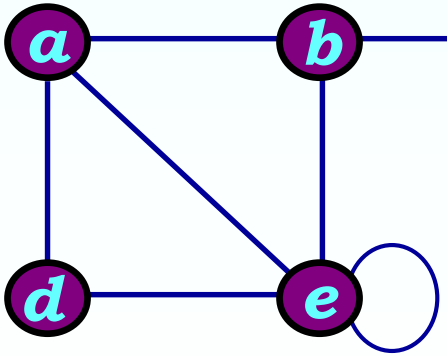
# Degree of Vertices

## Definition and Notation

Let $G$ be an undirected graph or multigraph.

For each vertex $v$ of $G$, the degree of $v$, denoted by ***deg(v)***, is the number of edges in $G$ that are incident with $v$

A **loop** at vertex $v$ : **two** incident edges for $v$




**Degree of each node**

**deg(a) = 3, deg(b) = 3**
**deg(c) = 1, deg(d) = 2**
**deg(e) = 5**

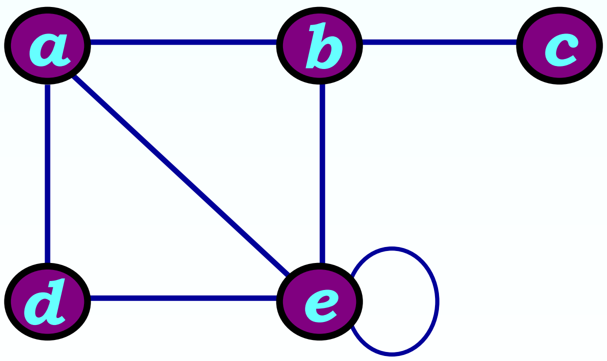
# Theorem 11.2

- **Total sum of degrees**

  If $G=(V,E)$ is an undirected graph or multigraph, then

$$\sum_{v \in V} \deg(v) = 2|E|$$

- **Proof :** Each edge contributes a count of 2




$\deg(a) = 3$
$\deg(b) = 3$
$\deg(c) = 1$
$\deg(d) = 2$
$\deg(e) = 5$
-------------- $2 \times$ #edges
14 = 2|E|

# Corollary 11.1

- **Corollary 11.1**

  For any undirected graph or multigraph, the number of odd degree vertices must be even

  ( Proof )

  The total sum of degrees is even since it is $2|E|$.

  Two kinds of vertices

  Even degree : $\sum deg(v_{even})$ is even

  Odd degree : $\sum deg(v_{odd})$ should be also even

  Therefore, the number of odd degree vertices must be even

# Regular Graphs

- Definition

An undirected graph (or multigraph) where each vertex has the same degree is called a regular graph. If deg($v$) = $k$ for all vertices, then the graph is called $k$-regular

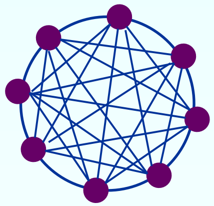
(Ex.) $K_5$, (Mesh), Hypercube, etc.

(Question) Is it possible to have a 4-regular graph with 15 edges?

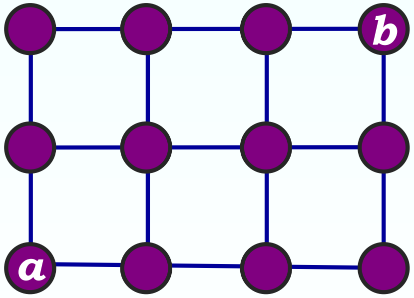
- **Interconnection Models** in Parallel Computers

  CPU : Vertex

  Connection for Communication : Edge



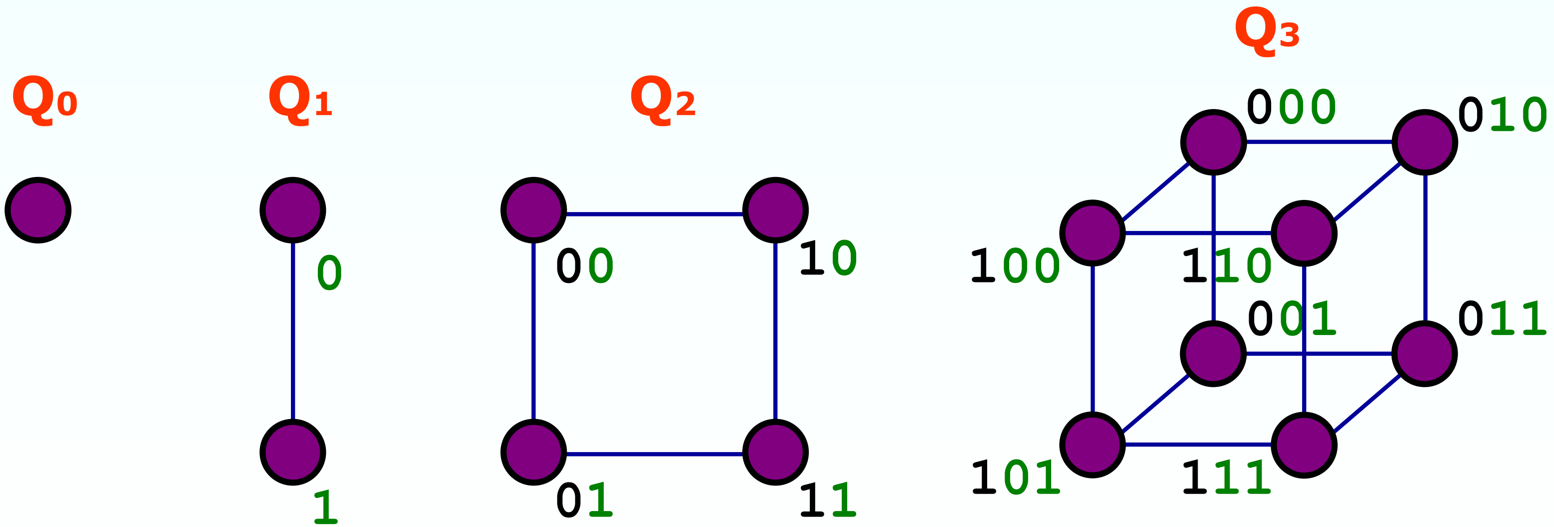
- Complete Graph : ideal but very expensive
- Linear or Cycle Graph : inefficient communication
- Mesh : 2D, 3D
  - Distance between processors increases as the number of processors does




- Hypercube
  - A compromised model that weighs the number of edges against the distance between processors

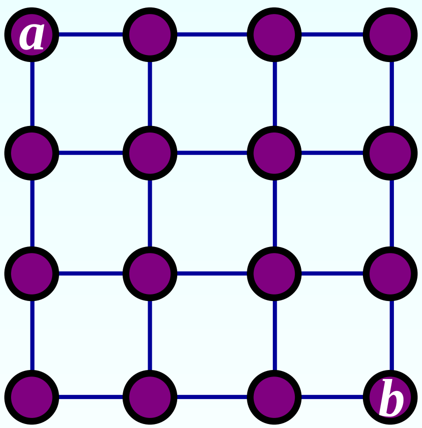
## Hypercube $Q_n$

- An $n$-regular loop-free undirected graph with $2^n$ vertices which are labeled by $n$-bit sequences
- Two vertices $v_1$, $v_2$ of $Q_n$ are joined by an edge when their binary labels differ in exactly one position



# Distances in interconnection models

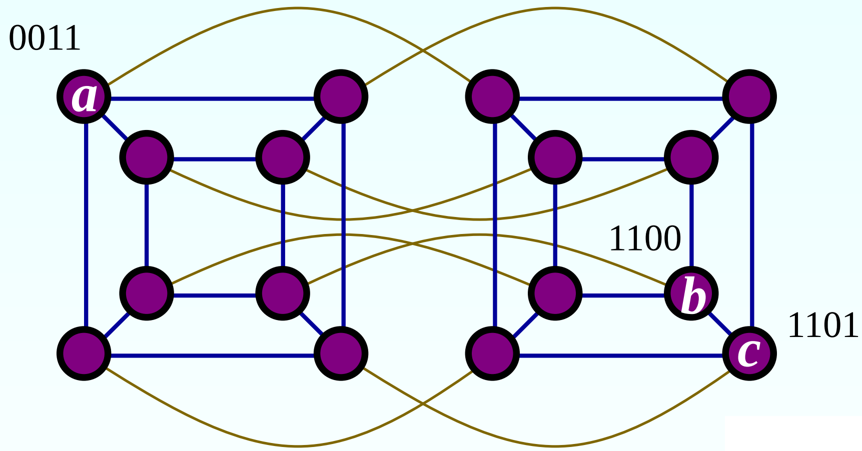
**Mesh**




$d(a,b) = 6$

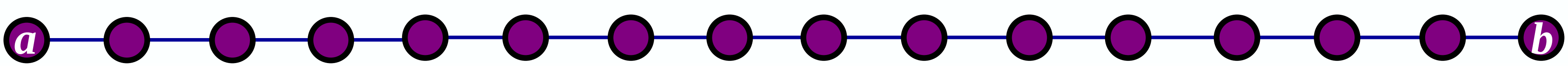
**$Q_4$**

$d(x,y)$ = Hamming distance between $x$ and $y$




$d(a,b) = 4$

0011
1100



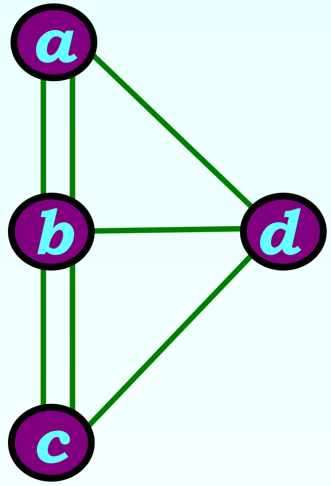

$d(a,b) = 15$

# Euler Circuit

Let $G$ an undirected graph with no isolated vertices

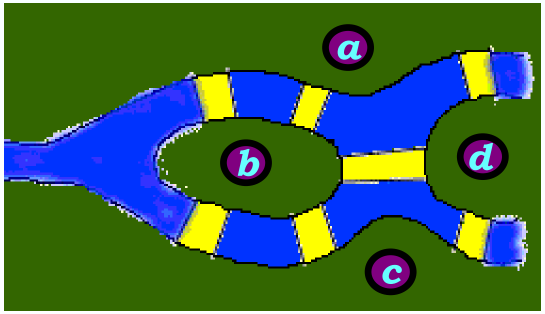
- **Euler circuit**

  A circuit in $G$ that traverses every edge of $G$ exactly once




- **Euler trail**

  An open trail from a vertex to another vertex in $G$, which traverses each edge in $G$ exactly once



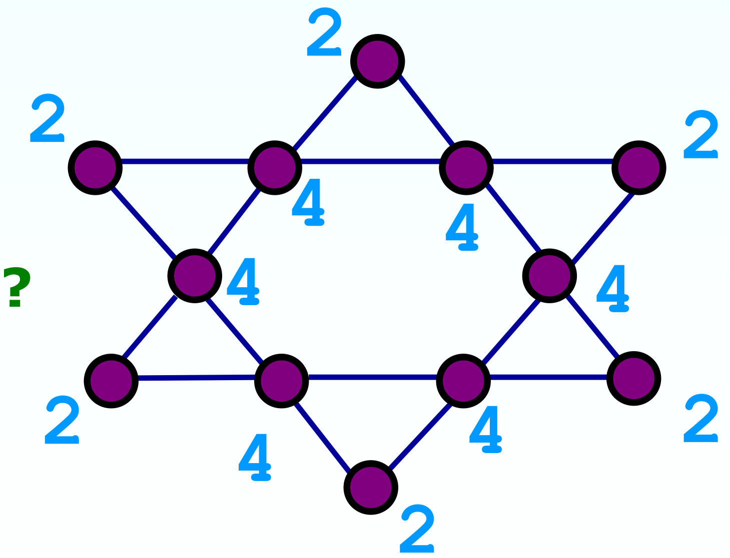
# Theorem 11.3

- Existence of Euler circuit

  Let $G=(V,E)$ is an undirected graph <u>or multigraph</u> with no isolated vertices. $G$ has an Euler circuit

  $\Leftrightarrow$ $G$ is connected and every vertex in $G$ has even degree

Euler Circuit ?



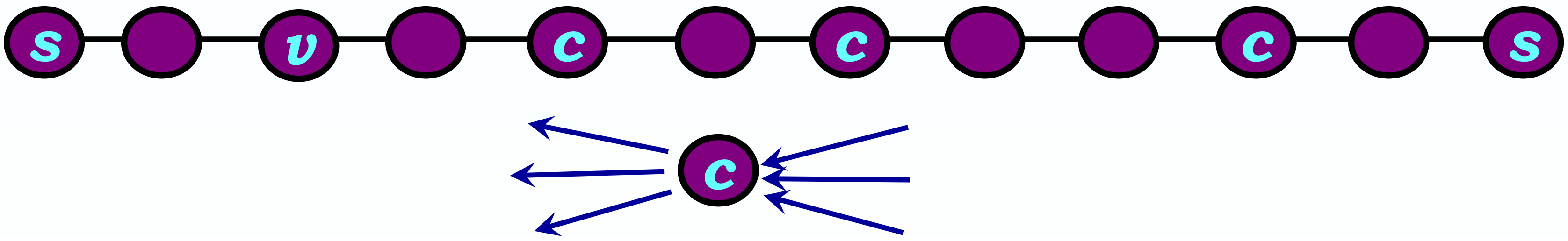
# Proof (➔)

- Only if (➔)
  - ∃ Euler circuit in $G$ ➔ connected, even degrees for all vertices

(1) For all $a, b \in V$, there is a trail from $a$ to $b$. By Theorem (∃ trail → ∃ path), $G$ is connected

(2) Let $s$ be the starting vertex. For any other vertex $v$, a count of 2 is obtained each time the circuit passes through $v$. Thus *deg(v)* is even.

