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High dimensional time series analysis



5. Forecast reconciliation

robjhyndman.com/hdtsa

Outline

- 1 Hierarchical and grouped time series
- 2 Forecast reconciliation
- 3 Example: Australian tourism
- 4 Lab Session 10

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Australian Pharmaceutical Benefits Scheme



PBS sales

PBS

```
## # A tsibble: 65,219 x 9 [1M]
## # Key:      Concession, Type, ATC1, ATC2 [336]
##           Month Concession Type ATC1 ATC1_desc ATC2 ATC2_desc Scripts
##           <mt> <chr>      <chr> <chr> <chr>      <chr> <chr>      <dbl>
## 1 1991 Jul Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 18228
## 2 1991 Aug Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 15327
## 3 1991 Sep Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 14775
## 4 1991 Oct Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 15380
## 5 1991 Nov Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 14371
## 6 1991 Dec Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 15028
## 7 1992 Jan Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 11040
## 8 1992 Feb Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 15165
## 9 1992 Mar Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 16898
## 10 1992 Apr Concessio~ Co-p~ A Alimenta~ A01 STOMATOL~ 18141
## # ... with 65,209 more rows, and 1 more variable: Cost <dbl>
```

ATC drug classification

- A Alimentary tract and metabolism
- B Blood and blood forming organs
- C Cardiovascular system
- D Dermatologicals
- G Genito-urinary system and sex hormones
- H Systemic hormonal preparations, excluding sex hormones and insulins
- J Anti-infectives for systemic use
- L Antineoplastic and immunomodulating agents
- M Musculo-skeletal system
- N Nervous system
- P Antiparasitic products, insecticides and repellents
- R Respiratory system
- S Sensory organs
- V Various

ATC drug classification

ATC1: 14 classes

A

Alimentary tract and metabolism

ATC2: 84 classes

A10

Drugs used in diabetes

A10B

Blood glucose lowering drugs

A10BA

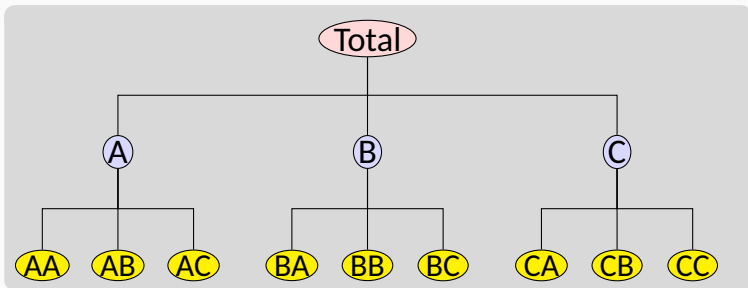
Biguanides

A10BA02

Metformin

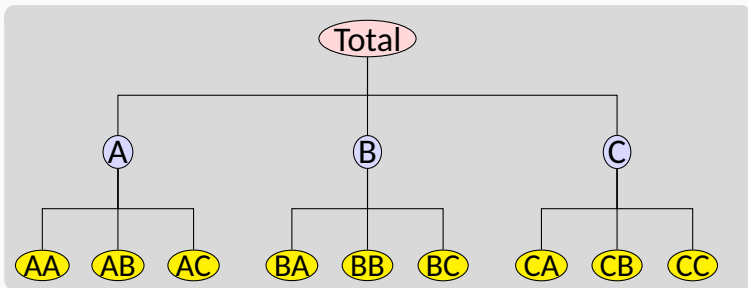
Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



