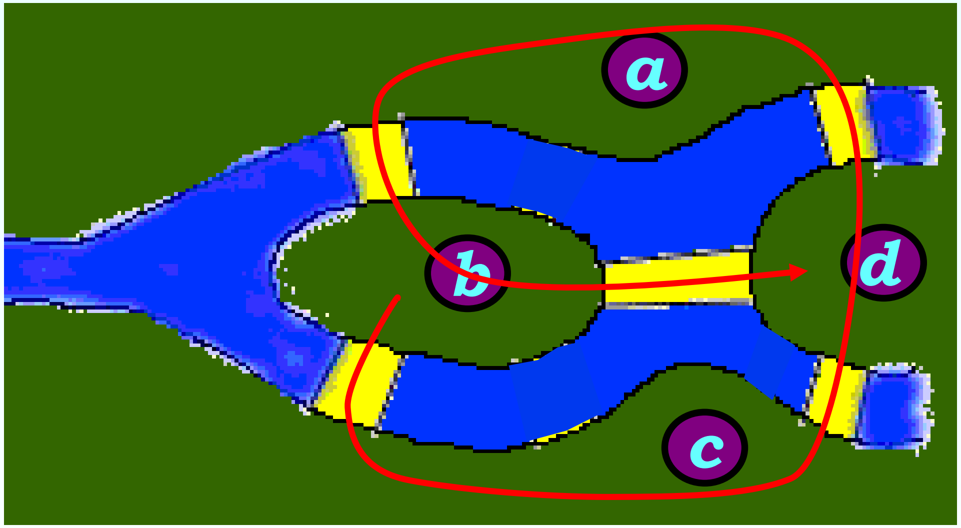


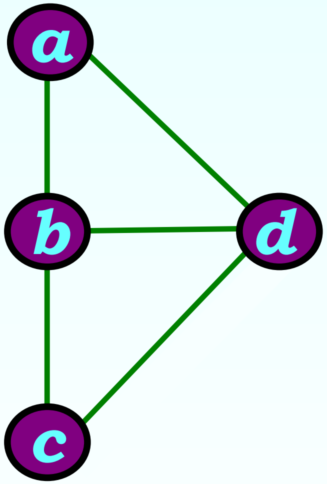
Topology: Rubber Sheet Geometry

Study the properties of shapes that remain the same when the shapes are stretched or compressed

Graph Application (1)

Five bridges of Königsberg



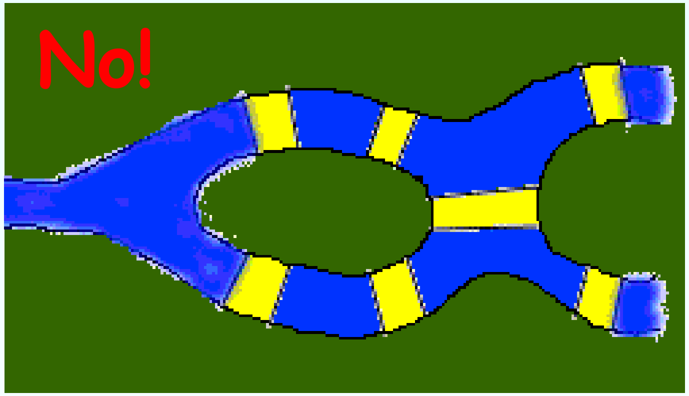


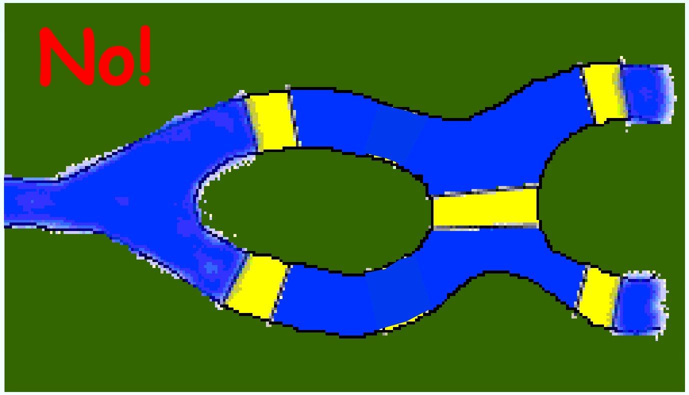
Theorem (1)

If a graph has exactly two odd vertices, it has an Euler trail.

Graph Application (1)

Seven bridges of Königsberg Five ?

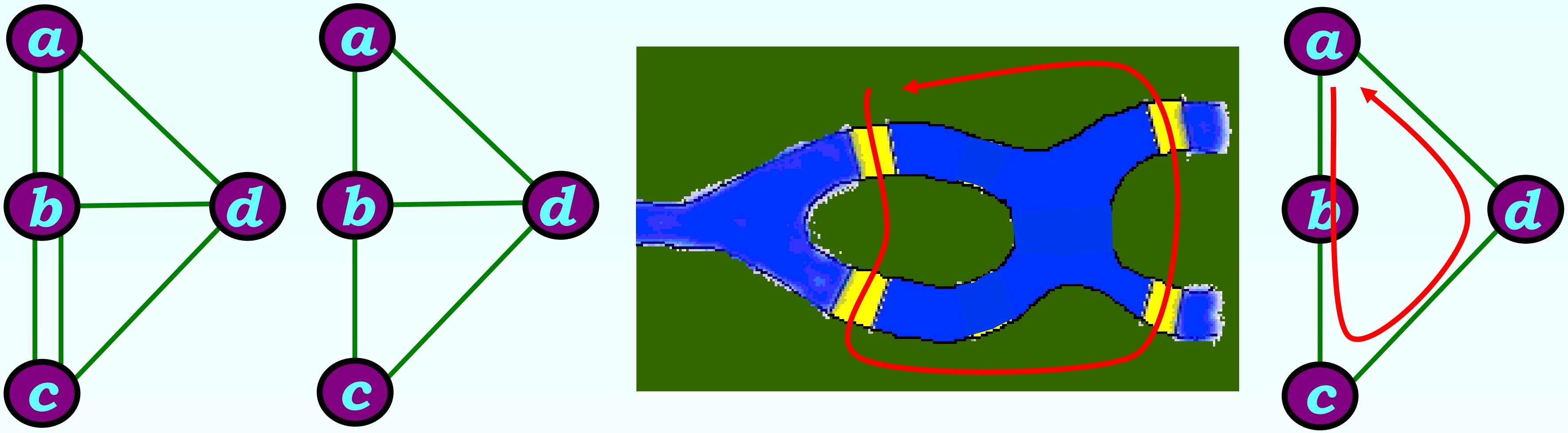




Problem (2)

Can we cross each bridge exactly once and then return the starting point?

Graph Application (1)

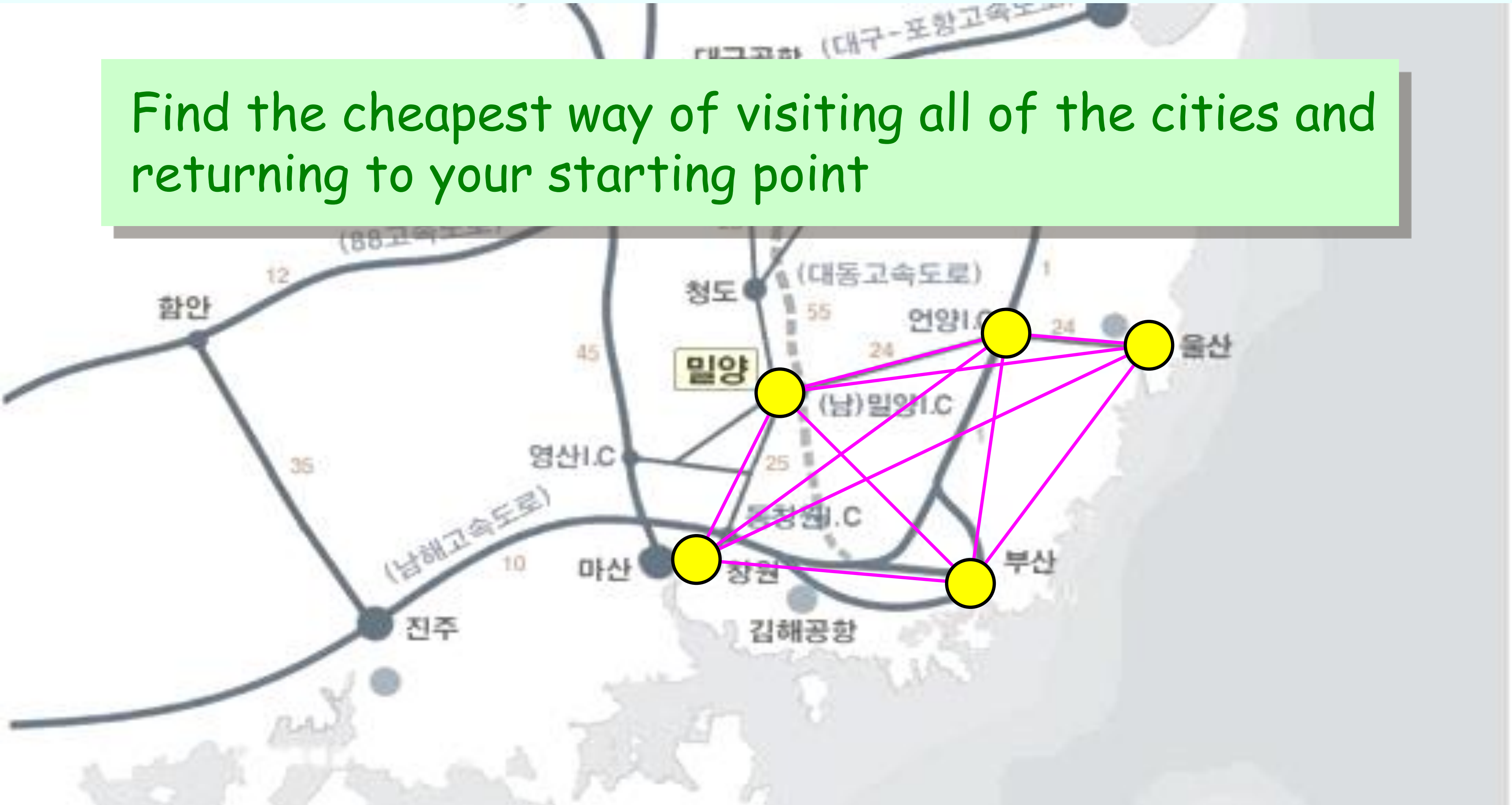


Theorem (2)

If every vertices of a graph have even degree, it has an Euler circuit.

