

# Determinants

# Overview and Learning Outcomes

- Properties of determinants
  - Interpret properties of determinants
- Determinants as area/volume
  - Interpret geometric properties of determinants
- Linear Transformations
  - Interpret geometry of linear transformations by determinants
  - Compute change of area/volume using determinants

## 3.1 Properties of determinants

1. **The determinant of the  $n \times n$  identity matrix is 1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \text{ and } \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1$$

2. **The determinant changes sign when two rows are exchanged.**

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

If  $P$  is a permutation matrix with  $r$  row exchanges, then  $|P| = 1$  for even  $r$  and  $|P| = -1$  for odd  $r$ .

### 3. The determinant is a linear function of each row separately.

If 1 row of a matrix  $A$  is multiplied by  $t$  to get  $A'$ , then  $|A'| = t|A|$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

If one row of  $A$  is added to one row of  $A'$ , then the determinants add.

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Important: *This rule applies only when the other rows do not change.*

### 4. If two rows of $A$ are equal, then $|A| = 0$ .

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$$

This follows from Rule 2 (Show!).

**5. Subtracting a multiple of one row from another row leaves  $|A|$  unchanged.**

$$\begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

This follows from Rule 3 and Rule 4.

$|A| = |U|$  without row exchanges and  $|A| = \pm|U|$  with row exchanges.

**6. A matrix with a row of zeros has  $|A| = 0$ .**

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0.$$

This follows from Rule 4 and Rule 5.

**7. If  $A$  is triangular, then  $|A| = a_{11}a_{22} \dots a_{nn} = \text{product of diagonal matrices.}$**

Consider the determinant of a diagonal matrix:

$$\begin{vmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{vmatrix} = D$$

Factor  $a_{11}$  from the first row. By rule 3,  $D = a_{11}D'$ .

Factor  $a_{22}$  from the second row. By rule 3,  $D = a_{11}a_{22}D''$ .

Finally, factor  $a_{nn}$  from the last row. By rule 3,  $D = a_{11}a_{22} \dots a_{nn}|I|$ .

From rule 1,  $|I| = 1$ . So,  $D = a_{11}a_{22} \dots a_{nn}$ .

Now, consider the determinants for the following triangular matrices

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = D_1 \text{ and } \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = D_2.$$

Make the off diagonal elements 0 through elimination.

$$R_1 \leftarrow R_1 - \frac{b}{d}R_2 : D'_1 = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} = ad$$

$$R_2 \leftarrow R_2 - \frac{c}{a}R_1 : D'_2 = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} = ad$$

If an  $a_{ii} = 0$ , elimination produces a zero row.

By Rule 5, determinant is unchanged and by Rule 6, determinant = 0.

Such matrices are called **singular**.

8. If  $A$  is singular, then  $|A| = 0$ . If  $A$  is invertible, then  $|A| \neq 0$ .

Transform  $A$  to  $U$  through elimination.

If  $A$  is singular:

- $U$  has a zero row
- From previous rules,  $|A| = |U| = 0$

If  $A$  is invertible:

- $U$  has pivots along its diagonal
- From Rule 7, product of non-zero pivots  $\Rightarrow$  non zero determinant
- $|A| = \pm|U| = \pm(\text{product of pivots})$

[+ for even number of row exchanges and  $-$  for odd number of row exchanges]

Pivots of a  $2 \times 2$  matrix ( $a \neq 0$ ):

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - (c/b)a \end{vmatrix} = ad - bc \quad (\text{Finally, a formula for the determinant!!})$$



9.  $|AB| = |A||B|$ .

Consider the ratio  $D(A) = |AB|/|B|$ . If  $D(A)$  satisfies rules 1, 2 and 3, then it is a determinant.

