

CX1104: Linear Algebra for Computing

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{A \quad m \times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{x \quad n \times 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b \quad m \times 1}$$

Chap. No : **6.2.2**

Lecture : **Orthogonality**

Topic : **Orthogonality**

Concept : **Orthogonal Projections**

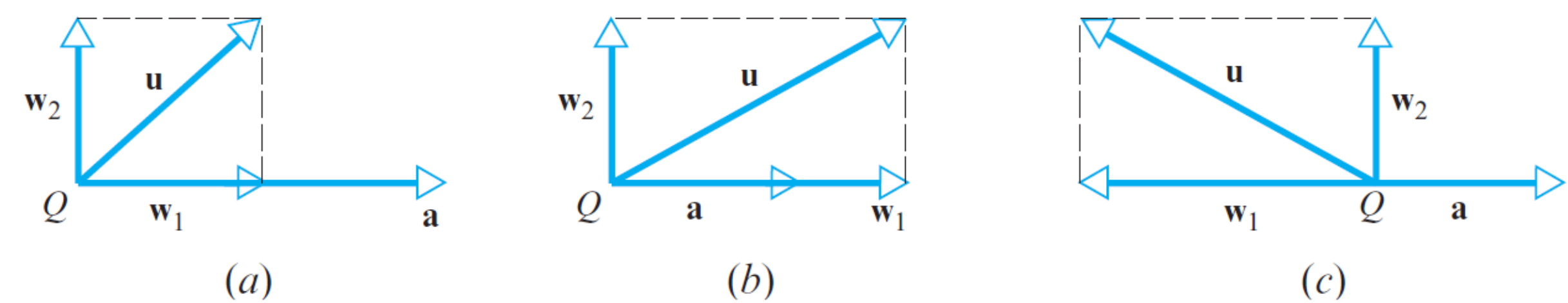
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Orthogonal Projections

Decomposition in R^2 space

In many applications it is necessary to “decompose” a vector \mathbf{u} into a sum of two terms, one term being a scalar multiple of a specified nonzero vector \mathbf{a} and the other term being orthogonal to \mathbf{a} . For example, if \mathbf{u} and \mathbf{a} are vectors in R^2 that are positioned so their initial points coincide at a point Q , then we can create such a decomposition as follows (Figure 3.3.2):



▲ Figure 3.3.2 Three possible cases.

- Drop a perpendicular from the tip of \mathbf{u} to the line through \mathbf{a} .
- Construct the vector \mathbf{w}_1 from Q to the foot of the perpendicular.
- Construct the vector $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.

Since

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 + (\mathbf{u} - \mathbf{w}_1) = \mathbf{u}$$

we have decomposed \mathbf{u} into a sum of two orthogonal vectors, the first term being a scalar multiple of \mathbf{a} and the second being orthogonal to \mathbf{a} .

Decomposing vectors into orthogonal basis has many advantages!
The standard basis is an orthogonal basis.
Orthogonal basis is explained in 6.2.3.

► EXAMPLE 1 The Standard Basis for R^n

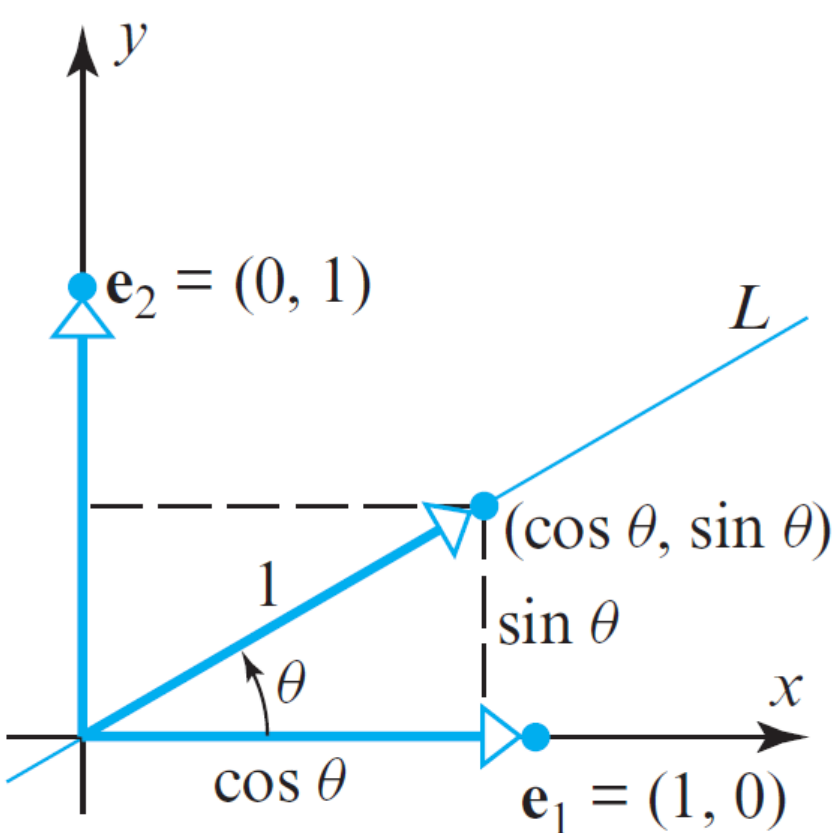
Recall from Example 11 of Section 4.2 that the standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