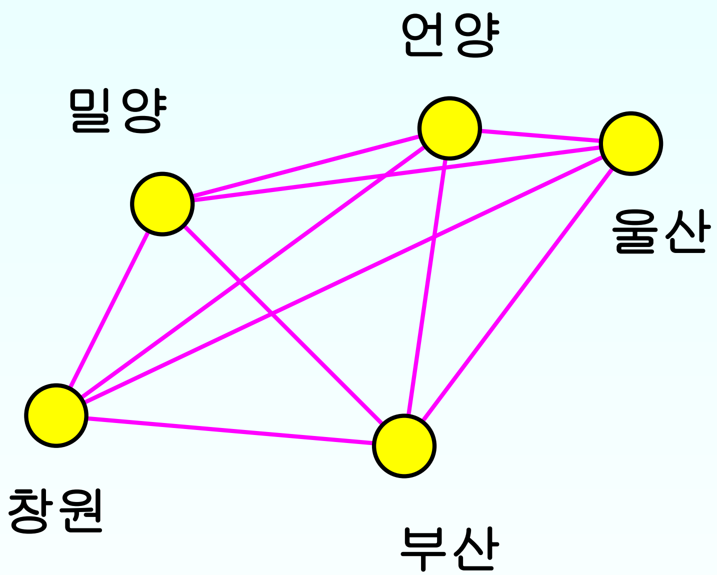
Graph Application (2)

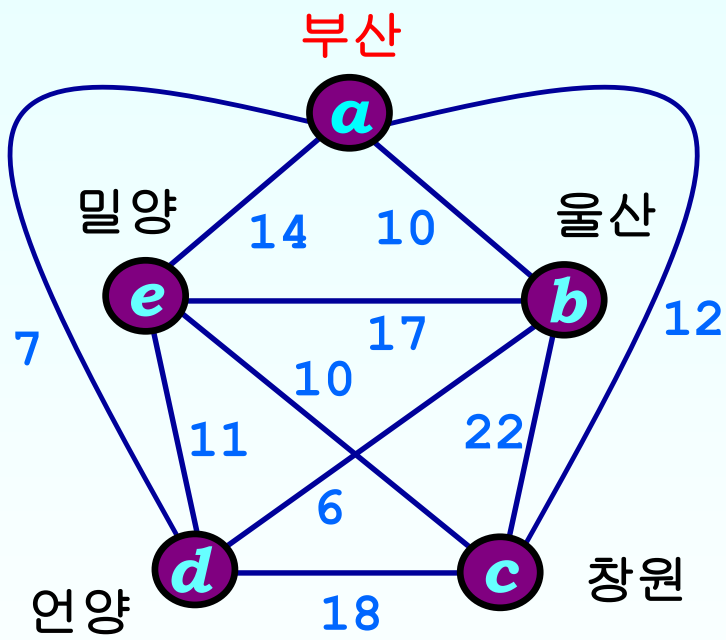
Traveling Salesman Problem



Find the cheapest way of visiting all of the cities and returning to your starting point

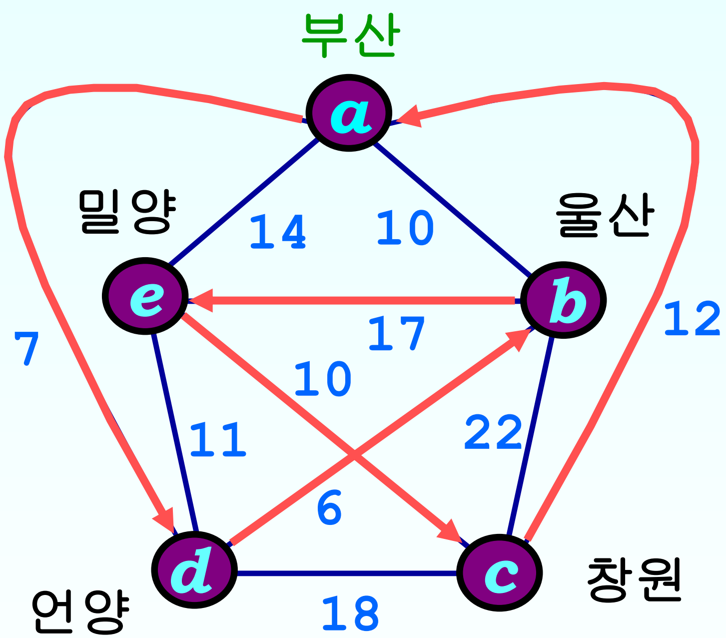
Graph Application (2)

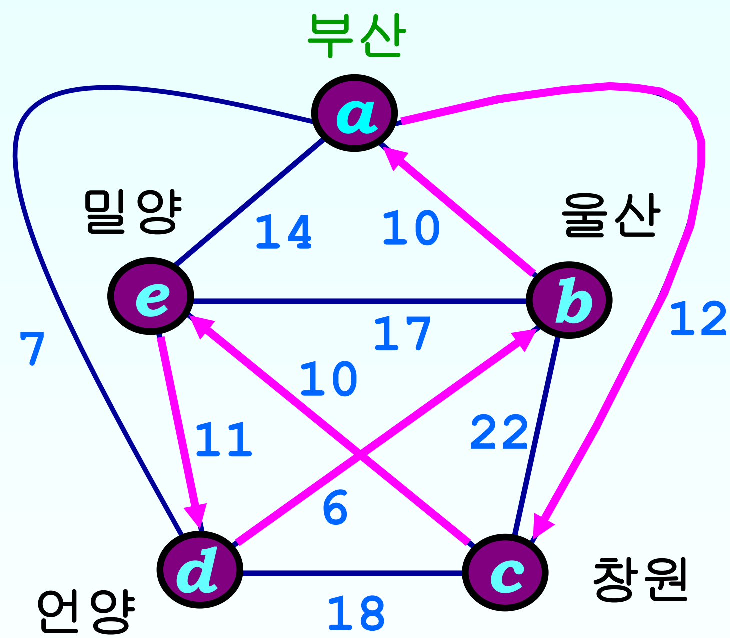




Find the cheapest way of visiting all of the nodes and returning to the starting node

Graph Application (2)





A Greedy Solution (52)

; a → d → b → e → c → a

7 6 17 10 12

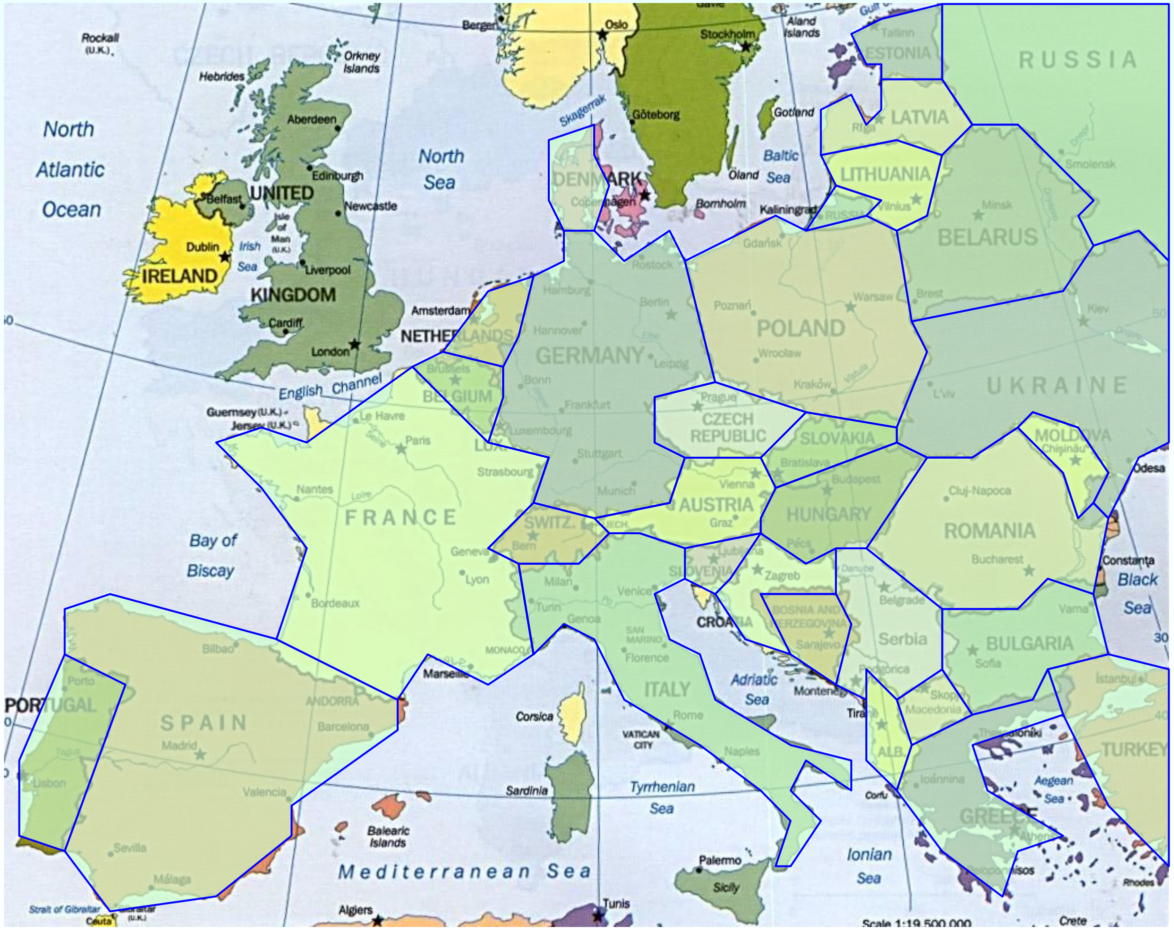
A Better Solution (49)

; a → c → e → d → b → a

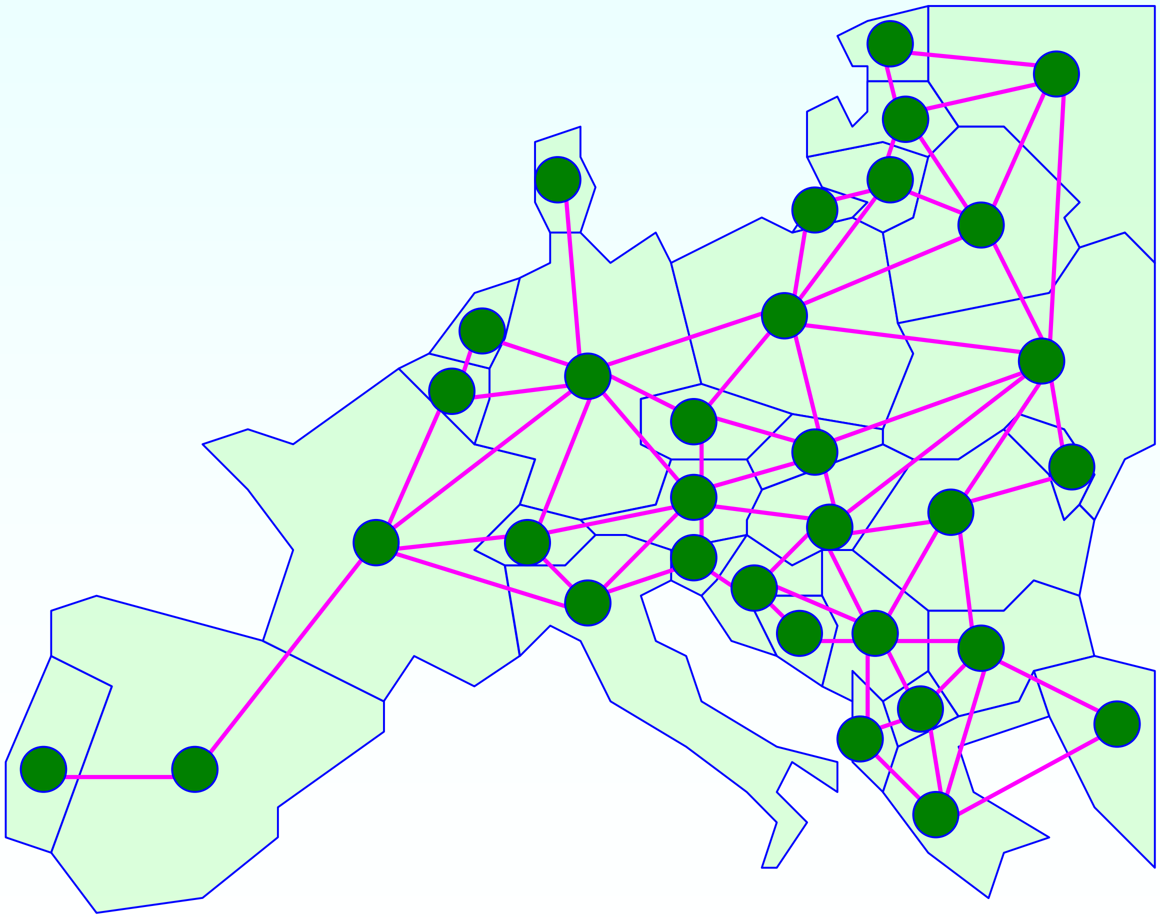
12 10 11 6 10

Graph Application (3)

