MONASH University

MONASH BUSINESS SCHOOL

High dimensional time series analysis

4. Automatic forecasting algorithms

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Outline

Outline

Historical perspective

- Developed in the 1950s and 1960s as methods (algorithms) to produce point forecasts.
- Combine a “level”, “trend” (slope) and “seasonal” component to describe a time series.
- The rate of change of the components are controlled by “smoothing parameters”: α, β and γ respectively.
- Need to choose best values for the smoothing parameters (and initial states).
- Equivalent ETS state space models developed in the 1990s and 2000s.

A model for levels, trends, and seasonalities

We want a model that captures the level (ℓ_t), trend (b_t) and seasonality (s_t).

How do we combine these elements?

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Perhaps a mix of both?

$$y_t = (\ell_{t-1} + b_{t-1}) s_{t-m} + \varepsilon_t$$

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How do the level, trend and seasonal components evolve over time?

ETS models

General notation

$$\underset{\textbf{Error}}{E}\ \underset{\textbf{Trend}}{T}\ \underset{\textbf{Season}}{S}$$

Exponen**T**ial **S**moothing

Error: Additive (`"A"`) or multiplicative (`"M"`)

ETS models

General notation E T S : **E**xponen**T**ial **S**moothing

Error **T**rend **S**eason

Error: Additive (`"A"`) or multiplicative (`"M"`)

Trend: None (`"N"`), additive (`"A"`), multiplicative (`"M"`), or damped (`"Ad"` or `"Md"`).

ETS models

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Error: Additive (`"A"`) or multiplicative (`"M"`)

Trend: None (`"N"`), additive (`"A"`), multiplicative (`"M"`), or damped (`"Ad"` or `"Md"`).

Seasonality: None (`"N"`), additive (`"A"`) or multiplicative (`"M"`)

ETS(A,N,N): SES with additive errors

$$
\begin{aligned}
\text{Forecast equation} \qquad \hat{y}_{T+h|T} &= \ell_T \\
\text{Measurement equation} \qquad y_t &= \ell_{t-1} + \varepsilon_t \\
\text{State equation} \qquad \ell_t &= \ell_{t-1} + \alpha\varepsilon_t
\end{aligned}
$$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

ETS(A,N,N): SES with additive errors

Forecast equation	$\hat{y}_{T+h\|T} = \ell_T$
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where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- "innovations" or "single source of error" because equations have the same error process, ε_t.
- Measurement equation: relationship between observations and states.
- Transition/state equation(s): evolution of the state(s) through time.

ETS(M,N,N): SES with multiplicative errors

Forecast equation	$\hat{y}_{T+h\|T} = \ell_T$
Measurement equation	$y_t = \ell_{t-1}(1 + \varepsilon_t)$
State equation	$\ell_t = \ell_{t-1}(1 + \alpha\varepsilon_t)$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

ETS(M,N,N): SES with multiplicative errors

$$\begin{aligned} \text{Forecast equation} \quad \hat{y}_{T+h|T} &= \ell_T \\ \text{Measurement equation} \quad y_t &= \ell_{t-1}(1 + \varepsilon_t) \\ \text{State equation} \quad \ell_t &= \ell_{t-1}(1 + \alpha\varepsilon_t) \end{aligned}$$

where $\varepsilon_t \sim \text{NID}(0, \sigma^2)$.

- Models with additive and multiplicative errors with the same parameters generate the same point forecasts but different prediction intervals.

ETS(A,A,N): Holt's linear trend

Additive errors

$$\begin{aligned}
\text{Forecast equation} \quad \hat{y}_{T+h|T} &= \ell_T + hb_T \\
\text{Measurement equation} \quad y_t &= \ell_{t-1} + b_{t-1} + \varepsilon_t \\
\text{State equations} \quad \ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= b_{t-1} + \beta\varepsilon_t
\end{aligned}$$

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Multiplicative errors

$$\begin{aligned} \text{Forecast equation} \quad \hat{y}_{T+h|T} &= \ell_T + hb_T \\ \text{Measurement equation} \quad y_t &= (\ell_{t-1} + b_{t-1})(1 + \varepsilon_t) \\ \text{State equations} \quad \ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\ b_t &= b_{t-1} + \beta\varepsilon_t \end{aligned}$$

Example: Australian population

```r
aus_economy <- global_economy %>% filter(Code == "AUS") %>%
  mutate(Pop = Population/1e6)
fit <- aus_economy %>% model(AAN = ETS(Pop))
report(fit)
```

```
## Series: Pop
## Model: ETS(A,A,N)
##   Smoothing parameters:
##     alpha = 1
##     beta  = 0.327
##
##   Initial states:
##     l     b
##  10.1 0.222
##
##   sigma^2:  0.0041
##
##   AIC  AICc   BIC
## -77.0 -75.8 -66.7
```

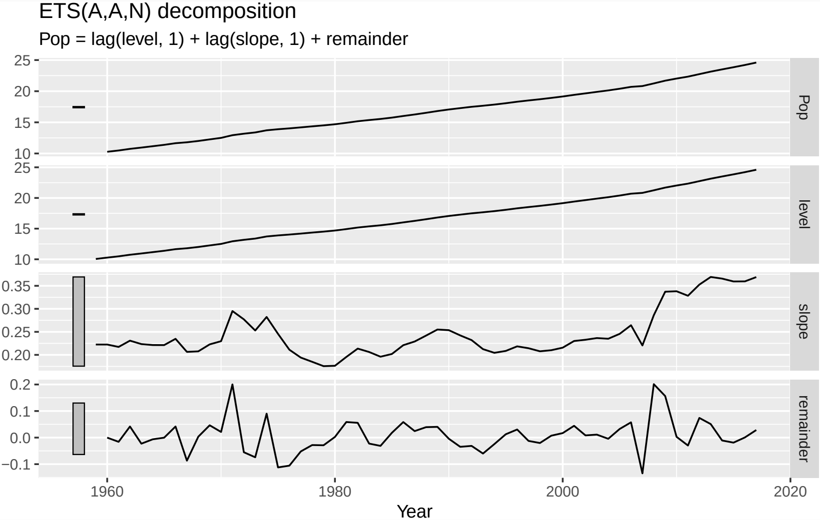
Example: Australian population

```r
components(fit)
```

```
## # A dable:                       59 x 7 [1Y]
## # Key:                           Country, .model [1]
## # ETS(A,A,N) Decomposition: Pop = lag(level, 1) + lag(slope, 1)
## #   remainder
##     Country   .model  Year   Pop level slope remainder
##     <fct>     <chr>  <dbl> <dbl> <dbl> <dbl>     <dbl>
##   1 Australia AAN     1959  NA    10.1 0.222 NA
##   2 Australia AAN     1960  10.3  10.3 0.222 -0.000145
##   3 Australia AAN     1961  10.5  10.5 0.217 -0.0159
##   4 Australia AAN     1962  10.7  10.7 0.231  0.0418
##   5 Australia AAN     1963  11.0  11.0 0.223 -0.0229
##   6 Australia AAN     1964  11.2  11.2 0.221 -0.00641
##   7 Australia AAN     1965  11.4  11.4 0.221 -0.000314
##   8 Australia AAN     1966  11.7  11.7 0.235  0.0418
##   9 Australia AAN     1967  11.8  11.8 0.206 -0.0869
##  10 Australia AAN     1968  12.0  12.0 0.208  0.00350
```

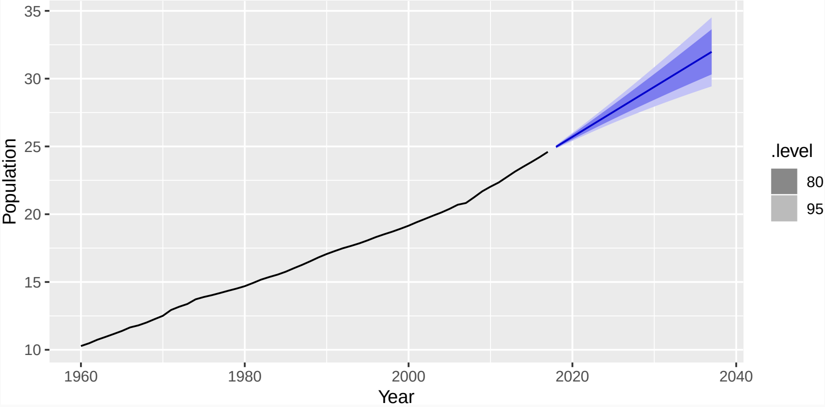
Example: Australian population

```
components(fit) %>% autoplot()
```