Hierarchical time series

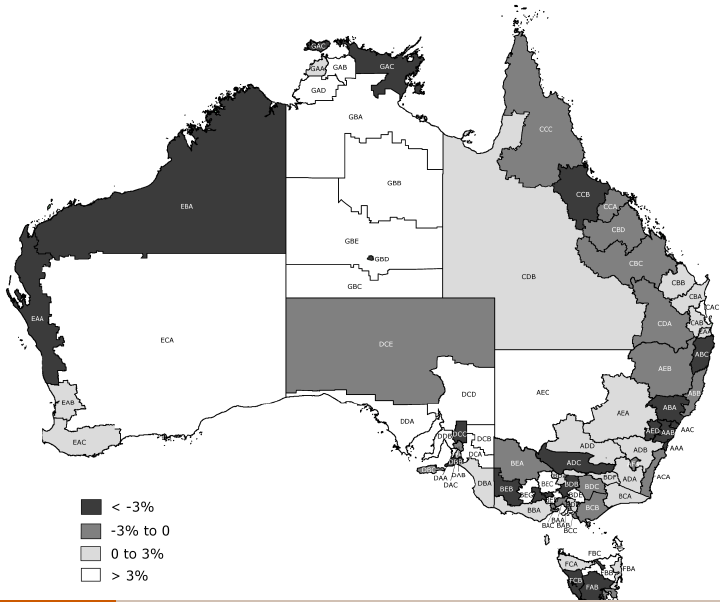
A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



Examples

- PBS sales by ATC groups
- Tourism demand by states, zones, regions

Australian tourism



Australian tourism

tourism

```
## # A tsibble: 24,320 x 5 [1Q]
```

```
## # Key:           Region, State, Purpose [304]
```

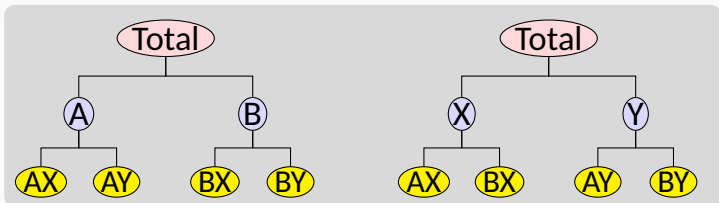
##	Quarter	Region	State	Purpose	Trips	
##	<qtr>	<chr>	<chr>	<chr>	<dbl>	
##	1	1998 Q1	Adelaide	South Australia	Business	135.
##	2	1998 Q2	Adelaide	South Australia	Business	110.
##	3	1998 Q3	Adelaide	South Australia	Business	166.
##	4	1998 Q4	Adelaide	South Australia	Business	127.
##	5	1999 Q1	Adelaide	South Australia	Business	137.
##	6	1999 Q2	Adelaide	South Australia	Business	200.
##	7	1999 Q3	Adelaide	South Australia	Business	169.
##	8	1999 Q4	Adelaide	South Australia	Business	134.
##	9	2000 Q1	Adelaide	South Australia	Business	154.
##	10	2000 Q2	Adelaide	South Australia	Business	169.

Australian tourism

- Quarterly data on visitor night from 1998:Q1 – 2013:Q4
- From: *National Visitor Survey*, based on annual interviews of 120,000 Australians aged 15+, collected by Tourism Research Australia.
- Split by 7 states, 27 zones and 76 regions (a geographical hierarchy)
- Also split by purpose of travel
 - ▶ Holiday
 - ▶ Visiting friends and relatives (VFR)
 - ▶ Business
 - ▶ Other
- 304 bottom-level series

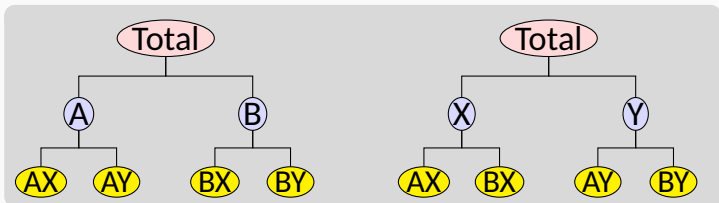
Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



