EECS 55 – Engineering Probability
Homework #8
Solutions

(1)

The expected value of S is given by
$E[S] = E[3X - 2Y + Z] = 3\,E[X] - 2\,E[Y] + E[Z] = (3 \times 3) - \big(2 \times (-2)\big) + 1 = 14$

Also, since X, Y, Z are independent, we can calculate $VAR(S)$ as
$VAR(S) = VAR(3X - 2Y + Z) = 9\,VAR(X) + 4\,VAR(Y) + VAR(Z) = (9 \times 4) + (4 \times 6) + 3 = 63$

Therefore, considering that X, Y, Z are independent normal random variables, S = 3X-2Y+Z will also be a normal random variable with mean 14 and variance 63. S~N(14, 63).

(2)

The CDF of Z is given by

$F_Z(a) = P\{Z \le a\} = P\{X+Y \le a\} = \int_0^a \int_0^{a-x} e^{-y}\,dy\,dx = a - 1 + e^{-a}, \; 0 \le a \le 1$

$F_Z(a) = \int_0^1 \int_0^{a-x} e^{-y}\,dy\,dx = 1 - e^{-a}(e-1), \; a > 1$

Therefore, the pdf of Z can be obtained by differentiating $F_Z(a)$

$$f_Z(a) = \frac{d}{da}F_Z(a) = f(x) = \begin{cases} 1 - e^{-a}, & 0 \le a \le 1 \\ e^{-a}(e-1), & a > 1 \end{cases}$$

(3)

(a) $f(x,y) = xe^{-x(y+1)}$ cannot be written as the product of two functions g(x) and h(y). Therefore, X and Y are not independent.

(b) $f_{X|Y}(x \mid y) = \dfrac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}\,dx} = (y+1)^2\,xe^{-x(y+1)}, 0 < x$

(c) $f_{Y|X}(y \mid x) = \dfrac{xe^{-x(y+1)}}{\int_0^\infty xe^{-x(y+1)}\,dy} = xe^{-xy}, 0 < y$

(4)

(a) Let $u = xy$ and $v = x/y$. Then,

$$J(x,y) = \begin{vmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = -\frac{x}{y} - \frac{x}{y} = -\frac{2x}{y}$$

and $x = \sqrt{uv}$, $y = \sqrt{u/v}$. Hence,

$$f_{U,V}(u,v) = f\left(\sqrt{uv}, \sqrt{u/v}\right) \cdot \left| -\frac{2\sqrt{uv}}{\sqrt{u/v}} \right|^{-1} = \frac{1}{(uv)(u/v)} \cdot \frac{\sqrt{u/v}}{2\sqrt{uv}} = \frac{1}{2u^2 v},$$

$$u \geq 1, \qquad u \geq v \geq \frac{1}{u}$$

and $f_{U,V}(u,v) = 0$ otherwise.

(b) The marginal density of U is given by

$f_U(u) = \int_{1/u}^{u} \frac{1}{2u^2 v}\, dv = \frac{1}{u^2} \log u, \quad u \geq 1$

$f_U(u) = 0, \quad u < 1$

Similarly, the marginal density of V is given by

$f_V(v) = \int_{v}^{\infty} \frac{1}{2u^2 v}\, du = \frac{1}{2v^2}, \quad v \geq 1$

$f_V(v) = \int_{1/v}^{\infty} \frac{1}{2u^2 v}\, du = \frac{1}{2}, \quad 0 < v < 1$

$f_V(v) = 0, \quad v \leq 0$

(5)

$$\text{E[XY]} = \int_0^\infty \int_o^x 2xye^{-x}e^{-y}\,dydx = \int_0^\infty 2xe^{-x}(1 - e^{-x} - xe^{-x})dx = 1$$

$$\text{E[X]} = \int_0^\infty \int_o^x 2xe^{-x}e^{-y}\,dydx = \int_0^\infty 2xe^{-x}(1 - e^{-x})dx = 3/2$$

$$\text{E[Y]} = \int_0^\infty \int_y^\infty 2ye^{-x}e^{-y}\,dxdy = \int_0^\infty 2ye^{-2y}dy = 1/2$$

Cov(X,Y) = E[XY] – E[X]E[Y] = 1-(3/2)(1/2) = 1/4

$$\text{E}[\text{X}^2] = \int_0^\infty \int_o^x 2x^2e^{-x}e^{-y}\,dydx = \int_0^\infty 2x^2e^{-x}(1 - e^{-x})dx = 14/4$$

$$\text{E}[\text{Y}^2] = \int_0^\infty \int_y^\infty 2y^2e^{-x}e^{-y}\,dxdy = \int_0^\infty 2y^2e^{-2y}dy = 1/2$$

In order to find E[X], E[X^2], E[Y], E[Y^2] you could have also found $f_X(x)$, $f_Y(y)$ and used them to find the expectations.

Var(X) = E[X^2] – $(E[X])^2$ = 5/4
Var(Y) = E[Y^2] – $(E[Y])^2$ = ¼

$$\rho_{X,Y} = \frac{\frac{1}{4}}{\sqrt{\frac{5}{4}}\sqrt{\frac{1}{4}}} = \frac{1}{\sqrt{5}}$$