Example: Australian population

```
fit %>%
  forecast(h = 20) %>%
  autoplot(aus_economy) +
  ylab("Population") + xlab("Year")
```



ETS(A,Ad,N): Damped trend method

Additive errors

$$
\begin{aligned}
\text{Forecast equation} \quad \hat{y}_{T+h|T} &= \ell_T + (\phi + \cdots + \phi^{h-1})b_T \\
\text{Measurement equation} \quad y_t &= (\ell_{t-1} + \phi b_{t-1}) + \varepsilon_t \\
\text{State equations} \quad \ell_t &= (\ell_{t-1} + \phi b_{t-1}) + \alpha\varepsilon_t \\
b_t &= \phi b_{t-1} + \beta\varepsilon_t
\end{aligned}
$$

ETS(A,Ad,N): Damped trend method

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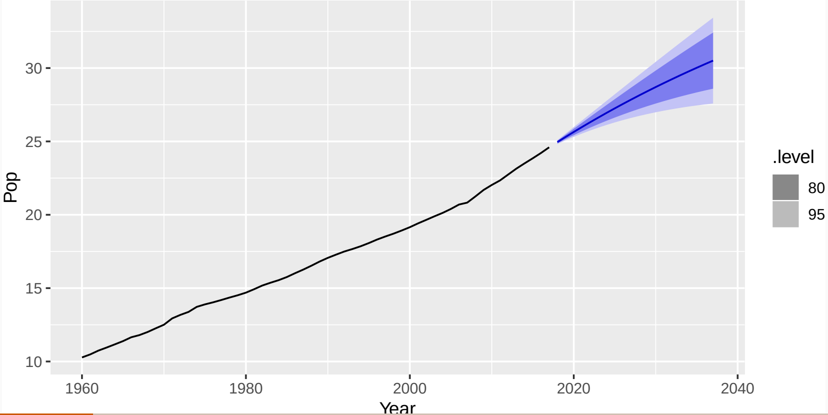
$$
\begin{aligned}
\text{Forecast equation} \quad \hat{y}_{T+h|T} &= \ell_T + (\phi + \cdots + \phi^{h-1})b_T \\
\text{Measurement equation} \quad y_t &= (\ell_{t-1} + \phi b_{t-1}) + \varepsilon_t \\
\text{State equations} \quad \ell_t &= (\ell_{t-1} + \phi b_{t-1}) + \alpha\varepsilon_t \\
b_t &= \phi b_{t-1} + \beta\varepsilon_t
\end{aligned}
$$

- Damping parameter $0 < \phi < 1$.
- If $\phi = 1$, identical to Holt's linear trend.
- As $h \rightarrow \infty$, $\hat{y}_{T+h|T} \rightarrow \ell_T + \phi b_T/(1 - \phi)$.
- Short-run forecasts trended, long-run forecasts constant.

Example: Australian population

```r
aus_economy %>%
  model(holt = ETS(Pop ~ trend("Ad"))) %>%
  forecast(h = 20) %>%
  autoplot(aus_economy)
```



Example: National populations

```
fit <- global_economy %>%
  mutate(Pop = Population/1e6) %>%
  model(ets = ETS(Pop))
fit
```

```
## # A mable: 263 x 2
## # Key:     Country [263]
##    Country             ets
##    <fct>               <model>
##  1 Afghanistan         <ETS(A,A,N)>
##  2 Albania             <ETS(M,A,N)>
##  3 Algeria             <ETS(M,A,N)>
##  4 American Samoa      <ETS(M,A,N)>
##  5 Andorra             <ETS(M,A,N)>
##  6 Angola              <ETS(M,A,N)>
##  7 Antigua and Barbuda <ETS(M,A,N)>
##  8 Arab World          <ETS(M,A,N)>
##  9 Argentina           <ETS(A,A,N)>
## 10 Armenia             <ETS(M,A,N)>
## # ... with 253 more rows
```

Example: National populations

```r
fit %>%
  forecast(h = 5)
```

```
## # A fable: 1,315 x 5 [1Y]
## # Key:     Country, .model [263]
##    Country     .model  Year   Pop .distribution
##    <fct>       <chr>  <dbl> <dbl> <dist>
##  1 Afghanistan ets     2018 36.4  N(36, 0.012)
##  2 Afghanistan ets     2019 37.3  N(37, 0.059)
##  3 Afghanistan ets     2020 38.2  N(38, 0.164)
##  4 Afghanistan ets     2021 39.0  N(39, 0.351)
##  5 Afghanistan ets     2022 39.9  N(40, 0.644)
##  6 Albania     ets     2018  2.87 N(2.9, 0.00012)
##  7 Albania     ets     2019  2.87 N(2.9, 0.00060)
##  8 Albania     ets     2020  2.87 N(2.9, 0.00169)
##  9 Albania     ets     2021  2.86 N(2.9, 0.00362)
```

ETS(A,A,A): Holt-Winters additive method

Forecast equation	$\hat{y}_{t+h\|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$
Observation equation	$y_t = \ell_{t-1} + b_{t-1} + s_{t-m} + \varepsilon_t$
State equations	$\ell_t = \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t$
	$b_t = b_{t-1} + \beta\varepsilon_t$
	$s_t = s_{t-m} + \gamma\varepsilon_t$

- k = integer part of $(h-1)/m$.
- $\sum_i s_i \approx 0$.
- Parameters: $0 \le \alpha \le 1$, $0 \le \beta^* \le 1$, $0 \le \gamma \le 1-\alpha$ and m = period of seasonality (e.g. $m = 4$ for quarterly data).

ETS(M,A,M): Holt-Winters multiplicative method

$$
\begin{aligned}
\text{Forecast equation} \quad \hat{y}_{t+h|t} &= (\ell_t + hb_t)s_{t+h-m(k+1)} \\
\text{Observation equation} \quad y_t &= (\ell_{t-1} + b_{t-1})s_{t-m}(1 + \varepsilon_t) \\
\text{State equations} \quad \ell_t &= (\ell_{t-1} + b_{t-1})(1 + \alpha\varepsilon_t) \\
b_t &= b_{t-1}(1 + \beta\varepsilon_t) \\
s_t &= s_{t-m}(1 + \gamma\varepsilon_t)
\end{aligned}
$$

- k is integer part of $(h-1)/m$.
- $\sum_i s_i \approx m$.
- Parameters: $0 \le \alpha \le 1$, $0 \le \beta^* \le 1$, $0 \le \gamma \le 1 - \alpha$ and m = period of seasonality (e.g. $m = 4$ for quarterly data).

Example: Australian holiday tourism

```
holidays <- tourism %>%
  filter(Purpose == "Holiday")
fit <- holidays %>% model(ets = ETS(Trips))
fit
```

```
## # A mable: 76 x 4
## # Key:     Region, State, Purpose [76]
##    Region                     State              Purpose ets
##    <chr>                      <chr>              <chr>   <model>
##  1 Adelaide                   South Australia    Holiday <ETS(A,N,A)>
##  2 Adelaide Hills             South Australia    Holiday <ETS(A,A,N)>
##  3 Alice Springs              Northern Territory Holiday <ETS(M,N,A)>
##  4 Australia's Coral Coast    Western Australia  Holiday <ETS(M,N,A)>
##  5 Australia's Golden Outback Western Australia  Holiday <ETS(M,N,M)>
##  6 Australia's North West     Western Australia  Holiday <ETS(A,N,A)>
##  7 Australia's South West     Western Australia  Holiday <ETS(M,N,M)>
##  8 Ballarat                   Victoria           Holiday <ETS(M,N,A)>
##  9 Barkly                     Northern Territory Holiday <ETS(A,N,A)>
## 10 Barossa                    South Australia    Holiday <ETS(A,N,N)>
```

