Fundamentals of Logic
(Predicate Calculus)

Contents

Predicate Calculus

- Introduction

  Propositions cannot capture important relationships between objects.

  Nancy is a female. → N or $Nancyfemale$?

  Sandy is a female. → S or $Sandyfemale$?

  We need to introduce predicate!

  x is a female ($F(x)$) where x is Sandy ($F(Sandy)$).

  The property "is a female" is represented by a predicate.

One place predicate: $F(x)$

Two place predicate: $M(x, y)$

Three place predicate: $S(x, y, z)$

Predicate variables, constants: F, M, S

Individual variables, constants: x, y, z

- The set of allowable choices for individual objects is called *universe* or *domain of discourse*.

(*Ex*) For the universe of all integers,

$E(x)$: the number x is an even integer.

$S(x, y)$: the number $x - y$ and $x + 2y$ are even integers.

$E(5)$: False, $\neg E(7)$: True, $S(4, 2)$: True, $S(5, 2)$: False

(*Question*)

Is the statement "for *every* x, $E(x)$" true?

Is the statement "for *some* x, $E(x)$" a proposition?

- Universal Quantifier $(\forall x)$

  For a universe D, $(\forall x)p(x)$ means that for every $a \in D$, $p(a)$ is true.

  $(\forall x)$ is read as "for all x", "for any x", "for each x", or "for every x".

- Existential Quantifier $(\exists x)$

  For a universe D, $(\exists x)p(x)$ means that there exists $a \in D$ such that $p(a)$ is true.

  $(\exists x)$ is read as "for some x" or "for at least one x".

  It can also be expressed as "there exists an x such that".

(*Note*)

The predicate itself has no truth value. It has truth value when substituted by individual constants or quantified.

$D = \{a_1, a_2, a_3\}$

$(\forall x)F(x) \quad F(a_1) \land F(a_2) \land F(a_3)$

$(\exists x)F(x) \quad F(a_1) \lor F(a_2) \lor F(a_3)$

(Examples)

1. Every student in this class is bright.

   D: the set of individuals in this class

   $S(x)$: x is a student $B(x)$: x is bright

   For every x in this class, if x is a student then x is bright.

   $(\forall x)(S(x) \rightarrow B(x))$ Is $(\forall x)(S(x) \rightarrow B(x))$ true?

2. There is a student who is bright. (Some students are bright.)

   There exists an x such that x is a student and x is bright.

   $(\exists x)(S(x) \wedge B(x))$

   $(\exists x)(S(x) \rightarrow B(x))$?

3. Borogoves are mimsy whenever it is brillig.

   It is now brillig and this thing is a borogove.

   Therefore, this thing is mimsy.

$B(x)$: x is a borogove $M(x)$: x is mimsy R: it is brillig

$$R \to (\forall x)\,(B(x) \to M(x)),\quad R \wedge B(a) \;\models\; M(a)$$

First-Order Predicate Calculus

- Well-Formed Formula (wff)
  1. (Basis clause) A predicate is a wff.
  2. (Inductive clause) If ***A*** and ***B*** are wffs and x is a variable, then so are the following: $(\neg \boldsymbol{A})$, $(\boldsymbol{A} \wedge \boldsymbol{B})$, $(\boldsymbol{A} \vee \boldsymbol{B})$, $(\boldsymbol{A} \rightarrow \boldsymbol{B})$, $(\boldsymbol{A} \leftrightarrow \boldsymbol{B})$, $(\forall x)(\boldsymbol{A})$, $(\exists x)(\boldsymbol{A})$.
  3. (Extremal clause) Only the strings formed by finite applications of 1 and 2 are wffs.

- Precedence
  1. The precedence of $(\forall x)$ and $(\exists x)$ is the same as that of $\neg$.
  2. If these three symbols appear together, the rightmost symbol is grouped with the smallest wff to its right.

     (*Ex*) $\forall x \neg \exists x P(x, y)$ means $\forall x(\neg(\exists x P(x, y)))$.

- Scope of Quantifiers
  1. The scope of quantifier is the *smallest sub-formula* immediately following the quantifier.

$$(\exists x)(L(x) \wedge (\forall y)(J(y) \rightarrow A(x, y)))$$ vs.

$$(\exists x)(L(x) \wedge (\forall y)J(y) \rightarrow A(x, y))$$

2. An *occurrence* of a variable in a wff is said to be ***bound*** if it is within the scope of the quantifier employing the variable or if it is the occurrence of the variable in the quantifier itself.

