$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.1**

Lecture : **Eigen and Singular Values**

Topic : **Introducing Eigenvectors and Eigenvalues**

Concept :

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Rev: 3rd July 2020

Eigenvectors and Eigenvalues

DEFINITION

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

[1] Note that an eigenvector must be *nonzero*, by definition, but an eigenvalue may be zero

The requirement that an eigenvector be nonzero is imposed to avoid the unimportant case $A\mathbf{0} = \lambda\mathbf{0}$, which holds for every A and λ.

Important note:
Eigenvalues and **eigenvectors** are only for **square** matrices.

266 CHAPTER 5 Eigenvalues and Eigenvectors

Lay, Linear Algebra and its Applications (4th Edition)

Eigenvectors and Eigenvalues

In general, the image of a vector $\mathbf{x}$ under multiplication by a square matrix A differs from $\mathbf{x}$ in both magnitude and direction. However, in the special case where $\mathbf{x}$ is an eigenvector of A, multiplication by A leaves the direction unchanged. For example,

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of $\mathbf{u}$ and $\mathbf{v}$ under multiplication by A are shown in Fig. 1. In fact, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only "stretches," or dilates, $\mathbf{v}$. ■

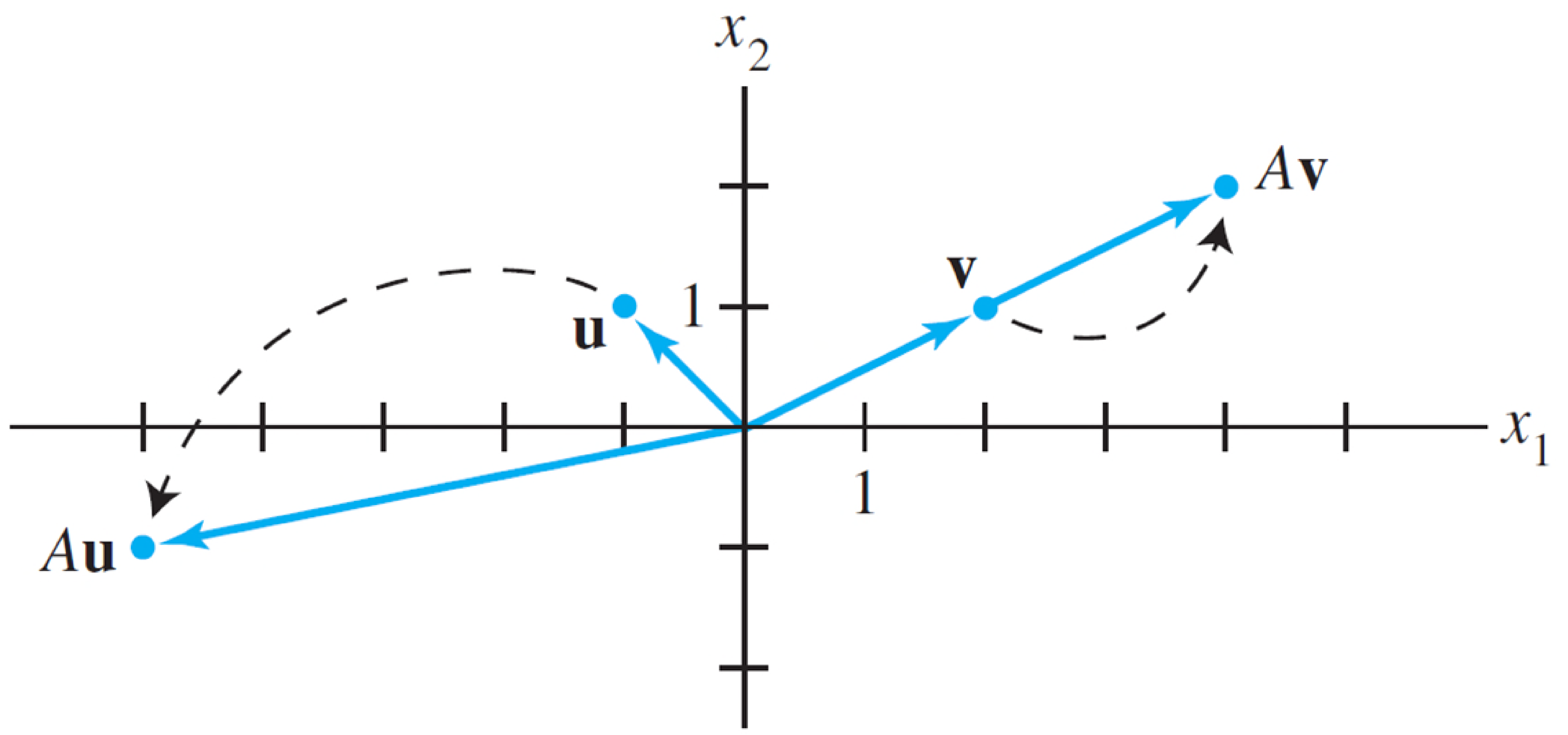


FIGURE 1 Effects of multiplication by A.

EigenVectors and Values

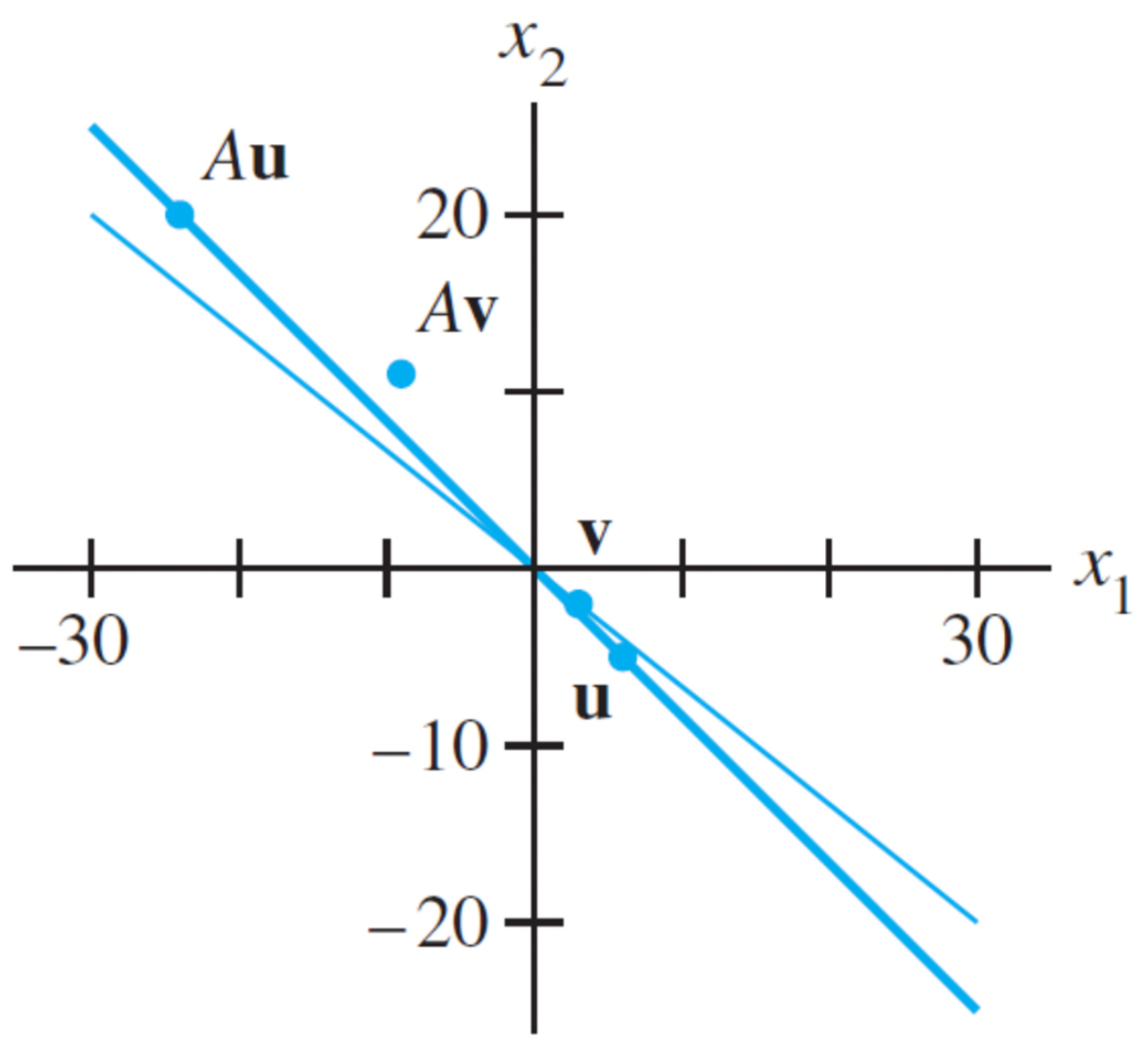
EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