Example: Australian holiday tourism

```r
fit %>% filter(Region=="Snowy Mountains") %>% report()
```

```
## Series: Trips
## Model: ETS(M,N,A)
##   Smoothing parameters:
##     alpha = 0.157
##     gamma = 1e-04
##
##   Initial states:
##     l   s1  s2    s3    s4
##   142 -61 131 -42.2 -27.7
##
##   sigma^2:  0.0388
##
##   AIC  AICc  BIC
##   852  854   869
```

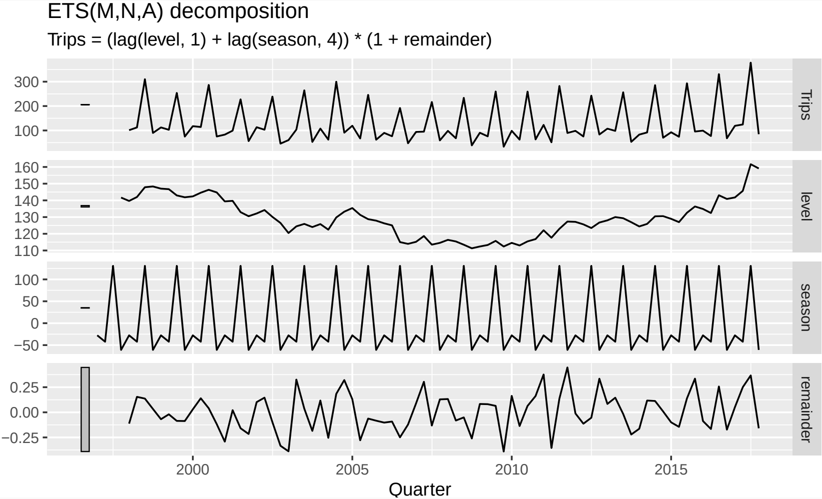
Example: Australian holiday tourism

```r
fit %>% filter(Region=="Snowy Mountains") %>% components(fit)
```

```
## # A dable:                      84 x 9 [1Q]
## # Key:                          Region, State, Purpose, .model [1]
## # ETS(M,N,A) Decomposition: Trips = (lag(level, 1) + lag(season, 4))
## #   * (1 + remainder)
##    Region State Purpose .model    Quarter Trips level season
##    <chr>  <chr> <chr>   <chr>       <qtr> <dbl> <dbl>  <dbl>
##  1 Snowy~ New ~ Holiday ets       1997 Q1   NA     NA   -27.7
##  2 Snowy~ New ~ Holiday ets       1997 Q2   NA     NA   -42.2
##  3 Snowy~ New ~ Holiday ets       1997 Q3   NA     NA   131.
##  4 Snowy~ New ~ Holiday ets       1997 Q4   NA    142.  -61.0
##  5 Snowy~ New ~ Holiday ets       1998 Q1 101.    140.  -27.7
##  6 Snowy~ New ~ Holiday ets       1998 Q2 112.    142.  -42.2
##  7 Snowy~ New ~ Holiday ets       1998 Q3 310.    148.  131.
##  8 Snowy~ New ~ Holiday ets       1998 Q4  89.8   148.  -61.0
##  9 Snowy~ New ~ Holiday ets       1999 Q1 112.    147.  -27.7
## 10 Snowy~ New ~ Holiday ets       1999 Q2 103.    147.  -42.2
## # ... with 74 more rows, and 1 more variable: remainder <dbl>
```

Example: Australian holiday tourism

```r
fit %>% filter(Region=="Snowy Mountains") %>%
  components(fit) %>% autoplot()
```



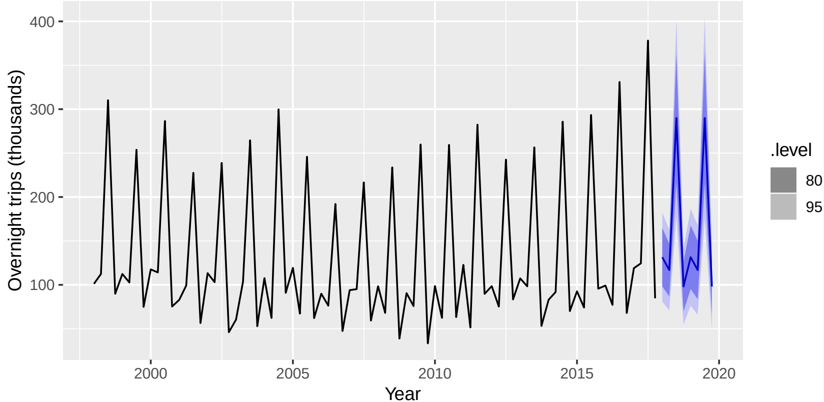
Example: Australian holiday tourism

```r
fit %>% forecast()
```

```
## # A fable: 608 x 7 [1Q]
## # Key:     Region, State, Purpose, .model [76]
##    Region      State       Purpose .model  Quarter Trips .distribution
##    <chr>       <chr>       <chr>   <chr>     <qtr> <dbl> <dist>
##  1 Adelaide    South Aus~ Holiday ets      2018 Q1 210.  N(210, 457)
##  2 Adelaide    South Aus~ Holiday ets      2018 Q2 173.  N(173, 473)
##  3 Adelaide    South Aus~ Holiday ets      2018 Q3 169.  N(169, 489)
##  4 Adelaide    South Aus~ Holiday ets      2018 Q4 186.  N(186, 505)
##  5 Adelaide    South Aus~ Holiday ets      2019 Q1 210.  N(210, 521)
##  6 Adelaide    South Aus~ Holiday ets      2019 Q2 173.  N(173, 537)
##  7 Adelaide    South Aus~ Holiday ets      2019 Q3 169.  N(169, 553)
##  8 Adelaide    South Aus~ Holiday ets      2019 Q4 186.  N(186, 569)
##  9 Adelaide H~ South Aus~ Holiday ets      2018 Q1  19.4 N(19, 36)
## 10 Adelaide H~ South Aus~ Holiday ets      2018 Q2  19.6 N(20, 36)
## # ... with 598 more rows
```

Example: Australian holiday tourism

```
fit %>% forecast() %>%
  filter(Region=="Snowy Mountains") %>%
  autoplot(holidays) +
    xlab("Year") + ylab("Overnight trips (thousands)")
```



Exponential smoothing models

Additive Error

Trend Component		Seasonal Component: N (None)	A (Additive)	M (Multiplicative)
N	(None)	A,N,N	A,N,A	~~A,N,M~~
A	(Additive)	A,A,N	A,A,A	~~A,A,M~~
A_d	(Additive damped)	A,A_d,N	A,A_d,A	~~A,A_d,M~~

Multiplicative Error

Trend Component		Seasonal Component: N (None)	A (Additive)	M (Multiplicative)
N	(None)	M,N,N	M,N,A	M,N,M
A	(Additive)	M,A,N	M,A,A	M,A,M
A_d	(Additive damped)	M,A_d,N	M,A_d,A	M,A_d,M

Estimating ETS models

- Smoothing parameters α, β, γ and ϕ, and the initial states ℓ_0, b_0, $s_0, s_{-1}, \dots, s_{-m+1}$ are estimated by maximising the "likelihood" = the probability of the data arising from the specified model.
- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

Model selection

Akaike's Information Criterion

$$\text{AIC} = -2\log(\text{L}) + 2k$$

where L is the likelihood and k is the number of parameters initial states estimated in the model.

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Corrected AIC

$$\text{AIC}_\text{c} = \text{AIC} + \frac{2(k+1)(k+2)}{T-k}$$

which is the AIC corrected (for small sample bias).

Model selection

Akaike's Information Criterion

$$\text{AIC} = -2\log(\text{L}) + 2k$$

where L is the likelihood and k is the number of parameters initial states estimated in the model.

Corrected AIC

$$\text{AIC}_\text{c} = \text{AIC} + \frac{2(k+1)(k+2)}{T-k}$$

which is the AIC corrected (for small sample bias).

Bayesian Information Criterion

$$\text{BIC} = \text{AIC} + k(\log(T) - 2).$$

AIC and cross-validation

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

Automatic forecasting

From Hyndman et al. (IJF, 2002):

1. Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE.
2. Select best method using AICc.
3. Produce forecasts using best method.
4. Obtain forecast intervals using underlying state space model.

