MXM 2023 – Tangled Labelings on Posets

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1 Introduction

1.1 Posets and Promotion Algorithms

Definition: A **partially ordered set**, or **poset**, is a set P with a partial ordering $>_P$ such that

1. $\forall x, y \in P, x >_P y \implies y \not>_P x$
2. $\forall x, y, z \in P, x >_P y, y >_P z \implies x >_P z$

When convenient, we use the $<_P$ symbol such that $\forall x, y \in P, x >_P y \iff y <_P x$

We can also use the $\geq_P$ symbol such that $\forall x, y \in P, x \geq_P y \iff x >_P y$ or $x = y$. The symbol $\leq_P$ is analogous.

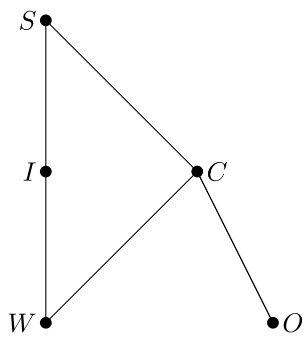
Note that a partially ordered set differs from a "totally ordered set" since it does **not** feature the axiom

$$3. \forall x \neq y \in P, \text{ either } x >_P y \text{ or } y >_P x$$

and so it is possible to have two incomparable elements within a poset.

Posets are often visualized as **Hasse diagrams**[1], with elements of the poset at nodes and (undirected) edges denoting the partial ordering relation. If $x >_P y$, you can travel strictly upwards from y to x along the edges of the Hasse diagram.

For instance, if we consider the poset on five elements $\{W, I, S, C, O\}$ with the relations $I >_P W, S >_P I, C >_P W, C >_P O, S >_P C$, its Hasse diagram is as follows.



This project only considers **finite, connected posets**, which are posets whose Hasse diagram is a finite connected graph.

[1]Named after German mathematician Helmut Hasse.

1.2 Definitions

We now introduce several definitions important to the rest of the paper.

Definition 2: Let $x \in P$. The **up-set** of x, denoted $x\uparrow$, is defined as $x\uparrow = \{y \in P | y >_P x\}$ Similarly, the **down-set** of x is $x\downarrow = \{y \in P | y <_P x\}$

Definition 3: An element $x \in P$ is called **maximal** if $x\uparrow = \emptyset$ or **minimal** if $x\downarrow = \emptyset$

Definition 4: For $x, y \in P$, we say y **covers** x if

1. $y >_P x$
2. $\nexists z \in P, x <_P z <_P y$

Definition 5: A **labeling** of a poset P with n elements is a bijective function

$$L : P \to \{1, 2, ..., n\}$$

which assigns a natural number to each element of P.

Definition 6: A **natural** labeling L of a poset P is a labeling such that

$$\forall x, y \in P, x >_P y \implies L(x) > L(y)$$

where $>$ is the usual greater-than relation for natural numbers.

Definition 7: A **frozen element** in a given labeling L on an n-element poset P is an $x \in P$ such that

1. $\forall y \in x\uparrow, L(y) > L(x)$.
2. $\forall m > L(x), m \in \{1, 2, ..., n\}, L^{-1}(m)$ is frozen.

2 Promotion Algorithm

Definition: We call a promotion algorithm ∂ on a set labelings of a poset P such that:

1. Find the element labeled "1" in the poset.
2. Find the smallest number (label) above the element labeled "1," where "above" refers to the poset structure.
3. Swap the element labeled "1" with this smallest element.
4. Repeat until the element labeled "1" has no elements above it.
5. Relabel the element labeled "1" as n, and subtract 1 from all other elements.

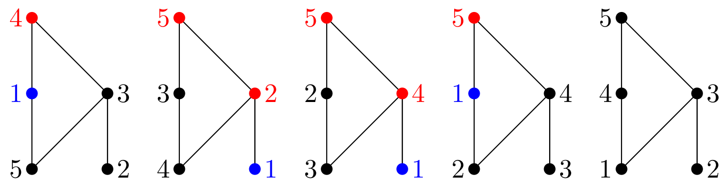
Sometimes we mention a "promotion chain". This is simply the set of elements that are swapped in a given application of ∂

Fact: Applying ∂ to any labeling on any n-element poset $n - 1$ times will result in a natural labeling (See Lemma 5.4)

Definition 7: A **tangled** labeling L of a poset P is a labeling such that $\partial^{n-2}(L)$ is *not* natural.

Big Conjecture: For all n-element posets, there are at most $(n-1)!$ tangled labelings.

Example of a tangled labeling (left) and promotion, resulting in a natural labeling (right):



3 Demotion Algorithms

As we seek to find the number of tangled labelings of a given poset, a useful technique is to take a natural labeling and identify which tangled labelings promote toward it. Toward this end, we use a *demotion algorithm*, which we describe in this section.

3.1 The Demotion ∂^{-1} Algorithm

This describes how to find *all* preimages under a single iteration of ∂ of a given labeling L on an n-element poset.