span R^n and from Example 1 of Section 4.3 that they are linearly independent. Thus, they form a basis for R^n that we call the *standard basis for R^n* . In particular,

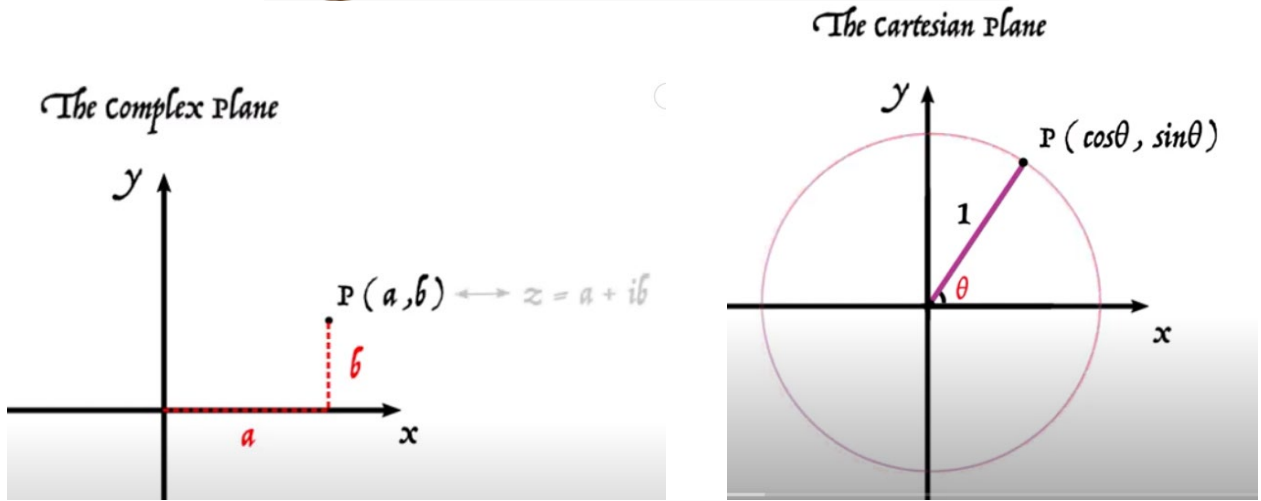
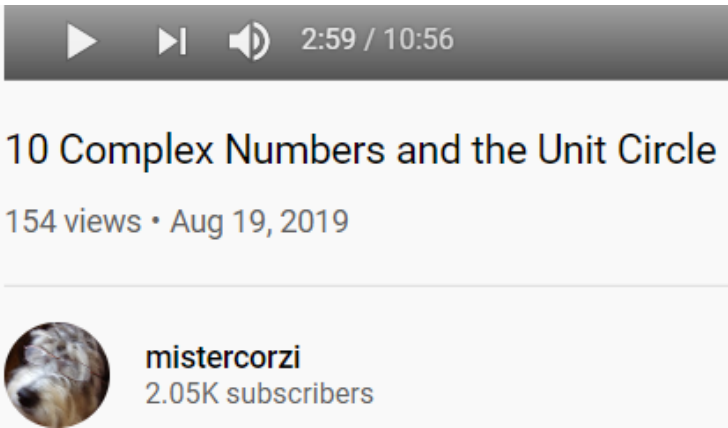
$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

is the standard basis for R^3 .



