EECS 50 - Discrete Time Signals and Systems - Spring 2016

Homework III

Student Name:
Student ID:

Instructions: Each problem is worth 20 points. Please attach this title page as a cover-sheet with your homework submission. Use extra sheets if needed to present your solutions. The table below is for grading purposes only.

Problem No.	Points
Problem 1	
Problem 2	
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Total Score	

1. The frequency response of an LTI system is given by

$$H(e^{j\Omega}) \quad = \quad (1+4j\sin(2\Omega))\cos\left(\frac{3\Omega}{2}\right)e^{-j\Omega/2}$$

(a) Find $h[n]$.

(b) Find the output $y_1[n]$ if the input to the system is $x_1[n] = 1 + \sin\left(\frac{\pi n}{2}\right)$

(c) Find the output $y_2[n]$ if the input to the system is $x_2[n]$ which has DTFT

$$X_2(e^{j\Omega}) \quad = \quad 1 - 2\cos(\Omega)$$

Solution:

(a)

$$\begin{aligned}
H(e^{j\Omega}) &= (1+4j\sin(2\Omega))\cos\left(\frac{3\Omega}{2}\right)e^{-j\Omega/2} \\
&= \left(1+4j\left(\frac{e^{j2\Omega}-e^{-j2\Omega}}{2j}\right)\right)\left(\frac{e^{j\frac{3\Omega}{2}}+e^{-j\frac{3\Omega}{2}}}{2}\right)e^{-j\Omega/2} \\
&= \frac{1}{2}\left(1+2e^{j2\Omega}-2e^{-j2\Omega}\right)\left(e^{j\Omega}+e^{-j2\Omega}\right) \\
&= \frac{1}{2}e^{j\Omega}+\frac{1}{2}e^{-j2\Omega}+e^{j3\Omega}+1-e^{-j\Omega}-e^{-j4\Omega} \\
\Rightarrow h[n] &= \frac{1}{2}\delta[n+1]+\frac{1}{2}\delta[n-2]+\delta[n+3]+\delta[n]-\delta[n-1]-\delta[n-4]
\end{aligned}$$

(b) We will use the fact that complex exponentials are eigenfunctions of LTI systems. $x_1[n] = 1 + \sin(\pi n/2) = e^{j0n} + \frac{1}{2j}e^{j\pi n/2} - \frac{1}{2j}e^{-j\pi n/2}$, so that

$$\begin{aligned}
y_1[n] &= H(e^{j0})e^{j0n}+\frac{1}{2j}H(e^{j\pi/2})e^{j\pi n/2}-\frac{1}{2j}H(e^{-j\pi/2})e^{-j\pi n/2} \\
&= 1+\frac{1}{2j}\cos(3\pi/4)e^{-j\pi/4}e^{j\pi n/2}-\frac{1}{2j}\cos(3\pi/4)e^{j\pi/4}e^{-j\pi n/2} \\
&= 1+\cos(3\pi/4)\sin(\pi n/2-\pi/4) \\
&= 1-\frac{1}{2}\sin(\pi n/2)+\frac{1}{2}\cos(\pi n/2)
\end{aligned}$$

(c)

$$\begin{aligned}
X_2(e^{j\Omega}) &= 1-2\cos(\Omega) \\
&= 1-(e^{j\Omega}+e^{-j\Omega}) \\
\Rightarrow x_2[n] &= \delta[n]-\delta[n+1]-\delta[n-1]
\end{aligned}$$

$$\begin{aligned}
y_2[n] &= x_2[n]*h[n] \\
&= (-\delta[n+1]+\delta[n]-\delta[n-1])*(\delta[n+3]+\frac{1}{2}\delta[n+1]+\delta[n]-\delta[n-1]+\frac{1}{2}\delta[n-2]-\delta[n-4]) \\
&= -\delta[n+4]+\delta[n+3]-\frac{3}{2}\delta[n+2]-\frac{1}{2}\delta[n+1]+\frac{3}{2}\delta[n]-\frac{5}{2}\delta[n-1] \\
&\quad +\frac{3}{2}\delta[n-2]+\frac{1}{2}\delta[n-3]-\delta[n-4]+\delta[n-5]
\end{aligned}$$

2. Consider the LTI system T_1 with input $x_1[n]$ and output $y_1[n] = \frac{1}{2}(x_1[n] - x_1[n-1])$.

   (a) Find the impulse response $h_1[n]$.