- Check if n is maximal. If not, no preimages
- Find all elements that are smaller than all labels in their up-set, call these"demotable"
- Choose n to be to top of the demotion chain
- Repeatedly choose the next element in the demotion chain from elements that are demotable and in the down-set of the previous element picked. Stop at any time.
- Run demotion on the demotion chain. That is, swap the top element (n) with the next element in the demotion chain. Repeat until n is at the bottom of the demotion chain. Relabel n as 0, and increment all labels in the poset by 1.

Fact: All elements of a natural labeling are demotable. During demotion, once an element becomes not demotable, it stays not demotable.

3.1.1 Cold Demotion $\tilde{\partial}^{-1}$

It is natural to restrict demotion to not make too many frozen elements, as this is counterproductive to finding tangled labelings. Thus, "cold demotion" maximizes the number of frozen elements of the preimages.

- Start at n.
- While the lowest element in the demotion chain covers at least one frozen element, choose the next element in the demotion chain to be the largest label that is covered by the current lowest in the demotion chain.
- When the lowest element in the demotion chain covers no frozen elements, repeatedly choose the next element in the demotion chain as in regular demotion, that is, from elements that are demotable and in the down-set of the previous element picked. Stop at any time.
- Run demotion on the demotion chain.

Note that one can enumerate all the tangled labelings by

1. Finding natural labelings such that 2 covers 1 (See Theorem 5.8)
2. Swapping the $1, 2$
3. Applying cold demotion $n-2$ times

3.1.2 Maximal Demotion $\overline{\partial^{-1}}$

It is perhaps desirable to have a bijective notion of inverse. Additionally, $\overline{\partial^{-1}}$ doesn't decrease the amount of demotable elements, which is in some sense good. To perform maximal demotion,

1. Start at maximal n.
2. While the lowest element in the demotion chain covers at least one demotable element, choose the next element in the demotion chain to be the largest label that is covered by the current lowest in the demotion chain.
3. Run demotion on the demotion chain.

Facts:

- $\overline{\partial^{-1}}$: {natural labelings} $\to$ {natural labelings} is bijective
- $\overline{\partial^{-1}}$: {labelings with $n-i$ frozen elements} $\to$ {labelings with $n-i-1$ frozen elements}, $i \geq 2$ is bijective

Application: One can arrive at a (weak) upper bound for the number of tangled labelings in the following manner:

1. Enumerate the natural labelings such that 2 covers 1 (see Theorem 5.8). Call the number of such labelings x.
2. Run maximal demotion on a single one of the above labelings. After each application, multiply your running total (x) by the number of possible demotion chains within the down-set of the last frozen element in the demotion chain (this is conceptually all the demotion chains that could happen if no element were made not demotable in previous demotions).
3. After we run maximal demotion to the end, that is $n-1$ times, x is an upper bound on the number of tangled labelings.

Question: How can this be improved? We seek to incorporate the idea that elements can become non-demotable, essentially removing them from the demotion chain at all future steps.

4 Permuting Minimal Elements

The case when a poset has 1 minimal element is known (See Lemma 5.10).

Permuting minimal elements on a tangled labeling can still yield a tangled labeling. Consider a poset with 2 minimal elements x, y. We partition all labelings according to the following criteria.

A labeling is called **(i,j)-k-ambidextrous** (abbreviated **(i,j)-k-ambi**) if it has minimal elements labeled i, j and k of the permutations of minimal elements result in tangled labelings. Thus, a (3,8)-2-ambi labeling has minimal elements **3** and **8**, and both permutations of them are tangled. This is only possible on an 8-element poset (See Lemma 5.7).

We can show the conjecture is true for posets with 2 minimal elements if we show that there are more (n,i)-0-ambi labelings than (n,i)-2-ambi labelings. This is computationally supported from a handful of examples on posets with less than 10 elements.

We may be able to do better. We can show the conjecture is true for posets with 2 minimal elements if we show that there are at least as many (n,1)-0-ambi labelings as (n,1)-2-ambi labelings. This may be true because a (n,i)-k-ambi labeling reduces to a (n-i+1,1)-k-ambi labeling on a smaller poset, so we may be able to use induction to take care of it.

How to construct (n,1)-2-ambi labelings? Well, we know that tangled labelings are the $n-2$ preimages of non-natural labelings such that the 1 covers the 2. Since they are 2-ambi, the n-1 label must get swapped into the up-set of both minimal element. Since they are (n,1)-2-ambi, the non-n minimal element must remain demotable throughout the demotion process.

How to construct (n,1)-0-ambi labelings? Well, we know that they are the $n-2$ preimages of natural labelings such that the 2 is minimal, 1 is *not* above the 2. Since they are 0-ambi, the n-1 label must get swapped into the up-set of both minimal element. Since they are (n,1)-0-ambi, the non-n minimal element must remain demotable throughout the demotion process. Further, the n minimal element must not be involved in any demotion steps.

Note the similarities betwee 0-ambi and 2-ambi. Both start with a 2 minimal. Both require the minimal element not labeled 2 must remain demotable during the entire process. The differ in whether the element being swapped into the shared up-set originated in the basin with or without the 2.

