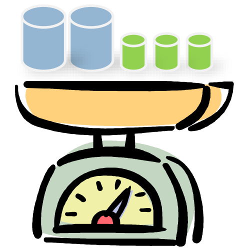
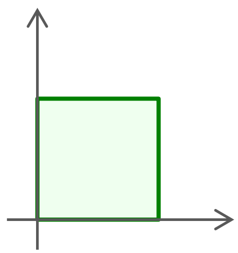
Linear Transformations

정보컴퓨터공학부 김민환 교수

Linear Transformation ?

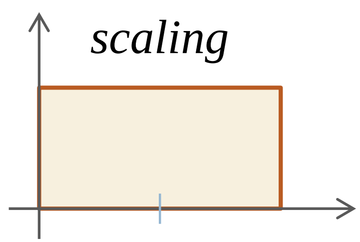


$$T(a\mathbf{x} + b\mathbf{y}) = a\,T(\mathbf{x}) + b\,T(\mathbf{y})$$

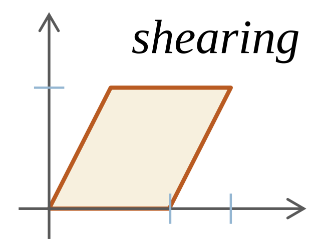


$$T(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

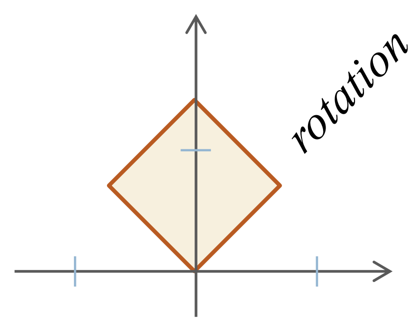
$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix}$$



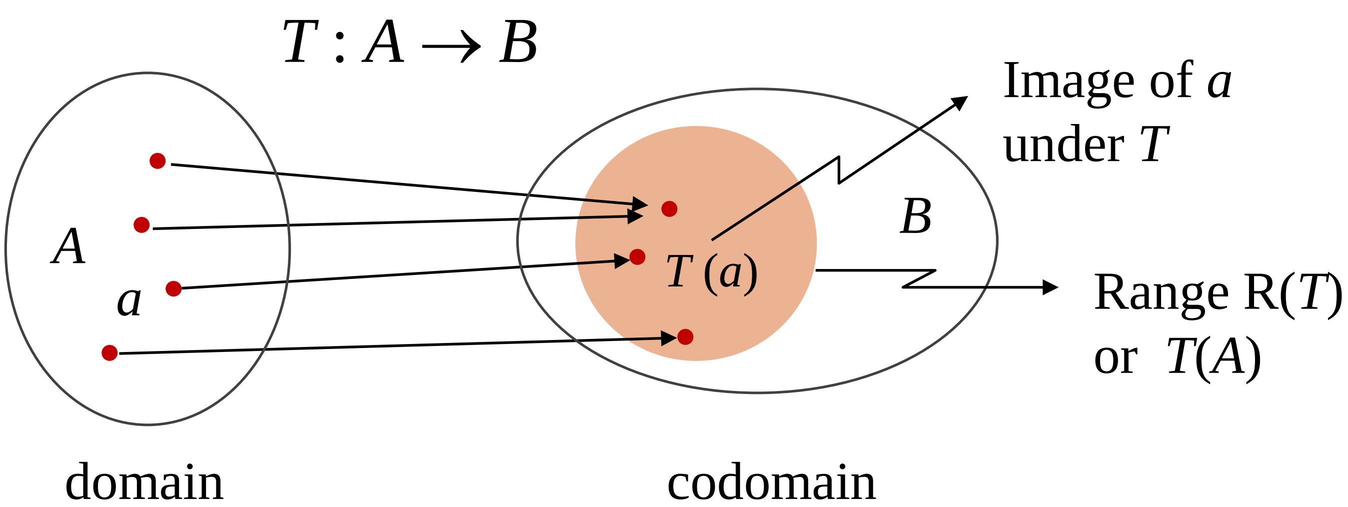
Contents

Definition and Examples

To really understand the definition and how to test for a linear transformation from a vector space to another

Transformations (from a functional point of view)

- A transformation $T : A \to B$ is a rule that associates *each* element of the set A with a *unique* element of the set B



Def. Linear Transformation

A mapping L from *a vector space* V into *a vector space* W is said to be a *linear transformation* $L : V \to W$ if

$$L(a\mathbf{u} + b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v})$$

for all vectors $\mathbf{u}$, $\mathbf{v} \in V$ and for all scalars a and b

- A linear transformation $L : V \to V$ is called a *linear operator* on V

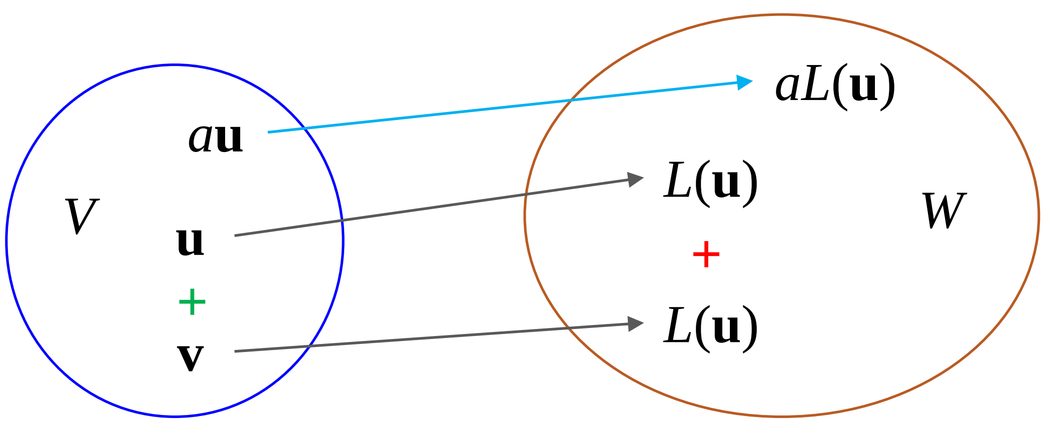
(Another Definition with Two Conditions)

$$L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v}) \quad \text{and} \quad L(a\mathbf{u}) = aL(\mathbf{u})$$

If $L(a\mathbf{u}+b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v})$, then $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ for $a = b = 1$ and $L(a\mathbf{u}) = aL(\mathbf{u})$ for $b = 0$

Conversely, if $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ and $L(a\mathbf{u}) = aL(\mathbf{u})$,

$$L(a\mathbf{u}+b\mathbf{v}) = L(a\mathbf{u}) + L(b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v})$$



$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \Rightarrow T\begin{bmatrix} a \\ b \end{bmatrix}, \quad T((a,b)) \Rightarrow T(a,b)$$

(*Ex*.1)

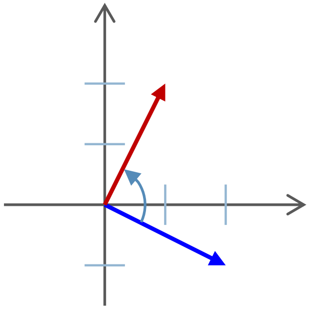
Show that the transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ is linear

$$T(\mathbf{x}) = T(x, y) = (-y, x)$$

(*Solution*)

Yes. It has the effect of rotating each vector in $\mathbf{R}^2$ by 90°

in the counterclockwise direction



$$T(a\mathbf{x} + b\mathbf{y}) = T(a(x_1, x_2) + b(y_1, y_2))$$

$$= T(ax_1 + by_1, ax_2 + by_2)$$

$$= \begin{bmatrix} -(ax_2 + by_2) \\ ax_1 + by_1 \end{bmatrix} = a\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + b\begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

$$= aT(\mathbf{x}) + bT(\mathbf{y})$$

(*Ex.*2)

Show that the transformation $T: \mathbf{R}^2 \rightarrow \mathbf{R}$ is linear

$$T(\mathbf{x}) = T(x, y) = x + y - 2$$

(*Solution*)

It dose not satisfy the linear condition when $a + b - 1 \neq 0$.

