EECS 50 - Discrete Time Signals and Systems - Spring 2015

Midterm Exam II Solution

Student Name:
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Discussion Section:

Instructions: This exam contains 6 problems for a total of 60 points. The table below is for grading purposes only.

Problem No.	Points
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Problem 6	
Total Score	

1. (8 pts) Simplify each of the following as much as possible.

   (a) $\delta[n-1] * 2^n u[n-2] * \delta[n-3]$

   Sol: $2^{n-4}u[n-6]$

   (b) $\Big(\delta[n-1] * 2^n u[n-2]\Big) \times \delta[n-3]$

   Sol: $2^{n-1}u[n-3] \times \delta[n-3] = 2^{3-1}u[3-3]\delta[n-3] = 4\delta[n-3]$

   (c) $\delta[n-1] * \Big(2^n u[n-2]) \times \delta[n-3]\Big)$

   Sol: $\delta[n-1] * 2^3 u[3-2]\delta[n-3] = \delta[n-1] * 8\delta[n-3] = 8\delta[n-4]$

   (d) $\delta[n-1] \times 2^n u[n-2] \times \delta[n-3]$

   Sol: $0\ \forall n.$

2. (8 pts)

(a) (4 pts) Given that $3^n u[n] * 2^n u[n] = (3^{n+1} - 2^{n+1})u[n]$, evaluate

$$3^{n-2}u[n-1] * 2^{n+1}u[n+2]$$

Sol:

$$\begin{aligned}
3^n u[n] * 2^n u[n] &= (3^{n+1} - 2^{n+1})u[n] \\
3^{n-1}u[n-1] * 2^{n+2}u[n+2] &= (3^{(n-1+2)+1} - 2^{(n-1+2)+1})u[n-1+2] \\
&= (3^{n+2} - 2^{n+2})u[n+1] \\
\frac{3^{n-1}}{3}u[n-1] * \frac{2^{n+2}}{2}u[n+2] &= \frac{1}{6}(3^{n+2} - 2^{n+2})u[n+1] \\
3^{n-2}u[n-1] * 2^{n+1}u[n+2] &= \frac{1}{6}(3^{n+2} - 2^{n+2})u[n+1]
\end{aligned}$$

(b) (4 pts) Find a signal $h[n]$ such that $h[n] * 2^n u[n] = 3^n u[n]$.

Sol: Taking z-transforms

$$\begin{aligned} H(z)\frac{z}{z-2} &= \frac{z}{z-3} \\ \Rightarrow H(z) &= \frac{z-2}{z-3} \\ &= 1 + \frac{1}{z-3} \\ \Rightarrow h[n] &= \delta[n] + 3^{n-1}u[n-1] \end{aligned}$$

Optional Verification Step:

$$\begin{aligned} h[n] * 2^n u[n] &= (\delta[n] + 3^{n-1}u[n-1]) * 2^n u[n] \\ &= 2^n u[n] + 3^{n-1}u[n-1] * 2^n u[n] \\ &= 2^n u[n] + (3^n - 2^n)u[n-1] \\ &= \begin{cases} 0, & n \leq -1 \\ 1, & n = 0 \\ 2^n + 3^n - 2^n = 3^n, & n \geq 1 \end{cases} \\ &= 3^n u[n] \end{aligned}$$

3. (12 pts) Suppose $g[n]$ is the impulse response of an LTI system that is known to be stable and causal. For each of the following systems, determine if it must be (i) LTI, (ii) Causal, (iii) Stable, and (iv) find its impulse response in terms of $g[n]$ (simplify your answer as much as possible).

   (a) (4 pts) $T_1(x[n]) = x[n] * g[2n]$

   (b) (4 pts) $T_2(x[n]) = x[2n] * g[n]$

   (c) (4 pts) $T_3(x[n]) = x[n+1] * (g[n]u[n-2])$

Sol:

(a) LTI: Yes
Causal: Yes
Stable: Yes
$h_1[n] = g[2n]$.

(b) LTI: No
Causal: No
Stable: Yes
$h_2[n] = g[n]$.

(c) LTI: Yes
Causal: Yes
Stable: Yes
$h_3[n] = g[n+1]u[n-1]$.

Explanation (Optional):

(a) LTI: Yes, because output is a convolution of input with another function, $g[2n]$, so it is LTI and the impulse response is $h_1[n] = g[2n]$. Since $g[n] = 0$ for $n < 0$, $h_1[n] = g[2n] = 0$ for $n < 0$. So the system is causal. Since $\sum_{n=-\infty}^{\infty} |g[n]| < \infty$, and $\sum_{n=-\infty}^{\infty} |h_1[n]| = \sum_{n=-\infty}^{\infty} |g[2n]| < \sum_{n=-\infty}^{\infty} |g[n]| < \infty$, the system is stable.

(b) LTI: No, proof by counterexample: Suppose $g[n] = \delta[n]$, then $T_2(x[n]) = x[2n]$ which is not a time-invariant system. Causal: No, same counterexample. Stable: Yes, if $x[n]$ is bounded, then $x[2n]$ is also bounded, and the convolution of a bounded signal with the impulse response of a stable LTI system must produce a bounded signal.

