Fundamental Principles of Counting

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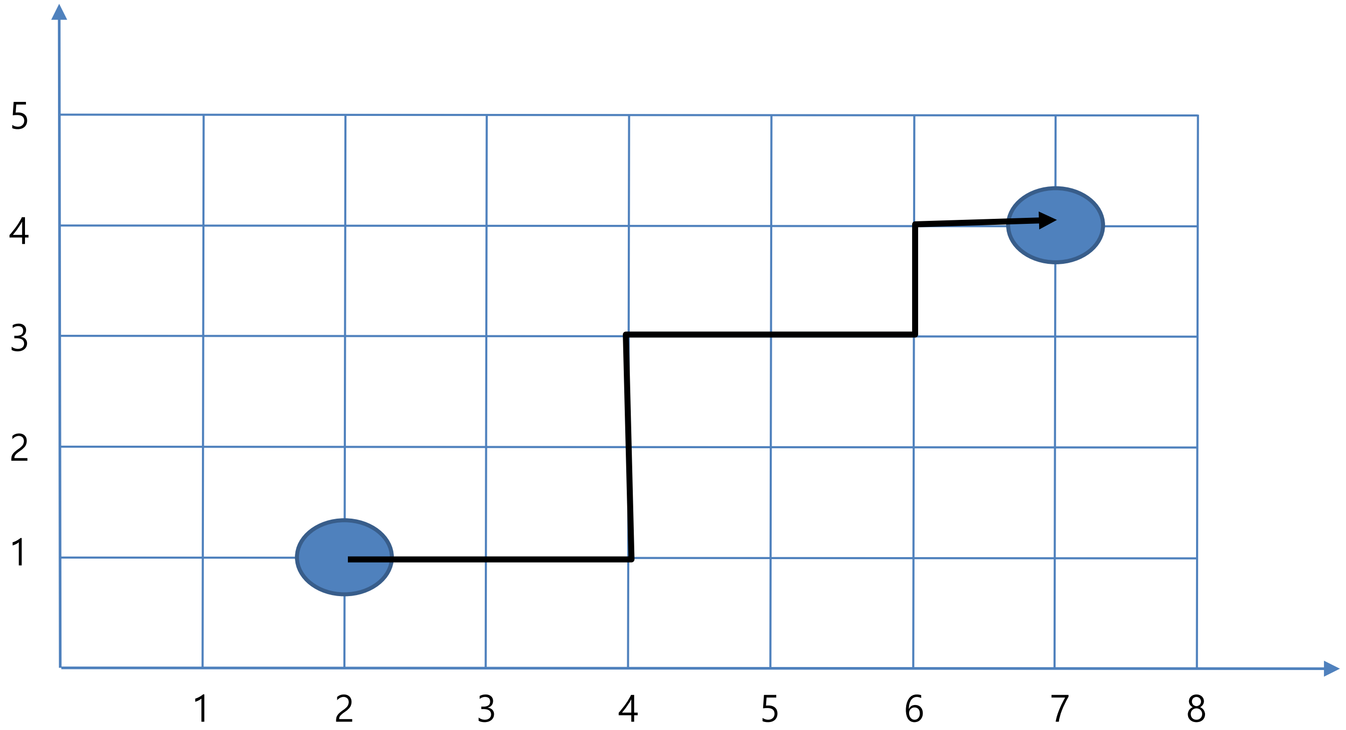
Note) Some open-source fonts, such as Naver Nanum fonts and Google Raleway fonts, are used in this slide. Download and Install them, if you see broken characters

❖ How many paths from (2,1) to (7,4) in the xy-plane?



❖ Do any two people in Busan have the same number of hairs on their head ?

- A typical human head has an average of around 150,000 hairs, it is reasonable to assume (as an upper bound) that no one has more than 1,000,000 hairs on their head
- The population of Busan is about 3.4 millions

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Contents

Rules of Sum and Product

- Useful for analyzing complicated problems through decomposing into parts and piecing together partial solutions in order to arrive at the final answer

- Rule of Sum

If a first task can be performed in $\boldsymbol{m}$ ways (1), while a second task in $\boldsymbol{n}$ ways (2), and

the two tasks cannot be performed simultaneously (3),

then performing either task can be accomplished in any one of $\boldsymbol{m}+\boldsymbol{n}$ ways.