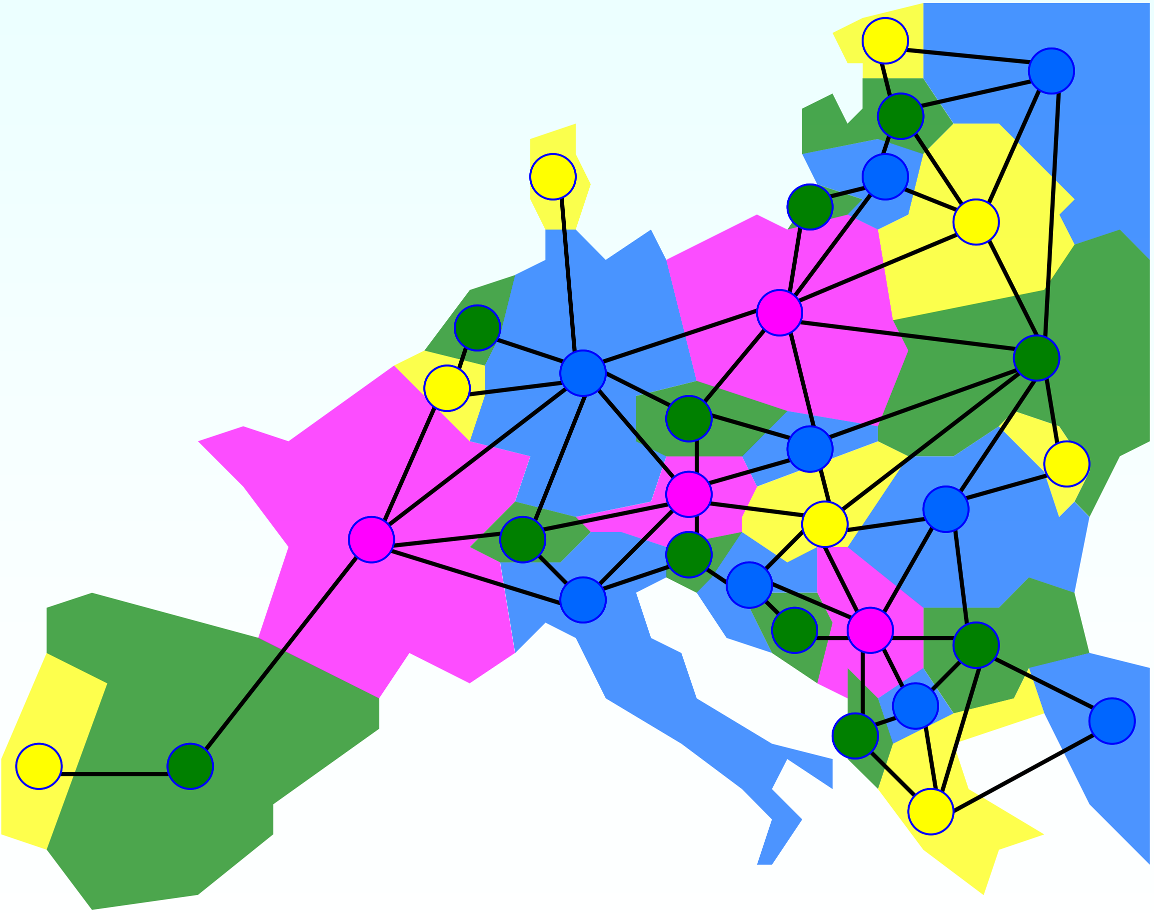
Map Maker's Problem: Graph Coloring



Graph Application (3)

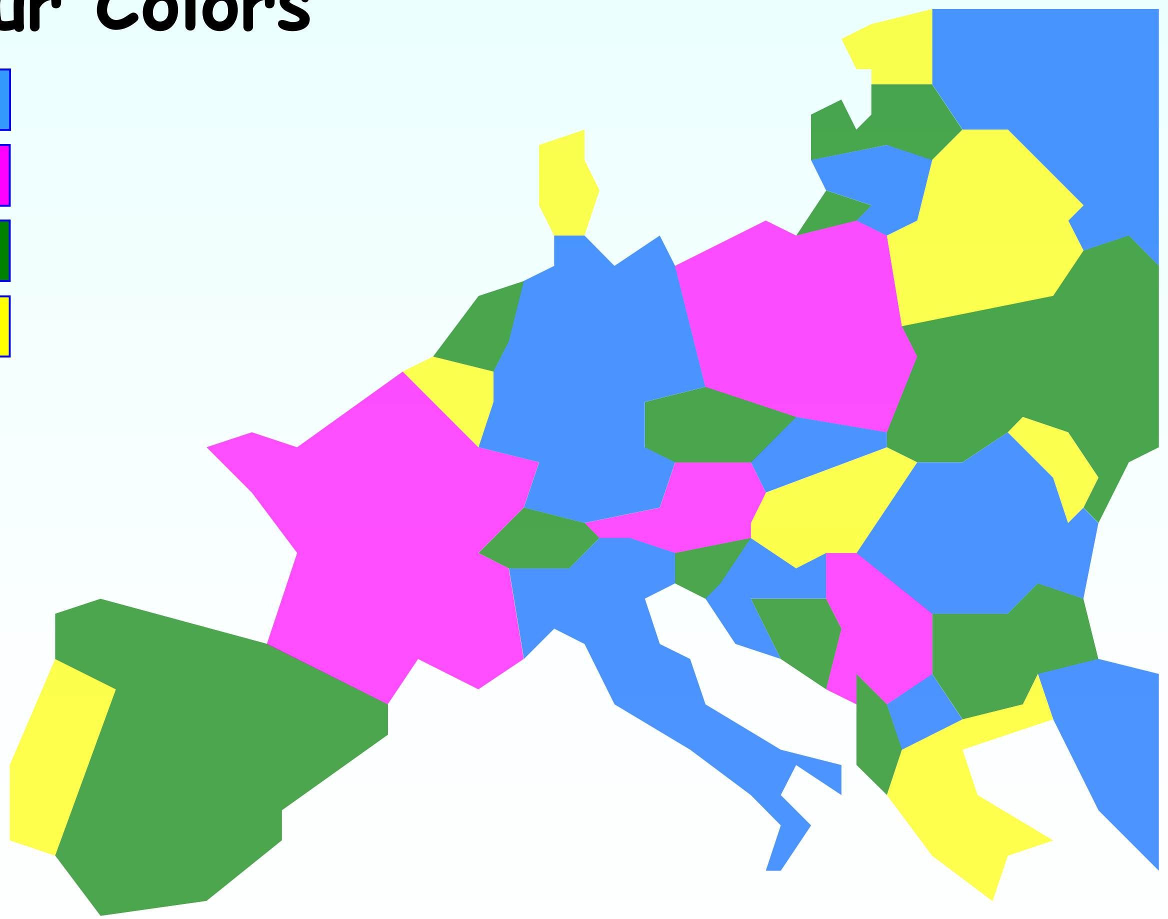


Graph Application (3)



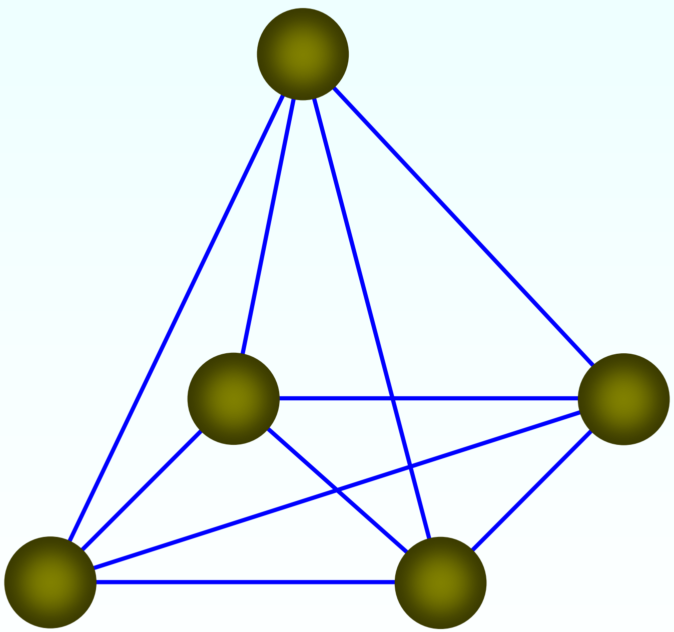
Graph Application (3)

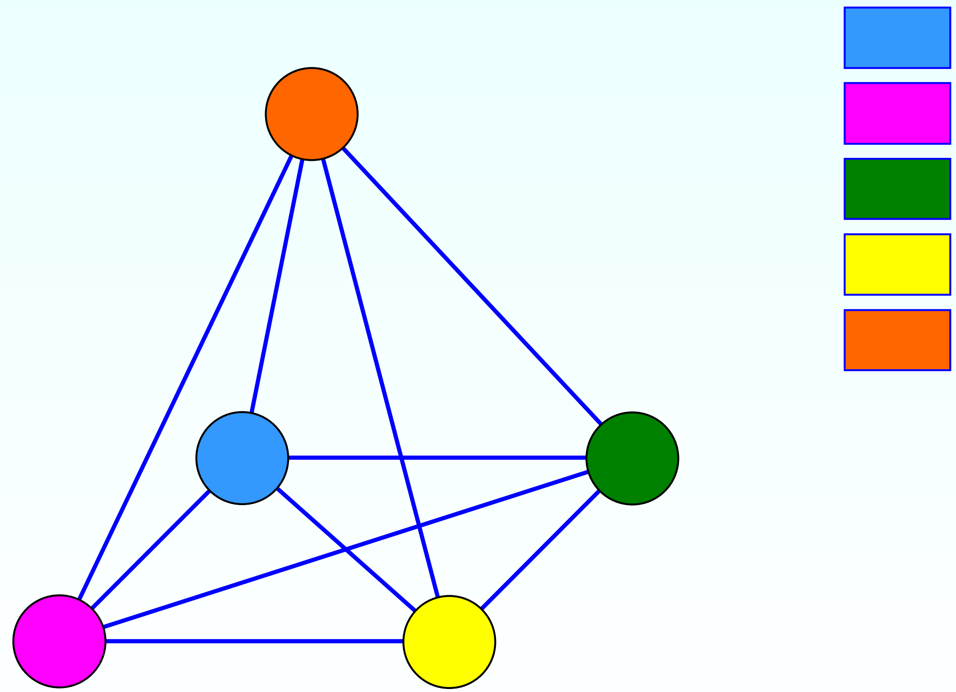
Four Colors



Graph Application (3)