- Rule 1 (*Determinant of I*)
  - If  $A = I$ , then  $D(A) = |B|/|B| = 1$
- Rule 2 (*Sign reversal*)
  - Two rows of  $A$  are exchanged  $\Rightarrow$  Same two rows of  $|AB|$  are exchanged  $\Rightarrow |AB|$  changes sign  $\Rightarrow D(A)$  changes sign
- Rule 3 (*Linearity*)
  - When 1 row of  $A$  is multiplied by  $t \Rightarrow$  so is 1 row of  $AB \Rightarrow |AB|$  is multiplied by  $t \Rightarrow D(A)$  is multiplied by  $t$
  - When 1 row of  $A$  is added to 1 row of  $A' \Rightarrow$  1 row of  $AB$  is added to 1 row of  $A'B \Rightarrow$  determinants add  $\Rightarrow$  dividing by  $B$ , the ratios add

The ratio  $|AB|/|B|$  has the same properties that define  $|A|$ .

Therefore,  $|AB|/|B| = |A| \Rightarrow |AB| = |A||B|$

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If  $|B| = 0$ ,  $B$  is singular  $\Rightarrow AB$  is singular  $\Rightarrow |AB| = 0$

$$|A||B| = 0$$

Therefore  $|AB| = |A||B|$

$$10. |A^T| = |A|$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

All the above properties apply to *columns* also.

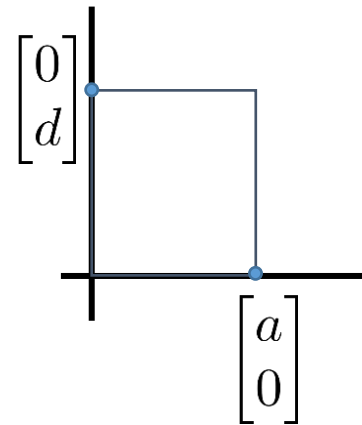
## 3.2 Determinants as Area or Volume

- Geometric interpretation of determinants

**Theorem 3.1.** *If  $A$  is a  $2 \times 2$  matrix, the area of the parallelogram determined by the columns of  $A$  is  $|A|$ . If  $A$  is a  $3 \times 3$  matrix, the volume of the parallelepiped determined by the columns of  $A$  is  $|A|$ .*

*Proof.* True for a  $2 \times 2$  diagonal matrix:

$$\text{abs}\left(\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}\right) = \text{abs}(ad) = \text{area of rectangle}$$



Can we transform any  $2 \times 2$  matrix  $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$  into a diagonal matrix without change in area of the associated parallelogram or in  $|A|$ ?

$A$  can be transformed into a diagonal matrix by:

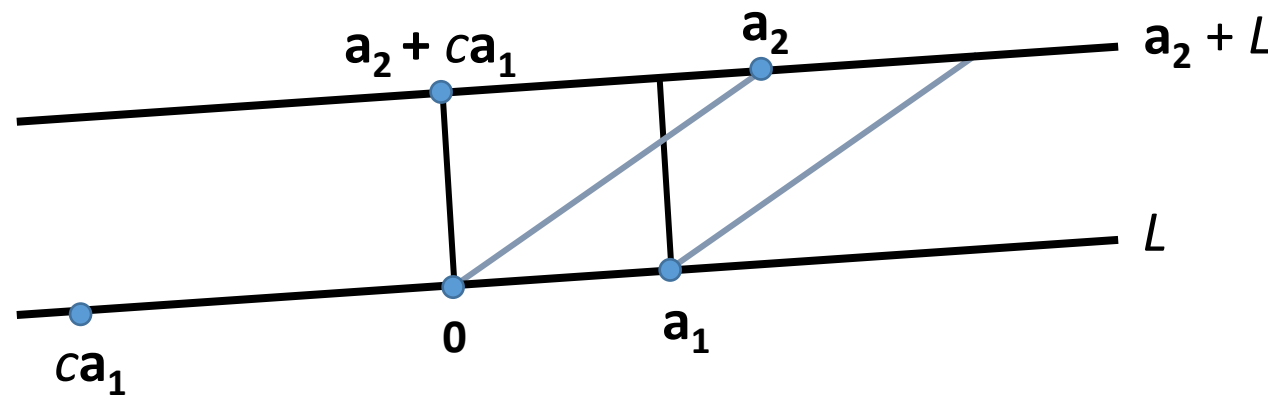
- Interchanging two columns
  - Does not change the parallelogram
  - From property 2,  $|A|$  is unchanged

Remember: properties apply to *columns* also.