SOLUTION

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Thus $\mathbf{u}$ is an eigenvector corresponding to an eigenvalue (-4), but $\mathbf{v}$ is not an eigenvector of A, because $A\mathbf{v}$ is not a multiple of $\mathbf{v}$. ■



$A\mathbf{u} = -4\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$.

Example: How to find EigenVectors

EXAMPLE 3 Show that 7 is an eigenvalue of matrix A in Example 2, and find the corresponding eigenvectors.

SOLUTION The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathbf{x} = 7\mathbf{x} \tag{1}$$

has a nontrivial solution. But (1) is equivalent to $A\mathbf{x} - 7\mathbf{x} = \mathbf{0}$, or

$$(A - 7I)\mathbf{x} = \mathbf{0} \tag{2}$$

To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

The columns of $A - 7I$ are obviously linearly dependent, so (2) has nontrivial solutions. Thus 7 *is* an eigenvalue of A. To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$. ■

Example: How to find EigenVectors

EigenSpace

The equivalence of equations (1) and (2) obviously holds for any λ in place of $\lambda = 7$. Thus λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3}$$

has a nontrivial solution. The set of *all* solutions of (3) is just the null space of the matrix $A - \lambda I$. So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ. The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

Note:
The $\lambda, and\ x$ that are solutions of equation (3) are the eigenvalue and vector respectively.

The eigenvector x that is a solution of the above is tied to the λ eigenvalue.

The eigenvalue could be repeated and
there could be more than 1 independent eigenvector.
(See Example 4)

Example: EigenSpace

EigenSpace

Example 3 shows that for matrix A in Example 2, the eigenspace corresponding to $\lambda = 7$ consists of *all* multiples of $(1, 1)$, which is the line through $(1, 1)$ and the origin. From Example 2, you can check that the eigenspace corresponding to $\lambda = -4$ is the line through $(6, -5)$. These eigenspaces are shown in Fig. 2, along with eigenvectors $(1, 1)$ and $(3/2, -5/4)$ and the geometric action of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ on each eigenspace.

