EECS 50 - Discrete Time Signals and Systems - Spring 2016

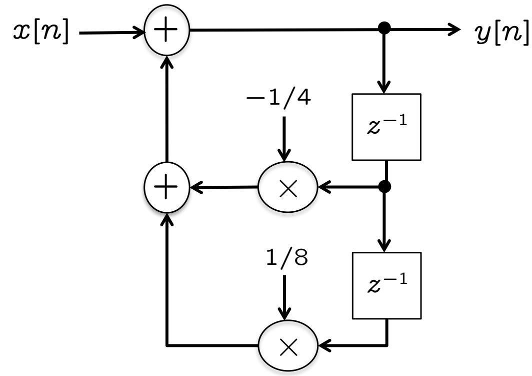
Homework II

Student Name:
Student ID:

Instructions: Each problem is worth 20 points. Please attach this title page as a cover-sheet with your homework submission. Use extra sheets if needed to present your solutions. The table below is for grading purposes only.

Problem No.	Points
Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	
Total Score	

1. The LTI system, T_o, shown below, is known to be causal.

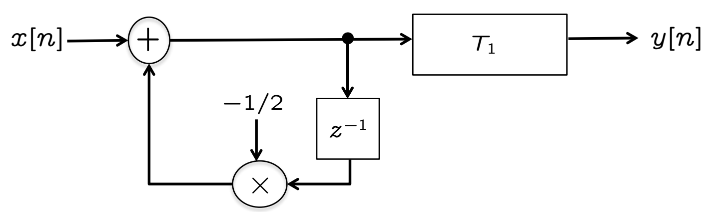


(a) Write the difference equation describing T_o.

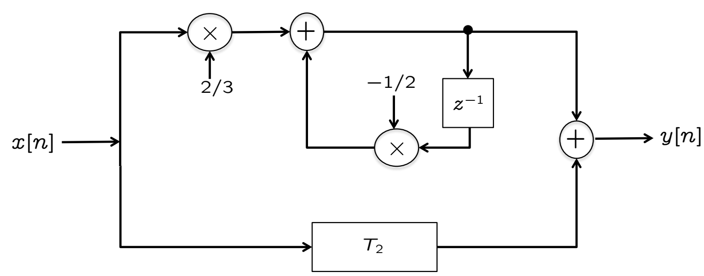
(b) Write the transfer function $H_o(z)$ for T_o.

(c) Is T_o a stable system?

(d) The system T_o is realized as a series connection of two systems as shown below. Draw the realization of system T_1.



(e) The system T_o is realized as a parallel connection of two systems as shown below. Draw the realization of system T_2.



Sol:

(a) Difference equation describing T_o:

$$y[n] = x[n] - \frac{1}{4}y[n-1] + \frac{1}{8}y[n-2]$$

(b) Taking z-transforms:

$$\begin{aligned} Y(z) &= X(z) - \frac{1}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) \\ \Rightarrow \frac{Y(z)}{X(z)} &= \frac{1}{1+\frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} = H_o(z) \end{aligned}$$

(c)

$$\begin{aligned} H_o(z) &= \frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} \\ &= \frac{1}{(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} \end{aligned}$$

So the poles are at $z=-\frac{1}{2}$ and at $z=\frac{1}{4}$. Since the system is causal, $h_o[n]$ is a right-sided signal. Therefore the Region of Convergence (ROC) must be $|z|>\frac{1}{2}$. Thus, the ROC includes the unit circle, and the system must be stable.