Thus it is *not* linear.

$$\begin{aligned} T(a\mathbf{x}+b\mathbf{y}) &= (ax_1+by_1)+(ax_2+by_2)-2 \\ &= a(x_1+x_2)+b(y_1+y_2)-2 \\ &= a(x_1+x_2-2)+b(y_1+y_2-2)+(2a+2b-2) \\ &= aT(\mathbf{x})+bT(\mathbf{y})+2(a+b-1) \end{aligned}$$

(*Ex*.3) Show that the transformation $T : \mathbf{R}^2 \rightarrow M_{22}$ is linear, which is defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & 2y \\ y & 3x \end{bmatrix}$$

(*Solution*)

Let $\mathbf{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then

$$T(a\mathbf{u} + b\mathbf{v}) = T(a\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}) = T\begin{bmatrix} ax_1 + bx_2 \\ ay_1 + by_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & 2(ay_1 + by_2) \\ ay_1 + by_2 & 3(ax_1 + bx_2) \end{bmatrix} = a\begin{bmatrix} x_1 & 2y_1 \\ y_1 & 3x_1 \end{bmatrix} + b\begin{bmatrix} x_2 & 2y_2 \\ y_2 & 3x_2 \end{bmatrix}$$

$$= aT\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + bT\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = aT(\mathbf{u}) + bT(\mathbf{v})$$

Def. Matrix Transformation

A *matrix transformation* is a transformation $L : \mathbf{R}^n \to \mathbf{R}^m$ for which there is an $m \times n$ matrix $\mathbf{A}$ such that

$$L(\mathbf{x}) = \mathbf{Ax}$$

for all $\mathbf{x} \in \mathbf{R}^n$. $\mathbf{A}$ is called the *matrix* of L

- $L : \mathbf{R}^2 \to \mathbf{R}^3$

$$L(\mathbf{x}) = (x-y, 0, y) = \begin{bmatrix} x-y \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{Ax}$$

- Th. 4.1.1

  Any matrix transformation $L : \mathbf{R}^n \to \mathbf{R}^m,\ L(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is *linear*, since it satisfies

$$L(a\mathbf{x} + b\mathbf{y}) = aL(\mathbf{x}) + bL(\mathbf{y})$$

  for all $\mathbf{x}, \mathbf{y}$ in $\mathbf{R}^n$ and for all scalars a, b

$$\mathbf{A}(a\mathbf{x} + b\mathbf{y}) = a\mathbf{A}\mathbf{x} + b\mathbf{A}\mathbf{y}$$

Th. 4.1.2

$L : V \to W$ is a linear transformation. Then

1. $L(\mathbf{0}_V) = \mathbf{0}_W$
2. If $\mathbf{v}_1,\ldots,\mathbf{v}_n$ are elements of V and $a_1,\ldots,a_n$ are scalars, then

$$L(a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n) = a_1 L(\mathbf{v}_1) + \ldots + a_n L(\mathbf{v}_n)$$

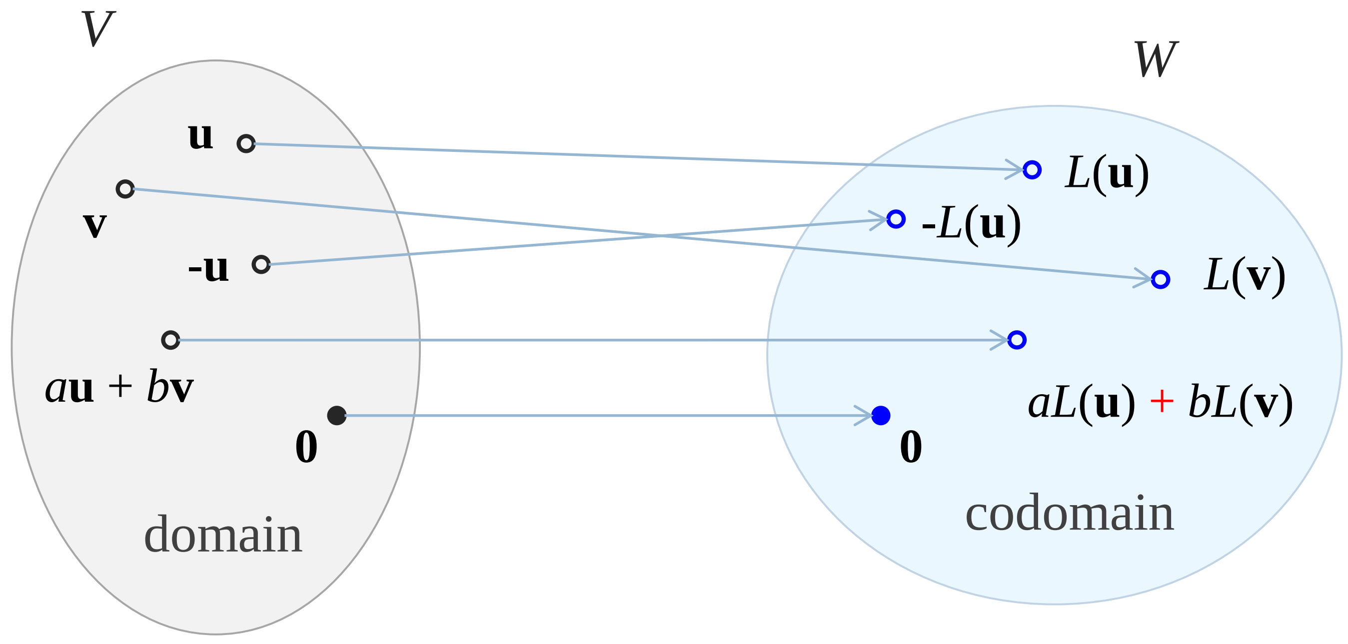
3. $L(-\mathbf{v}) = -L(\mathbf{v})$ for all $\mathbf{v} \in V$

1. $L(\mathbf{0}_V) = L(0\mathbf{v}) = 0L(\mathbf{v}) = \mathbf{0}_W$ for some $\mathbf{v} \in V$
2. By the mathematical induction
3. $\mathbf{0}_W = L(\mathbf{0}_V) = L(\mathbf{v}+\underline{(-\mathbf{v})}) = L(\mathbf{v}) + L(-\mathbf{v}) \quad \therefore L(-\mathbf{v}) = \underline{-L(\mathbf{v})}$

Negative elements

Linear Transformation $L : V \to W$

$$L(a\mathbf{u} + b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v})$$



(1st Review)

(*Ex*.4)

Is $f: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $f(x,y) = (x,1)$ a linear transformation ?

(*Answer*)

Since $f(0,0) = (0,1) \neq \mathbf{0}$, f is not linear

(*Note*) Is $g(x,y) = (xy,0)$ a linear transformation ?

$$g(0,0) = (0,0). \quad A \to B \quad \Rightarrow \quad B \to A \quad ?$$

click

(*Ex*.5) *Zero Transformation*

Show that the zero transformation $0 : V \to W$ is linear

(*Solution*) The zero transformation maps all $\mathbf{v} \in V$ to $\mathbf{0}_W$.

For $\mathbf{u}$, $\mathbf{v} \in V$, $0(a\mathbf{u} + b\mathbf{v}) = \mathbf{0}_W = (a+b)\mathbf{0}_W = a\mathbf{0}_W + b\mathbf{0}_W = a0(\mathbf{u}) + b0(\mathbf{v})$

(*Ex*.6) *Identity Transformation*

Show that the identity transformation $I : V \to V$ is linear

(*Solution*) The identity transformation maps each $\mathbf{v} \in V$ to $\mathbf{v}$. For $\mathbf{u}$, $\mathbf{v} \in V$, $I(a\mathbf{u} + b\mathbf{v}) = a\mathbf{u} + b\mathbf{v} = aI(\mathbf{u}) + bI(\mathbf{v})$

(*Ex*.7) Is the transformation $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ linear ?

(*Answer*) No. Since $f(x+y) = (x+y)^2$ and $f(x) + f(y) = x^2 + y^2$, $f(x+y) \neq f(x) + f(y)$ if $xy \neq 0$

(*Ex*.8) Is $f : \mathbf{R} \to \mathbf{R}$, $f(x) = 3x+1$ linear ?

(*Answer*) $f(ax + by) = 3(ax + by) + 1 = a(3x + 1) + b(3y + 1) + (-a - b + 1) = a\,f(x) + b\,f(y) + (-a - b + 1)$. Since $(-a - b + 1)$ is not 0 for some $a, b \in \mathbf{R}$, f is not linear

(*Ex*.9)

Let $C[0,1]$ be the vector space of all continuous real-valued differential functions defined on the interval $[0,1]$. Show that the transformation $T : C[0,1] \to \mathbf{R}$ defined by (Riemann) integration is linear

$$T(f) = \int_0^1 f(x)dx$$

(*Solution*)

If $f, g \in C[0,1]$ and $a, b \in \mathbf{R}$, then

$$T(a\,f + b\,g) = \int_0^1 (a\,f(x) + b\,g(x))dx = a\int_0^1 f(x)dx + b\int_0^1 g(x)dx$$

$$= aT(f) + bT(g)$$

(*Ex*.10)

Let $T:\mathbf{R}^2 \rightarrow \mathbf{R}^3$ be a linear transformation such that $T(1,1) = (-1,3,1)$ and $T(-1,2) = (-8,-6,5)$. Compute $T(1,-2)$, $T(-9,6)$, and $T(x,y)$

(*Solution*)

$T(1,-2) = T(-(-1,2)) = -T(-1,2) = -(-8,-6,5) = (8,6,-5)$.

