# Set Theory

# Sets and Subsets

- Set
  - A collection of objects
  - $A = \{a_1, a_2, a_3\}$
  - $a_2 \in A$:  $a_2$ is an element of $A$ ($a_2$ is in $A$).
  - $a_2 \notin A$:  $a_2$ is not an element of $A$ ($a_2$ is not in $A$).
  - $a_i \in A,\ \ 1 \le i \le 3$.
  - Another representation

    $A = \{x \mid B(x)\}$,     $B(x)$: $x$ has blue eyes

    $A = \{x \mid x$ is an integer and $1 \le x \le 5\}$

    "the set of all $x$ such that …"

- Notations for sets frequently referred to
  - $\mathbf{N} = \{0, 1, 2, \ldots\}$ set of natural numbers
  - $\mathbf{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ set of integers
  - $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ set of positive integers
  - $\mathbf{R}$: set of real numbers
  - $\mathbf{R}^+$: set of positive real numbers
  - $\mathbf{Q}$: set of rational numbers
  - $\varnothing$, $\{\}$: empty set or null set
  - $U$: universal set or universe of discourse
- Cardinality (size) of a set
  - $|A|$: the number of elements in $A$ (when it is finite).

- Subset
  - A set $B$ is a subset of $A$ ($B \subseteq A$) if every element of $B$ is an element of $A$.
  - $B \subseteq A \Leftrightarrow (\forall x)(x \in B \rightarrow x \in A)$
  - $B \subseteq A \Rightarrow |B| \leq |A|$

- Theorem

  For every set $A$, $A \subseteq U$, $A \subseteq A$, *and* $\varnothing \subseteq A$.

  $$(\forall x)(x \in \varnothing \rightarrow x \in A)$$

- Set Equality
  - A set $A$ is equal to a set $B$ iff $(A \subseteq B) \wedge (B \subseteq A)$.
  - $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$

- Proper Subset
  - If $A \subseteq B$ and $A \neq B$, then $A$ is called a proper subset of $B$ ($A \subset B$).
  - Note that $A \not\subset A$.
  - $B \subset A \;\Rightarrow\; |B| < |A|$
  - $B \subset A \;\Rightarrow\; B \subseteq A$

- Power Set
  - If $A$ is a set then the power set of $A$, denoted by $\wp(A)$, is the collection (or set) of all subsets of $A$.

    $$\wp(A) = \{B \mid B \subseteq A\}$$

    E.g., for a set $A = \{a, b\}$, $\wp(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$

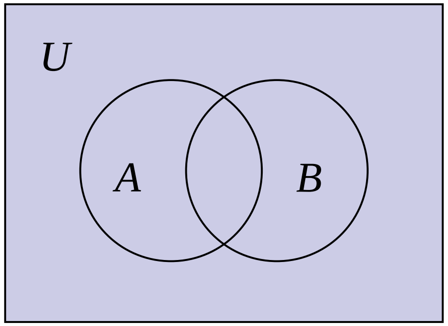
    $$|A| = 2, \ |\wp(A)| = 2^2$$

- Theorem

  In general, $|\wp(A)| = 2^{|A|}$.

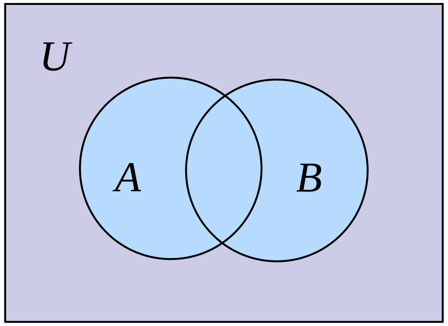
## Set Operations

- Venn Diagram
  - Represents relations of sets
  - Does not constitute a proof



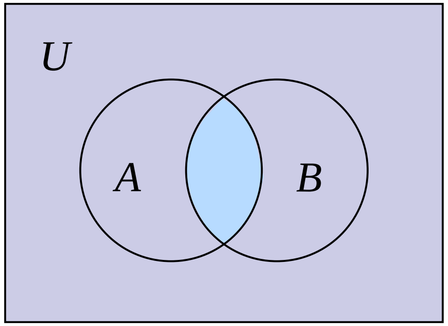

- Let $A$ and $B$ be two sets, then
  - $A \cup B = \{ x \mid x \in A \vee x \in B \}$