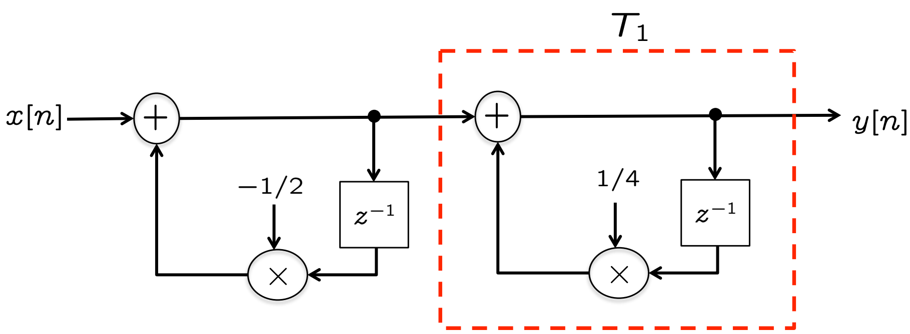
(d) Since

$$H_o(z) = \left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

and the given series realization already shows the part corresponding to $\frac{1}{1+\frac{1}{2}z^{-1}}$, the remaining part corresponds to system T_1. The transfer function of T_1 is therefore,

$$H_1(z) = \left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

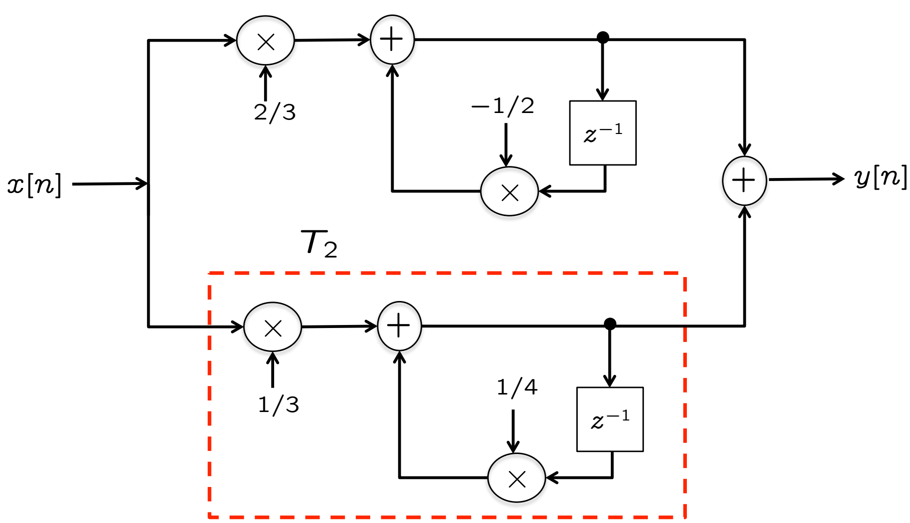
which is realized as shown below.



(e) A partial-fraction expansion of $H_o(z)$ yields

$$H_o(z) = \frac{\frac{2}{3}}{1+\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}}$$

which leads to the parallel form implementation as shown in the next figure.



2. Given the following z-transform of the impulse response $h_1[n]$, of a causal LTI system T_1.

$$H_1(z) = \frac{0.5z^2}{(z-1)(z-0.5)}$$

(a) Find $h_1[n]$.

(b) Verify the first three non-zero values of $h_1[n]$ using long division.

(c) Find the z transform $H_2(z)$, of $h_2[n] = h_1[n]u[n-3]$, and specify the ROC.

(d) Find the z transform $H_3(z)$ of $h_3[n] = 2^n h_1[n]$, and specify the ROC.

(e) Find the z transform $H_4(z)$ of $h_4[n] = 2^{n+1} h_1[n-1]$, and specify the ROC.

(f) Find the impulse response, $h_5[n]$, of the system T_5, which is the inverse of T_1. Verify that $T_5(T_1(\delta[n])) = \delta[n]$.

