EEСS 50 - Discrete Time Signals and Systems - Spring 2016

Midterm Exam II Solution

Student Name:
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Instructions: This exam contains 7 problems for a total of 50 points. The table below is for grading purposes only.

Problem No.	Points Scored	Maximum Possible
Problem 1		7
Problem 2		7
Problem 3		6
Problem 4		10
Problem 5		6
Problem 6		9
Problem 7		5
Total Score		50

1. (7 pts) For the given signal $x[n]$, find the z-transform $X(z)$ and specify its region of convergence.

$$x[n] \quad = \quad \left(\frac{1}{3}\right)^n u[n-1] + 2^n u[-n]$$

Sol:

$$\left(\frac{1}{3}\right)^n u[n] \longrightarrow \frac{1}{1-(\frac{1}{3})z^{-1}}, \qquad |z| > 1/3$$

Shifting $n \to n-1$ in time domain corresponds to multiplication by z^{-1} in z-domain

$$\Rightarrow \left(\frac{1}{3}\right)^{n-1} u[n-1] \longrightarrow \frac{z^{-1}}{1-(\frac{1}{3})z^{-1}}, \qquad |z| > 1/3$$

Scaling both sides by $1/3$

$$\Rightarrow \left(\frac{1}{3}\right)^{n} u[n-1] \longrightarrow \frac{\frac{1}{3}z^{-1}}{1-(\frac{1}{3})z^{-1}}, \qquad |z| > 1/3$$

Similarly,

$$-2^n u[-n-1] \longrightarrow \frac{1}{1-2z^{-1}}, \qquad |z| < 2$$

Shifting $n \to n-1$ in time domain corresponds to multiplication by z^{-1} in z-domain

$$\Rightarrow -2^{n-1} u[-(n-1)-1] \longrightarrow \frac{z^{-1}}{1-2z^{-1}}, \qquad |z| < 2$$

$$\Rightarrow -2^{n-1} u[-n] \longrightarrow \frac{z^{-1}}{1-2z^{-1}}, \qquad |z| < 2$$

Scaling both sides by -2

$$\Rightarrow 2^{n} u[-n] \longrightarrow \frac{-2z^{-1}}{1-2z^{-1}}, \qquad |z| < 2$$

Finally, combining everything

$$\left(\frac{1}{3}\right)^n u[n-1] + 2^n u[-n] \longrightarrow \frac{\frac{1}{3}z^{-1}}{1-(\frac{1}{3})z^{-1}} - \frac{2z^{-1}}{1-2z^{-1}}, \qquad \frac{1}{3} < |z| < 2$$

Other equivalent forms are also acceptable, such as

$$
\begin{aligned}
X(z) &= \frac{1}{3z-1} - \frac{2}{z-2}, && \frac{1}{3} < |z| < 2 \\
&= \frac{-5z}{(3z-1)(z-2)}, && \frac{1}{3} < |z| < 2 \\
&= \frac{-5z}{3z^2-7z+2}, && \frac{1}{3} < |z| < 2
\end{aligned}
$$

2. (7 pts) An LTI system has impulse response $h[n]$, which is a right-sided signal. The transfer function of this LTI system is the following.

$$H(z) \quad = \quad \frac{1 + 3z^6}{2 + z^3}$$

(a) (5 pts) Find the values of $h[n]$ for all $n < 0$.

(b) (2 pts) Is this a stable system? Explain your answer.

Sol:

(a) Since the signal is right-sided, let us arrange the numerator and denominator of $H(z)$ in decreasing powers of z, i.e.,

$$H(z) \quad = \quad \frac{3z^6 + 1}{z^3 + 2}$$

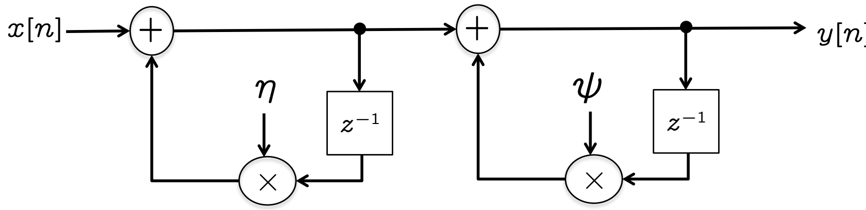
and use long-division as follows

$$\begin{aligned} H(z) &= \frac{3z^6 + 1}{z^3 + 2} \\ &= 3z^3 + \left(\frac{-6z^3 + 1}{z^3 + 2} \right) \\ &= 3z^3 - 6 + \left(\frac{13}{z^3 + 2} \right) \end{aligned}$$