    set union



- $A \cap B = \{ x \mid x \in A \wedge x \in B \}$

  set intersection

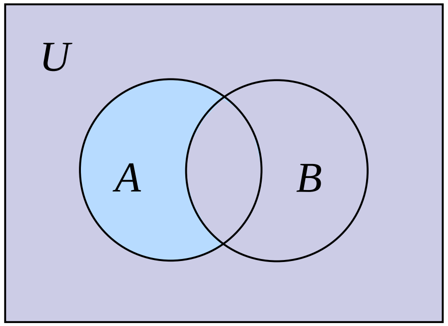



- $A - B = \{ x \mid x \in A \wedge x \notin B \}$

  set difference

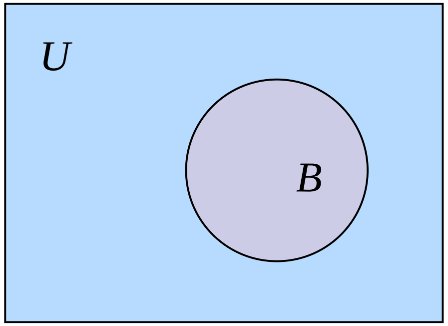
  (relative complement of $B$

  with respect to $A$)



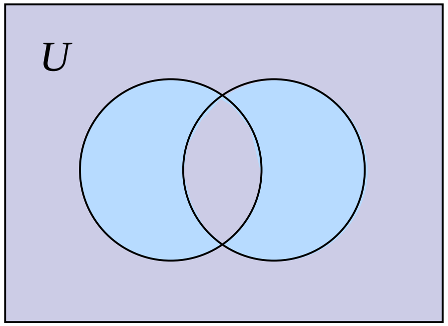
- $\overline{B} = U - B = \{ x \mid x \notin B \}$

  complement of a set $B$



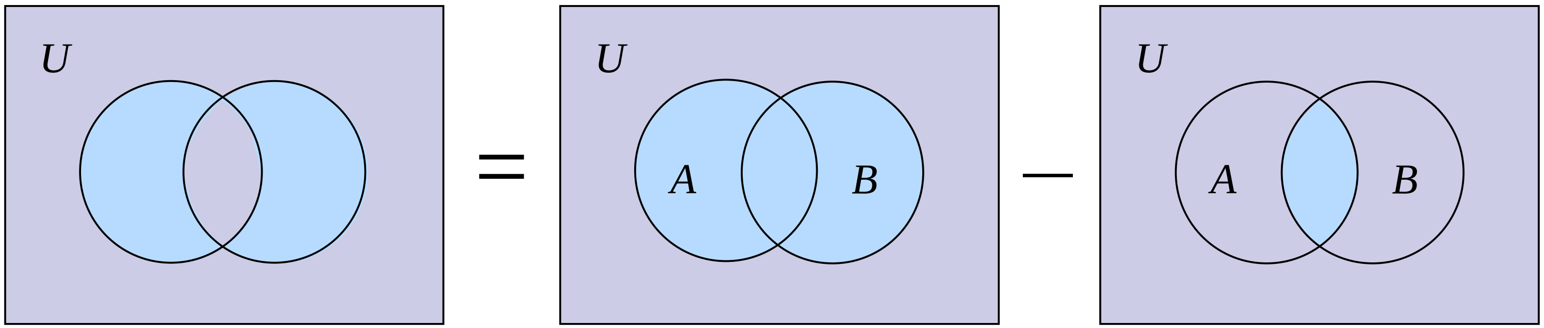

- $A \,\Delta\, B = \{ x \mid x \in A - B \vee x \in B - A \}$

  symmetric difference (of $A$ and $B$)