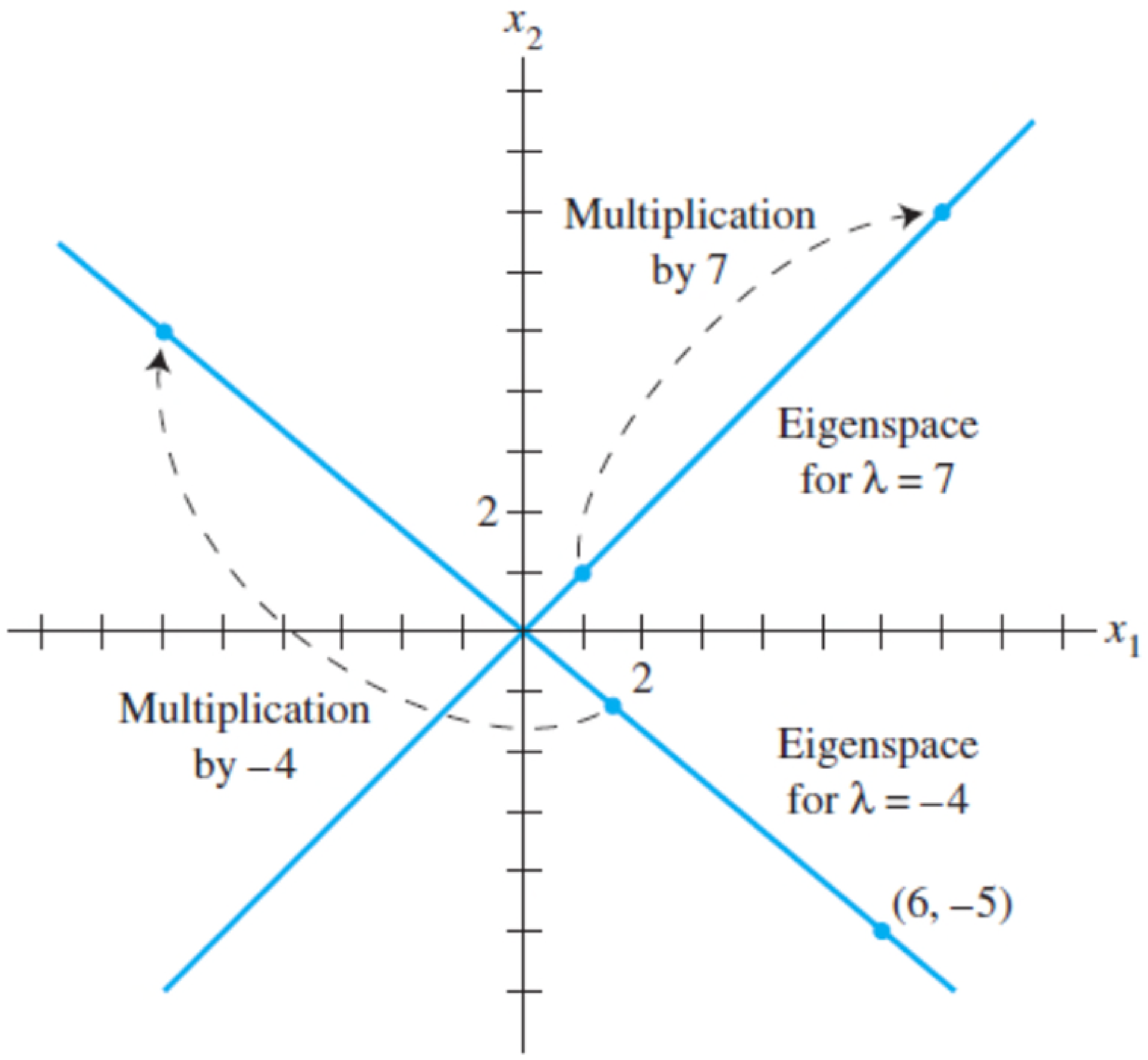


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

EigenSpace: Example two-dimensional subspace of R^3

Example:

In this example, there is repeated eigenvalue.

When this occur, it is possible to have more than 1 eigenvector for that eigenvalue.
I.e, the eigenspace's dimension is greater than 1.

