EECS 50 - Discrete Time Signals and Systems - Spring 2014

Midterm Exam II

Student Name:
Student ID:
Discussion Section:

Instructions: This exam contains 5 problems, worth 10 points each, for a total of 50 points.
The z-transforms refer to bilateral z-transforms throughout this exam. The table below is for grading purposes only.

Problem No.	Points
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Total Score	

1. (10 pts)

   (a) (1 pts) For the $x_o[n]$ of Problem 5, find $X_o(z)$ and its region of convergence.

   (b) (3 pts) $x_1[n] = \frac{3^n}{2^{n+1}}u[-n-2] + \frac{1}{2^n}u[n]$. Find $X_1(z)$ and its region of convergence.

   (c) (2 pts) $H(z) = \frac{z}{z-0.5} + \frac{z}{z-2}$ is the transfer function of a stable system. Find $h[n]$.

   (d) (2 pts) Given that a causal system has transfer function

   $$H(z) = \frac{z^2 - 2z + 1}{z^3 + 3z^2 - 2z - 1}$$

   use long division to find $h[0], h[1], h[2]$.

   (e) (2 pts) Signal $f[n]$ has z-transform $F(z) = (1+z)e^{1/z}$. Find $f[3]$.
   (Hint: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$)

Solution:

(a) $X_o(z) = 1 - z^{-1} - z^{-2} + z^{-3} - z^{-4}$, ROC: $|z| \neq 0$.

(b)

$$\begin{aligned}
x_1[n] &= -\frac{1}{3}\left(-\left(\frac{3}{2}\right)^{n+1} u[-n-2]\right) + 0.5^n u[n] \\
&= -\frac{1}{3}\left(-\left(\frac{3}{2}\right)^{n} u[-n-1]\right)_{n\longrightarrow n+1} + 0.5^n u[n] \\
\Rightarrow X_1(z) &= \left(-\frac{1}{3}\right)\left(\frac{z}{z-1.5}\right)z + \frac{z}{z-0.5}, \quad 0.5 < |z| < 1.5 \\
&= \frac{z^2}{4.5-3z} + \frac{z}{z-0.5}, \quad 0.5 < |z| < 1.5
\end{aligned}$$

(c) Stable system must include unit circle in the ROC of its transfer function. So the pole at 0.5 corresponds to a right sided signal and the pole at 2 corresponds to a left sided signal.

$$h[n] = 0.5^n u[n] - 2^n u[-n-1]$$

(d) Using long division we find $h[n] = z^{-1} - 5z^{-2} + \cdots$, from which we have $h[0] = 0, h[1] = 1$, and $h[2] = -5$.

(e) $f[3]$ is the coefficient of z^{-3}, i.e., $1/z^3$ in $F(z) = e^{1/z} + ze^{1/z}$.
From $e^{1/z}$ we have the coefficient of $1/z^3$ as $1/3! = 1/6$.
From $ze^{1/z}$ we have the coefficient of $1/z^3$ as $1/4! = 1/24$.
So the answer is $1/6 + 1/24 = 5/24 = f[3]$.

2. (10 pts) A causal LTI system is described by the difference equation

$$y[n] \quad = \quad 2x[n-1] - 4x[n-2] + 4y[n-1] - 3y[n-2]$$

(a) (3 pts) Find the transfer function $H(z)$, specify its poles and zeros and its region of convergence.

(b) (1 pt) Evaluate the following sum.

$$\sum_{n=-\infty}^{\infty} \frac{h[n]}{4^n}$$

(c) (1 pts) Find the output if the input is the signal $x[n] = 4^n$.

(d) (1 pts) Find the output if the input is the signal $x[n] = 2^n$.

(e) (3 pts) Find the impulse response $h[n]$.

(f) (1 pt) Does this system have an inverse system that is both causal and stable?

Solution:

(a) Taking the z-transform of the difference equation we have

$$\begin{aligned}
Y(z) &= 2z^{-1}X(z) - 4z^{-2}X(z) + 4z^{-1}Y(z) - 3z^{-2}Y(z) \\
\Rightarrow Y(z)\left(1 - 4z^{-1} + 3z^{-2}\right) &= X(z)\left(2z^{-1} - 4z^{-2}\right) \\
\frac{Y(z)}{X(z)} = H(z) &= \frac{2z^{-1} - 4z^{-2}}{1 - 4z^{-1} + 3z^{-2}} \\
&= \frac{2z - 4}{z^2 - 4z + 3} \\
&= \frac{2(z-2)}{(z-1)(z-3)}
\end{aligned}$$

The poles are at $z = 1$ and $z = 3$ and the zero is at $z = 2$.
Since this is a causal system, the ROC is outside the outermost pole. So, the ROC is $|z| > 3$.