- Method performed very well in M3 competition.
- Used as a benchmark in the M4 competition.

Outline

Lab Session 7

1. Find an ETS model for the Gas data from `aus_production`.
    - Why is multiplicative seasonality necessary here?
    - Experiment with making the trend damped.
2. Use `ETS()` on some of these series: `tourism`, `gafa_stock`, `pelt`.
    - Does it always give good forecasts?
    - Find an example where it does not work well. Can you figure out why?

Outline

ARIMA models

AR: autoregressive (lagged observations as inputs)

I: integrated (differencing to make series stationary)

MA: moving average (lagged errors as inputs)

ARIMA models

AR: autoregressive (lagged observations as inputs)
I: integrated (differencing to make series stationary)
MA: moving average (lagged errors as inputs)

An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

Stationarity

Definition

If $\{y_t\}$ is a stationary time series, then for all s, the distribution of $(y_t, \ldots, y_{t+s})$ does not depend on t.

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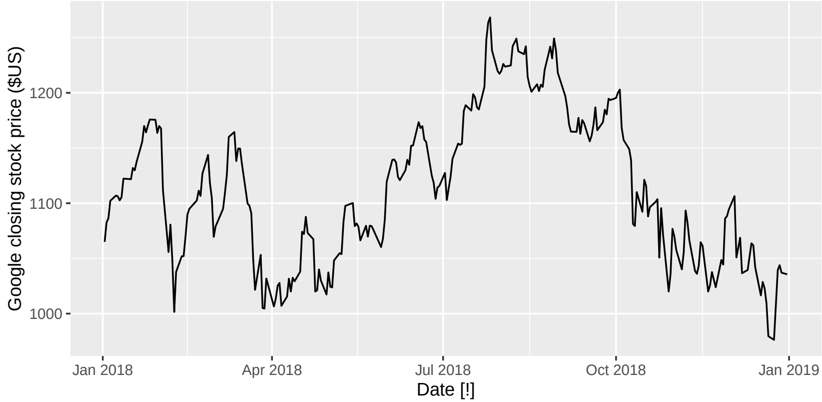
If $\{y_t\}$ is a stationary time series, then for all s, the distribution of $(y_t, \dots, y_{t+s})$ does not depend on t.

A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

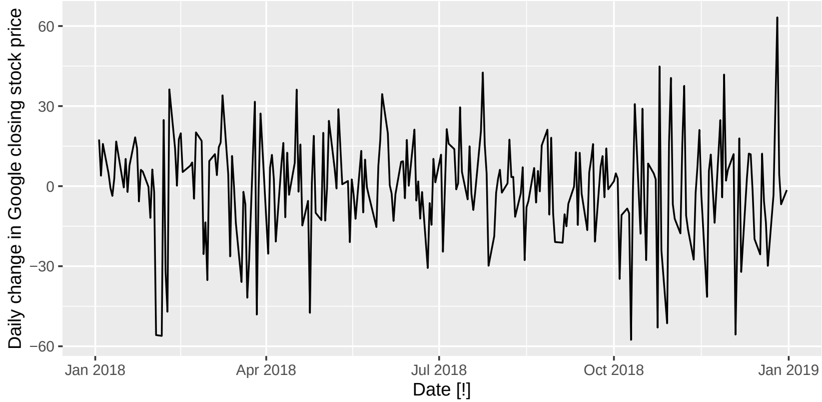
Stationary?

```
gafa_stock %>%
  filter(Symbol == "GOOG", year(Date) == 2018) %>%
  autoplot(Close) +
    ylab("Google closing stock price ($US)")
```



Stationary?

```
gafa_stock %>%
  filter(Symbol == "GOOG", year(Date) == 2018) %>%
  autoplot(difference(Close)) +
    ylab("Daily change in Google closing stock price")
```



Differencing

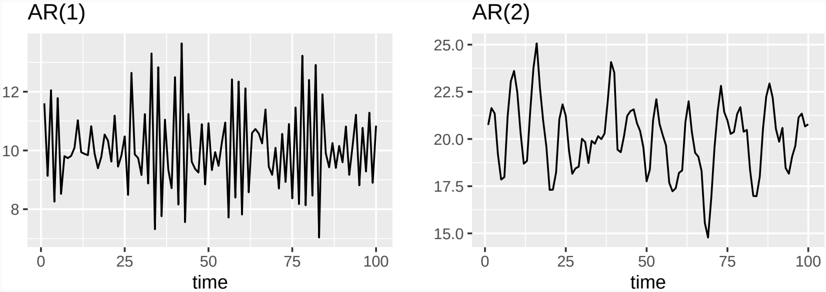
- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series.
- Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time.
- In practice, it is almost never necessary to go beyond second-order differences.

Autoregressive models

Autoregressive (AR) models:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where ε_t is white noise. This is a multiple regression with **lagged values** of y_t as predictors.



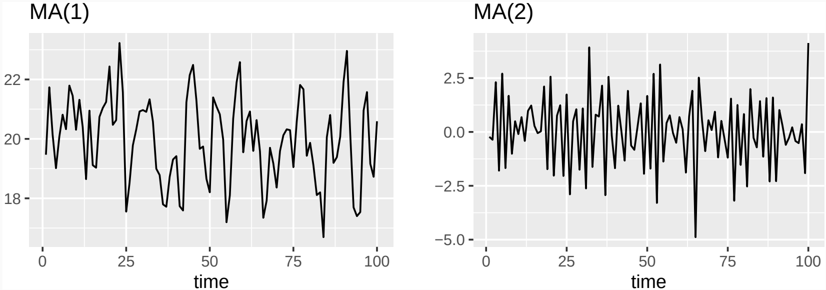
- Cyclic behaviour is possible when $p \geq 2$.

Moving Average (MA) models

Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \theta_q\varepsilon_{t-q},$$

where ε_t is white noise. This is a multiple regression with **lagged *errors*** as predictors. *Don't confuse this with moving average smoothing!*



ARIMA models

Autoregressive Moving Average models:

$$
\begin{aligned}
y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\
+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.
\end{aligned}
$$

ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p}$$
$$+ \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of y_t and lagged errors.**

ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of y_t and lagged errors.**

Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing**.
- d-differenced series follows an ARMA model.
- Need to choose p, d, q and whether or not to include c.

ARIMA models

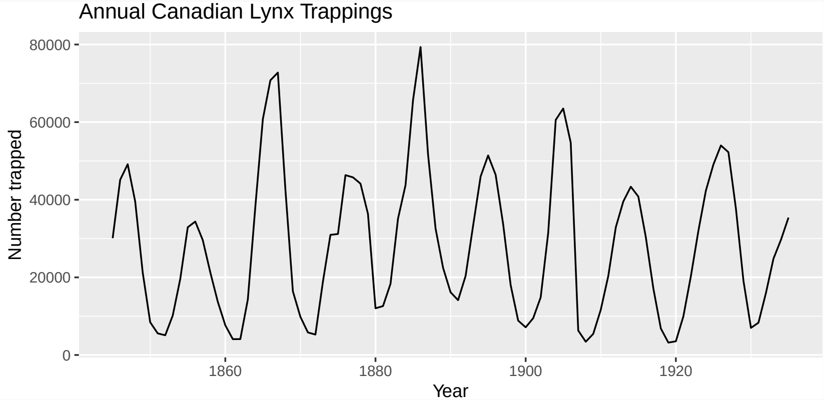
ARIMA(p, d, q) model

AR:	p = order of the autoregressive part
I:	d = degree of first differencing involved
MA:	q = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA(p,0,0)
- MA(q): ARIMA(0,0,q)

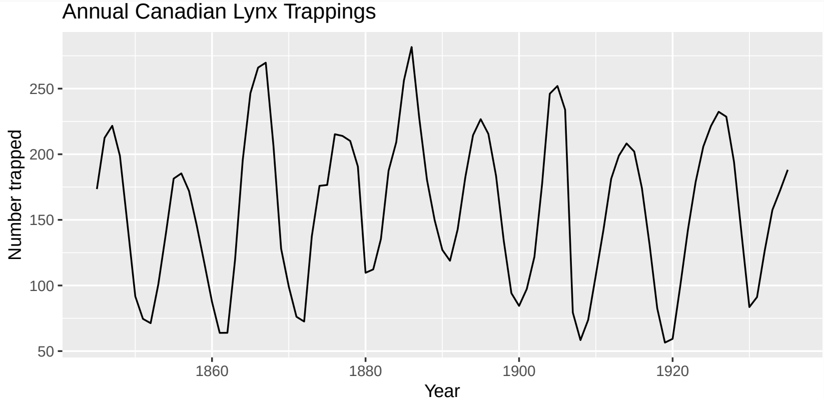
Annual lynx trappings

```
pelt %>% autoplot(Lynx) +
  xlab("Year") + ylab("Number trapped") +
  ggtitle("Annual Canadian Lynx Trappings")
```