   (b) Find the frequency response $H_1(e^{j\Omega})$.

   (c) Identify the poles and zeros of the filter.

   (d) Sketch the magnitude response $|H_1(e^{j\Omega})|$.

   (e) What type of filter is this?

   (f) Sketch the phase response $\angle H_1(e^{j\Omega})$.

   (g) Write the difference equation for an LTI system T_2 whose frequency response is

$$H_2(e^{j\Omega}) = H_1\left(e^{j(\Omega-\pi)}\right)$$

Solution:

(a) Plug in $x_1[n] = \delta[n]$, we have $h_1[n] = \frac{1}{2}\delta[n] - \frac{1}{2}\delta[n-1]$.

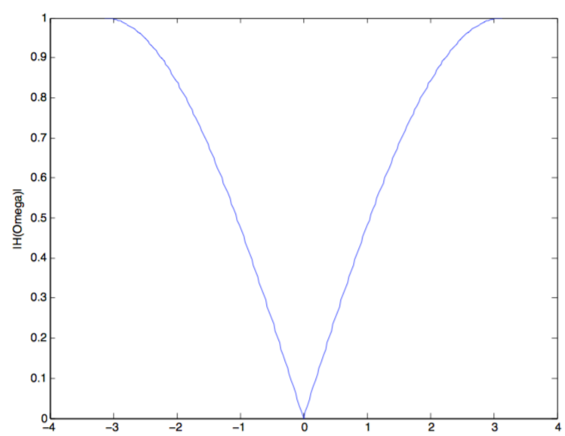
(b) $H_1(e^{j\Omega}) = \frac{1}{2} - \frac{1}{2}e^{-j\Omega}$

(c) $H_1(z) = \frac{1}{2} - \frac{1}{2}z^{-1} = \frac{z-1}{z}$. The zero is at $z = 1$ and the pole is at $z = 0$.

(d)

$$\begin{aligned} H_1(e^{j\Omega}) &= \frac{1}{2}e^{-j\Omega/2}(e^{j\Omega/2} - e^{-j\Omega/2}) \\ &= je^{-j\Omega/2}\sin(\Omega/2) \\ \Rightarrow |H_1(e^{j\Omega})| &= |\sin(\Omega/2)| \end{aligned}$$

which is plotted as follows.

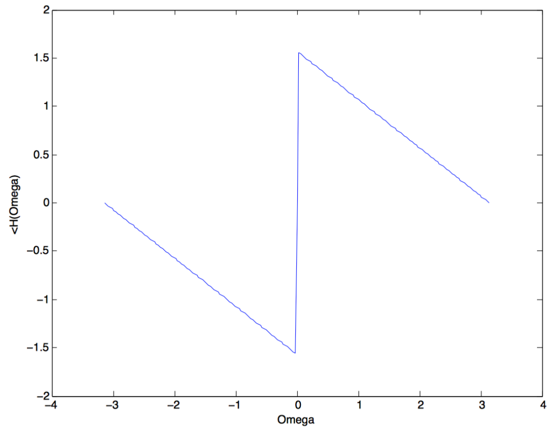


(e) This is a non-ideal high pass filter.

(f) As $H_1(e^{j\Omega}) = je^{-j\Omega/2}\sin(\Omega/2) = e^{j(\pi/2-\Omega/2)}\sin(\Omega/2)$, depending on whether $\sin(\Omega/2)$ is positive or negative, we have

$$\angle H_1(e^{j\Omega}) = \begin{cases} \pi/2 - \Omega/2, & 0 \le \Omega \le \pi \\ \pi/2 - \Omega/2 - \pi, & -\pi \le \Omega \le 0 \end{cases}$$

The phase response is plotted next.