▲ Figure 3.3.3

Example above:
Commonly found in complex number of unit circle.
Where e_1 is the real line, and e_2 is the imaginary line

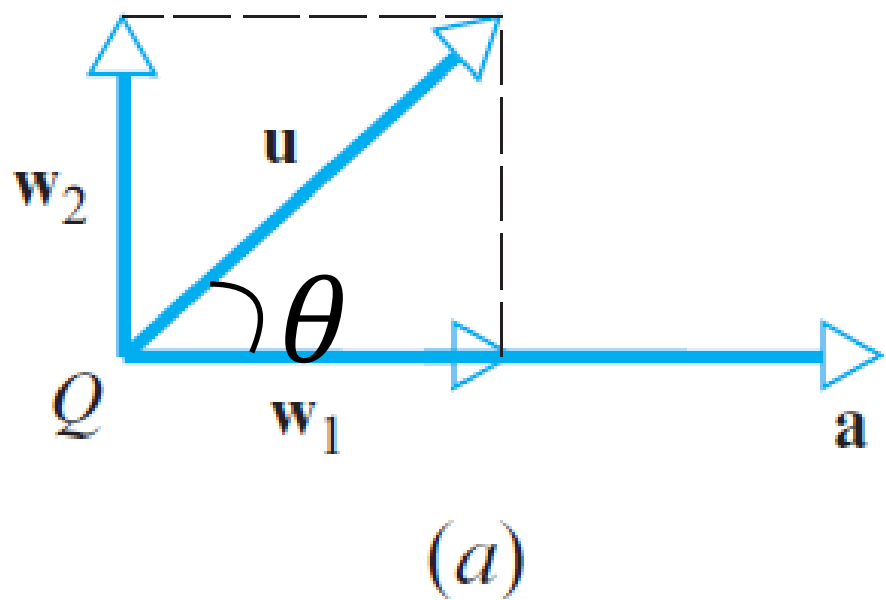


Mister Corzi: <https://www.youtube.com/watch?v=UF172DZOiWo>

Orthogonal Projections

THEOREM 3.3.2 Projection Theorem

If \mathbf{u} and \mathbf{a} are vectors in R^n , and if $\mathbf{a} \neq \mathbf{0}$, then \mathbf{u} can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where \mathbf{w}_1 is a scalar multiple of \mathbf{a} and \mathbf{w}_2 is orthogonal to \mathbf{a} .



The vectors \mathbf{w}_1 and \mathbf{w}_2 in the Projection Theorem have associated names—the vector \mathbf{w}_1 is called the *orthogonal projection of \mathbf{u} on \mathbf{a}* or sometimes *the vector component of \mathbf{u} along \mathbf{a}* , and the vector \mathbf{w}_2 is called the vector *component of \mathbf{u} orthogonal to \mathbf{a}* . The vector \mathbf{w}_1 is commonly denoted by the symbol $\text{proj}_{\mathbf{a}}\mathbf{u}$, in which case it follows from (8) that $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$. In summary,

$$\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} \quad (\text{vector component of } \mathbf{u} \text{ along } \mathbf{a}) = \|\mathbf{u}\| \cos \theta \quad (10)$$

$$\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} \quad (\text{vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{a}) \quad (11)$$

EXAMPLE 5 Vector Component of \mathbf{u} Along \mathbf{a}

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector component of \mathbf{u} along \mathbf{a} and the vector component of \mathbf{u} orthogonal to \mathbf{a} .

Solution

$$\mathbf{u} \cdot \mathbf{a} = (2)(4) + (-1)(-1) + (3)(2) = 15$$

$$\|\mathbf{a}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

Thus the vector component of \mathbf{u} along \mathbf{a} is

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} = \frac{15}{21} (4, -1, 2) = \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$

and the vector component of \mathbf{u} orthogonal to \mathbf{a} is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (2, -1, 3) - \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right) = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right)$$

As a check, you may wish to verify that the vectors $\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$ and \mathbf{a} are perpendicular by showing that their dot product is zero. ◀