- Theorem

$$A \,\Delta\, B = (A \cup B) - (A \cap B)$$




( *Note* )

$$A \,\Delta\, B = \{\, x \mid x \in A - B \vee x \in B - A \}$$

$$A - B = \{\, x \mid x \in A \wedge x \notin B \}$$

$$(A \cup B) - (A \cap B) = \{\, x \mid x \in A \cup B \wedge x \notin A \cap B \}$$

***Proof*:**

By the definition of set equality, we need to show

(1) $A \Delta B \subseteq (A \cup B) - (A \cap B)$ and

(2) $(A \cup B) - (A \cap B) \subseteq A \Delta B$.

Part 1:

Let $x \in A \Delta B$.

Then, $x \in A - B$ or $x \in B - A$.

Suppose $x \in A - B$.

Then, $x \in A$ and $x \notin B$.

This implies that $x \in A \cup B$ and $x \notin A \cap B$.

Therefore, $x \in (A \cup B) - (A \cap B)$.

Now, suppose $x \in B - A$.

*Proof*:

We can similarly show that

$$x \in (B \cup A) - (B \cap A) = (A \cup B) - (A \cap B).$$

From these two cases, we conclude that

$$A \,\Delta\, B \subseteq (A \cup B) - (A \cap B).$$

Part 2: (left as an exercise)

Let $x \in (A \cup B) - (A \cap B)$.

. . . . .

$$(A \cup B) - (A \cap B) \subseteq A \,\Delta\, B.$$

From the proofs of Part 1 and Part 2, we finally conclude that

$$A \,\Delta\, B = (A \cup B) - (A \cap B).$$

## Formal proof of **part 2**: $(A \cup B) - (A \cap B) \subseteq A \,\Delta\, B$

| No. | Formula | Rule | Just. | Taut. |
|---|---|---|---|---|
| 1 | $x \in (A \cup B) - (A \cap B)$ | AP | | |
| 2 | $x \in (A \cup B) \wedge x \notin (A \cap B)$ | T | 1 | Def. of $-$ |
| 3 | $x \in (A \cup B)$ | T | 2 | $I_1$ |
| 4 | $x \notin (A \cap B)$ | T | 2 | $I_2$ |
| 5 | $x \in A \vee x \in B$ | T | 3 | Def. of $\cup$ |
| 6 | $\neg(x \in A \wedge x \in B)$ | T | 4 | Def. of $\cap$ |
| 7 | $x \notin A \vee x \notin B$ | T | 6 | $E_8$ |
| 8 | $(x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B)$ | T | 5, 7 | $I_9$ |
| 9 | $\{(x \in A \vee x \in B) \wedge x \notin A\}$ | T | 8 | $E_6$ |
| | $\vee \{(x \in A \vee x \in B) \wedge x \notin B\}$ | | | |
| 10 | $(x \in A \wedge x \notin A) \vee (x \in B \wedge x \notin A)$ | T | 9 | $E_6$ |
| | $\vee (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin B)$ | | | |

## Formal proof of part 2: $(A \cup B) - (A \cap B) \subseteq A \Delta B$

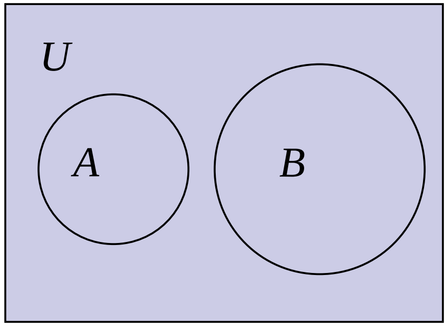
| No. | Formula | Rule | Just. | Taut. |
|---|---|---|---|---|
| 11 | $(x \in B \wedge x \notin A) \vee (x \in A \wedge x \notin B)$ | T | 10 | $E_{12}$ |
| 12 | $(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$ | T | 11 | $E_3$ |
| 13 | $(x \in A - B) \vee (x \in B - A)$ | T | 12 | Def. of $-$ |
| 14 | $x \in A \Delta B$ | T | 13 | Def. of $\Delta$ |
| 15 | $x \in (A \cup B) - (A \cap B) \rightarrow x \in A \Delta B$ | CP | 1, 14 | |
| 16 | $(\forall x)\ \{x \in (A \cup B) - (A \cap B) \rightarrow x \in A \Delta B\}$ | UG | 15 | |
| 17 | $(A \cup B) - (A \cap B) \subseteq A \Delta B$ | T | 16 | Def. of $\subseteq$ |

