EECS 50 - Discrete Time Signals and Systems - Spring 2016

Homework I Solution

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Instructions: Please attach this title page as a cover-sheet with your homework submission. Use extra sheets if needed to present your solutions. The table below is for grading purposes only.

Problem No.	Points
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Problem 7	
Total Score	

1. A continuous time sinusoidal signal $x(t)$ is sampled at the rate $f_s = 10$ samples/second to produce the discrete-time signal $x[n] = 10\cos\left(\frac{22\pi n}{5} + \frac{\pi}{7}\right)$.

   (a) (3 pts) What are the possible signals $x(t)$ that could produce this $x[n]$.
   Sol: Sampling
   $$x(t) = 10\cos(2\pi f t + \pi/7)$$
   at $f_s = 10$ samples/sec produces
   $$x[n] = 10\cos(2\pi(f/10)n + \pi/7) = 10\cos(22\pi n/5 + \pi/7)$$
   $\Rightarrow f = 22$ Hz. But all values $f' = f + kf_s = 22 + 10k$ for integer k, produce the same $x[n]$ as well. So the possible signals are
   $$x(t) = 10\cos(2\pi(22 + 10k)t + \pi/7)$$

   (b) (3 pts) Is $x[n]$ a periodic signal? If so, then find its fundamental period.
   Sol: Yes, $x[n]$ is periodic because $\Omega = 22\pi/5$ is a rational multiple of π. For this signal
   $$F = \Omega/2\pi = 11/5$$
   is a rational number. The fundamental period
   $$N_o = k/F = 5k/11$$
   where k is the smallest positive integer value which makes $5k/11$ an integer. In this case, that would be $k = 11$ and therefore $N_o = 5$ samples.

   (c) (3 pts) Find $F \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ such that $10\cos\left(2\pi F n + \frac{\pi}{7}\right) = 10\cos\left(\frac{22\pi n}{5} + \frac{\pi}{7}\right)$.
   Sol: Adding any integer value to F does not change the discrete time sinusoid. So the equivalent values of F are $11/5 + k$. Choosing $k = -2$, we have $11/5 - 2 = 1/5$ which lies in the interval $(-1/2, 1/2)$.

   (d) (3 pts) Assuming the Nyquist criterion was satisfied find $x(t)$.
   Sol: From the possible values $f' = f + kf_s = 22 + 10k$ the only value, $\bar{f}$, that satisfies the Nyquist criterion, i.e., lies in the range $(-f_s/2, f_s/2) = (-5, 5)$ Hz is when $k = -2$, i.e., $\bar{f} = 22 - 20 = 2$. So the $x(t)$ that satisfies the Nyquist criterion is
   $$x(t) = 10\cos(4\pi t + \pi/7)$$

   (e) (2 pt) What is the Nyquist sampling rate for the $x(t)$ found in part (d)?
   Sol: The $x(t)$ found in part (d) is a sinusoid of frequency 2 Hz. So the Nyquist rate is 4 Hz, i.e., any sampling frequency higher than 4 samples/second will avoid aliasing.

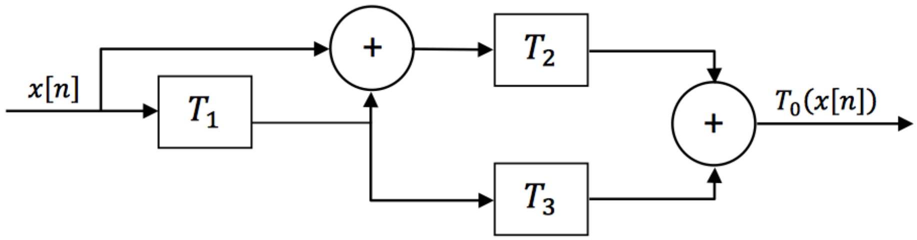
   (f) (3 pts) If the same $x(t)$ was sampled at the rate of $f_s' = 1.5$ samples/sec, to produce the discrete time signal $x'[n]$, find $x'[n]$.
   Sol: Sampling $x(t) = 10\cos(4\pi t + \pi/7)$ at $f_s' = 1.5$ samples/sec, produces the discrete time signal
   $$x'[n] = 10\cos(2\pi(2/1.5)n + \pi/7) = 10\cos(8\pi n/3 + \pi/7)$$