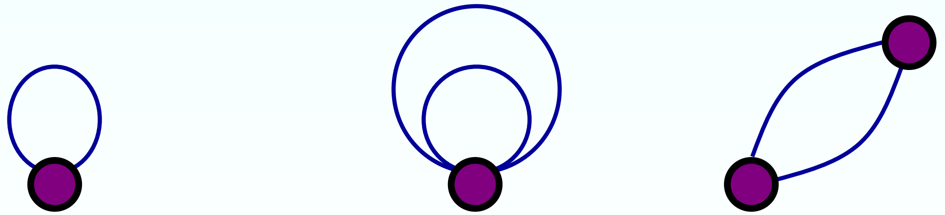
Meanwhile, *deg(s)* is also even. (Why?)



# Proof (←)

- if (←)
  - ∃ Euler circuit in *G* ← connected, even degrees for all vertices

If the number of edges in *G* is 1 or 2, then *G* must be shown as follows. Euler circuits are immediate in these cases.



We proceed now by induction and assume the result *true* for all situations where there are *fewer than n edges*.

# Proof (←)

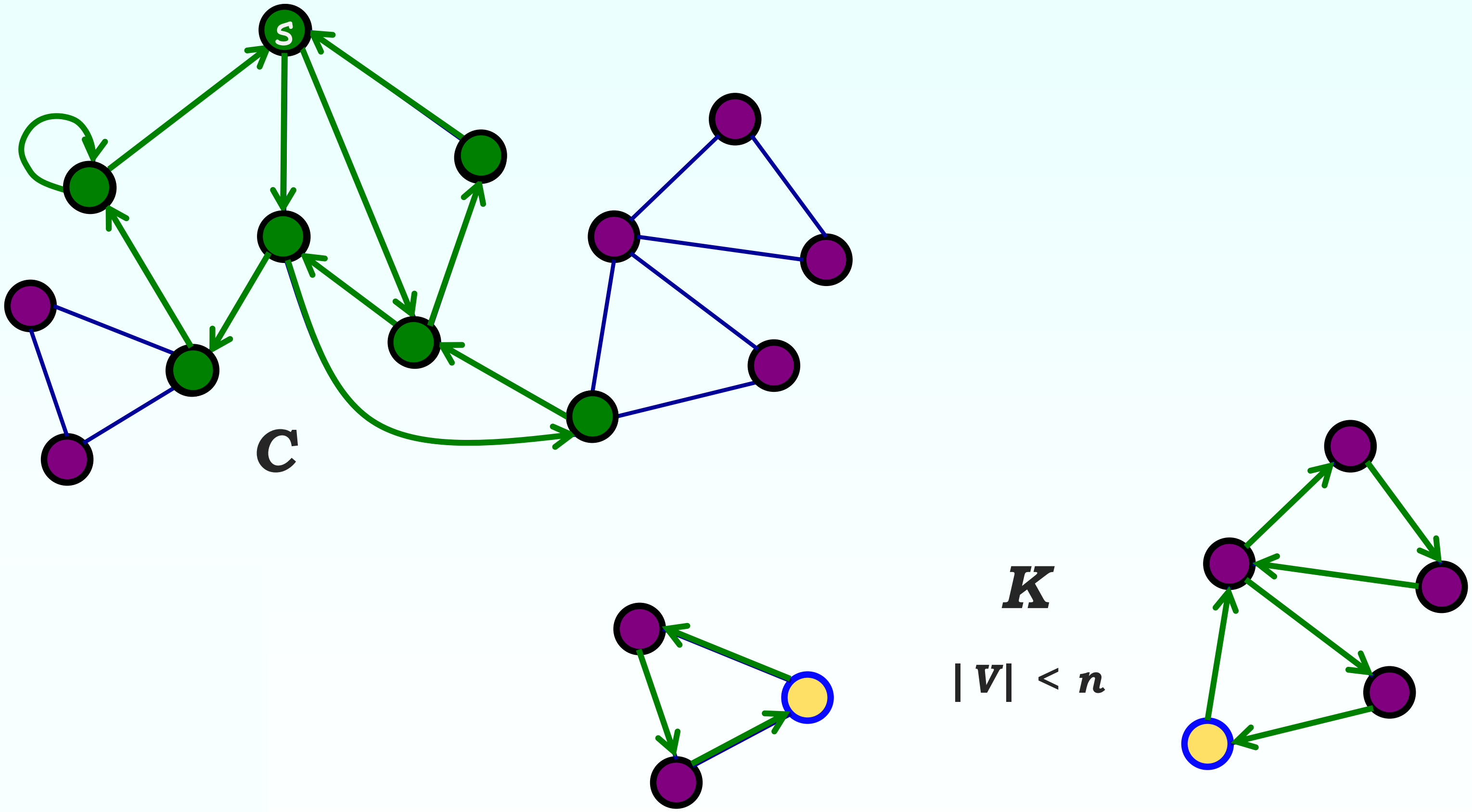
- We can at least *construct a circuit* C starting at a vertex *s* in G with *n* edges ( Why ? )

  ( Considering the longest trail in G that starts at *s*. )

- If the circuit contains every edge of G, we are finished. If not, *remove* the edges of the circuit from G, making sure to remove any vertex that would become isolated. The *remaining graph K* has all vertices of even degree, but it may not be connected. *Each connected component of K* has fewer than *n* edges and thus it will *have an Euler circuit*. In addition, each of these Euler circuits has a vertex that is on C.

If we enter a non-starting vertex during traversing from s, then we can get out of the vertex. Thus, ultimately, we can return to the starting vertex.

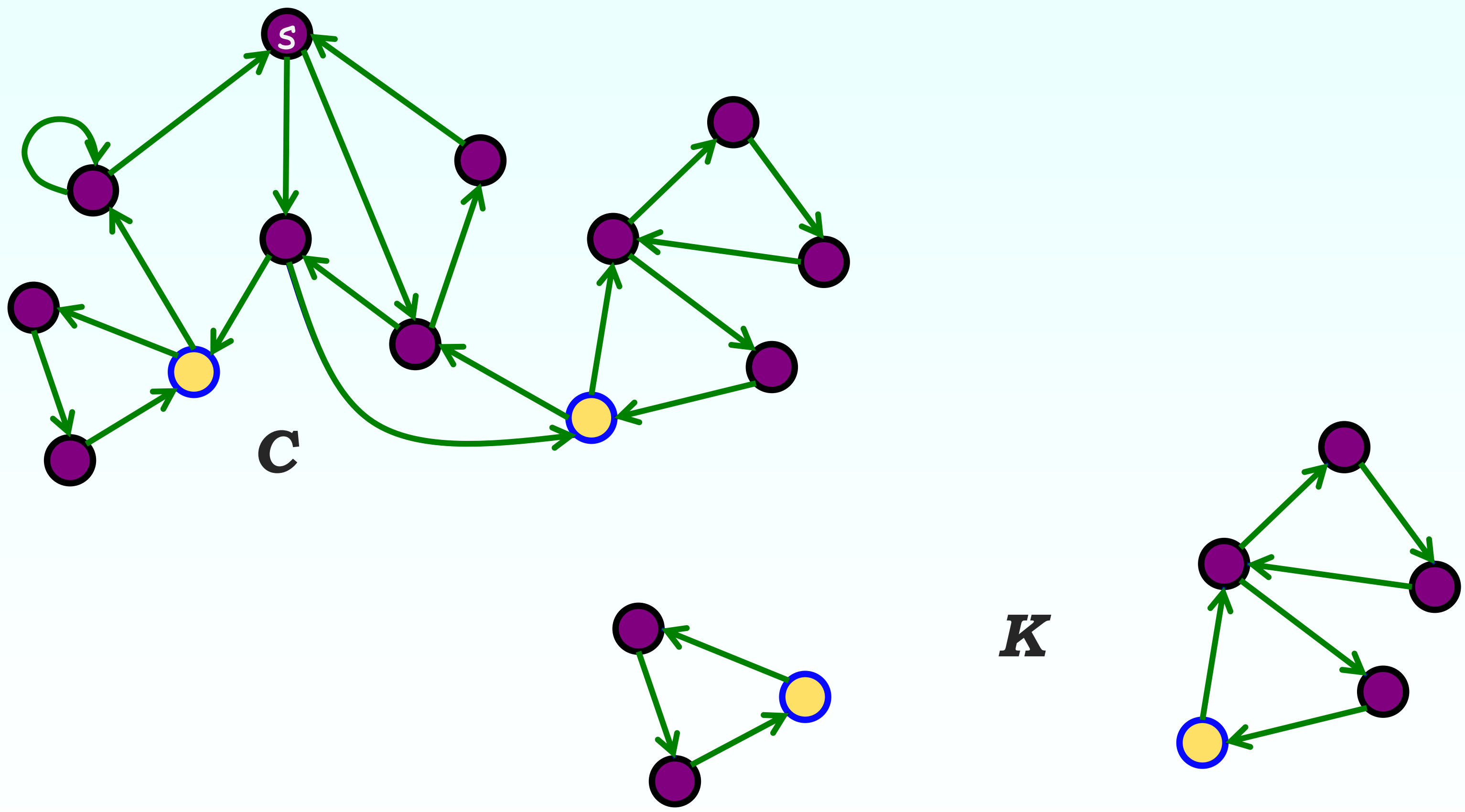
# Proof (🡐)



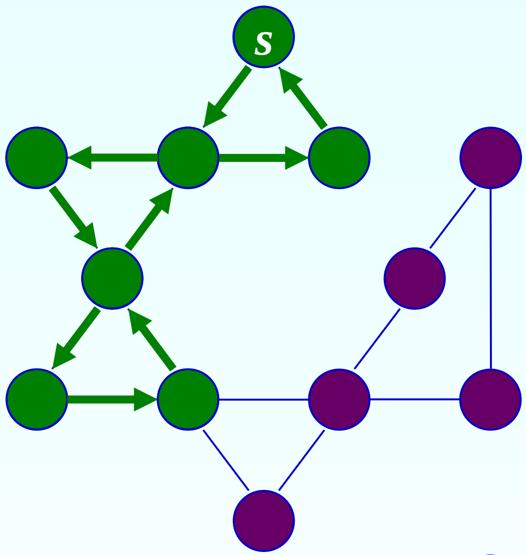
# Proof (🡐)

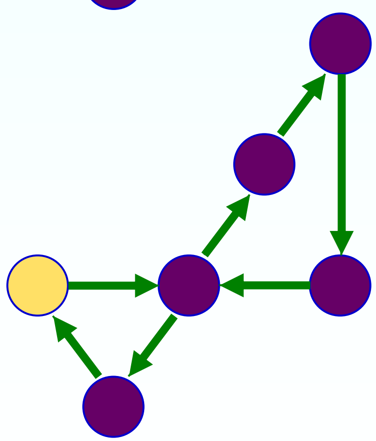
Consequently, starting at $s$ we *travel on C* until we arrive at a vertex $s_1$ that is on the Euler circuit of a connected component $C_1$ of $K$. Then we traverse this Euler circuit and, returning to $s_1$, continue on $C$ until we reach a vertex $s_2$ that is on the Euler circuit of connected component $C_2$ of $K$. Since the graph $G$ is finite, as we continue this process we construct an Euler circuit for $G$.

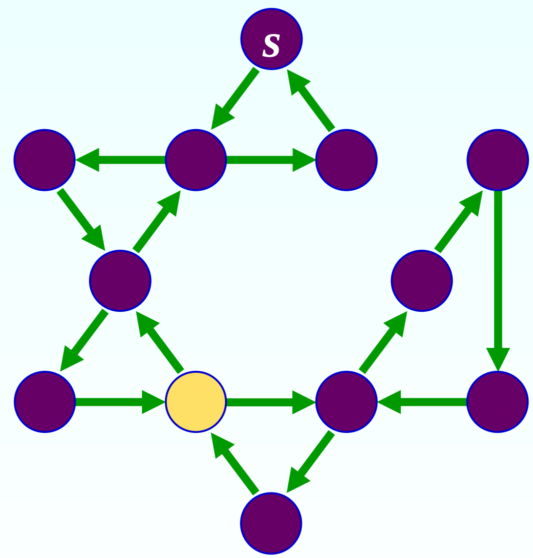
# Proof (⇐)



# Another Example





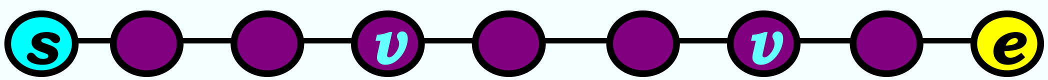
# Corollary 11.2

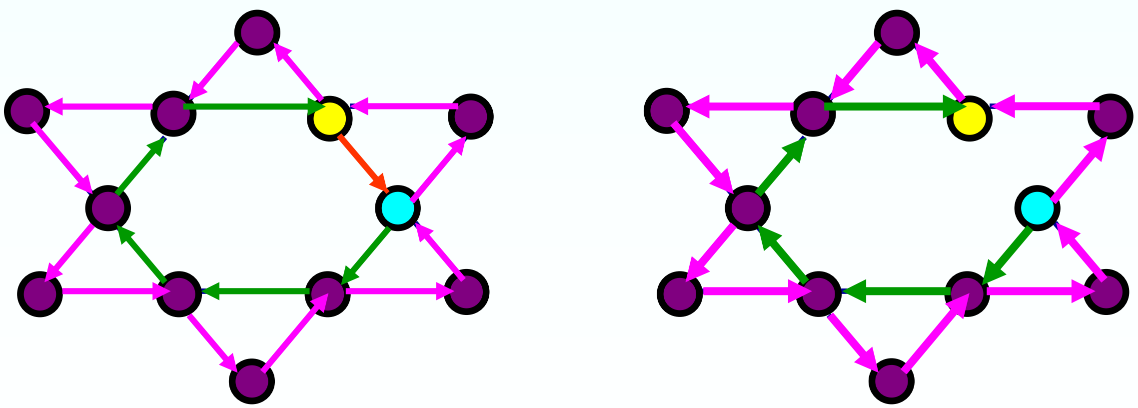
- Existence of Euler trail

  If $G=(V,E)$ is an undirected graph with no isolated vertices, then we can construct an Euler trail ⇔ $G$ is connected and exactly two vertices of odd degrees