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

SOLUTION Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lay,4thEd, pg 269

EigenSpace: Example two-dimensional subspace of R^3

At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A-2I)\mathbf{x} = \mathbf{0}$ has free variables. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 \text{ and } x_3 \text{ free}$$

The eigenspace, shown in Fig. 3, is a two-dimensional subspace of $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

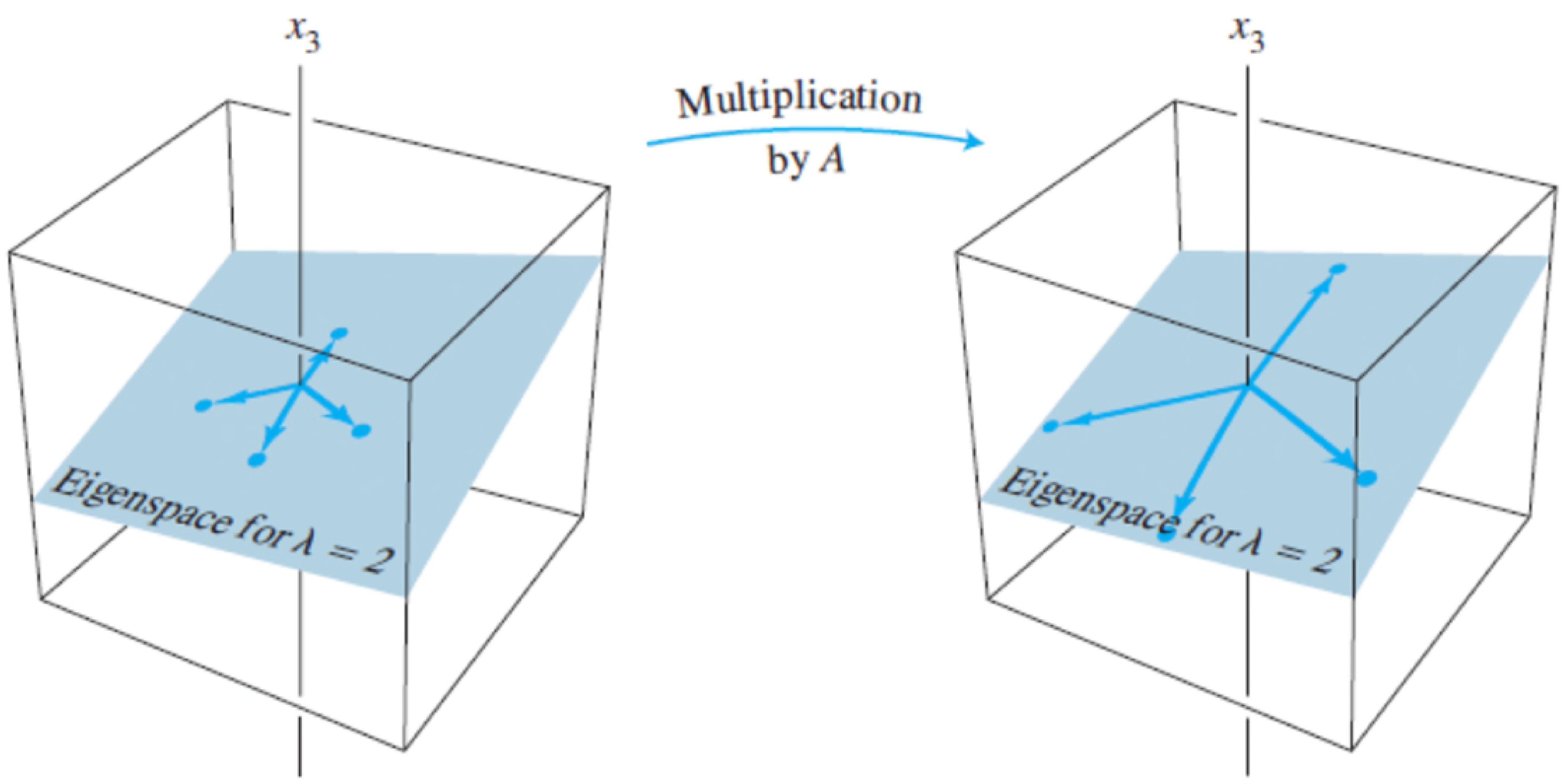


FIGURE 3 A acts as a dilation on the eigenspace.

