Maths/LA/Tut8 EigenValue/Vectors

23 Oct 2021

Chng Eng Siong

Tutorial 8 Help links

Youtube link: playlist (Tut 8)

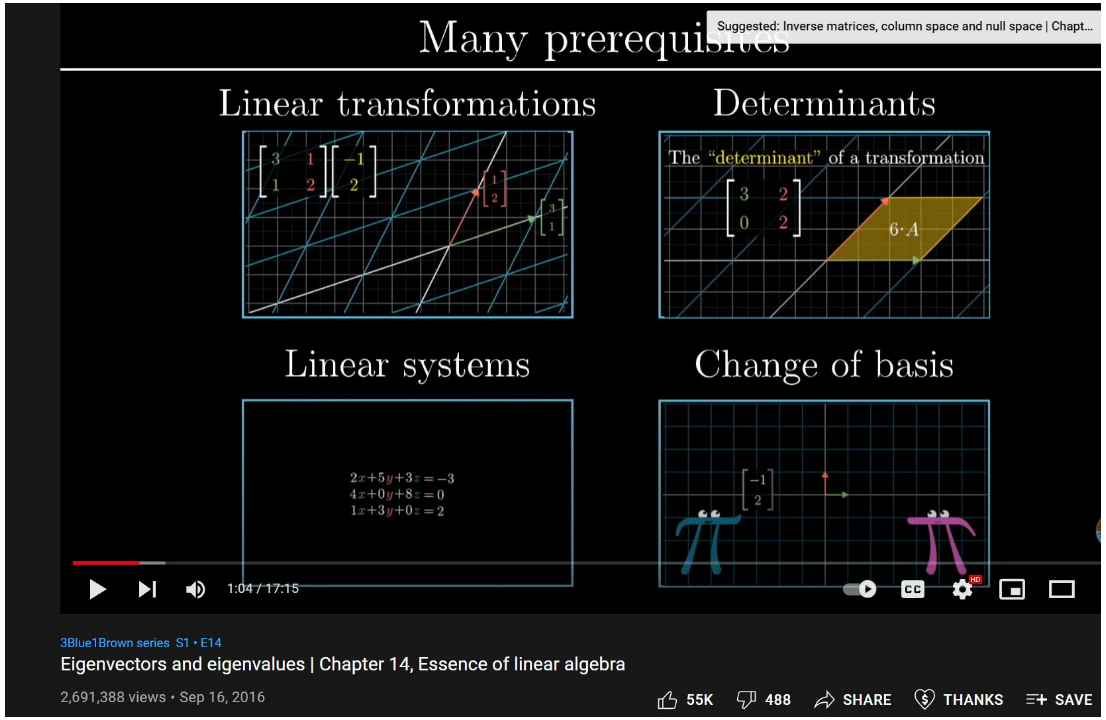
https://www.youtube.com/playlist?list=PLki3aFwg-9ewVB6R252cfd2QpJWLmtuGA

Solutions:

PDF:
https://www.dropbox.com/s/k1odexxe2jhs15r/Tut8revA_withSol.docx?dl=0

Insight to EigenValue/Vectors

- 3Blue1Bro Ch14
- https://www.youtube.com/watch?v=PFDu9oVAE-g



Q1) Intro to EigenValue/Vectors and examples how to calculate them

1) Prof Dave Explans (overview and 2x2 example)

https://www.youtube.com/watch?v=TQvxWaQnrqI

2) PatrickJMT (2x2 example)

https://www.youtube.com/watch?v=IdsV0RaC9jM

3) Adam Panagos: "Eigenvalue and Eigen Vector computation" 3x3 example:
https://www.youtube.com/watch?v=cHOsd2PhkqE&feature=youtu.be

