Variations on a linear parabolic PDE

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Abstract. The goal of this short note is to show how one linear parabolic PDE can be solved using three complementary methods: analytic, probabilistic, and spectral.

Problem. Let us consider the PDE

$$u_t = \frac{1}{2}u_{xx} - xu, -\infty < x < \infty \text{ and } t > 0$$

with the initial value condition $u(x,0) = f(x) \in L^1(\mathbb{R})$. Find the general solution u.

First solution. To reduce the problem to the heat equation, we introduce the transformation

$$u(t,x) = e^{A(x,t)}v(x + B(t), t),$$

where v satisfies the heat equation $v_t = \frac{1}{2}v_{xx}$ and $v(x,0) = f(x)$. Straightforward computation shows that the PDE for u holds if and only if

$$A_t v + v_x B' + v_t = \frac{1}{2}A_{xx}v + \frac{1}{2}A_x^2 v + A_x v_x + \frac{1}{2}v_{xx} - xv.$$

Thus, from the equation $B' = A_x$, we obtain that $A(x,t) = a(t)x + b(t)$. Furthermore, from the equation $A_t = \frac{1}{2}A_{xx} + \frac{1}{2}A_x^2 - x$, we have $a'(t)x + b'(t) = \frac{1}{2}a^2(t) - x$. Hence, $a'(t) = -1$ and $a(t) = -t$, and $b(t) = t^3/6$ must hold. That is, $A(x,t) = -xt + t^3/6$. From $B' = a(t)$, we also have $B(t) = -t^2/2$. This implies

$$u(x,t) = e^{-xt+t^3/6}v(x - t^2/2, t).$$

From the heat kernel representation of v,

$$u(x,t) = \frac{e^{-xt+t^3/6}}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(s)e^{-\frac{(s-x+t^2/2)^2}{2t}}\, ds.$$

Second solution. This approach is based on the Feynman-Kac theory. The general solution to the diffusion equation is provided by

$$u(x,t) = \mathbb{E}\Big(e^{-\int_0^t B_s\, ds} f(B_t) | B_0 = x\Big),$$

where B_s is the Brownian motion. We now compute the expected value, which is a bit technical. Let us introduce the random variable

$$Y = \int_0^t B_s ds | B_0 = x.$$

From the tower property of the expectation,

$$u(x,t) = \mathbb{E}(\mathbb{E}(e^{-Y}f(B_t)|B_t = s) = \int_{-\infty}^{\infty} \mathbb{E}(e^{-Y}|B_t = s)f(s)p_{B_t}(s)\, ds,$$

where p_{B_t} denotes the PDF of the variable $B_t \sim N(x,t)$.
Let us find the conditional expectation. It is simple to check that $\mathbb{E}(Y) = \displaystyle\int_0^t x\, ds = tx$, and

$$\mathbb{E}(Y^2) = \mathbb{E}\Big(\int_0^t \int_0^t B_s B_r\, dr\, ds\Big) = \int_0^t \int_0^t \mathbb{E}(B_s B_r)\, dr\, ds = \int_0^t \int_0^t \min(s,r)\, dr\, ds = \frac{t^3}{3}.$$

Similarly, we also obtain that

$$\mathbb{E}(YB_t) = \int_0^t \mathbb{E}(B_s B_t)\, ds = \frac{t^2}{2}.$$

Hence, (B_t, Y) has a joint normal distribution

$$(\mathrm{B}_t, Y) \sim N\left(\begin{bmatrix} x \\ xt \end{bmatrix}, \begin{bmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{bmatrix}\right).$$

We recall that the conditional distribution of $Y|B_t = s$ is normal with expectation $t(x+s)/2$ and variance $t^3/12$. Thus, from the moment generating function of the normal variable,

$$\mathbb{E}(e^{-Y}|B_t = s) = e^{-t(x+s)/2 + t^3/24}.$$

We obtain

$$\int_{-\infty}^{\infty} \mathbb{E}(e^{-Y}|B_t = s)f(s)p_{B_t}(s)\, ds = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-(x-s)^2/(2t)} e^{-t(x+s)/2 + t^3/24} f(s) ds.$$

Now, simple computation shows that the exponent can be written as a complete square, adding a few correction terms. Indeed,

$$\begin{aligned} &-(x-s)^2/(2t) - t(x+s)/2 + t^3/24 \\ &\qquad = -(s - x + t^2/2)^2/(2t) + (x - t^2/2)^2/(2t) - x^2/(2t) - tx/2 + t^3/24 \\ &\qquad = -(s - x + t^2/2)^2/(2t) - tx + t^3/6. \end{aligned}$$

Thus,

$$u(x,t) = \frac{e^{-xt + t^3/6}}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(s-x+t^2/2)^2}{2t}}\, ds.$$

This expression coincides exactly with the result obtained via the gauge transformation method.

Third solution. We now apply Fourier theory to find the solution to the PDE. The Fourier transform of a function $g \in L^1(\mathbb{R})$ is defined by

$$G(\omega) = \int_{-\infty}^{\infty} g(x)e^{-i\omega x}\,dx.$$

Let us take the Fourier transform of the PDE with respect to the spatial variable x. We obtain the first-order semilinear equation

$$U_t(\omega,t) + iU_\omega(\omega,t) = -\frac{\omega^2}{2}U(\omega,t) \text{ and } U(\omega,0) = F(\omega).$$

We now solve the PDE by means of the method of characteristics. Formally, the system of characteristic ODEs is given by

$$\frac{dt}{1} = \frac{d\omega}{i} = \frac{dU}{-(\omega^2/2)U}.$$

Straightforward computation shows that

$$U(\omega,t) = e^{i\omega^3/6}F(\omega - it)e^{-i(\omega-it)^3/6}.$$

Hence,

$$U(\omega,t) = F(\omega - it)e^{t^3/6}e^{-(\omega-it)^2t/2}e^{-i(\omega-it)t^2/2}.$$

Let us recall that how shift and exponential multiplication change the Fourier transform of an arbitrary function h:

$$h(x-a) \mapsto e^{-ia\omega}H(\omega) \qquad \text{and} \qquad e^{ax}h(x) \mapsto H(\omega + ia).$$

Moreover, the Fourier transform of the Gaussian distribution $\frac{1}{\sqrt{2\pi t}}e^{-x^2/(2t)}$ is $e^{-w^2t/2}$. Thus, the inverse Fourier of U is given by the rescaled convolution

$$u(x,t) = \quad e^{t^3/6}e^{-xt}\left(\frac{1}{\sqrt{2\pi t}}e^{-(x-t^2/2)^2/(2t)} * f\right).$$

This representation agrees with the heat kernel expression derived in the previous two approaches, confirming the equivalence of the analytic, probabilistic, and spectral methods.