Annual lynx trappings

```
pelt %>% autoplot(sqrt(Lynx)) +
  xlab("Year") + ylab("Number trapped") +
  ggtitle("Annual Canadian Lynx Trappings")
```



Annual lynx trappings

```r
pelt %>%
  model(lynx = ARIMA(sqrt(Lynx))) %>%
  report()
```

```
## Series: Lynx
## Model: ARIMA(2,0,1) w/ mean
## Transformation: sqrt(.x)
##
## Coefficients:
##           ar1      ar2     ma1  constant
##        1.5059  -0.8645  -0.331     56.33
## s.e.   0.0583   0.0522   0.108      1.56
##
## sigma^2 estimated as 507.9:  log likelihood=-412
## AIC=834   AICc=835   BIC=847
```

Annual lynx trappings

```r
pelt %>%
  model(lynx = ARIMA(sqrt(Lynx))) %>%
  report()
```

```
## Series: Lynx
## Model: ARIMA(2,0,1) w/ mean
## Transformation: sqrt(
##
## Coefficients:
##          ar1      ar2     ma1  constant
##       1.5059  -0.8645  -0.331     56.33
## s.e.  0.0583   0.0522   0.108      1.56
##
## sigma^2 estimated as 507.9:  log likelihood=-412
## AIC=834   AICc=835   BIC=847
```

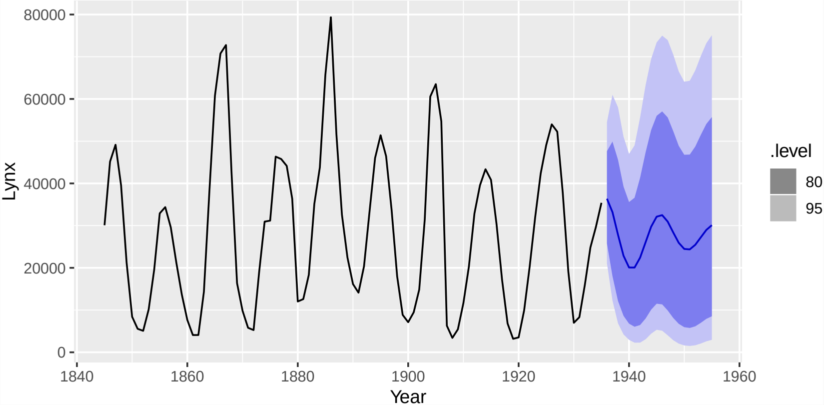
$y_t = \sqrt{(\text{lynx in year } t)}$

$y_t = 56.3 + 1.5 y_{t-1} - 0.9 y_{t-2} - 0.3 \varepsilon_{t-1} + \varepsilon_t$

$\varepsilon_t \sim \text{NID}(0, 508)$

Annual lynx trappings

```r
pelt %>%
  model(lynx = ARIMA(sqrt(Lynx))) %>%
  forecast(h=20) %>%
  autoplot(pelt)
```



Example: National populations

```r
fit <- global_economy %>%
  model(arima = ARIMA(Population))
fit
```

```
## # A mable: 263 x 2
## # Key:     Country [263]
##    Country             arima
##    <fct>               <model>
##  1 Afghanistan         <ARIMA(4,2,1)>
##  2 Albania             <ARIMA(0,2,2)>
##  3 Algeria             <ARIMA(2,2,2)>
##  4 American Samoa      <ARIMA(2,2,2)>
##  5 Andorra             <ARIMA(2,1,2) w/ drift>
##  6 Angola              <ARIMA(4,2,1)>
##  7 Antigua and Barbuda <ARIMA(2,1,2) w/ drift>
##  8 Arab World          <ARIMA(0,2,1)>
```

Example: National populations

```r
fit %>% filter(Country=="Australia") %>% report()
```

```
## Series: Population
## Model: ARIMA(0,2,1)
##
## Coefficients:
##           ma1
##        -0.661
## s.e.    0.107
##
## sigma^2 estimated as 4.063e+09:  log likelihood=-699
## AIC=1401   AICc=1402   BIC=1405
```

Example: National populations

```r
fit %>% filter(Country=="Australia") %>% report()
```

```
## Series: Population
## Model: ARIMA(0,2,1)
##
## Coefficients:
##           ma1
##        -0.661
## s.e.    0.107
##
## sigma^2 estimated as 4.063e+09:  log likelihood=-699
## AIC=1401   AICc=1402   BIC=1405
```

$$y_t = 2y_{t-1} - y_{t-2} - 0.7\varepsilon_{t-1} + \varepsilon_t$$
$$\varepsilon_t \sim \text{NID}(0, 4 \times 10^9)$$

Understanding ARIMA models

- If $c = 0$ and $d = 0$, the long-term forecasts will go to zero.
- If $c = 0$ and $d = 1$, the long-term forecasts will go to a non-zero constant.
- If $c = 0$ and $d = 2$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 0$, the long-term forecasts will go to the mean of the data.
- If $c \neq 0$ and $d = 1$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 2$, the long-term forecasts will follow a quadratic trend.

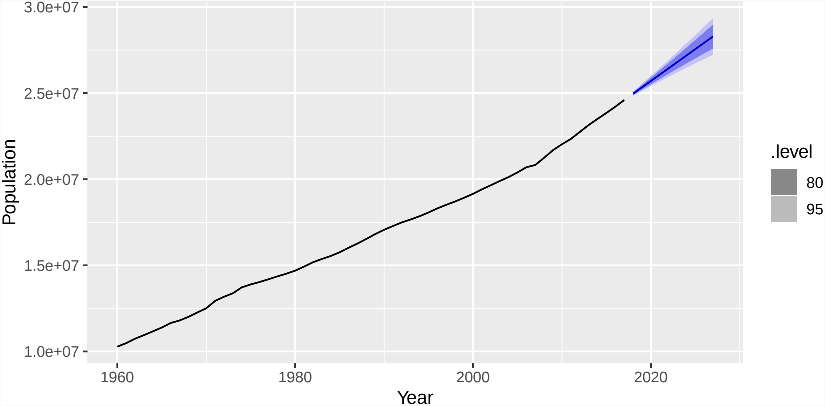
Understanding ARIMA models

Forecast variance and d

- The higher the value of d, the more rapidly the prediction intervals increase in size.
- For $d = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Example: National populations

```
fit %>% forecast(h=10) %>%
  filter(Country=="Australia") %>%
  autoplot(global_economy)
```



How does ARIMA() work?

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d via KPSS test.
- Select p, q and inclusion of c by minimising AICc.
- Use stepwise search to traverse model space.

How does ARIMA() work?

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d via KPSS test.
- Select p, q and inclusion of c by minimising AICc.
- Use stepwise search to traverse model space.

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \frac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

How does ARIMA() work?

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d via KPSS test.
- Select p, q and inclusion of c by minimising AICc.
- Use stepwise search to traverse model space.

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \frac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

Note: Can't compare AICc for different values of d.

How does ARIMA() work?

Step1: Select current model (with smallest AICc) from:

ARIMA($2, d, 2$)
ARIMA($0, d, 0$)
ARIMA($1, d, 0$)
ARIMA($0, d, 1$)

How does ARIMA() work?

Step1: Select current model (with smallest AICc) from:

ARIMA($2, d, 2$)

ARIMA($0, d, 0$)

ARIMA($1, d, 0$)

ARIMA($0, d, 1$)

Step 2: Consider variations of current model:

- vary one of $p, q,$ from current model by ± 1;
- p, q both vary from current model by ± 1;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

How does ARIMA() work?