- Adding a multiple of one column to another

Prove the following geometric observation:

Let  $\mathbf{a}_1$  and  $\mathbf{a}_2$  be nonzero vectors. Then for any scalar  $c$ , the area of a parallelogram determined by  $\mathbf{a}_1$  and  $\mathbf{a}_2$  equals the area of the parallelogram determined by  $\mathbf{a}_1$  and  $\mathbf{a}_2 + c\mathbf{a}_1$ .

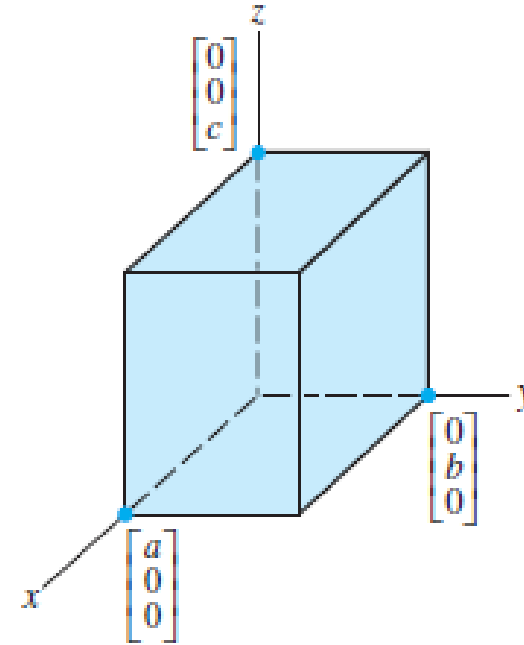


Assume  $\mathbf{a}_2$  is not a multiple of  $\mathbf{a}_1$ .

- $L$  is the line through  $\mathbf{0}$  and  $\mathbf{a}_1 \Rightarrow \mathbf{a}_2 + L$  is the line through  $\mathbf{a}_2$  and parallel to  $L$
- Points  $\mathbf{a}_2$  and  $\mathbf{a}_2 + c\mathbf{a}_1$  have the same perpendicular distance to  $L$
- Hence, two parallelograms have the same area (base  $\times$  height)

Proof for  $\mathbb{R}^3$  (i.e.,  $3 \times 3$  matrix):

True for a  $3 \times 3$  diagonal matrix  $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$



Can we transform any  $3 \times 3$  matrix  $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$  into a diagonal matrix without change in volume of the associated parallelepiped or in  $|A|$ ?

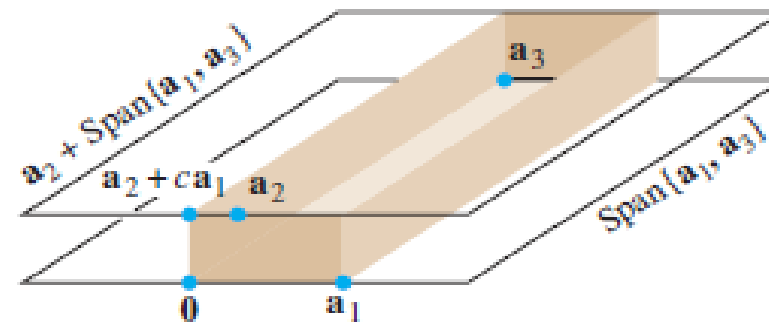
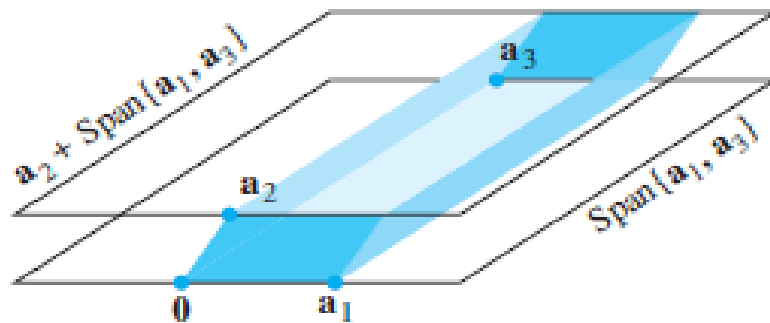
$A$  can be transformed into a diagonal matrix by:

- Interchanging two columns (same as row operations on  $A^T$ )
  - Does not change the parallelepiped

- Adding a multiple of one column to another

In the figure below:

- Volume of parallelepiped = area of base  $\times$  height
- Base is in  $\text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$       Height =  $\mathbf{a}_2 + \text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$
- $\mathbf{a}_2 + c\mathbf{a}_1$  lies in the plane  $\mathbf{a}_2 + \text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$ , which is parallel to  $\text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$
- Hence, any vector  $\mathbf{a}_2 + c\mathbf{a}_1$  has the same height as  $\mathbf{a}_2$
- Therefore, the volume of the parallelepiped is unchanged when  $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$  is changed to  $[\mathbf{a}_1 \quad \mathbf{a}_2 + c\mathbf{a}_1 \quad \mathbf{a}_3]$

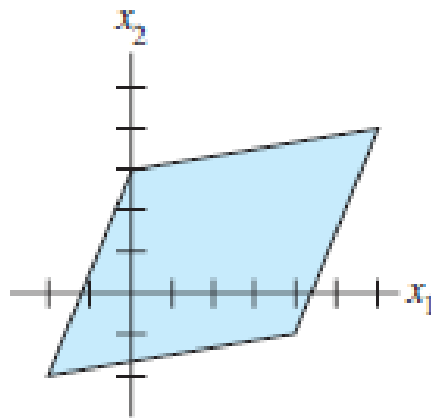


Exercise 3.2.1

Calculate the area of the parallelogram determined by the points  $(-2, -2)$ ,  $(0, 3)$ ,  $(4, -1)$ ,  $(6, 4)$ .

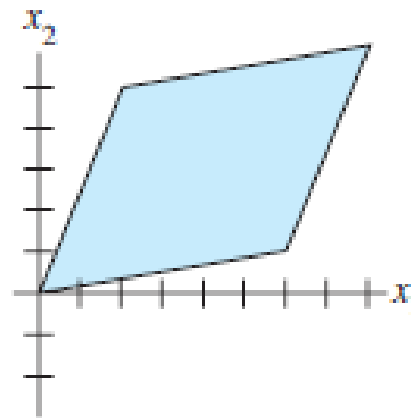
Solution

Translate the parallelogram to one having the origin as a vertex, e.g., subtract  $(-2, -2)$  from each of the four vertices.



(a)

Translating a parallelogram does not change its area



(b)

New vertices are at  $(0, 0)$ ,  $(2, 5)$ ,  $(6, 1)$ ,  $(8, 6)$ .

This parallelogram is determined by the columns of  $A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$

$$\text{abs}(|A|) = |-28|$$

Therefore, area of the parallelogram is 28.



### 3.3 Linear Transformations

- How does the area (or volume) of a transformed set compare with the area (or volume) of the original

**Theorem 3.2.** *Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation determined by a  $2 \times 2$  matrix  $A$ . If  $S$  is a parallelogram in  $\mathbb{R}^2$ , then area of  $T(S) = \text{abs}(|A|) \times \text{area of } S$ .*

*If  $T$  is determined by a  $3 \times 3$  matrix  $A$ , and if  $S$  is a parallelepiped in  $\mathbb{R}^3$ , then volume of  $T(S) = \text{abs}(|A|) \times \text{volume of } S$ .*

*Proof.*

Consider the  $2 \times 2$  case,  $A = [\mathbf{a}_1 \quad \mathbf{a}_2]$

A parallelogram at the origin in  $\mathbb{R}^2$  determined by the vectors  $\mathbf{b}_1$  and  $\mathbf{b}_2$  has the form

$$S = \{s_1\mathbf{b}_1 + s_2\mathbf{b}_2 : 0 \leq s_1 \leq 1, 0 \leq s_2 \leq 1\}$$

The image of  $S$  under  $T$  consists of the points of the form

$$T(s_1\mathbf{b}_1 + s_2\mathbf{b}_2) = s_1T(\mathbf{b}_1) + s_2T(\mathbf{b}_2) = s_1A\mathbf{b}_1 + s_2A\mathbf{b}_2,$$

where  $0 \leq s_1 \leq 1, 0 \leq s_2 \leq 1$ .