https://www.adampanagos.org/courses/ala/eigenvaluesandeigenvectors

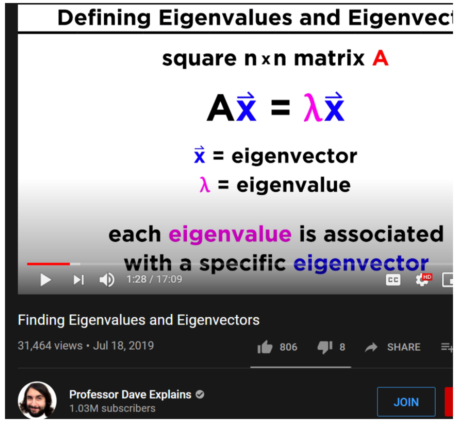
4) Scot Annin: EigenValue/Vector Intro

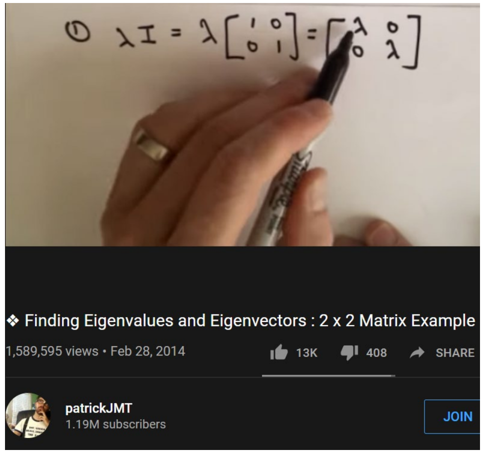
Part1: https://www.youtube.com/watch?v=Z3ULegjcArA

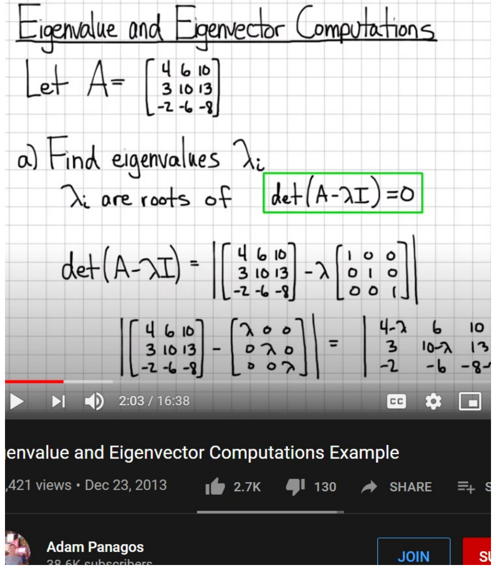
Part2: https://www.youtube.com/watch?v=o4nH1JgyfII

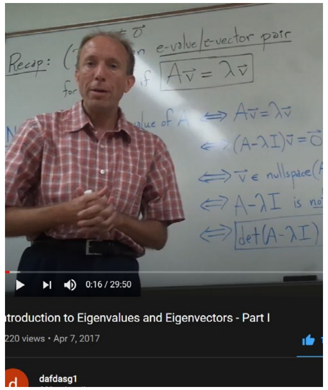
5) MIT Strang: Lecture 21 (18.06, Spring 2005)

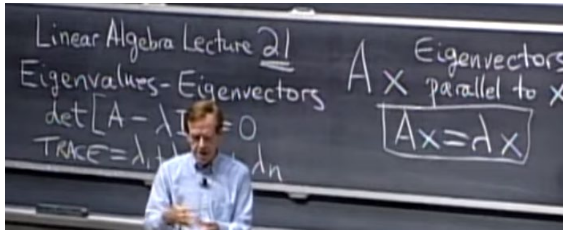
https://www.youtube.com/watch?v=cdZnhQjJu4I











Linear Algebra — Part 6: eigenvalues and eigenvectors

Sho Nakagome Following
Oct 14, 2018 · 5 min read

Q1) Characteristic Eqn/Polynomial

Characteristic Equation

The characteristic equation is the equation which is solved to find a matrix's eigenvalues, also called the characteristic polynomial. For a general $k \times k$ matrix $\mathbf{A}$, the characteristic equation in variable λ is defined by

$$\det(\mathbf{A} - \lambda\,\mathbf{I}) = 0,$$

Ref:

1)https://mathworld.wolfram.com/CharacteristicEquation.html

2)https://textbooks.math.gatech.edu/ila/characteristic-polynomial.html

Definition. Let A be an $n \times n$ matrix. The ***characteristic polynomial*** of A is the function $f(\lambda)$ given by

$$f(\lambda) = \det(A - \lambda I_n).$$

Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Solution

We have

$$\begin{aligned} f(\lambda) = \det(A - \lambda I_2) &= \det\left(\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) \\ &= \det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \\ &= (5-\lambda)(1-\lambda) - 2 \cdot 2 = \lambda^2 - 6\lambda + 1. \end{aligned}$$

The roots for $\lambda^2 - 6\lambda + 1$, are the eigen values of A.

Solution

In the above example we computed the characteristic polynomial of A to be $f(\lambda) = \lambda^2 - 6\lambda + 1$. We can solve the equation $\lambda^2 - 6\lambda + 1 = 0$ using the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}.$$

Therefore, the eigenvalues are $3 + 2\sqrt{2}$ and $3 - 2\sqrt{2}$.

Q1) Why determinant $(A - \lambda I) = 0$? To find eigenvalues?

Ref: https://math.stackexchange.com/questions/2619022/why-can-the-determinant-be-assumed-to-be-0

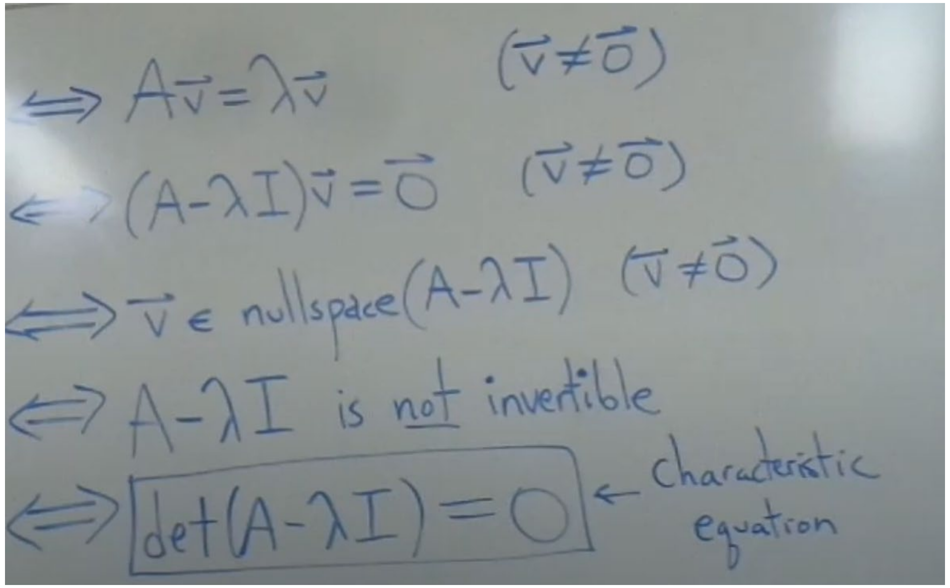
For a square matrix like $M = (A - \lambda I)$, the equation $Mx = 0$ will have a non-zero solution x if and only if M doesn't have an inverse, which is true if and only if the determinant of M is 0.

share cite edit follow

answered Jan 24 '18 at 12:27
Ben Grossmann
172k 10 118 236

Scott Annin's board: last eqn showing det(A-λI) $= 0$

Part1: https://www.youtube.com/watch?v=Z3ULegjcArA



As for *why* you are interested in the values of λ that make the determinant equal to 0, remember that

$$rank(A - \lambda I) = n \iff det(A - \lambda I) \neq 0$$

So, if $det(A - \lambda I) \neq 0$, you will find that the *only* solution to $(A - \lambda I)x = 0$ is $x = 0$ (due to the fact that the rank of the matrix is full, hence the kernel only contains the 0 vector). This means that the *only* x such that $Ax = \lambda x$ is $x = 0$, which means that x is *not* an eigenvector.