(c) LTI: Yes, because $T_3(x[n]) = x[n] * \delta[n+1] * (g[n]u[n-2]) = x[n] * g[n+1]u[n-1]$. The impulse response is $h_3[n] = g[n+1]u[n-1]$. Causal: Yes, because system is LTI and the impulse response $h_3[n] = g[n+1]u[n-1] = 0$ for $n < 0$. Stable: Yes, because if $g[n]$ is absolutely summable, then so is $g[n+1]u[n-1]$.

4. (14 pts) A **causal** LTI system with impulse response $h_1[n]$ has the transfer function

$$H_1(z) \quad = \quad \frac{6z^3 - 5z^2 + z}{z^4 - z + 1}$$

(a) (3 pts) Write the difference equation that describes this system.

(b) (3 pts) Draw the Direct Form II realization of this system.

(c) (3 pts) Find $h_1[0], h_1[1], h_1[2]$.

(d) (3 pts) Does this system have an inverse system which is causal and stable? Explain your reasoning.

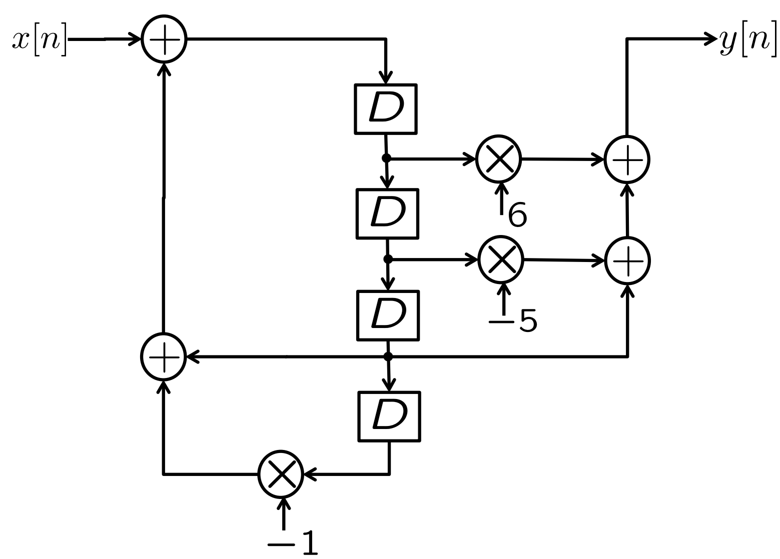
(e) (2 pts) Find the transfer function $H_2(z)$ for the LTI system with impulse response $h_2[n] = h_1[n]/2^n$.

Sol:

(a) Starting from $Y(z) = H_1(z)X(z)$ we have

$$\begin{aligned}
Y(z) &= H_1(z)X(z) \\
\Rightarrow Y(z)(z^4 - z + 1) &= (6z^3 - 5z^2 + z)X(z) \\
\Rightarrow y[n+4] - y[n+1] + y[n] &= 6x[n+3] - 5x[n+2] + x[n+1] \\
\text{Equivalently, } y[n] - y[n-3] + y[n-4] &= 6x[n-1] - 5x[n-2] + x[n-3] \\
\text{or, } y[n] &= 6x[n-1] - 5x[n-2] + x[n-3] + y[n-3] - y[n-4]
\end{aligned}$$

(b) Block diagram for direct form II representation



(c) Using long division, we proceed as follows.

$$\begin{aligned}
H_1(z) &= \frac{6z^3 - 5z^2 + z}{z^4 - z + 1} \\
&= 6z^{-1} + \frac{-5z^2 + z + 6 - 6z^{-1}}{z^4 - z + 1} \\
&= 6z^{-1} - 5z^{-2} + \frac{z + 6 - 11z^{-1} + 5z^{-2}}{z^4 - z + 1}
\end{aligned}$$

Thus, $h_1[0] = 0, h_1[1] = 6, h_1[2] = -5$.

(d) The inverse system must have transfer function $H_0(z) = \frac{1}{H_1(z)} = \frac{z^4-z+1}{6z^3-5z^2+z}$. By long division, it is clear that this system has $h_0[-1] = \frac{1}{6}$. Thus, it is not causal. Therefore, a causal and stable inverse does not exist for this system.

(e) The transform for $h_1[n]a^n$ is $H_1(z/a)$. Therefore, the transform for $h_1[n]/2^n$ is $H_1(2z) = \frac{48z^3-20z^2+2z}{16z^4-2z+1}$

5. (12 pts) A **stable** LTI system with input $x[n]$, output $y[n]$, and impulse response $h[n]$, is described by the difference equation

$$y[n] \quad = \quad x[n] - y[n-1] + 6y[n-2]$$

(a) (4 pts) Find $H(z)$ and its region of convergence.

(b) (4 pts) Find $h[n]$.

(c) (1 pt) Is this system causal?

(d) (1 pt) Is this an FIR filter?

(e) (1 pt) Find the output of this system for the input $x[n] = \left(\frac{1}{2}\right)^n$.