Sanity check: Matlab

```
% Lay 4th edition, pg 268, Example 4,
% a matrix with repeated roots = -2

A = [4 -1 6; 2 1 6; 2 -1 8]
[P,D] = eig(A)
A_tst = P*D*inv(P)
```

```
A =

     4    -1     6
     2     1     6
     2    -1     8
```

```
A_tst =

    4.0000   -1.0000    6.0000
    2.0000    1.0000    6.0000
    2.0000   -1.0000    8.0000
```

```
P =

   -0.5774   -0.6122    0.3205
   -0.5774   -0.7873   -0.9112
   -0.5774    0.0728   -0.2587


D =

    9.0000         0         0
         0    2.0000         0
         0         0    2.0000
```

Matlab’s EIG function

eig

Eigenvalues and eigenvectors

Syntax

```
e = eig(A)
[V,D] = eig(A)
[V,D,W] = eig(A)
```

Description

`e = eig(A)` returns a column vector containing the eigenvalues of square matrix `A`. example

`[V,D] = eig(A)` returns diagonal matrix `D` of eigenvalues and matrix `V` whose columns are the corresponding right eigenvectors, so that `A*V = V*D`. example

Warning:

1) D is the eigenvalue BUT they are typically not sorted.
2) V are columns of the right eigenvectors AND they (the columsn) are normalized to norm 1. Hence they may be different to Lay’s example BUT their direction will be correct.

Matlab – sanity check

Note: Here, $\lambda_2 = \lambda_3 = 2$. Hence, there are two eigenvectors corresponding to the eigenvalue of 2. The eigen space can be formed by any set of basis vectors that span it. Hence, the eigen space spanned by the basis on LHS is the same eigen space spanned by the 2nd and 3rd columns of matrix U.

MATLAB

```
A =

     4    -1     6
     2     1     6
     2    -1     8

>> [U,D] = eig(A)

U =

   -0.5774   -0.6122    0.3205
   -0.5774   -0.7873   -0.9112
   -0.5774    0.0728   -0.2587
```

Eigenvector corresponding to λ_1

Eigenvector corresponding to λ_2

Eigenvector corresponding to λ_3

```
D =

    9.0000         0         0
         0    2.0000         0
         0         0    2.0000
```