So the only way to have eigenvectors is to have the determinant of $A - \lambda I$ be equal to zero, so that's why to find eigenvalues you look for the values of λ that make $det(A - \lambda I) = 0$

share cite edit follow

edited Jan 24 '18 at 14:41

answered Jan 24 '18 at 12:32
Ant
19.9k 3 33 91

Diagonalization

Ref:

1)Ref: http://www2.math.uconn.edu/~troby/math2210f16/LT/sec5_3.pdf

$$\begin{aligned} AP &= PD \\ APP^{-1} &= PDP^{-1} \\ A &= PDP^{-1} \end{aligned}$$

Where P = eigenvector,
D = diagonal matrix of eigen values

5.3 Diagonalization

The goal here is to develop a useful factorization $A = PDP^{-1}$, when A is $n \times n$. We can use this to compute A^k quickly for large k.

The matrix D is a *diagonal* matrix (i.e. entries off the main diagonal are all zeros).

D^k is trivial to compute as the following example illustrates.

EXAMPLE: Let $D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$. Compute D^2 and D^3. In general, what is D^k, where k is a positive integer?

Solution:

$$D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} & 0 \\ 0 & \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} 5^2 & 0 \\ 0 & 4^2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} & 0 \\ 0 & \end{bmatrix}$$

and in general,

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix}$$

THEOREM 5 The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

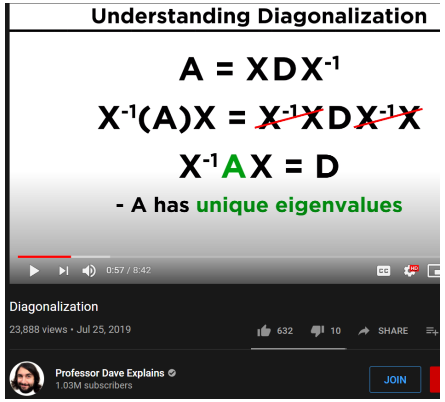
Q7) Diagonalization and Powers of A

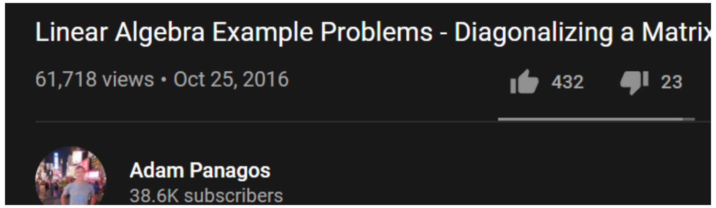
1) Prof Dave Expains: “Diagonalization”
https://www.youtube.com/watch?v=WTLl03D4TNA

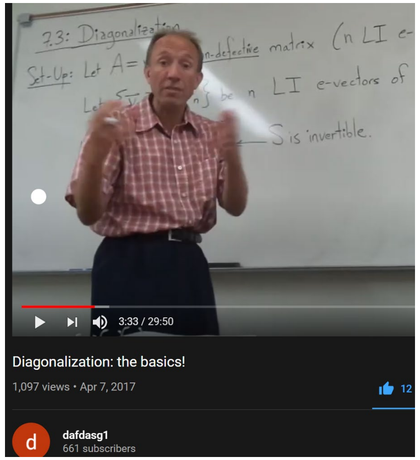
2) Adams Panagos:
https://www.youtube.com/watch?v=zEoHJfiQvt8

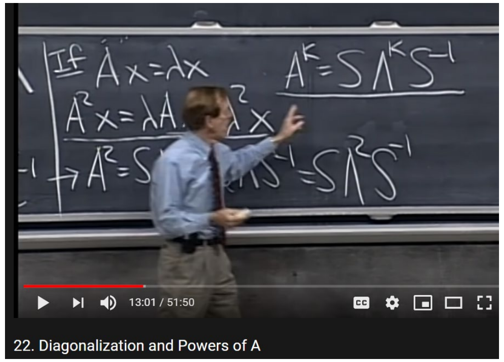
3) Prof Scott Annin, Cal State U Fullerton:
Dafdasg1 video: Diagonalization:
https://www.youtube.com/watch?v=6FknM_bPhUk

