Poisson process

Poisson process – definition

- A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of events that occur by time t.

- The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if the following axioms hold:
  1. $N(0) = 0$
  2. $\{N(t), t \geq 0\}$
  3. $P\{N(t+h) - N(t) = 1\} = \lambda h + o(h)$
  4. $P\{N(t+h) - N(t) \geq 2\} = o(h)$

- If $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, the number of events in any interval of length t, i.e. $N(t+h) - N(t)$ is a Poisson random variable with mean λt.

Interarrival and Waiting time distribution

- T_n: the elapse time between the $(n-1)\mathrm{st}$ and the $n\mathrm{th}$ event. The sequence $\{T_n, n = 1, 2, \ldots\}$ is called the sequence of interarrival times.
  - $P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$: exponential with mean $\frac{1}{\lambda}$
  - $P\{T_2 > t \mid T_1\} = P\{0\ events\ in\ (s, s+t] \mid T_1 = s\} = P\{0\ events\ in\ (s, s+t]\} = e^{-\lambda t}$
  - $T_n, n = 1, 2, \ldots,$ are independent identically distributed exponential with mean $\frac{1}{\lambda}$

- Waiting time $S_n = \sum_{i=1}^{n} T_i,\ n \geq 1 \sim$ gamma with parameters n and λ
  - pdf of $S_n = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

Example 5.13

- Suppose that people immigrate into a territory at a Poisson rate $\lambda = 1$ per day.
  - (Question 1) what is the expected time until the tenth immigrant arrives?
    - ✓ (Solution) $E[\,S_{10}] = \frac{10}{\lambda} = 10$
  - (Question 2) what is the probability that the elapsed time between the tenth and the eleventh arrival exceeds two days?
    - ✓ (Solution) $P[\,T_{11} > 2] = e^{-2\lambda} = e^{-2}$