Joint Distribution Functions

Joint CDF (continuous case)

- The joint CDF of a pair of random variables, $\{X, Y\}$, is

$$F_{X,Y}(a, b) = \mathrm{P}(X \le a, Y \le b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y) dy dx$$

- Properties of joint CDFs

  - $F_{X,Y}(-\infty, -\infty) = F_{X,Y}(-\infty, y) = F_{X,Y}(x, -\infty) = 0$

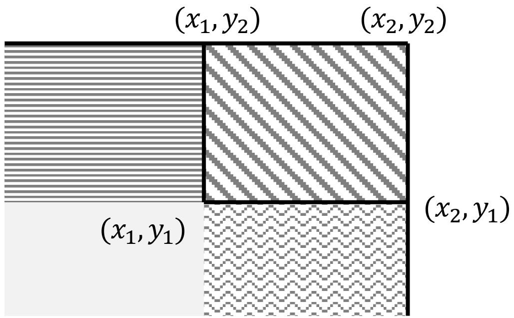
  - $F_{X,Y}(\infty, \infty) = 1$

  - $0 \le F_{X,Y}(x, y) \le 1$

- Relation of the joint PDF & the joint CDF

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$$

Joint CDF (continuous case)



❖ Properties of joint CDFs

- $F_{X,Y}(x, \infty) = F_X(x)$, $F_{X,Y}(\infty, y) = F_Y(y)$ → Marginal CDF
- $\mathrm{P}(x_1 < X_1 \leq x_2, y_1 < Y_1 \leq y_2) = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1) \geq 0$
- Given random variables X and Y, and a function g of X and Y,

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$

e.g. $E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)dxdy$

Joint PMF (discrete case)

- The joint PMF of a pair of random variables, $\{X, Y\}$, is

$$p(x, y) = \mathrm{P}(X = x, Y = y)$$

  - $E[g(X,Y)] = \sum_y \sum_x g(x,y)p(x,y)$

e.g. $E[X+Y] = \sum_y \sum_x (x+y)p(x,y)$

$$= \sum_y \sum_x xp(x,y) + \sum_y \sum_x yp(x,y) = E(X) + E(Y)$$

$$E[a_1X_1 + a_2X_2 + \cdots + a_nX_n] = a_1E(X_1) + a_2E(X_2) + \cdots + a_nE(X_n)$$

Examples

- For three fair dice, what is the expected sum obtained?
  $E(\mathrm{X}) = E(X_1 + X_2 + X_3) =$

- X_i ~ Bernoulli (p)
  $E(\mathrm{X}) = E(X_1 + X_2 + \cdots + X_n) =$

- At a party N men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one. Find the expected number of men who select their own hats.

$$X_i = \begin{cases} 1, \text{if the } i\text{th man selects its own hat} \\ 0, \text{otherwise} \end{cases}$$

$$P(X_i) =$$

$$E(\mathrm{X}) = E(X_1 + X_2 + \cdots + X_n)$$

Examples

- Given 25 different types of coupons, one coupon is obtained each time. What is the expected number of different types that are contained in a set of 10 coupons?

$$E(\mathrm{X}) = E(X_1 + X_2 + \cdots + X_{25})$$

$$X_i = \begin{cases} 1, \text{if at least one type } i \text{ coupon is in the set of 10} \\ 0, \text{otherwise} \end{cases}$$

$$P(X_i) =$$

$$E(X) =$$

Independent random variables

- X and Y are independent if, for all a, b, $P\{X \le a, Y \le b\} = P\{X \le a\}P\{Y \le b\}$
  - $F(a, b) = F_X(a)F_Y(b)$ and $f(a, b) = f_X(a)f_Y(b)$
  - $p(x, y) = p_X(x)p_Y(y)$
- If X and Y are independent, then for any functions h and g

$$E[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy$$

$$= \int_{-\infty}^{\infty} h(y)f_Y(y)dy \int_{-\infty}^{\infty} g(x)f_X(x)dx = E[g(X)]E[h(Y)]$$

Covariance

- For random variables X and Y,
  $$Cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E(XY - XE[Y] - YE[X] + E[X]E[Y])$$
  $$= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y] = E[XY] - E[X]E[Y]$$
  - If X and Y are independent, $Cov(X,Y) = 0$

- Example 2.33
  - $f(x,y) = \frac{1}{y}e^{-(y+\frac{x}{y})}, 0 < x, y < \infty$
    - Verify that this is a joint density function
    - $Cov(X,Y) =$

Covariance – example

- $f(x,y) = \frac{1}{y} e^{-\left(y+\frac{x}{y}\right)}, 0 < x, y < \infty$
  - Verify that this is a joint density function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{y} e^{-\left(y+\frac{x}{y}\right)} dxdy =$$

$$\int_{0}^{\infty} e^{-y} \int_{0}^{\infty} \frac{1}{y} e^{-\frac{x}{y}} dxdy = \int_{0}^{\infty} e^{-y}\, dy = 1$$

  - $Cov(X,Y) = ?$

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dxdy = \int_{0}^{\infty} e^{-y} \underbrace{\int_{0}^{\infty} \frac{x}{y} e^{-\frac{x}{y}} dx}_{\text{Expected value of an exponential random variable with parameter } 1/y} dy = \int_{0}^{\infty} ye^{-y} dy = 1$$