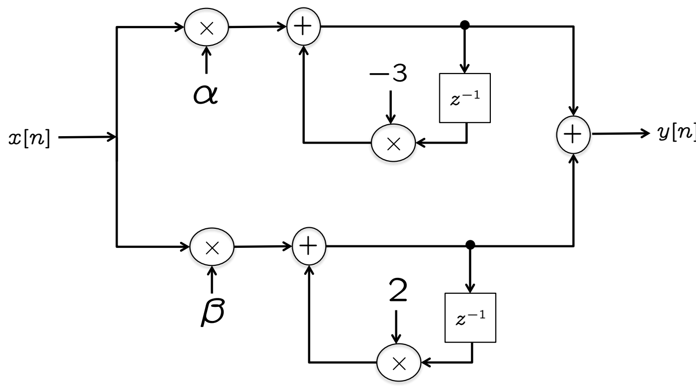
Thus, $h[-3] = 3$ and $h[n] = 0$ for all other negative values of n.

(b) No. The poles of $H(z)$ are the roots of $z^3 + 2$, so all the poles have magnitude $2^{1/3}$ which is greater than 1. Since $h[n]$ is right-sided, the region of convergence is $|z| > 2^{1/3}$, which does not include the unit circle. Therefore the system is not stable.

3. (6 pts) A realization of a causal LTI system is shown below for some $\eta > 0$ and $\psi < 0$.



Another realization of a causal LTI system is shown below for some α, β.



Find $\alpha, \beta, \eta, \psi$ such that the two realizations represent the same system.

Sol: The transfer function of the first realization is

$$H(z) = \left(\frac{1}{1-\eta z^{-1}}\right)\left(\frac{1}{1-\psi z^{-1}}\right)$$

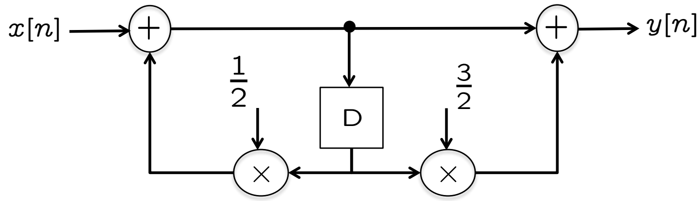
For the second realization, the transfer function is

$$\begin{aligned} H(z) &= \alpha\left(\frac{1}{1+3z^{-1}}\right) + \beta\left(\frac{1}{1-2z^{-1}}\right) \\ &= \frac{\alpha - 2\alpha z^{-1} + \beta + 3\beta z^{-1}}{(1+3z^{-1})(1-2z^{-1})} \\ &= \frac{\alpha + \beta + (-2\alpha + 3\beta)z^{-1}}{(1+3z^{-1})(1-2z^{-1})} \end{aligned}$$

Comparing the two representations we have,

$$\begin{aligned} &\eta = 2, && \alpha + \beta = 1 \\ &\psi = -3, && -2\alpha + 3\beta = 0, && \Rightarrow \alpha = \frac{3}{5}, \quad \beta = \frac{2}{5} \end{aligned}$$

4. (10 pts) A **stable** LTI system is realized as follows.



(a) (2 pts) Is this a causal system? Explain your answer.

(b) (4 pts) Find its impulse response $h[n]$. Show your work.

(c) (2 pts) Find the output of this LTI system if the input is $x[n] = 2^n$.

(d) (2 pts) Find the output of this LTI system if the input is $x[n] = \left(\frac{1}{4}\right)^n$.

Sol: From the Direct Form II block diagram, the transfer function of the system is

$$H(z) = \frac{1 + \frac{3}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}} = -3 + \frac{4}{1 - \frac{1}{2}z^{-1}}$$

Pole at $z = 1/2$, and since the system is stable, the unit circle $|z| = 1$ is in the region of convergence. Therefore the ROC is

$$|z| > 1/2$$

and the impulse response must be right-sided.

$$h[n] = -3\delta[n] + 4\left(\frac{1}{2}\right)^n u[n]$$

(a) Yes it is causal, because $h[n] = 0$ for $n < 0$.

(b) $h[n] = -3\delta[n] + 4\left(\frac{1}{2}\right)^n u[n]$.

(c) $2^n * h[n] = 2^n H(2) = 2^n \left(-3 + \frac{4}{1-\frac{1}{4}}\right) = \left(\frac{7}{3}\right) 2^n$. Note that $H(2)$ is defined because $z = 2$ is in the ROC.

(d) $\left(\frac{1}{4}\right)^n * h[n] = \left(\frac{1}{4}\right)^n H(\frac{1}{4})$. But $H(\frac{1}{4})$ is not defined because $z = \frac{1}{4}$ is not in the ROC. So in this case, the output is not defined.

5. (6 pts) T_1 and T_2 are causal LTI systems. For T_1, the input $x_1[n]$ is related to the output $y_1[n]$ through the following difference equation.

$$y_1[n] \quad = \quad x_1[n] + \frac{1}{2}y_1[n-2]$$

For T_2, the input $x_2[n]$ is related to the output $y_2[n]$ through the following difference equation.

$$y_2[n] \quad = \quad x_2[n] - \frac{1}{2}y_2[n-2]$$

From these systems, an overall system T is composed as follows.

$$T(x[n]) \quad = \quad T_1(T_2(x[n]))$$

Find the difference equation for the overall system T.

Sol: Writing out the transfer functions for the two systems,

$$H_1(z) = \frac{1}{1 - \frac{1}{2}z^{-2}}, \qquad H_2(z) = \frac{1}{1 + \frac{1}{2}z^{-2}}$$

Since the two are connected in series to form the overall system T, the transfer function of the overall system is,

$$\begin{aligned} H(z) &= H_1(z)H_2(z) \\ &= \frac{1}{1 - \frac{1}{4}z^{-4}} \end{aligned}$$

Therefore the difference equation of the overall system T is

$$y[n] = x[n] + \frac{1}{4}y[n-4]$$

6. (9 pts) Consider the sequence: $1, 5, 19, 65, \cdots$, which follows the rule $y[n] = 5y[n-1] - 6y[n-2]$, so, e.g., the 5^{th} term is $5(65) - 6(19) = 211$. Note that the sequence starts with the first term, i.e., $y[1] = 1, y[2] = 5, \ldots$ Find a closed form expression for the n^{th} term of this sequence.

Sol: Let us set it up as the difference equation for an LTI system that is initially at rest ($x[n] = 0, y[n] = 0$ for $n < 0$), as follows:

$$y[n] = x[n] + 5y[n-1] - 6y[n-2]$$

Solving for the first few values of n

$$\begin{array}{lll}
y[0] = x[0] + 5y[-1] - 6y[-2] & \Rightarrow 0 = x[0] + 5(0) - 6(0) & \Rightarrow x[0] = 0 \\
y[1] = x[1] + 5y[0] - 6y[-1] & \Rightarrow 1 = x[1] + 5(0) - 6(0) & \Rightarrow x[1] = 1 \\
y[2] = x[2] + 5y[1] - 6y[0] & \Rightarrow 5 = x[2] + 5(1) - 6(0) & \Rightarrow x[2] = 0 \\
y[3] = x[3] + 5y[2] - 6y[1] & \Rightarrow 19 = x[3] + 5(5) - 6(1) & \Rightarrow x[3] = 0 \\
\quad \vdots & &
\end{array}$$

Therefore $x[n] = \delta[n-1]$. With this choice of input, let us go to the z-domain

$$\begin{aligned}
Y(z) &= z^{-1} + 5z^{-1}Y(z) - 6z^{-2}Y(z) \\
Y(z) &= \frac{z^{-1}}{1 - 5z^{-1} + 6z^{-2}} \\
&= \frac{z^{-1}}{(1 - 3z^{-1})(1 - 2z^{-1})} \\
&= \frac{1}{1 - 3z^{-1}} - \frac{1}{1 - 2z^{-1}} \\
\Rightarrow y[n] &= 3^n u[n] - 2^n u[n]
\end{aligned}$$

Therefore, for $n > 0$, the n^{th} term of the sequence is $3^n - 2^n$.

7. (5 pts) For each of the following statements, mark it as true or false.

True/False	Statement
True	$2^n * u[n] = 2^{n+1}$
False	All the poles and zeros of the transfer function of a stable and causal LTI system must be inside the unit circle.
True	If $h[n]$ is an even signal then $H(z) = H(1/z)$.
False	The impulse response of a stable LTI system must be right-sided.
True	$\left(\left(\frac{1}{2}\right)^n u[n]\right) * (\delta[n] - \frac{1}{2}\delta[n-1]) = \delta[n]$