Step1: Select current model (with smallest AICc) from:

ARIMA$(2, d, 2)$

ARIMA$(0, d, 0)$

ARIMA$(1, d, 0)$

ARIMA$(0, d, 1)$

Step 2: Consider variations of current model:

- vary one of $p, q,$ from current model by ± 1;
- p, q both vary from current model by ± 1;
- Include/exclude c from current model.

Model with lowest AICc becomes current model.

Repeat Step 2 until no lower AICc can be found.

Outline

Lab Session 9

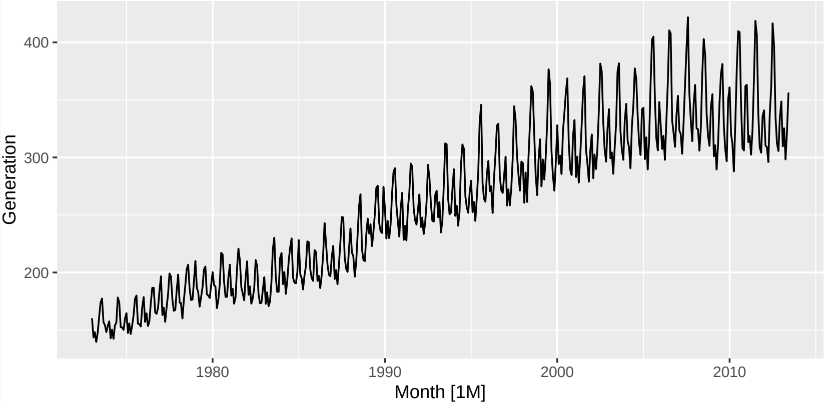
For the United States GDP data (from `global_economy`):

- Fit a suitable ARIMA model for the logged data.
- Produce forecasts of your fitted model. Do the forecasts look reasonable?

Outline

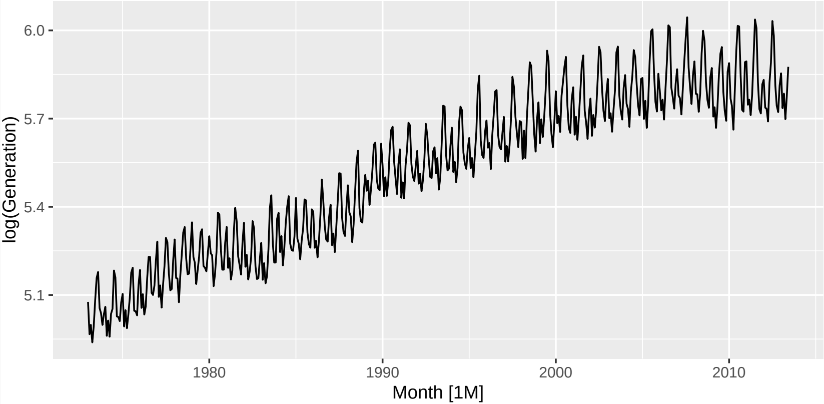
Electricity production

```r
usmelec %>% autoplot(
  Generation
)
```



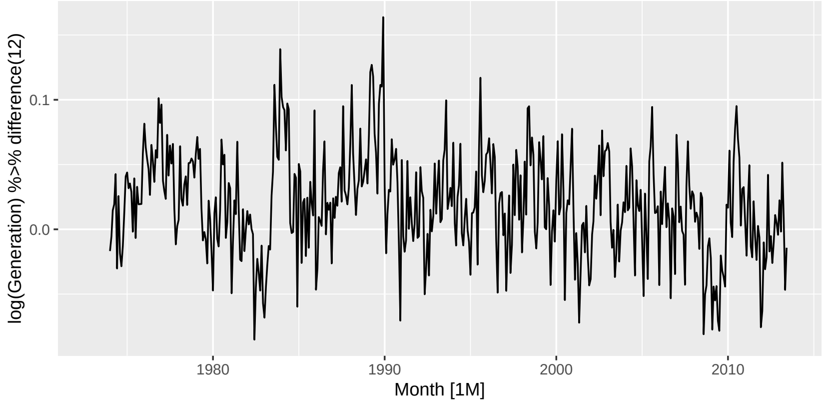
Electricity production

```
usmelec %>% autoplot(
  log(Generation)
)
```



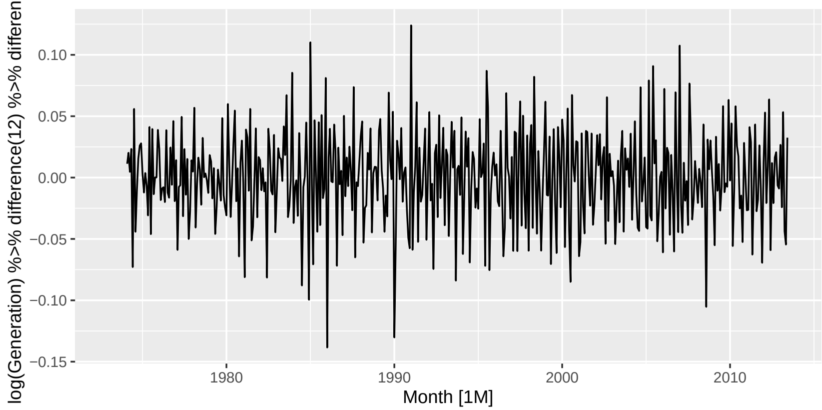
Electricity production

```r
usmelec %>% autoplot(
  log(Generation) %>% difference(12)
)
```



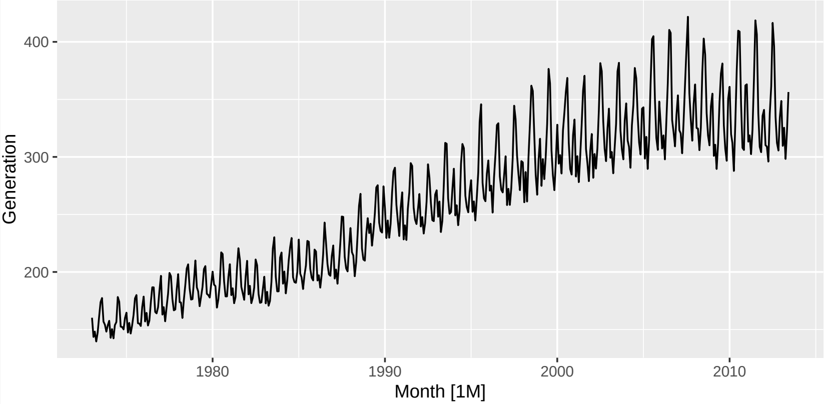
Electricity production

```r
usmelec %>% autoplot(
  log(Generation) %>% difference(12) %>% difference()
)
```



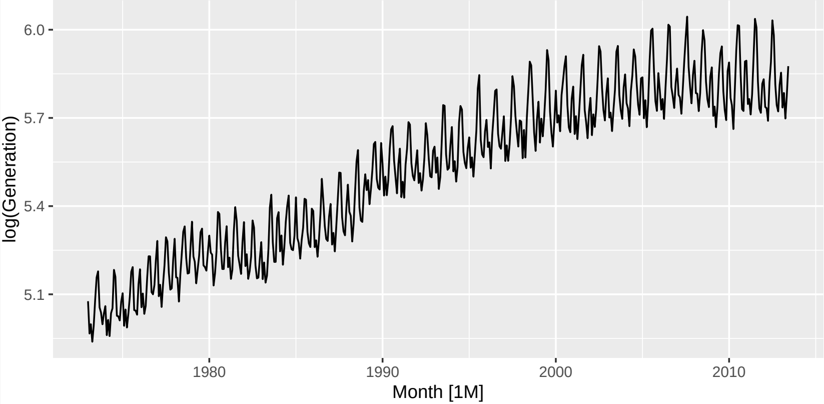
Example: US electricity production

```r
usmelec %>% autoplot(Generation)
```



Example: US electricity production

```
usmelec %>% autoplot(log(Generation))
```



