

From Black-Scholes to the Heat Equation

Zoltan Leka

We consider the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + rs \frac{\partial V}{\partial s} + \frac{\sigma^2 s^2}{2} \frac{\partial^2 V}{\partial s^2} - rV = 0,$$

defined for $0 \leq t < T$ and $0 < s$, with terminal condition $V(s, T) = F(s)$. We can readily prove that the gauge transform $V(s, t) = e^{ax+b\tau} U(x, \tau)$ solves the PDE, where $x = \log s$, $\tau = \frac{\sigma^2}{2}(T-t)$ and U is the solution to the heat equation

$$\frac{\partial U}{\partial \tau} = \frac{1}{2} \frac{\partial^2 U}{\partial x^2} \text{ with } U(x, 0) = e^{-ax} F(e^x),$$

where constants a and b need to be defined properly. Although the transformation is ultimately a straightforward computation, the sequence of changes of variables makes the derivation nontrivial. For completeness, we record the full calculation here.

After substituting the expressions for V , V_t and V_{ss} , a straightforward computation shows that

$$V_t = b\tau' e^{ax+b\tau} U + e^{ax+b\tau} \tau' U_\tau, \quad V_s = ax' e^{ax+b\tau} + e^{ax+b\tau} \tau' U_\tau,$$

and

$$V_{ss} = ax'' e^{ax+b\tau} U + a^2(x')^2 e^{ax+b\tau} U + ax' e^{ax+b\tau} U_{xx} x' \\ + e^{ax+b\tau} U_{xx} (x')^2 + e^{ax+b\tau} U_{xx}'' + ae^{ax+b\tau} U_x (x')^2.$$

Clearly, $x' = 1/s$ and $\tau' = -\sigma^2/2$. Plugging these formulae into the Black-Scholes PDE leads to a somewhat lengthy equation

$$-\frac{b\sigma^2}{2} U - U_\tau \frac{\sigma^2}{2} + raU + rU_x - \frac{a\sigma^2}{2} U + \frac{\sigma^2 a^2}{2} U + \frac{\sigma^2 a}{2} U_x + \frac{\sigma^2}{2} U_{xx} - \frac{\sigma^2}{2} U_x + \frac{a\sigma^2}{2} U_x - r = 0$$

We recall that $U_\tau = \frac{1}{2} U_{xx}$ holds. Moreover, to determine the constants a and b , we require that the coefficients of both U_x and U vanish. In particular, the coefficient of U must be

$$-\frac{b\sigma^2}{2} + ra - \frac{a\sigma^2}{2} + \frac{a^2\sigma^2}{2} - r = 0$$

and that of U_x must be

$$r + \sigma^2 a - \frac{\sigma^2}{2} = 0.$$

That is,

$$a = \frac{1}{2} - \frac{r}{\sigma^2}.$$

This implies

$$b = - \left(\frac{1}{2} + \frac{r}{\sigma^2} \right)^2.$$

Summarizing up all these computations, under the gauge (exponential) transform

$$x = \log s, \tau = \frac{\sigma^2}{2}(T - t) \text{ and } V(s, t) = e^{ax+b\tau} U(x, \tau)$$

with

$$a = \frac{1}{2} - \frac{r}{\sigma^2} \text{ and } b = - \left(\frac{1}{2} + \frac{r}{\sigma^2} \right)^2,$$

the transformed function U satisfies the heat equation

$$\frac{\partial U}{\partial \tau} = \frac{1}{2} \frac{\partial^2 U}{\partial x^2} \text{ with } U(x, 0) = e^{-ax} F(e^x).$$

Now, the heat kernel representation

$$U(x, \tau) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-ay} F(e^y) e^{-(x-y)^2/(2\tau)} dy$$

defines the solution to the Black-Scholes PDE as well. Indeed, the corresponding Black-Scholes solution $V(s, t)$ follows immediately by substitution in the above integral formula.