Linear Algebra: Projection onto a Subspace

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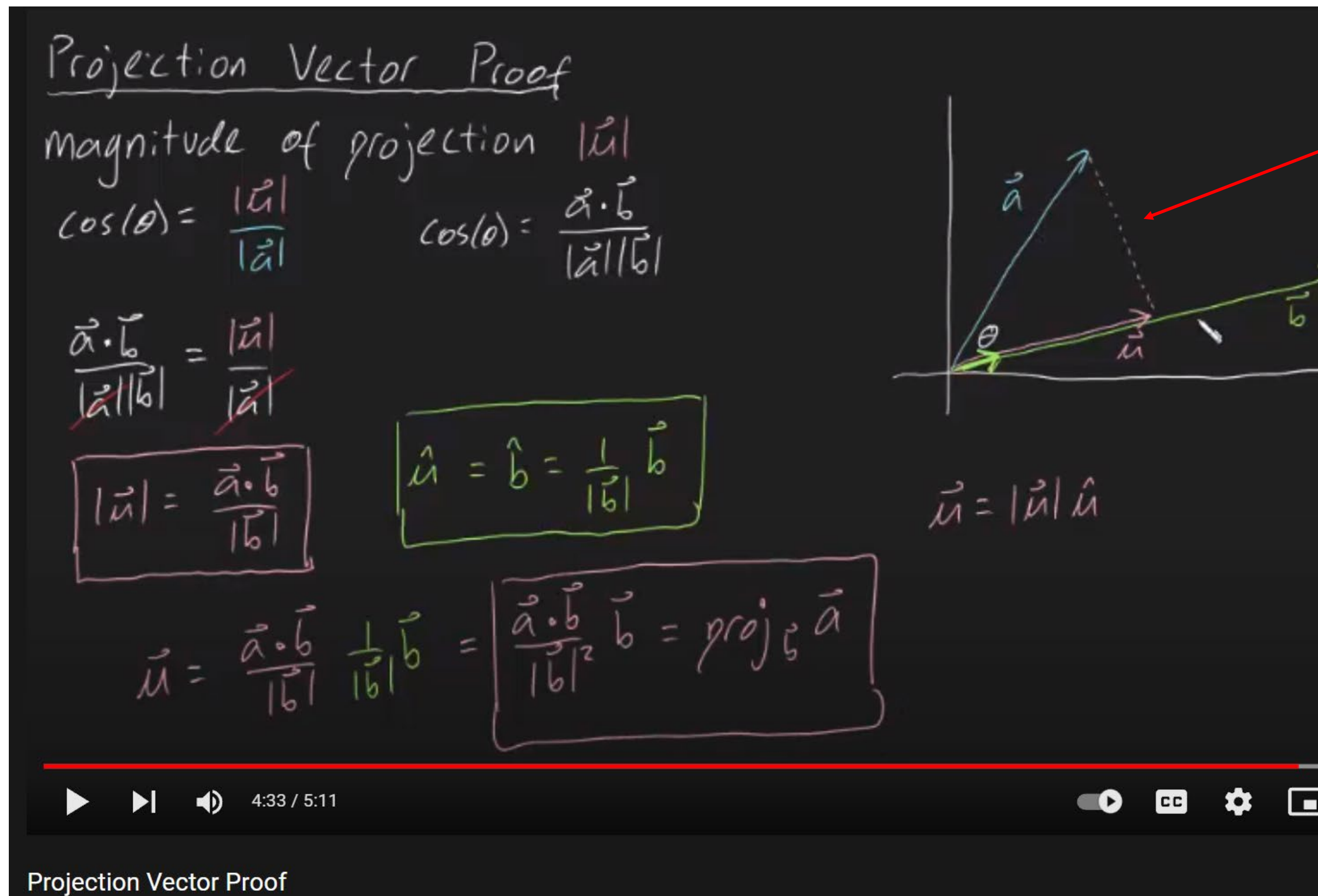
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Projection onto Line: <https://www.youtube.com/watch?v=GnvYEbaSBoY>
Projection onto Subspace: <https://www.youtube.com/watch?v=zZW6JV4yA54>

Proof: $u = Proj_b a = \left(\frac{a \cdot b}{||b||^2} \right) b$

Method 1:

In this example, u is the projection of a onto b



Method 2

Let $e = a - u$ /* the residual */

The residual e is orthogonal to b
Therefore,

$$e \cdot b = 0$$

$$(a - u) \cdot b = 0$$

$$a \cdot b - u \cdot b = 0$$

$$a \cdot b = u \cdot b$$

multiply with b on rhs

$$(a \cdot b)b = (u \cdot b)b$$

$$(a \cdot b)b = u(b \cdot b)$$

$$u = \frac{(a \cdot b)}{b \cdot b} b = \frac{(a \cdot b)}{||b||^2} b$$

Ref:

1) <https://www.youtube.com/watch?v=aTBtgW7U-Y8>

Orthogonal Projections

Fig. 2.5 shows two vectors \mathbf{u}, \mathbf{v} in R^2 space. The orthogonal projection of \mathbf{u} on \mathbf{v} , i.e. $Proj_{\mathbf{v}}\mathbf{u}$, can be thought of as the shadow formed by vector \mathbf{u} onto \mathbf{v} when an imaginary light is directed to \mathbf{u} along the normal of \mathbf{v} ; The vector $Proj_{\mathbf{v}}\mathbf{u}$ is the best approximation of \mathbf{u} using \mathbf{v} . The following are the definitions of orthogonal projection and the equation to find the orthogonal residual.

$$Proj_{\mathbf{v}}\mathbf{u} = \mathbf{v} \left(\frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{v}\|^2} \right), \quad (2.34)$$

$$\mathbf{w}_{\mathbf{v}} = \mathbf{u} - Proj_{\mathbf{v}}\mathbf{u}, \quad (2.35)$$

where $\|\mathbf{v}\| = \sqrt{\mathbf{v}^T \mathbf{v}}$ is the Euclidean norm of \mathbf{v} and the vector $\mathbf{w}_{\mathbf{v}}$ is the residual when \mathbf{v} is used to approximate \mathbf{u} .

Why is the denominator $\|\mathbf{v}\|^2$? (eqn 2.34).

Ans: it is to normalize vector \mathbf{v} to have unit length.

Notice that \mathbf{v} occurs twice, to get a vector to have unit length,

We need to divide \mathbf{v} by $\|\mathbf{v}\|$

$$\left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right)^T \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) = \frac{\mathbf{v}^T \mathbf{v}}{\|\mathbf{v}\|^2} = 1$$