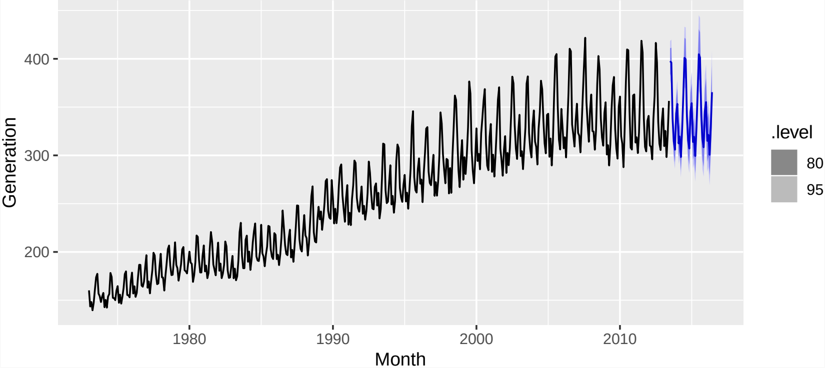
Example: US electricity production

```
usmelec %>%
  model(arima = ARIMA(log(Generation))) %>%
  report()
```

```
## Series: Generation
## Model: ARIMA(1,1,1)(2,1,1)[12]
## Transformation: log(.x)
##
## Coefficients:
##          ar1      ma1    sar1     sar2     sma1
##       0.4116  -0.8483  0.0100  -0.1017  -0.8204
## s.e.  0.0617   0.0348  0.0561   0.0529   0.0357
##
## sigma^2 estimated as 0.0006841:  log likelihood=1047
## AIC=-2082   AICc=-2082   BIC=-2057
```

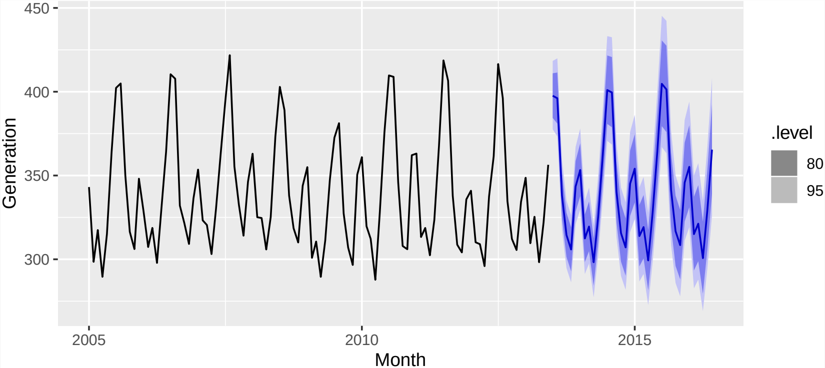
Example: US electricity production

```r
usmelec %>%
  model(arima = ARIMA(log(Generation))) %>%
  forecast(h="3 years") %>%
  autoplot(usmelec)
```



Example: US electricity production

```
usmelec %>%
  model(arima = ARIMA(log(Generation))) %>%
  forecast(h="3 years") %>%
  autoplot(filter_index(usmelec, 2005 ~ .))
```



Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	$\uparrow$	$\uparrow$
	Non-seasonal part of the model	Seasonal part of of the model

where m = number of observations per year.

- d first differences
- D seasonal differences
- p AR lags
- q MA lags
- P seasonal AR lags

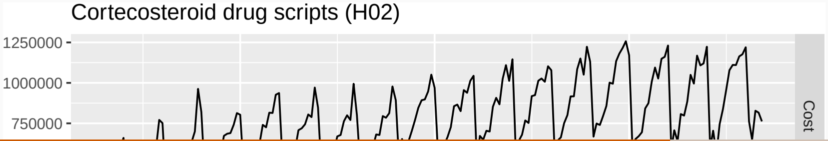
Common ARIMA models

The US Census Bureau uses the following models most often:

ARIMA$(0,1,1)(0,1,1)_m$	with log transformation
ARIMA$(0,1,2)(0,1,1)_m$	with log transformation
ARIMA$(2,1,0)(0,1,1)_m$	with log transformation
ARIMA$(0,2,2)(0,1,1)_m$	with log transformation
ARIMA$(2,1,2)(0,1,1)_m$	with no transformation

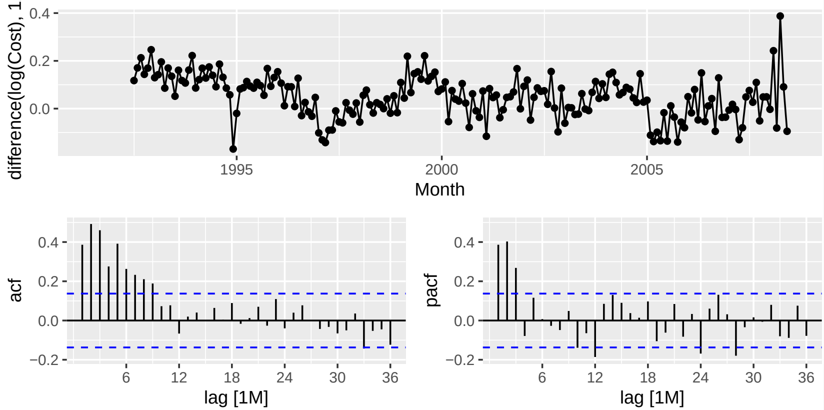
Cortecosteroid drug sales

```r
h02 %>%
  mutate(log(Cost)) %>%
  gather() %>%
  ggplot(aes(x = Month, y = value)) +
  geom_line() +
  facet_grid(key ~ ., scales = "free_y") +
  xlab("Year") + ylab("") +
  ggtitle("Cortecosteroid drug scripts (H02)")
```



Cortecosteroid drug sales

```
h02 %>% gg_tsdisplay(difference(log(Cost),12),
```



Cortecosteroid drug sales

- Choose $D = 1$ and $d = 0$.
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model: ARIMA(3,0,0)(2,1,0)$_{12}$.

Cortecosteroid drug sales

.model	AICc
ARIMA(3,0,1)(0,1,2)[12]	-485
ARIMA(3,0,1)(1,1,1)[12]	-484
ARIMA(3,0,1)(0,1,1)[12]	-484
ARIMA(3,0,1)(2,1,0)[12]	-476
ARIMA(3,0,0)(2,1,0)[12]	-475
ARIMA(3,0,2)(2,1,0)[12]	-475
ARIMA(3,0,1)(1,1,0)[12]	-463

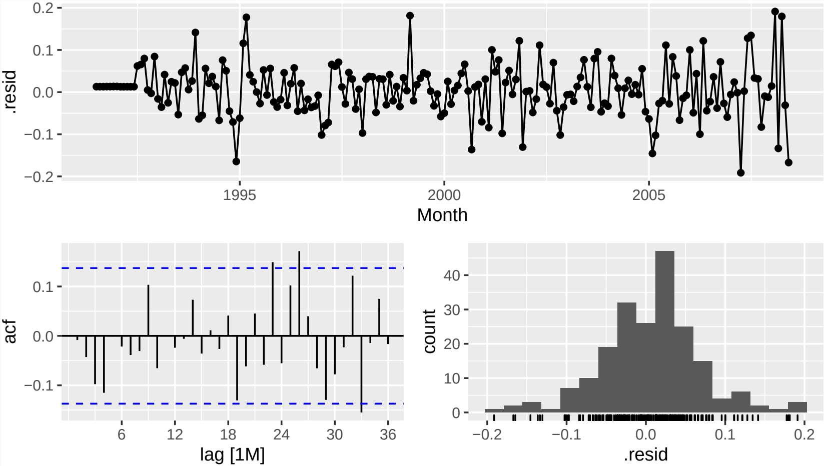
Cortecosteroid drug sales

```r
fit <- h02 %>%
  model(best = ARIMA(log(Cost) ~ 0 + pdq(3,0,1) + PDQ(0,1,2)))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(3,0,1)(0,1,2)[12]
## Transformation: log(.x)
##
## Coefficients:
##          ar1     ar2     ar3    ma1     sma1     sma2
##       -0.160  0.5481  0.5678  0.383  -0.5222  -0.1769
## s.e.   0.164  0.0878  0.0942  0.190   0.0861   0.0872
##
## sigma^2 estimated as 0.004289:  log likelihood=250
## AIC=-486   AICc=-485   BIC=-463
```

Cortecosteroid drug sales

```r
augment(fit) %>%
  gg_tsdisplay(.resid, lag_max=36, plot_type = "hist")
```