λ_1 λ_2 λ_3

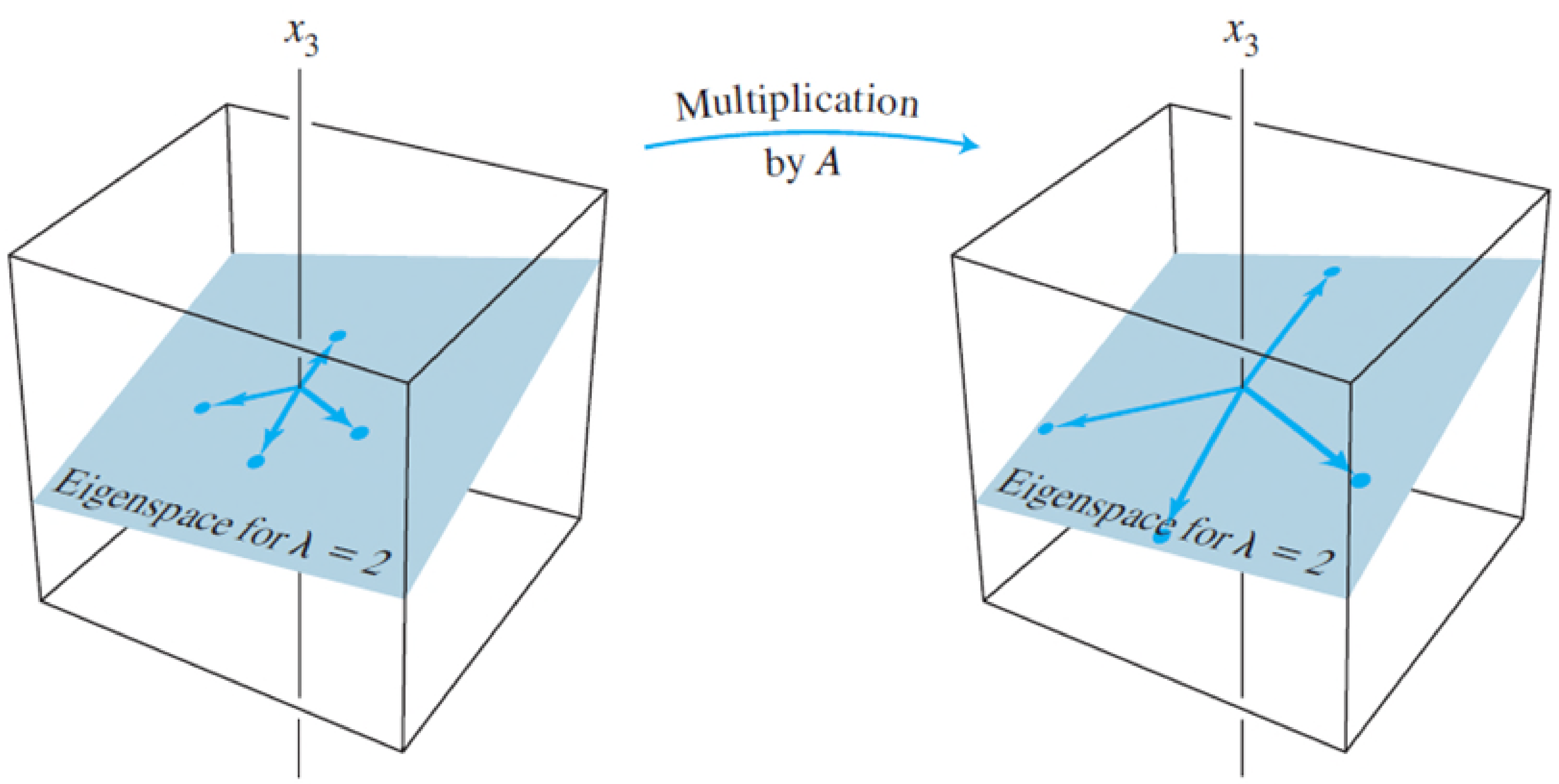


FIGURE 3 A acts as a dilation on the eigenspace.

Matlab provide solutions of eigenvectors such that the norm of the eigenvectors == 1.
When the matrix A is not symmetrical, the eigenvectors will not be orthogonal to each other.
In the special case when matrix is symmetric, eigen vectors will be orthogonal to each other.

Summary

To find the eigenvalue and vectors of a matrix, 2 steps:

a) First find the eigen value of the matrix.

- slide 8.1.2 shows how to find the eigen values.

b) Second, use Gaussian Elimination (row reduction) to find the solution of the homogenous equation

$(A - \lambda I)x = 0$

for each eigenvalue λ.

Example:

1) https://www.scss.tcd.ie/Rozenn.Dahyot/CS1BA1/SolutionEigen.pdf
2) https://lpsa.swarthmore.edu/MtrxVibe/EigMat/MatrixEigen.html

References Videos

Ritvik Math
- overview https://youtu.be/KTKAp9Q3yWg
- how to compute 2x2 eigvalue and vector https://youtu.be/glaiP222JWA

Eigen vector 33 mins: Brunton: https://youtu.be/ZSGrJBS_qtc

MIT Strang,
https://youtu.be/cdZnhQjJu4I lecture 21: (51 mins)
https://youtu.be/U8R54zOTVLw (11 mins)

Trevor Bazett:
https://youtu.be/4wTHFmZPhT0 (9. Min - geometric)
Examples -
https://youtu.be/LsZ-nNy0ZRs (4.5min example)
https://youtu.be/4u55V_Yp-QI (7 min)
https://youtu.be/EZkDtcyPP6Q (9 mins - repeated eigenvalues, complex eigenvalues,)

Zach Star: applications of eigen vectors values
https://youtu.be/i8FukKfMKCI