Computing expectations by conditioning

Conditional PDF (continuous case)

- The conditional PDF of X given that $Y = y$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$

- (Example 3.5) $f(x,y) = \begin{cases} 6xy(2-x-y), & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$
  - (Question) $E[X|Y = y] = ?$
  - $f_Y(y) = \int_0^1 6xy(2-x-y)dx = -6y\int_0^1 (x^2 + (y-2)x)dx = -6y\frac{x^3}{3} + \frac{(y-2)x^2}{2}\Big|_0^1 = y(4-3y)$
  - $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{6xy(2-x-y)}{y(4-3y)}$
  - $E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx = \int_0^1 x\frac{6xy(2-x-y)}{y(4-3y)}dx = \frac{6}{4-3y}\int_0^1 x^2(2-x-y)\,dx = \frac{5-4y}{8-6y}$

Conditional PDF – example

- (Example 3.7) $f(x,y) = \begin{cases} \frac{1}{2} y e^{-xy}, 0 < x < \infty, 0 < y < 2 \\ 0, \quad \text{otherwise} \end{cases}$
  - (Question) $E[e^{X/2}|Y = 1] = ?$
  - (Solution) $f_Y(1) = \int_0^\infty \frac{1}{2} e^{-x} dx = \frac{-e^{-x}}{2}\Big|_0^\infty = \frac{1}{2}$
  - $f_{X|Y}(x|1) = \frac{f(x,1)}{f_Y(1)} = \frac{\frac{1}{2}e^{-x}}{\frac{1}{2}} = e^{-x}$
  - $E\left[e^{X/2}|Y = 1\right] = \int_0^\infty e^{\frac{x}{2}} f_{X|Y}(x|1)\, dx = \int_0^\infty e^{-\frac{x}{2}}\, dx = -2\int_0^\infty e^{-\frac{x}{2}}\, dx = 2$

Computing expectations by conditioning

- $\mathrm{E}[\mathrm{X}] = \sum_y \mathrm{E}[X|Y = y]\mathrm{P}[Y = y]$ (discrete case)
- $\mathrm{E}[\mathrm{X}] = \int_{-\infty}^{\infty} \mathrm{E}[X|Y = y] f_Y(y)\, dy$

- (Example 3.9)

  The number of misprints in Sam's probability chapter ~ Poisson (2), the number in his history chapter ~ Poisson (5), either book is equally likely chosen
- (Question) What is the expected number of misprints?
  - (Solution) Let X denote the number of misprints and

$$\mathrm{Y} = \begin{cases} 1, \text{if his history book is chosen} \\ 2, \text{if his probability book is chosen} \end{cases}$$

$$\mathrm{E}[\mathrm{X}] = \sum_y \mathrm{E}[X|Y = y]\mathrm{P}[Y = y] = \mathrm{E}[\mathrm{X}|\mathrm{Y} = 1]\mathrm{P}[\mathrm{Y} = 1] + \mathrm{E}[X|\mathrm{Y} = 2]\mathrm{P}[\mathrm{Y} = 2]$$

$$= 5 * \frac{1}{2} + 2 * \frac{1}{2} = \frac{7}{2}$$

Computing expectations by conditioning

- $\mathrm{E}[\mathrm{X}] = \sum_y \mathrm{E}[X|Y = y]\mathrm{P}[Y = y]$ (discrete case)
- $\mathrm{E}[\mathrm{X}] = \int_{-\infty}^{\infty} \mathrm{E}[X|Y = y] f_Y(y)\, dy$

- (Example 3.11) a coin, having probability *p* of coming up heads. Let *N* denote the number of flips until the first head appears.
- (Question) E[N] = ?
  - (Solution)

$$\text{Let } \mathrm{Y} = \begin{cases} 1, \text{if the first flip results in a head} \\ 0, \text{ if the first flip results in a tail} \end{cases}$$

$$\mathrm{E}[N] = \sum_y \mathrm{E}[N|Y = y]\mathrm{P}[Y = y] = \mathrm{E}[N|Y = 0]\mathrm{P}[Y = 0] + \mathrm{E}[N|Y = 1]\mathrm{P}[Y = 1] = p\mathrm{E}[N|Y = 1] + (1 - p)\mathrm{E}[N|Y = 0]$$

$$\mathrm{E}[N|Y = 1] = 1, \mathrm{E}[N|Y = 0] = 1 + \mathrm{E}[N]$$

$$\mathrm{E}[N] = p + (1 - p)(1 + \mathrm{E}[N]) \Rightarrow \mathrm{E}[N] = \frac{1}{p}$$