(g) $H_2(e^{j\Omega}) = H_1(e^{j(\Omega-\pi)} = \frac{1}{2}(1 - e^{-j(\Omega-\pi)} = \frac{1}{2}(1 - e^{j\pi}e^{-j\Omega}) = \frac{1}{2}(1 + e^{-j\Omega})$. Replacing $e^{j\Omega}$ with z, we have $H_2(z) = \frac{1}{2}(1 + z^{-1})$. As $H_2(z) = Y_2(z)/X_2(z)$, we have

$$\begin{aligned} Y_2(z) &= \frac{1}{2}(X_2(z) + z^{-1}X_2(z)) \\ \Rightarrow y_2[n] &= \frac{1}{2}(x_2[n] + x_2[n-1]) \end{aligned}$$

3. Suppose that the frequency response of an LTI system is given by

$$H(e^{j\Omega}) = (1+e^{-j\Omega})(1-e^{-j(\Omega+\pi/3)})(1-e^{-j(\Omega-\pi/3)})$$

(a) What is the impulse response $h[n]$ of the system?

(b) Suppose that the input signal has the form $x[n] = Ae^{j\Omega_o n}$. For what values of Ω_o in the range $-\pi \le \Omega_o \le \pi$ will we have $y[n] = 0$ for all n?

(c) Find the output $y[n]$ of the system for the input $x[n] = 3 + \delta[n-2] + \cos(0.5\pi n + 0.25\pi)$.

Solution:

(a)

$$\begin{aligned}
H(e^{j\Omega}) &= (1+e^{-j\Omega})(1-e^{-j(\Omega+\pi/3)} - e^{-j(\Omega-\pi/3)} + e^{-j2\Omega}) \\
&= (1+e^{-j\Omega})[1-e^{-j\Omega}(e^{-j\pi/3}+e^{j\pi/3}) + e^{-j2\Omega}] \\
&= (1+e^{-j\Omega})[1-e^{-j\Omega}\; 2\cos(\pi/3) + e^{-j2\Omega}] \\
&= (1+e^{-j\Omega})(1-e^{-j\Omega} + e^{-j2\Omega}) \\
&= 1-e^{-j\Omega}+e^{-j2\Omega}+e^{-j\Omega}-e^{-j2\Omega}+e^{-j3\Omega} \\
&= 1+e^{-j3\Omega} \\
\Rightarrow h[n] &= \delta[n]+\delta[n-3]
\end{aligned}$$

(b) Complex exponentials are eigenfunctions of LTI systems.

$$y[n] = Ae^{j\Omega_o n} * h[n] = Ae^{j\Omega_o n}H(e^{j\Omega_o}) = (1+e^{-j3\Omega_o})Ae^{j\Omega_o n}$$

In order for $y[n] = 0$ for all n, we must have

$$\begin{aligned}
1+e^{-j3\Omega_o} &= 0 \\
\Rightarrow e^{-j3\Omega_o} &= -1 \\
\Rightarrow e^{-j3\Omega_o} &= e^{j(\pi+2k\pi)} \\
\Rightarrow \Omega_o &= -\pi/3 - 2k\pi/3, \text{ for all integer } k.
\end{aligned}$$

Choosing values only in the range $-\pi \le \Omega_o \le \pi$, we have $\Omega_o = \pm\pi/3, \pm\pi$.

(c)

$$\begin{aligned}
y[n] = x[n] * h[n] &= \{3+\delta[n-2]+\cos(0.5\pi n+0.25\pi)\} * \{\delta[n]+\delta[n-3]\} \\
&= 3+\delta[n-2]+\cos(0.5\pi n+0.25\pi)+3+\delta[n-5]+\cos(0.5\pi(n-3)+0.25\pi) \\
&= 6+\delta[n-2]+\delta[n-5]+\cos(0.5\pi n+0.25\pi)-\sin(0.5\pi n+0.25\pi) \\
&= 6+\delta[n-2]+\delta[n-5]-\sqrt{2}\sin(0.5\pi n)
\end{aligned}$$