Sol:

(a)

$$\begin{aligned} H_1(z) &= \frac{0.5z^2}{(z-1)(z-0.5)} \\ &= z\left(\frac{z}{z-1} - \frac{z}{z-0.5}\right) \end{aligned}$$

Since the LTI system is causal, the impulse response $h[n]$ must be right-sided. Therefore,

$$\begin{aligned} h_1[n] &= \Big(u[n] - (0.5)^n u[n]\Big)_{n \to n+1} \\ &= u[n+1] - (0.5)^{n+1} u[n+1] \\ &= (1 - (0.5)^{n+1}) u[n+1] \end{aligned}$$

Note that $h_1[n] = 0$ for $n = -1$. So we can also equivalently write

$$h_1[n] = (1 - (0.5)^{n+1}) u[n]$$

Both answers are correct, and equivalent, although the second answer is preferable because it makes it explicit that $h_1[n] = 0$ for $n < 0$.

(b) Representing long division as follows

$$\begin{aligned} H_1(z) &= \frac{0.5z^2}{(z-1)(z-0.5)} \\ &= \frac{0.5z^2}{z^2 - 1.5z + 0.5} \\ &= 0.5 + \frac{0.75z - 0.25}{z^2 - 1.5z + 0.5} \\ &= 0.5 + 0.75z^{-1} + \frac{0.875 - 0.375z^{-1}}{z^2 - 1.5z + 0.5} \\ &= 0.5 + 0.75z^{-1} + 0.875z^{-2} + \frac{0.9375z^{-1} - 0.4375z^{-2}}{z^2 - 1.5z + 0.5} \\ &= \cdots \end{aligned}$$

Thus, we have $h_1[0] = 0.5$, $h_1[1] = 0.75$, $h_1[2] = 0.875$. The values match what we get from the closed form expression we found in the previous part of this problem, $h_1[n] = (1 - (0.5)^{n+1}) u[n]$.

(c) Multiplication by $u[n-3]$ nulls the first three terms corresponding to $h_1[0], h_1[1], h_1[[2]$.

$$\begin{aligned} h_2[n] &= h_1[n]u[n-3] \\ &= h_1[n] - h_1[0]\delta[n] - h_1[1]\delta[n-1] - h_1[2]\delta[n-2] \\ &= h_1[n] - 0.5\delta[n] - 0.75\delta[n-1] - 0.875\delta[n-2] \end{aligned}$$

$$\begin{aligned} \Rightarrow H_2(z) &= H_1(z) - 0.5 - 0.75z^{-1} - 0.875z^{-2} \\ &= \frac{0.9375z^{-1} - 0.4375z^{-2}}{z^2 - 1.5z + 0.5} \\ &= \frac{0.9375z^{-1} - 0.4375z^{-2}}{(z-1)(z-0.5)} \\ &= \frac{0.9375z - 0.4375}{z^2(z-1)(z-0.5)} \end{aligned}$$

Since $h_2[n]$ is still a right-sided signal, the ROC is outside the outermost pole, i.e., $|z| > 1$.

(d) Use the property shown in class, that

$$\begin{aligned} h[n] &\xrightarrow{\mathcal{Z}} H(z) \\ \Rightarrow a^n h[n] &\xrightarrow{\mathcal{Z}} H(z/a) \end{aligned}$$

Therefore, using this property we have

$$\begin{aligned} H_3(z) &= H_1(z/2) \\ &= \frac{0.125z^2}{(0.5z-1)(0.5z-0.5)} \\ &= \frac{0.5z^2}{(z-2)(z-1)} \end{aligned}$$

Since this is still a right-sided signal, the ROC is $|z| > 2$.

(e)

$$\begin{aligned} \text{As shown in previous part,}\quad 2^n h_1[n] &\xrightarrow{\mathcal{Z}} H_3(z) = \frac{0.5z^2}{(z-2)(z-1)} \\ \text{Shifting in time,}\quad 2^{n-1}h_1[n-1] &\xrightarrow{\mathcal{Z}} z^{-1} \times H_3(z) \\ \text{Multiplying by 4,}\quad 4 \times 2^{n-1}h_1[n-1] &\xrightarrow{\mathcal{Z}} 4 \times z^{-1}H_3(z) \\ &= \frac{2z}{(z-2)(z-1)} \end{aligned}$$

ROC: $|z| > 2$.