   (g) (3 pts) Will aliasing occur in recovering $x(t)$ from $x'[n]$? If so, then find the aliased frequency.
   Sol: Yes, because f_s' does not satisfy the Nyquist criterion, there will be aliasing. All frequencies $f' = f + kf_s' = 2 + 1.5k$ produce equivalent discrete time signals, and the reconstructed signal will choose the value that lies in the interval $(-f_s'/2, f_s'/2) = (-0.75, 0.75)$. The only value of $2 + 1.5k$ that lies in this interval is obtained when $k = -1$, which gives us $2 - 1.5 = 0.5$ Hz. This is the frequency of the reconstructed signal, i.e., the aliased frequency.

2. (a) (5 pts) Draw the block diagram for the system described by

$$T_0(x[n]) \quad = \quad T_2\left(x[n] + T_1(x[n])\right) + T_3\left(T_1(x[n])\right)$$

Sol:



(b) (10 pts) With the systems T_1, T_2, T_3 defined as follows,

$$\begin{aligned} T_1\left(x[n]\right) &= x[-n-2] \\ T_2\left(x[n]\right) &= x[n+1] \\ T_3\left(x[n]\right) &= x[2n] \end{aligned}$$

give a direct definition of the system $T_o(x[n])$.
Sol:

$$\begin{aligned} \text{Define } a[n] &= T_1(x[n]) = x[-n-2] && (1) \\ T_3(T_1(x[n])) &= T_3(a[n]) && (2) \\ &= a[2n] && (3) \\ &= x[-2n-2] && (4) \\ \text{Define } b[n] &= x[n] + T_1(x[n]) && (5) \\ &= x[n] + x[-n-2] && (6) \\ T_2(b[n]) &= b[n+1] && (7) \\ &= x[n+1] + x[-(n+1)-2] && (8) \\ &= x[n+1] + x[-n-3] && (9) \\ T_o(x[n]) &= T_2(b[n]) + T_3(T_1(x[n])) && (10) \\ &= x[n+1] + x[-n-3] + x[-2n-2] && (11) \end{aligned}$$

(c) (5 points) For the input signal

$$x[n] \quad = \quad 2\delta[n+1] - \delta[n] + 3\delta[n-1] + \delta[n-2]$$

Plot $T_0(x[n])$ over the range of n that includes all non-zero values.
Sol: For the given $x[n]$,

$$\begin{aligned} x[n+1] = x[n]|_{n\to n+1} &= 2\delta[n+2] - \delta[n+1] + 3\delta[n] + \delta[n-1] && (12) \\ x[-n-3] = x[n]|_{n\to -n-3} &= 2\delta[-n-2] - \delta[-n-3] + 3\delta[-n-4] + \delta[-n-5] && (13) \\ &= 2\delta[n+2] - \delta[n+3] + 3\delta[n+4] + \delta[n+5] && (14) \\ x[-2n-2] = x[n]|_{n\to -2n-2} &= 2\delta[-2n-1] - \delta[-2n-2] + 3\delta[-2n-3] + \delta[-2n-4] && (15) \\ &= 2\delta[2n+1] - \delta[2n+2] + 3\delta[2n+3] + \delta[2n+4] && (16) \\ &= -\delta[n+1] + \delta[n+2] && (17) \end{aligned}$$

Note that $\delta[n'] = 0$ whenever $n' \neq 0$. But $n' = 2n + 1$ is always non-zero for integer values of n. Therefore $\delta[n'] = \delta[2n + 1]$ is always zero. Similarly, $\delta[2n + 3]$ is also always zero. Because $\delta[n]$ is an even function, $\delta[n] = \delta[-n]$, $\delta[-n - 2] = \delta[n + 2]$. Finally, $\delta[2n + 4] = \delta[n + 2]$. Adding up the resulting expressions we have.

$$T_o(x[n]) \quad = \quad 5\delta[n + 2] + 3\delta[n] - 2\delta[n + 1] + \delta[n - 1] - \delta[n + 3] + 3\delta[n + 4] + \delta[n + 5] \quad (18)$$

which is easily plotted as it takes the value $+1$ at $n = -5$, $+3$ at $n = -4$, -1 at $n = -3$, $+5$ at $n = -2$, -2 at $n = -1$, $+3$ at $n = 0$, and $+1$ at $n = 1$. It is zero everywhere else.