(b) From the definition of z-transform, $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$. Substituting $z = 4$ gives us this sum. Since $z = 4$ is in the ROC, the answer is

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{h[n]}{4^n} &= H(4) \\
&= 4/3
\end{aligned}$$

(c) We use the property that complex exponentials are eigenfunctions of LTI systems. i.e.,

$$a^n * h[n] \quad = \quad H(a)a^n$$

So if the input is $x[n] = 4^n$, then the output is $y[n] = H(4)4^n = \left(\frac{4}{3}\right)4^n$.

(d) Similarly, when the input is $x[n] = 2^n$, the output is $y[n] = H(2)2^n$ but $H(2)$ is not defined because $z = 2$ is not in the ROC. Therefore, the output in this case is not defined.

(e) Doing partial fractions for $H(z)$ we have

$$\begin{aligned}
H(z) &= \frac{1}{z-1} + \frac{1}{z-3} = z^{-1}\left(\frac{z}{z-1} + \frac{z}{z-3}\right) \\
\Rightarrow h[n] &= \left(u[n] + 3^n u[n]\right)_{n \to n-1} = u[n-1] + 3^{n-1}u[n-1]
\end{aligned}$$

(f) For the inverse to be causal and stable, all poles and zeros of $H(z)$ must be inside the unit circle. Since this is not the case, this system does not have an inverse that is both causal and stable.

3. (10 pts) Consider the LTI system, T_1, with the impulse response

$$h_1[n] \quad = \quad 3^{-n}u[n]$$

(a) (1 pt) Find the transfer function $H_1(z)$.

(b) (2 pts) Find the difference equation that describes this LTI system.

(c) (3 pts) Find the output $y_1[n]$ produced in response to the input $x_1[n] = 3^n u[n]$.

(d) (2 pts) Find the impulse response function $h_2[n]$ of the LTI system T_2 that is the inverse of T_1.

(e) (2 pts) Find $h_1[n] * u[n] * h_2[n]$.

Solution:

(a) $H_1(z) = z/(z - 1/3)$, for $|z| > 1/3$.

(b) Since $H_1(z) = Y(z)/X(z)$ we have

$$\begin{aligned} Y(z)(z - 1/3) &= X(z)z \\ \Rightarrow y[n+1] - \frac{1}{3}y[n] &= x[n+1] \\ \text{We can also write: } y[n] &= \frac{1}{3}y[n-1] + x[n] \end{aligned}$$

(c) The output is

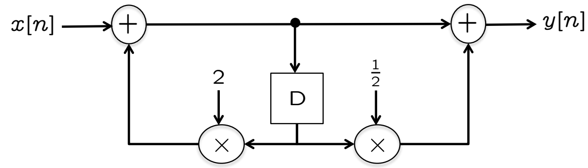
$$\begin{aligned} y[n] &= h_1[n] * x_1[n] \\ &= (1/3)^n u[n] * 3^n u[n] \\ &= \left(\frac{1/3}{1/3 - 3}\right)(1/3)^n u[n] + \left(\frac{3}{3 - 1/3}\right)3^n u[n] \\ &= -\frac{1}{8}(1/3)^n u[n] + \frac{9}{8}(3^n)u[n] \end{aligned}$$

(d)

$$\begin{aligned} H_2(z) &= \frac{1}{H_1(z)} \\ &= \frac{z - 1/3}{z} \\ &= 1 - \frac{1}{3}z^{-1} \\ \Rightarrow h_2[n] &= \delta[n] - \frac{1}{3}\delta[n-1] \end{aligned}$$

(e) $h_1[n] * u[n] * h_2[n] = h_1[n] * h_2[n] * u[n] = \delta[n] * u[n] = u[n]$ because convolution is commutative and associative, and because $h_2[n]$ is the inverse of $h_1[n]$, i.e., $h_1[n] * h_2[n] = \delta[n]$.

4. (10 pts) The direct form 2 realization of a causal LTI system is shown below.



(a) (2 pts) Sketch the direct form 1 realization of this LTI system.

(b) (3 pts) Find the difference equation that describes this LTI system.

(c) (2 pts) Is this a BIBO stable system?

(d) (1 pt) Is this a finite impulse response (FIR) system?