(f)

$$\begin{aligned} H_5(z) &= \frac{1}{H_1(z)} \\ &= \frac{z^2 - 1.5z + 0.5}{0.5z^2} \\ &= \frac{2z^2 - 3z + 1}{z^2} \\ &= 2 - 3z^{-1} + z^{-2} \\ \Rightarrow h_5[n] &= 2\delta[n] - 3\delta[n-1] + \delta[n-2] \end{aligned}$$

Next we verify if $T_5(T_1(\delta[n])) = \delta[n]$.

$$
\begin{aligned}
T_5(T_1(\delta[n])) &= T_5(h_1[n]) \\
&= h_5[n] * h_1[n] \\
&= \Big(2\delta[n] - 3\delta[n-1] + \delta[n-2]\Big) * \Big(1 - (0.5)^{n+1}\Big)u[n] \\
&= 2\Big(1 - (0.5)^{n+1}\Big)u[n] - 3\Big(1 - (0.5)^{n}\Big)u[n-1] + \Big(1 - (0.5)^{n-1}\Big)u[n-2]
\end{aligned}
$$

For $\mathbf{n} < 0$,

$$T_5(T_1(\delta[n])) = 0,$$

For $\mathbf{n} = 0$,

$$
\begin{aligned}
T_5(T_1(\delta[n])) &= 2(1 - 0.5) \\
&= 1
\end{aligned}
$$

For $\mathbf{n} = 1$,

$$
\begin{aligned}
T_5(T_1(\delta[n])) &= 2(1 - 0.25) - 3(1 - 0.5) \\
&= 0
\end{aligned}
$$

For $\mathbf{n} \geq 2$,

$$
\begin{aligned}
T_5(T_1(\delta[n])) &= 2(1 - 0.5^{n+1}) - 3(1 - 0.5^{n}) + (1 - 0.5^{n-1}) \\
&= 2 - 0.5^{n} - 3 + 3(0.5)^{n} + 1 - 2(0.5^{n}) \\
&= 0
\end{aligned}
$$

Therefore, since $T_5(T_1(\delta[n]))$ takes the value 1 when $n = 0$, and is zero everywhere else, it is equal to $\delta[n]$.

3. Consider an LTI system described by the difference equation

$$y[n] - 2.5y[n-1] + y[n-2] = 3x[n] - 3x[n-1]$$

(a) Draw a direct form I representation for this system.

(b) Draw a direct form II representation for this system.

(c) From the difference equation, use z-transform to find the transfer function and all possible Regions of Convergence.

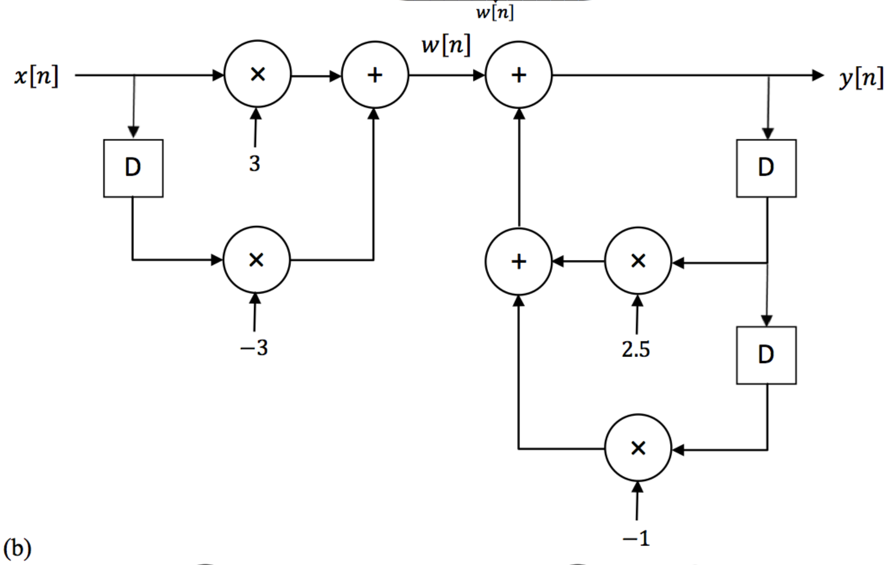
(d) Find the impulse response of the LTI system corresponding to each ROC. In each case identify if the system is stable, causal.

