Tangled Labelings on Posets

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Posets

Definition: A **poset**, or partially ordered set, is a set P with a partial ordering $<_p, >_p$ such that

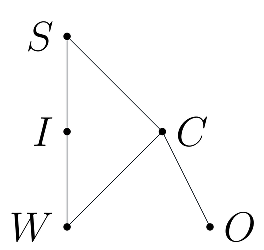
1. $\forall x, y \in P, x <_p y \iff y >_p x$
2. $\forall x, y, z \in P, x <_p y, y <_p z \implies x <_p z$

Note that this differs from a "totally ordered set" since it lacks the axiom

$$\forall x \neq y \in P, \text{ either } x <_p y \text{ or } y <_p x$$

and so it is possible to have two incomparable elements in a poset.

Posets are often visualized as **Hasse diagrams**, with elements of the poset at nodes and edges denoting the partial ordering relation. If $x <_p y$, you can travel strictly upwards from x to y along the edges of the Hasse diagram.
For instance, if we consider the poset on five elements $\{W, I, S, C, O\}$ with the relations $W <_p I, I <_p S, W <_p C, O <_p C, C <_p S$, its Hasse diagram looks like



This project only considers **finite, connected posets**, which are posets whose Hasse diagram is a connected graph.

Definition: We call an element $x \in P$ **maximal** if $\nexists y \in P, y >_p x$ or **minimal** if $\nexists y \in P, y <_p x$

Labelings

Consider a poset P with n elements. A **labeling** L is a bijection between the natural numbers $\{1, 2, ..., n\}$ and elements of the poset.

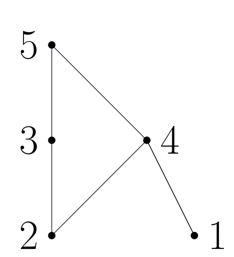
$$L : P \to \{1, 2, ..., n\}$$

For any n-element poset, there are $n!$ total labelings.

Definition: We call a labeling $L(P)$ **natural** if

$$\forall x, y \in P, \quad x <_p y \implies L(x) < L(y)$$

where $<$ is the usual ordering on natural numbers.
For instance, the following is one of 5 natural labelings of the poset P:



P

There is no convenient formula for the number of natural labelings – also called *linear extensions* – of an arbitrary poset.

Definition: A **frozen** element $x \in P$ is an element such that

$$\forall i \geq L(x), \forall y >_p L^{-1}(i), L(y) > i$$

that is, they have a smaller label then all elements above them, and all elements with larger labels satisfy the same property.
Note that every element in a natural labeling is frozen.

Promotion

The "Promotion Algorithm" ∂ on any poset P maps labelings of P to labelings of P. We let $\mathcal{L}(P) = \{L(P)\}$. Then

$$\partial : \mathcal{L}(P) \to \mathcal{L}(P)$$

The promotion function ∂ works as follows:

1. Find the element labeled "1" in the poset.
2. Find the element with smallest label which is immediately greater than "1" in the poset structure.
3. Swap "1" with this element.
4. Repeat Steps 1-3 until "1" has no elements above it.
5. Relabel "1" as "n", and subtract 1 from all other labels.

∂ applied to a natural labeling results in a natural labeling. Further, repeatedly applying ∂ to any labeling will eventually result in a natural labeling. It is known [Defant-Kravitz, Proposition 2.7] that it takes at most $n-1$ promotions to reach such a labeling, i.e. $\partial^{n-1}(L(P))$ is natural for any poset P and labeling $L(P)$. The proof of this relies on the fact that ∂ increases the number of frozen elements of a non-natural labeling by at least 1.

Definition: A labeling L of P is **tangled** if $\partial^{n-2}(L(P))$ is *not* a natural labeling.

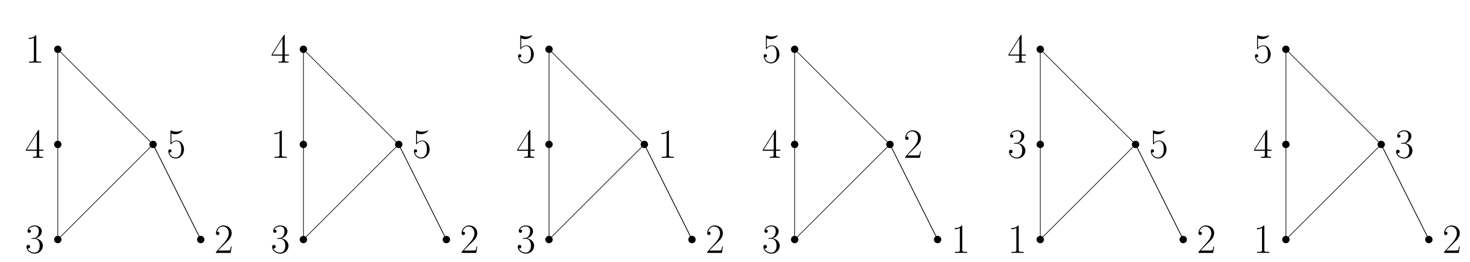
One way to find tangled labelings is to apply the promotion algorithm in reverse, given a natural labeling. This is what we call the **Demotion Algorithm**.