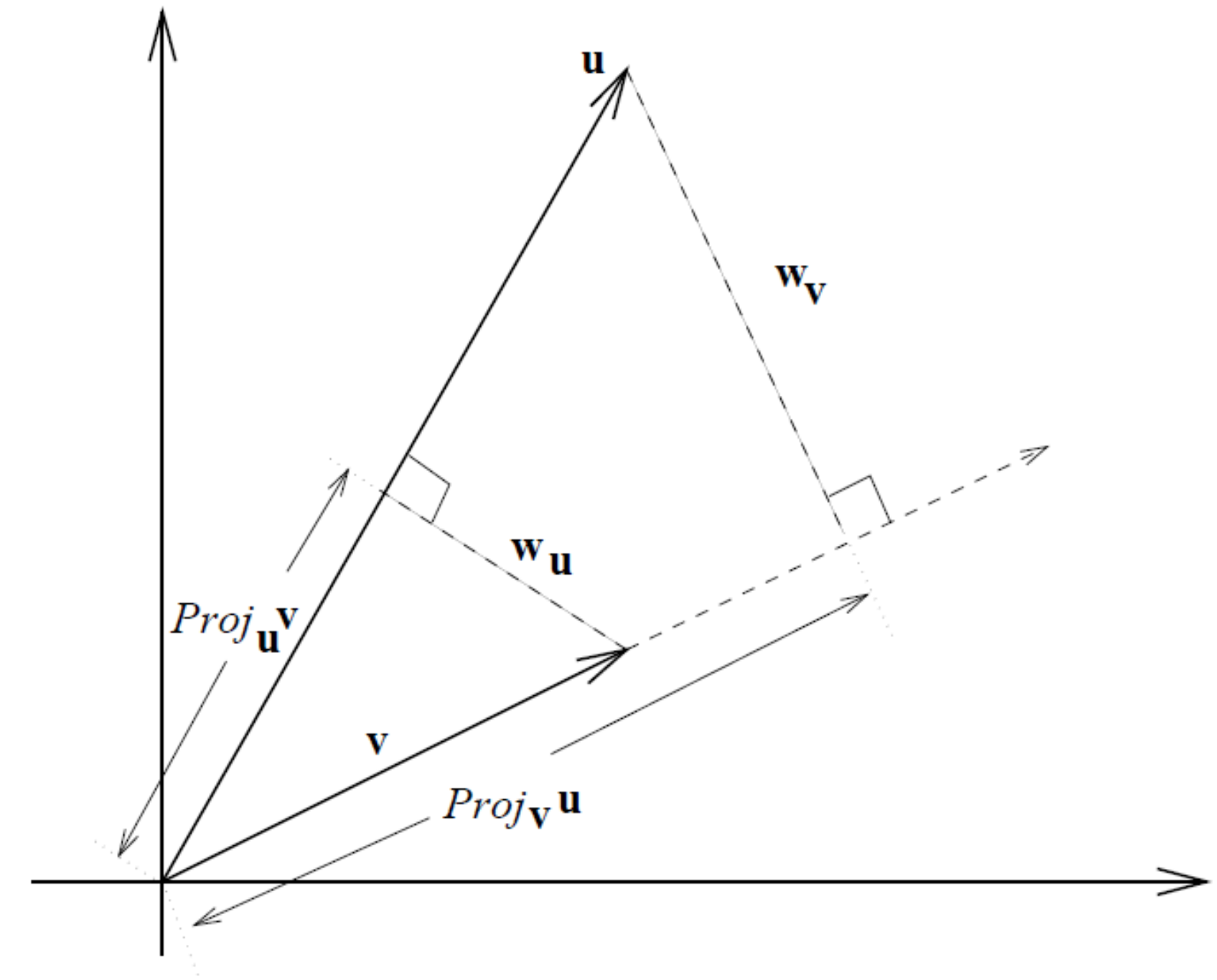
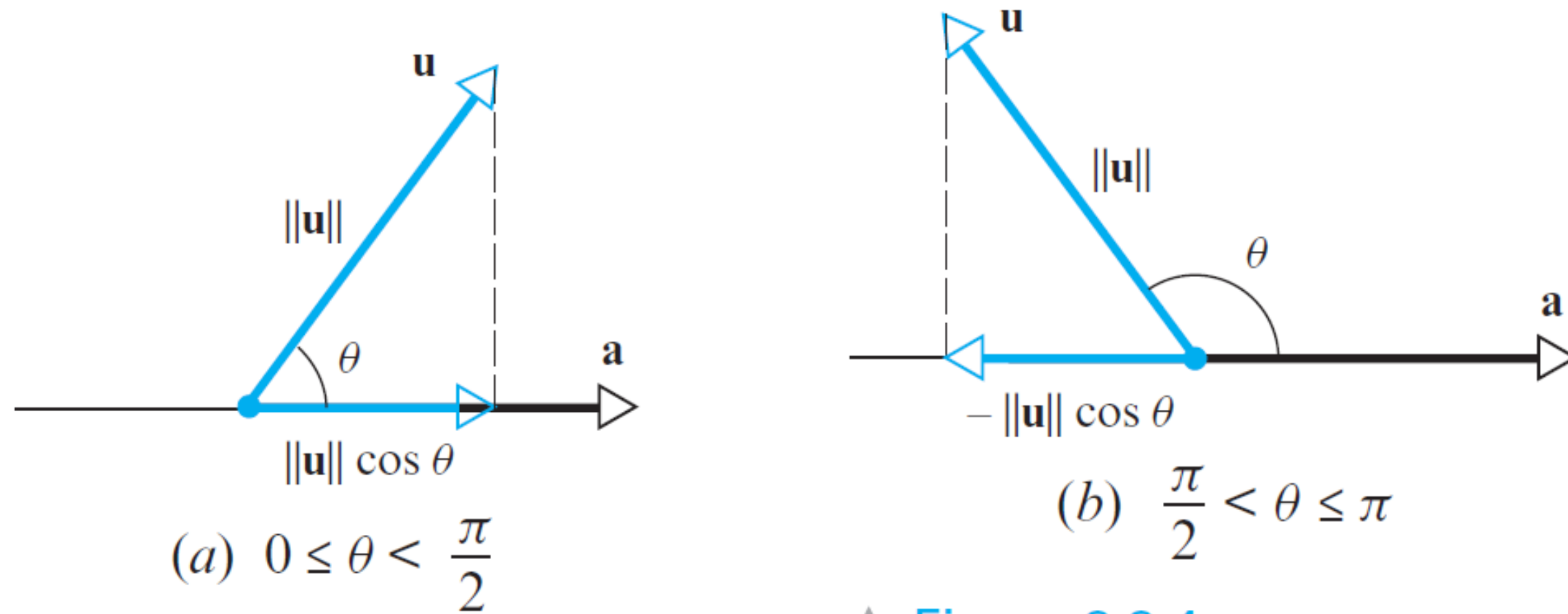


Figure 2.5: Orthogonal projections

Orthogonal Projections

Finding the norm of the projected component.