(e) Assuming that the LTI system is causal, find the output, $y_1[n]$, if the input is $x_1[n] = 3^n$.

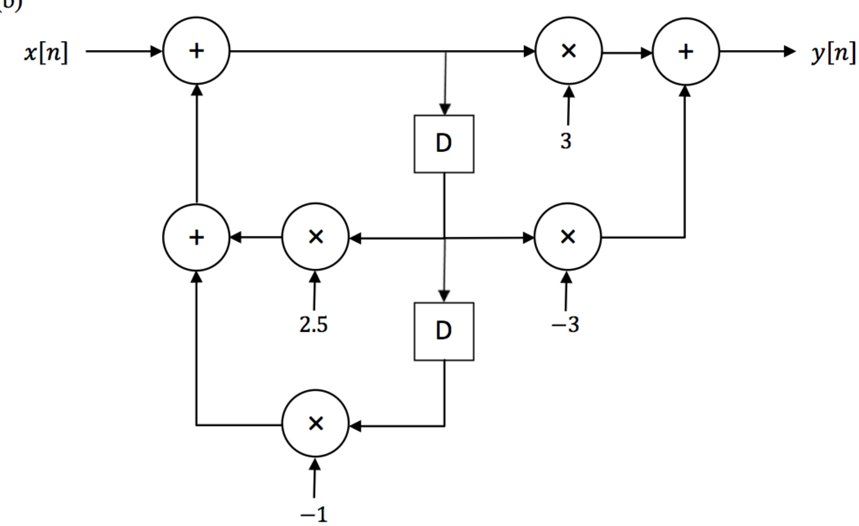
(f) Assuming that the LTI system is causal, find the output, $y_2[n]$, if the input is $x_2[n] = 3^n u[n]$.

Sol:

(a) $y[n] = 2.5\,y[n-1] - y[n-2] + \underbrace{3\,x[n] - 3\,x[n-1]}_{w[n]}$



(b)



(c) Taking the z-transform of the difference equation, we have.

$$y[n] - 2.5y[n-1] + y[n-2] = 3x[n] - 3x[n-1]$$

$$
\begin{aligned}
\Rightarrow Y(z) - 2.5z^{-1}Y(z) + z^{-2}Y(z) &= 3X(z) - 3z^{-1}X(z) \\
\Rightarrow Y(z)\left(1 - 2.5z^{-1} + z^{-2}\right) &= X(z)\left(3 - 3z^{-1}\right) \\
\Rightarrow H(z) = \frac{Y(z)}{X(z)} &= \frac{3 - 3z^{-1}}{1 - 2.5z^{-1} + z^{-2}} \\
&= \frac{3z^2 - 3z}{z^2 - 2.5z + 1} \\
&= \frac{3z^2 - 3z}{(z - 0.5)(z - 2)}
\end{aligned}
$$

$H(z)$ has poles at $z = 0.5$, $z = 2$. Therefore, three different ROCs are possible.

$$
\begin{aligned}
\mathrm{ROC}_1 : &\quad |z| > 2 \\
\mathrm{ROC}_2 : &\quad 2 > |z| > 0.5 \\
\mathrm{ROC}_3 : &\quad 0.5 > |z|
\end{aligned}
$$

(d) To find $h[n]$ we will first do a partial fraction expansion of $H(z)$.

$$
\begin{aligned}
H(z) &= \frac{3z^2 - 3z}{(z - 0.5)(z - 2)} \\
&= z\left(\frac{3z - 3}{(z - 0.5)(z - 2)}\right) \\
&= z\left(\frac{1}{z - 0.5} + \frac{2}{z - 2}\right) \\
&= \frac{z}{z - 0.5} + \frac{2z}{z - 2}
\end{aligned}
$$

From this point on, we will have three different answers depending on ROC.