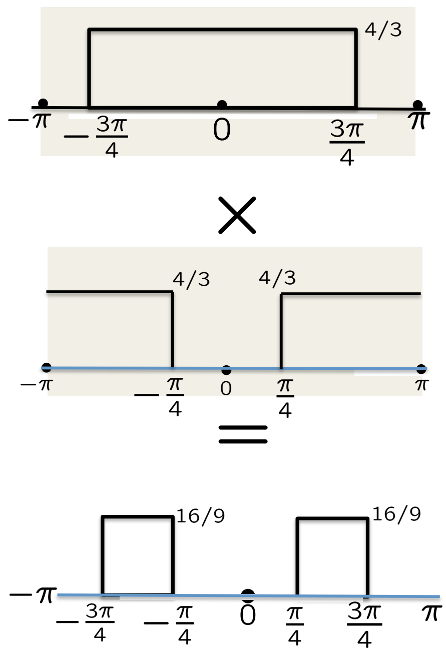
4. (a) Consider the LTI system with impulse response $h[n] = \text{sinc}(3\pi n/4) * ((-1)^n \text{sinc}(3\pi n/4))$. Find $H(e^{j\Omega})$ and roughly sketch $|H(e^{j\Omega})|$ and $\angle H(e^{j\Omega})$. What type of filter is this?

(b) Consider a causal LTI system whose transfer function $H(z) = 1 - \frac{0.1}{1-0.9z^{-1}}$. Find the pole and zero of the transfer function and roughly plot $|H(e^{j\Omega})|$. What type of filter is this?

(c) Consider the LTI system with input $x[n]$ and output $y[n] = \frac{x[n]+x[n+1]+x[n+2]}{3}$. Find $H(e^{j\Omega})$ and roughly sketch $|H(e^{j\Omega})|$ and $\angle H(e^{j\Omega})$.

Solution:

(a) Recognizing the terms as scaled versions of low pass and high pass filters, and noting that the convolution becomes a multiplication in frequency domain, we proceed as follows.

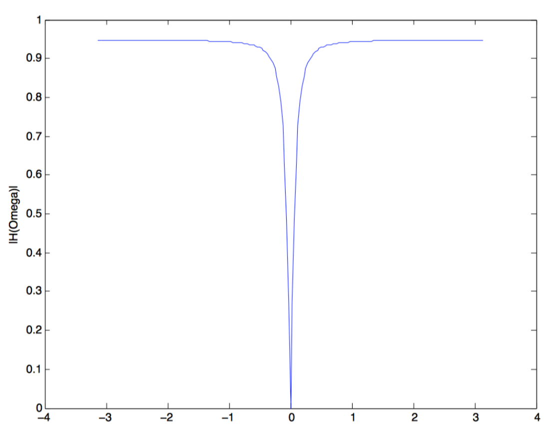


This is an ideal band pass filter. Since $H(e^{j\Omega})$ only takes non-negative real values, the magnitude response is the same as the frequency response. The phase is 0 throughout.

(b) $H(z) = \frac{1-0.9z^{-1}-0.1}{1-0.9z^{-1}} = 0.9\frac{z-1}{z-0.9}$.

The pole is at $z = 0.9$ and the zero is at $z = 1$.

$|H(e^{j\Omega})|$ is plotted below. It is a notch filter that filters out $\Omega = 0$.



(c) Taking the z-transform of the difference equation we have

$$
\begin{aligned}
Y(z) &= \frac{1}{3}(1+z+z^2)X(z) \\
\Rightarrow \frac{Y(z)}{X(z)} = H(z) &= \frac{z^2+z+1}{3} \\
\Rightarrow H(e^{j\Omega}) &= \frac{e^{2j\Omega}+e^{j\Omega}+1}{3} \\
&= \frac{e^{j\Omega}(e^{j\Omega}+1+e^{-j\Omega})}{3} \\
&= e^{j\Omega}\left(\frac{1+2\cos(\Omega)}{3}\right)
\end{aligned}
$$

