Computing variances and probabilities by conditioning

Computing variances by conditioning

- $Var(X) = E[X^2] - (E[X])^2$
- (Example 3.18) Variance of the Geometric random variable with *p*. Let *N* be the trial number of the first success.
  - Let $Y = \begin{cases} 1, \text{if the first trial results in a successe} \\ 0, \quad \text{otherwise} \end{cases}$
  - $E[N] = \frac{1}{p}$ (shown in Example 3.11)
  - $E[N^2|Y=1] = 1, E[N^2|Y=0] = E[(N+1)^2]$
  - $E[N^2] = E\big[E[N^2|Y]\big] = E[N^2|Y=1]P\{Y=1\} + E[N^2|Y=0]P\{Y=0\} = p + (1-p)E[(N+1)^2]$
  - Therefore, $E[N^2] = \frac{2-p}{p^2}$ and $\mathrm{Var}[\mathrm{N}] = E[N^2] - (E[N])^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-\mathrm{p}}{p^2}$

Computing probabilities by conditioning

- $P(E) = \sum_y P(E|Y = y)P(Y = y)$, if Y is discrete
- $P(E) = \int_{-\infty}^{\infty} P(E|Y = y)\, f_Y(y)dy$, if Y is continuous
- (Example 3.22) Let X denote the number of accidents that a randomly chosen policyholder has next year. Let Y denote the Poisson mean number of accidents for this policyholder.
  - $P[X = n|Y = \lambda] = e^{-\lambda}\frac{\lambda^n}{n!}$
  - $g(\lambda) = \lambda e^{-\lambda}, \lambda \geq 0$
  - $P[X = n] = \int_{-\infty}^{\infty} P(X = n|Y = \lambda)\, g(\lambda)d\lambda = \int_0^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\lambda e^{-\lambda}\, d\lambda = \frac{1}{n!}\int_0^{\infty} \lambda^{n+1}e^{-2\lambda}d\lambda$
  - $h(\lambda) = \frac{2e^{-2\lambda}(2\lambda)^{n+1}}{(n+1)!}, \lambda > 0 \Leftrightarrow \lambda \sim$ gamma $(n + 2, 2)$
  - $\int_0^{\infty} \frac{2e^{-2\lambda}(2\lambda)^{n+1}}{(n+1)!}d\lambda = 1 \rightarrow \int_0^{\infty} \lambda^{n+1}e^{-2\lambda}d\lambda = \frac{(n+1)!}{2^{n+2}}$
  - $P[X = n] = \frac{1}{n!}\int_0^{\infty} \lambda^{n+1}e^{-2\lambda}d\lambda = \frac{n+1}{2^{n+2}}$

The Ballot problem

- $P(E) = \sum_y P(E|Y=y)P(Y=y)$, if Y is discrete
- (Example 3.27) In an election, candidate A receives *n* votes and candidate B *m* votes where *n* > *m*. Assuming that all orderings are equally likely, prove the probability that A is always ahead in the count of votes is $\frac{n-m}{n+m}$ (Let $P_{n,m}$ denote this probability)?
  - (Solution) Condition on which candidate receives the last vote counted
  - Then, $P_{n,m} = P\{\text{A always ahead} \mid \text{A receives last vote}\} \frac{n}{n+m} +$ $P\{\text{A always ahead} \mid \text{B receives last vote}\} \frac{m}{n+m}$

    $P\{\text{A receives a vote}\}$
  - $P_{n,m} = \frac{n}{n+m} P_{n-1,m} + \frac{m}{n+m} P_{n,m-1}$ (Equation 1)
  - It is true when $n + m = 1$ ➔ $P_{1,0} = 1$
  - Assume it is true on $n + m - 1$, *i.e.* $P_{n-1,m} = \frac{n-1-m}{n-1+m}$ and $P_{n,m-1} = \frac{n-(m-1)}{n+m-1}$
  - By induction on $n + m$ and (Equation 1), $P_{n,m} = \frac{n}{n+m}\frac{n-1-m}{n-1+m} + \frac{m}{n+m}\frac{n-m+1}{n+m-1} = \frac{n-m}{n+m}$