$T(S)$  is the parallelogram determined by columns of  $[A\mathbf{b}_1 \quad A\mathbf{b}_2] = AB$  where  $B = [\mathbf{b}_1 \quad \mathbf{b}_2]$ .

$$\text{area of } T(S) = \text{abs}(|AB|) = (\text{abs}|A|)(\text{abs}|B|) = (\text{abs}|A|)(\text{area of } S)$$

Now for the general case:

An arbitrary parallelogram has the form  $\mathbf{p} + S$   
 where  $\mathbf{p}$  is a vector and  $S$  is a parallelogram at the origin.

$$T(\mathbf{p} + S) = T(\mathbf{p}) + T(S)$$

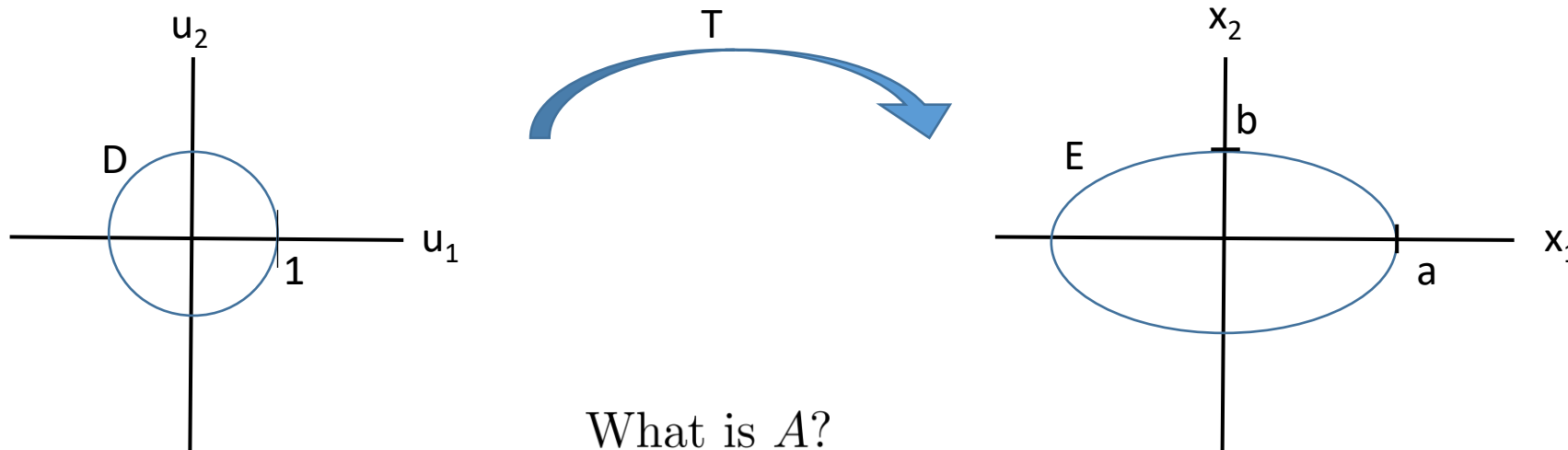
Translation does not affect the area of a set

$$\begin{aligned} \text{area of } T(\mathbf{p} + S) &= \text{area of } (T(\mathbf{p}) + T(S)) \\ &= \text{area of } T(S) \\ &= \text{abs}(|A|) \times \text{area of } S \\ &= \text{abs}(|A|) \times \text{area of } \mathbf{p} + S \end{aligned}$$

Proof for  $3 \times 3$  is analogous.

Theorem 3.2 is applicable for arbitrary shapes also.

### Example



What is  $A$ ?

$$\begin{aligned}
 \text{area of ellipse} &= \text{area of } T(D) \\
 &= \text{abs}(|A|) \times \text{area of } D \\
 &= ab \times \pi 1^2 = \pi ab
 \end{aligned}$$

If  $\mathbf{x} = A\mathbf{u}$  with  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ,  
 equation of ellipse given by  
 $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$ , what is  $\mathbf{u}$ ?

\*\*\*\*\* END OF CHAPTER \*\*\*\*\*