Since $(-9,6) = -4(1,1) + 5(-1,2)$, we can get $T(-9,6) = -4T(1,1) + 5T(-1,2) = -4(-1,3,1) + 5(-8,-6,5) = (-36,-42,21)$

We can find the image of every element if T is defined on an entire basis. Note that $\{(1,1), (-1,2)\}$ is a basis.

$$\begin{bmatrix} 1 & -1 & : & x \\ 1 & 2 & : & y \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & : & \dfrac{2x+y}{3} = a \\ 0 & 1 & : & \dfrac{-x+y}{3} = b \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = T\left[a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right] = aT\begin{bmatrix} 1 \\ 1 \end{bmatrix} + bT\begin{bmatrix} -1 \\ 2 \end{bmatrix} = a\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} -8 \\ -6 \\ 5 \end{bmatrix}$$

$$= \left(\frac{2x+y}{3}\right)\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + \left(\frac{-x+y}{3}\right)\begin{bmatrix} -8 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2x-3y \\ 4x-y \\ -x+2y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\begin{bmatrix} -9 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -9 \\ 6 \end{bmatrix} = \begin{bmatrix} -36 \\ -42 \\ 21 \end{bmatrix}$$

Def. Kernel and Range

Let $L : V \to W$ be a linear transformation. The *kernel* of L, denoted $\mathrm{Ker}(L)$, consists of all vectors in V that *map to zero* in W

$$\mathrm{Ker}(L) = \{\ \mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0} \in W\ \}$$

Let S be a subspace of V. The *image* of S, denoted $L(S)$, is defined by

$$L(S) = \{\ \mathbf{w} \in W \mid \mathbf{w} = L(\mathbf{v}) \text{ for some } \mathbf{v} \in S\ \}$$

The image of the entire vector space, $L(V)$, is called the *range* of L, denoted $\mathrm{Range}(L)$ or $\mathrm{R}(L)$

(*Ex*.11)

Compute the kernel and range of

(a) Zero linear transformation $0 : V \to W$

(b) Identity linear transformation $I : V \to V$

(c) Projection $p : \mathbf{R}^2 \to \mathbf{R}^2$, $p(x,y) = (x,0)$

(*Ex*.12)

Find the kernel of $L(x,y,z) = (x - z, y + z)$

(*Solution*)

Solving the system $x - z = 0$ and $y + z = 0$, we get $(r, -r, r)$, $r \in \mathbf{R}$. Hence,

$$\mathrm{Ker}(L) = \left\{ \begin{bmatrix} r \\ -r \\ r \end{bmatrix}, r \in \mathbf{R} \right\} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(*Ex*.13) Find the range of $T : \mathbf{R}^2 \to \mathbf{R}^3$, $T(x,y) = (x\text{-}y,\ 0,\ y)$

(*Solution*)

A 3-vector $\mathbf{w} = (a,b,c)$ is in $\mathrm{R}(T)$ if and only if there is a 2-vector $\mathbf{x} = (x,y)$ such that $T(x,y) = (x\text{-}y,\ 0,\ y) = (a,b,c)$

Thus the following linear system should be consistent.

$$\begin{bmatrix} 1 & -1 & : & a \\ 0 & 0 & : & b \\ 0 & 1 & : & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & : & a+c \\ 0 & 1 & : & c \\ 0 & 0 & : & b \end{bmatrix}$$

Therefore, $b = 0$ and

$\mathrm{R}(T) = \{\ (a,0,c) \mid a,c \in \mathbf{R}\ \}$

Conversely, for any $\mathbf{v} = (\alpha,0,\beta)$ in $\mathrm{R}(T)$, there exists a vector $(\alpha+\beta,\beta)$ that is mapped into $\mathbf{v}$

Th. 4.1.3

Let $L : V \to W$ be a linear transformation and S is a subspace of V, then

1. $\mathrm{Ker}(L)$ is subspace of V
2. $L(S)$ is a subspace of W

(*Proof* of 1)

Let $\mathbf{u}, \mathbf{v} \in \mathrm{Ker}(L)$ and let $c \in \mathbf{R}$. Then

$$L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v}) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W$$
$$L(c\mathbf{u}) = c\,L(\mathbf{u}) = \mathbf{0}_W$$

Thus $\mathrm{Ker}(L)$ is a subspace of V

(*Proof* of 2)

Since $\mathbf{0}_V \in S$ and $\mathbf{0}_W = L(\mathbf{0}_V) \in L(S)$, $L(S)$ is nonempty.

If $\mathbf{w} \in L(S)$, then $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in S$. For any scalar c, $c\mathbf{w} = cL(\mathbf{v}) = L(c\mathbf{v})$. Since $c\mathbf{v} \in S$, $c\mathbf{w} \in L(S)$.

If $\mathbf{w}_1, \mathbf{w}_2 \in L(S)$, then there exist $\mathbf{v}_1, \mathbf{v}_2 \in S$ such that $L(\mathbf{v}_1) = \mathbf{w}_1$ and $L(\mathbf{v}_2) = \mathbf{w}_2$. Thus,

$$\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1) + L(\mathbf{v}_2) = L(\mathbf{v}_1 + \mathbf{v}_2) \in L(S)$$

Therefore, $L(S)$ is a subspace of W

Def. Nullity and Rank

Let L be a linear transformation.

The *dimension* of the *kernel* of L is called the *nullity* of L.

The *dimension* of the *range* of L is called the *rank* of L

- Th. 4.1.4 *Dimension Theorem*

  If $L : V \to W$ is a linear transformation from a finite-dimensional vector space V into a vector space W,

$$\text{Nullity}(L) + \text{Rank}(L) = \dim(V)$$

(*Ex*.14)

Find the nullity and rank of $L : \mathbf{R}^3 \to \mathbf{R}^2$,

$$L(x,y,z) = (x+y,\ y+z) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{Ax}$$

(*Solution*)

$L(x,y,z)$ is a matrix transform. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

From $\mathbf{Ax} = \mathbf{0}$, $\mathrm{Ker}(L) = \{\ (r,-r,r) \mid r \in \mathbf{R}\ \} = \mathrm{Span}(\ (1,-1,1)\ )$ and the nullity of L is 1. From the dimension theorem, $\mathrm{Rank}(L) = \dim(V) - \mathrm{Nullity}(L) = 2$. Thus $\mathrm{R}(L) = \mathbf{R}^2$

(*Ex*.15)

Find the nullity and rank of $L : \mathbf{R}^n \to \mathbf{R}$, $L(\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}$ for a fixed nonzero vector $\mathbf{u}$

(*Solution*)

The kernel is the *hyperplane* through the origin with the normal $\mathbf{u}$. Since $L(\mathbf{u}) = \mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2 > 0$, the range contains the span of $\|\mathbf{u}\|^2 \in \mathbf{R}$. Thus, $\mathrm{R}(L) = \mathbf{R}$ and the rank of $L = 1$

From the dimension theorem, the nullity of $L = n - 1$

Th. 4.1.5

Let $L : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a matrix transformation with standard matrix $\mathbf{A}$, that is, $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Then

1. $\mathrm{Ker}(L) = \mathrm{Null}(\mathbf{A})$
2. $\mathrm{Range}(L) = \mathrm{Col}(\mathbf{A})$
3. $\mathrm{Nullity}(L) = \mathrm{Nullity}(\mathbf{A})$
4. $\mathrm{Rank}(L) = \mathrm{Rank}(\mathbf{A})$

(*Proof*)

Because $L(\mathbf{x}) = \mathbf{Ax}$, $L(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{Ax} = \mathbf{0}$. Hence,

$\mathrm{Ker}(L) = \mathrm{Null}(\mathbf{A})$

Likewise, if $\mathbf{b} \in \mathrm{R}(L)$, then there is an $\mathbf{x}$ in $\mathbf{R}^n$ such that

$L(\mathbf{x}) = \mathbf{b}$ if and only if $\mathbf{Ax} = \mathbf{b}$. Hence, $\mathrm{R}(L) = \mathrm{Col}(\mathbf{A})$

$$\mathbf{b} \in \mathrm{R}(L) \Leftrightarrow L(\mathbf{x}) = \mathbf{b} \text{ for some } \mathbf{x} \Leftrightarrow \mathbf{Ax} = \mathbf{b} \text{ for some } \mathbf{x}$$

$$\Leftrightarrow [\mathbf{A} : \mathbf{b}] \text{ is consistent} \Leftrightarrow \mathbf{b} \in \mathrm{Col}(\mathbf{A})$$

The claims on the nullities and ranks follow

(*Ex*.16)

Find bases for the kernel and range and compute the nullity and rank of $L : \mathbf{R}^4 \to \mathbf{R}^3$

$$L(x, y, z, w) = (x + 3z, y - 2z, w)$$

(*Solution*)

$$L(\mathbf{x}) = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

The vectors $\{(1,0,0),(0,1,0),(0,0,1)\}$ form a basis for $\mathrm{Col}(\mathbf{A}) = \mathrm{R}(L) = \mathbf{R}^3$ and $\{(-3,2,1,0)\}$ is a basis for $\mathrm{Null}(\mathbf{A}) = \mathrm{Ker}(L)$. Hence, $\mathrm{Rank}(L) = 3$ and $\mathrm{Nullity}(L) = 1$

(*Ex*.17)

Determine the range of $L: \mathbf{R}^4 \to P_3$,

$$L(a, b, c, d) = (a - b) + (c + d)x + (2a + b)x^2$$

(*Solution*)

The kernel of L is spanned by $(0,0,-1,1)$, since $a - b = 0$, $c + d = 0$, and $2a + b = 0$.