Proof (⇒)

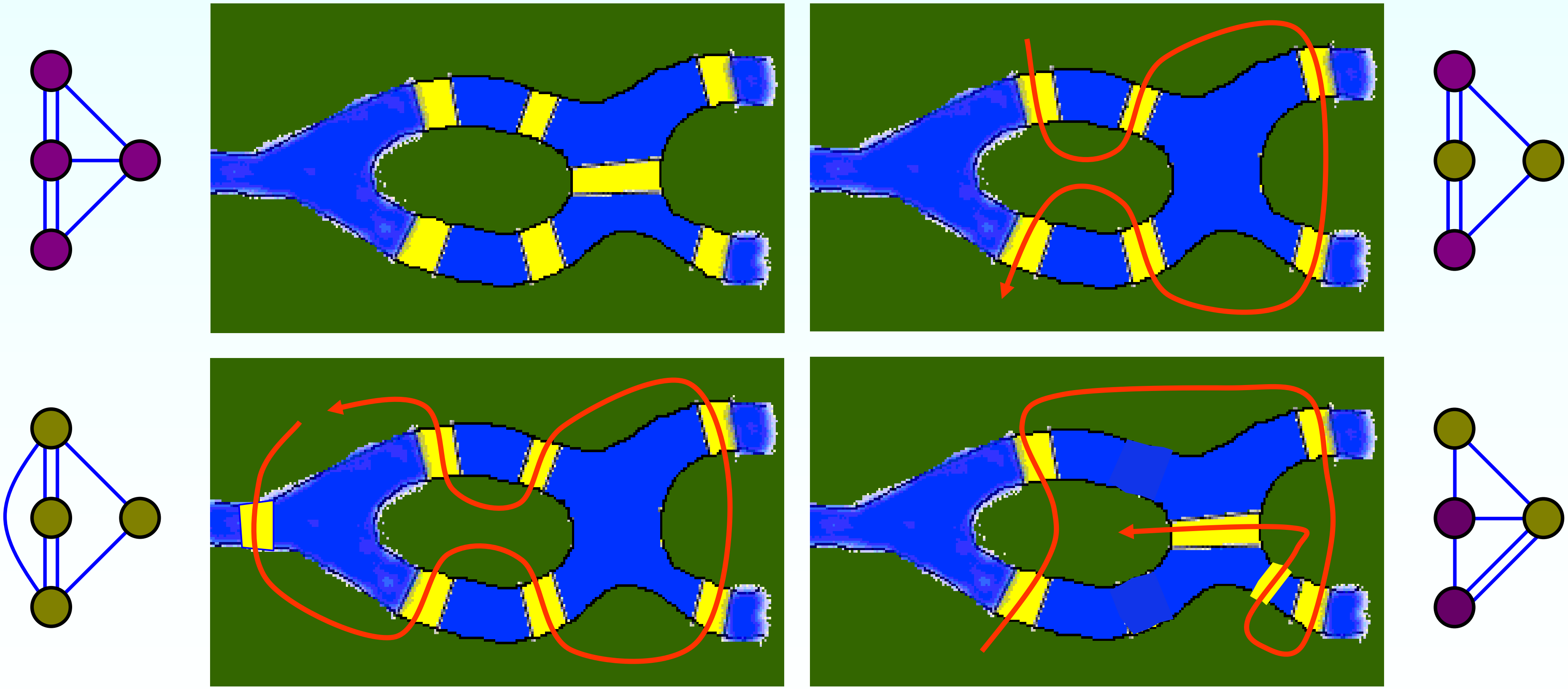
Proof (⇐)



# Examples

- Königsberg Bridges Problem



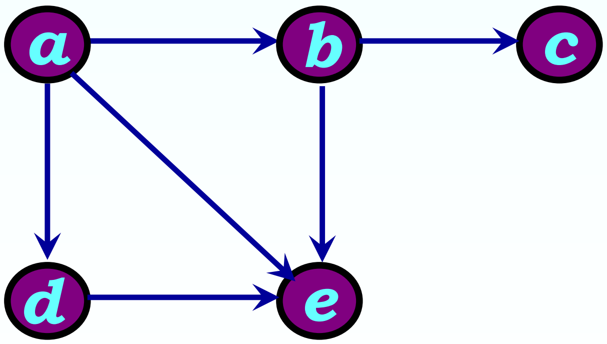
# Degrees in directed graphs

- ***id*($v$) & *od*($v$)**

  Let $G$ be a directed graph or multigraph. For each $v \in V$,

  (a) ***id*($v$)** : the **incoming**, or **in**, degree of $v$

  (b) ***od*($v$)** : the **outgoing**, or **out**, degree of $v$




**Indegree and Outdegree**

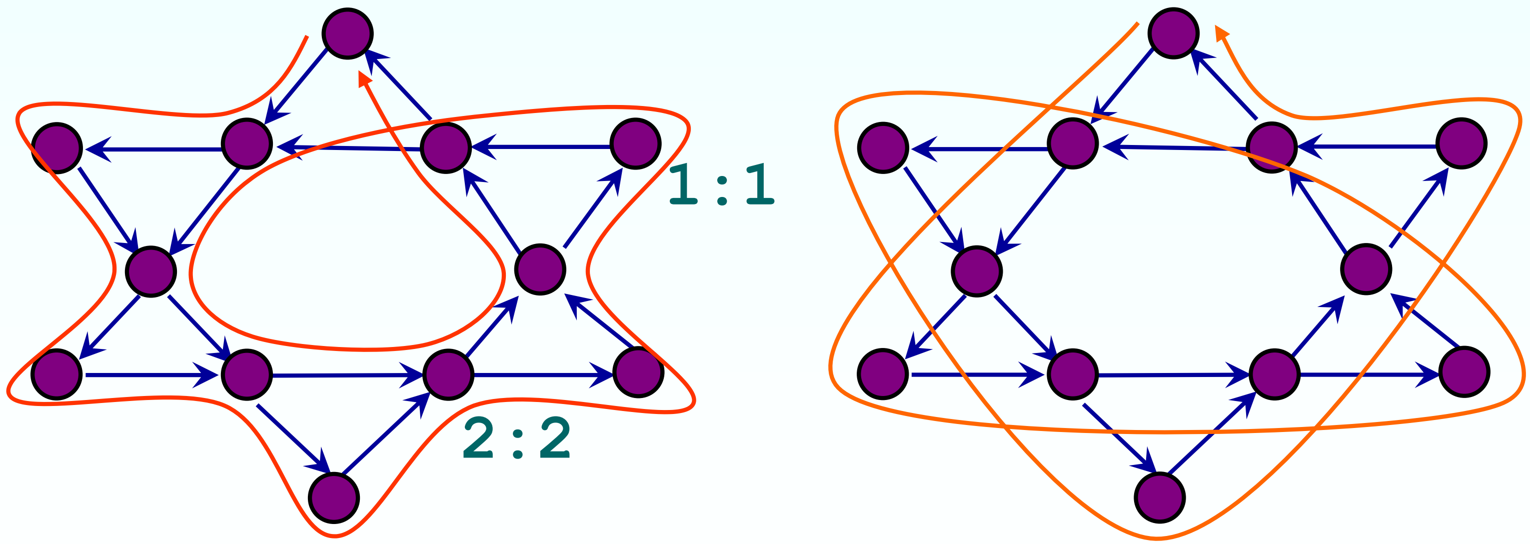
**id($a$) = 0, od($a$) = 3**
**id($b$) = 1, od($b$) = 2**

# Theorem 11.4

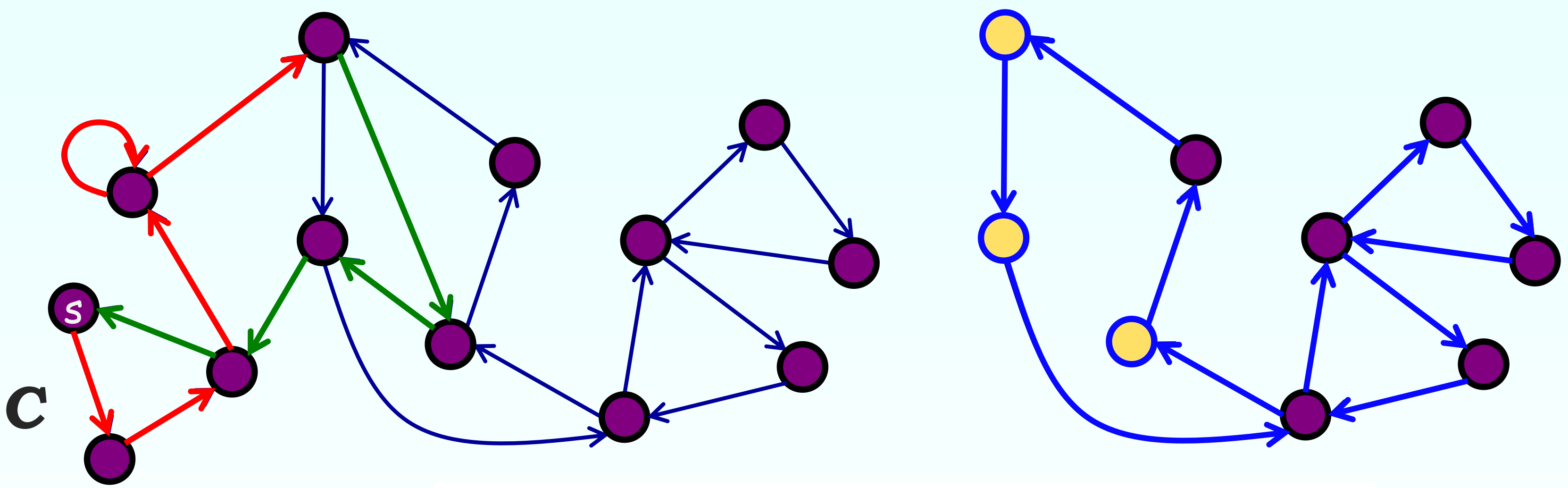
- **Directed graphs : Euler circuit**

  Let $G=(V,E)$ is a directed graph with no isolated vertices. The graph $G$ has a direct Euler circuit

  ⇔ $G$ is connected and $id(v) = od(v)$ for all $v \in V$




(Note) *A directed graph is called connected, if it is weakly connected*


s
C

# Summary of Euler Circuits

- Degree of Vertex
  - $\sum deg(v) = 2|e|$
  - Regular Graphs : Mesh, Hypercube

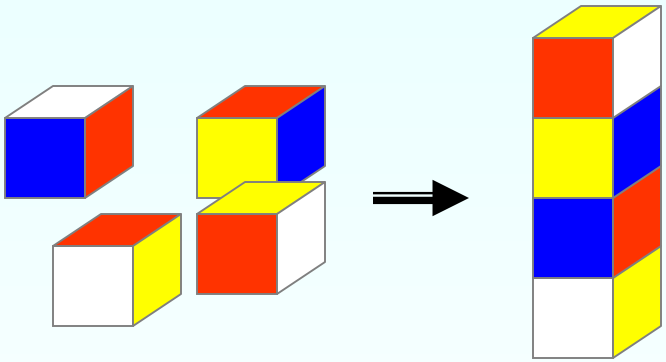
- Euler Circuit and Trail
  - Euler Circuit $\leftrightarrow$ Connected & All even-degree V's
  - Euler Trail

    $\leftrightarrow$ Connected & Only Two odd-degree V's
  - Directed Euler Circuit

    $\leftrightarrow$ Connected & $id(v) = od(v)$ for all V's

# 이산수학 쉼터

## Instant Insanity Game



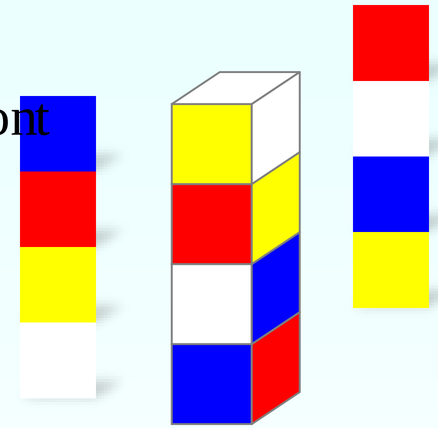
$24\,(=6\times 4)$ ; bottom $\times$ front