Grouped time series

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Examples

- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

Creating aggregates

```
PBS %>%  
  aggregate_key(ATC1/ATC2, Scripts = sum(Scripts)) %>%  
  filter(Month == yearmonth("1991 Jul")) %>% print(n=18)
```

```
## # A tsibble: 98 x 4 [?]  
## # Key:      ATC1, ATC2 [98]  
##   ATC1      ATC2      Month Scripts  
##   <chr>    <chr>    <mtm>   <dbl>  
## 1 <aggregated> <aggregated> 1991 Jul 8090395  
## 2 A          <aggregated> 1991 Jul  799025  
## 3 B          <aggregated> 1991 Jul  109227  
## 4 C          <aggregated> 1991 Jul 1794995  
## 5 D          <aggregated> 1991 Jul  299779  
## 6 G          <aggregated> 1991 Jul  300931  
## 7 H          <aggregated> 1991 Jul  112114  
## 8 J          <aggregated> 1991 Jul 1151681  
## 9 L          <aggregated> 1991 Jul   24580  
## 10 M         <aggregated> 1991 Jul  562956  
## 11 N         <aggregated> 1991 Jul 1546023  
## 12 P         <aggregated> 1991 Jul   47661  
## 13 R         <aggregated> 1991 Jul  859273  
## 14 S         <aggregated> 1991 Jul  391639  
## 15 V         <aggregated> 1991 Jul   38705  
## 16 Z         <aggregated> 1991 Jul   51806  
## 17 A         A01          1991 Jul   22615  
## 18 A         A02          1991 Jul  299251  
## # ... with 80 more rows
```

Creating aggregates

```
tourism %>%  
  aggregate_key(Purpose * (State / Region), Trips = sum(Trips)) %>%  
  filter(Quarter == yearquarter("1998 Q1")) %>% print(n=15)
```

```
## # A tibble: 425 x 5 [?]  
## # Key:      Purpose, State, Region [425]  
##   Purpose      State      Region      Quarter  Trips  
##   <chr>        <chr>        <chr>        <qtr>    <dbl>  
## 1 <aggregated> <aggregated> <aggregated> 1998 Q1 23182.  
## 2 Business    <aggregated> <aggregated> 1998 Q1  3599.  
## 3 Holiday     <aggregated> <aggregated> 1998 Q1 11806.  
## 4 Other       <aggregated> <aggregated> 1998 Q1   680.  
## 5 Visiting    <aggregated> <aggregated> 1998 Q1  7098.  
## 6 <aggregated> ACT          ~ <aggregated> 1998 Q1   551.  
## 7 <aggregated> New South Wale~ <aggregated> 1998 Q1  8040.  
## 8 <aggregated> Northern Terri~ <aggregated> 1998 Q1   181.  
## 9 <aggregated> Queensland    ~ <aggregated> 1998 Q1  4041.  
## 10 <aggregated> South Australi~ <aggregated> 1998 Q1  1735.  
## 11 <aggregated> Tasmania      ~ <aggregated> 1998 Q1   982.  
## 12 <aggregated> Victoria      ~ <aggregated> 1998 Q1  6010.  
## 13 <aggregated> Western Austra~ <aggregated> 1998 Q1  1641.  
## 14 <aggregated> ACT          ~ Canberra    ~ 1998 Q1   551.  
## 15 <aggregated> New South Wale~ Blue Mounta~ 1998 Q1   196.  
## # ... with 410 more rows
```


Creating aggregates

- Similar to `summarise()` but using the key structure
- A grouped structure is specified using `grp1 * grp2`
- A nested structure is specified via `parent / child`.
- Groups and nesting can be mixed:

```
(country/region/city) * (brand/product)
```

- All possible aggregates are produced.
- These are useful when forecasting at different levels of aggregation.

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The problem

- 1 How to forecast time series at all nodes such that the forecasts add up in the same way as the original data?
- 2 Can we exploit relationships between the series to improve the forecasts?

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- 1 How to forecast time series at all nodes such that the forecasts add up in the same way as the original data?
- 2 Can we exploit relationships between the series to improve the forecasts?

The solution

- 1 Forecast all series at all levels of aggregation using an automatic forecasting algorithm.
(e.g., ETS, ARIMA, ...)
- 2 Reconcile the resulting forecasts so they add up correctly using least squares optimization (i.e., find closest reconciled forecasts to the original forecasts).
- 3 This is available using `reconcile()`.