   Otherwise, the occurrence of a variable is said to be ***free***.

3. A variable in a wff is said to be bound (free) if there is at least one occurrence of the variable in the formula that is bound (free).

bound occurrences

$$(\exists x)(L(x) \wedge (\forall y)J(y) \rightarrow A(x, y)) \quad ?$$

free occurrence

x: bound variable

y: bound variable, free variable

(Examples)

- All judges are lawyers.

  For every x, if x is a judge then x is a lawyer:

$$(\forall x)(J(x) \rightarrow L(x))$$

- Some lawyers are shysters.

  There exists an x such that x is a lawyer and x is a shyster:

$$(\exists x)(L(x) \wedge S(x))$$

- No judge is a shyster.

  1. It is not the case that there exists an x such that x is a judge and x is a shyster (It is not the case that some judges are shysters)

  $$\neg(\exists x)(J(x) \wedge S(x))$$

  2. For all x if x is a judge then x is not a shyster:

  $$(\forall x)(J(x) \rightarrow \neg S(x))$$

$$\neg(\exists x)P(x) \Leftrightarrow (\forall x)\neg P(x)$$

E.g., $\neg(\exists x)(J(x) \wedge S(x)) \Leftrightarrow (\forall x)\neg(J(x) \wedge S(x))$

$$\Leftrightarrow (\forall x)(\neg J(x) \vee \neg S(x))$$

$$\Leftrightarrow (\forall x)(J(x) \rightarrow \neg S(x))$$

- Some lawyers admire only judges.

  There exists an x such that x is a lawyer and for every y if x admires y then y is a judge:

  $$(\exists x)(L(x) \wedge (\forall y)(A(x,y) \rightarrow J(y)))$$

- Only judges admire lawyers. (Lawyers are admired by only judges.)

  For every x if x is a lawyer then for every y if y admires x then y is a judge: $(\forall x)(L(x) \rightarrow (\forall y)(A(y,x) \rightarrow J(y)))$

- $(\forall x)(W(x) \rightarrow \neg(P(x) \wedge H(x)))$ (W: woman, P: politician, H: housewife)

  For every x if x is a woman then x is neither a politician nor a housewife.

  No women are both politician and housewife:

  $$\neg(\exists x)(W(x) \wedge P(x) \wedge H(x))$$

- Quantified Equivalences and Quantified Implications

E_{23}: $(\exists x)(A(x) \vee B(x)) \;\Leftrightarrow\; (\exists x)A(x) \vee (\exists x)B(x)$

E_{24}: $(\forall x)(A(x) \wedge B(x)) \;\Leftrightarrow\; (\forall x)A(x) \wedge (\forall x)B(x)$

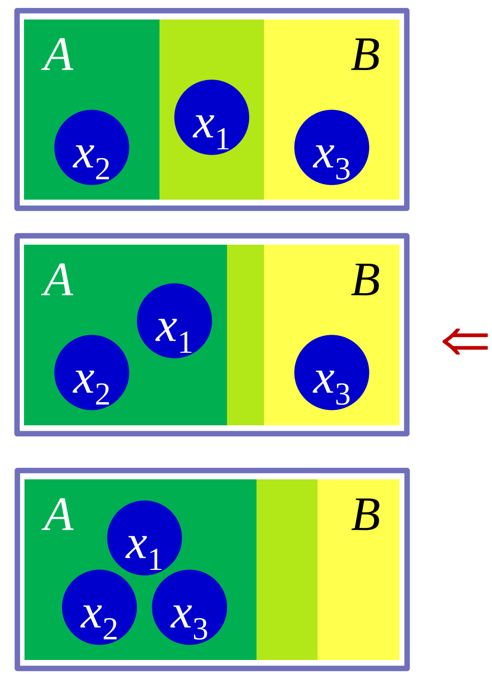
E_{25}: $\neg(\exists x)P(x) \;\Leftrightarrow\; (\forall x)\neg P(x)$

E_{26}: $\neg(\forall x)P(x) \;\Leftrightarrow\; (\exists x)\neg P(x)$

I_{15}: $(\forall x)A(x) \vee (\forall x)B(x) \Rightarrow (\forall x)(A(x) \vee B(x))$

i.e., $(\forall x)(A(x) \vee B(x)) \Leftarrow (\forall x)A(x) \vee (\forall x)B(x)$