Computing expectations by conditioning – example

- $\mathrm{E}[\mathrm{X}] = \sum_y \mathrm{E}[X|Y = y]\mathrm{P}[Y = y]$ (discrete case)
- $\mathrm{E}[\mathrm{X}] = \int_{-\infty}^{\infty} \mathrm{E}[X|Y = y] f_Y(y)\, dy$

- (Example 3.12) a miner, having three doors. The first door → safety after two hours of travel. The second → the mine after three hours of travel. The third → the mine after five hours of travel. The miner chooses any one of the doors equally likely. Let *X* denote the time until the miner reaches safety and *Y* the door he initially chooses.
- (Question) E[X] = ?
  - (Solution)

$$\mathrm{E}[X] = \sum_y \mathrm{E}[X|Y = y]\mathrm{P}[Y = y] = \mathrm{E}[X|Y = 1]\mathrm{P}[Y = 1] + \mathrm{E}[X|Y = 2]\mathrm{P}[Y = 2] + \mathrm{E}[X|Y = 3]\mathrm{P}[Y = 3] = \frac{1}{3} * 2 + \frac{1}{3}(3 + \mathrm{E}[\mathrm{X}]) + \frac{1}{3}(5 + \mathrm{E}[\mathrm{X}])$$

$$\Leftrightarrow \frac{1}{3}\mathrm{E}[X] = \frac{10}{3} \Leftrightarrow \mathrm{E}[X] = 10$$

Example 3.15

- Let N_k denote the number of necessary trials to obtain k consecutive successes, and let $M_k = E[N_k]$. Each trial ~ Bernoulli (p)
  - $N_k = N_{k-1} + A_{k-1,k}$ where $A_{k-1,k}$ is the number of additional trials needed to go from $k-1$ successes to k in a row
  - $M_k = M_{k-1} + E[A_{k-1,k}]$
  - $E\left[A_{k-1,k}\right] = p * 1 + (1-p)(1+M_k)$
  - Thus, $M_k = M_{k-1} + p + (1-p)(1+M_k)$

  $\Leftrightarrow pM_k = M_{k-1} + 1$

  $\Leftrightarrow M_k = \frac{1}{p} M_{k-1} + \frac{1}{p}$

  - N_1 ~ Geometric (p), so $M_1 = \frac{1}{p}$
  - $M_2 = \frac{1}{p^2} + \frac{1}{p}, \ldots, M_k = \frac{1}{p^k} + \frac{1}{p^{k-1}} + \cdots + \frac{1}{p}$

Analyzing Quick-Sort Algorithm (Example 3.16)

