Exponential distribution

Exponential distribution

- $f(x) = \lambda e^{-\lambda x}, x \geq 0$
- $F(x) = P\{X \leq x\} = 1 - e^{-\lambda x}, x \geq 0$
- $\mathrm{E}[X] = \int_0^\infty x\lambda e^{-\lambda x} = \frac{1}{\lambda}$
- $\mathrm{Var}[X] = \frac{1}{\lambda^2}$

- Memoryless
  - $P\{X > s + t \mid X > t\} = P\{X > s\}$ for all $s, t \geq 0$
  - $\Leftrightarrow \frac{P\{X>s+t\ ,X>t\}}{P\{X>t\}} = \frac{P\{X>s+t\}}{P\{X>t\}} = P\{X > s\}$

Example 5.2

- X: the amount of time that the customer spends in the bank
  - $X \sim Exp(\frac{1}{10})$

(Question) what is the probability that a customer will spend more than 15 min in the bank?

(Solution)

- $P\{X > 15\} = 1 - P\{X \leq 15\} = 1 - \left(1 - e^{-15\lambda}\right) = e^{-15*\frac{1}{10}} = e^{-\frac{3}{2}}$

Example 5.5

- X: the demand, $X \sim Exp(\lambda)$. The store orders t pounds, it costs $c \times t$ and is sold at a price of $s \times t$. Left-over is worthless and no penalty if the store cannot meet all the demand.

  (Question) how much should be ordered so as to maximize the store's expected profit?

  (Solution)

  - Profit: $Y = s\min(X,t) - ct$ where $\min(X,t) = X - (X-t)^+$
  - $E[(X-t)^+] = E[(X-t)^+|X > t]P(X > t) + E[(X-t)^+|X \leq t]P(X \leq t) = E[(X-t)^+|X > t]P(X > t) = E[(X-t)^+|X > t]e^{-\lambda t} = \frac{1}{\lambda}e^{-\lambda t}$

    Using the lack of memory property

  - $E[\min(X,t)] = \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda t}$, $E[P] = \frac{s}{\lambda} - \frac{s}{\lambda}e^{-\lambda t} - ct$ (maximal when $se^{-\lambda t} - c = 0$)

Further properties of the exponential distribution

- Let $X_1, \dots, X_n$ be independent and identically distributed exponential with mean $\frac{1}{\lambda}$
  - $X_1 + \cdots + X_n \sim$ Gamma(n, λ), i.e. $f_{X_1+\cdots+X_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$
    - Proved using mathematical induction
    - Trivial when $n = 1$
    - Assume $X_1 + \cdots + X_{n-1} \sim$ Gamma$(n-1, \lambda)$, i.e. $f_{X_1+\cdots+X_{n-1}}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-2}}{(n-2)!}$
    - $f_{X_1+\cdots+X_n}(t) = \int_0^t f_{X_n}(t-s) f_{X_1+\cdots+X_{n-1}}(s) ds = \int_0^t \lambda e^{-\lambda(t-s)} \frac{\lambda e^{-\lambda s}(\lambda s)^{n-2}}{(n-2)!} ds = \lambda e^{-\lambda t} \frac{\lambda^{n-1}}{(n-2)!} \int_0^t s^{n-2} ds = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

Further properties of the exponential distribution

- $X_1 \sim Exp(\lambda_1)$, and $X_2 \sim Exp(\lambda_2)$
  - $P\{X_1 < X_2\} = \int_0^\infty P\{X_1 < X_2 | X_1 = x\}\lambda_1 e^{-\lambda_1 x} dx = \int_0^\infty P\{x < X_2\}\lambda_1 e^{-\lambda_1 x} dx = \int_0^\infty e^{-\lambda_2 x}\lambda_1 e^{-\lambda_1 x} dx = \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x} dx = \frac{\lambda_1}{\lambda_1+\lambda_2}$
- $P\{\min(X_1, \ldots, X_n) > x\} = P\{X_i > x \text{ for each } i\} = \prod_{i=1}^n P\{X_i > x\} = e^{-\sum_i \lambda_i x}$

Greedy algorithms for the assignment problem

$$
\begin{array}{cc}
 & \\
n \text{ people} & \begin{bmatrix}
C_{1,1} & C_{1,2} & \cdots & C_{1,(n-1)} & C_{1,n} \\
C_{2,1} & C_{2,2} & \cdots & C_{2,(n-1)} & C_{2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
C_{(n-1),1} & C_{(n-1),2} & \cdots & C_{(n-1),(n-1)} & C_{(n-1),n} \\
C_{n,1} & C_{n,2} & \cdots & C_{n,(n-1)} & C_{n,n}
\end{bmatrix} \\
 & n \text{ jobs}
\end{array}
$$

  - $C_{i,j}$ (cost of *i* → *j*) constitutes a set of n^2 independent exponential with mean 1

- Assignment problem: the set of assignments that minimize the sum of *n* costs incurred
  - Greedy algorithm 1: assign person *i* to the job that results in the least cost
  - Greedy algorithm 2: choose the pair *i*, *j* for which C(*i*, *j*) is minimal

Example 5.7

- Greedy Algorithm 1
  - Let C_i denote the cost associated with person i
  - C_1 is the minimum of n independent exponential each having rate 1
  - $E[C_1] = \frac{1}{n}$ since the minimum of n independent exponential variables $\sim \exp(\text{sum of } \lambda_i))$
  - $E[C_1 + C_2 + \cdots + C_n] = \frac{1}{n} + \frac{1}{n-1} + \cdots + 1$
- Greedy Algorithm 2
  - Let C_i denote the cost of the ith person-job pair assigned by the algorithm
  - $E[C_1] = \frac{1}{n^2}$
  - $E[C_2] = E[C_1] + \frac{1}{(n-1)^2}$
  - ...
  - $E[C_n] = E[C_{n-1}] + 1 = \frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + 1 \rightarrow E[C_1 + C_2 + \cdots + C_n] = \frac{1}{n} + \frac{1}{n-1} + \cdots + 1$

- $(\text{the other } C_{i,j}) - C_1 \sim$ independent exponentials with rates 1
- $C_2 = \min\left(\text{the other } C_{i,j}\right) = C_1 + \min((n-1)^2$ independent exponentials with rate 1)