Forecast reconciliation

```
tourism %>%  
  aggregate_key(Purpose*(State/Region), Trips=sum(Trips)) %>%  
  model(ets = ETS(Trips)) %>%  
  reconcile(ets_adjusted = min_trace(ets)) %>%  
  forecast(h = 2)
```

```
## # A tibble: 1,700 x 7 [1Q]
```

```
## # Key:      Purpose, State, Region, .model [850]
```

```
##   Purpose      State      Region      .model      Quarter Trips  
##   <chr>        <chr>        <chr>        <chr>        <qtr> <dbl>
```

```
## 1 Business    ACT          ~ Canberra ~ ets      2018 Q1 144.
```

```
## 2 Business    ACT          ~ Canberra ~ ets      2018 Q2 203.
```

```
## 3 Business    ACT          ~ <aggregat~ ets      2018 Q1 144.
```

```
## 4 Business    ACT          ~ <aggregat~ ets      2018 Q2 203.
```

```
## 5 Business    New South~ Blue Moun~ ets      2018 Q1 19.7
```

```
## 6 Business    New South~ Blue Moun~ ets      2018 Q2 19.7
```

```
## 7 Business    New South~ Capital C~ ets      2018 Q1 36.1
```

Hierarchical and grouped time series

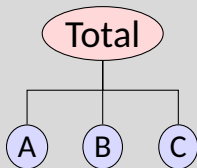
Every collection of time series with aggregation constraints can be written as

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

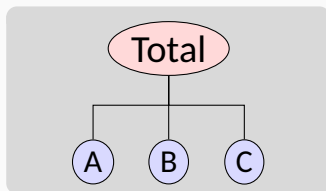
where

- \mathbf{y}_t is a vector of all series at time t
- \mathbf{b}_t is a vector of the most disaggregated series at time t
- \mathbf{S} is a “summing matrix” containing the aggregation constraints.

Hierarchical time series



Hierarchical time series

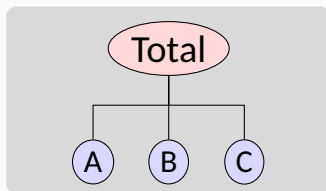


y_t : observed aggregate of all series at time t .

$y_{X,t}$: observation on series X at time t .

b_t : vector of all series at bottom level in time t .

Hierarchical time series



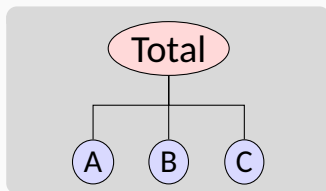
y_t : observed aggregate of all series at time t .

$y_{X,t}$: observation on series X at time t .

b_t : vector of all series at bottom level in time t .

$$y_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

Hierarchical time series



y_t : observed aggregate of all series at time t .

$y_{X,t}$: observation on series X at time t .

b_t : vector of all series at bottom level in time t .

$$y_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_S \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{b_t}$$

$$y_t = S b_t$$

Forecasting notation

Let $\hat{\mathbf{y}}_n(h)$ be vector of initial h -step forecasts, made at time n , stacked in same order as \mathbf{y}_t .

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Reconciled forecasts must be of the form:

$$\tilde{\mathbf{y}}_n(h) = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_n(h)$$

for some matrix \mathbf{G} .

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for some matrix \mathbf{G} .

- \mathbf{G} extracts and combines base forecasts $\hat{\mathbf{y}}_n(h)$ to get bottom-level forecasts.
- \mathbf{S} adds them up

Optimal combination forecasts

Main result

The best (minimum sum of variances) unbiased forecasts are obtained when $\mathbf{G} = (\mathbf{S}'\Sigma_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\Sigma_h^{-1}$, where Σ_h is the h -step base forecast error covariance matrix.

Optimal combination forecasts

Main result

The best (minimum sum of variances) unbiased forecasts are obtained when $\mathbf{G} = (\mathbf{S}'\Sigma_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\Sigma_h^{-1}$, where Σ_h is the h -step base forecast error covariance matrix.

$$\tilde{\mathbf{y}}_n(h) = \mathbf{S}(\mathbf{S}'\Sigma_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\Sigma_h^{-1}\hat{\mathbf{y}}_n(h)$$

Problem: Σ_h hard to estimate, especially for $h > 1$.