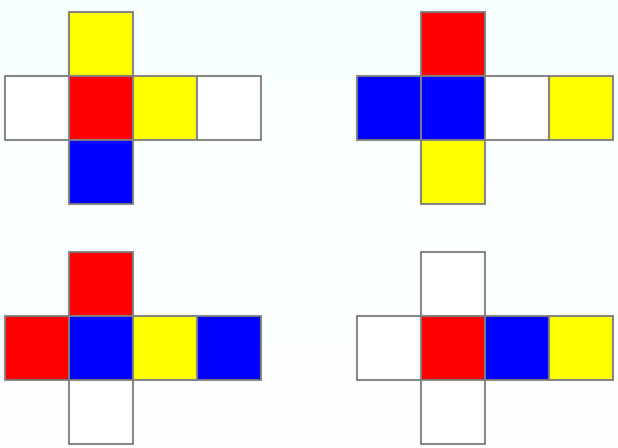
$24\,(=6\times 4)$

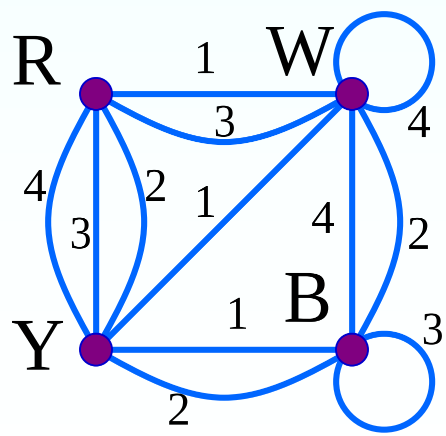
$24\,(=6\times 4)$

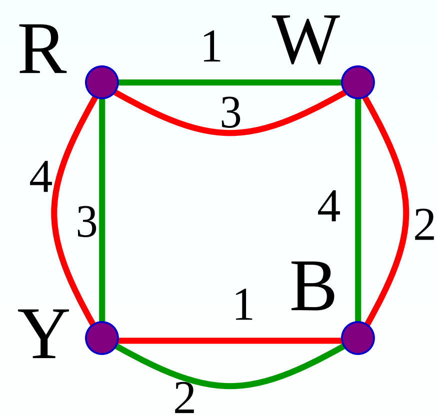
$\times\ 3$ ; top - bottom

41,472 cases







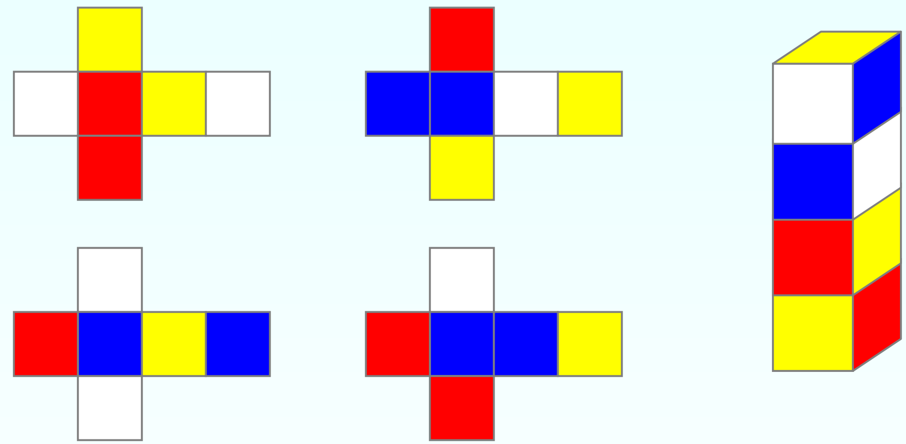



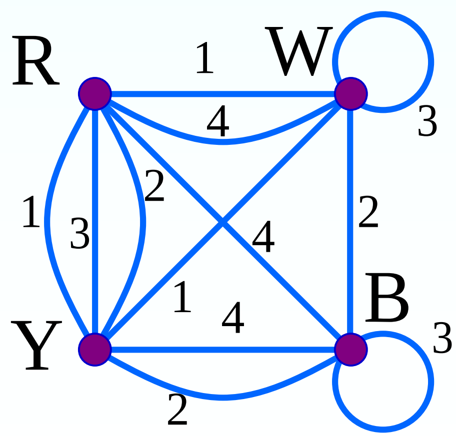
## Selection Rule

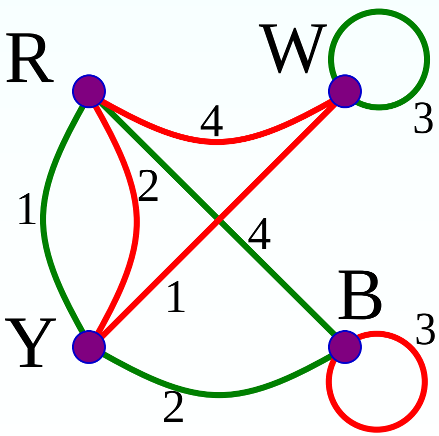
(1) Each subgraph contains all four vertices, four edges, one for each label

(2) In each subgraph, each vertex is incident with two edges ( a loop is counted twice)

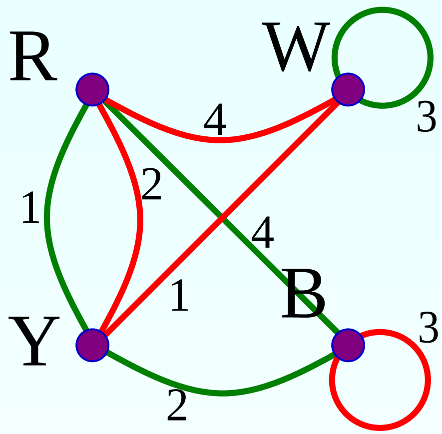
(3) No edge appears in both subgraph



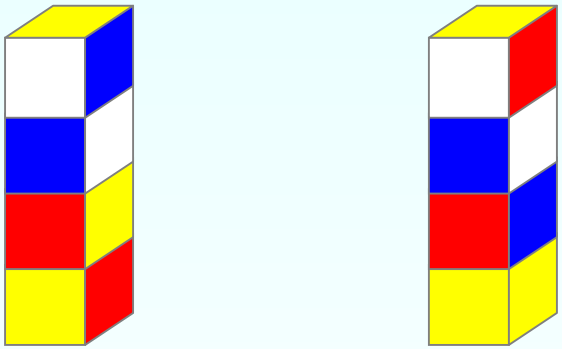


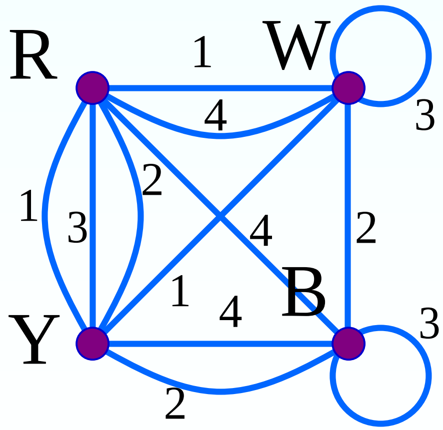



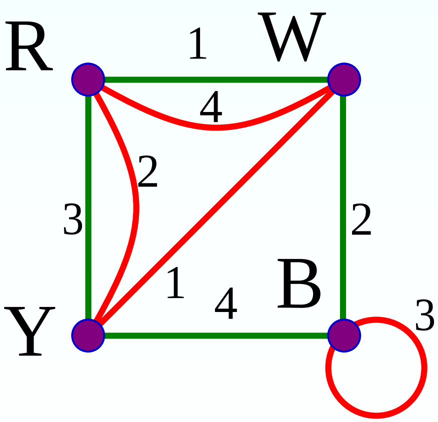
# 이산수학 쉼터




Another interpretation






Another selection

# Planar Graphs

Section 11.4

부산대학교
PUSAN NATIONAL UNIVERSITY

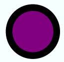
# Planar Graphs

## Definition

*A graph G is called **planar** if G can be drawn in a plane with its edges intersecting only at vertices.*

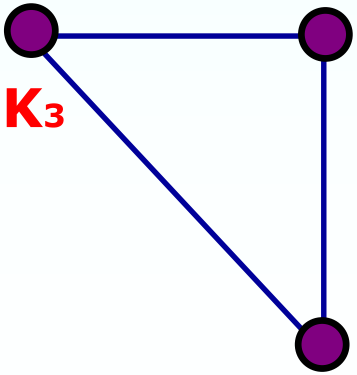
*Such a drawing of G is called an **embedding** of G in the plane.*

$K_1$

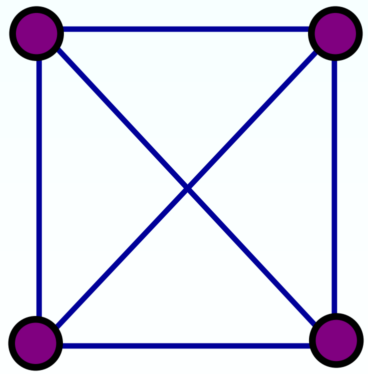


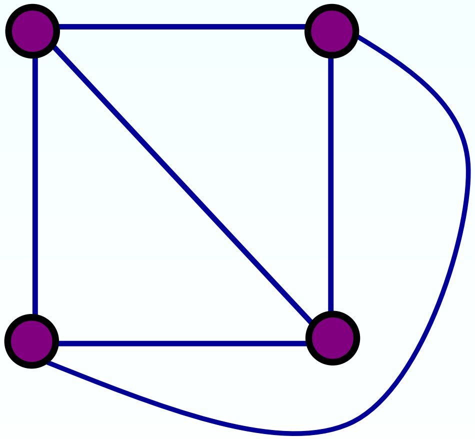
$K_2$

$K_3$



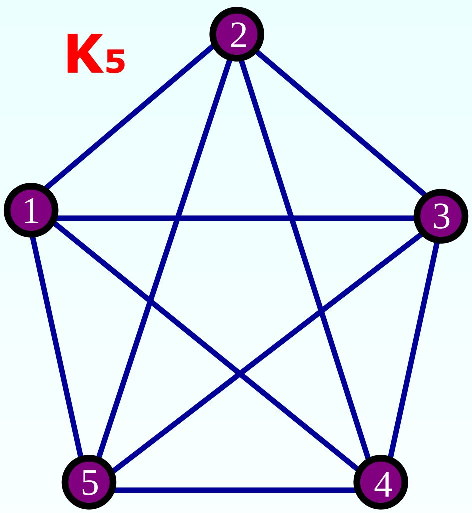
$K_4$

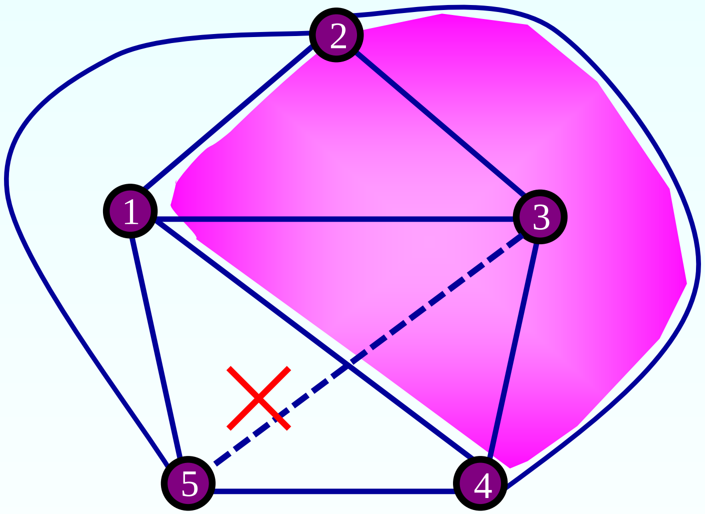




embedding

# Non-planar Graph $K_5$



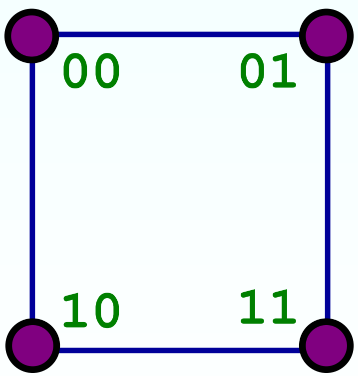

(Theorem) A graph is nonplanar if and only if it contains a subgraph that is homeomorphic to either $K_5$ or $K_{3,3}$

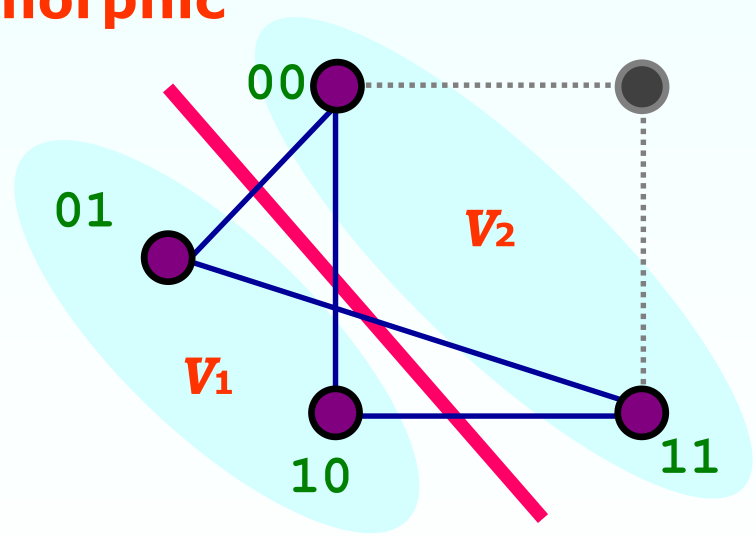
# Bipartite Graphs

## Definition

A graph $G = (V,E)$ is called **bipartite**, if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \varnothing$ , and every edge of $G$ is of the form $\{a, b\}$ with $a \in V_1$ and $b \in V_2$.