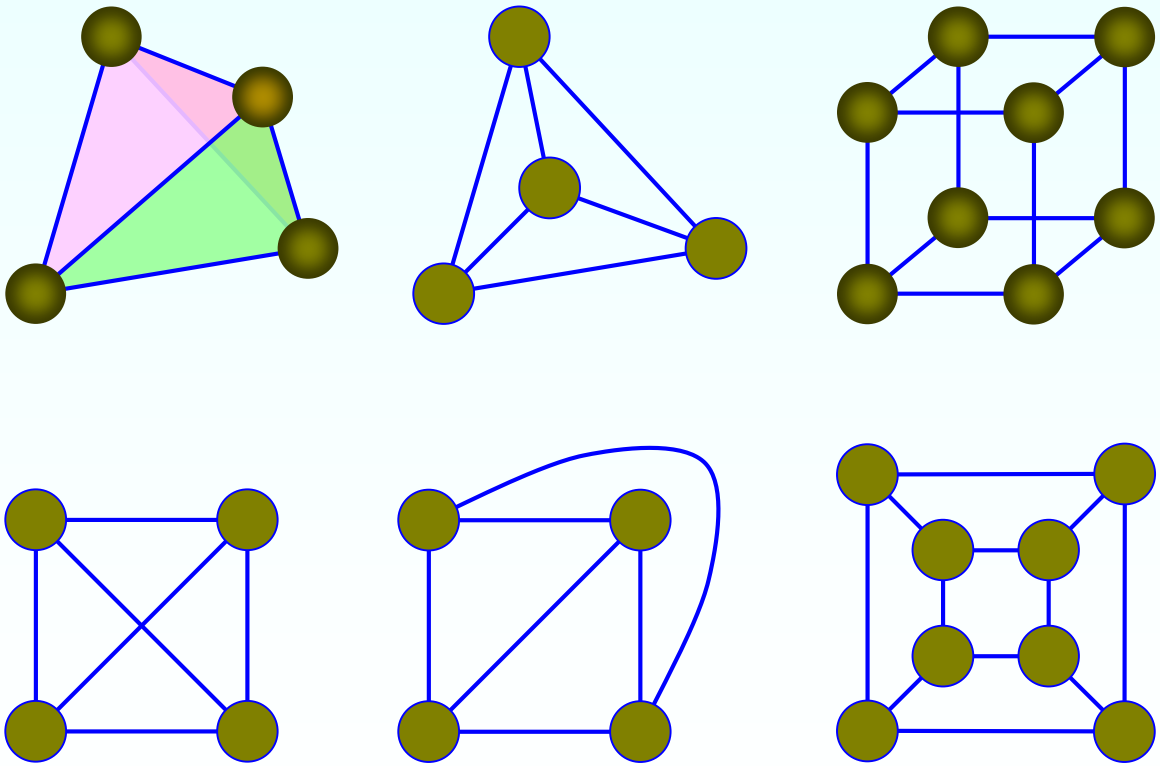
Five Colors



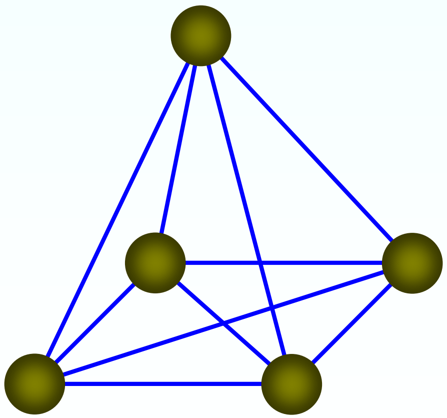


Graph Application (4)

Planar Graph

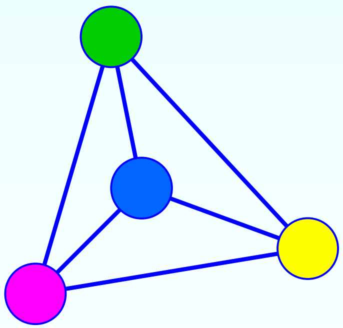


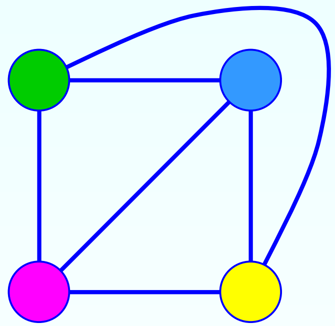
Planar?

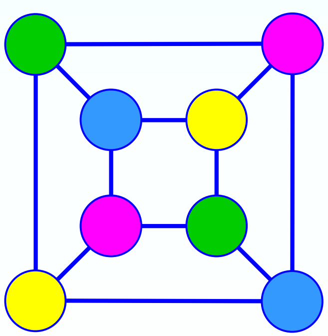


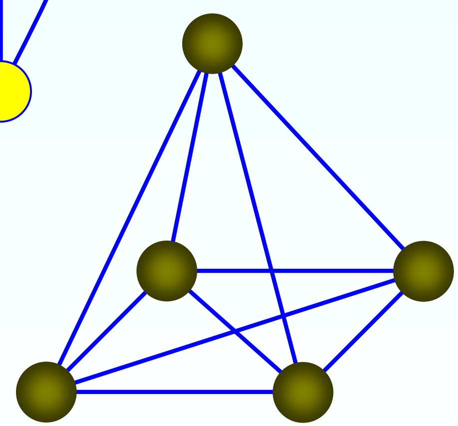
Graph Application (4)

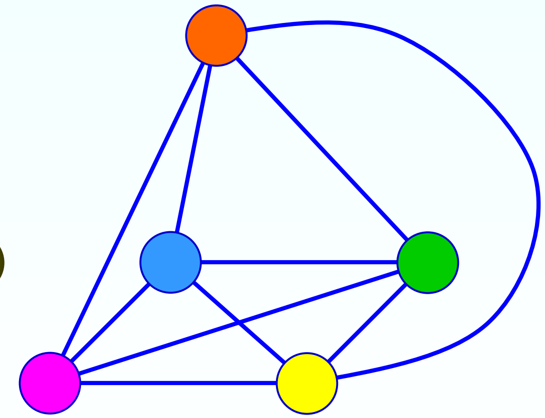
Planar graph is 4-colorable











Basic Definitions

Section 11.1

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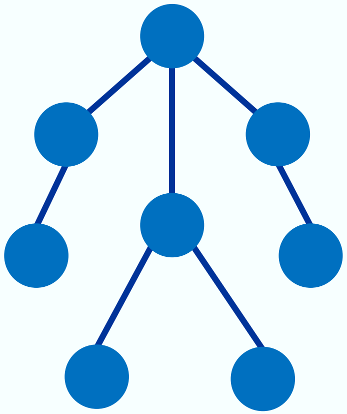
Definition !!!



A tree is an acyclic connected graph.

in Wikipedia

Definition of Graphs

- **Graph** $G = (V, E)$: a pair of V and E

  $V = \{v_1, v_2, \ldots, v_n\}$: a set of vertices

  - v_i : vertex, node, 정점

  $E = \{e_1, e_2, \ldots, e_n\}$: a set of edges between two vertices

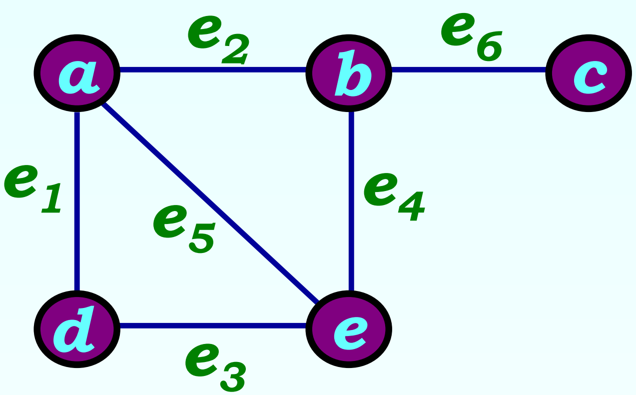
  - e_i : edge, arc, 간선

- **Directed** graph vs. **Undirected** graph

  Depending on the definition of E

  - Ordered pair $e_k = (v_i, v_j)$ for a directed graph (with direction)
  - Unordered pair $e_k = \{v_i, v_j\}$ for an undirected graph

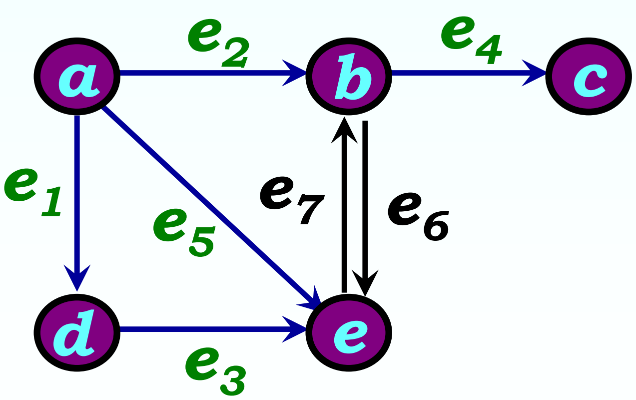
Examples of Graphs



Undirected Graph $G = (V, E)$

$V = \{ a, b, c, d, e \}$

$E = \{ \{a,d\}, \{a,b\}, \{d,e\}, \{b,e\}, \{a,e\}, \{b,c\} \}$



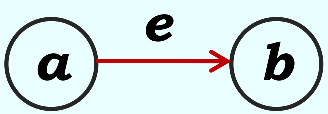
Directed Graph $G = (V, E')$

$V = \{ a, b, c, d, e \}$

$E' = \{ (a,d), (a,b), (d,e), (b,c), (a,e), (b,e), (e,b) \}$

Definitions

- An edge e = (a,b)



  - e is incident with vertices a and b
    e is incident from (out of) a ~로 부터 투사된다.
    e is incident into (on) b ~로 입사한다.
  - a is adjacent to b 인접하다.
    b is adjacent from a
  - a is the source (origin) of edge e
    b is the terminus (terminating vertex) of edge e
  - (a,a), $\{a,a\}$: a loop
  - Isolated vertex : it has no incident edges

Walk

- x-y walk

  Let x,y be vertices in an undirected graph $G=(V,E)$.