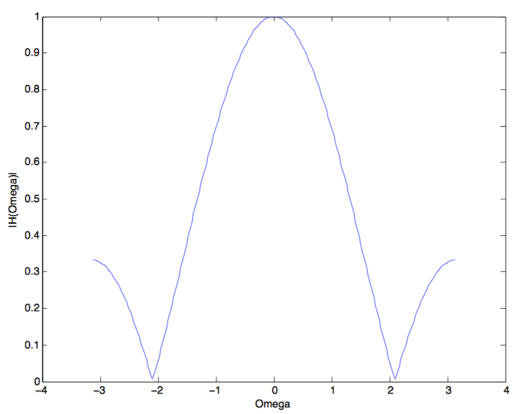
Then

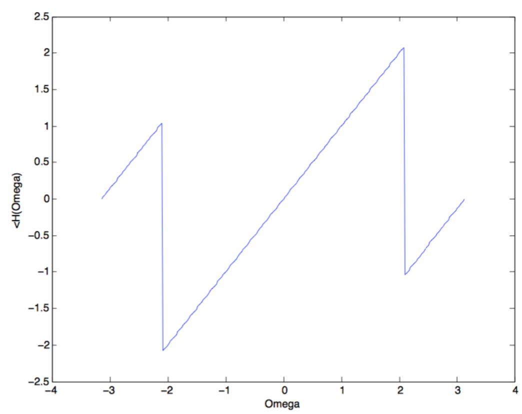
$$
\begin{aligned}
|H(e^{j\Omega})| &= \left|\frac{1+2\cos(\Omega)}{3}\right| \\
&= \begin{cases} \frac{1+2\cos(\Omega)}{3}, & 0 \le |\Omega| \le 2\pi/3 \\ \frac{-1-2\cos(\Omega)}{3}, & 2\pi/3 \le |\Omega| \le \pi \end{cases}
\end{aligned}
$$

and

$$
\angle H(e^{j\Omega}) = \begin{cases} \Omega, & 0 \le |\Omega| \le 2\pi/3 \\ \Omega \pm \pi, & 2\pi/3 \le |\Omega| \le \pi \end{cases}
$$

$|H(e^{j\Omega})|$ and $\angle H(e^{j\Omega})$ are plotted as follows.





5. Consider a discrete-time LTI system T that generates an output $y[n]$ according to

$$y[n] = bx[n] - ay[n-1] - \frac{a^2}{4}y[n-2]$$

where a, b are non-negative real constants.

(a) Find the poles of the z-transform of the impulse response $h[n]$ of T.

(b) Let $H(e^{j\Omega})$ be the frequency response of T. Find a, b so that the system is stable, $|H(e^{j0})| = 0.04$, and $|H(e^{j\pi})| = 1$.

Sol:

(a) Taking z-transforms

$$\begin{aligned} Y(z) &= bX(z) - az^{-1}Y(z) - \frac{a^2}{4}z^{-2}Y(z) \\ \Rightarrow H(z) = \frac{Y(z)}{X(z)} &= \frac{b}{1 + z^{-1}a + z^{-2}a^2/4} \\ &= \frac{bz^2}{z^2 + za + a^2/4} \\ &= \frac{bz^2}{(z + a/2)^2} \end{aligned}$$

Therefore the poles are at $z = -a/2$.

(b)

$$H(e^{j\Omega}) = \frac{be^{j2\Omega}}{(e^{j\Omega} + a/2)^2}$$

Since $|H(e^{j0})| = 0.04$, setting $\Omega = 0$, i.e., $e^{j\Omega} = 1$, we have

$$\begin{aligned} 0.04 &= \frac{b}{(1 + a/2)^2} \\ \Rightarrow b &= \frac{(1 + a/2)^2}{25} \end{aligned}$$

Similarly, since $|H(e^{j\pi})| = 1$, setting $\Omega = \pi$, i.e., $e^{j\Omega} = -1$, we have

$$\begin{aligned} 1 &= \frac{b}{(-1 + a/2)^2} \\ \Rightarrow b &= (-1 + a/2)^2 \\ \Rightarrow \frac{(1 + a/2)^2}{25} &= (-1 + a/2)^2 \\ \Rightarrow 1 + a/2 &= \pm 5(-1 + a/2) \\ \Rightarrow a &= 3, \text{ or} \\ a &= 4/3 \end{aligned}$$

For a causal and stable system the poles should be inside the unit circle. So we choose $a = 4/3$, which places the poles at $-a/2 = -2/3$, i.e., inside the unit circle. The value of $b = (-1 + a/2)^2 = (-1 + 2/3)^2 = 1/9$.