Thus $\mathrm{Nullity}(L) = 1$ and $\mathrm{Rank}(L) = 3$. Because $\dim(P_3) = 3$, the range $\mathrm{R}(L) = P_3$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & 1 & : & 0 \\ 2 & 1 & 0 & 0 & : & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & 0 & : & 0 \\ 0 & 0 & 1 & 1 & : & 0 \end{bmatrix}$$

Def. One-to-One and Onto

A transformation $T : A \to B$ is called *one-to-one* if for each element b of the range, there is *exactly one* element a with image $b = T(a)$. This can be rephrased as

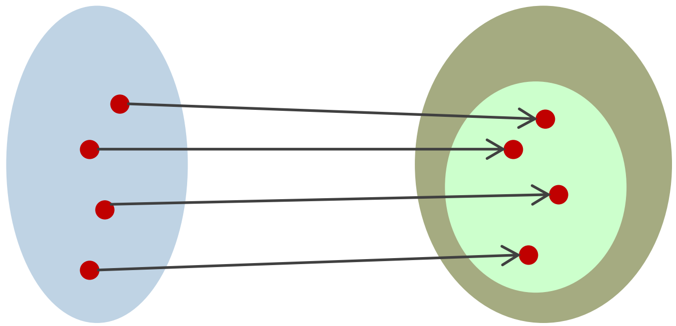
$$T(a_1) = T(a_2) \quad \Rightarrow \quad a_1 = a_2$$

or, equivalently,

$$a_1 \neq a_2 \quad \Rightarrow \quad T(a_1) \neq T(a_2)$$

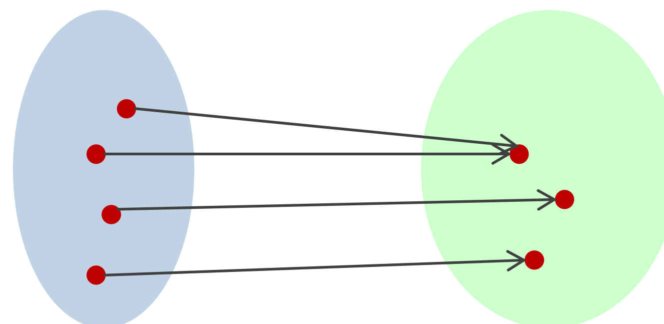
A transformation $T : A \to B$ is called *onto* if its range equals its codomain, i.e.,

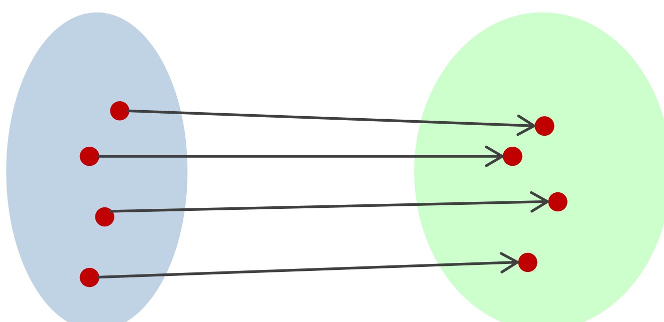
$$\mathrm{R}(T) = B$$



One-to-One

Onto





One-to-One & Onto

(*Ex*.18) Show that the transformations are one-to-one or onto?

(a) $T : \mathbf{R}^2 \rightarrow \mathbf{R}^3,\ T(x, y) = (x + y, y, 0)$

(b) $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2,\ T(x, y, z) = (x, z)$

(*Answer*)

(a) If $T(x_1,y_1) = T(x_2,y_2)$, then $(x_1+y_1,y_1,0) = (x_2+y_2,y_2,0)$. Therefore, $y_1 = y_2$ and $x_1 = x_2$. So $(x_1,y_1) = (x_2,y_2)$ and T is one-to-one. Since $(0,0,1) \notin T(\mathbf{R}^2)$, T is not onto

(b) T is not one-to-one, because $T(0,0,0) = (0,0) = T(0,1,0)$. For any 2-vector (a,b) there is a 3-vector $(a,0,b)$, so T is onto

□ Th. 4.1.6

Let $L : V \to W$ be a linear transformation. Then

$$L \text{ is one-to-one} \iff \mathrm{Ker}(L) = \{\mathbf{0}\}$$

(*Proof*)

Suppose L is one-to-one. Since $L(\mathbf{0}) = \mathbf{0}$, $\mathrm{Ker}(L) = \{\mathbf{0}\}$

Conversely, suppose $\mathrm{Ker}(L) = \{\mathbf{0}\}$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors of V such that $L(\mathbf{u}) = L(\mathbf{v})$.

$$L(\mathbf{u}) = L(\mathbf{v}) \Rightarrow L(\mathbf{u}) + (-L(\mathbf{v})) = L(\mathbf{v}) + (-L(\mathbf{v})) = \mathbf{0}$$

$$\Rightarrow L(\mathbf{u}) + (-1)L(\mathbf{v}) = \mathbf{0} \Rightarrow L(\mathbf{u} + (-1)\mathbf{v}) = \mathbf{0} \Rightarrow L(\mathbf{u} + (-1)\mathbf{v}) = \mathbf{0}$$

$$\Rightarrow L(\mathbf{u} + (-\mathbf{v})) = \mathbf{0} \Rightarrow \mathbf{u} + (-\mathbf{v}) = \mathbf{0} \Rightarrow \mathbf{u} + (-\mathbf{v}) + \mathbf{v} = \mathbf{v} \Rightarrow \mathbf{u} = \mathbf{v}$$

Therefore, L is one-to-one

(*Ex*.19)

Show that $L : \mathbf{R}^2 \to \mathbf{R}^2$, $L(x,y) = (x-y, x+2y)$ is one-to-one and onto

(*Solution*)

$(x-y, x+2y) = (0,0)$ implies that $x = y = 0$. So $\mathrm{Ker}(L) = \{\mathbf{0}\}$ and L is one-to-one. $\mathrm{Rank}(L) = 2 - \mathrm{Nullity}(L) = 2$ by the dimension theorem. Thus, $\mathrm{R}(L) = \mathbf{R}^2$ and L is onto

- Th. 4.1.7

  A one-to-one linear transformation maps linearly independent sets to linearly independent sets

  In other words, if $L : V \to W$ is linear and one-to-one and $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent subset of V, then $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$ is a linearly independent subset of W

(*Proof*)

Let $c_1 L(\mathbf{v}_1) + \ldots + c_k L(\mathbf{v}_k) = \mathbf{0}_W$. Then $L(c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k) = \mathbf{0}_W$ and $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{0}_V$. Because $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent, $c_1 = \ldots = c_k = 0$ and $\{L(\mathbf{v}_1), \ldots, L(\mathbf{v}_k)\}$ is also linearly independent

Th. 4.1.8

Let $L : V \to W$ be a linear transformation between two finite-dimensional vector spaces V, W with $\dim(V) = \dim(W)$. Then L is one-to-one if and only if it is onto

(*Proof*)

Suppose L is one-to-one. Then, $\text{Nullity}(L) = 0$, because $\text{Ker}(L) = \{\mathbf{0}\}$. So, $\text{Rank}(L) = \dim(V) - \text{Nullity}(L) = \dim(V) = \dim(W)$. Therefore, L is onto

Conversely, suppose L is onto. Since $\dim(W) = \text{Rank}(L) = \dim(V)$, $\text{Nullity}(L) = 0$. Therefore, L is one-to-one

Def. Isomorphism

A linear transformation between two vector spaces that is one-to-one and onto is called an *isomorphism*

Two vector spaces are called *isomorphic* if there is an isomorphism between them

- We consider isomorphic spaces to be the same because their elements correspond one for one and *the structure of the vector space operations is preserved through linearity*

(*Ex*.20)