3. For each of the discrete-time systems defined below, determine if it is (i) (1 pt) memoryless, (ii) (1 pt) causal, (iii) (2 pts) invertible, (iv) (2 pts) BIBO stable, (v) (2 pts) linear, and (vi) (2 pts) time-invariant. Provide a proof in each case.

$$\begin{aligned}
\text{(a) (10 pts) } y[n] &= 2^{x[1-n]} \\
\text{(b) (10 pts) } y[n] &= (n-1)x[n-1]
\end{aligned}$$

Sol:

(a) $y[n] = 2^{x[1-n]}$ is

(i) not memoryless because, e.g., at $n = 0$, $y[0] = 2^{x[1]}$ depends on $x[1]$, which is a future input.

(ii) not causal for the same reason.

(iii) Not invertible if $x[n]$ is complex in general, because complex exponentials are periodic signals. If $x[n]$ is only allowed to be a real signal, then this system would be invertible, because the exponential function is invertible, and one can uniquely identify the input from the output as follows: $x[1-n] = \log_2(y[n])$, so $x[n] = \log_2(y[1-n])$.

(iv) BIBO stable because for arbitrary bounded $x[n]$ such that $|x[n]| \leq M$ for all n, where M is a finite positive constant, the output value $|y[n]| \leq M'$ where $M' = 2^M$ is also a positive finite constant. Thus bounded inputs produce bounded outputs, and the system must be BIBO stable.

(v) not linear, because

$$\begin{aligned}
a_1T(x_1[n]) + a_2T(x_2[n]) &= a_1 2^{x_1[1-n]} + a_2 2^{x_2[1-n]} && (19) \\
T(a_1x_1[n] + a_2x_2[n]) &= 2^{a_1x_1[1-n]+a_2x_2[1-n]} && (20) \\
\Rightarrow T(a_1x_1[n] + a_2x_2[n]) &\neq a_1T(x_1[n]) + a_2T(x_2[n]) && (21)
\end{aligned}$$

(vi) not time-invariant because

$$\begin{aligned}
y[n-n_o] &= 2^{x[1-(n-n_o)]} && (22) \\
&= 2^{x[1-n+n_o]} && (23) \\
T(x[n-n_o]) &= 2^{x[1-n-n_o]} && (24) \\
\Rightarrow y[n-n_o] &\neq T(x[n-n_o]) && (25)
\end{aligned}$$

(b) $y[n] = (n-1)x[n-1]$ is

(i) not memoryless because e.g., at time $n = 0$, the output $y[0] = -x[-1]$ depends on $x[-1]$ which is a past input value.

(ii) causal because $y[n]$ always depends only on $x[n-1]$ which is a past input value.

(iii) not invertible because, e.g., $x_1[n] = \delta[n]$ and $x_2[n] = 2\delta[n]$ are two distinct input signals that produce outputs $y_1[n] = (n-1)\delta[n-1]$ and $y_2[n] = 2(n-1)\delta[n-1]$. However, $y_1[n] = y_2[n] = 0$ for all n, because whenever $n \neq 1$ we have $\delta[n-1] = 0$ and when $n = 1$ we have $(n-1) = 0$. Since there exist distinct input signals that produce the same output signal, the system is not invertible.

(iv) not BIBO stable, because there exists a bounded input signal, e.g., $x[n] = 1$ for all n, which produces the unbounded output signal $y[n] = (n-1)$ for all n, which will exceed any constant value for sufficiently large n.