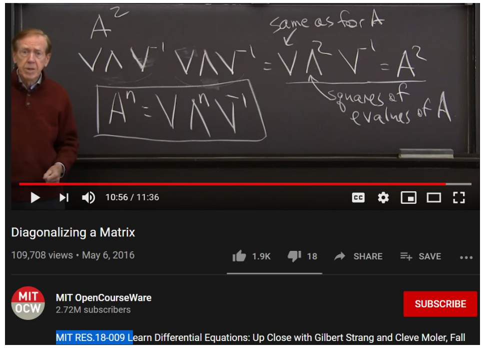
4) Strang:
   A) MIT Strang L22 (Spring 2005, 18.06):
   Diagonalization and Powers of A
   https://www.youtube.com/watch?v=13r9QY6cmjc
   B) MIT 2015 “Diagonalizing a Matrix”,
   https://www.youtube.com/watch?v=U8R54zOTVLw











Q7) Lay's example, 5th edition, pg 280
Application of diagonalization and power of A

Application to Dynamical Systems

Eigenvalues and eigenvectors hold the key to the discrete evolution of a dynamical system, as mentioned in the chapter introduction.

EXAMPLE 5 Let $A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$. Analyze the long-term behavior of the dynamical system defined by $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \dots)$, with $\mathbf{x}_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

SOLUTION The first step is to find the eigenvalues of A and a basis for each eigenspace. The characteristic equation for A is

$$\begin{aligned} 0 &= \det \begin{bmatrix} .95 - \lambda & .03 \\ .05 & .97 - \lambda \end{bmatrix} = (.95 - \lambda)(.97 - \lambda) - (.03)(.05) \\ &= \lambda^2 - 1.92\lambda + .92 \end{aligned}$$

By the quadratic formula

$$\begin{aligned} \lambda &= \frac{1.92 \pm \sqrt{(1.92)^2 - 4(.92)}}{2} = \frac{1.92 \pm \sqrt{.0064}}{2} \\ &= \frac{1.92 \pm .08}{2} = 1 \quad \text{or} \quad .92 \end{aligned}$$

It is readily checked that eigenvectors corresponding to $\lambda = 1$ and $\lambda = .92$ are multiples of

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

respectively.

The next step is to write the given $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. This can be done because $\{\mathbf{v}_1, \mathbf{v}_2\}$ is obviously a basis for $\mathbb{R}^2$. (Why?) So there exist weights c_1 and c_2 such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{3}$$

In fact,

$$\begin{aligned} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= [\,\mathbf{v}_1 \quad \mathbf{v}_2\,]^{-1}\mathbf{x}_0 = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} .60 \\ .40 \end{bmatrix} \\ &= \frac{1}{-8} \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} .60 \\ .40 \end{bmatrix} = \begin{bmatrix} .125 \\ .225 \end{bmatrix} \end{aligned} \tag{4}$$

Because $\mathbf{v}_1$ and $\mathbf{v}_2$ in (3) are eigenvectors of A, with $A\mathbf{v}_1 = \mathbf{v}_1$ and $A\mathbf{v}_2 = .92\mathbf{v}_2$, we easily compute each $\mathbf{x}_k$:

$$\begin{aligned} \mathbf{x}_1 &= A\mathbf{x}_0 = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 && \text{Using linearity of } \mathbf{x} \mapsto A\mathbf{x} \\ &= c_1\mathbf{v}_1 + c_2(.92)\mathbf{v}_2 && \mathbf{v}_1 \text{ and } \mathbf{v}_2 \text{ are eigenvectors.} \\ \mathbf{x}_2 &= A\mathbf{x}_1 = c_1 A\mathbf{v}_1 + c_2(.92)A\mathbf{v}_2 \\ &= c_1\mathbf{v}_1 + c_2(.92)^2\mathbf{v}_2 \end{aligned}$$

and so on. In general,

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(.92)^k\mathbf{v}_2 \quad (k = 0, 1, 2, \dots)$$