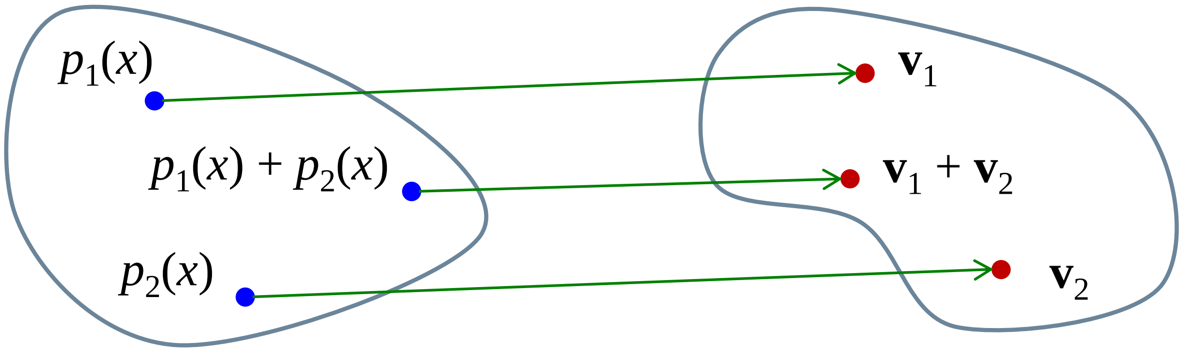
Show that P_n and $\mathbf{R}^n$ are isomorphic

(*Solution*)

There is a linear transformation $L : P_n \rightarrow \mathbf{R}^n$,

$$L(a_0 + a_1 x + \ldots + a_{n\text{-}1} x^{n\text{-}1}) = (a_0, a_1, \ldots, a_{n\text{-}1})$$

This transformation is one-to-one and onto, so it is the isomorphism. Therefore, $P_{n\text{-}1}$ and $\mathbf{R}^n$ are isomorphic



□ Th. 4.1.9

Let V and W be finite-dimensional vector spaces. Then

V and W are isomorphic $\Leftrightarrow$ $\dim(V) = \dim(W)$

(*Ex*.21)

Show that $\mathbf{R}^{mn}$ and M_{mn} are isomorphic

(*Solution*)

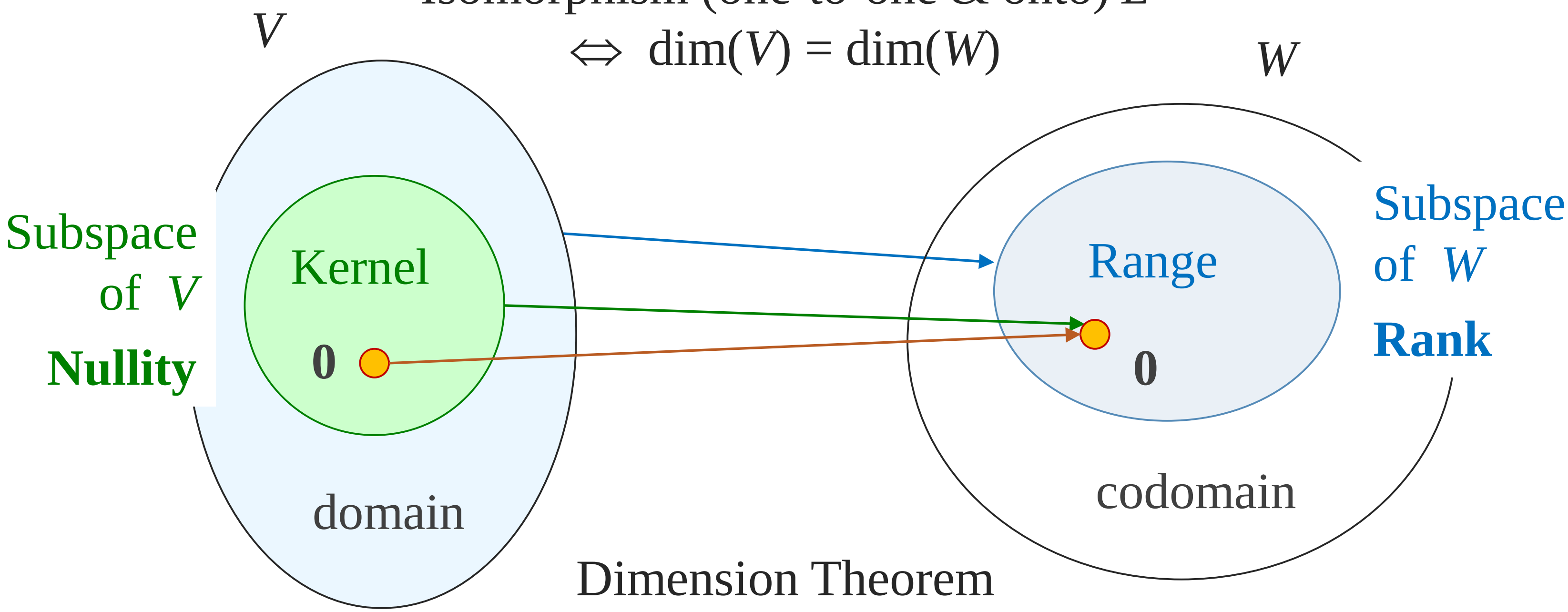
They are isomorphic because $\dim(\mathbf{R}^{mn}) = \dim(M_{mn}) = mn$

Linear Transformation $L : V \to W$

$$L(a\mathbf{u} + b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v})$$

One-to-one $L \Leftrightarrow \mathrm{Ker}(L) = \{\mathbf{0}\}$

Isomorphism (one-to-one & onto) L
$\Leftrightarrow \dim(V) = \dim(W)$



Dimension Theorem

$\mathrm{Nullity}(L) + \mathrm{Rank}(L) = \dim(V)$

(2nd Review)

Matrix Representation of Linear Transformations

- To know how to compute the matrix of a linear transformation
- To be able to evaluate a linear transformation from its matrix
- To know how to compute the matrix of a linear transformation with respect to a new basis

Th. 4.2.1

If L is a linear transformation mapping $\mathbf{R}^n$ into $\mathbf{R}^m$, then there is an $m \times n$ matrix $\mathbf{A}$ such that, for each $\mathbf{x} \in \mathbf{R}^n$,

$$L(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

In fact, the jth column vector $\mathbf{a}_j$ of $\mathbf{A}$ is given by

$$\mathbf{a}_j = L(\mathbf{e}_j) \quad j = 1, 2, \ldots, n$$

For any $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \Lambda + x_n\mathbf{e}_n \in \mathbf{R}^n$,

$$L(\mathbf{x}) = L(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \Lambda + x_n\mathbf{e}_n) = x_1L(\mathbf{e}_1) + x_2L(\mathbf{e}_2) + \Lambda + x_nL(\mathbf{e}_n)$$

$$= \begin{bmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) & \cdot\cdot & L(\mathbf{e}_n) \end{bmatrix}\mathbf{x} = \mathbf{A}\mathbf{x}$$

(*Ex*.21)

Find the standard matrix of the linear transformation L : $\mathbf{R}^3 \to \mathbf{R}^2$, $L(x,y,z) = (x + y, y + z)$

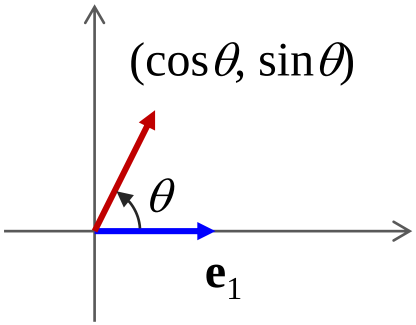
(*Solution*)

Since $L(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, L(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, L(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Therefore, $L(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}\mathbf{x}$

(*Ex.*22)

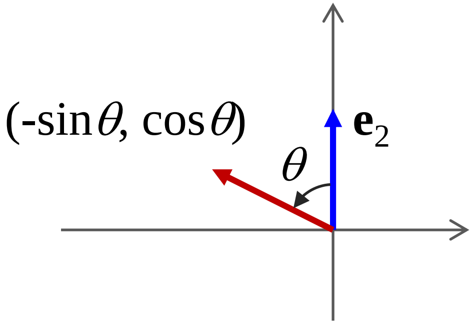
Find the matrix transformation $L : \mathbf{R}^2 \to \mathbf{R}^2$ that has the effect of rotating each vector $\mathbf{x}$ in $\mathbf{R}^2$ by θ in the counterclockwise direction



(*Solution*)

