Expectation of a Random Variable

The Discrete Case

Given a discrete random variable X and its probability mass function $p(x)$, the expected value of X is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x)$$

Example. $p(0) = p(1) = \frac{1}{2}$. $E[X] = ?$

Example 2.15. We roll a fair die. Let X the outcome. $E[X] = ?$

Expectation of a Bernoulli Random Variable

- Example 2.16
- $p(x) = \begin{cases} 1 - p, & x = 0 \\ p, & x = 1 \end{cases}$
- $E(X) = ?$

Expectation of a Binomial Random Variable

- Example 2.17. $X \sim B(n, p)$

  $p(x = k) = \binom{n}{k} p^k (1 - p)^{n-k}$

- $E(X) = ?$

Expectation of a Binomial Random Variable

- Example 2.17. $X \sim B(n,p)$

  $p(x=k) = \binom{n}{k} p^k (1-p)^{n-k}$

- $E(X) = \sum_{i=0}^{n} ip(i) = \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=1}^{n} i \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i} =$

  $\sum_{i=1}^{n} \frac{n!}{(n-i)!(i-1)!} p^i (1-p)^{n-i} = np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!(i-1)!} p^{i-1} (1-p)^{n-i} =$

  Let $k = i - 1$

  $np \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-(k+1))!k!} p^k (1-p)^{n-(k-1)} =$

  $$np \sum_{k=0}^{n-1} \frac{(n-1)!}{((n-1)-k)!\,k!} p^k (1-p)^{(n-1)-k}$$

  $$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np[p + (1-p)]^{n-1} = np$$

Expectation of a Geometric Random Variable

- Example 2.18. $X \sim Geom(p),\ p(x=k) = \sum_{n=1}^{\infty} p(1-p)^{k-1}$

- $E(X) = \sum_{i=1}^{\infty} ip(1-p)^{i-1} = p\sum_{i=1}^{\infty} iq^{i-1} = p\sum_{i=1}^{\infty} \frac{d}{dq} q^i = p\frac{d}{dq}\sum_{i=1}^{\infty} q^i$

Let $q = 1-p$

$$= p\frac{d}{dq}\left(\frac{q}{1-q}\right) = p\frac{1*(1-q)-q*(-1)}{(1-q)^2} = \frac{1}{p}$$

Expectation of a Poisson Random Variable

- Example 2.19. $p(x = k) = \frac{e^{-\lambda}\lambda^k}{k!}$

- $E(X) = \sum_{i=1}^{\infty} i \frac{e^{-\lambda}\lambda^i}{i!} = e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} = e^{-\lambda}\lambda \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} = e^{-\lambda}\lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} =$

Let $k = i - 1$

$= e^{-\lambda}\lambda \frac{1}{e^{-\lambda}} \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = \lambda$

The Continuous Case

Given a continuous random variable X and its probability density function $f(x)$, the expected value of X is defined by

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Expectation of a Uniform Random Variable

- $X \sim Unif(a, b),$ $\quad f(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & elsewhere \end{cases}$
- $E(X) = ?$

Expectation of a Uniform Random Variable

- $X \sim Unif\ (a, b)$, $\qquad f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & elsewhere \end{cases}$

- $E(X) = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \frac{1}{2} x^2 |_a^b = \frac{1}{2(b-a)} (b^2 - a^2) = \frac{a+b}{2}$

Expectation of a Exponential Random Variable

- $X \sim Exp(\lambda), \ \ f(x) = \lambda \cdot e^{-\lambda x}, \text{if } x \geq 0$

- $E(X) = \int_0^{\infty} x\lambda \cdot e^{-\lambda x} dx = ?$

Expectation of a Exponential Random Variable

- $X \sim Exp(\lambda)$, $f(x) = \lambda \cdot e^{-\lambda x}$, if $x \geq 0$

- $E(X) = \int_0^\infty x\lambda \cdot e^{-\lambda x} dx = \lambda \int_0^\infty x e^{-\lambda x} dx$ (using integration by parts)

$$= -xe^{-\lambda x}|_0^\infty + \int_0^\infty e^{-\lambda x} dx = 0 - \frac{e^{-\lambda x}}{\lambda}|_0^\infty = \frac{1}{\lambda}$$

Expectation of a Normal Random Variable

- $X \sim N(\mu, \sigma)$, $f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, -\infty < x < \infty$

- $E(X) = \mu$

Expectation of a function of a random variable

- $E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$
  - Example 2.25.

    $p(0) = 0.2, p(1) = 0.5, p(2) = 0.3$. Calculate $E[X]$ and $E[X^2]$.

- $E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$
  - Example 2.26.

    $X \sim Unif\ (0,1)$. $E[X^3]$.

- $E[aX + b] = aE[X] + b$
- $E[X^n]$ is called the n^{th} moment of X

Variance

- Let $E[X] = \mu$.
- $Var(X) = E[(X - E[X])^2] = E[X^2 - 2XE[X] + (E[X])^2]$
  $= E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - \mu^2 = E[X^2] - (E[X])^2$

- Example 2.28. Calculate Var(X) of the outcome X for rolling a fair die.
  - $Var(X) = E[X^2] - (E[X])^2$
  - $E[X^2] = \sum_{k=1}^{6} k^2 p(X = k) = ?$