Covariance – example

- $f(x,y) = \frac{1}{y} e^{-\left(y+\frac{x}{y}\right)}, 0 < x, y < \infty$
  - $Cov(X,Y) = ?$

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)dxdy = \int_0^{\infty} e^{-y} \int_0^{\infty} \frac{x}{y} e^{-\frac{x}{y}} dxdy = \int_0^{\infty} y e^{-y} dy = 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = = e^{-y} \underbrace{\int_0^{\infty} \frac{1}{y} e^{-\frac{x}{y}} dx}_{\text{p.d.f. of an exponential random variable with parameter 1/y}} = e^{-y}$$ (exponential with parameter 1)

$$E(Y) = 1$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)dxdy = \int_0^{\infty} y e^{-y} \int_0^{\infty} \frac{x}{y} e^{-\frac{x}{y}} dxdy = \int_0^{\infty} y^2 e^{-y} dy$$

$$= -y^2 e^{-y}\Big|_0^{\infty} + \int_0^{\infty} 2y e^{-y} dy = 2\mathrm{E(Y)} = 2$$

$$Cov(X,Y) = \mathrm{E(XY)} - \mathrm{E(X)E(Y)} = 2 - 1 = 1$$

Covariance – properties

- For random variables X, Y, and Z,
  - $Cov(X,X) = Var(X)$
  - $Cov(X,Y) = Cov(Y,X)$
  - $Cov(cX,Y) = cCov(X,Y)$
  - $Cov(X,Y+Z) = Cov(X,Y) + Cov(X,Z)$

$$Cov(X,Y+Z) = \mathrm{E[X(Y+Z)]} - \mathrm{E[X]E[Y+Z]} = \mathrm{E[XY+XZ]} - \mathrm{E[X](E[Y]+E[Z])}$$
$$= \mathrm{E[XY]} + \mathrm{E[XZ]} - \mathrm{E[X](E[Y]+E[Z])} = (\mathrm{E[XY]} - \mathrm{E[X]E[Y]}) + (\mathrm{E[XZ]} - \mathrm{E[X]E[Z]})$$

Generalized to $Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$

Covariance – properties

- For random variables X, Y, and Z,
  - $Cov(X, X) = Var(X)$
  - $Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(X_i, Y_j)$
  - $Var(\sum_{i=1}^{n} X_i) = Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(X_i, X_j)$

$$= \sum_{i=1}^{n} Cov(X_i, X_i) + \sum_{i=1}^{n} \sum_{j \neq i} Cov(X_i, X_j) = \sum_{i=1}^{n} Var(X_i) + 2 \sum_{i=1}^{n} \sum_{j<i} Cov(X_i, X_j)$$

If X_i, i = 1, 2, …, n are independent, $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$

Proposition 2.4

- If X_i, i = 1, 2, …, n are independent and identically distributed (*i.i.d.*) with expected value μ and variance σ^2 and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then,
  - $E(\bar{X}) = \mu$
  - $Var(\bar{X}) = \frac{1}{n}\sigma^2$
  - $Cov(\bar{X}, X_i - \bar{X}) = 0$

$$Cov(\bar{X}, X_i - \bar{X}) = Cov(\bar{X}, X_i) - Cov(\bar{X}, \bar{X})$$

$$= \frac{1}{n} Cov(X_i + \sum_{j \neq i}^{n} X_j, X_i) - Var(\bar{X})$$

$$= \frac{1}{n} Cov(X_i, X_i) + \frac{1}{n} Cov(\sum_{j \neq i}^{n} X_j, X_i) - Var(\bar{X}) = \frac{1}{n}\sigma^2 - \frac{1}{n}\sigma^2 = 0$$

Example – Variance of a binomial random variable

- X_i ~ Bernoulli (p)

$$X_i = \begin{cases} 1, \text{if the } i\text{th trial is a suc}ess \\ 0, \text{otherwise} \end{cases}$$

$$Var(X_i) = E[{X_i}^2] - E[X_i]^2 = E[X_i] - E[X_i]^2$$

X ~ Binomial (n, p)

$$Var(X) = Var(X_1 + X_2 + \cdots + X_n) =$$

Convolution

- X and Y are independent. pdf of X is f and Y has pdf g.

$$F_{X+Y}(a) = \mathrm{P}(X + Y \le a) = \int\int_{x+y\le a} f(x)g(y)dydx$$

$$= \int\int_{-\infty}^{a-y} f(x)g(y)\, dydx = \int_{-\infty}^{\infty} F_X(a-y)g(y)\, dy$$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} \frac{d}{da} F_X(a-y)g(y)\, dy = \int_{-\infty}^{\infty} f(a-y)g(y)\, dy$$

- $X, Y \sim$ Uniform (0, 1)

  The probability density of $X + Y$ = ?

$$f_{X+Y}(a) = \int_0^1 f(a-y)g(y)\, dy = \int_{-a}^{1-a} f(-y)\, dy = \int_{a-1}^{a} f(y)\, dy$$