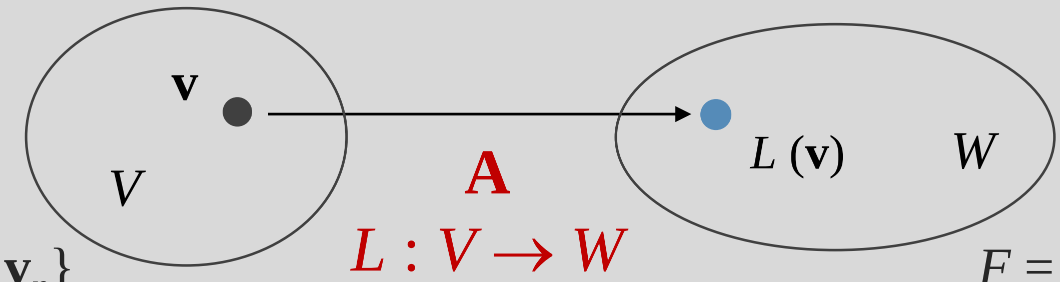
$$L(\mathbf{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad L(\mathbf{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\therefore \mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$L : \mathbf{R}^n \to \mathbf{R}^m, \quad L(\mathbf{x}) = \mathbf{A}\mathbf{x} = [L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ \ldots\ L(\mathbf{e}_n)]\mathbf{x}$$

Basis
$E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$



Basis
$F = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$

$$[\mathbf{v}]_E \in \mathbf{R}^n \to [L(\mathbf{v})]_F \in \mathbf{R}^m \ \ !!!$$

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n \ \Rightarrow\ [\mathbf{v}]_E = (x_1, x_2, \ldots, x_n)$$

$$L(\mathbf{v}) = x_1L(\mathbf{v}_1) + x_2L(\mathbf{v}_2) + \ldots + x_nL(\mathbf{v}_n) \ \Rightarrow$$

$$[L(\mathbf{v})]_F = x_1\,[L(\mathbf{v}_1)]_F + x_2\,[L(\mathbf{v}_2)]_F + \ldots + x_n\,[L(\mathbf{v}_n)]_F$$

$$= [\ [L(\mathbf{v}_1)]_F\ [L(\mathbf{v}_2)]_F \ldots [L(\mathbf{v}_n)]_F\ ]\,[\mathbf{v}]_E = \mathbf{A}\,[\mathbf{v}]_E$$

□ Th. 4.2.2 *Matrix Representation Theorem*

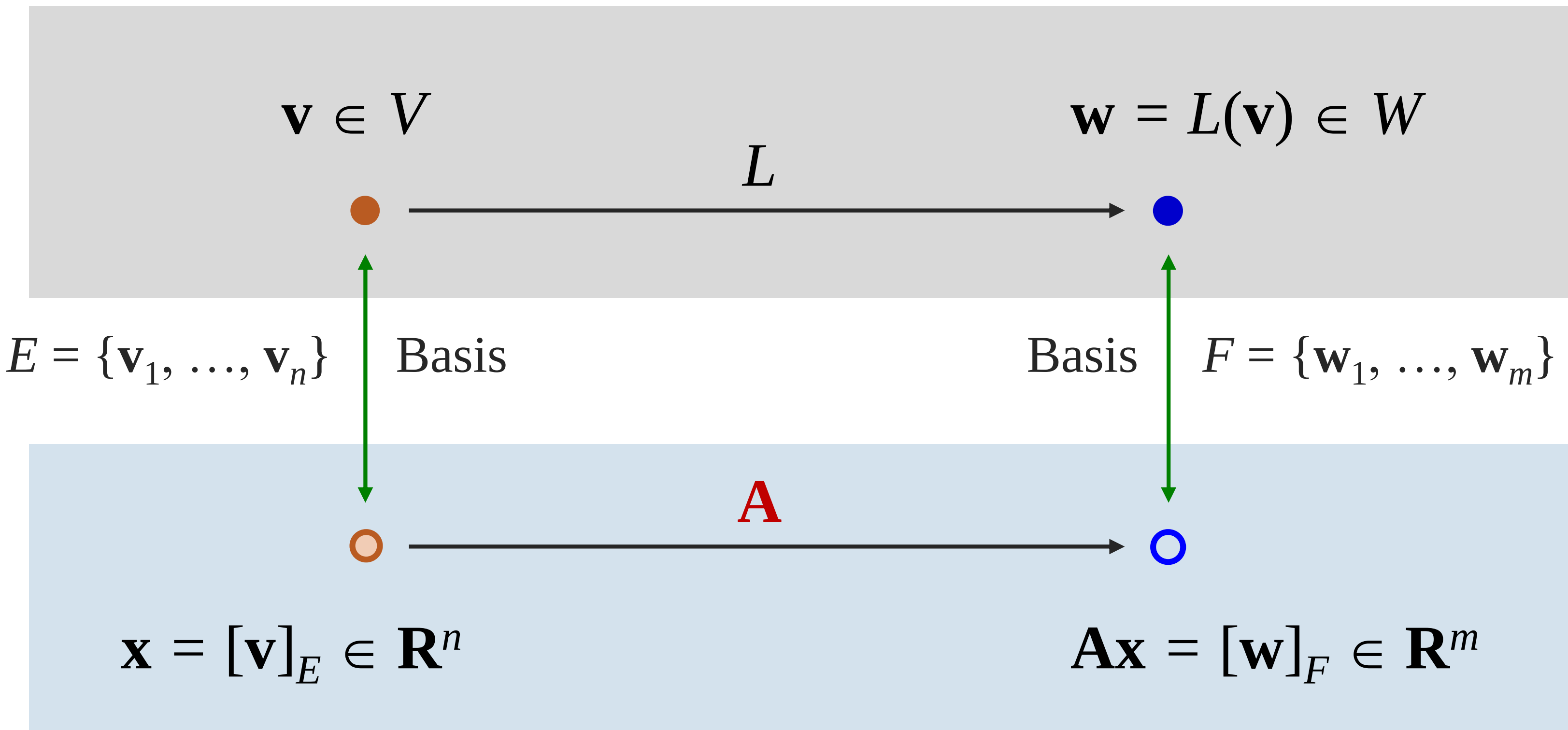
If $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $F = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ are ordered bases for vector spaces V and W, respectively, then corresponding to each linear transformation $L : V \to W$, there is an $m \times n$ matrix $\mathbf{A}$ such that

$$[L(\mathbf{v})]_F = \mathbf{A}\,[\mathbf{v}]_E \qquad \text{for each } \mathbf{v} \in V$$

$\mathbf{A}$ is the matrix representing L relative to the ordered bases E and F. In fact,

$$\mathbf{a}_j = [L(\mathbf{v}_j)]_F \quad j = 1, 2, \ldots, n$$

$$\mathbf{A} = [\ [L(\mathbf{v}_1)]_F\ \ [L(\mathbf{v}_2)]_F\ \ \ldots\ [L(\mathbf{v}_n)]_F\ ]$$



$$\mathbf{A} = [\, [L(\mathbf{v}_1)]_F \,, \ldots ,\, [L(\mathbf{v}_n)]_F \,]$$

$$\mathbf{A} = [\; L(\mathbf{v}_1) \;, \ldots ,\; L(\mathbf{v}_n) \;]$$

$W = \mathbf{R}^m$ &
Stand. B. S

$V = \mathbf{R}^n$ &
Stand. B. S

$$\mathbf{A} = [\; L(\mathbf{e}_1) \;, \ldots ,\; L(\mathbf{e}_n) \;]$$

$W = \mathbf{R}^m$ &
Stand. B. S

Remark

- Th. 4.2.2 is very useful. If we know $\mathbf{A}$, we can evaluate $L(\mathbf{v})$ by computing $\mathbf{A}[\mathbf{v}]_E$, which is just matrix multiplication
- The matrix of L depends on L, E, and F. Even if the order of the vectors in one of the basis changes, the matrix of L changes

(*Ex.*23)

Let $L : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation defined by

$$L(\mathbf{x}) = x\mathbf{b}_1 + (y+z)\mathbf{b}_2 \quad \text{for each } \mathbf{x} \in \mathbf{R}^3,$$

where $\mathbf{b}_1 = (1,1)$ and $\mathbf{b}_2 = (-1,1)$

Find the matrix $\mathbf{A}$ representing L with respect to the ordered bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{b}_1,\mathbf{b}_2\}$

(*Solution*)

$$L(\mathbf{e}_1) = 1\mathbf{b}_1 + 0\mathbf{b}_2$$
$$L(\mathbf{e}_2) = 0\mathbf{b}_1 + 1\mathbf{b}_2 \qquad \therefore \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$
$$L(\mathbf{e}_3) = 0\mathbf{b}_1 + 1\mathbf{b}_2$$

(*Ex.*24)

Let $L : P_3 \to P_3$ be the linear transformation defined by $L(a+bx+cx^2) = b+2cx$. Evaluate $L(5\text{-}7x\text{-}3x^2)$ using the matrix of L with respect to the standard basis S of P_3

