EECS 50 - Discrete Time Signals and Systems - Spring 2017

Midterm Exam II Solution

Student Name:
Student ID:

Instructions: This exam contains 5 problems for a total of 50 points. The table below is for grading purposes only.

Problem No.	Points Scored	Maximum Possible
Problem 1		10
Problem 2		10
Problem 3		10
Problem 4		10
Problem 5		10
Total Score		50

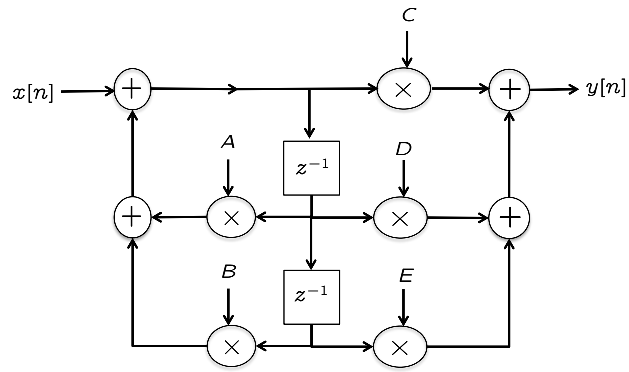
1. (10 pts) The LTI system $T_1(x[n])$ has impulse response

$$h_1[n] = 2\delta[n] - 2^n u[n].$$

Causal LTI system T_2 has input $x_2[n]$ and output $y_2[n] = T_2(x_2[n])$ which are related according to the difference equation:

$$y_2[n] = x_2[n] - x_2[n-1] + 3y_2[n-1]$$

Find the values of constants A, B, C, D, E for the following Direct Form II representation of $T(x[n])$.



such that

$$T(x[n]) \quad = \quad T_1\Big(T_2(x[n])\Big)$$

Sol:

$$\begin{aligned}
H_1(z) &= 2 - \frac{z}{z-2} = \frac{z-4}{z-2} \\
H_2(z) &= \frac{1-z^{-1}}{1-3z^{-1}} = \frac{z-1}{z-3} \\
H(z) &= H_1(z)H_2(z) = \frac{(z-1)(z-4)}{(z-2)(z-3)} = \frac{z^2-5z+4}{z^2-5z+6} = \frac{1-5z^{-1}+4z^{-2}}{1-5z^{-1}+6z^{-2}} = \frac{Y(z)}{X(z)} \\
Y(z) &= X(z) - 5z^{-1}X(z) + 4z^{-2}X(z) + 5z^{-1}Y(z) - 6z^{-2}Y(z) \\
y[n] &= x[n] - 5x[n-1] + 4x[n-2] + 5y[n-1] - 6y[n-2]
\end{aligned}$$

Therefore,

$$\begin{aligned}
A &= 5 \\
B &= -6 \\
C &= 1 \\
D &= -5 \\
E &= 4
\end{aligned}$$

2. (10 pts) A **causal** LTI system has impulse response $h[n]$ and transfer function

$$H(z) = \frac{5 + z + 2z^2}{1 + 3z^3}$$

(a) (2 pts) Is this a stable system? Explain why or why not.

(b) (3 pts) Find $h[0], h[1], h[2]$.

(c) (2 pts) Find the output of this LTI system if the input is chosen to be $x[n] = 1 + 2^n$.

(d) (3 pts) Write the difference equation that describes this system.

Sol:

(a) Poles z_i are solutions of $z_i^3 = -1/3 \Rightarrow |z_i| < 1$. So all the poles are strictly inside the unit circle. Since the system is causal, the ROC lies outside the outermost pole. Thus, the ROC includes the unit circle and the LTI system is stable.

(b) By long division:

$$\begin{aligned} H(z) &= \frac{2z^2 + z + 5}{3z^3 + 1} \\ &= \frac{2}{3}z^{-1} + \frac{1}{3}z^{-2} + \ldots \end{aligned}$$

Therefore,

$$\begin{aligned} h[0] &= 0 \\ h[1] &= 2/3 \\ h[2] &= 1/3 \end{aligned}$$

(c) Using the eigenfunction property, if input $x[n] = (1)^n + 2^n$, then the output is $y[n] = (1)^n H(1) + 2^n H(2) = 2 + 2^n(3/5)$.