Either English or Korean

3 textbooks in English

2 textbooks in Korean

How many ways?

* In combinatorics, the rule of sum is a basic counting principle. Stated simply, it is the idea that if we have $\boldsymbol{a}$ ways of doing something and $\boldsymbol{b}$ ways of doing another thing and we can not do both at the same time, then there are $\boldsymbol{a}+\boldsymbol{b}$ ways to choose one of the actions.

Examples of Rule of Sum

❖ Travel Options

- You need to travel in between city A and B.
- You can either fly, take a train, or a bus.
- There are 12 different flights in between A and B, 5 different trains and 10 buses.
- How many options do you have to get from A to B?

❖ # of Possible Password

- The minimum length of the password is 1 and the maximum is 3.
- The password can consist of only the uppercase letters (A~Z).
- How many cases are there for the possible password ?

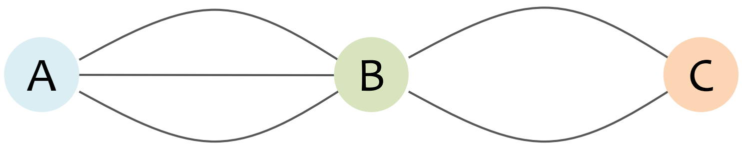
Rule of Product

- If a procedure can be broken down into first & second stages (1),

  and if there are $\boldsymbol{m}$ possible outcomes for the first stage (2) and

  if, for each of these outcomes, there are $\boldsymbol{n}$ possible outcomes for the second stage (3),

  then the total procedure can be carried out, in the designated order, in $\boldsymbol{m} \cdot \boldsymbol{n}$ ways.



How many ways for A $\rightarrow$ C ?

* In combinatorics, the rule of product is a basic counting principle. Stated simply, it is the idea that if we have $\boldsymbol{a}$ ways of doing something and $\boldsymbol{b}$ ways of doing another thing, then there are $\boldsymbol{a} \cdot \boldsymbol{b}$ ways of performing both actions.

Examples of Rule of Product

- How many different bit strings of length 7 are there ? (for example, 1011010)

- The license plate of the car is consisted of two alphabet letters followed by four digits

Permutations (순열:順列)

❖ **Definition**

Given a collection of n distinct objects, any (linear) arrangement of these objects is called a *permutation* of the collection.

❖ In general, if there are n distinct objects and r is an integer, with $1 \le r \le n$,

then by the rule of product, the number of permutations of size r for the n objects is

$$\boldsymbol{P(n, r) = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}}$$

* An *m*-permutation of a set S with *n* elements is a non-repeating ordered selection of *m* elements of S, that is, a sequence of *m* distinct elements of S.
An *n*-permutation is simply called a permutation of S.

- The number of permutations of the letters in the word “COMPUTER”

- If only five of the letters are used

- if ‘C’ and ‘O’ should be appeared side by side. (Both 'CO' and 'OC' are possible)

❖ The number of permutations of the letter “BALL” ?

- The number of permutations of this letter of size 4 is not $4!$, but $2 \cdot 3!$
- We can list them as the following Table.

A	B	L	L	A	B	L_1	L_2	A	B	L_2	L_1
A	L	B	L	A	L_1	B	L_2	A	L_2	B	L_1
A	L	L	B	A	L_1	L_2	B	A	L_2	L_1	B
B	A	L	L	B	A	L_1	L_2	B	A	L_2	L_1
B	L	A	L	B	L_1	A	L_2	B	L_2	A	L_1
B	L	L	A	B	L_1	L_2	A	B	L_2	L_1	A
L	A	B	L	L_1	A	B	L_2	L_2	A	B	L_1
L	A	L	B	L_1	A	L_2	B	L_2	A	L_1	B
L	B	A	L	L_1	B	A	L_2	L_2	B	A	L_1
L	B	L	A	L_1	B	L_2	A	L_2	B	L_1	A
L	L	A	B	L_1	L_2	A	B	L_2	L_1	A	B
L	L	B	A	L_1	L_2	B	A	L_2	L_1	B	A