6. Let $X(e^{j\Omega})$ be the discrete-time Fourier Transform of a signal $x[n]$ which is known to be zero for $n < 0$ and $n > 3$. We know $X(e^{j\Omega})$ for four values of Ω as follows.

$$X(e^{j0}) = 10, \quad X(e^{j\pi/2}) = 5 - 5j, \quad X(e^{j\pi}) = 0, \quad X(e^{j3\pi/2}) = 5 + 5j$$

(a) Find $x[n]$. (Hint: Compute the IDFT)

(b) Find $X(e^{j\Omega})$.

Sol:

(a) $x[n]$ has length 4, i.e., only $x[0], x[1], x[2], x[3]$ are non-zero. To find $x[n]$ from $X(e^{j\Omega})$, we compute the IDFT. Note that $N = 4, W = e^{-j2\pi/N} = e^{-j\pi/2} = -j$. And $X(0) = X[1], X(\pi/2) = X(2\pi/N) = X[1], X(\pi) = X[2], X(3\pi/2) = X[3]$.

$$\begin{aligned}
\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} &= \frac{1}{4}\begin{bmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{bmatrix}^* \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}^* \begin{bmatrix} 10 \\ 5-5j \\ 0 \\ 5+5j \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 10 \\ 5-5j \\ 0 \\ 5+5j \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ 5 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

(b) $X(e^{j\Omega}) = \sum_{n=0}^{3} x[n]e^{-j\Omega} = 5 + 5e^{-j\Omega}$.

7. Consider the signal $x_1[n]$ which is zero for $n < 0$ and for $n > 3$. We know its DTFT values $X_1(e^{j\Omega})$ at certain values of Ω as follows.

$$X_1(e^{j0}) = 0, \quad X_1(e^{-j\pi/2}) = 2 + 4j,$$
$$X_1(e^{j\pi}) = 0, \quad X_1(e^{j\pi/2}) = 2 - 4j.$$

(a) Find the DTFT value $X_1(e^{j3\pi/2})$.

(b) Find the DFT values $X_1[0], X_1[1], X_1[2], X_1[3]$.

(c) Find $x_1[n]$

(d) Find $X_1(e^{j\Omega})$

(e) Explain in words the difference between the DTFT, the DFT and the FFT.

Solution:

(a) Because the DTFT is periodic in Ω with period 2π, $X_1(e^{j3\pi/2}) = X_1(e^{j(3\pi/2-2\pi)}) = X_1(e^{-j\pi/2}) = 2 + 4j$.

(b) $X_1[0] = X_1(e^{j0}) = 0$, $X_1[1] = X_1(e^{j\pi/2}) = 2-4j$, $X_1[2] = X_1(e^{j\pi}) = 0$, $X_1[3] = X_1(3\pi/2) = 2+4j$.

(c) Note that $N = 4$, $W = e^{-j2\pi/N} = e^{-j\pi/2} = -j$.

$$\begin{aligned}
\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} &= \frac{1}{4} \begin{bmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^1 & W^2 & W^3 \\ W^0 & W^2 & W^4 & W^6 \\ W^0 & W^3 & W^6 & W^9 \end{bmatrix}^* \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}^* \begin{bmatrix} 0 \\ 2-4j \\ 0 \\ 2+4j \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 0 \\ 2-4j \\ 0 \\ 2+4j \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}
\end{aligned}$$

(d) $X_1(e^{j\Omega}) = \sum_{n=0}^{3} x_1[n]e^{-j\Omega n} = 1 + 2e^{-j\Omega} - e^{-j2\Omega} - 2e^{-j3\Omega}$.

(e) DTFT is the Discrete-Time Fourier Transform, i.e., the Fourier Transform of a Discrete Time signal, which is generally a continuous and periodic function of Ω. DFT, or the Discrete Fourier Transform, is obtained by the sampling of DTFT in frequency, so it is discrete and periodic. FFT, or the Fast Fourier Transform, is an efficient algorithm to compute the DFT.