(v) linear, because

$$\begin{aligned}
a_1T(x_1[n]) + a_2T(x_2[n]) &= a_1(n-1)x_1[n-1] + a_2(n-1)x_2[n-1] && (26) \\
T(a_1x_1[n] + a_2x_2[n]) &= (n-1)(a_1x_1[n-1] + a_2x_2[n-2]) && (27) \\
\Rightarrow T(a_1x_1[n] + a_2x_2[n]) &= a_1T(x_1[n]) + a_2T(x_2[n]) && (28)
\end{aligned}$$

(vi) not time-invariant because

$$\begin{aligned}
y[n-n_o] &= (n-n_o-1)x[n-n_o-1] && (29) \\
T(x[n-n_o]) &= (n-1)x[n-1-n_o] && (30) \\
\Rightarrow y[n-n_o] &\neq T(x[n-n_o]) && (31)
\end{aligned}$$

4. For a linear system $y[n] = T(x[n])$, we are given the following input-output pairs.

$$x_1[n] = \delta[n-1] + \delta[n+1] \xrightarrow{T} y_1[n] = \delta[n+1] \tag{32}$$
$$x_2[n] = \delta[n] + \delta[n+1] \xrightarrow{T} y_2[n] = \delta[n-1] \tag{33}$$
$$x_3[n] = \delta[n] + \delta[n-1] \xrightarrow{T} y_3[n] = \delta[n] \tag{34}$$

Find the outputs of this linear system for the following inputs.

(a) $x_4[n] = \delta[n] \xrightarrow{T} y_4[n] =?$

(b) $x_5[n] = \delta[n-1] \xrightarrow{T} y_5[n] =?$

(c) $x_6[n] = \delta[n+1] \xrightarrow{T} y_6[n] =?$

Based on your answers, determine if it is possible that the system is also time-invariant.

Sol:

(a) Let us try to express the new input signal $x_4[n]$ as a linear combination of the old input signals $x_1[n], x_2[n], x_3[n]$. Suppose we write

$$\begin{aligned}
x_4[n] &= a_1x_1[n] + a_2x_2[n] + a_3x_3[n] && (35)\\
\Rightarrow \delta[n] &= a_1\delta[n-1] + a_1\delta[n+1] + a_2\delta[n] + a_2\delta[n+1] + a_3\delta[n] + a_3\delta[n-1] && (36)\\
\Rightarrow \delta[n] &= (a_1+a_3)\delta[n-1] + (a_2+a_3)\delta[n] + (a_1+a_2)\delta[n+1] && (37)\\
\Rightarrow & \quad (a_1+a_3) = 0, (a_2+a_3) = 1, (a_1+a_2) = 0 && (38)\\
\Rightarrow & \quad a_1 = -0.5, a_2 = 0.5, a_3 = 0.5 && (39)\\
\Rightarrow x_4[n] &= -0.5x_1[n] + 0.5x_2[n] + 0.5x_3[n] && (40)\\
\Rightarrow y_4[n] &= -0.5y_1[n] + 0.5y_2[n] + 0.5y_3[n] && (41)\\
\Rightarrow y_4[n] &= -0.5\delta[n+1] + 0.5\delta[n-1] + 0.5\delta[n] && (42)
\end{aligned}$$

(b) Let us try to express the new input signal $x_5[n]$ as a linear combination of the old input signals $x_1[n], x_2[n], x_3[n]$. Suppose we write

$$\begin{aligned}
x_5[n] &= a_1x_1[n] + a_2x_2[n] + a_3x_3[n] && (43)\\
\Rightarrow \delta[n-1] &= a_1\delta[n-1] + a_1\delta[n+1] + a_2\delta[n] + a_2\delta[n+1] + a_3\delta[n] + a_3\delta[n-1] && (44)\\
\Rightarrow \delta[n-1] &= (a_1+a_3)\delta[n-1] + (a_2+a_3)\delta[n] + (a_1+a_2)\delta[n+1] && (45)\\
\Rightarrow & \quad (a_1+a_3) = 1, (a_2+a_3) = 0, (a_1+a_2) = 0 && (46)\\
\Rightarrow & \quad a_1 = 0.5, a_2 = -0.5, a_3 = 0.5 && (47)\\
\Rightarrow x_5[n] &= 0.5x_1[n] - 0.5x_2[n] + 0.5x_3[n] && (48)\\
\Rightarrow y_5[n] &= 0.5y_1[n] - 0.5y_2[n] + 0.5y_3[n] && (49)\\
\Rightarrow y_5[n] &= 0.5\delta[n+1] - 0.5\delta[n-1] + 0.5\delta[n] && (50)
\end{aligned}$$

(c) Let us try to express the new input signal $x_6[n]$ as a linear combination of the old input signals

$x_1[n], x_2[n], x_3[n]$. Suppose we write

$$\begin{aligned}
x_6[n] &= a_1x_1[n] + a_2x_2[n] + a_3x_3[n] && (51)\\
\Rightarrow \delta[n+1] &= a_1\delta[n-1] + a_1\delta[n+1] + a_2\delta[n] + a_2\delta[n+1] + a_3\delta[n] + a_3\delta[n-1] && (52)\\
\Rightarrow \delta[n+1] &= (a_1+a_3)\delta[n-1] + (a_2+a_3)\delta[n] + (a_1+a_2)\delta[n+1] && (53)\\
\Rightarrow & \quad (a_1+a_3) = 0, (a_2+a_3) = 0, (a_1+a_2) = 1 && (54)\\
\Rightarrow & \quad a_1 = 0.5, a_2 = 0.5, a_3 = -0.5 && (55)\\
\Rightarrow x_5[n] &= 0.5x_1[n] + 0.5x_2[n] - 0.5x_3[n] && (56)\\
\Rightarrow y_5[n] &= 0.5y_1[n] + 0.5y_2[n] - 0.5y_3[n] && (57)\\
\Rightarrow y_5[n] &= 0.5\delta[n+1] + 0.5\delta[n-1] - 0.5\delta[n] && (58)
\end{aligned}$$

Based on our answers, the system cannot be time-invariant. This is because $x_5[n] = x_4[n-1]$ but $y_5[n] \neq y_4[n-1]$, i.e., shifting the input to the right by 1 did not shift the output to the right by 1.

5. The system $T(x[n])$ is known to be both linear and time-invariant. We are given the information that

$$x_1[n] = \delta[n-1] \xrightarrow{T} y_1[n] = \delta[n] + 2\delta[n-1]$$

i.e., the input $x_1[n] = \delta[n-1]$ produces the output $y_1[n] = \delta[n] + 2\delta[n-1]$. Find the output for each of the following input signals.

(a) (5 pts) $x_2[n] = \delta[n] \xrightarrow{T} \quad y_2[n] = ?$

(b) (5 pts) $x_3[n] = 2\delta[n-1] - \delta[n] + 3\delta[n+1] \xrightarrow{T} \quad y_3[n] = ?$

(c) (5 pts) $x_4[n] = \begin{cases} 1, & \text{if } 0 \le n \le 2 \\ 0, & \text{otherwise.} \end{cases} \xrightarrow{T} \quad y_4[n] = ?$

(d) (5 pts) $x_5[n] = \delta[2-n] \xrightarrow{T} \quad y_5[n] = ?$

Sol:

(a) Since the system is time-invariant,

$$\begin{aligned} x_2[n] &= x_1[n+1] \\ \Rightarrow y_2[n] &= y_1[n+1] = \delta[n+1] + 2\delta[n] \end{aligned}$$

(b) Using linearity and time-invariance

$$\begin{aligned} x_3[n] &= 2\delta[n-1] - \delta[n] + 3\delta[n+1] \\ &= 2x_2[n-1] - x_2[n] + 3x_2[n+1] \\ \Rightarrow y_3[n] &= 2y_2[n-1] - y_2[n] + 3y_2[n+1] \\ &= 2\delta[n] + 4\delta[n-1] - \delta[n+1] - 2\delta[n] + 3\delta[n+2] + 6\delta[n+1] \\ &= 4\delta[n-1] + 5\delta[n+1] + 3\delta[n+2] \end{aligned}$$

(c) Using linearity and time-invariance

$$\begin{aligned} x_4[n] &= \delta[n] + \delta[n-1] + \delta[n-2] \\ \Rightarrow y_4[n] &= y_2[n] + y_2[n-1] + y_2[n-2] \\ &= \delta[n+1] + 2\delta[n] + \delta[n] + 2\delta[n-1] + \delta[n-1] + 2\delta[n-2] \\ &= \delta[n+1] + 3\delta[n] + 3\delta[n-1] + 2\delta[n-2] \end{aligned}$$