  An x-y walk in G is a (loop-free) finite alternating sequence of vertices and edges from G

  $$x \equiv x_0 \quad e_1 \quad x_1 \quad e_2 \quad x_2 \ldots e_{n-1} \quad x_{n-1} \quad e_n \quad x_n \equiv y$$

- Length of walk

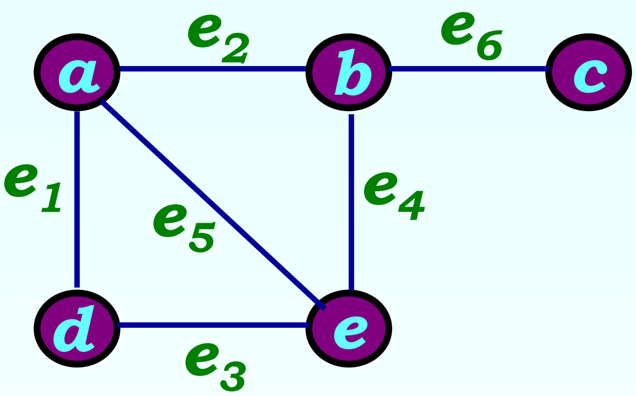
  The number of edges in the walk

  (a walk with zero length ➔ a trivial walk)

- Closed walk vs. Open walk

  Depending on whether $x = y$ or not

Examples of *x*-*y* walk



a-*c* walks

- a e_5 e e_4 b e_6 c
- a e_2 b e_6 c

Lengths of the *a*-*c* walks

- 3
- 2

A closed walk

- a e_5 e e_4 b e_2 a

Trail, Circuit, Path, Cycle

- x-y **trail**

  No edge in the x-y walk is repeated

- **Circuit**

  A closed x-x trail is called a circuit

  It should have at least one edge

- x-y **path**

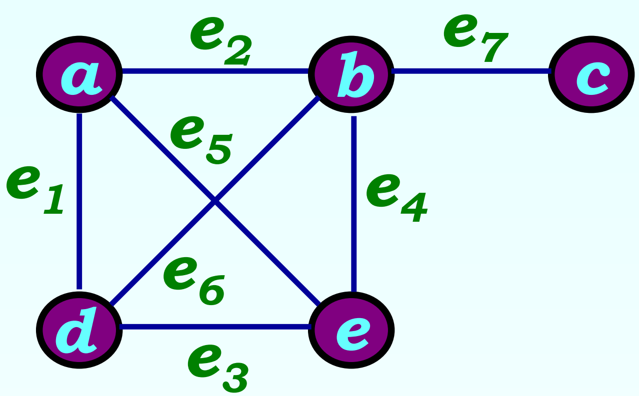
  No vertex in the x-y walk occurs more than once

- **Cycle**

  A closed x-x path is called a cycle

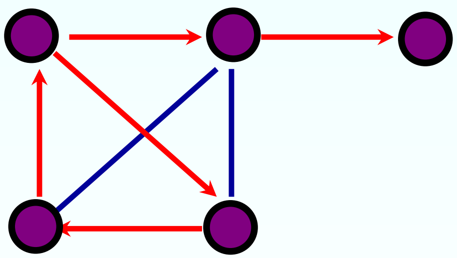
  It should have at least three distinct edges

Examples of Circuit & Cycle



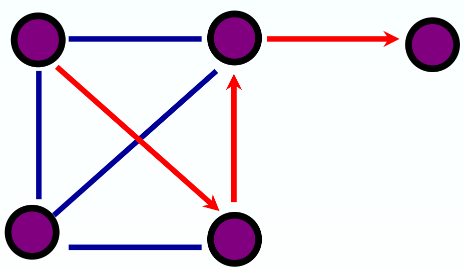
A a-c walk (not a-c trail)

; $a \;\; e_5 \;\; e \;\; e_3 \;\; d \;\; e_1$
$a \;\; e_5 \;\; e \;\; e_4 \;\; b \;\; e_7 \;\; c$



A a-c trail (not a-c path)

; $a \;\; e_5 \;\; e \;\; e_3 \;\; d \;\; e_1$
$a \;\; e_2 \;\; b \;\; e_7 \;\; c$



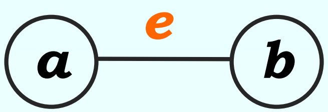
A a-c path

; $a \;\; e_5 \;\; e \;\; e_4 \;\; b \;\; e_7 \;\; c$

A circuit and a cycle

; $a \;\; e_1 \;\; d \;\; e_3 \;\; e \;\; e_4 \;\; b \;\; e_2 \;\; a$

Conventions

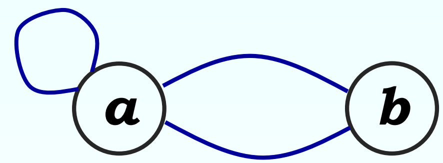


a e *b* e *a*

Circuit ?
Cycle ?

Circuits

- Presence of at least one edge
- Circuit with only one edge → loop
- Circuit with two edges → may occur only in multigraphs



Cycles

- Presence of at least three distinct edges

Summary (1)

	Repeated Vertices	Repeated Edges
Open Walk	yes	yes
Closed Walk	yes	yes
Trail	yes	no
Circuit	yes	no
Path	no	no
Cycle	no	no

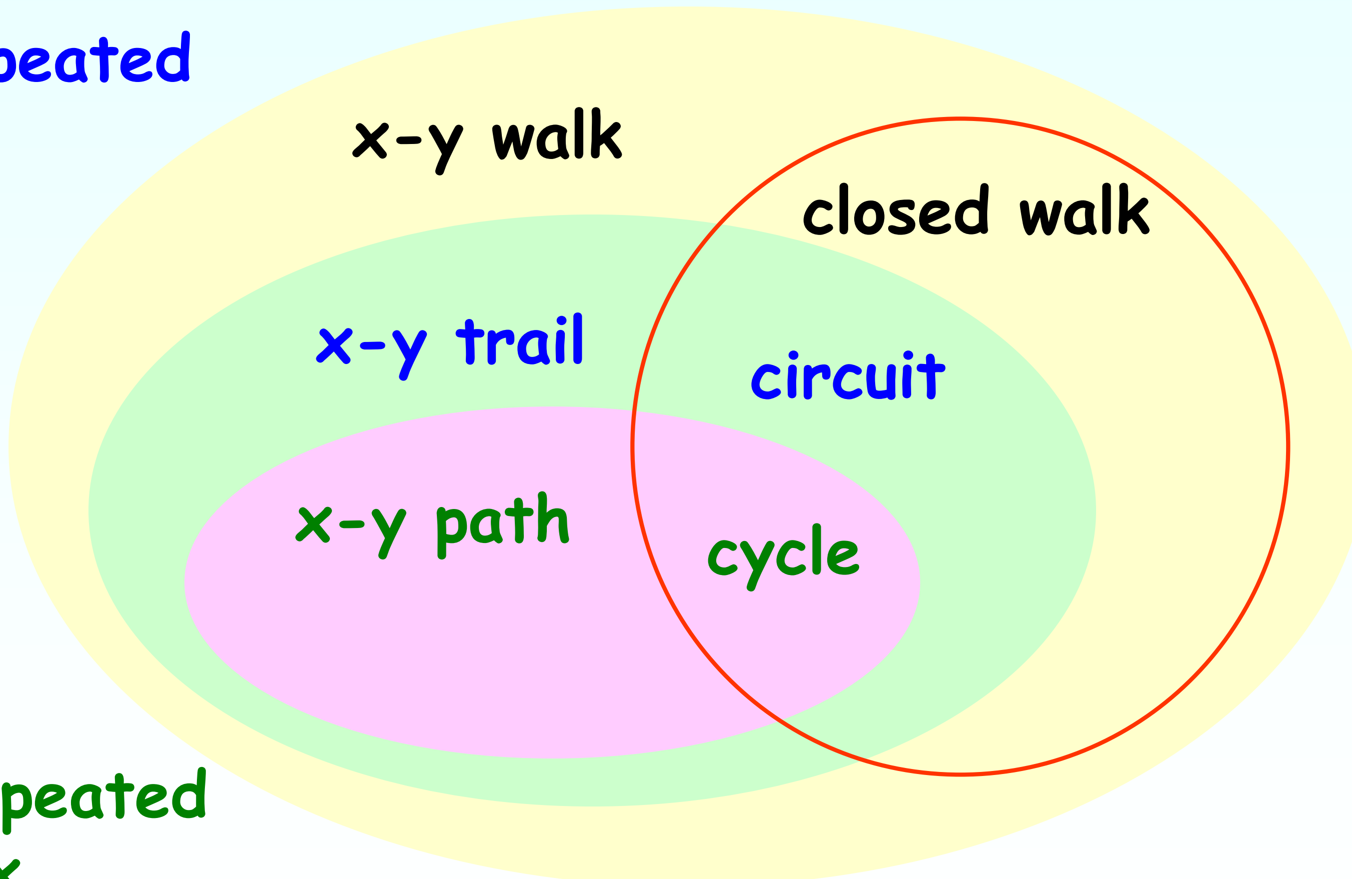
A path is a trail. Is it ?

- **In directed graph**

  directed walk, directed trail, directed path, ……

Summary (2)

No repeated edge



No repeated vertex

Theorem 11.1

- $\exists_{(exist)}$ trail $\Rightarrow$ $\exists_{(exist)}$ path

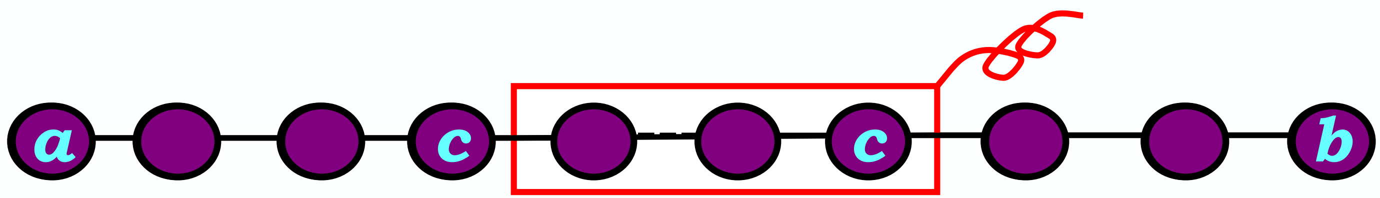
  Let $G = (V, E)$ be an undirected graph, with $a, b \in V$, $a \neq b$. If there is a trail from a to b, then there exists a path from a to b

Proof : Consider one of *shortest* a-b trail.