(a) (b)

4!=24

- If we can distinguish the two Ls as L1 and L2, then the number of permutations is 4! (shown in (b))
- 2 × (Number of Arrangements of the letters B, A, L, L)
  = (Number of Permutations of the symbols, B, A, L1, L2)

❖ Note that if repetitions are allowed, then by the rule of product there are n^r possible arrangements, with $r \geq 0$.

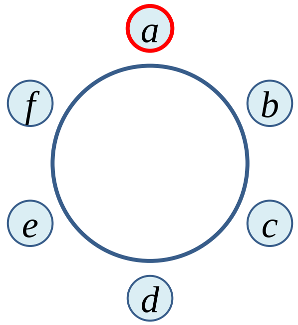
❖ If there are n objects with n_1 indistinguishable objects of a first type, n_2 of a second type, … , n_r of an r-th type, where $n_1 + n_2 + \cdots + n_r = n$, then there are

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

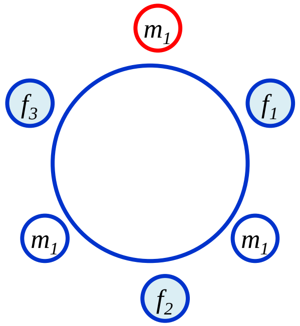
(linear) arrangements of the given n objects. (Objects of the same type are indistinguishable.)

(Ex.) PEPPER vs. P1E1P2P3E2R

❖ How many circular arrangements of six people at a round table ?



❖ How many sexually alternate arrangements of three males and three females at a round table ?



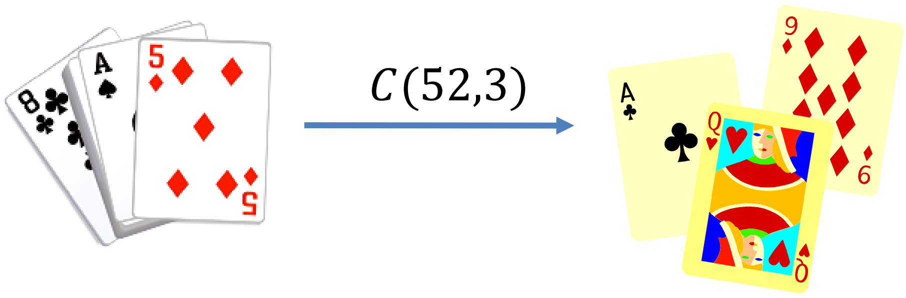
Combinations: 조합(組合)

❖ Combination: Unordered Selection

- The number of combinations of size r from a collection of n distinct objects is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}, \qquad 0 \le r \le n$$

- $C(n,r)$ is sometimes read as "n choose r".
- Note that $C(n,r) = 1$, for all $n \ge 0$, and $C(n,r) = C(n, n-r)$.



Binomial Theorem(이항정리 :二項定理)

❖ Binomial coefficients

- $\boldsymbol{\binom{n}{k}}$ is also known as binomial coefficient
  - it is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$
  - $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- Example) Counting bit strings with exactly k 0s: There are $\binom{n}{k}$ bit strings of length n with exactly k 0s, since each such bit string is determined by choosing a subset of size k from the n positions; 0s are placed in these k positions, and 1s in the remaining positions.
- Ex) what is the coefficient of x^5y^2 in the expansion of $(x+y)^7$?