For $\mathrm{ROC}_1 : |z| > 2$, $h_1[n]$ must be right-sided.

$$h_1[n] = 0.5^n u[n] + 2(2^n)u[n]$$

This LTI system is causal because $h_1[n] = 0$ for $n < 0$. It is not stable because ROC_1 does not include the unit circle $|z| = 1$.

For $\mathrm{ROC}_2 : 2 > |z| > 0.5$, the part with the pole at $z = 0.5$ must be right-sided and the part with the pole at $z = 2$ must be left-sided.

$$h_2[n] = 0.5^n u[n] - 2(2^n)u[-n-1]$$

This LTI system is not causal because $h[n] \neq 0$ for $n < 0$. It is stable because ROC_2 includes the unit circle $|z| = 1$.

For $\mathrm{ROC}_3 : 0.5 > |z|$, $h_3[n]$ must be left-sided.

$$h_3[n] = -0.5^n u[-n-1] - 2(2^n)u[-n-1]$$

This LTI system is not causal because $h[n] \neq 0$ for $n < 0$. It is not stable because ROC_3 does not include the unit circle $|z| = 1$.

(e) Exponentials (**without u[n]**) are eigenfunctions of LTI systems. The input a^n to an LTI system with transfer function $H(z)$ will produce the output $H(a)a^n$ if a is in the region of convergence of $H(z)$.

For causal LTI system, ROC must be ROC_1, i.e., $|z| > 2$. Here $a = 3$, which is in the ROC of $H(z)$. So the output is

$$\begin{aligned}
x_1[n] = 3^n \longrightarrow y_1[n] &= H(3)3^n \\
&= \left(\frac{3(3^2) - 3(3)}{3^2 - 2.5(3) + 1}\right) 3^n \\
&= \left(\frac{36}{5}\right) 3^n
\end{aligned}$$

(f) Using z-transforms

$$\begin{aligned}
Y_2(z) &= H(z)X_2(z) \\
&= \left(\frac{z}{z - 0.5} + \frac{2z}{z - 2}\right)\left(\frac{z}{z - 3}\right) \\
&= z\left(\frac{z}{(z - 0.5)(z - 3)}\right) + 2z\left(\frac{z}{(z - 2)(z - 3)}\right) \\
&= \frac{z}{5}\left(\frac{6}{z - 3} - \frac{1}{z - 0.5}\right) + 2z\left(\frac{3}{z - 3} - \frac{2}{z - 2}\right) \\
&= \frac{1}{5}\left(\frac{6z}{z - 3} - \frac{z}{z - 0.5}\right) + 2\left(\frac{3z}{z - 3} - \frac{2z}{z - 2}\right) \\
&= \frac{36}{5}\left(\frac{z}{z - 3}\right) - \frac{1}{5}\left(\frac{z}{z - 0.5}\right) - 4\left(\frac{z}{z - 2}\right) \\
\Rightarrow y_2[n] &= \left(\frac{36}{5}\right) 3^n u[n] - \frac{1}{5}(0.5)^n u[n] - 4(2^n)u[n]
\end{aligned}$$

4. T_1, T_2, T_3 are causal LTI systems. For T_1, the input $x_1[n]$ is related to the output $y_1[n]$ as follows.

$$y_1[n] = 0.5y_1[n-1] + x_1[n-1]$$

T_2 has the transfer function $H_2(z) = \frac{z^{-1}}{1-2z^{-1}}$, and T_3 has the impulse response $h_3[n] = \delta[n] + \delta[n-1]$. From these systems, an overall LTI system T is composed as follows.

$$T(x[n]) = T_3\left(T_1(x[n]) + T_2(x[n])\right)$$

(a) Find the difference equation for the overall system T.

