Discrete Fourier Transform and Signal Processing

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Introduction

- **Signal:** An function $f : \mathbb{Z}_N \to \mathbb{C}$ of length N. We can also identity a signal with with its values $(x_0, x_1, x_2, x_3, \dots, x_{N-1})$ where $x_j = f(j)$.
- **Discrete Fourier transform:** The Fourier transform of a signal $\hat{f} : \mathbb{Z}_N \to \mathbb{C}$ has values

$$\hat{f}(m) = \frac{1}{N} \sum_{x \in \mathbb{Z}_N} e^{\frac{-2\pi i m x}{N}} f(x).$$

We call $(\omega_0, \omega_1, \dots, \omega_{N-1})$ the frequencies where $\omega_j = \hat{f}(j)$.

- **Fourier inversion:** We can recover a signal given the full set of frequencies by the inversion formula

$$f(x) = \sum_{x \in \mathbb{Z}_N^d} \chi(x \cdot m) \hat{f}(m)$$

- **Plancherel's identity:** The signal and its frequencies are related with the identity

$$\sum_{x \in \mathbb{Z}_N} |f(x)|^2 = N \sum_{m \in \mathbb{Z}_N} |\hat{f}(m)|^2$$

- **Support of a signal:**

$$supp f = \{x \in \mathbb{Z}_N \mid f(x) \neq 0\}.$$

For any signal f, we have $|supp f| \leq N$. Equivalently, $|supp f|$ is the number of nonzero entries in (x_n).

Fast Fourier Transform (FFT)

- The **fast Fourier transform** is a method that was developed as a means of efficiently calculating the Fourier coefficients of a signal F on $\mathbb{Z}_N$.
- We begin with a naive approach to the problem. Fix N, and suppose that we are given $F(0), \dots, F(N-1)$ and $\omega_N = e^{-2\pi i/N}$. If we denote by $a_k^N(F)$ the k-th Fourier coefficient of F on $\mathbb{Z}(N)$, then by definition

$$a_k^N(F) = \frac{1}{N} \sum_{r=0}^{N-1} F(r) \omega_N^{kr},$$

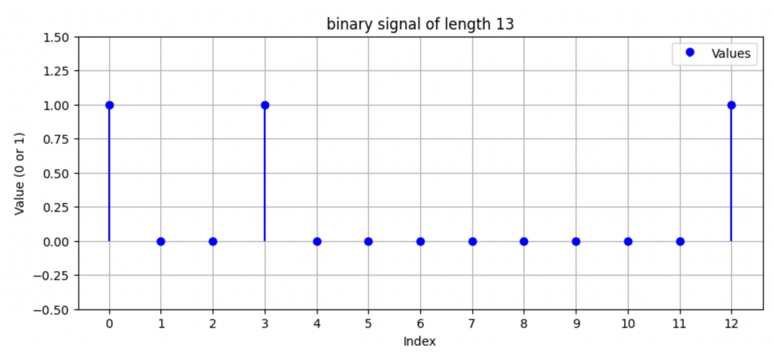
- **Naive Computation:** For a signal F defined on $\mathbb{Z}_N$ with N values, directly calculating Fourier coefficients requires $\leq 2N^2 + N$operations.
- **Lemma(Recursive Calculation):** If we are given $\omega_{2M} = e^{-2\pi i/(2M)}$, then $\#(2M) \leq 2\#(M) + 8M$.
- **Theorem:** Given $\omega_N = e^{-2\pi i/N}$ with $N = 2^n$, it is possible to calculate the Fourier coefficients of a function on $\mathbb{Z}(N)$ with at most

