Chapman-Kolmogorov Equations

N-step transition probabilities

- ${P_{i,j}}^{n} = P\{X_{n+k} = j | X_k = i\},\ n \geq 0, i, j \geq 0$
- Chapman-Kolmogorov Equations
  - ${P_{i,j}}^{n+m} = \sum_{k=0}^{\infty} {P_{i,k}}^{n} {P_{k,j}}^{m}$ for all $n, m \geq 0$, all i, j
  - ${P_{i,j}}^{n+m} = P\{X_{n+m} = j | X_0 = i\} = \sum_{k=0}^{\infty} P\{X_{n+m} = j, X_n = k | X_0 = i\} = \sum_{k=0}^{\infty} P\{X_{n+m} = j | X_n = k, X_0 = i\} P\{X_n = k | X_0 = i\} = \sum_{k=0}^{\infty} {P_{i,k}}^{n} {P_{k,j}}^{m}$
- Let $P^{(n)}$ denote the matrix of n-step transition probabilities ${P_{i,j}}^{n}$, then

$$P^{(n+m)} = P^{(n)} P^{(m)}$$

$$P^{(2)} = P \cdot P$$

Example 4.8

- In Example 4.1, $P = \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix}$
- Probability that it will rain four days from today given that it is raining today ? (*i.e.* $P_{0,0}{}^{4}$)
- (Sol.) $P^2 = \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix} \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix}$

$$P^4 =$$

Example 4.9

- In Example 4.4, given that it rained on Monday and Tuesday, what is the probability that it will rain on Thursday ?
- $P = \left\|\begin{matrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{matrix}\right\|$
- (Sol.) $P^2 =$

$${P_{0,0}}^2 + {P_{0,1}}^2 =$$

Example 4.10

- 2 balls (red or blue) in an urn. A ball is randomly chosen and replaced by a new ball. The same color with probability 0.8 and the opposite color with probability 0.2.
- The probability that the fifth ball selected is red if initially both balls are red ?
- (Sol.)
  - States 0, 1, 2 (defined to be the number of red balls)
  - Denote X_n as the number of red balls in the run after the *n*th selection
  - $P = \begin{Vmatrix} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{Vmatrix}$, $P^4 = ?$
  - P(fifth selection is red) = $\sum_{i=0}^{2} P(\text{fifth selection is red} | X_4 = i) P(X_4 = i | X_0 = 2) = (0) {P_{2,0}}^4 + (0.5) {P_{2,1}}^4 + (1) {P_{2,2}}^4$

Classification of States

- State *j* is accessible from state *i*
  - *i* → *j* if $P_{ij}{}^{(n)} > 0$ for some $n \geq 0$
- Two states *i* and *j* are accessible to each other if *i* → *j* and *j* → *i*
  - They are said to communicate
  - "Communication" is an equivalent relation.
    1. $i \leftrightarrow i$ (Reflective)
    2. $i \leftrightarrow j$ ➔ $j \leftrightarrow i$ (Symmetric)
    3. $i \leftrightarrow j$, $j \leftrightarrow k$ ➔ $i \leftrightarrow k$ (Transitive)

    $\exists n \geq 0 \; s.t. P_{i,j}{}^{n} > 0, \text{ and } \exists m \geq 0 \; s.t. P_{j,k}{}^{m} > 0$ ➔ $P_{i,k}{}^{(n+m)} = \sum_{l=0}^{\infty} P_{i,l}{}^{n} P_{l,k}{}^{m} \geq P_{i,j}{}^{n} P_{j,k}{}^{m} > 0$
  - Two states that communicate are said to be in the same *class*. The state space is divided into a number of separate classes. The Markov chain is <u>*irreducible*</u> if there is only one class, *i.e*. all states communicate each other.

Example 4.14

- Three states 0, 1, 2
- $P = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{Vmatrix}$
- (Question) Is this Markov chain irreducible?
- (Sol.)

{0} {1} {2}

➔ {0, 1, 2}

Example 4.15

- Four states 0, 1, 2, 3
- $P = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{Vmatrix}$
- (Question) Is this Markov chain irreducible?
- (Sol.)

{0} {1} {2} {3}

➔ {0, 1}, {2}, {3}

Recurrent vs Transient

- State *i* is recurrent if $f_i = P\{X_n = i \text{ for some } n \geq 1 |\ X_0 = i\} = 1$

  $\Leftrightarrow P\{X_n \text{ will visit } i \mid X_0 = i\} = 1$

  $\Leftrightarrow$ E(the number of visits to state $i \mid X_0 = i) = \sum_{n=1}^{\infty} {P_{i,i}}^n = \infty$

  0 visits: $1 - f_i$

  1 visit: $f_i(1 - f_i)$

  2 visits: ${f_i}^2(1 - f_i)$

  …

  k visits: ${f_i}^k(1 - f_i)$

  Thus E(the number of visits to state $i \mid X_0 = i) = \frac{1}{1-f_i} = \infty$ if $f_i = 1$

- State *i* is transient if $f_i < 1$

  $\Leftrightarrow \sum_{n=1}^{\infty} {P_{i,i}}^n < \infty$

Corollary 4.2

- If state *i* is recurrent, state *i* communicates with state *j*, then state *j* is recurrent
  - (Proof) ${P_{j,j}}^{m+n+k} \geq {P_{j,i}}^{m}{P_{i,i}}^{n}{P_{i,j}}^{k}$

    $\sum_{n=1}^{\infty} {P_{j,j}}^{m+n+k} \geq \sum_{n=1}^{\infty} {P_{j,i}}^{m}{P_{i,i}}^{n}{P_{i,j}}^{k} = {P_{j,i}}^{m}{P_{i,j}}^{k} \sum_{n=1}^{\infty} {P_{i,i}}^{n} = \infty$

Example 4.16

- $P = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
  - $f_0 = \frac{1}{2} * 1 * 1 + \frac{1}{2} * 1 * 1 = 1$
  - All states communicate
  - ➔ all states are recurrent

Example 4.17

- $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{2} \end{pmatrix}$
  - Three classes $\{0, 1\}$, $\{2, 3\}$, and $\{4\}$
  - $f_0 = \frac{1}{2} + \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} * \frac{1}{2} + \cdots = 1$