Solutions:

- Ignore Σ_h (OLS) [`min_trace(method='ols')`]
- Assume $\Sigma_h = k_h \Sigma_1$ is diagonal (WLS) [`min_trace(method='wls')`]
- Assume $\Sigma_h = k_h \Sigma_1$ and estimate it (GLS) [`min_trace(method='shrink')` (the default)]

Features

- Covariates can be included in initial forecasts.
- Adjustments can be made to initial forecasts at any level.
- Very simple and flexible method. Can work with *any* hierarchical or grouped time series.
- Conceptually easy to implement: regression of base forecasts on structure matrix.

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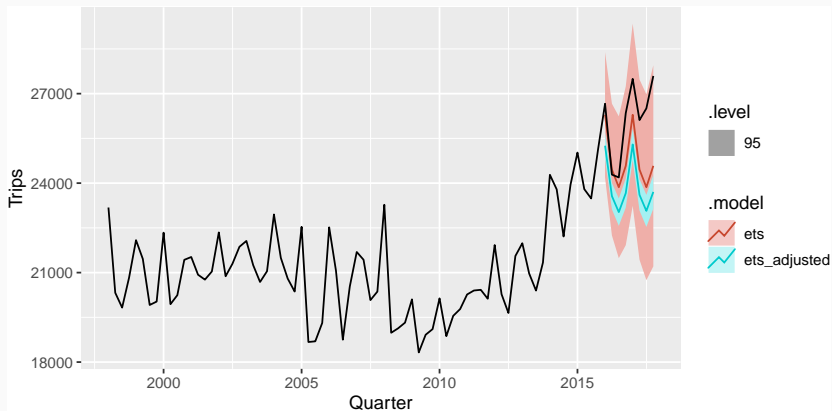
Example: Australian tourism

```
tourism_agg <- tourism %>%  
  aggregate_key(Purpose * (State / Region),  
               Trips = sum(Trips))  
fc <- tourism_agg %>%  
  filter_index(. ~ yearquarter("2015 Q4")) %>%  
  model(ets = ETS(Trips)) %>%  
  reconcile(ets_adjusted = min_trace(ets)) %>%  
  forecast(h = "2 years")
```

Example: Australian tourism

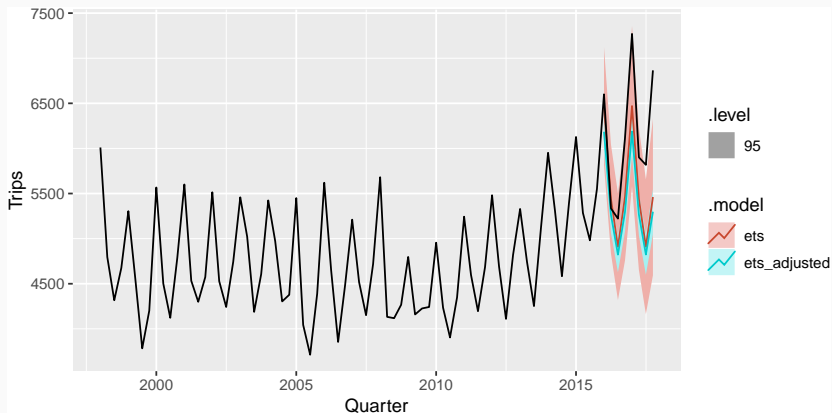
```
fc %>%
```

```
  filter(is_aggregated(Purpose) & is_aggregated(State)) %>%  
  autoplot(tourism_agg, level=95)
```



