Matrices

Overview and Learning Outcomes

- Matrix Notations and Terminology
  - Determine the size of a given matrix
  - Identify the row vectors and column vectors of a given matrix
- Operations on Matrices
  - Perform matrix addition, subtraction, scalar multiplication and matrix multiplication
- Matrix Multiplication by Columns and by Rows
  - Compute matrix products using the column method and the row method
  - Express the product of a matrix and a column vector as a linear combination of the columns of a matrix

Overview and Learning Outcomes

- Transpose and Trace of a Matrix
  - Compute transpose of a matrix
  - Compute trace of a square matrix
- Inverse of a 2×2 Square Matrix

I. Matrix Notation and Terminology

A $m \times n$ matrix is a rectangular array of numbers with m rows and n columns written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

a_{11}, a_{12} and a_{mn} are called the elements/entries of the matrix. a_{ij} is the element at the i^{th} row and j^{th} column of the matrix. In simplified notation, an $m \times n$ matrix is written as $[a_{ij}]$.

A matrix with only one column (row) is called a **column (row) vector**.

A matrix with n rows and n columns is called a **square matrix of order n** and the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are said to be on the **main/leading diagonal** of the matrix.

I. Matrix Notation and Terminology(contd.)

The sum of the elements of the leading diagonal is called the **trace** of the square matrix A, i.e.,

$$\text{trace } \; A = a_{11} + a_{22} + \cdots + a_{nn}.$$

A **diagonal matrix** is a square matrix that has its only non-zero elements along the leading diagonal.

A **unit matrix** or **identity matrix** is a diagonal matrix for which $a_{11} = a_{22} = \cdots = a_{nn} = 1$.

I. Matrix Notation and Terminology(contd.)

The **transpose** of an $m \times n$ matrix A, denoted by A^T is the $n \times m$ matrix that results by interchanging rows and columns of A,

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{mn} \end{bmatrix}$$

Note that $(A^T)^T = A$.

If a square matrix A is such that $A^T = A$, then $a_{ij} = a_{ji}$. Such a matrix is called a **symmetric matrix**. If $A^T = -A$, so that $a_{ij} = -a_{ji}$, the matrix is called **skew-symmetric**.

***** End of Matrix Notation and Terminology *****

II. Operations on Matrices

Let $A = [a_{ij}], B = [b_{ij}]$ and $C = [c_{ij}]$ be $m \times n$ matrices.

Equality $(A = B)$: Two matrices A and B are defined to be **equal** if they have the same size and their corresponding elements are equal, i.e., $a_{ij} = b_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$.

Addition and Subtraction $(C = A + B,\ C = A - B)$: The **sum** $A + B$ is obtained by adding the entries of B to the corresponding entries of A, i.e., $c_{ij} = a_{ij} + b_{ij}$. Similar definition for subtraction.

Multiplication by scalar $(C = sA)$: If s is any scalar, then the **product** sA is the matrix obtained by multiyplying each entry of the matrix A by s. If $s = -1$, then $(-1)B$ is denoted by $-B$.

II. Operations on Matrices (contd.)

Let $A = [a_{ij}], B = [b_{ij}]$ and $C = [c_{ij}]$ be $m \times n$ matrices and s and q be scalars. The following properties can be proved from basic defintions.

- $A + B = B + A$
- $A + (B + C) = (A + B) + C = A + B + C$
- $(s + q)A = sA + qA$
- $q(sA) = (qs)A$
- $s(A + B) = sA + sB$
- $(A + B)^T = A^T + B^T$ and $(sA)^T = sA^T$

II. Operations on Matrices (contd.)

Matrix Multiplication $(C = AB)$: Let $A = [a_{ij}]$ be an $m \times r$ matrix and $B = [b_{ij}]$ be an $r \times n$ matrix, then the product $C = AB$ is the $m \times n$ matrix with elements

$$c[ij] = \sum_{k=1}^{r} = a_{ik}b_{kj} \quad \text{for} \quad i = 1, \ldots, m \;\; \text{and} \;\; j = 1, \ldots, n.$$

$$\begin{array}{ccccccccc} A & & & & B & & & AB & \\ m & \times & r & & r & \times & n = & m \times & n \end{array}$$

$$C = AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ \boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{ir}} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

Properties of Matrix Multiplication

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ but } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \underbrace{AB \neq BA}_{\text{in general}}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \underbrace{(AB = 0 \not\Rightarrow A = 0 \vee B = 0)}_{\text{in general}}$$

- $(kA)B = k(AB) = A(kB)$
- $A(BC) = (AB)C$
- $(A + B)C = AC + BC$
- $A(B + C) = AB + AC$
- $I_m A = AI_n = A$ where I_m is an $m \times m$ Identity Matrix
- $(AB)^T = B^T A^T$

Matrix Multiplication by Columns and by Rows

- AB computed column by column:

$$AB = A\left[\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{array}\right] = \left[\begin{array}{cccc} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{array}\right]$$

where $\mathbf{b}_j$ is the j^{th} colummn vector of B.

- AB computed row by row:

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix} \text{ where } \mathbf{a}_j \text{ is the } i^{th} \text{ row vector of A.}$$

Matrix Products as Linear Combinations

- AB computed as linear combinations:

$$A = \begin{bmatrix} a_{11} & a_{11} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ then } A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Theorem. *If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the elements of A.*

Inverse of a Matrix

Definition. If A is a square matrix, and if a matrix B can be constructed such that $AB = BA = I$, then A is said to be **invertible** and B is called the **inverse** of A. If no such matrix can be found, then A is not invertible and is said to be **singular**.

The inverse of A is denoted by A^{-1} so that $AA^{-1} = A^{-1}A = I$.

The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The quantity $ad - bc$ is called the **determinant** of the 2×2 matrix and is denoted by $|A|$ or $\det(A)$.

Theorem. *If A and B are invertible matrices of the same size then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

Theorem. *If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.*