Using c_1 and c_2 from (4),

$$\mathbf{x}_k = .125 \begin{bmatrix} 3 \\ 5 \end{bmatrix} + .225(.92)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (k = 0, 1, 2, \dots) \tag{5}$$

This explicit formula for $\mathbf{x}_k$ gives the solution of the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$. As $k \to \infty$, $(.92)^k$ tends to zero and $\mathbf{x}_k$ tends to $\begin{bmatrix} .375 \\ .625 \end{bmatrix} = .125\mathbf{v}_1$. ■

Q8 (21a) You can generate your own $A = PDP^{-1}$

You can generate your own A, and use [P,D] = eig(A) (Matlab) to check.
As long as your P is invertible and your D is diagonal.

E.g,

```
P =

    -1     1     1
    -1    -1     1
    -1     0    -1

>> D

D =

   -3.0000         0         0
         0    2.0000         0
         0         0    1.0000

>> A=P*D*inv(P)

A =

    0.5000   -1.5000   -2.0000
   -1.5000    0.5000   -2.0000
   -1.0000   -1.0000   -1.0000
```

```
>> A*P

ans =

    3.0000    2.0000    1.0000
    3.0000   -2.0000    1.0000
    3.0000         0   -1.0000
```

```
>> [P1,D1]=eig(A)

P1 =

    0.5774    0.7071   -0.5774
    0.5774   -0.7071   -0.5774
    0.5774   -0.0000    0.5774


D1 =

   -3.0000         0         0
         0    2.0000         0
         0         0    1.0000
```

Note, Matlab produced normalized columns of P, and the sorting of D and P may be arbitrary (not sorted according to eigenvalues)

Additionally, the eigenvector can be –ve direction!
Eig P(:,1) is –P1(:,1)
And P(:,3) is – P1(:,3)

Q8 (21c) When is a matrix diagonalizable

When there is a basis of eigenvectors, we can diagonalize the matrix. Where there is not, we can't. We can come close, but that's another very complicated story. So the good case is when the geometric multiplicity of each eigenvalue equals its algebraic multiplicity because then there are just enough eigenvectors to make a basis.

Definition: the *algebraic multiplicity* of an eigenvalue e is the power to which $(\lambda - e)$ divides the characteristic polynomial.

Definition: the *geometric multiplicity* of an eigenvalue is the number of linearly independent eigenvectors associated with it. That is, it is the dimension of the nullspace of $A - eI$.

In the example above, 1 has algebraic multiplicity two and geometric multiplicity 1. It is always the case that the algebraic multiplicity is at least as large as the geometric:

Theorem: if e is an eigenvalue of A then its algebraic multiplicity is at least as large as its geometric multiplicity.