Example: Australian tourism

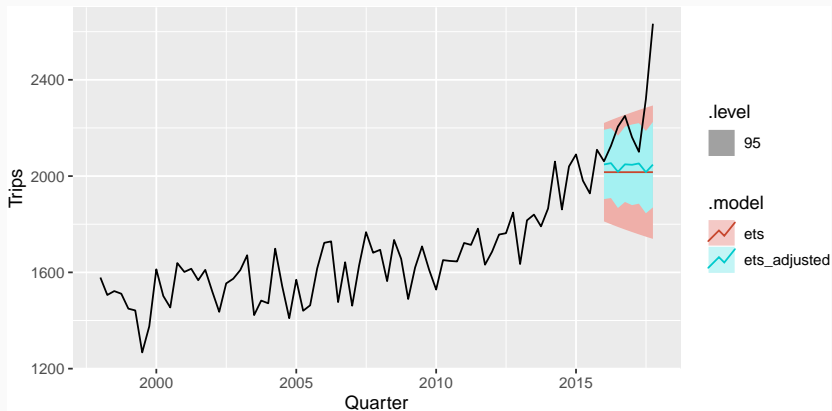
```
fc %>%  
  filter(is_aggregated(Purpose) & State=="Victoria" &  
         is_aggregated(Region)) %>%  
  autoplot(tourism_agg, level=95)
```



Example: Australian tourism

```
fc %>%
```

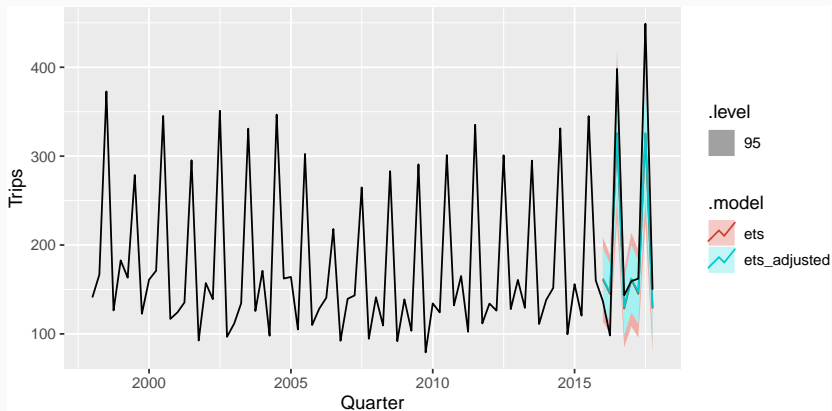
```
  filter(is_aggregated(Purpose) & Region=="Melbourne") %>%  
  autoplot(tourism_agg, level=95)
```



Example: Australian tourism

```
fc %>%
```

```
  filter(is_aggregated(Purpose) & Region=="Snowy Mountains") %>%  
  autoplot(tourism_agg, level=95)
```

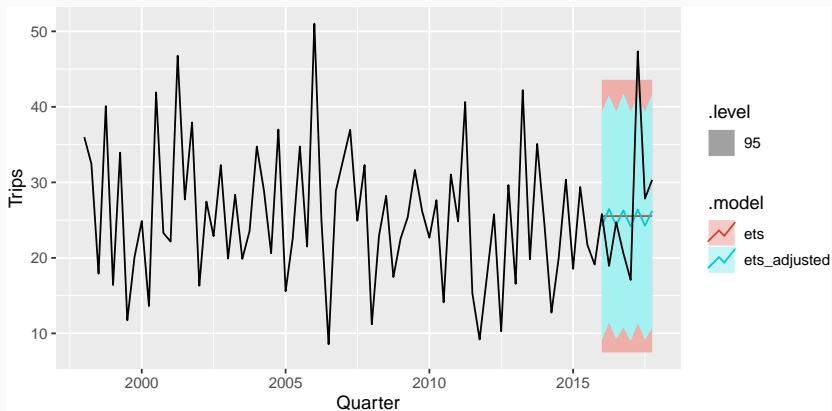


Example: Australian tourism

```
fc %>%
```

```
  filter(Purpose=="Holiday" & Region=="Barossa") %>%
```