❖ Binomial Theorem

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are variables and $\boldsymbol{n}$ is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \cdots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0$$

$$= \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k} = \sum_{k=0}^{n}\binom{n}{n-k}x^ky^{n-k}$$

❖ Corollary (따름 정리, 계: 系)

For each integer $\boldsymbol{n > 0}$,

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

$$(-1+1)^n = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n} = 0$$

❖ Multinomial Theorem

For positive integers $\boldsymbol{n}, \boldsymbol{t}$,

the multinomial coefficient of $x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3} \cdots x_t^{n_t}$

in the expansions of $(x_1 + x_2 + \cdots + x_t)^n$ is

$$\frac{\boldsymbol{n!}}{\boldsymbol{n_1!\, n_2!\, n_3! \cdots n_t!}} = \binom{\boldsymbol{n}}{\boldsymbol{n_1, n_2, n_3, \cdots, n_t}}$$

Where $0 \le n_i \le n$, for all $1 \le i \le t$ and $n_1 + n_2 + \cdots + n_t = n$

Proof :

$$\binom{\boldsymbol{n}}{\boldsymbol{n_1}}\binom{\boldsymbol{n - n_1}}{\boldsymbol{n_2}}\binom{\boldsymbol{n - n_1 - n_2}}{\boldsymbol{n_3}} \cdots \binom{\boldsymbol{n - n_1 - n_2 - \cdots - n_{t-1}}}{\boldsymbol{n_t}} = \frac{\boldsymbol{n!}}{\boldsymbol{n_1!\, n_2!\, n_3! \cdots n_t!}}$$

Combinations with Repetition

❖ An Example :

- On their way home from track practice, **seven high school freshmen** stop at a restaurant, where each of them has one of the following:
  a **c**heese-burger, a **h**ot dog, a **t**aco, or a **f**ish sandwich.

- How many different purchases are possible?

❖ Answer :

- The number of ways for selecting $7(\boldsymbol{r})$ of $4(\boldsymbol{n})$ different objects, with repetition, is $\boldsymbol{C(r+n-1, r) = C(10, 7)}$

- When we wish to select, with repetition, $\boldsymbol{r}$ of $\boldsymbol{n}$ distinct objects, we find that we are considering all arrangements of $\boldsymbol{r}$ x's and $(\boldsymbol{n-1})$ | 's and that their number is

$$\frac{(\boldsymbol{n+r-1})!}{\boldsymbol{r}!\,(\boldsymbol{n-r})!} = \binom{\boldsymbol{n+r-1}}{\boldsymbol{r}}$$

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Combination with Repetition

$\boldsymbol{n}$ different types → $\boldsymbol{n-1}$ bar(|)s to divide them, $\boldsymbol{r}$ selections → $\boldsymbol{r}$ Xs to put-in

Cheese-Burger		Hot dog		Taco		Fish Sandwich
	|		|		|	

X X X X X X X

X X X	|	X X	|	X	|	X
X	|	X X X	|	X X	|	X
X X	|	X X	|		|	X X X

- How many different arrangements are possible for the 5 characters A, B, C, D, E ?
  - Select **All** from $\boldsymbol{N}$ **different Objects**, Linear Arrangement (Order) : $\boldsymbol{N}!$

- How many different arrangements of 3 characters are possible from the 5 characters A, B, C, D, E ?
  - Select $\boldsymbol{R}$ from $\boldsymbol{N}$ **different Objects**, Arrangement, $\frac{\boldsymbol{N}!}{(\boldsymbol{N}-\boldsymbol{R})!} = \boldsymbol{P}(\boldsymbol{N}, \boldsymbol{R})$

- How many different arrangements of 3 characters are possible from the 5 characters A, B, C, D, E if characters can be used repeatedly ?
  - Select $\boldsymbol{R}$ from $\boldsymbol{N}$ **different (Object) Types (Repetition)**, Arrangement,

$$\boldsymbol{N} \times \boldsymbol{N} \times \cdots \times \boldsymbol{N} = \boldsymbol{N}^{\boldsymbol{R}}$$

- How many different arrangements are possible for the 5 characters A, A, C, D, D ?
  - Select All from $\boldsymbol{N}$ Objects where some of them are indistinguishable.

❖ How may ways can we select 3 characters from the 5 characters A, B, C, D, E ?