▲ Figure 3.3.4

Sometimes we will be more interested in the *norm* of the vector component of \mathbf{u} along \mathbf{a} than in the vector component itself. A formula for this norm can be derived as follows:

$$\|\text{proj}_{\mathbf{a}} \mathbf{u}\| = \left\| \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} \right\| = \left| \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \right| \|\mathbf{a}\| = \frac{|\mathbf{u} \cdot \mathbf{a}|}{\|\mathbf{a}\|^2} \|\mathbf{a}\|$$

where the second equality follows from part (c) of Theorem 3.2.1 and the third from the fact that $\|\mathbf{a}\|^2 > 0$. Thus,

$$\|\text{proj}_{\mathbf{a}} \mathbf{u}\| = \frac{|\mathbf{u} \cdot \mathbf{a}|}{\|\mathbf{a}\|} \quad (12)$$

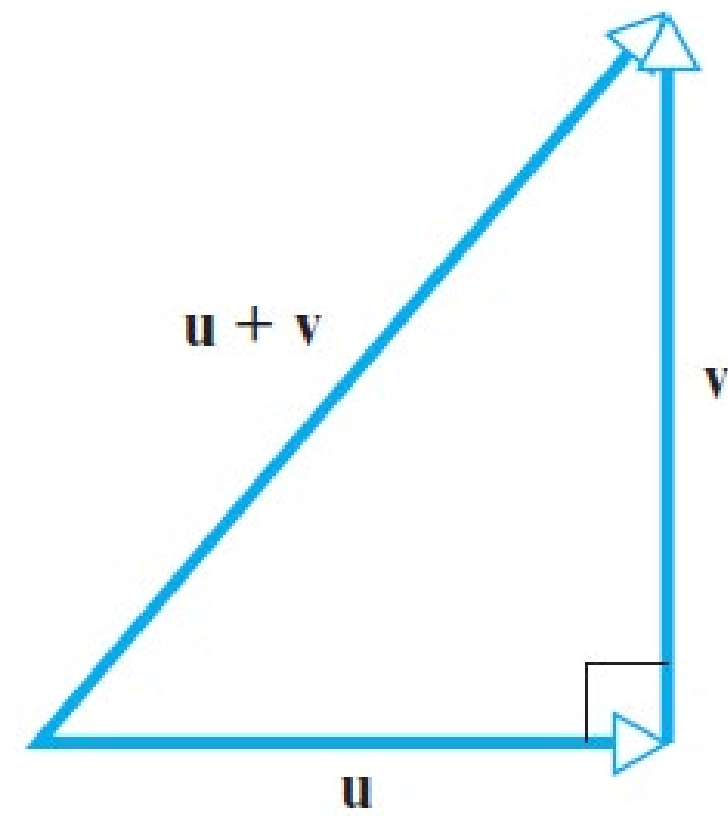
If θ denotes the angle between \mathbf{u} and \mathbf{a} , then $\mathbf{u} \cdot \mathbf{a} = \|\mathbf{u}\| \|\mathbf{a}\| \cos \theta$, so (12) can also be written as

$$\|\text{proj}_{\mathbf{a}} \mathbf{u}\| = \|\mathbf{u}\| |\cos \theta| \quad (13)$$

(Verify.) A geometric interpretation of this result is given in Figure 3.3.4.

Pythagoras Theorem

The Theorem of Pythagoras



▲ Figure 3.3.5

THEOREM 3.3.3 Theorem of Pythagoras in R^n

If \mathbf{u} and \mathbf{v} are orthogonal vectors in R^n with the Euclidean inner product, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \quad (14)$$

Proof Since \mathbf{u} and \mathbf{v} are orthogonal, we have $\mathbf{u} \cdot \mathbf{v} = 0$, from which it follows that

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \quad \blacktriangleleft$$

► EXAMPLE 6 Theorem of Pythagoras in R^4

We showed in Example 1 that the vectors

$$\mathbf{u} = (-2, 3, 1, 4) \quad \text{and} \quad \mathbf{v} = (1, 2, 0, -1)$$

are orthogonal. Verify the Theorem of Pythagoras for these vectors.

Solution We leave it for you to confirm that

$$\mathbf{u} + \mathbf{v} = (-1, 5, 1, 3)$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = 36$$

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = 30 + 6$$

Thus, $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \quad \blacktriangleleft$