Demotion

The demotion operator ∂^{-1} finds all preimages of ∂. Let P be an n-element poset with some labeling $L(P)$, ∂^{-1} works as follows:

1. Identify $L^{-1}(n)$. If $L^{-1}(n)$ not a maximal element in P, there are no possible preimages
2. Identify all $x \in P$ such that $\forall y >_p x, L(y) > L(x)$. Call these elements "demotable"
3. Optionally, choose an element $x \in P$ such that $x <_p L^{-1}(n)$, and swap the labelings on $x, L^{-1}(n)$
4. Repeat step 3 as many times as desired, or until you are unable to do so further.
5. Relabel $L^{-1}(n)$ as 1, and add 1 to all other labels.

There are often a large number of preimages of a single labeling. For instance, the natural labeling in the left column has 8 preimages under ∂, 6 are shown below.



For this reason, we define a specific preimage by using "Maximal Demotion", denoted by $\bar{\partial}^{-1}$. This algorithm differs from ∂^{-1} in steps 3 and 4.

Step 3 now requires that the chosen $x \in P$ is such that $\nexists y \in P$ such that $x <_p y <_p L^{-1}(n)$ and that $\forall z \in P$ that satisfy this property $L(x) > L(z)$.
Step 4 requires that step 3 be repeated until you are unable to do so further.

$\bar{\partial}^{-1}$ always takes natural labelings to natural labelings.
Maximal demotion is so-called because the number of frozen elements is kept as large as possible, and we keep repeating step 3 as many times as possible. In particular, if $L(P)$ has n frozen elements, there are still n frozen elements in $\bar{\partial}^{-1}(L(P))$. If there are $k < n$ frozen elements in $L(P)$, there are $k-1$ frozen elements in $\bar{\partial}^{-1}(L(P))$.

Conjecture

The primary conjecture is as follows.

~**Big Conjecture**~: Any poset with n elements has no more than $(n-1)!$ tangled labelings.

There are many posets (e.g. any poset with a unique minimal element) for which this conjectured bound is sharp.

Results

Result 1: In general, not every natural labeling can arise as the $\partial^{n-1}(L(P))$ of a tangled labeling $L(P)$. The number of natural labelings that can arise is the same as the number of natural labelings such that $L^{-1}(2) >_p L^{-1}(1)$

Proof Outline:
Let $T(P)$ be a tangled labeling. $\partial^{n-2}(T(P))$ is not natural, but $\partial^{n-1}(T(P))$ is. There exists a natural labeling $N(P)$ that has a preimage $\partial^{-1}(N(P))$ with $n-2$ frozen elements, which is only possible when $(\partial^{-1}(N))^{-1}(2) >_p (\partial^{-1}(N))^{-1}(1)$. Applying ∂ then $\bar{\partial}^{-1}$, we arrive at a natural labeling L such that $L^{-1}(2) >_p L^{-1}(1)$. All other labels $i \geq 3$ are such that $L^{-1}(i) = (\partial^{-1}(N))^{-1}(i)$.

Corollary 1: A labeling $L(P)$ is tangled if and only if, in $(\partial^{n-2}(L))(P)$, the element labeled "1" is above the element labeled "2".

Corollary 2: A tangled labeling $L(P)$ must have the label n at a minimal element.

Result 2: A tangled labeling $L(P)$ must have the label n at a minimal element $x \in P$ such that $\exists y \in P, y >_p x$ and $y \not>_p$ any other minimal element.

Proof Outline: By Corollary 2, n must be minimal to start. By inspection of ∂, we see that n never influences ∂ until ∂^n. So, $P \setminus \{L^{-1}(n)\}$ contains $n-1$ elements and thus is natural after just $n-2$ promotions, so 1 is minimal. But 1 must be above 2 by corollary 1.

Future Investigations

1. Is there a way to use demotion to conveniently estimate or bound the number of $n-1$th level preimages of a natural labeling?
2. Find some sort of greedy algorithm.
3. Look at some generating functions and find patterns.

Acknowledgements

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References

[1] Colin Defant and Noah Kravitz, Promotion Sorting, *Order* 58 (2022), 1–18. arXiv:2005.07187