(*Solution*)

$$[L(1)]_S = [0]_S = (0,0,0)$$
$$[L(x)]_S = [1]_S = (1,0,0)$$
$$[L(x^2)]_S = [2x]_S = (0,2,0)$$

$$\therefore \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[L(5-7x-3x^2)]_S = \mathbf{A}[5-7x-3x^2]_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ -7 \\ -3 \end{bmatrix} = \begin{bmatrix} -7 \\ -6 \\ 0 \end{bmatrix}$$

$$\therefore L(5-7x-3x^2) = (-7)1 + (-6)x + 0x^2 = -7-6x$$

Th. 4.2.3

Let $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $F = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ be ordered bases for $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. If $L : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation and $\mathbf{A}$ is the $m \times n$ matrix with respect to E and F, then

$$\mathbf{a}_j = \mathbf{B}^{-1} L(\mathbf{u}_j) \quad j = 1, 2, \ldots, n$$

where $\mathbf{B} = [\ \mathbf{b}_1 \ \ldots \ \mathbf{b}_m \ ]$

$$\mathbf{A} = [\ [L(\mathbf{u}_1)]_F \ \ldots \ [L(\mathbf{u}_j)]_F \ \ldots \ [L(\mathbf{u}_n)]_F \ ] = [\ \mathbf{a}_1 \ \ldots \ \mathbf{a}_j \ \ldots \ \mathbf{a}_n \ ]$$

$$\mathbf{a}_j = [L(\mathbf{u}_j)]_F = (a_{1j}, a_{2j}, \Lambda\ , a_{mj}) \quad for \quad j = 1,2,\ldots,n$$

$$L(\mathbf{u}_j) = a_{1j}\mathbf{b}_1 + a_{2j}\mathbf{b}_2 + \Lambda\ + a_{mj}\mathbf{b}_m = \mathbf{B}\mathbf{a}_j \qquad \therefore \mathbf{a}_j = \mathbf{B}^{-1}L(\mathbf{u}_j)$$

$$\text{Note that } \ \mathbf{A} = \mathbf{B}^{-1}[\ L(\mathbf{u}_1) \ \Lambda \ \ L(\mathbf{u}_j) \ \Lambda \ \ L(\mathbf{u}_n) \ ]$$

$$\mathbf{v} \in \mathbf{R}^n \xrightarrow{\quad L \quad} L(\mathbf{v}) \in \mathbf{R}^m$$

Basis $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ Basis $F = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$

Transition Matrix from F to S

$$\mathbf{B} = [\mathbf{b}_1 \ \ldots \ \mathbf{b}_m]$$

$$\mathbf{A} = [\,\mathbf{B}^{-1}L(\mathbf{u}_1)\,, \ldots, \mathbf{B}^{-1}L(\mathbf{u}_n)\,]$$

$W = \mathbf{R}^m$ & Basis F

$$= \mathbf{B}^{-1}[\,L(\mathbf{u}_1), \ldots, L(\mathbf{u}_n)\,] = \mathbf{B}^{-1}\mathbf{L}$$

$$[L(\mathbf{v})]_F = \mathbf{A}\,[\mathbf{v}]_E = \mathbf{B}^{-1}\mathbf{L}\,[\mathbf{v}]_E = \mathbf{B}^{-1}[L(\mathbf{v})]_S$$

$$\mathbf{A} = [\ [L(\mathbf{u}_1)]_F\,, \ldots, [L(\mathbf{u}_n)]_F\ ]$$

$$\mathbf{A} = [\ \ L(\mathbf{u}_1)\quad, \ldots, \ \ L(\mathbf{u}_n)\quad]$$

$W = \mathbf{R}^m$ & Stand. B. S

Corollary 4.2.4

If $\mathbf{A}$ is the matrix representing the linear transformation $L : \mathbf{R}^n \to \mathbf{R}^m$ with respect to the bases $E = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $F = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$, then the reduced echelon form of $[\ \mathbf{b}_1 \ \ldots\ \mathbf{b}_m \mid L(\mathbf{u}_1) \ \ldots\ L(\mathbf{u}_n)\ ]$ is $[\ \mathbf{I} \mid \mathbf{A}\ ]$

$$\mathbf{B} = [\ \mathbf{b}_1\ \mathbf{b}_2\ \Lambda\ \ \mathbf{b}_n\ ]$$

$$[\ \mathbf{B} \mid L(\mathbf{u}_1)\ L(\mathbf{u}_2)\ \Lambda\ \ L(\mathbf{u}_n)\ ]$$

$$\sim [\ \mathbf{I} \mid \mathbf{B}^{-1}L(\mathbf{u}_1)\ \ \mathbf{B}^{-1}L(\mathbf{u}_2)\ \ \Lambda\ \ \mathbf{B}^{-1}L(\mathbf{u}_n)\ ] = [\ \mathbf{I} \mid \mathbf{A}]$$

(*Ex*.25)

Let $L : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation defined by

$$L(\mathbf{x}) = (y, x+y, x-y)$$

Find the matrix representation of L with respect to the ordered bases $E = \{\mathbf{u}_1,\mathbf{u}_2\}$ and $F = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, where

$\mathbf{u}_1 = (1,2)$, $\mathbf{u}_2 = (3,1)$, $\mathbf{b}_1 = (1,0,0)$, $\mathbf{b}_2 = (1,1,0)$, $\mathbf{b}_3 = (1,1,1)$

(*Solution*)

$$\left[\, \mathbf{B} \mid L(\mathbf{u}_1) \;\; L(\mathbf{u}_2) \,\right] = \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

(*Another Solution* from Th.4.2.3)

$$\begin{tikzcd} & {[L(\mathbf{x})]_S} \arrow[dr, "\mathbf{B}^{-1}"] & \\ {[\mathbf{x}]_E} \arrow[ur, "\mathbf{L}"] & & {[L(\mathbf{x})]_F} \end{tikzcd}$$

$$\mathbf{L} = [\, L(\mathbf{u}_1)\ L(\mathbf{u}_2)] = \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 2 \end{bmatrix}$$
; Transformation Matrix w.r.t E and S

$$\mathbf{B} = [\, \mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
; Transition Matrix from F to S

$$\therefore \mathbf{A} = \mathbf{B}^{-1}\mathbf{L} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$$

Similarity

- To consider different matrix representations of a *linear operator* $L : V \rightarrow V$
- To characterize the relationship between matrices representing the same linear operator

Matrices of L with respect to different bases

Let $L(\mathbf{x}) = (2x, x+y)$. Then the matrix of L with respect to $S = \{\mathbf{e}_1,\mathbf{e}_2\}$ is

$$\mathbf{A} = [\,\mathbf{e}_1\ \mathbf{e}_2\,]^{-1}[\,L(\mathbf{e}_1)\ L(\mathbf{e}_2)\,] \overset{\star}{=} [\,L(\mathbf{e}_1)\ L(\mathbf{e}_2)\,] = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

If we use a basis $U = \{\mathbf{u}_1 = (1,1),\ \mathbf{u}_2 = (-1,1)\}$ for $\mathbf{R}^2$,

$$\mathbf{B} = [\,\mathbf{u}_1\ \mathbf{u}_2\,]^{-1}[\,L(\mathbf{u}_1)\ L(\mathbf{u}_2)\,] = \mathbf{U}^{-1}[\,\mathbf{A}\mathbf{u}_1\ \mathbf{A}\mathbf{u}_2\,] = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$$

since $L(\mathbf{x}) = [L(\mathbf{x})]_S = \mathbf{A}[\mathbf{x}]_S = \mathbf{A}\mathbf{x}$

Therefore, $$\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$\star$ From Th. 4.2.3, $\mathbf{A} = [\,\mathbf{b}_1 \ldots \mathbf{b}_m\,]^{-1}[\,L(\mathbf{u}_1) \ldots L(\mathbf{u}_n)\,]$ from $\{\mathbf{u}_1 \ldots \mathbf{u}_n\}$ to $\{\mathbf{b}_1 \ldots \mathbf{b}_m\}$

(Another Interpretation of Computing $\mathbf{B}$)

Now we want to get the matrix $\mathbf{B}$ such that

$[L(\mathbf{x})]_U = \mathbf{B}\,[\mathbf{x}]_U$. Notice that $[L(\mathbf{x})]_S = \mathbf{A}\,[\mathbf{x}]_S$

Since the *transition matrix* from $U = \{\mathbf{u}_1, \mathbf{u}_2\}$ to S is $\mathbf{U} = [\ \mathbf{u}_1\ \mathbf{u}_2\ ]$, the transition matrix from S to U is $\mathbf{U}^{-1}$

Thus

$$[L(\mathbf{x})]_U = \mathbf{U}^{-1}\,[L(\mathbf{x})]_S = \mathbf{U}^{-1}\mathbf{A}\,[\mathbf{x}]_S = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}\,[\mathbf{x}]_U$$