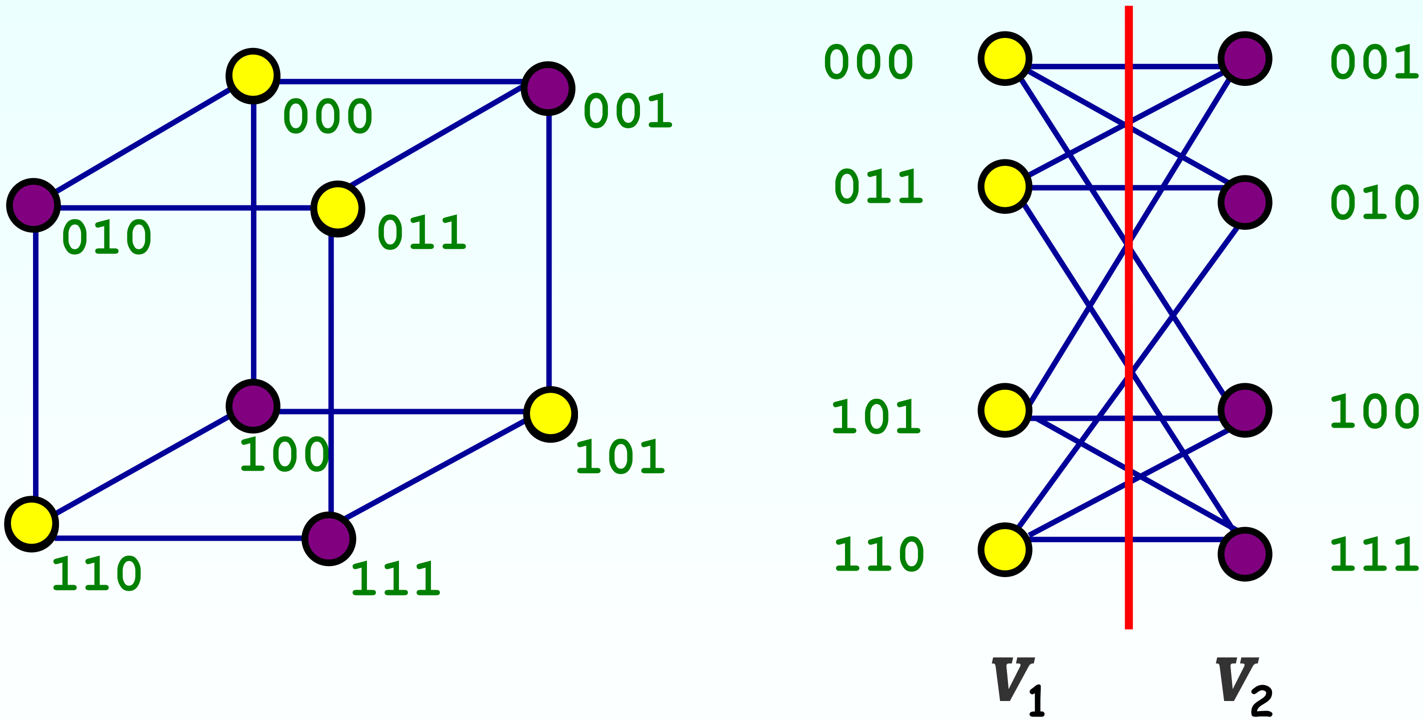
**Isomorphic**



# An Example

Hypercube $Q_3$

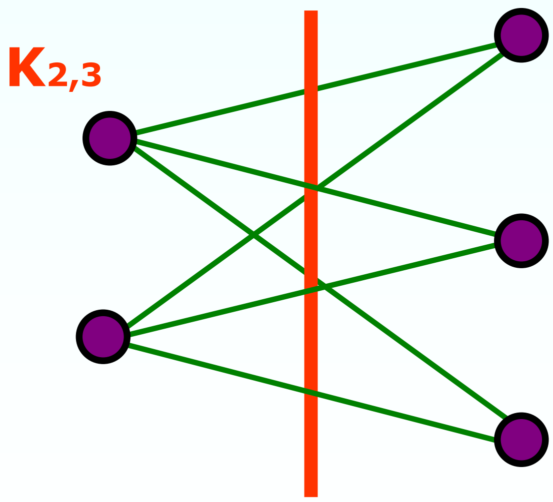


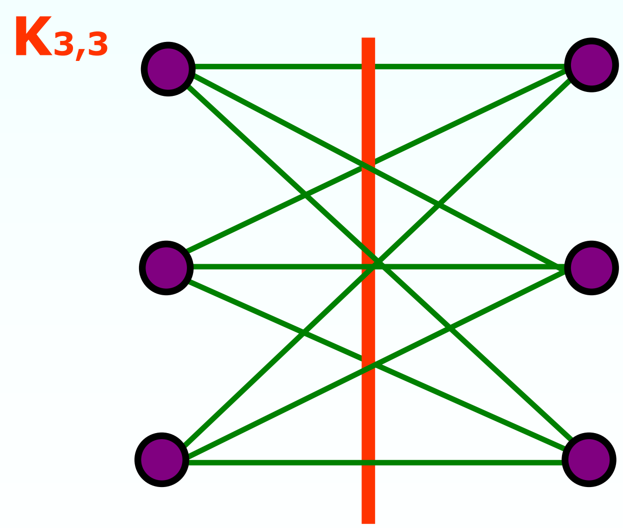
# Complete Bipartite Graphs

- $K_{m,n}$

  If each vertex in $V_1$ is joined with every vertex in $V_2$, we have a complete bipartite graph.

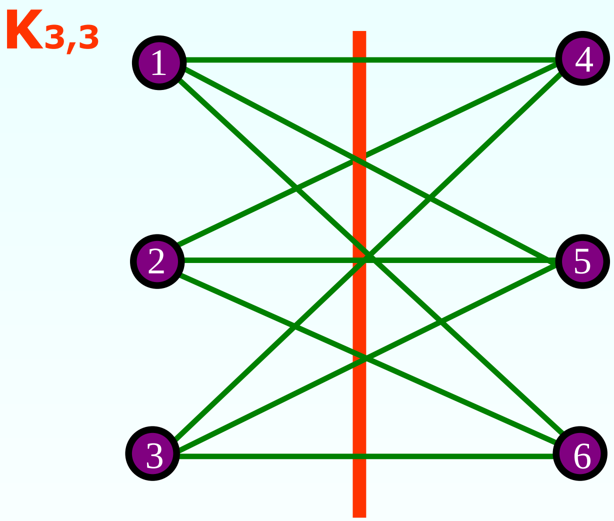
  If $|V_1| = m$, $|V_2| = n$, it is denoted by $K_{m,n}$

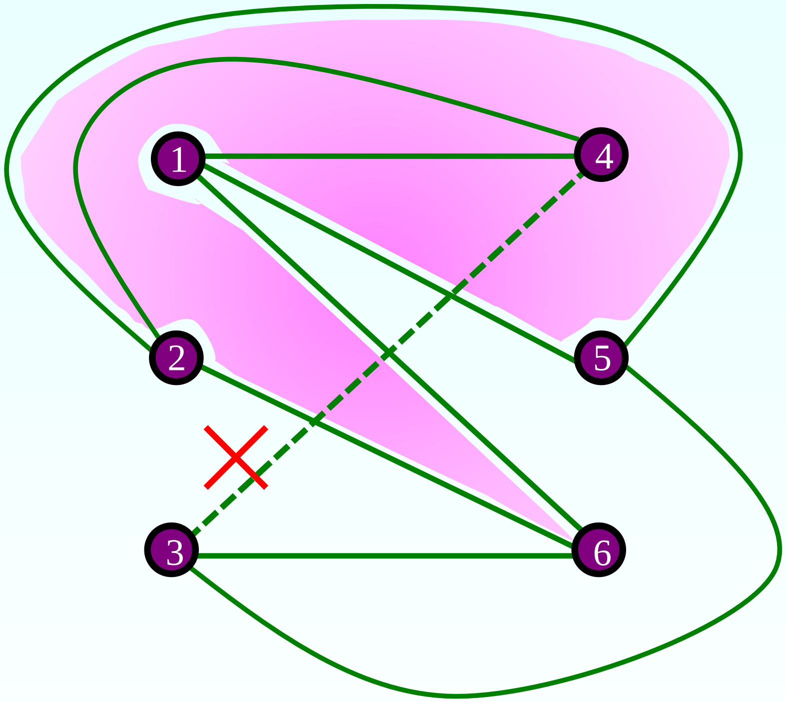



9 edges

# Non-planar Graph $K_{3,3}$



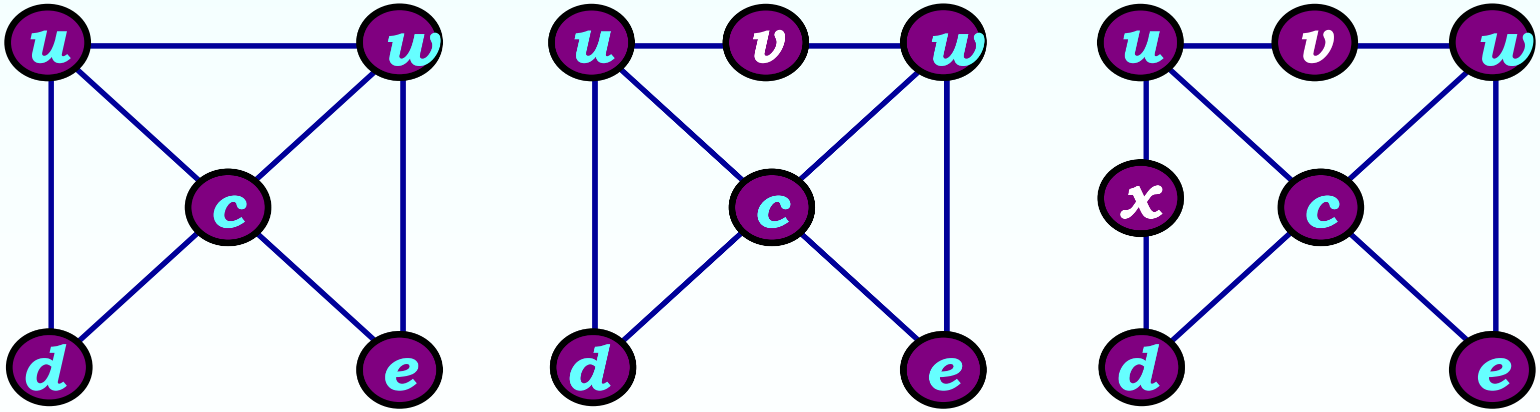




(Theorem) *A graph is nonplanar if and only if it contains a subgraph that is* homeomorphic *to either $K_5$ or $K_{3,3}$*

# Subdivision

A graph $G = (V,E)$ be a loop-free undirected graph.

An elementary subdivision of $G$ results when an edge $e = \{u, w\}$ is removed from $G$ and then the edges $\{u, v\}, \{v, w\}$ are added to $G - e$ where $v \notin V$.

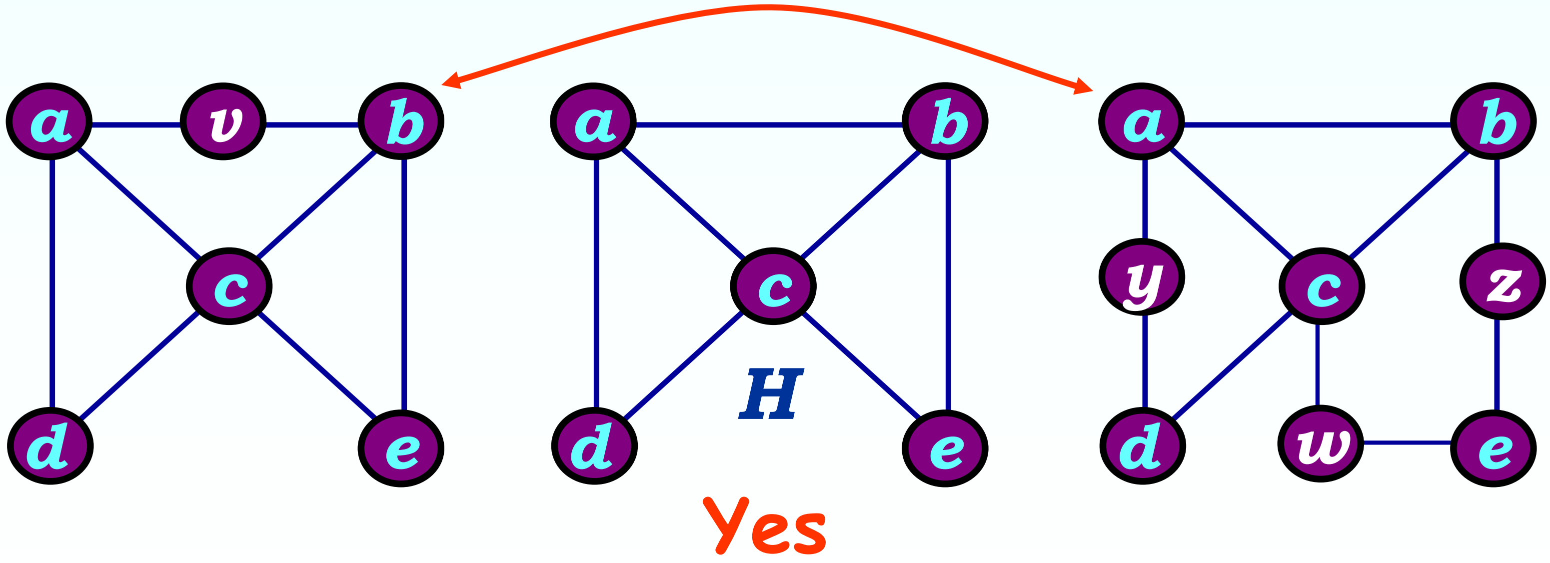


# Homeomorpic

The loop-free undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic, if

(a) they are isomorphic or

(b) they can both be obtained from the same loop-free undirected graph $H$ by a sequence of elementary subdivisions.

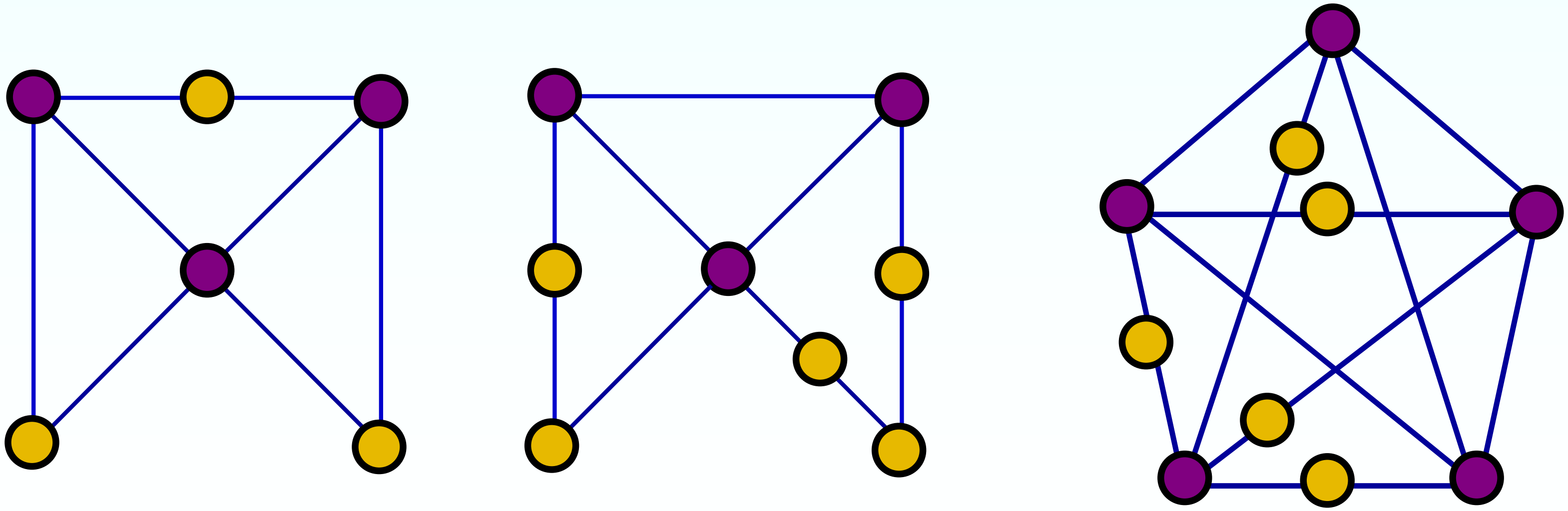


# 혹시! 쩝쩝 ??

Homeomorphic graphs may be isomorphic except, possibly, for vertices of degree 2.