If the trail is not a path, then at least one vertex is repeated

Then the trail is not one of shortest trail

This contradiction shows that the trail is a path



Connected Components

- **Connected Graph** (*cf.* disconnected one)

  We call an undirected graph $G = (V,E)$ connected if there is a path between any two distinct vertices of G

- **Connected Components**

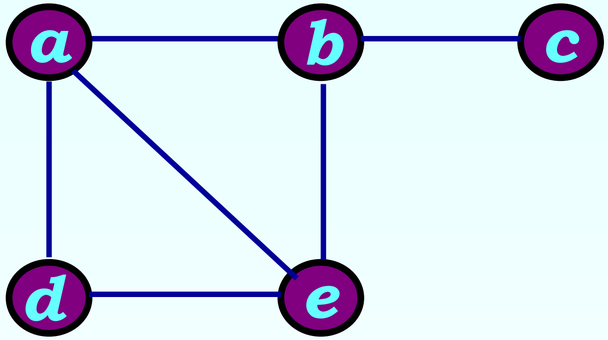
  For an undirected graph G, the maximally connected pieces of G are called its connected components

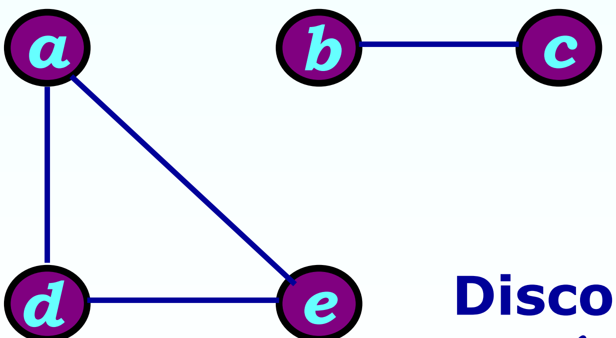
- $\kappa(G)$: the number of connected components of G

  If $\kappa(G) = 1$, the graph G is connected

Examples

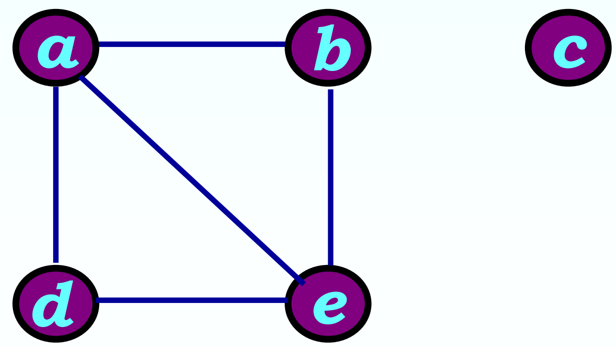
Connected
$\kappa(G) = 1$



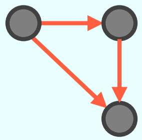


Disconnected
$\kappa(G) = 2$

Isolated vertex



Connectivity in directed G



- **Strongly connected**

  For each distinct vertex pair $\{a, b\}$ in G, there is a path from a to b as well as a path from b to a

- **Unilaterally connected**

  For each $\{a, b\}$, there is at least one path between a and b

- **Weakly connected**

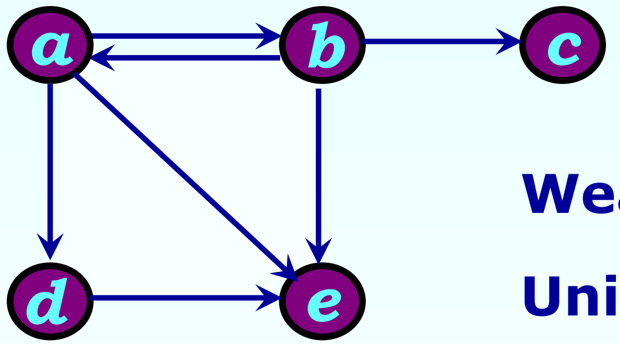
  When the associated graph of G is connected

  (Direction on each edge of G is ignored and only one edge is drawn when multiple edges between any two vertices exist)

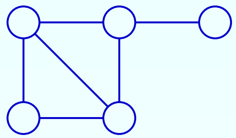
We consider G connected, when it is weakly connected

Examples

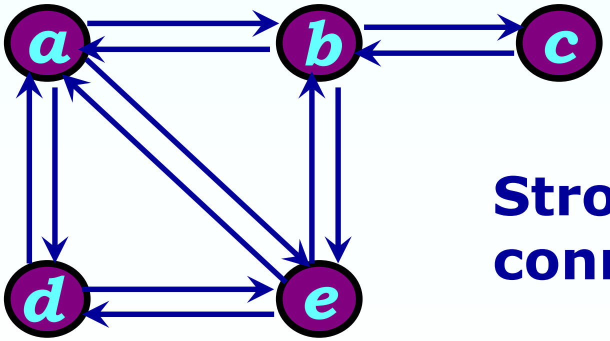
Directed Graph $G = (V,E)$



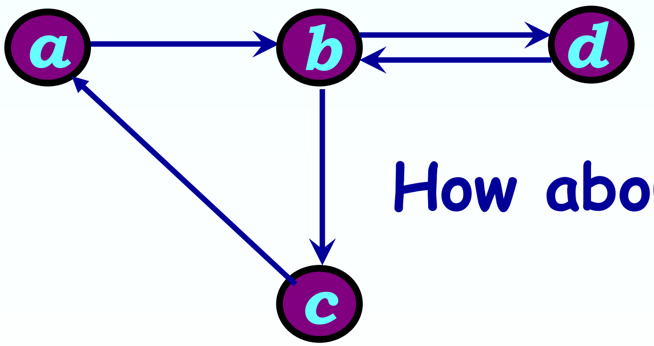
Weakly connected ?



Unilaterally connected ?



Strongly connected



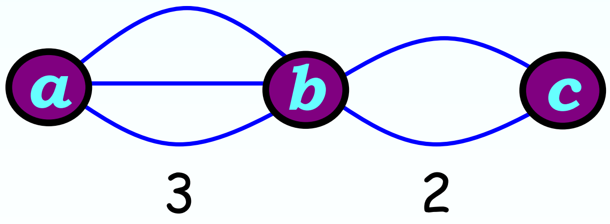
How about ?

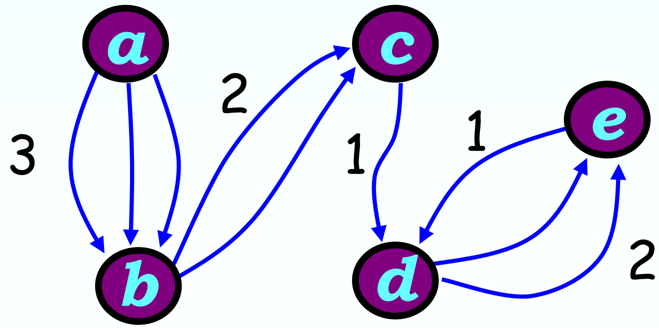
Multigraph

- **Multigraph** $G = (V,E)$

  For some $x, y \in V$, there are two or more edges in E of the form (x,y) or $\{x,y\}$

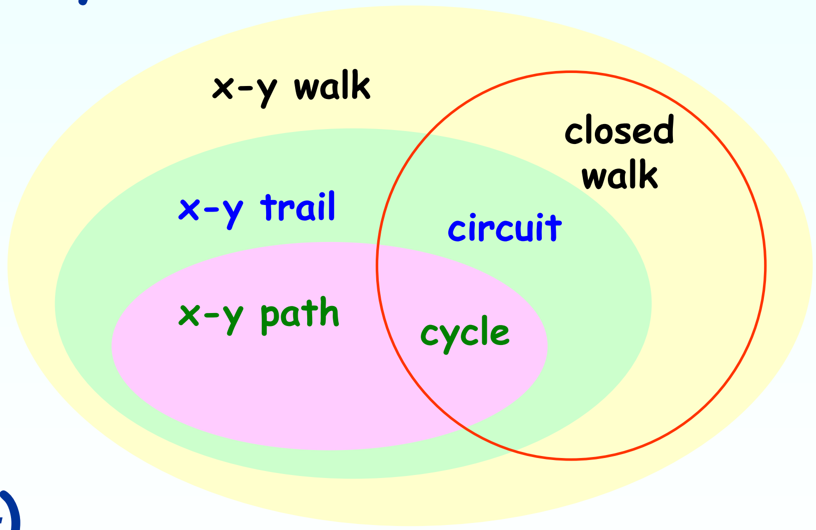
- **Multiplicity** : the number of edges (x,y) or $\{x,y\}$





Summary of Basic Definitions

- Graph $G = (V,E)$
  - Undirected vs. Directed
  - Incident, Adjacent, Loop, Isolated Vertex
  - Walk, Trail, Circuit, Path, Cycle
  - Th. ∃Trail → ∃Path
  - Connected Graph
    - Strongly, Unilaterally, Weakly connected *directed graph*
  - Connected Component, $\kappa(G)$
  - Multigraph, Multiplicity



Relations on Graphs

Section 11.2

Subgraph

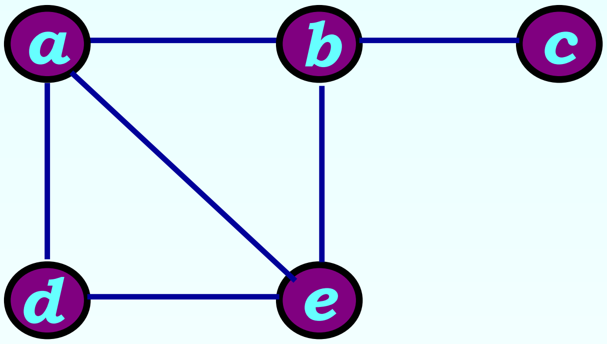
- **Subgraph**

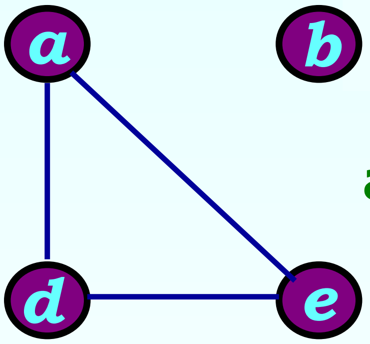
  Let $G=(V,E)$ be a graph. A graph $G_1=(V_1, E_1)$ is called a subgraph of G, if non-empty $V_1 \subseteq V$ and $E_1 \subseteq E$ where each edge in E_1 is incident with vertices in V_1

- **Spanning subgraph**

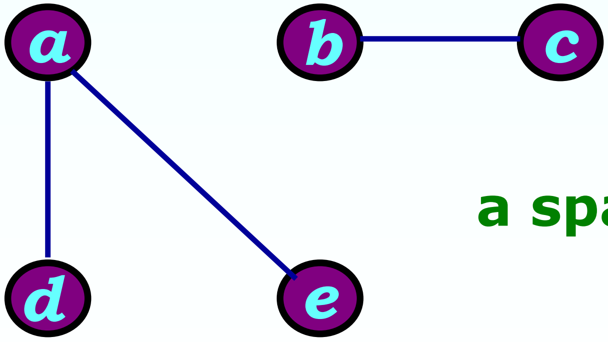
  Given a graph $G=(V,E)$, let $G_1=(V_1, E_1)$ be a subgraph of G. If $V_1 = V$, then G_1 is called a spanning subgraph of G

Examples of Subgraph





a subgraph



a spanning subgraph

Induced Subgraph

- **Definition**

Let $G = (V,E)$ be a graph and a non-empty $U \subseteq V$.