- Disjoint Set
  - The sets $A$ and $B$ are said to be disjoint, or mutually disjoint if $A \cap B = \varnothing$.

- Theorem

  Let $A, B \subseteq U$.

  $A$ and $B$ are disjoint iff $A \cup B = A \,\Delta\, B$.

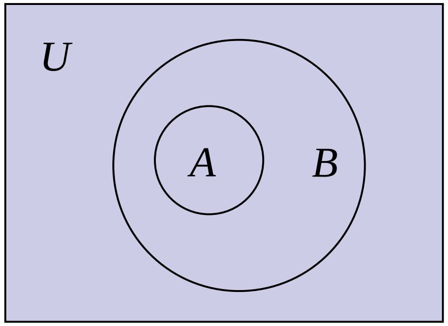



- Theorem

  If $A, B \subseteq U$, then the following are equivalent:

  (a) $A \subseteq B$ (b) $A \cap B = A$

  (c) $A \cup B = B$ (d) $\overline{B} \subseteq \overline{A}$



***Proof***:

To prove this theorem we need to show

(a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) (The proof consists of six parts).

Alternatively, we can just show the following:

(a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), (d) $\Rightarrow$ (a) (Only four parts)

Part (a) $\Rightarrow$ (b): $(A \subseteq B) \Rightarrow (A = A \cap B)$

Assume $A \subseteq B$.

Let $x$ be an arbitrary element of $A$, i.e., $x \in A$.

Then, $x \in B$ by the definition of subset and the assumption.

Since $x \in A$ and $x \in B$, we know $x \in A \cap B$ by the definition of set intersection.

***Proof***:

Since $x$ is an arbitrary element of $A$, every element of $A$ is an element of $A \cap B$. [UG]

Hence, by the definition of subset, $A \subseteq A \cap B$.

Let $x$ be an arbitrary element of $A \cap B$, i.e., $x \in A \cap B$.

Then, by the definition of set intersection, $x \in A$ and $x \in B$.

Obviously, $x \in A$. [$I_1$]

Since $x$ is an arbitrary element of $A \cap B$, every element of $A \cap B$ is an element of $A$. [UG]

Hence, by the definition of subset, $A \cap B \subseteq A$.

Therefore, $A \cap B = A$ by the definition of set equality.

## Formal proof of <u>part (c) ⇒ (d)</u>: $A \cup B = B \Rightarrow \overline{B} \subseteq \overline{A}$

| No. | Formula | Rule | Just. | Taut. |
|---|---|---|---|---|
| 1 | $A \cup B = B$ | P | | |
| 2 | $x \in \overline{B}$ | AP | | |
| 3 | $x \notin B$ | T | 2 | Def. of Comp. |
| 4 | $x \notin A \cup B$ | T | 1, 3 | Equal sets |
| 5 | $\neg(x \in A \vee x \in B)$ | T | 4 | Def. of $\cup$ |
| 6 | $x \notin A \wedge x \notin B$ | T | 5 | $E_9$ |
| 7 | $x \notin A$ | T | 6 | $I_1$ |
| 8 | $x \in \overline{A}$ | T | 7 | Def. of Comp. |
| 9 | $x \in \overline{B} \rightarrow x \in \overline{A}$ | CP | 2, 8 | |
| 10 | $\forall x\,(x \in \overline{B} \rightarrow x \in \overline{A})$ | UG | 9 | |
| 11 | $\overline{B} \subseteq \overline{A}$ | T | 10 | Def. of $\subseteq$ |