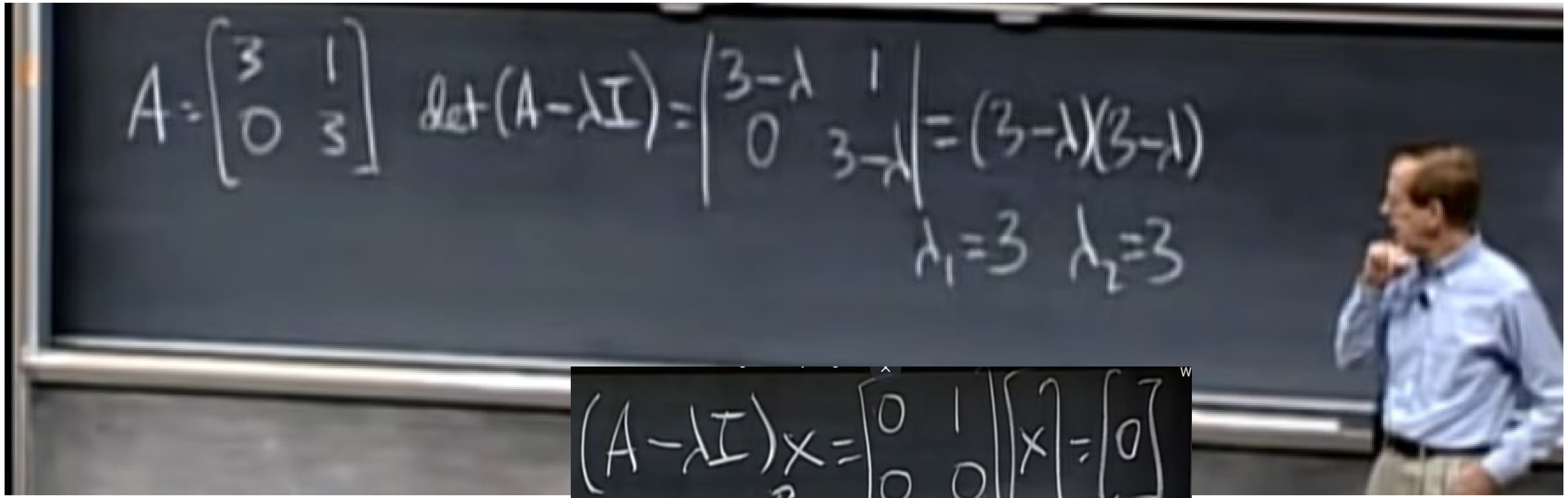
- https://staff.imsa.edu/~fogel/LinAlg/PDF/44%20Multiplicity%20of%20Eigenvalues.pdf

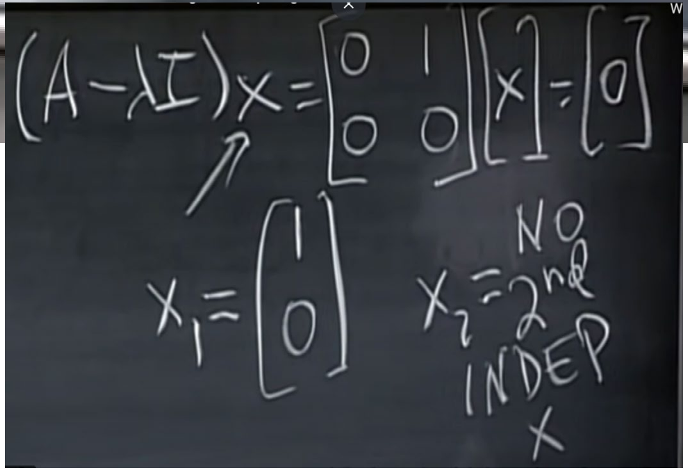
Q8 Degenerate Matrix – Strang
Not enough eigenvectors

- http://www.teachingtree.co/watch/degenerate-matrix-eigenvectors
- Strang: Lect **Degenerate Matrix Eigenvectors**
- Lec 21 | MIT 18.06 Linear Algebra, Spring 2005

Note, Matlab produced the same eigenvector! P(:,1) vs P(:,2)
P(:,2) = -ve P(:,1) which is dependent! Hence useless eigenvector

```
>> A = [3 1; 0 3]

A =

     3     1
     0     3

>> [P,D] = eig(A)

P =

    1.0000   -1.0000
         0    0.0000


D =

     3     0
     0     3
```





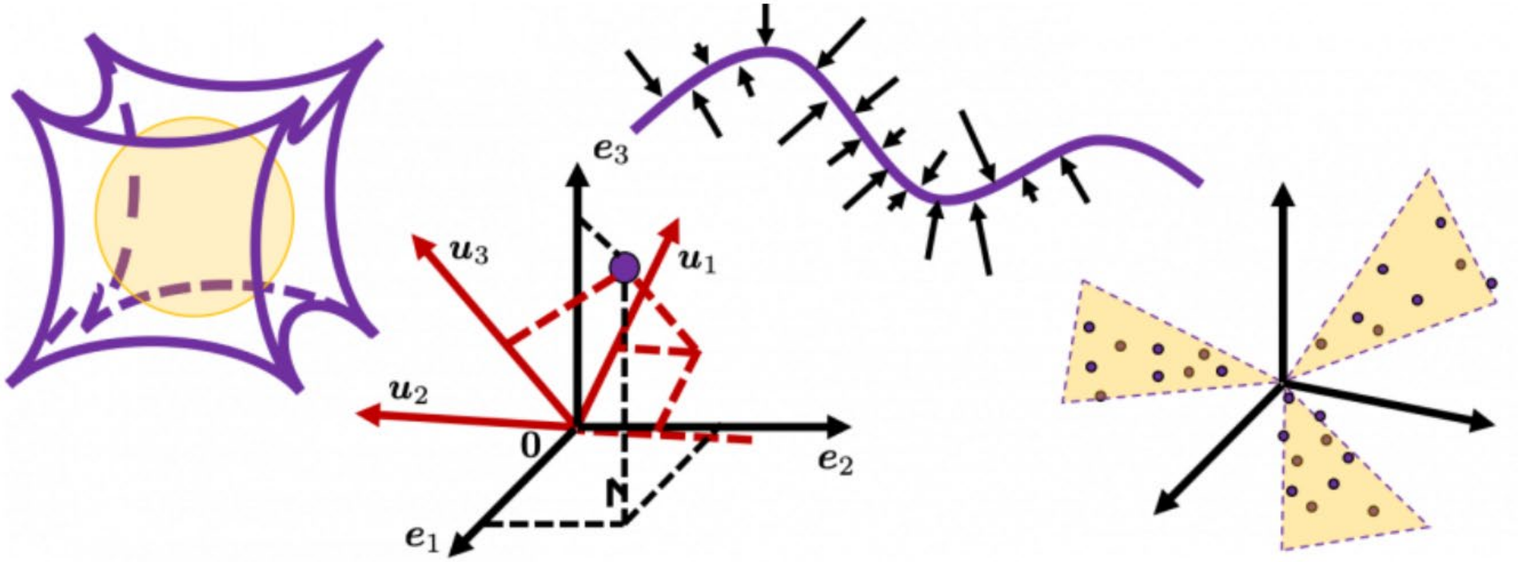
Q9,10,11) Eigenvectors and Linear Transformation

1) Paul Cartie: https://www.youtube.com/watch?v=DI6BU1sMaLs
2) Purdue Math:https://www.math.purdue.edu/~bkrummel/ma265_lecture5_4.pdf
3) U.of Michigan: http://www.math.lsa.umich.edu/~kesmith/CoordinateChange.pdf

4) Worked examples:
   a) https://math.stackexchange.com/questions/16386/eigenvalues-and-eigenvectors-of-linear-transformations
   b) https://math.stackexchange.com/questions/2560162/change-of-basis-for-linear-transformation-linear-algebra
   b) https://yutsumura.com/linear-algebra/eigenvalues-and-eigenvectors-of-linear-transformations/

5) CS Blog in DataScience by Yasuto Tamura:
https://data-science-blog.com/blog/2020/10/27/10360/

Q9,10,11) Eigenvectors and Linear Transformation



Rethinking linear algebra: visualizing linear transformations and eigenvectors

October 27, 2020 / in Data Mining, Data Science, Machine Learning, Main Category, Mathematics / by Yasuto Tamura

4) CS Blog in DataScience by Yasuto Tamura:
https://data-science-blog.com/blog/2020/10/27/10360/

Q9,10,11) Matrix of Linear Transformation with respect to a change of basis

Ref

1) Arnold Yim: Basis, coordinate system, and change of basis
   https://www.youtube.com/watch?v=IpKUPhNHFQA&list=PLUhmPGIKgSbIlTyKCvvZ6BLOcIgyHMLTr

2) Adams Panagos: https://www.youtube.com/watch?v=VG4-8yW3Ce8