(f) (1 pt) Find the output of this system for the input $x[n] = 4^n$.

Sol:

(a) Taking z-transform of each term we have

$$\begin{aligned} Y(z) &= X(z) - z^{-1}Y(z) + 6z^{-2}Y(z) \\ \Rightarrow \frac{Y(z)}{X(z)} &= \frac{1}{1 + z^{-1} - 6z^{-2}} \\ \Rightarrow H(z) &= \frac{z^2}{z^2 + z - 6} \\ &= \frac{z^2}{(z-2)(z+3)} \end{aligned}$$

and since the LTI system is stable, the ROC must include the unit circle. So the ROC is $|z| < 2$.

(b) Using partial fractions

$$\begin{aligned} H(z) &= \frac{z^2}{2-(-3)} \left(\frac{1}{z-2} - \frac{1}{z+3} \right) \\ &= \frac{z}{5} \left(\frac{z}{z-2} - \frac{z}{z+3} \right) \\ \Rightarrow h[n] &= \frac{1}{5} \Big(-2^n u[-n-1] + (-3)^n u[-n-1] \Big)_{n \to n+1} \\ &= -\left(\frac{2}{5}\right) 2^n u[-n-2] - \left(\frac{3}{5}\right) (-3)^n u[-n-2] \end{aligned}$$

(c) No, the system is not causal.

(d) No, this is not an FIR filter.

(e) In response to the exponential input $\left(\frac{1}{2}\right)^n$, an LTI system with transfer function $H(z)$ must produce output $H(\frac{1}{2}) \left(\frac{1}{2}\right)^n$. Here, we check that $z = 1/2$ is in the ROC, and therefore

$$\begin{aligned} H(1/2) &= \frac{(1/2)^2}{(1/2-2)(1/2+3)} \\ &= \frac{1}{(1-4)(1+6)} = -\frac{1}{21} \end{aligned}$$

and the output is $y[n] = -\frac{1}{21}(\frac{1}{2})^n$.

(f) Since $z = 4$ is outside the ROC, $H(4)$ is not defined. Hence the output does not converge to a finite value in this case.

6. (6 pts) Consider the sequence: $3, 3, 9, 15, 33, \cdots$, which follows the rule $y[n] = y[n-1] + 2y[n-2]$, so e.g., the 6^{th} term is $33 + 2(15) = 63$. Find a closed form expression for the n^{th} term of this sequence.

Sol: Let us set it up as an LTI system that is initially at rest, so $x[n] = y[n] = 0$ for $n < 0$, and then plug in various values of n to determine what $x[n]$ is needed.

$$\begin{aligned}
y[n] &= y[n-1] + 2y[n-2] + x[n] \\
0 = y[0] &= y[-1] + 2y[-2] + x[0] = x[0], \Rightarrow x[0] = 0 \\
3 = y[1] &= y[0] + 2y[-1] + x[1] = x[1], \Rightarrow x[1] = 3 \\
3 = y[2] &= y[1] + 2y[0] + x[2] = 3 + x[2], \Rightarrow x[2] = 0 \\
9 = y[3] &= y[2] + 2y[1] + x[3] = 3 + 6 + x[2], \Rightarrow x[3] = 0 \\
&\vdots
\end{aligned}$$

Proceeding this way, we determine that $x[n] = 3\delta[n-1]$. Now let us use this to find $y[n]$ by employing the z-transform.

$$\begin{aligned}
y[n] &= y[n-1] + 2y[n-2] + 3\delta[n-1] && (1) \\
\Rightarrow Y(z) &= z^{-1}Y(z) + 2z^{-2}Y(z) + 3z^{-1} && (2) \\
\Rightarrow Y(z)\Big(1 - z^{-1} - 2z^{-2}\Big) &= 3z^{-1} && (3) \\
\Rightarrow Y(z) &= \frac{3z^{-1}}{1 - z^{-1} - 2z^{-2}} && (4) \\
&= \frac{3z}{z^2 - z - 2} && (5) \\
&= \frac{3z}{(z-2)(z+1)} && (6) \\
&= \frac{3z}{2-(-1)}\Big(\frac{1}{z-2} - \frac{1}{z+1}\Big) && (7) \\
&= \Big(\frac{z}{z-2} - \frac{z}{z+1}\Big) && (8) \\
& && (9) \\
\Rightarrow y[n] &= \Big(2^n - (-1)^n\Big)u[n] && (10)
\end{aligned}$$

Therefore, for $n = 1, 2, 3, \cdots$, the n^{th} term of the sequence is

$$2^n - (-1)^n \qquad (11)$$

Optional Verification Step:

$$\begin{aligned}
n = 1 &\Rightarrow y[1] = (2+1) = 3 \\
n = 2 &\Rightarrow y[2] = (4-1) = 3 \\
n = 3 &\Rightarrow y[3] = (8+1) = 9 \\
n = 4 &\Rightarrow y[4] = (16-1) = 15 \\
n = 5 &\Rightarrow y[5] = (32+1) = 33 \\
&\vdots
\end{aligned}$$