(b) Find the impulse response $h[n]$ of the overall system T and find its z-transform $H(z)$.

(c) Is the overall system BIBO stable?

Sol:

(a) and (b) First we obtain the transfer function $H(z)$ of the overall system T. Since $T(x[n]) = T_3(T_1(x[n]) + T_2(x[n]))$, for any input $x[n]$ we have

$$\begin{aligned} h[n] * x[n] &= h_3[n] * (T_1(x[n]) + T_2(x[n])) \\ &= h_3[n] * (h_1[n] * x[n] + h_2[n] * x[n]) \\ &= h_3[n] * (h_1[n] + h_2[n]) * x[n] \end{aligned}$$

By taking z-transforms of the two sides of the above equality, we obtain

$$\begin{aligned} H(z)X(z) &= H_3(z)\Big[H_1(z) + H_2(z)\Big]X(z) \\ \Rightarrow H(z) &= H_3(z)\Big[H_1(z) + H_2(z)\Big] \qquad (1) \end{aligned}$$

Now, let us find $H_1(z)$. By taking z-transform of the two sides of the difference equation for T_1, we obtain

$$\begin{aligned} Y_1(z) &= 0.5z^{-1}Y_1(z) + z^{-1}X_1(z) \\ \Rightarrow (1-0.5z^{-1})Y_1(z) &= z^{-1}X_1(z) \\ \Rightarrow H_1(z) &= \frac{Y_1(z)}{X_1(z)} = \frac{z^{-1}}{1-0.5z^{-1}} \qquad (2) \end{aligned}$$

The transfer functions for T_2, T_3 are given by

$$\begin{aligned} H_2(z) &= \frac{z^{-1}}{1-2z^{-1}} \qquad (3) \\ H_3(z) &= 1 + z^{-1} \qquad (4) \end{aligned}$$

By plugging in (2), (3), (4) into (1) we find that $H(z)$ is given by

$$\begin{aligned} H(z) &= (1+z^{-1})\left(\frac{z^{-1}}{1-0.5z^{-1}} + \frac{z^{-1}}{1-2z^{-1}}\right) \qquad (5) \\ &= \frac{(z+1)(2z-2.5)}{z(z-0.5)(z-2)} \text{ (upon simplification)} \\ &= \frac{-2.5-0.5z+2z^2}{z-2.5z^2+z^3} \qquad (6) \end{aligned}$$

Since $H(z) = Y(z)/X(z)$, from the transfer function we can find the difference equation for the overall system as follows.

$$\begin{aligned}
\frac{Y(z)}{X(z)} &= \frac{-2.5 - 0.5z + 2z^2}{z - 2.5z^2 + z^3} \\
\Rightarrow (z - 2.5z^2 + z^3)Y(z) &= (-2.5 - 0.5z + 2z^2)X(z) \\
\Rightarrow zY(z) - 2.5z^2Y(z) + z^3Y(z) &= -2.5X(z) - 0.5zX(z) + 2z^2X(z) \\
\Rightarrow y[n+1] - 2.5y[n+2] + y[n+3] &= -2.5x[n] - 0.5x[n+1] + 2x[n+2] \qquad (7) \\
\text{Equivalently, } y[n-2] - 2.5y[n-1] + y[n] &= -2.5x[n-3] - 0.5x[n-2] + 2x[n-1] \qquad (8)
\end{aligned}$$

Note that there are multiple equivalent (correct) ways to write the difference equation.