(d)

$$\begin{aligned} H(z) &= \frac{2z^2 + z + 5}{3z^3 + 1} = \frac{2z^{-1} + z^{-2} + 5z^{-3}}{3 + z^{-3}} = \frac{Y(z)}{X(z)} \\ \Rightarrow 3y[n] &= 2x[n-1] + x[n-2] + 5x[n-3] - y[n-3] \end{aligned}$$

3. LTI systems T_1 and T_2 have impulse response functions $h_1[n]$ and $h_2[n]$, and transfer functions $H_1(z)$ and $H_2(z)$, respectively. The two transfer functions have the same form

$$H_1(z) = H_2(z) \quad = \quad \frac{3z}{3z-1} + \frac{3}{z-3}$$

but they have different regions of convergence (ROC).

(a) T_1 **is known to be causal.**

i. (1 pt) What is the ROC of $H_1(z)$?

ii. (3 pts) Find $h_1[n]$.

iii. (1 pt) Is T_1 stable?

(b) T_2 **is known to be stable.**

i. (1 pt) What is the ROC of $H_2(z)$?

ii. (3 pts) Find $h_2[n]$.

iii. (1 pt) Is T_2 causal?

Sol:

(a) T_1 **is known to be causal**

i. Poles at $z = 1/3$ and at $z = 3$. For causal LTI system, ROC is outside the outermost pole, i.e., $|z| > 3$.

ii.

$$\begin{aligned} H_1(z) &= \frac{3z}{3z-1} + \frac{3}{z-3} \\ &= \frac{z}{z-1/3} + 3z^{-1}\left(\frac{z}{z-3}\right) \\ h_1[n] &= \left(\frac{1}{3}\right)^n u[n] + 3\Big(3^n u[n]\Big)_{n\to n-1} \\ &= \left(\frac{1}{3}\right)^n u[n] + 3^n u[n-1] \end{aligned}$$

iii. ROC does not include unit circle, so T_1 is not stable.

(b) T_2 **is known to be stable.**

i. Poles at $z = 1/3$ and at $z = 3$. For stable LTI system, ROC must include unit circle, i.e., $1/3 < |z| < 3$.

ii.

$$\begin{aligned} H_2(z) &= \frac{z}{z-1/3} + 3z^{-1}\left(\frac{z}{z-3}\right) \\ h_2[n] &= \left(\frac{1}{3}\right)^n u[n] + 3\Big(-3^n u[-n-1]\Big)_{n\to n-1} \\ &= \left(\frac{1}{3}\right)^n u[n] - 3^n u[-n] \end{aligned}$$

iii. No, $h_2[n] \neq 0$ for all $n < 0$, so T_2 is not causal.

4. (10 pts) Consider the sequence $y[n]$ that begins as $2, 1, 5, 7, 17, \ldots$ and follows the rule

$$y[n] = y[n-1] + 2y[n-2].$$

For example, the 3^{rd} term is $y[3] = 1 + 2 \times 2 = 5$, the 4^{th} term is $y[4] = 5 + 2 \times 1 = 7$, the 5^{th} term is $y[5] = 7 + 2 \times 5 = 17$, and so on. Find a closed form expression for the n^{th} term of this sequence.

Sol: Let us set it up as the difference equation for an LTI system that is initially at rest ($x[n] = 0, y[n] = 0$ for $n < 0$), as follows:

$$y[n] = x[n] + y[n-1] + 2y[n-2]$$

Solving for the first few values of n

$$\begin{aligned}
y[0] &= x[0] + y[-1] + 2y[-2] & &\Rightarrow 0 = x[0] + 0 + 2(0) & &\Rightarrow x[0] = 0 \\
y[1] &= x[1] + y[0] + 2y[-1] & &\Rightarrow 2 = x[1] + 0 + 2(0) & &\Rightarrow x[1] = 2 \\
y[2] &= x[2] + y[1] + 2y[0] & &\Rightarrow 1 = x[2] + 2 + 2(0) & &\Rightarrow x[2] = -1 \\
y[3] &= x[3] + y[2] + 2y[1] & &\Rightarrow 5 = x[3] + 1 + 2(2) & &\Rightarrow x[3] = 0 \\
y[4] &= x[4] + y[3] + 2y[2] & &\Rightarrow 7 = x[4] + 5 + 2(1) & &\Rightarrow x[4] = 0 \\
&\vdots
\end{aligned}$$