- Select **R** from **N** different Objects, $\frac{\boldsymbol{N}!}{(\boldsymbol{N}-\boldsymbol{R})!\cdot\boldsymbol{R}!} = \boldsymbol{C(N, R)}$

❖ How may ways can we select 3 characters from the 5 characters A, B, C, D, E if characters can be selected <u>repeatedly</u> ?

- Select **R** from **N** different Types, $\frac{(\boldsymbol{R}+\boldsymbol{N}-\mathbf{1})!}{(\boldsymbol{N}-\mathbf{1})!\cdot\boldsymbol{R}!} = \boldsymbol{C(N+R-1, R)}$

❖ How may ways can we select 5 characters from the 3 characters A, B, C with repetition ?

❖ Select *R* objects with repetition from *N* different object Types

- **R → X, N → |**

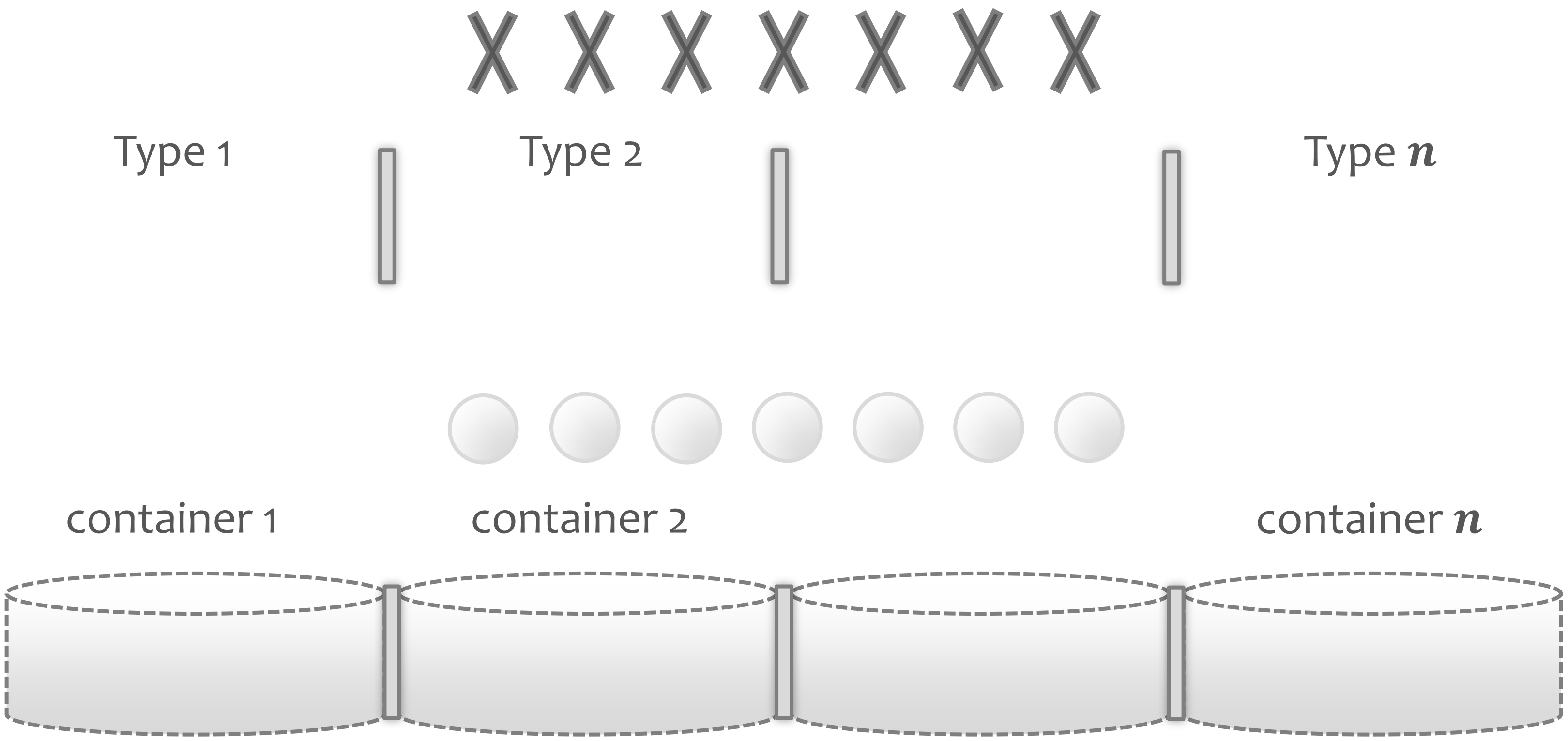
Type 1 *Type 2* *Type 3* *Type N*

X | X X X | X X - - - X | X X

Equivalent Problems #1

1. The number of selections, with repetition, of size r from n different object types.

2. The number of ways r identical objects can be distributed among n distinct containers.



Equivalent Problems #2

3. The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \qquad x_i \geq 0, 1 \leq i \leq n$$

X X X X X X X

Type 1 | Type 2 | | Type n

1 1 1 1 1 1 1

x_1 | x_2 | | x_n

❖ How many non-negative integer solutions are there to the equation $a + b + c + d = 100$.

(Ex.) Determine the number of integer combinations $\langle x_1, x_2, x_3, x_4 \rangle$ that satisfying the following inequalities

- $\mathbf{0 < x_1 < x_2 < x_3 < x_4 < 9}$

- $\mathbf{0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 9}$

(Ex.) Consider the following program segment, where i, j, and k are integer variables.

How many times is the print statement executed?

```
for i=1 to 20 do
    for j=1 to i do
        for k=1 to j do
            print(i*j+k);
```

❖ Note that the answer is $\boldsymbol{C(n+3-1,3)}$ in general.

```
for i=1 to 20 do
    for j=1 to i do
        for k=1 to j do
            print(i*j+k);
```

❖ Another Approach :

- The print statement is executed $\boldsymbol{T}$ times that can be represented as follows;

$$T = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j = \sum_{i=1}^{n} \frac{i\cdot(i+1)}{2} = \frac{1}{2}\sum_{i=1}^{n}(i^2+i)$$

$$= \frac{1}{2}\left\{\frac{n(n+1)(2n+1)}{6} + \frac{\big(n(n+1)\big)}{2}\right\} = \frac{1}{6}n(n+1)(n+2) = 1540$$

❖ (*Ex*. 2) In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?

 $\times$ **7**

 $\times$ **6**

- (*Ex*. 3) A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

A_ _ _ _ B _ _ _ _ C _ _ _ D _ _ _ E_ _ _ _ _ F _ _ _ _ _ G _ _ _ _H_ _ _ I _ _ _J _ _ _ _ K_ _ _ _ _ _ _ L

The Pigeonhole Principle

- If m pigeons occupy n pigeonholes and $m > n$,

  then at least one pigeonhole has two or more pigeons roosting in it.

- Example : Sock-picking
  - Sock-picking : Assume a drawer contains a mixture of black socks and blue socks, each of which can be worn on either foot, and that you are pulling a number of socks from the drawer without looking.

    What is the minimum number of pulled socks required to guarantee a pair of the same color?
  - # of Holes ? - holes per one color

❖ Hand-Shaking Example

- There are n (> 1) people who can shake hands randomly with one another (no repeat handshakes).
- Show that there are always a pair of people who will shake hands with the same number of people.
- Holes : The number of hands shaken by a person

  $0 \sim n-1$: n possible holes;

  Can both "0" and "n-1" holes can be occupied ?

❖ The birthday problem

- For a set of n randomly chosen people, what is the probability that some pair of them will have the same birthday?
  - If the group is as small as 23 individuals, the probability that there is a pair of people with the same birthday is still above 50%

Do you need a Calculator ? Use Python

❖ https://www.python.org/downloads/

- Version : Python 3.8.* or above

- import math
- math.factorial(5)
- math.perm(5,3)
- math.comb(5,3)

❖ Python's Arbitrary Precision

❖ Floating Point Number