For a causal system, the impulse response is right-sided. Using this, we find the impulse response $h[n]$ starting from the partial-fractions in (5) as follows.

$$\begin{aligned}
H(z) &= (1 + z^{-1})\left(\frac{z^{-1}}{1 - 0.5z^{-1}} + \frac{z^{-1}}{1 - 2z^{-1}}\right) \qquad (9) \\
&= (z^{-1} + z^{-2})\left(\frac{z}{z - 0.5} + \frac{z}{z - 2}\right) \qquad (10) \\
\Rightarrow h[n] &= \Big(0.5^n u[n] + 2^n u[n]\Big)_{n \to n-1} + \Big(0.5^n u[n] + 2^n u[n]\Big)_{n \to n-2} \qquad (11) \\
&= 0.5^{n-1}u[n-1] + 2^{n-1}u[n-1] + 0.5^{n-2}u[n-2] + 2^{n-2}u[n-2] \qquad (12) \\
&= \begin{cases} 0, & n < 0 \\ 2, & n = 1 \\ 0.5^{n-1} + 2^{n-1} + 0.5^{n-2} + 2^{n-2}, & n \geq 2 \\ = 6(0.5)^n + 0.75(2)^n & \end{cases} \qquad (13) \\
\Rightarrow h[n] &= 2\delta[n-1] + \left(\frac{6}{2^n} + \left(\frac{3}{4}\right)2^n\right)u[n-2] \qquad (14)
\end{aligned}$$

Note that there are many other (equivalent, correct) ways to represent $h[n]$ as well, such as

$$h[n] = 2\delta[n-1] + 3(2^{1-n} + 2^{n-2})u[n-2]$$

(c) For a causal LTI system to be stable, all the poles must lie inside the unit circle. However, $H(z)$ has a pole at $z = 2$. So the overall system is not BIBO stable.

5. Using z-transforms, find a closed form expression for the n^{th} term of the sequence: $1, 2, 5, 12, 29, \cdots$, which follows the rule $y[n] = 2y[n-1] + y[n-2]$.

Sol: Let us set it up as an LTI system that is initially at rest, so $x[n] = y[n] = 0$ for $n < 0$, and then plug in various values of n to determine what $x[n]$ is needed.

$$\begin{aligned}
y[n] &= 2y[n-1] + y[n-2] + x[n] \\
0 = y[0] &= 2y[-1] + y[-2] + x[0] = x[0], \Rightarrow x[0] = 0 \\
1 = y[1] &= 2y[0] + y[-1] + x[1] = x[1], \Rightarrow x[1] = 1 \\
2 = y[2] &= 2y[1] + y[0] + x[2] = 2 + x[2], \Rightarrow x[2] = 0 \\
5 = y[3] &= 2y[2] + y[1] + x[3] = 4 + 1 + x[2], \Rightarrow x[3] = 0 \\
&\vdots
\end{aligned}$$

Proceeding this way, we determine that $x[n] = \delta[n-1]$. Now let us use this to find $y[n]$ by employing the z-transform.

$$\begin{aligned}
y[n] &= 2y[n-1] + y[n-2] + \delta[n-1] && (15) \\
\Rightarrow Y(z) &= 2z^{-1}Y(z) + z^{-2}Y(z) + z^{-1} && (16) \\
\Rightarrow Y(z)\left(1 - 2z^{-1} - z^{-2}\right) &= z^{-1} && (17) \\
\Rightarrow Y(z) &= \frac{z^{-1}}{1 - 2z^{-1} - z^{-2}} && (18) \\
&= \frac{z}{z^2 - 2z - 1} && (19) \\
&= \frac{z}{(z - (1+\sqrt{2}))(z - (1-\sqrt{2}))} && (20) \\
&= \frac{z}{(1+\sqrt{2}) - (1-\sqrt{2})}\left(\frac{1}{z - (1+\sqrt{2})} - \frac{1}{z - (1-\sqrt{2})}\right) && (21) \\
&= \frac{1}{2\sqrt{2}}\left(\frac{z}{z - (1+\sqrt{2})} - \frac{z}{z - (1-\sqrt{2})}\right) && (22) \\
& && (23) \\
\Rightarrow y[n] &= \frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right)u[n] && (24)
\end{aligned}$$

Therefore, for $n = 1, 2, 3, \cdots$, the n^{th} term of the sequence is

$$\frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right) \qquad (25)$$