Conditional Probability

Conditional PMF (discrete case)

- The conditional PMF of X given that $Y = y$

$$p_{X|Y}(x|y) = \mathrm{P}(X = x|Y = y) = \frac{P\{X = x, Y = y)}{P(Y = y)} = \frac{p(x,y)}{p_Y(y)}$$

$$E[X|Y = y] = \sum_x xP\{X = x|Y = y\} = \sum_x xp_{X|Y}(x|y)$$

- (Example 3.1) $p(1,1) = 0.5$, $p(1,2) = 0.1$, $p(2,1) = 0.1$, $p(2,2) = 0.3$
  - (Question) $p_{X|Y}(x|1) = ?$
  - $p_Y(1) = \sum_x p(x, 1) = p(1,1) + p(2,1) = 0.6$
  - $p_{X|Y}(1|1) = \frac{p(1,1)}{p_Y(1)} = \frac{0.5}{0.6}$, $p_{X|Y}(2|1) = \frac{p(2,1)}{p_Y(1)} = \frac{0.1}{0.6}$

Conditional PMF – Binomial

- (Example 3.2) X_1 ~ Binomial (n_1, p), X_2 ~ Binomial (n_2, p)

  - (Question) What is the conditional PMF of X_1 given $X_1 + X_2 = m$?
  - (Solution) Let $q = 1 - p$

$$\mathrm{P}(X_1 = k | X_1 + X_2 = m) = \frac{P\{X_1 = k, X_2 = m - k)}{P(X_1 + X_2 = m)} =$$

$$\binom{n_1}{k} p^k q^{n_1 - k} \binom{n_2}{m-k} p^{m-k} q^{n_2 - m + k} / \binom{n_1 + n_2}{m} p^m q^{n_1 + n_2 - m} =$$

$$\binom{n_1}{k}\binom{n_2}{m-k} / \binom{n_1 + n_2}{m}$$ ➔ hypergeometric distribution

Conditional PMF – Poisson

- (Example 3.3) X_1 ~ Poisson (λ_1), X_2 ~ Poisson (λ_2)
  - (Question) What is the conditional PMF of X_1 given $X_1 + X_2 = n$?
  - (Solution)

$$\mathrm{P}(X_1 = k | X_1 + X_2 = n) = \frac{P\{X_1 = k, X_2 = n - k)}{P(X_1 + X_2 = n)} = \frac{P\{X_1 = k\}P\{X_2 = n - k)}{P(X_1 + X_2 = n)}$$

Conditional PMF – Poisson

- (Example 2.37) X_1 ~ Poisson (λ_1), X_2 ~ Poisson (λ_2)

$$\mathrm{P}(X_1 + X_2 = n) = \sum_{k=0}^{n} P\{X_1 = k, X_2 = n - k\}$$

$$= \sum_{k=0}^{n} P\{X_1 = k\} P\{X_2 = n - k\} = \sum_{k=0}^{n} e^{-\lambda_1} \frac{{\lambda_1}^k}{k!} e^{-\lambda_2} \frac{{\lambda_2}^{n-k}}{(n-k)!}$$

$$= e^{-\lambda_1} e^{-\lambda_2} \sum_{k=0}^{n} \frac{{\lambda_1}^k}{k!} \frac{{\lambda_2}^{n-k}}{(n-k)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} {\lambda_1}^k {\lambda_2}^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n = \frac{(\lambda_1 + \lambda_2)^n}{n!} e^{-(\lambda_1+\lambda_2)}$$

i.e. $X_1 + X_2$ ~ Poisson $(\lambda_1 + \lambda_2)$

Conditional PMF – Poisson

- (Example 3.3) X_1 ~ Poisson (λ_1), X_2 ~ Poisson (λ_2)

  - (Question) What is the conditional PMF of X_1 given $X_1 + X_2 = n$?
  - (Solution)

$$\mathrm{P}(X_1 = k | X_1 + X_2 = n) = \frac{P\{X_1 = k, X_2 = n - k)}{P(X_1 + X_2 = n)} = \frac{P\{X_1 = k\}P\{X_2 = n - k)}{P(X_1 + X_2 = n)}$$

$$= \frac{e^{-\lambda_1}\frac{{\lambda_1}^k}{k!}e^{-\lambda_2}\frac{{\lambda_2}^{n-k}}{(n-k)!}}{\frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)}} = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

➔ Binomial (n, $\frac{\lambda_1}{\lambda_1+\lambda_2}$)

➔ $\mathrm{E}[X_1 | X_1 + X_2 = n] = n\frac{\lambda_1}{\lambda_1+\lambda_2}$