In particular, if two graphs are homeomorphic, they are either both planar or both nonplanar.

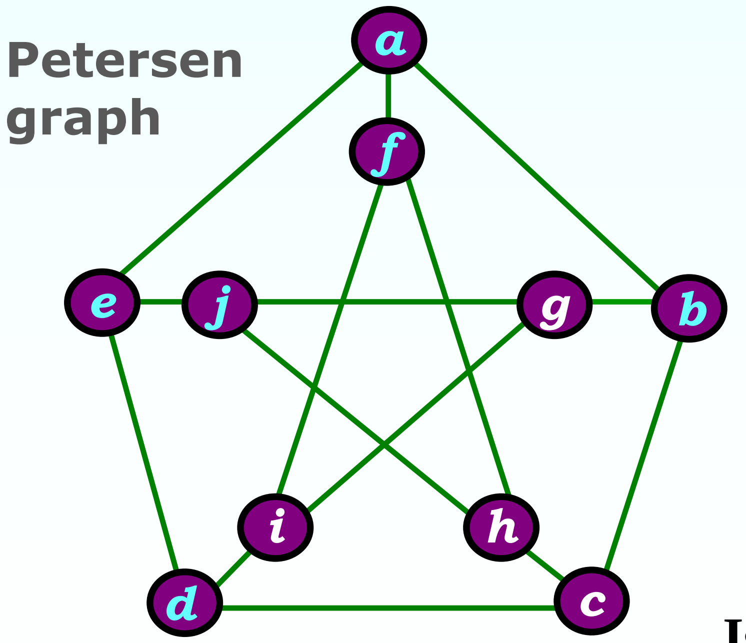
These lead us to the following theorem.



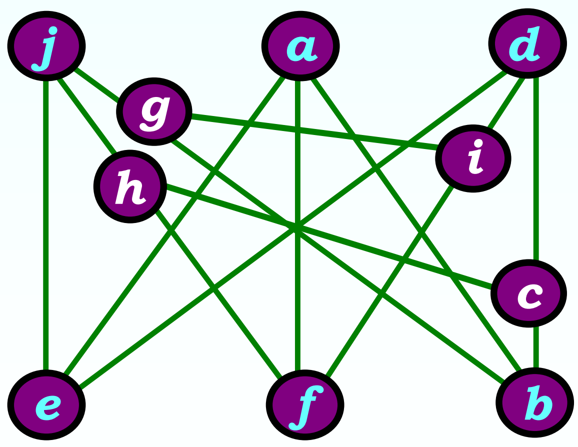
# Theorem 11.5

## Kuratowski's theorem

A graph is nonplanar <u>if and only if</u> it contains a subgraph that is homeomorphic to either $K_5$ or $K_{3,3}$.



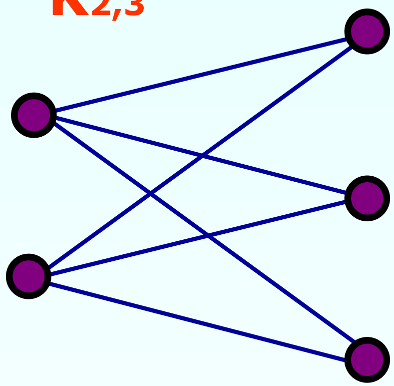

Isomorphic




$K_{3,3}$

# An Example

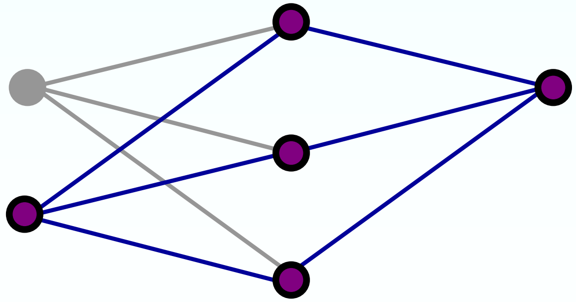
$K_{2,3}$



Is it planar ?

**Kuratowski's theorem**

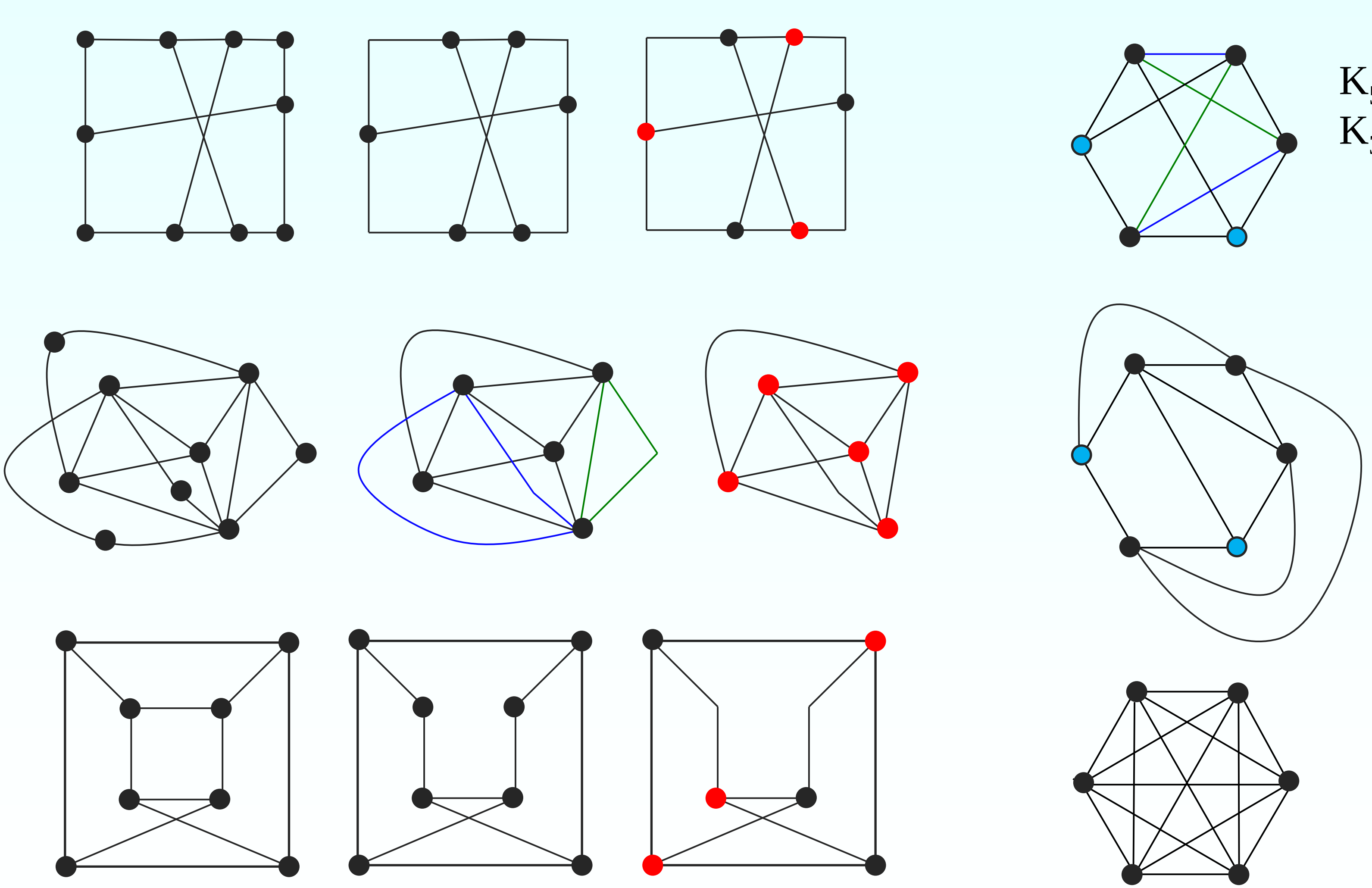
**A graph is nonplanar if and only if it contains a subgraph that is homeomorphic to either $K_5$ or $K_{3,3}$.**



**A graph is planar if and only if it dose not contains a subgraph that is homeomorphic to either $K_5$ or $K_{3,3}$.**

# Examples

## Is it planar ?

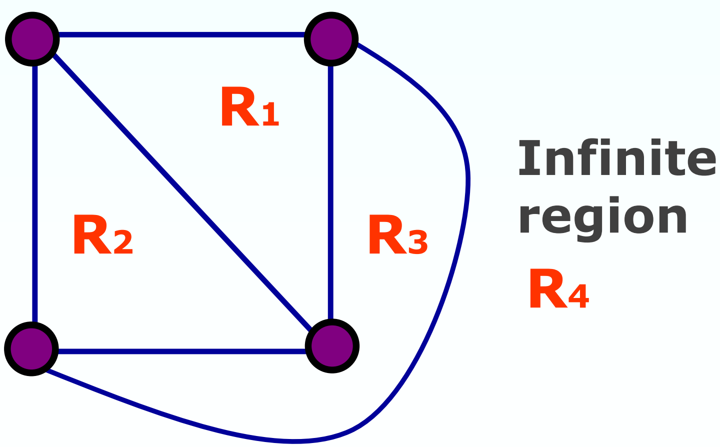


$K_5$ (X)

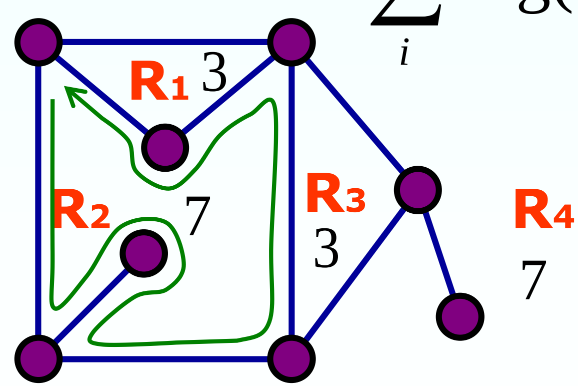
$K_{3,3}$ (X)

# Regions

- Region: edge에 의하여 둘러싸인 영역
  - * Embedded planar graph에서 정의
- Degree of region R, deg(R)
  - The number of edges in a (shortest) closed walk about the boundary of R

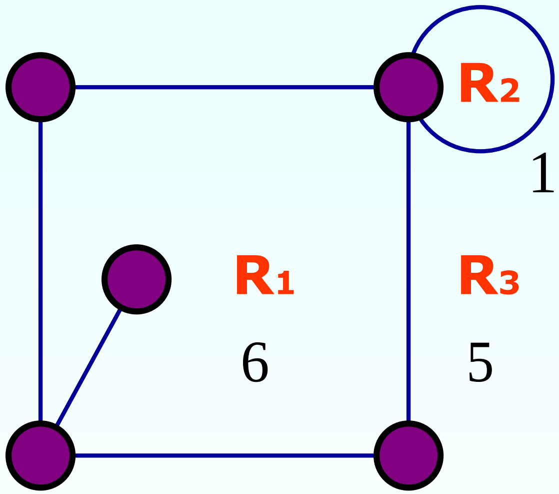


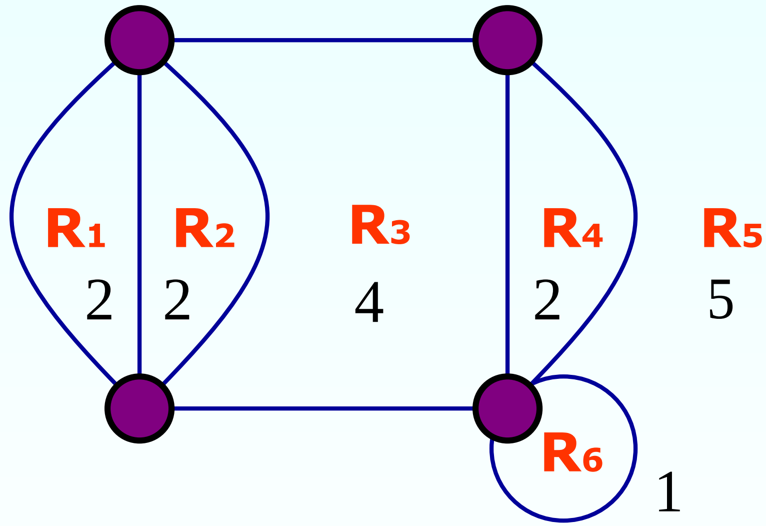

$$\sum_i \deg(R_i) = 2e$$



# Examples

## Even for a graph with loops or a multigraph




$$\sum_i \deg(R_i) = 6+5+1 = 12 = 2\times 6 = 2e$$

$$\sum_i \deg(R_i) = 16 = 2\times 8 = 2e$$

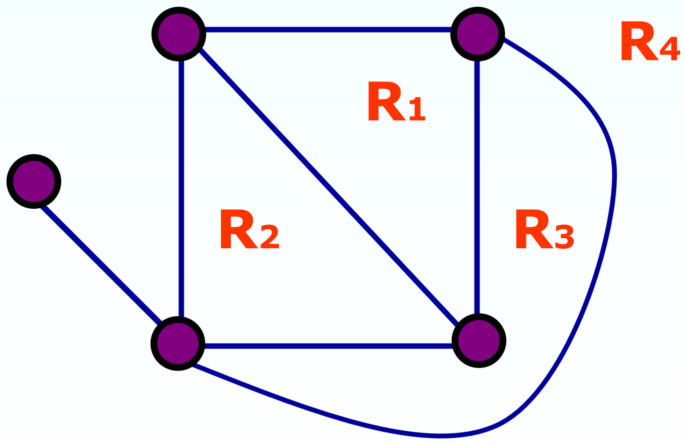
# Theorem 11.6

- Euler's formula ($v - e + r = 2$)

Let $G = (V,E)$ be a connected planar graph or multigraph with $|V| = v$ and $|E| = e$.