Therefore $x[n] = 2\delta[n-1] - \delta[n-2]$. With this choice of input, let us go to the z-domain

$$\begin{aligned}
Y(z) &= 2z^{-1} - z^{-2} + z^{-1}Y(z) + 2z^{-2}Y(z) \\
Y(z) &= \frac{2z^{-1} - z^{-2}}{1 - z^{-1} - 2z^{-2}} \\
&= \frac{2z - 1}{z^2 - z - 2} \\
&= \frac{2z - 1}{(z-2)(z+1)} \\
&= \frac{1}{z-2} + \frac{1}{z+1} \\
&= z^{-1}\left(\frac{z}{z-2} + \frac{z}{z+1}\right) \\
\Rightarrow y[n] &= 2^{n-1}u[n-1] + (-1)^{n-1}u[n-1]
\end{aligned}$$

Therefore, for $n > 0$, the n^{th} term of the sequence is $2^{n-1} + (-1)^{n-1}$.

Optional Verification Step:

$$\begin{aligned}
y[1] &= 2^{1-1} + (-1)^{1-1} = 1 + 1 = 2 \\
y[2] &= 2^{2-1} + (-1)^{2-1} = 2 - 1 = 1 \\
y[3] &= 2^{3-1} + (-1)^{3-1} = 4 + 1 = 5 \\
y[4] &= 2^{4-1} + (-1)^{4-1} = 8 - 1 = 7 \\
&\cdots
\end{aligned}$$

5. (10 pts) For each of the following statements, mark it as true or false.

True/False	Statement
T	If $y[n] = x[n] * h[n]$ then $2^n y[n] = (2^n x[n]) * (2^n h[n])$.
F	All the zeros of the transfer function of a stable LTI system must be inside the unit circle.
F	If $h[n]$ is an even signal then we must have $H(z) = H(-z)$.
F	If $h[n] = \sin\left(\frac{2\pi n}{3}\right)$ then $H(z)$ converges for all $\lvert z\rvert > 1$.
F	If $H(z) = \frac{z^2+z+18}{z^2-5z+6}$ is the transfer function of a stable LTI system with impulse response $h[n]$, then $\sum_{n=-\infty}^{\infty} h[n] = 3$.
T	If $H(z)$ is the z-transform of $h[n]$, then $H(-z)$ is the z-transform of $h[n]\cos(\pi n)$.
T	For a causal LTI system, $h[0] = \lim_{z\to\infty} H(z)$.
F	For the causal LTI system with $H(z) = \frac{z-2}{z-3}$, we must have $h[n] * 2^n = 0$.
T	For the stable LTI system with $H(z) = \frac{z-2}{z-3}$, we must have $h[n] * 2^n = 0$.
F	The system T with transfer function $H(z) = \frac{z^2}{z^2+3z+2}$ has an inverse system T' with impulse response $h'[n] = \delta[n-2] + 3\delta[n-1] + 2$.

Explanation (Optional):

(a) True, $Y(z/2) = X(z/2)H(z/2)$.

(b) False. For example, $H(z) = z - 2$, which corresponds to $h[n] = \delta[n+1] - 2\delta[n]$, is stable, but has zeros outside unit circle.

(c) False. For example, $h[n] = \delta[n-1] + \delta[n+1]$ is an even signal but $H(z) = z^{-1} + z$ is not equal to $H(-z) = -z^{-1} - z$.

(d) False. $h[n] = \sin(2\pi n/3) = \frac{1}{2j}e^{j2\pi n/3} - \frac{1}{2j}e^{-j2\pi n/3}$ is comprised of eternal exponentials, for which z-transform does not converge.

(e) False. $\sum_{n=-\infty}^{\infty} h[n] = H(1) = 20/2 = 10$.

(f) True. $h[n]\cos(\pi n) = (-1)^n h[n]$ which has z-transform $H(-z)$ because $a^n h[n]$ has z-transform $H(z/a)$.

(g) True. For a causal system, $H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \ldots$. The only term that survives the limit $z \to \infty$ is $h[0]$.

(h) False. By the eigenfunction property, $h[n] * 2^n = H(2)2^n$. For the causal system, the ROC is $|z| > 3$, so $H(2)$ is not defined.

(i) True. By the eigenfunction property, $h[n] * 2^n = H(2)2^n$. For the stable system, the ROC is $|z| < 3$, so $H(2) = (2-2)/(2-3) = 0$.

(j) False. The inverse system has transfer function $1+3z^{-1}+2z^{-2}$, so $h'[n] = \delta[n]+3\delta[n-1]+2\delta[n-2]$.