(d) Using time-invariance

$$\begin{aligned} x_5[n] &= \delta[2-n] \\ &= \delta[n-2] \\ &= x_2[n-2] \\ \Rightarrow y_5[n] &= y_2[n-2] \\ &= \delta[n-1] + 2\delta[n-2] \end{aligned}$$

6. (10 pts) A linear time-invariant (LTI) system, $T(x[n])$, is known to produce the output $y_1[n]$ in response to the input $x_1[n]$ as defined below.

$$\begin{aligned} x_1[n] &= u[n] \\ y_1[n] &= nu[n] \end{aligned}$$

(a) (5 pts) Find the impulse response, $h[n]$, of T.

Sol: The response to the unit step is called the step response $s[n]$. Here we are given that $s[n] = nu[n]$. Since the impulse $\delta[n] = u[n] - u[n-1]$, the impulse response $h[n] = s[n] - s[n-1]$. Therefore, $h[n] = nu[n] - (n-1)u[n-1]$. Since $h[0] = 0$, it is possible to further simplify the answer as $h[n] = nu[n-1] - (n-1)u[n-1] = u[n-1]$.

(b) (5 pts) Find the output, $y_2[n]$, of T for the input $x_2[n] = 2\delta[n] + \delta[n-1]$.

Sol: Since this is an LTI system, $y_2[n] = 2h[n] + h[n-1] = 2u[n-1] + u[n-2]$. There are many equivalent ways to express the same answer, e.g., $y_2[n] = 2\delta[n-1] + 3u[n-2]$, $y_2[n] = 3u[n-1] - \delta[n-1]$, etc.

7. (20 pts) Consider the LTI system $y[n] = T(x[n])$ defined as follows.

$$y[n] = \sum_{p=-\infty}^{n} \left(\sum_{q=-\infty}^{p} x[q-1] \right)$$

(a) (5 pts) Find the impulse response $h[n]$ of this LTI system.

Sol: To find the impulse response, replace $x[n] = \delta[n]$, and $y[n] = h[n]$, so that

$$\begin{aligned}
h[n] &= \sum_{p=-\infty}^{n} \left(\sum_{q=-\infty}^{p} \delta[q-1] \right) \\
&= \sum_{p=-\infty}^{n} (\delta[p-1] + \delta[p-2] + \delta[p-3] + \cdots) \\
&= \sum_{p=-\infty}^{n} u[p-1] \\
&= u[n-1] + u[n-2] + u[n-3] + \cdots \\
&= nu[n-1] \\
&= nu[n]
\end{aligned}$$

Note that $nu[n] = nu[n-1]$ because they are both equal to 0 if $n <= 0$ and equal to n if $n \geq 1$.

(b) (5 pts) Is the system stable?

Sol: $h[n] = nu[n]$ is not absolutely summable. So the LTI system is not stable.

(c) (5 pts) Is the system causal?

Sol: Yes, it is causal because $h[n] = 0$ for $n < 0$.

(d) (5 pts) Find the output of the LTI system if the input signal is $x[n] = u[-n-1] - u[-n-4]$.

Sol:

$$\begin{aligned}
x[n] &= u[-n-1] - u[-n-4] && (59) \\
&= \delta[n+1] + \delta[n+2] + \delta[n+3] && (60)
\end{aligned}$$

Output of an LTI system is the convolution of its input and impulse response. So

$$\begin{aligned}
y[n] &= x[n] * h[n] && (61) \\
&= (\delta[n+1] + \delta[n+2] + \delta[n+3]) * nu[n] && (62) \\
&= (n+1)u[n+1] + (n+2)u[n+2] + (n+3)u[n+3] && (63)
\end{aligned}$$

The answer can be equivalently expressed in many other forms, such as

$$\begin{aligned}
y[n] &= (n+1)u[n+1] + (2n+5)u[n+2] && (64) \\
&= \delta[n+2] + (3n+6)u[n+1] && (65)
\end{aligned}$$

In all cases,

$$y[n] = \begin{cases} 0, & n \leq -3 \\ 1, & n = -2 \\ 3n+6, & n \geq -1 \end{cases}$$