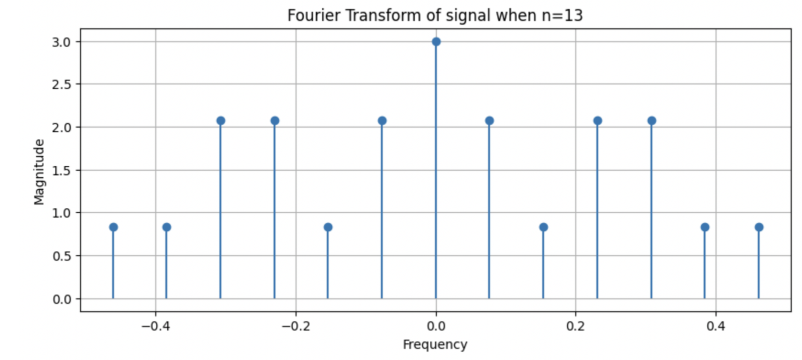
$$4 \cdot 2^n n = 4N \log_2(N) = O(N \log N)$$

operations.

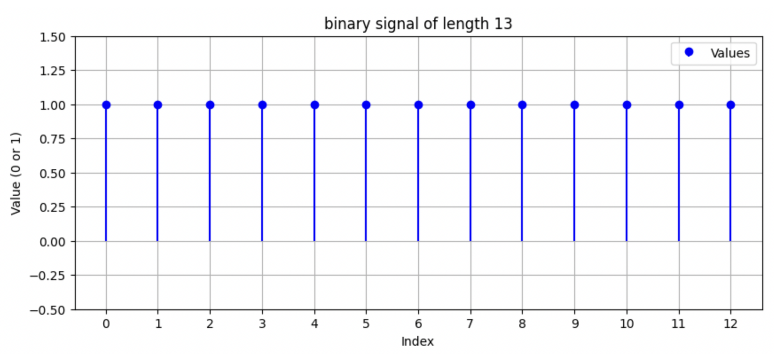
- **Foundation of FFT's Efficiency:** This recurrence relation is essential for proving that the FFT algorithm can achieve $O(N \log N)$ complexity instead of $O(N^2)$. By assuming $\#(M)$ operations for each half, we ensure that the total number of operations grows logarithmically with respect to the input size, making the algorithm much faster.

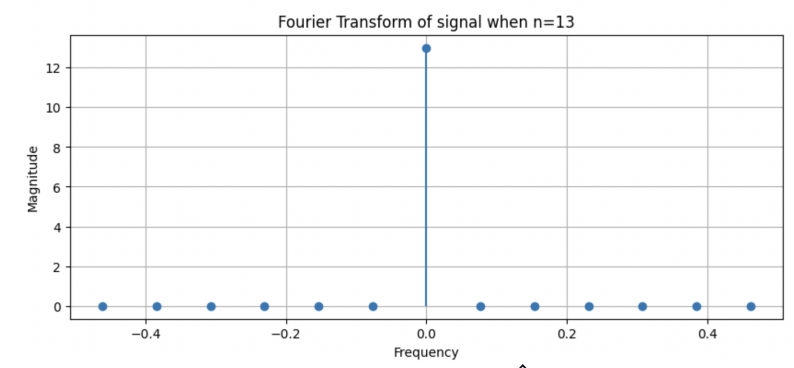
Visualization





Randomly generated binary signal with $N = 13$. We have $|supp f||supp \hat{f}| = 3 \cdot 13 > 13$.





Extremizers for UP with $N = 13$. We have $|supp f| = 13$, and $|supp \hat{f}| = 1$.

Signal Recovery and Uncertainty Principles

- **Motivation** Alice sends a signal to Bob by decomposing the signal (x_n) into its frequencies (ω_n). However, a subset of frequencies $W \subset \mathbb{Z}_N$ are unobserved on Bob's side due to white noise or interference. When is it possible for Bob to **reconstruct** the original signal sent by Alice?
- Suppose Bob knows there are N_t nonzero entries in the signal Alice sent (but not the positions or amplitudes!). Bob constructs a signal g given his observed frequencies satisfying
  1. g has also N_t nonzero entries.
  2. $\hat{g}(n) = \hat{f}(n)$ on the observed frequencies $n \notin W$.

