Exponential distribution and Poisson process

Exponential distribution

- $f(x) = \lambda e^{-\lambda x}, x \geq 0$
- $F(x) = P\{X \leq x\} = 1 - e^{-\lambda x}, x \geq 0$
- $\mathrm{E}[X] = \int_0^\infty x\lambda e^{-\lambda x} = \frac{1}{\lambda}$
- $\mathrm{Var}[X] = \frac{1}{\lambda^2}$

- Memoryless
  - $P\{X > s + t \mid X > t\} = P\{X > s\}$ for all $s, t \geq 0$
  - $\Leftrightarrow \frac{P\{X>s+t\,, X>t\}}{P\{X>t\}} = \frac{P\{X>s+t\}}{P\{X>t\}} = P\{X > s\}$

Example 5.2

- X: the amount of time that the customer spends in the bank
  - $X \sim Exp(\frac{1}{10})$

  (Question) what is the probability that a customer will spend more than 15 min in the bank?

  (Solution)

  - $P\{X > 15\} = 1 - P\{X \leq 15\} = 1 - \left(1 - e^{-15\lambda}\right) = e^{-15*\frac{1}{10}} = e^{-\frac{3}{2}}$

Example 5.5

- X: the demand, $X \sim Exp(\lambda)$. The store orders t pounds, it costs $c \times t$ and is sold at a price of $s \times t$. Left-over is worthless and no penalty if the store cannot meet all the demand.

  (Question) how much should be ordered so as to maximize the store's expected profit?

  (Solution)

  - Profit: $Y = s\min(X,t) - ct$ where $\min(X,t) = X - (X-t)^+$
  - $E[(X-t)^+] = E[(X-t)^+|X>t]P(X>t) + E[(X-t)^+|X \le t]P(X \le t) = E[(X-t)^+|X>t]P(X>t) = E[(X-t)^+|X>t]e^{-\lambda t} = \frac{1}{\lambda}e^{-\lambda t}$

    Using the lack of memory property

  - $E[\min(X,t)] = \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda t}$, $E[P] = \frac{s}{\lambda} - \frac{s}{\lambda}e^{-\lambda t} - ct$ (maximal when $se^{-\lambda t} - c = 0$)

Further properties of the exponential distribution

- Let $X_1, \ldots, X_n$ be independent and identically distributed exponential with mean $\frac{1}{\lambda}$
  - $X_1 + \cdots + X_n \sim$ Gamma(n, λ), i.e. $f_{X_1+\cdots+X_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$
    - Proved using mathematical induction
    - Trivial when $n = 1$
    - Assume $X_1 + \cdots + X_{n-1} \sim$ Gamma$(n-1, \lambda)$, i.e. $f_{X_1+\cdots+X_{n-1}}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-2}}{(n-2)!}$
    - $f_{X_1+\cdots+X_n}(t) = \int_0^t f_{X_n}(t-s) f_{X_1+\cdots+X_{n-1}}(s) ds = \int_0^t \lambda e^{-\lambda(t-s)} \frac{\lambda e^{-\lambda s}(\lambda s)^{n-2}}{(n-2)!} ds = \lambda e^{-\lambda t} \frac{\lambda^{n-1}}{(n-2)!} \int_0^t s^{n-2} ds = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

Further properties of the exponential distribution

- $X_1 \sim Exp(\lambda_1)$, and $X_2 \sim Exp(\lambda_2)$
  - $P\{X_1 < X_2\} = \int_0^\infty P\{X_1 < X_2 | X_1 = x\} \lambda_1 e^{-\lambda_1 x} dx = \int_0^\infty P\{x < X_2\} \lambda_1 e^{-\lambda_1 x} dx = \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx = \int_0^\infty \lambda_1 e^{-(\lambda_1 + \lambda_2) x} dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}$
- $P\{\min(X_1, \dots, X_n) > x\} = P\{X_i > x \text{ for each } i\} = \prod_{i=1}^n P\{X_i > x\} = e^{-\sum_i \lambda_i x}$

Poisson process – definition

- A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the total number of events that occur by time t.

- The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if the following axioms hold:
  1. $N(0) = 0$
  2. $\{N(t), t \geq 0\}$
  3. $P\{N(t+h) - N(t) = 1\} = \lambda h + o(h)$
  4. $P\{N(t+h) - N(t) \geq 2\} = o(h)$

- If $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, the number of events in any interval of length t, i.e. $N(t+h) - N(t)$ is a Poisson random variable with mean λt.

Interarrival and Waiting time distribution

- T_n: the elapse time between the $(n-1)\mathrm{st}$ and the $n\mathrm{th}$ event. The sequence $\{T_n, n = 1, 2, \dots\}$ is called the sequence of interarrival times.
  - $P\{T_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$: exponential with mean $\frac{1}{\lambda}$
  - $P\{T_2 > t \mid T_1\} = P\{0\ events\ in\ (s, s+t] \mid T_1 = s\} = P\{0\ events\ in\ (s, s+t]\} = e^{-\lambda t}$
  - $T_n, n = 1, 2, \dots,$ are independent identically distributed exponential with mean $\frac{1}{\lambda}$

- Waiting time $S_n = \sum_{i=1}^{n} T_i,\ n \geq 1 \sim$ gamma with parameters n and λ
  - pdf of $S_n = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

Example 5.13

- Suppose that people immigrate into a territory at a Poisson rate $\lambda = 1$ per day.
  - (Question 1) what is the expected time until the tenth immigrant arrives?
    - ✓ (Solution) $E[\,S_{10}] = \frac{10}{\lambda} = 10$
  - (Question 2) what is the probability that the elapsed time between the tenth and the eleventh arrival exceeds two days?
    - ✓ (Solution) $P[\,T_{11} > 2] = e^{-2\lambda} = e^{-2}$