Let $r$ be the number of regions in the plane determined by a planar embedding of $G$.

Then, $v - e + r = 2$.

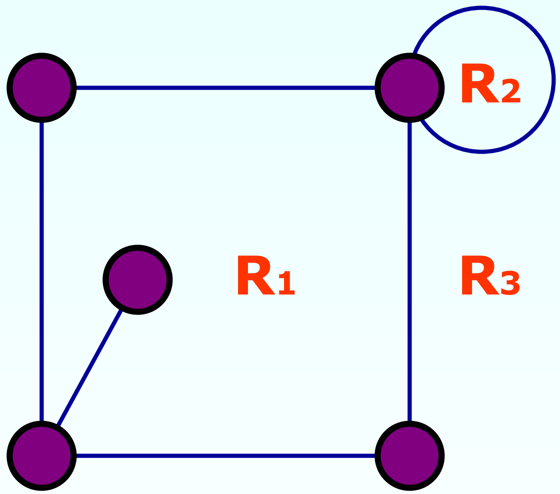


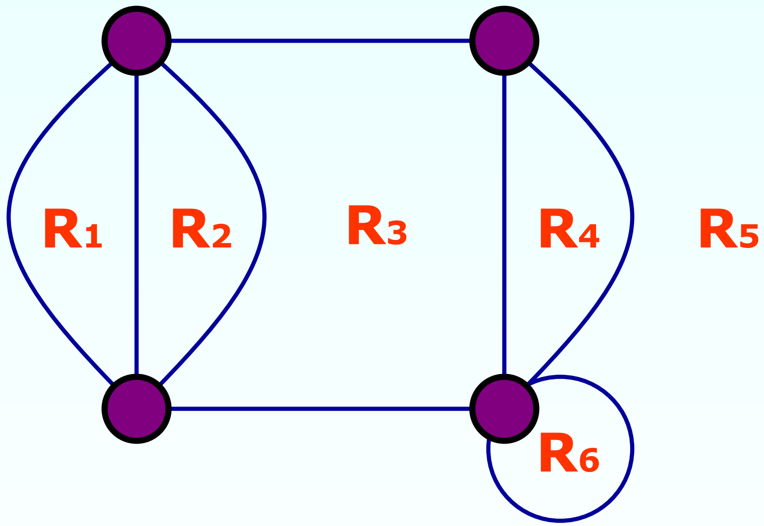

**Euler's formula**
; $v - e + r = 2$

$v = 5$
$e = 7$
$r = 4$

# Examples

Even for a **graph with loops** or a **multigraph**




**$v - e + r = 2$**
**$v = 5$**
**$e = 6$**
**$r = 3$**

**$v - e + r = 2$**
**$v = 4$**
**$e = 8$**
**$r = 6$**

# Corollary 11.3

- Two Inequalities for planar graphs

  Let $G = (V,E)$ be a loop-free connected planar graph with $|V| = v$ and $|E| = e > 2$, and $r$ regions. Then, $3r \le 2e$ and $e \le 3v-6$.

- **Proof**

  Since each region contains at least three edges, Degree of each region $\ge 3$

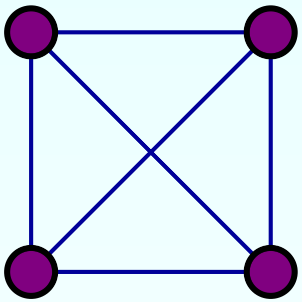
  $3r \le \sum \deg(R_i) = 2e \Rightarrow 3r \le 2e$

  $2 = v - e + r \le v - e + (2/3)\,e = v - (1/3)\,e$

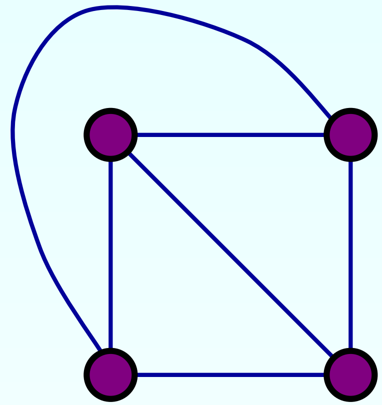
  $\Rightarrow e \le 3v-6$

# Examples

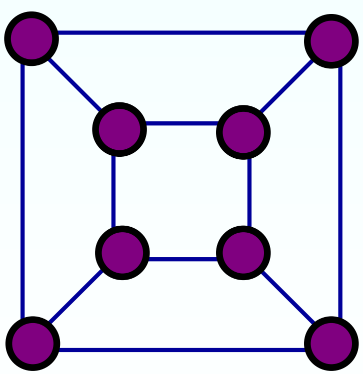
K$_4$



$v = 4$
$e = 6$
$\leq 6 = 3v - 6$

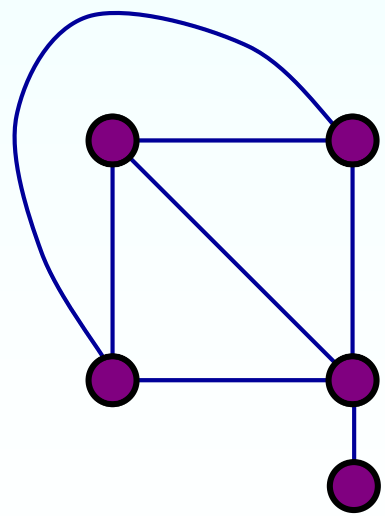


$r = 4 \quad e = 6$
$3r = 12$
$\leq 12 = 2e$



$v = 8$
$e = 12$
$\leq 18 = 3v - 6$

$r = 6 \quad e = 12$
$3r = 18$
$\leq 24 = 2e$



$v = 5 \quad e = 7$
$r = 4$
$e = 7$
$\leq 9 = 3v - 6$
$3r = 12$
$\leq 14 = 2e$

# Summary of Planar Graph

- Planar Embedding : Non-planar Graph $K_5$
- (Complete) Bipartite Graph : Non-planar Graph $K_{3,3}$
- Homeomorphic : Elementary Subdivision

## Kuratowski's Theorem

- A graph is nonplanar <u>if and only if</u> it contains a subgraph that is homeomorphic to $K_5$ or $K_{3,3}$

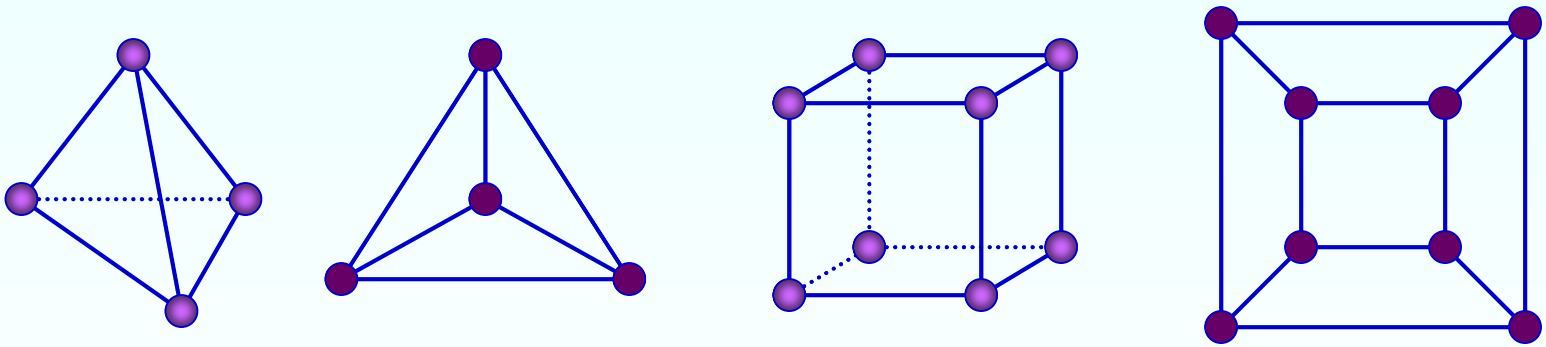
## Euler Formula

- Region (degree) in embedded planar graphs
- $v - e + r = 2$
- Inequalities in Planar Graph : $3r \leq 2e$ & $e \leq 3v-6$

# 이산수학 쉽터

## Platonic Solids

- All faces are congruent
- All (interior) solid angles are equal



$$\boxed{2e = mr = nv} \quad \Leftarrow \quad 2e = \sum \deg(R_i) = \sum \deg(v_i)$$

$m$ : degree of each region, deg(R) $\geq 3$
$n$ : # of regions that meet at each vertex v, deg(v) $\geq 3$

## From Euler's Theorem

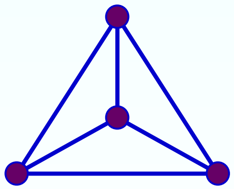
$$2e = mr = n\upsilon$$

$$0 < 2 = v - e + r = \frac{2e}{n} - e + \frac{2e}{m} = e\frac{2m - mn + 2n}{mn}$$

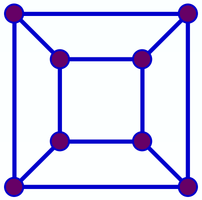
$$\Rightarrow \quad 0 < 2m - mn + 2n \quad \Rightarrow \quad mn - 2m - 2n + 4 < 4$$

$$\Rightarrow \quad (m-2)(n-2) < 4 \qquad (Note)\ \ m, n \geq 3$$

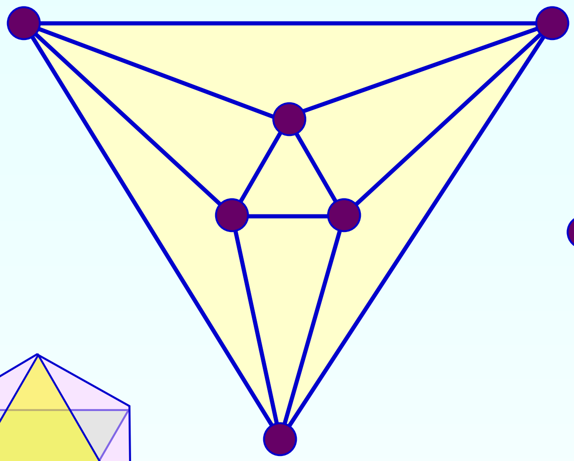
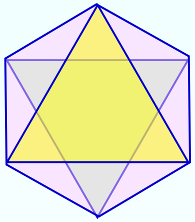
| $m$-2 | $n$-2 | $m$ | $n$ | Platonic Solid |
|---|---|---|---|---|
| 1 | 1 | 3 | 3 | Tetrahedron |
| 2 | 1 | 4 | 3 | Cube |
| 1 | 2 | 3 | 4 | Octahedron |
| 3 | 1 | 5 | 3 | Dodecahedron |
| 1 | 3 | 3 | 5 | Icosahedron |