Cortecosteroid drug sales

```r
augment(fit) %>%
  features(.resid, ljung_box, lag = 36, dof = 6)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>     <dbl>
## 1 best      50.5    0.0109
```

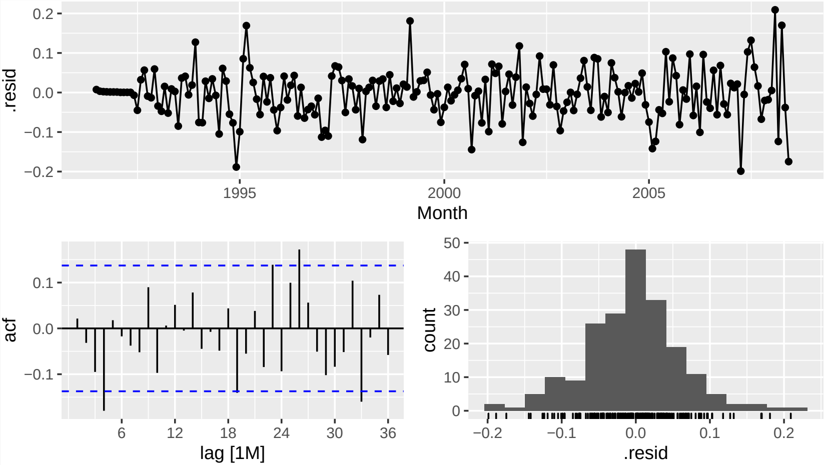
Cortecosteroid drug sales

```r
fit <- h02 %>% model(auto = ARIMA(log(Cost)))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(2,1,0)(0,1,1)[12]
## Transformation: log(.x)
##
## Coefficients:
##           ar1      ar2     sma1
##       -0.8491  -0.4207  -0.6401
## s.e.   0.0712   0.0714   0.0694
##
## sigma^2 estimated as 0.004399:  log likelihood=245
## AIC=-483   AICc=-483   BIC=-470
```

Cortecosteroid drug sales

```
augment(fit) %>%
  gg_tsdisplay(.resid, lag_max = 36, plot_type = "hist")
```



Cortecosteroid drug sales

```r
augment(fit) %>%
  features(.resid, ljung_box, lag = 36, dof = 5)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>     <dbl>
## 1 auto      57.5   0.00260
```

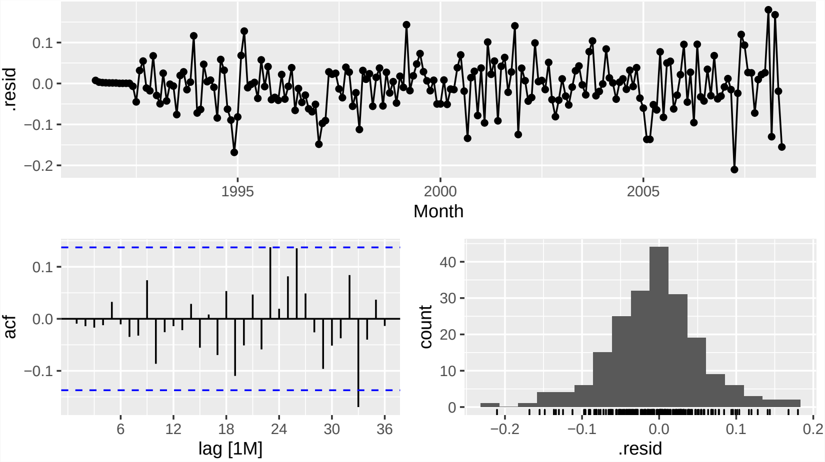
Cortecosteroid drug sales

```
fit <- h02 %>%
  model(best = ARIMA(log(Cost), stepwise = FALSE,
                    approximation = FALSE,
                    order_constraint = p + q + P + Q <= 9))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(4,1,1)(2,1,2)[12]
## Transformation: log(.x)
##
## Coefficients:
##            ar1    ar2    ar3     ar4     ma1   sar1    sar2    sma1
##        -0.0426  0.210  0.202  -0.227  -0.742  0.621  -0.383  -
1.202
## s.e.    0.2167  0.181  0.114   0.081   0.207  0.242   0.118   0.249
##          sma2
##         0.496
## s.e.    0.214
##
## sigma^2 estimated as 0.004061:  log likelihood=254
## AIC=-489   AICc=-487   BIC=-456
```

Cortecosteroid drug sales

```r
augment(fit) %>%
  gg_tsdisplay(.resid, lag_max = 36, plot_type = "hist")
```



Cortecosteroid drug sales

```
augment(fit) %>%
  features(.resid, ljung_box, lag = 36, dof = 9)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>     <dbl>
## 1 best      35.1     0.136
```

Cortecosteroid drug sales

Training data: July 1991 to June 2006

Test data: July 2006–June 2008

```
fit <- h02 %>%
  filter_index(~ "2006 Jun") %>%
  model(
    ARIMA(log(Cost) ~ pdq(3, 0, 0) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ pdq(3, 0, 1) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ pdq(3, 0, 2) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ pdq(3, 0, 1) + PDQ(1, 1, 0))
    # ... #
  )

fit %>%
  forecast(h = "2 years") %>%
  accuracy(h02 %>% filter_index("2006 Jul" ~ .))
```

Cortecosteroid drug sales

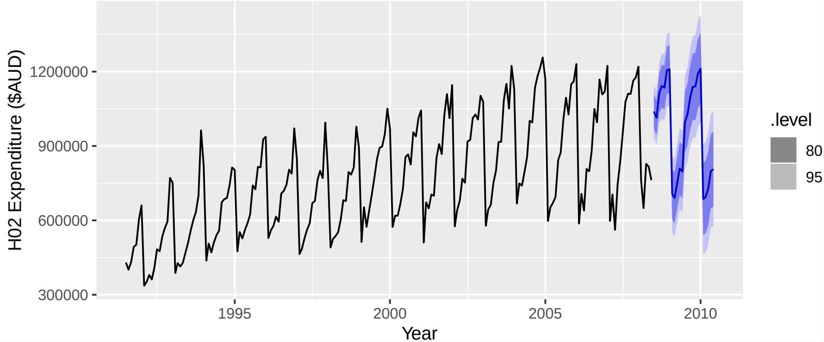
```r
models <- list(
  c(3,0,0,2,1,0),
  c(3,0,1,2,1,0),
  c(3,0,2,2,1,0),
  c(3,0,1,1,1,0),
  c(3,0,1,0,1,1),
  c(3,0,1,0,1,2),
  c(3,0,1,1,1,1),
  c(3,0,3,0,1,1),
  c(3,0,2,0,1,1),
  c(2,1,3,0,1,1),
  c(2,1,4,0,1,1),
  c(2,1,5,0,1,1),
  c(4,1,1,2,1,2))
```

Cortecosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- Use the best model available, even if it does not pass all tests.

Cortecosteroid drug sales

```
fit <- h02 %>%
  model(ARIMA(Cost ~ 0 + pdq(3,0,1) + PDQ(0,1,2)))
fit %>% forecast %>% autoplot(h02) +
  ylab("H02 Expenditure ($AUD)") + xlab("Year")
```



Outline

ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$
		$\theta_2 = 1 - \alpha$
ETS(A,A,N)	ARIMA(1,1,2)	$\phi_1 = \phi$
		$\theta_1 = \alpha + \phi\beta - 1 - \phi$
		$\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA$(0,0,m)(0,1,0)_m$	
ETS(A,A,A)	ARIMA$(0,1,m+1)(0,1,0)_m$	
ETS(A,A,A)	ARIMA$(1,0,m+1)(0,1,0)_m$	

Outline

Lab Session 10

For the `fma::condmilk` series:

- Do the data need transforming? If so, find a suitable transformation.
- Are the data stationary? If not, find an appropriate differencing which yields stationary data.
- Identify a couple of ARIMA models that might be useful in describing the time series.
- Which of your models is the best according to their AIC values?
- Estimate the parameters of your best model and do diagnostic testing on the residuals. Do the residuals resemble white noise? If not, try to find another ARIMA model which fits better.
- Forecast the next 24 months of data using your preferred model.