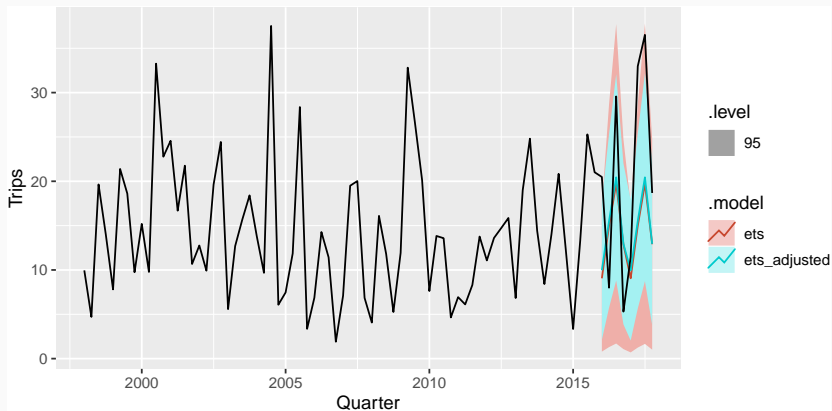
```
  autoplot(tourism_agg, level=95)
```



Example: Australian tourism

```
fc %>%
```

```
  filter(is_aggregated(Purpose) & Region=="MacDonnell") %>%  
  autoplot(tourism_agg, level=95)
```



Example: Australian tourism

```
fc <- tourism_agg %>%  
  filter_index(. ~ yearquarter("2015 Q4")) %>%  
  model(  
    ets = ETS(Trips),  
    arima = ARIMA(Trips)  
  ) %>%  
  mutate(  
    comb = (ets+arima)/2  
  ) %>%  
  reconcile(  
    ets_adj = min_trace(ets),  
    arima_adj = min_trace(arima),  
    comb_adj = min_trace(comb)  
  ) %>%  
  forecast(h = "2 years")
```

Forecast evaluation

```
fc %>% accuracy(tourism_agg)
```

```
## # A tibble: 2,550 x 12
```

##		.model	Purpose	State	Region	.type	ME	RMSE
##		<chr>	<chr>	<chr>	<chr>	<chr>	<dbl>	<dbl>
##	1	arma	Business	ACT	~ Canberra ~	Test	35.9	45.7
##	2	arma	Business	ACT	~ <aggregat~	Test	35.9	45.7
##	3	arma	Business	New South~	Blue Moun~	Test	1.93	10.6
##	4	arma	Business	New South~	Capital C~	Test	8.08	15.6
##	5	arma	Business	New South~	Central C~	Test	10.0	14.5
##	6	arma	Business	New South~	Central N~	Test	17.7	31.9
##	7	arma	Business	New South~	Hunter ~	Test	35.3	43.9
##	8	arma	Business	New South~	New Engla~	Test	23.1	31.8
##	9	arma	Business	New South~	North Coa~	Test	24.8	40.1
##	10	arma	Business	New South~	Outback N~	Test	6.87	11.0

```
## # ... with 2,540 more rows, and 5 more variables:
```

```
## #   MAE <dbl>, MPE <dbl>, MAPE <dbl>, MASE <dbl>,
```

Forecast evaluation

```
fc %>% accuracy(tourism_agg) %>%  
  group_by(.model) %>%  
  summarise(MASE = mean(MASE)) %>%  
  arrange(MASE)
```

```
## # A tibble: 6 x 2  
##   .model      MASE  
##   <chr>      <dbl>  
## 1 ets_adj    1.02  
## 2 comb_adj   1.02  
## 3 ets        1.04  
## 4 comb        1.04  
## 5 arima_adj  1.07  
## 6 arima      1.09
```

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Lab Session 10

- Prepare aggregations of the PBS data by Concession, Type, and ATC1.
- Use forecast reconciliation with the PBS data, using ETS, ARIMA and SNAIVE models, applied to all but the last 3 years of data.
- Which type of model works best?
- Does the reconciliation improve the forecast accuracy?
- Why doesn't the reconciliation make any difference to the SNAIVE forecasts?