Therefore, $\mathbf{B} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$

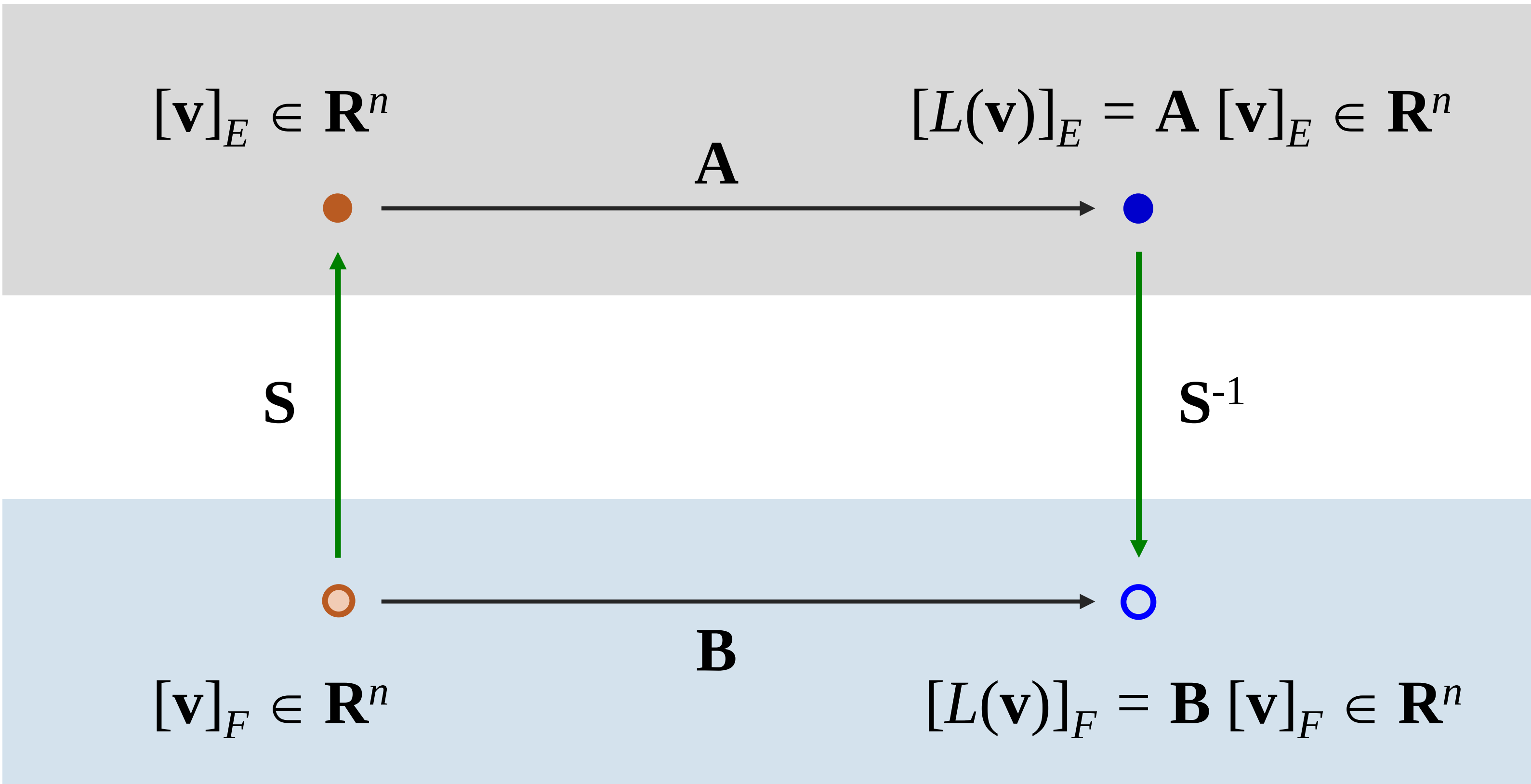
- Th. 4.3.1

  Let $E = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $F = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be two ordered bases for a vector space V, and let L be a linear operator on V. Let $\mathbf{S}$ be the transition matrix representing the change from F to E. If $\mathbf{A}$ is the matrix with respect to E, and $\mathbf{B}$ is the matrix with respect to F, then $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$

For $\mathbf{v} \in V, [L(\mathbf{v})]_E = \mathbf{A}[\mathbf{v}]_E, [L(\mathbf{v})]_F = \mathbf{B}[\mathbf{v}]_F, [\mathbf{v}]_E = \mathbf{S}[\mathbf{v}]_F$

$$[L(\mathbf{v})]_F = \mathbf{S}^{-1}[L(\mathbf{v})]_E = \mathbf{S}^{-1}\mathbf{A}[\mathbf{v}]_E = \mathbf{S}^{-1}\mathbf{AS}[\mathbf{v}]_F$$

$$\therefore \mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$$



For $\mathbf{v} \in V, [L(\mathbf{v})]_E = \mathbf{A}[\mathbf{v}]_E, [L(\mathbf{v})]_F = \mathbf{B}[\mathbf{v}]_F, [\mathbf{v}]_E = \mathbf{S}[\mathbf{v}]_F$

$$[L(\mathbf{v})]_F = \mathbf{S}^{-1}[L(\mathbf{v})]_E = \mathbf{S}^{-1}\mathbf{A}[\mathbf{v}]_E = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}[\mathbf{v}]_F$$

Def. Similar

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. $\mathbf{B}$ is said to be *similar* to $\mathbf{A}$ if there exist a nonsingular matrix $\mathbf{S}$ such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$

- $\mathbf{A}$ is also similar to $\mathbf{B}$ since $\mathbf{A} = (\mathbf{S}^{-1})^{-1}\mathbf{B}\mathbf{S}^{-1} = \mathbf{T}^{-1}\mathbf{B}\mathbf{T}$
- If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices representing the same operator L, then $\mathbf{A}$ and $\mathbf{B}$ are similar from Th. 4.3.1

- Conversely, suppose that $\mathbf{A}$ represents L with respect to the ordered basis $E = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$ for some nonsingular matrix $\mathbf{S}$.

  If each $\mathbf{w}_i$ $(i = 1,2,\ldots,n)$ is defined by

$$\mathbf{w}_i = s_{1i}\,\mathbf{v}_1 + s_{2i}\,\mathbf{v}_2 + \ldots + s_{ni}\,\mathbf{v}_n$$

  then $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is an ordered basis for V, and $\mathbf{B}$ is the matrix representing L with respect to the ordered basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$

(*Ref.*) $\mathbf{S} = [\ [\mathbf{w}_1]_E \ldots [\mathbf{w}_i]_E \ldots [\mathbf{w}_n]_E\ ]$

and $[\mathbf{w}_i]_E = (s_{1i}, s_{2i}, \ldots, s_{ni})$

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{bmatrix}$$

(*Ex*.26)

Let D be the differentiation operator on P_3. Find the matrix $\mathbf{A}$ representing D with respect to the standard basis $S = \{1, x, x^2\}$ and the matrix $\mathbf{B}$ representing D with respect to the ordered basis $E = \{1, 2x, 4x^2\text{-}2\}$

(*Solution*)

$$[D(1)]_S = [0]_S = (0,0,0)$$
$$[D(x)]_S = [1]_S = (1,0,0)$$
$$[D(x^2)]_S = [2x]_S = (0,2,0)$$

$$\therefore \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D(1) = 0 = 0\cdot 1 + 0\cdot 2x + 0\cdot(4x^2 - 2)$$
$$D(2x) = 2 = 2\cdot 1 + 0\cdot 2x + 0\cdot(4x^2 - 2)$$
$$D(4x^2 - 2) = 8x = 0\cdot 1 + 4\cdot 2x + 0\cdot(4x^2 - 2)$$

$$\therefore \mathbf{B} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

The transition matrix $\mathbf{S}$ from E to S is

$$\mathbf{S} = [\,[1]_s\ [2x]_s\ [4x^2-2]_s\,] = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Thus

$$\mathbf{B} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

(*Ex*.27)

Let L be the linear operator mapping $\mathbf{R}^3$ into $\mathbf{R}^3$ defined by $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

Thus the matrix $\mathbf{A}$ represents L with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Find the matrix $\mathbf{B}$ representing L with respect to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$, where

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{y}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(*Solution*)

Let the transition matrix from $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be $\mathbf{Y}$.

Then

$$\mathbf{B} = \mathbf{Y}^{-1}\mathbf{A}\mathbf{Y} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

On the one hand,

$$L(\mathbf{y}_1) = \mathbf{A}\mathbf{y}_1 = \mathbf{0} = 0\mathbf{y}_1 + 0\mathbf{y}_2 + 0\mathbf{y}_3$$
$$L(\mathbf{y}_2) = \mathbf{A}\mathbf{y}_2 = \mathbf{y}_2 = 0\mathbf{y}_1 + 1\mathbf{y}_2 + 0\mathbf{y}_3$$
$$L(\mathbf{y}_3) = \mathbf{A}\mathbf{y}_3 = 4\mathbf{y}_3 = 0\mathbf{y}_1 + 0\mathbf{y}_2 + 4\mathbf{y}_3$$

$$\therefore \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$