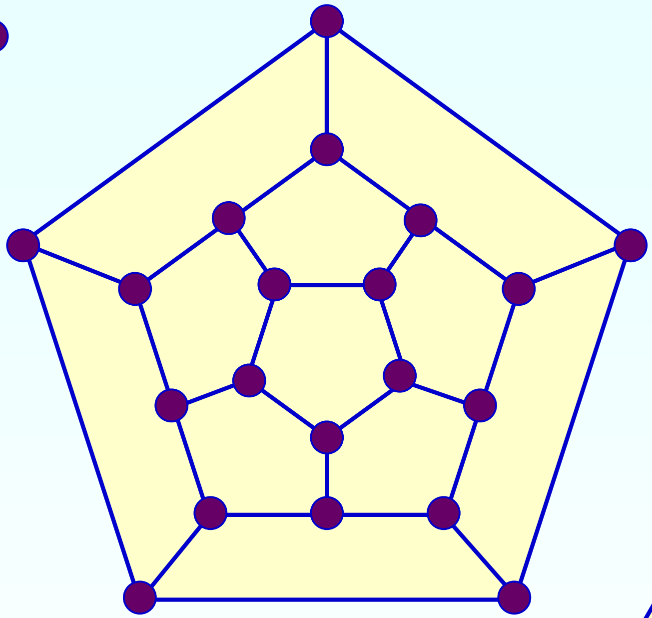
정4면체



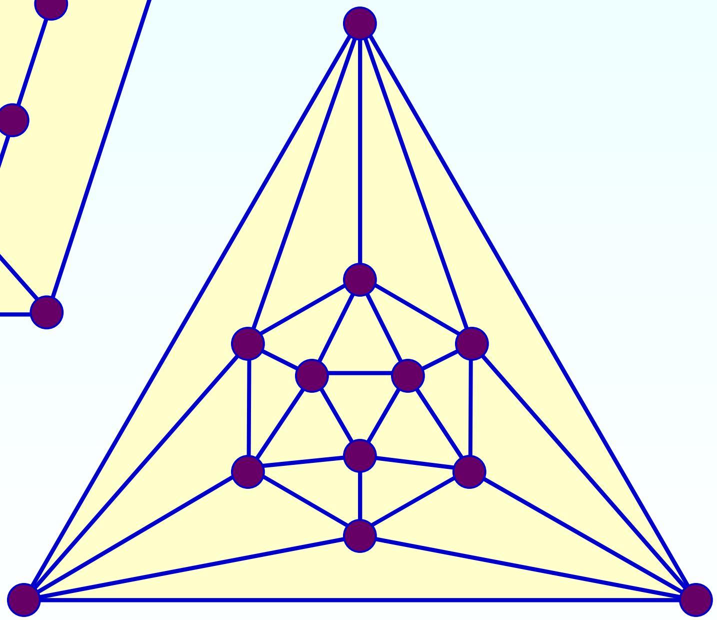
정6면체

정8면체



정12면체

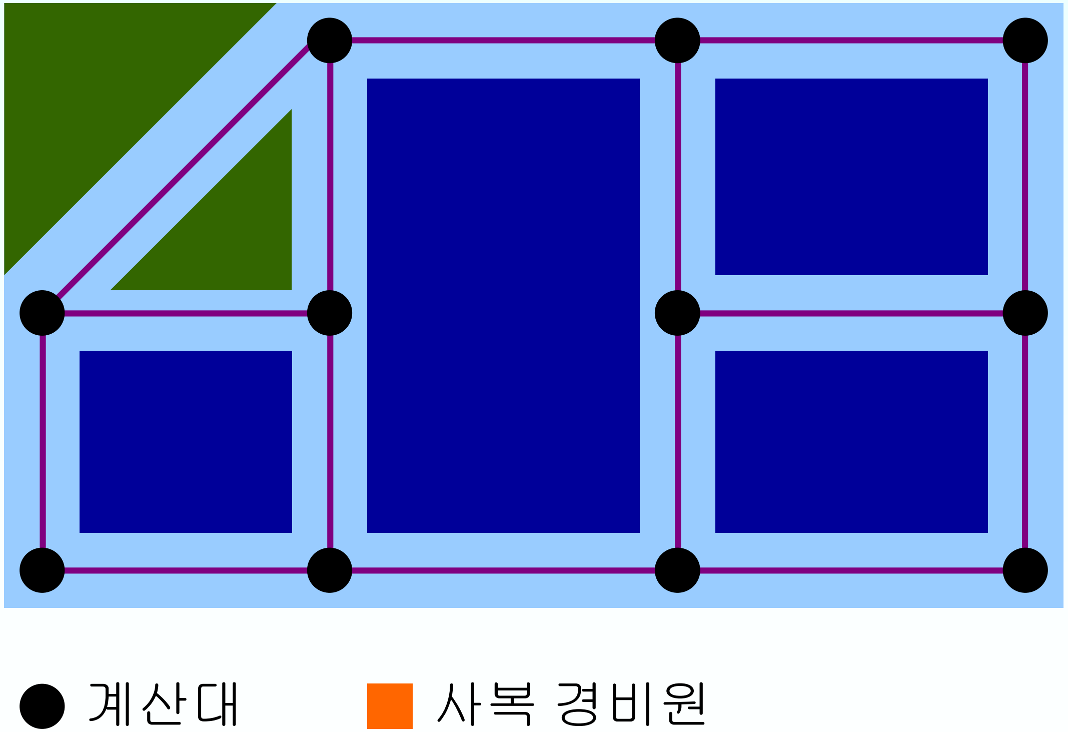


정20면체

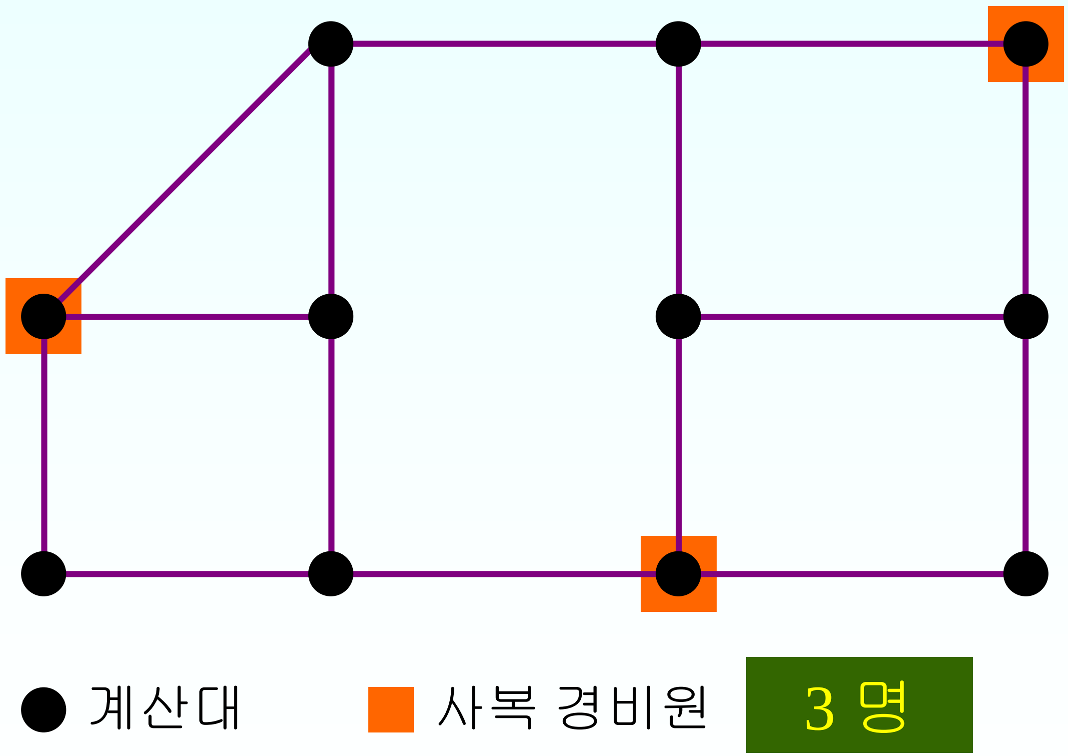
| $m$ | $n$ | |
|---|---|---|
| 3 | 4 | Octahedron |
| 5 | 3 | Dodecahedron |
| 3 | 5 | Icosahedron |

# 심심풀이 문제

- 백화점에 최소의 사복 경비원을 배치하는 문제

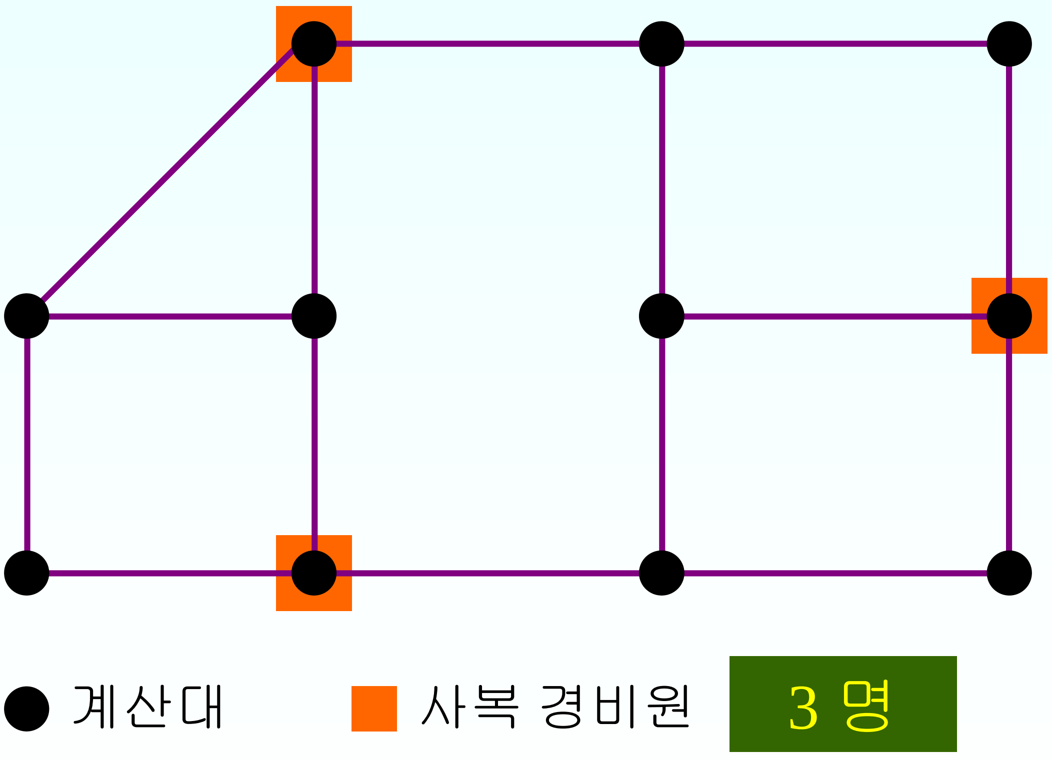


- 백화점에 최소의 사복 경비원을 배치하는 문제



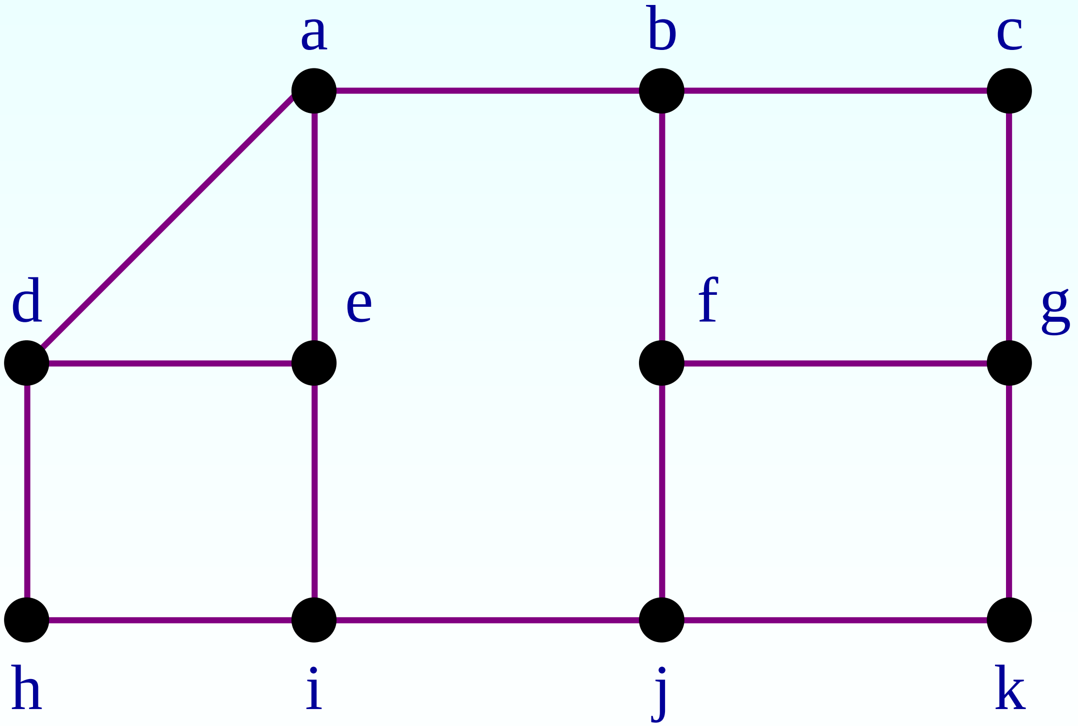
# 심심풀이 문제

- 백화점에 최소의 사복 경찰을 배치하는 문제



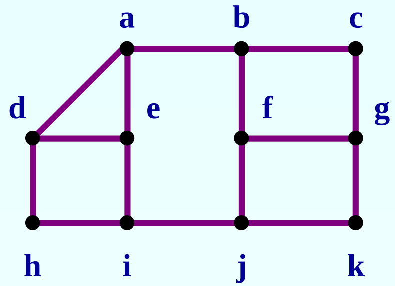
# 십십풀이 문제

- 구현 ?



# 심심풀이 문제

## ❑ 구현 ?




| | a | b | c | d | e | f | g | h | i | j | k |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **A** | √ | √ | | √ | √ | | | | | | |
| **B** | √ | √ | √ | | | √ | | | | | |
| **C** | | √ | √ | | | | √ | | | | |
| **D** | √ | | | √ | √ | | | √ | | | |
| **E** | √ | | | √ | √ | | | | √ | | |
| **F** | | √ | | | | √ | √ | | | √ | |
| **G** | | | √ | | | √ | √ | | | | √ |
| **H** | | | | √ | | | | √ | √ | | |
| **I** | | | | | √ | | | √ | √ | √ | |
| **J** | | | | | | √ | | | √ | √ | √ |
| **K** | | | | | | | √ | | | √ | √ |
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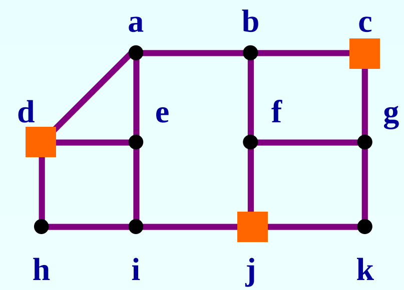
# 심심풀이 문제

## ❑ 구현 ?




| | a | b | c | d | e | f | g | h | i | j | k |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | √ | √ | | √ | √ | | | | | | |
| B | √ | √ | √ | | | √ | | | | | |
| C | | √ | √ | | | | √ | | | | |
| D | √ | | | √ | √ | | | √ | | | |
| E | √ | | | √ | √ | | | | √ | | |
| F | | √ | | | | √ | √ | | | √ | |
| G | | | √ | | | √ | √ | | | | √ |
| H | | | | √ | | | | √ | √ | | |
| I | | | | | √ | | | √ | √ | √ | |
| J | | | | | | √ | | | √ | √ | √ |
| K | | | | | | | √ | | | √ | √ |
| | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |

# 십십풀이 문제

## ❑ 구현 ?

| | a | b | c | d | e | f | g | h | i | j | k |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | √ | √ | | √ | √ | | | | | | |
| B | √ | √ | √ | | | √ | | | | | |
| C | | √ | √ | | | | √ | | | | |
| D | √ | | | √ | √ | | | √ | | | |
| E | √ | | | √ | √ | | | | √ | | |
| F | | √ | | | | √ | √ | | | √ | |
| G | | | √ | | | √ | √ | | | | √ |
| H | | | | √ | | | | √ | √ | | |
| I | | | | | √ | | | √ | √ | √ | |
| J | | | | | | √ | | | √ | √ | √ |
| K | | | | | | | √ | | | √ | √ |
| | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ | √ |