Long-Run Proportions and Limiting Probabilities

Positive recurrent

- $f_{i,j} = P\{X_n = j \text{ for some } n \geq 0 \mid X_0 = i\}$

  i.e. Probability that the Markov chain, starting in state *i*, will ever make a transition into state *j*
- If *i* is recurrent and I communicates with *j*, then $f_{i,j} = 1$
- State *j* is recurrent, let m_j denote the expected number of transitions that it takes the Markov chain when starting in state *j* to return to that state.
  - $N_j = \min\{n > 0 \colon X_n = j\}$
  - $m_j = E[N_j \mid X_0 = j]$
- The recurrent state *j* is positive recurrent if $m_j < \infty$, and it is null recurrent if $m_j = \infty$

Long-run proportions

- If the Markov chain is irreducible and recurrent, then for any initial state
  - π_j: the long-run proportion of time that the Markov chain is in state *j*
  - $\pi_j = 1/m_j$
- If *i* is positive recurrent and $i \leftrightarrow j$ $(P_{i,j}{}^{n} > 0)$, *j* is positive recurrent.
  - $\pi_i P_{i,j}{}^{n}$ = long-run proportion of time the chain is in i and will be in j after n transitions $\leq$ long-run proportion of time the chain is in $j = \pi_j$
  - $\pi_j \geq \pi_i P_{i,j}{}^{n} > 0$

Theorem 4.1

- If the Markov chain is irreducible and positive recurrent, then the long-run proportions are the unique solution of the equations
  - $\pi_j = \sum_i \pi_i P_{i,j}, \, j \geq 1$
  - $\sum_j \pi_j = 1$
- If there is no solution of the preceding linear equations, then the Markov chain is either transient or null recurrent and all $\pi_j = 0$.

Example 4.20

- $P = \begin{Vmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{Vmatrix}$

  (Sol.) $\pi_0 = \alpha\pi_0 + \beta\pi_1$

  - $\pi_1 = (1-\alpha)\pi_0 + (1-\beta)\pi_1$
  - $\pi_0 + \pi_1 = 1$
  - $\pi_0 =$
  - $\pi_1 =$

Example 4.21

- ❖ $P = \begin{Vmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{Vmatrix}$

  (Sol.) $\pi_0 = 0.5\pi_0 + 0.4\pi_1 + 0.1\pi_2$

  - $\pi_1 = 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2$
  - $\pi_2 = 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2$
  - $\pi_0 + \pi_1 + \pi_2 = 1$
  - $\pi_0 =$
  - $\pi_1 =$
  - $\pi_2 =$

Example 4.22

- $P = \begin{Vmatrix} 0.45 & 0.48 & 0.07 \\ 0.05 & 0.70 & 0.25 \\ 0.01 & 0.50 & 0.49 \end{Vmatrix}$

  (Sol.) $\pi_0 = 0.45\pi_0 + 0.48\pi_1 + 0.07\pi_2$

  - $\pi_1 = 0.05\pi_0 + 0.70\pi_1 + 0.25\pi_2$
  - $\pi_2 = 0.01\pi_0 + 0.50\pi_1 + 0.49\pi_2$
  - $\pi_0 + \pi_1 + \pi_2 = 1$
  - $\pi_0 =$
  - $\pi_1 =$
  - $\pi_2 =$

Limiting probabilities

- $d(i)$: period of state i
  - $P = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \ d(0) =$
  - $P = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{Vmatrix}, \ d(0) =$
  - $d(i) = 1$➔ $i: aperiodic$
- State *i* is said to be ***ergodic*** if positive recurrent and aperiodic
- Theorem
  - For an irreducible ergodic Markov Chain, $\alpha_j = \lim_{n\to\infty}(X_n = j)$ exists (independent of initial states)
  - $\alpha_j = \sum_i \alpha_i P_{i,j}, \ j \geq 1$
  - $\sum_j \alpha_j = 1$