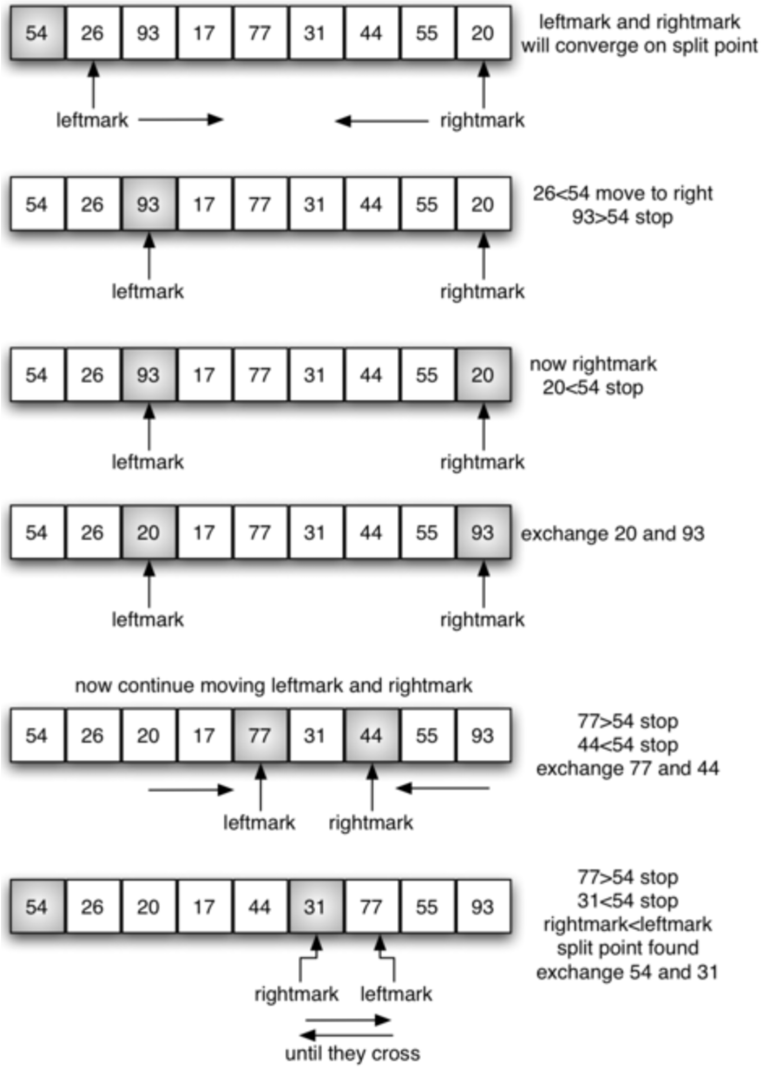
Quick-Sort Algorithm

- Quick-Sort (left, right)
  - Step 1 – Choose a pivot value
  - Step 2 – Partition the array using the pivot value
  - Step 3 – Quick-Sort the left partition recursively
  - Step 4 – Quick-Sort the right partition recursively

- Pseudocode

```
Quick-Sort(left, right)

if right <= left, return
else
    pivot = A[left] // or choose at random
    partition = partitioning(left, right, pivot)
    Quick-Sort(left, partition-1)
    Quick-Sort(partition+1, right)
```



The expected number of comparisons

- $\mathrm{E}[\mathrm{X}] = \sum_y \mathrm{E}[X|Y = y]\mathrm{P}[Y = y]$ (discrete case)

- (Example 3.16) Analyzing the Quick-Sort Algorithm
  - Assume the pivot value chosen at random
  - The number of comparisons vary depending on the pivot value
  - Let M_n the expected number of comparisons needed by the Quick-Sort algorithm to sort a set of n distinct values
  - $M_n = \sum_{j=1}^{n} E[\text{number of comparisons}|\text{value selected is } j\text{th smallest}]\frac{1}{n}$
  - $\Leftrightarrow M_n = \sum_{j=1}^{n}(n - 1 + M_{j-1} + M_{n-j})\frac{1}{n}$
  - $\Leftrightarrow M_n = n - 1 + \frac{2}{n}\sum_{k=1}^{n-1} M_k$
  - $\Leftrightarrow nM_n = n(n-1) + 2\sum_{k=1}^{n-1} M_k$

The expected number of comparisons

- $M_n = \sum_{j=1}^{n} E[\text{number of comparisons}|\text{value selected is } j\text{th smallest}]\,\frac{1}{n}$
- $\Leftrightarrow nM_n = n(n-1) + 2\sum_{k=1}^{n-1} M_k$ … (1)
- $\Leftrightarrow (n+1)M_{n+1} = (n+1)n + 2\sum_{k=1}^{n} M_k$ … (2)
- $(n+1)M_{n+1} - nM_n = 2n + 2M_n$ … by (2) – (1)
- $\Leftrightarrow (n+1)M_{n+1} = (n+2)M_n + 2n$
- $\Leftrightarrow \frac{M_{n+1}}{n+2} = \frac{M_n}{n+1} + \frac{2n}{(n+1)(n+2)}$
- $\Leftrightarrow \frac{M_{n+1}}{n+2} = \frac{M_{n-1}}{n} + \frac{2(n-1)}{n(n+1)} + \frac{2n}{(n+1)(n+2)} = \cdots = 2\sum_{k=0}^{n-1} \frac{n-k}{(n+1-k)(n+2-k)}$
- $\Leftrightarrow M_{n+1} = 2(n+2)\sum_{k=0}^{n-1} \frac{n-k}{(n+1-k)(n+2-k)} = 2(n+2)\sum_{i=1}^{n} \frac{i}{(i+1)(i+2)}$
- $\Leftrightarrow M_{n+1} = 2(n+2)\sum_{i=1}^{n}[\frac{2}{i+2} - \frac{1}{i+1}]$

Let n-k = i

$$\sim 2(n+2)\left[\int_3^{n+2} \frac{2}{x}\,dx - \int_2^{n+1} \frac{1}{x}\,dx\right] = 2(n+2)[2\log(n+2) - \log(n+1) + \log 2 - 2\log 3] = 2(n+2)[\log(n+2) + \log\frac{n+2}{n+1} + \log 2 - 2\log 3] \sim 2(n+2)\log(n+2)$$