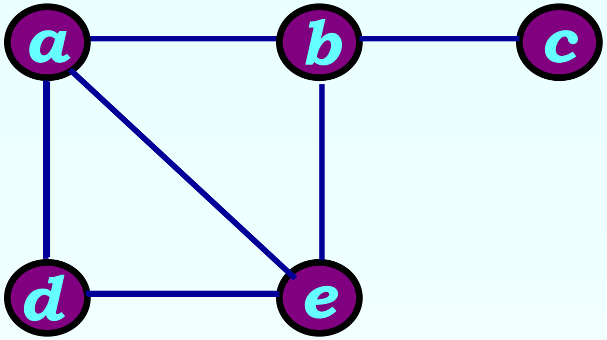
A subgraph of G induced by U is the subgraph $G' = (U, E')$ where E' contains all edges from G of the form

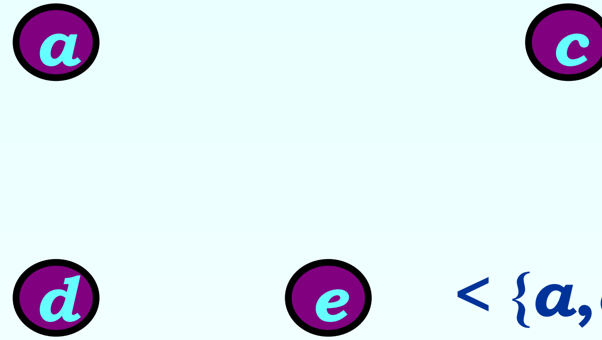
(x,y) for $x, y \in U$ (directed graph) or

$\{x,y\}$ for $x, y \in U$ (undirected graph)

Such a subgraph G', denoted by $<U>$, is called an induced subgraph of G

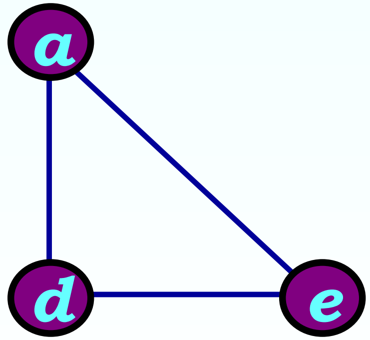
Examples

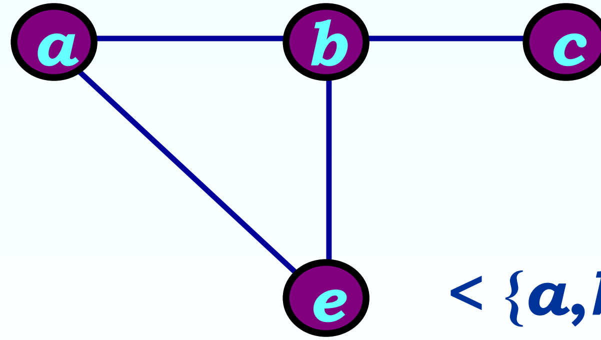




$< \{a,c,d,e\} >$

$< \{a,d,e\} >$





$< \{a,b,c,e\} >$

Examples of induced subgraphs

G-v & G-e

- $G - v$

  Let $G = (V,E)$ and $v \in V$. The subgraph $G - v$ has

  (a) the vertex set $V_1 = V - \{v\}$ and

  (b) the edge set $E_1 \subseteq E$ which contains all the edges in E except for those that are incident with the vertex v

  Hence, $G - v$ is the induced subgraph of G, $<V_1>$

- $G - e$

  (a) the unchanged vertex set V and

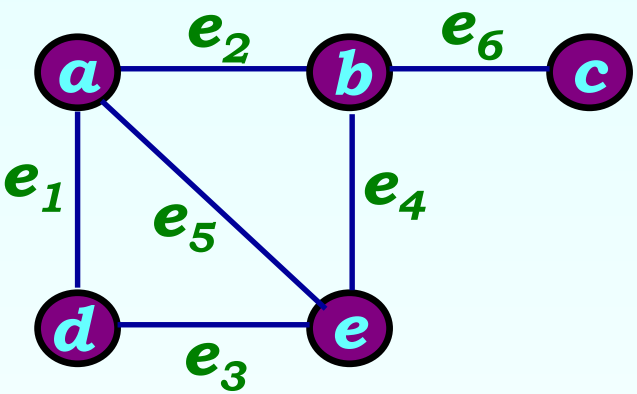
  (b) the edge set $E_1 = E - \{e\}$

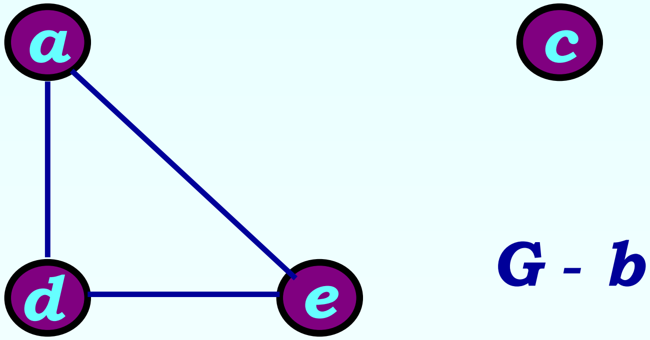
  spanning subgraph

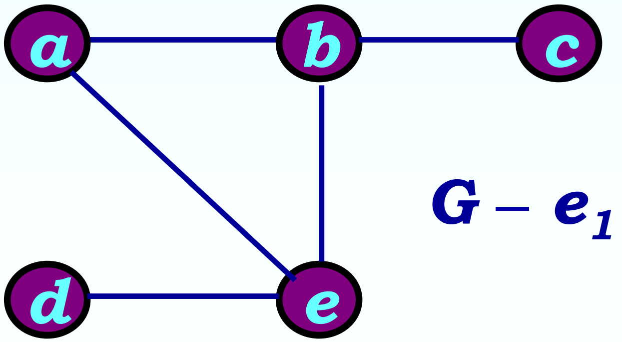
Relative difference of $\{v\}$ in V

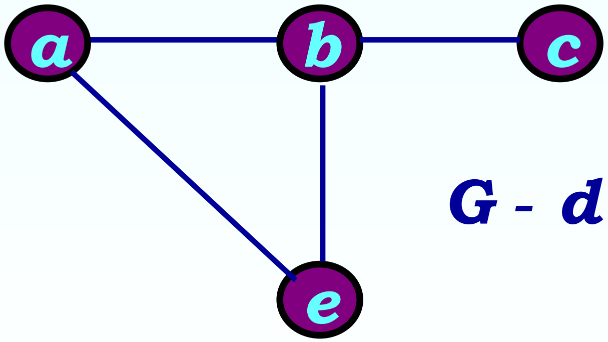
$V_1 = \{ x \mid x \in V \wedge x \notin \{v\}\} \equiv V \setminus \{v\}$

Examples



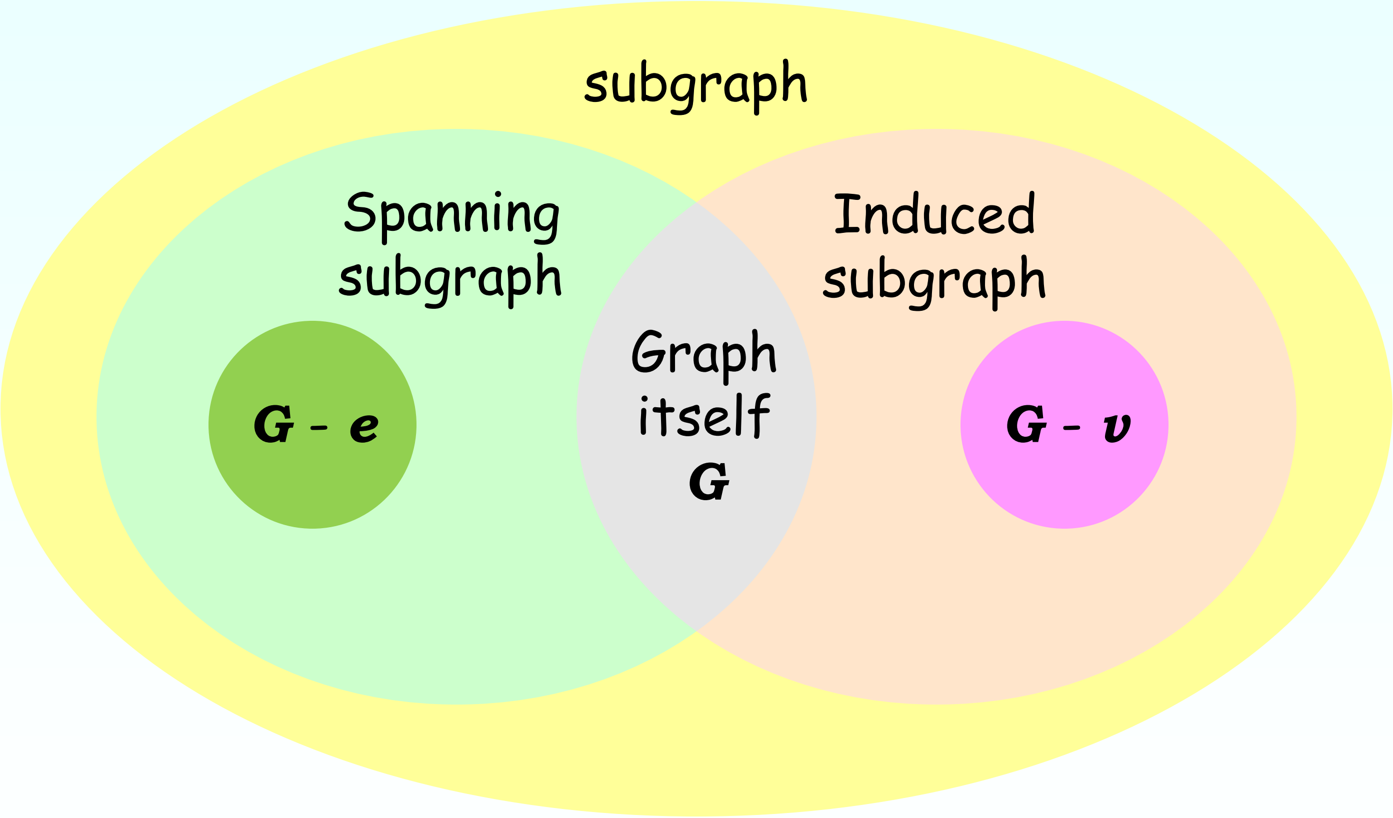






a spanning subgraph **an induced subgraph**

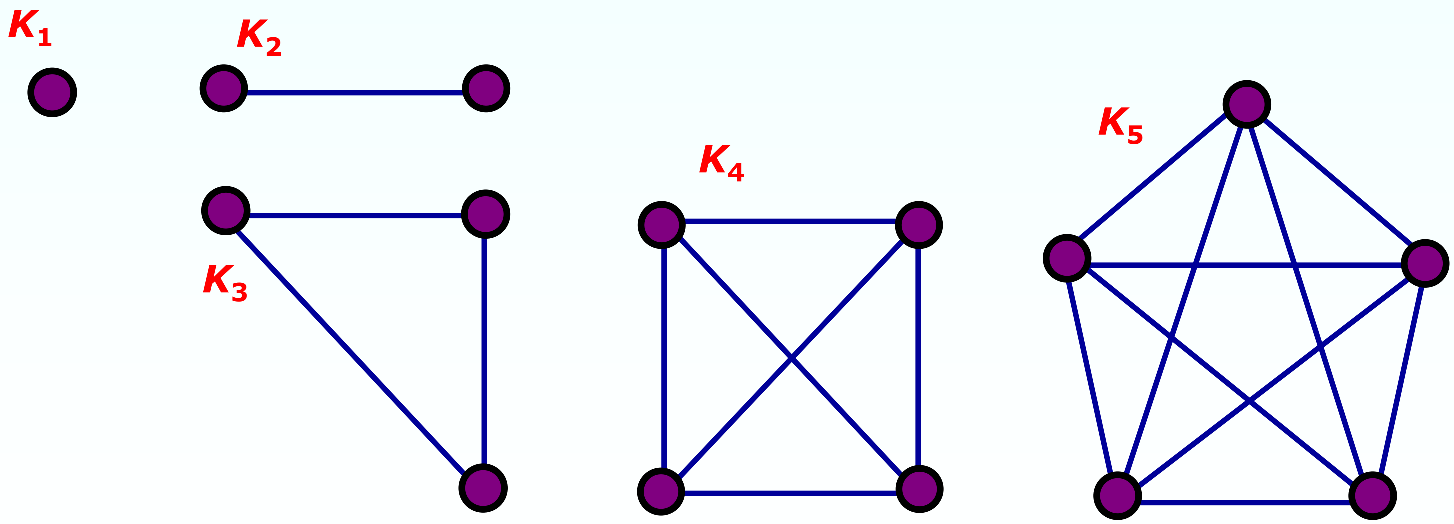
Summary of Subgraphs



Complete Graph

- **Def. of Complete graph K_n**

  Let V be a set of n vertices. The complete graph on V, denoted by K_n, is a loop-free undirected graph where for all $a, b \in V$, $a \neq b$, there is an edge $\{a,b\}$

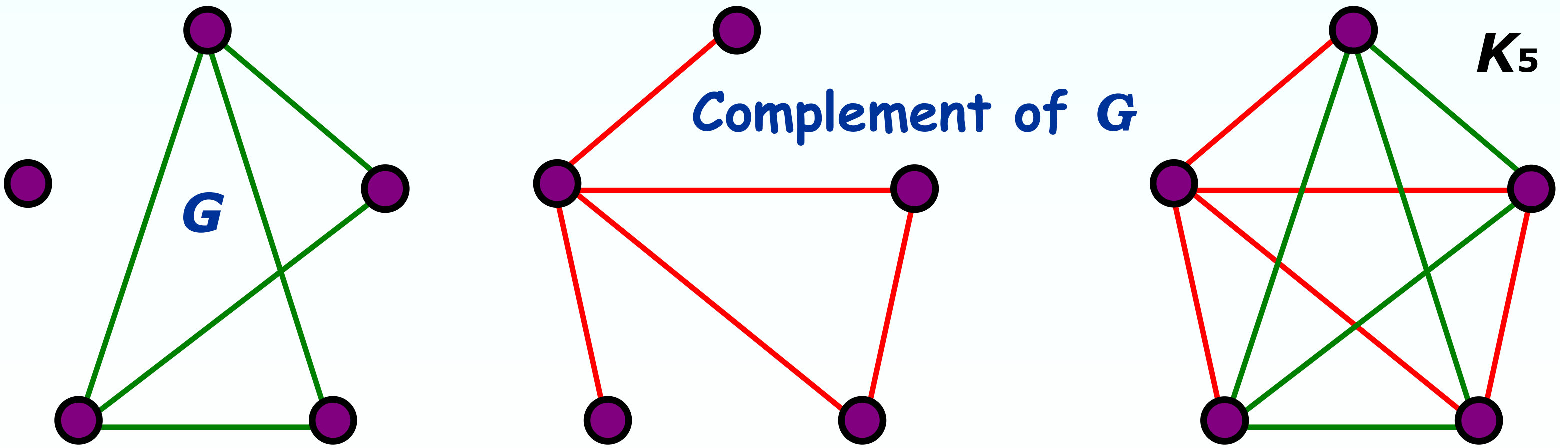


Complement Graph

- Def. of Complement graph

  Let G be a loop free undirected graph on n vertices. The complement of G is the subgraph of K_n consisting of the n vertices in G and all edges that are not in G

  The complement graph of K_n is called a null graph



Isomorphism

Graph Isomorphism

Let $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ be two graphs

A function $f : V_1 \rightarrow V_2$ is called a graph isomorphism if

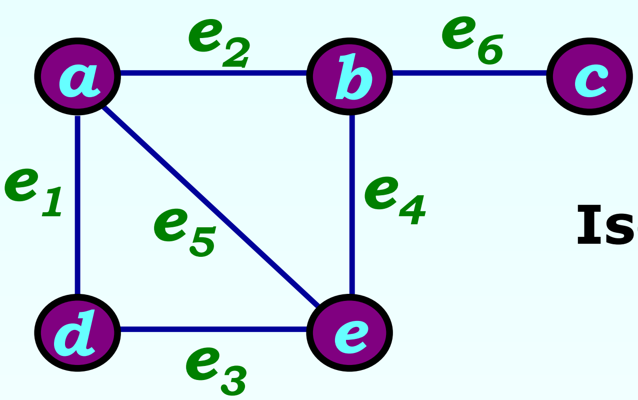
(a) f is one-to-one and onto, and

(b) for all $a,b \in V_1$, $\{a,b\} \in E_1 \Leftrightarrow \{f(a), f(b)\} \in E_2$

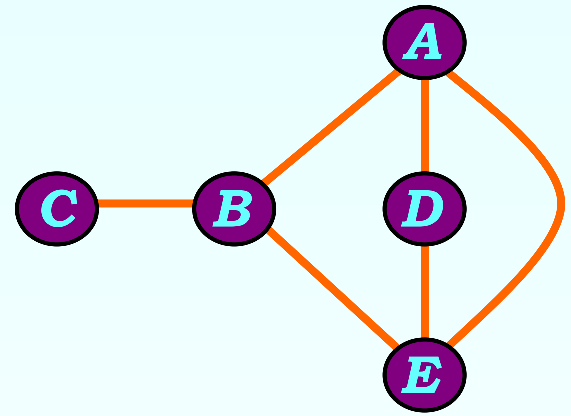
When such a function exists, G_1 and G_2 are called isomorphic (동형) graphs

G_1 과 G_2 의 vertex와 edge간의 incidence 관계가 보존되도록 일대일 대응이 성립할 경우.

An example of isomorphic graph



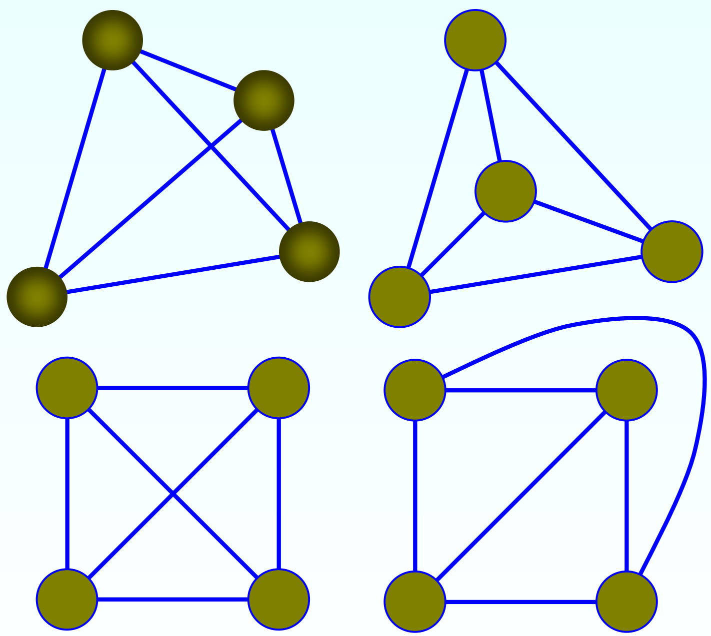
Isomorphic ?

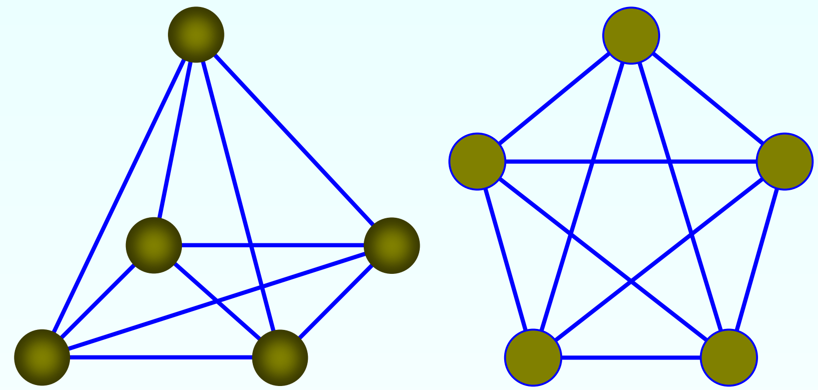


v	$f(v)$
a	A
b	B
c	C
d	D
e	E

edge	$\{v,w\}$	$\{f(v), f(w)\}$
e_1	$\{a,d\}$	$\{A,D\}$
e_2	$\{a,b\}$	$\{A,B\}$
e_3	$\{d,e\}$	$\{D,E\}$
e_4	$\{b,e\}$	$\{B,E\}$
e_5	$\{a,e\}$	$\{A,E\}$
e_6	$\{b,c\}$	$\{B,C\}$

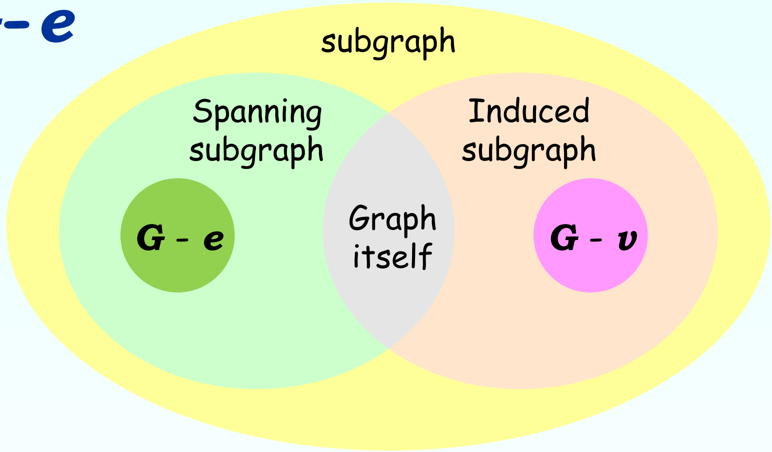
Examples of Isomorphic Graphs



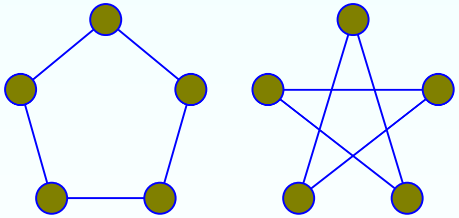


Summary of Graph Relations

- **Subgraph**
  - Spanning, Induced, $G-v$, $G-e$



- **Complement Graph**
  - Complete Graph K_n
  - Null Graph



- **Isomorphism**
  - $f : V_1 \rightarrow V_2$, One-to-one, onto
  - $a,b \in V_1$, $\{a,b\} \in E_1 \leftrightarrow \{f(a), f(b)\} \in E_2$