This seems to imply that there should be more 0-ambi labelings than 2-ambi. Constructing a direct injection is proving elusive, though.

5 Theorems and Proofs

Lemma 5.1. *The promotion map ∂ takes natural labelings to natural labelings.*

Proof:
Suppose it didn't. Then, after promotion, $\exists x >_P y \in P, L(x) < L(y)$. Exactly one of x, y must have been in the promotion chain.

If y was in the promotion chain, since now $L(y) > L(x)$, it must have swapped with a label higher than $L(x)$, contradicting the promotion algorithm.

If x was in the promotion chain, since now, it must have swapped with a label in its up-set and thus $y\uparrow$. After subtracting 1 from both, getting $L(x) < L(y)$ would imply that the original labeling was not natural, yielding a contradiction.

Lemma 5.2. *A labeling L of a poset P is natural if and only if all elements are frozen.*

Proof follows by inspection of definitions.

Lemma 5.3. *The map ∂ increases the number of frozen elements of a non-natural labeling by at least 1.*

Proof:
Note that frozen elements are still frozen under ∂. This can be seen a result of Lemma 5.1, since the set of frozen elements forms a natural labeling on the restriction of P to the frozen elements.

Since the labeling is not natural, 1 is not frozen. So it makes sense to consider the greatest non-frozen element in the promotion chain. After promotion, the labeling on this element was the smallest in its the up-set and belonged to a frozen element. Thus, this element is now frozen, increasing the count by at least 1.

Theorem 5.4. *Every labeling of an n-element poset, after applying ∂ $n-1$ times, is natural.*

Proof: After ∂^{n-1}, there are at least $n-1$ frozen elements by Lemma 5.3. It is impossible for only 1 element to be unfrozen, so there must be n frozen elements. By Lemma 5.2, this means the labeling is natural.

Lemma 5.5. *A labeling L is tangled if and only if, in $(\partial^{n-2}(L))(P)$, the element labeled "1" is above the element labeled "2".*

Proof:
Forward: Suppose L is a tangled labeling. Then ∂^{n-2} is not natural, but it has at least $n-2$ frozen elements. Thus, it has exactly $n-2$ frozen elements, so the element 2 is not frozen. Hence, the 1 covers the 2.

Reverse: Follows directly from the definition of being tangled.

Lemma 5.6. *Under demotion, once an element is not demotable, it stays not demotable.*

Proof:
If an element x is not demotable, then there is a label smaller than $L(x)$ in $x\uparrow$. Under demotion, labels get swapped up, so this smaller label will remain in $x\uparrow$ and x will continue to be not demotable.

Lemma 5.7. *A tangled labeling L must have the label n at a minimal element.*

Proof:
From Lemma 5.5, we see that all tangled labelings are $\partial^{-(n-2)}$ of a labeling such that the 1 covers the 2. This means 2 is not demotable. So, after $n-2$ preimages, the 2 will have label n.

Theorem 5.8. *In general, not every natural labeling can arise as the $(\partial^{n-1}(L))(P)$ of a tangled labeling L on a poset P. The number of natural labelings that can arise is the same as the number of natural labelings such that $L^{-1}(2) >_P L^{-1}(1)$*

Proof:
Start at a natural labeling and apply demotion. If this natural labeling is the result of a tangled labeling, it must have a preimage with exactly $n-2$ frozen elements. To do this, we perform cold demotion until the lowest element in the demotion chain covers the 1. If this doesn't happen, we don't have a preimage with exactly $n-2$ frozen elements. Run demotion on the demotion chain.

Note that, had we performed maximal demotion, we would have a labeling such that the 2 covered the 1.

Theorem 5.9. *A tangled labeling L must have the label n at a minimal element $x \in P$ such that $\exists y \in P, y >_P x$ and $y \not>_P$ any other minimal element.*

Proof:

By Lemma 5.7, L has n at a minimal element x. By inspection of ∂, x is never in the promotion chain, so after each iteration of promotion, $L(x)$ decreases by 1. So, applying promotion on the poset $P \setminus x$ has the same result for the first $n-1$ promotions, after adjusting the labels by nat most 1. Since $P \setminus x$ is an $(n-1)$-element poset, after ∂^{n-2} it is natural, so the 1 is minimal. In the labeling $\partial^{n-2}L$, x is labeled 2, so in order for L to be tangled, by Lemma 5.5, there must be a minimal element $y \in P \setminus x$ such that $y >_P x$.

Lemma 5.10. *If an n-element poset P has 1 minimal element, then there are exactly $(n-1)!$ tangled labelings of P.*

Proof:

From Lemmas 5.5, 5.7 we immediately see that having the n label minimal is necessary and sufficient to get a tangled labeling. So by simply counting, we see that there are $(n-1)!$ tangled labelings.

6 Future Thoughts

Future work can continue to explore cold demotion. In particular, finding expressions or bounds for number of cold preimages would be desirable.

7 References

[1] Colin Defant and Noah Kravitz, Promotion Sorting, *Order* 58 (2022), 1–18. `arXiv:2005.07187`