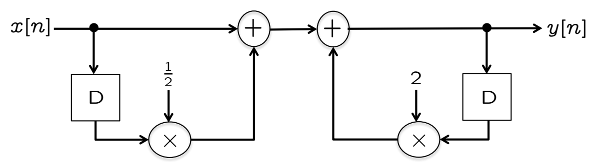
(e) (2 pts) Suppose we know that the input

$$\begin{aligned} x[n] &= 0 \text{ for } n < 0, \\ x[0] &= 2, \\ x[1] &= 1. \end{aligned}$$

We do not know the values of $x[n]$ for $n > 1$. Find $y[0]$ and $y[1]$.

Solution:

(a) The direct form 1 realization of the same LTI system is the following.



(b) The difference equation is the following

$$y[n] = x[n] + \frac{1}{2}x[n-1] + 2y[n-1]$$

(c) No, it is not a stable system. The transfer function is

$$\begin{aligned} H(z) &= \frac{1 + 0.5z^{-1}}{1 - 2z^{-1}} \\ &= \frac{z + 0.5}{z - 2} \end{aligned}$$

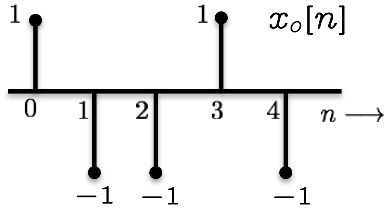
For this causal system with a pole at $z = 2$, the ROC is $|z| > 2$, which does not include the unit circle.

(d) This is not an FIR system. From $H(z)$ we have $h[n] = 2^n u[n] + 0.5(2^{n-1})u[n-1]$ which does not die out for any finite value of n.

(e) For a causal system, the output does not precede the input, so $y[n] = 0$ for $n < 0$. From the difference equation

$$\begin{aligned} y[0] &= x[0] + 0.5x[-1] + 2y[-1] = 2 + 0.5(0) + 2(0) = 2 \\ y[1] &= x[1] + 0.5x[0] + 2y[0] = 1 + 0.5(2) + 2(2) = 6 \end{aligned}$$

5. (10 pts) Suppose we have a long data sequence $x[n]$ consisting *only of* $+1$ *and* -1 *values*, and we are interested in detecting if and when the special pattern $+1, -1, -1, +1, -1$ (shown in the figure as $x_o[n]$) appears in the long data sequence. For this purpose we wish to use an LTI filter that will scan the input sequence and produce an output that will peak as soon as the desired pattern is detected.



(a) (2 pts) Find the impulse response of this filter.

(b) (2 pts) Find the peak value of the output if the special pattern $+1, -1, -1, +1, -1$ is present in the data sequence.

(c) (2 pts) Suppose the output takes this peak value at $n = 200$. This means that the pattern $+1, -1, -1, +1, -1$ appears in the long input sequence $x[n]$ starting at $n = n_1$ and ending at $n = n_2$. Find n_1, n_2.

(d) (2 pt) What is the maximum possible value of the output signal if the data sequence does not contain the special pattern $+1, -1, -1, +1, -1$? (Note that the input data sequence $x[n]$ consists only of $+1$ and -1 values.)

(e) (2 pt) What are such filters called? Name a practical application of these filters.

Solution:

(a) $h[n] = x_o[4 - n] = -\delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3] + \delta[n-4]$.

(b) When the pattern enters the matched filter, the output will be

$$\begin{aligned} y_{\max} &= h[0]x_o[4] + h[1]x_o[3] + h[2]x_o[2] + h[3]x_o[1] + h[4]x_o[0] \\ &= (-1)(-1) + (1)(1) + (-1)(-1) + (-1)(-1) + (1)(1) \\ &= 1 + 1 + 1 + 1 + 1 \\ &= 5 \end{aligned}$$

(c) The pattern fully enters the filter at $n = 200$, so it must start at $n_1 = 196$ and end at $n_2 = 200$.

(d) Consider the output at time n,

$$\begin{aligned} y[n] &= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + h[3]x[n-3] + h[4]x[n-4] \\ &= -x[n] + x[n-1] - x[n-2] - x[n-3] + x[n-4] \end{aligned}$$

If the special pattern is present, then all 5 of these terms become $+1$, to produce a sum of $+5$. However, if the input does not contain the special pattern, then at least one of the 5 values in the sum must produce a -1. Therefore, the maximum possible output is $1 + 1 + 1 + 1 - 1 = 3$.

(e) These filters are called matched filters. They are used e.g., in radar applications where a known sequence is transmitted and the received signal is scanned for the occurrence of the same sequence due to reflections from other objects. The delay in the reflected pattern is used to estimate the distance to the objects. Matched filters are also used with special sequences for timing synchronization between transmitter and receiver in communication systems.