I_{16}: $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$



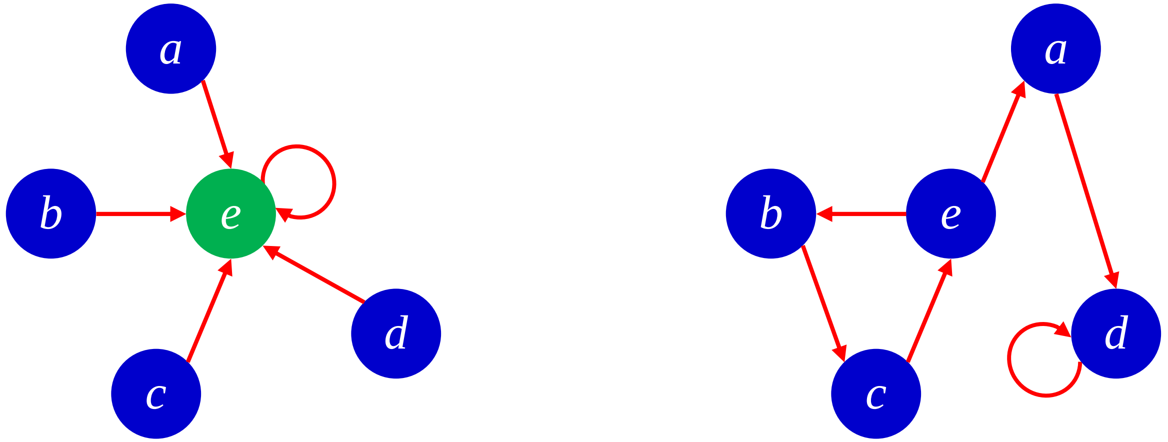
(4): $(\exists y)(\forall x)L(x, y) \Rightarrow (\forall x)(\exists y)L(x, y)$

(4)′: $(\forall x)(\exists y)L(x, y) \Rightarrow (\exists y)(\forall x)L(x, y)$??? No !!!

(*Ex*) $L(x, y)$: x loves y.

$(\exists y)(\forall x)L(x, y)$: There exists a y whom every x loves.

$(\forall x)(\exists y)L(x, y)$: For every x, there exists a y that x loves.



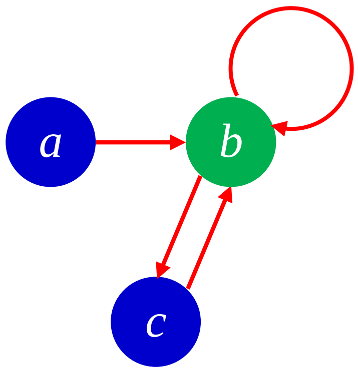
(*Ex*) $L(x, y)$: x loves y.

$$(\exists y)(\forall x)L(x, y)$$

$$\Leftrightarrow (\exists y)(L(a, y) \wedge L(b, y) \wedge L(c, y))$$

$$\Leftrightarrow (L(a, a) \wedge L(b, a) \wedge L(c, a)) \vee (L(a, b) \wedge L(b, b) \wedge L(c, b)) \vee (L(a, c) \wedge L(b, c) \wedge L(c, c))$$

$$\Leftrightarrow \mathrm{F} \vee \mathrm{T} \vee \mathrm{F} = \mathrm{T}$$



$$(\forall x)(\exists y)L(x, y)$$

$$\Leftrightarrow (\forall x)(L(x, a) \vee L(x, b) \vee L(x, c))$$

$$\Leftrightarrow (L(a, a) \vee L(a, b) \vee L(a, c)) \wedge (L(b, a) \vee L(b, b) \vee L(b, c)) \wedge (L(c, a) \vee L(c, b) \vee L(c, c))$$

$$\Leftrightarrow \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{T} = \mathrm{T}$$

(*Ex*) $L(x, y)$: x loves y.

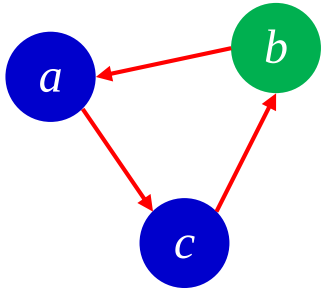
$(\exists y)(\forall x)L(x, y)$

$\Leftrightarrow (\exists y)(L(a, y) \wedge L(b, y) \wedge L(c, y))$

$\Leftrightarrow (L(a, a) \wedge L(b, a) \wedge L(c, a)) \vee (L(a, b) \wedge L(b, b) \wedge L(c, b))$

$\quad \vee (L(a, c) \wedge L(b, c) \wedge L(c, c))$

$\Leftrightarrow \mathrm{F} \vee \mathrm{F} \vee \mathrm{F} = \mathrm{F}$



$(\forall x)(\exists y)L(x, y)$

$\Leftrightarrow (\forall x)(L(x, a) \vee L(x, b) \vee L(x, c))$

$\Leftrightarrow (L(a, a) \vee L(a, b) \vee L(a, c)) \wedge (L(b, a) \vee L(b, b) \vee L(b, c))$

$\quad \wedge (L(c, a) \vee L(c, b) \vee L(c, c))$

$\Leftrightarrow \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{T} = \mathrm{T}$

(*Ex*) $L(x, y)$: x loves y.

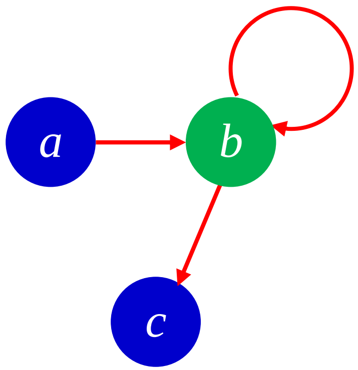
$(\exists y)(\forall x)L(x, y)$

$\Leftrightarrow (\exists y)(L(a, y) \wedge L(b, y) \wedge L(c, y))$

$\Leftrightarrow (L(a, a) \wedge L(b, a) \wedge L(c, a)) \vee (L(a, b) \wedge L(b, b) \wedge L(c, b))$

$\quad \vee (L(a, c) \wedge L(b, c) \wedge L(c, c))$

$\Leftrightarrow \mathrm{F} \vee \mathrm{F} \vee \mathrm{F} = \mathrm{F}$



$(\forall x)(\exists y)L(x, y)$

$\Leftrightarrow (\forall x)(L(x, a) \vee L(x, b) \vee L(x, c))$

$\Leftrightarrow (L(a, a) \vee L(a, b) \vee L(a, c)) \wedge (L(b, a) \vee L(b, b) \vee L(b, c))$

$\quad \wedge (L(c, a) \vee L(c, b) \vee L(c, c))$

$\Leftrightarrow \mathrm{T} \wedge \mathrm{T} \wedge \mathrm{F} = \mathrm{F}$

- $(\forall x)(\forall y)P(x, y) \Leftrightarrow (\forall y)(\forall x)P(x, y)$? Yes *!!!* $\cdots$ (1)

- $(\exists y)(\exists x)P(x, y) \Leftrightarrow (\exists x)(\exists y)P(x, y)$? Yes *!!!* $\cdots$ (8)

(2): $(\forall x)(\forall y)P(x, y) \Rightarrow (\exists y)(\forall x)P(x, y)$ $(\Rightarrow (\forall x)(\exists y)P(x, y))$

(3): $(\forall y)(\forall x)P(x, y) \Rightarrow (\exists x)(\forall y)P(x, y)$ $(\Rightarrow (\forall y)(\exists x)P(x, y))$

(4): $(\exists y)(\forall x)L(x, y) \Rightarrow (\forall x)(\exists y)L(x, y)$

(5): $(\exists x)(\forall y)P(x, y) \Rightarrow (\forall y)(\exists x)P(x, y)$

(6): $(\forall x)(\exists y)P(x, y) \Rightarrow (\exists y)(\exists x)P(x, y)$ $(\Leftrightarrow (\exists x)(\exists y)P(x, y))$

(7): $(\forall y)(\exists x)P(x, y) \Rightarrow (\exists x)(\exists y)P(x, y)$ $(\Leftrightarrow (\exists y)(\exists x)P(x, y))$

E_{27}: $(\forall x)(A \vee B(x)) \Leftrightarrow A \vee (\forall x)B(x)$

E_{28}: $(\exists x)(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x)B(x)$

E_{29}: $(\forall x)A(x) \to B \Leftrightarrow (\exists x)(A(x) \to B)$

$$\Leftrightarrow (\exists x)(\neg A(x) \vee B)$$

$$\Leftrightarrow (\exists x)\neg A(x) \vee (\exists x)B$$

$$\Leftrightarrow \neg(\forall x)\, A(x) \vee B$$

E_{30}: $(\exists x)A(x) \to B \Leftrightarrow (\forall x)(A(x) \to B)$

$$\Leftrightarrow (\forall x)(\neg B \to \neg A(x))$$

$$\Leftrightarrow \neg B \to (\forall x)\neg A(x) \Leftrightarrow \neg(\forall x)\neg A(x) \to B$$

E_{31}: $A \to (\forall x)B(x) \Leftrightarrow (\forall x)(A \to B(x)) \Leftrightarrow (\forall x)\, (\neg A \vee B(x))$

$$\Leftrightarrow \neg A \vee (\forall x)\, B(x)$$

E_{32}: $A \to (\exists x)B(x) \Leftrightarrow (\exists x)(A \to B(x))$

Formal Proofs in Predicate Calculus

All the inference rules of the propositional calculus can still be used for the predicate calculus. But we need more. We need to be able to *remove quantifiers* (*specification*) and to *restore quantifiers* (*generalization*).

- Universal Specification (US)

  The formula $P(x)$ is justified in a derivation if $(\forall y)P(y)$ precedes it in the derivation.

(Example) $M(x)$: x is a mathematics professor.

$C(x)$: x has studied calculus.

All mathematics professors have studied calculus. $(\forall x)(M(x) \rightarrow C(x))$

Judy is a mathematics professor. $M(J)$

Therefore, Judy has studied calculus. $C(J)$

No.	Formula	Rule	Just.	Taut.
1	$(\forall x)(M(x) \rightarrow C(x))$	P		
2	$M(J)$	P		
3	$M(J) \rightarrow C(J)$	US	1	
4	$C(J)$	T	2, 3	I_{11}

(I_{11}) $\boldsymbol{P}, \boldsymbol{P} \rightarrow \boldsymbol{Q} \Rightarrow \boldsymbol{Q}$; Modus Ponens

(Example) $G(x)$: x is a borogove. $M(x)$: x is mimsy.

B: It is brillig.

Borogoves are mimsy whenever it is brillig. $B \to (\forall x)(G(x) \to M(x))$

It is now brillig and this thing is a borogove. $B \wedge G(a)$

Therefore, this thing is mimsy. $M(a)$

No.	Formula	Rule	Just.	Taut.
1	$B \to (\forall x)(G(x) \to M(x))$	P		
2	$B \wedge G(a)$	P		
3	B	T	2	I_1
4	$(\forall x)(G(x) \to M(x))$	T	1, 3	I_{11}
5	$G(a) \to M(a)$	US	4	
6	$G(a)$	T	2	I_2
7	$M(a)$	T	5, 6	I_{11}

- Existential Generalization (EG)

  The formula $(\exists y)P(y)$ is justified in a derivation if $P(x)$ precedes it in the derivation.

- Existential Specification (ES)

  The formula $P(x)$ is justified in a derivation if $(\exists y)P(y)$ precedes it in the derivation and x is <u>not free</u> in any of the premises or in any preceding formulas.

  (<u>Choose a new variable each time ES is used</u>.)

- Universal Generalization (UG)

The formula $(\forall y)P(y)$ is justified in a derivation if $P(x)$ precedes it in the derivation under the condition that x is not free in any of the premises and that if $P(x)$ is the result of applying ES then it should not contain any variable introduced while applying ES.

$\boxed{(4)' \ (\forall x)(\exists y)L(x, y) \Rightarrow (\exists y)(\forall x)L(x, y)}$

This is not true !

1. $(\forall x)(\exists y)L(x, y)$; P
2. $(\exists y)L(a, y)$; US
3. $L(a, b)$; ES
4. $(\forall x)L(x, b)$; UG ⟵ Illegal because b is introduced by the previous ES
5. $(\exists y)(\forall x)L(x, y)$; EG

$\boxed{(4)\ (\exists y)(\forall x)L(x, y) \Rightarrow (\forall x)(\exists y)L(x, y)}$

1. $(\exists y)(\forall x)L(x, y)$	; P
2. $(\forall x)L(x, b)$	; ES
3. $L(a, b)$	; US
4. $(\exists y)L(a, y)$	; EG
5. $(\forall x)(\exists y)L(x, y)$	; UG

(Example)

All freshmen like all sophomores.

No freshmen likes any junior.

There are freshmen.

Therefore, no sophomore is a junior.

$(\forall x)(\forall y)(F(x) \wedge S(y) \rightarrow L(x, y))$ or $(\forall x)(F(x) \rightarrow (\forall y)(S(y) \rightarrow L(x, y)))$

$(\forall x)(\forall y)(F(x) \wedge J(y) \rightarrow \neg L(x, y))$ or $(\forall x)(F(x) \rightarrow (\forall y)(J(y) \rightarrow \neg L(x, y)))$

or $(\forall x)(F(x) \rightarrow \neg(\exists y)(J(y) \wedge L(x, y)))$

or $\neg(\exists x)(F(x) \wedge (\exists y)(J(y) \wedge L(x, y)))$

$(\exists x)F(x)$

$\models (\forall x)(S(x) \rightarrow \neg J(x))$

No.	Formula	Rule	Just.	Taut.
1	$(\forall x)(\forall y)(F(x) \wedge S(y) \rightarrow L(x, y))$	P		
2	$(\forall x)(\forall y)(F(x) \wedge J(y) \rightarrow \neg L(x, y))$	P		
3	$(\exists x)F(x)$	P		
4	$(\forall y)(F(a) \wedge S(y) \rightarrow L(a, y))$	US	1	
5	$F(a) \wedge S(b) \rightarrow L(a, b)$	US	4	
6	$(\forall y)(F(a) \wedge J(y) \rightarrow \neg L(a, y))$	US	2	
7	$F(a) \wedge J(b) \rightarrow \neg L(a, b)$	US	6	
8	$F(a)$	ES Illegal !!!	~~3~~	
9	$F(a) \rightarrow (S(b) \rightarrow L(a, b))$	T	5	E_{19}
10	$S(b) \rightarrow L(a, b)$	T	8, 9	I_{11}
11	$F(a) \rightarrow (J(b) \rightarrow \neg L(a, b))$	T	7	E_{19}
12	$J(b) \rightarrow \neg L(a, b)$	T	8, 11	I_{11}
13	$\neg\neg L(a, b) \rightarrow \neg J(b)$	T	12	E_{18}
14	$L(a, b) \rightarrow \neg J(b)$	T	13	E_{1}
15	$S(b) \rightarrow \neg J(b)$	T	10, 14	I_{13}
16	$(\forall x)(S(x) \rightarrow \neg J(x))$	UG	15	

(Note) ES before US!!

$(\exists x)F(x)$	; P	
$(\forall x)(\forall y)(F(x) \wedge S(y) \rightarrow L(x, y))$	; P	
$F(a)$	; ES	a: specific element
$(\forall y)(F(a) \wedge S(y) \rightarrow L(a, y))$	; US	a: specific element (○)

However,

$(\forall y)(F(a) \wedge S(y) \rightarrow L(a, y))$	; US	a: arbitrarily chosen element
$F(a)$	; ES	a: arbitrarily chosen element (×)

No.	Formula	Rule	Just.	Taut.
1	$(\forall x)(\forall y)(F(x) \wedge S(y) \rightarrow L(x, y))$	P		
2	$(\forall x)(\forall y)(F(x) \wedge J(y) \rightarrow \neg L(x, y))$	P		
3	$(\exists x)F(x)$	P		
4	$F(a)$	ES	3	
5	$(\forall y)(F(a) \wedge S(y) \rightarrow L(a, y))$	US	1	
6	$F(a) \wedge S(b) \rightarrow L(a, b)$	US	5	
7	$(\forall y)(F(a) \wedge J(y) \rightarrow \neg L(a, y))$	US	2	
8	$F(a) \wedge J(b) \rightarrow \neg L(a, b)$	US	7	
9	$F(a) \rightarrow (S(b) \rightarrow L(a, b))$	T	6	E_{19}
10	$S(b) \rightarrow L(a, b)$	T	4, 9	I_{11}
11	$F(a) \rightarrow (J(b) \rightarrow \neg L(a, b))$	T	8	E_{19}
12	$J(b) \rightarrow \neg L(a, b)$	T	4, 11	I_{11}
13	$\neg\neg L(a, b) \rightarrow \neg J(b)$	T	12	E_{18}
14	$L(a, b) \rightarrow \neg J(b)$	T	13	E_{1}
15	$S(b) \rightarrow \neg J(b)$	T	10, 14	I_{13}
16	$(\forall x)(S(x) \rightarrow \neg J(x))$	UG	15	