  If g is **unique**, then the signal f can be **uniquely recovered** and most importantly $f = g$, thus Bob reconstructed the original signal of Alice.
- Two natural questions arise:
  1. Under what conditions is the recovered signal unique?
  2. If recovery is unique, what is an algorithm to reconstruct the signal?
- **Donoho Stark Uncertainty Principle:** Let (x_n) be a length N signal with N_t nonzero entries. Suppose Bob can not observe N_ω of the frequencies (ω_k). Then the signal (x_k) can be **uniquely recovered** from the observed frequencies if

$$N_t N_\omega < \frac{N}{2}.$$

Notably, no more than half the frequencies can unobserved for recovery. This address the first question with quantitative conditions.

- **Classical Uncertainty Principle:** For any nonzero length N signal f, we have

$$|supp f||supp \hat{f}| \geq N.$$

- UP $\implies$ a signal and its frequencies all together can not have too many zeros entries.
- The UP inequality is extremized by taking a constant signal (see figure above) where $|supp f||supp \hat{f}| = N$. Extremizers are rare, we expect generic signal will have $|supp f||supp \hat{f}| > N$.

Binary Signals and Recovery By DRA

- A transmitted binary signal $(x_0, x_1, \dots, x_{N-1})$ can be realized as characteristic functions on subset of $E \subset \mathbb{Z}_N$ where the amplitude is 1;

$$E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E. \end{cases}$$

- **Donoho-Stark Uncertainty for Binary Signals:** Consider the following more restrictive conditions:
  1. Alice is sending an binary signal E with N_t nonzero entries.
  2. Bob can not observe N_ω of the frequencies.
  3. $N_t N_\omega < \frac{N}{4}$

  Then Bob can recover E by an direct rounding algorithm due to Donoho-Stark (**DRA**). We describe the rounding algorithm below.
- Alice attempts to transmit the signal E to Bob. Let $S \subset \mathbb{Z}_N$ be the set of **unobserved frequencies** for Bob, and for $B := \mathbb{Z}_N \setminus S$ define the frequency projected signal of E to B as the signal given by

$$P_B E(x) := \mathcal{F}^{-1}[B\hat{E}](x) = \sum_{m \in B} \hat{E}(m) \chi(m \cdot x).$$

- **DRA:** Bob has observed the frequencies at B hence Bob has full information on all the amplitudes of $P_B E(x)$. He attempts to reconstruct E through defining a binary signal G from rounding $P_B E$ as follows:

$$G_B(x) := \begin{cases} 1, & |P_B E(x)| \geq \frac{1}{2} \\ 0, & |P_B E(x)| < \frac{1}{2}. \end{cases}$$

so we need to prove $G_B(x) = E(x)$ for all $x \in \mathbb{Z}_N$.

- *Proof:* By Fourier inversion

$$E(x) = P_S E(x) + P_B E(x).$$

Applying Cauchy Schwartz on $P_S E$ and the assumptions give

$$\begin{aligned} |P_S E(x)|^2 &\leq |S| \Big(\sum_{m \in S} |\hat{E}(m)|^2 \Big) \\ &\leq |S| \Big(\sum_{m \in \mathbb{Z}_n} |\hat{E}(m)|^2 \Big) \\ &= |S| N^{-1} \sum_{m \in E} 1 && \text{(Planacharel)} \\ &= N^{-1} N_\omega N_t < \frac{1}{4}. && \text{(Assumption 3.)} \end{aligned}$$

Taking square roots we get $|P_S E(x)| < \frac{1}{2}$. We have by triangle inequality

$$|E(x)| \leq |P_B E(x)| + \frac{1}{2}.$$

Then since E is a binary signal, we have $|P_B E(x)| < \frac{1}{2}$ implying $E(x) = 0$, so

$$|G_B(x) - E(x)| = 0$$

for every $x \in \mathbb{Z}_N$.

References

- Donoho, David L., and Philip B. Stark. "Uncertainty Principles and Signal Recovery." SIAM Journal on Applied Mathematics, vol. 49, no. 3, 1989, pp. 906–31. JSTOR, http://www.jstor.org/stable/2101993. Accessed 17 Oct. 2024.
- Stein, E. M., and Shakarchi, R. Fourier analysis an introduction. Princeton Univ.