CX1104: **Linear Algebra for Computing**

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

Chap. No : **8A.1**

Lecture : **Complex Numbers**

Topic : **Overview of Complex Numbers**

Concept : **Definitions, Motivations and Arithmetic**

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Rev: 23rd July 2024

Content: Complex Numbers

8A.1: Introduction to complex numbers+ Arithmetic Operations (1.5 hour)

8A.1.1 - Definitions and Motivations
- number system
- solving quadratic equation for complex roots

8A.1.2 - The argand diagram, and geometric interpretation
- The polar form, The unit circle

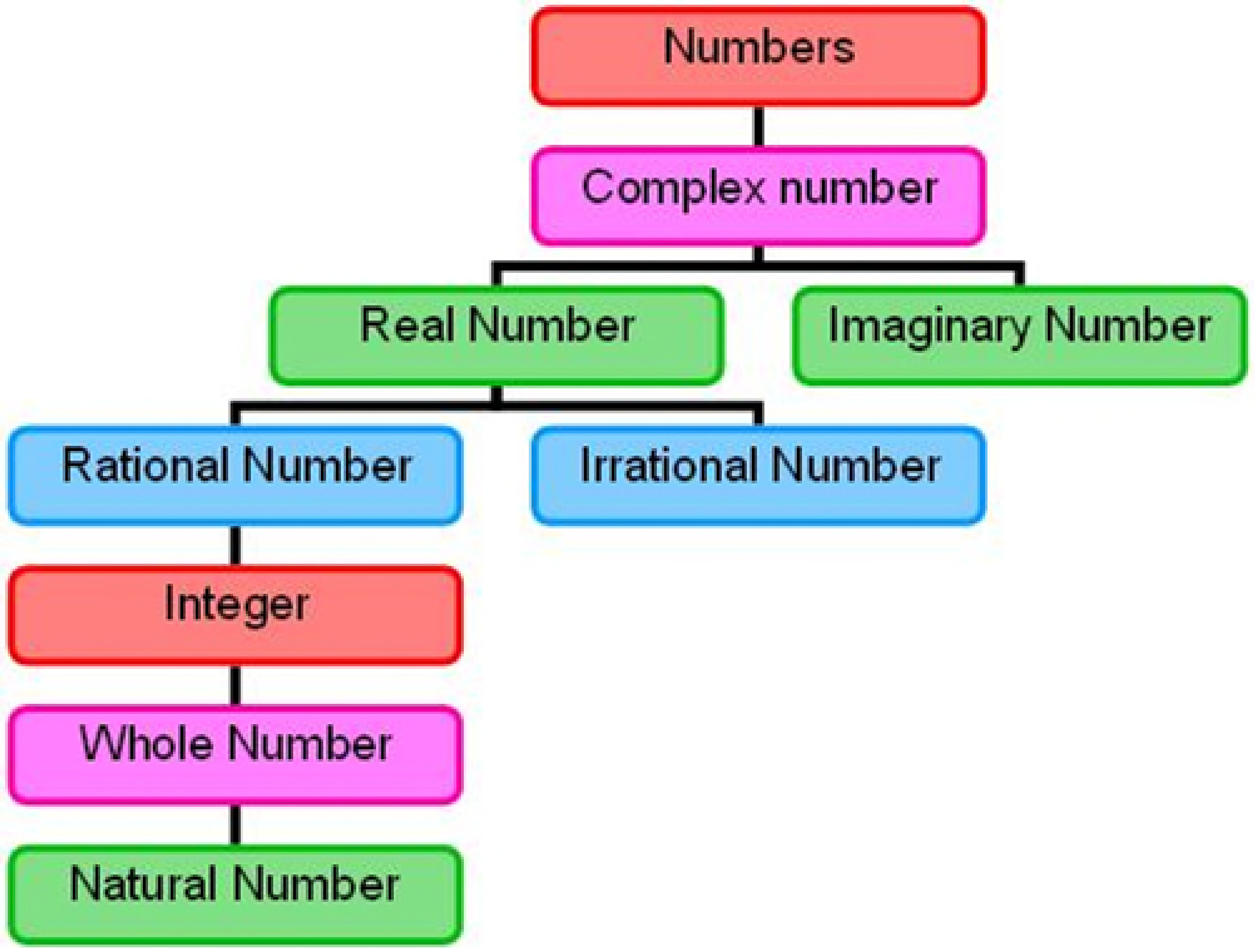
8A.1.3 - Arithmetic of Complex numbers
- Add, subtract, multiply, and divide,
- Properties
- Multiplication and Division in polar form

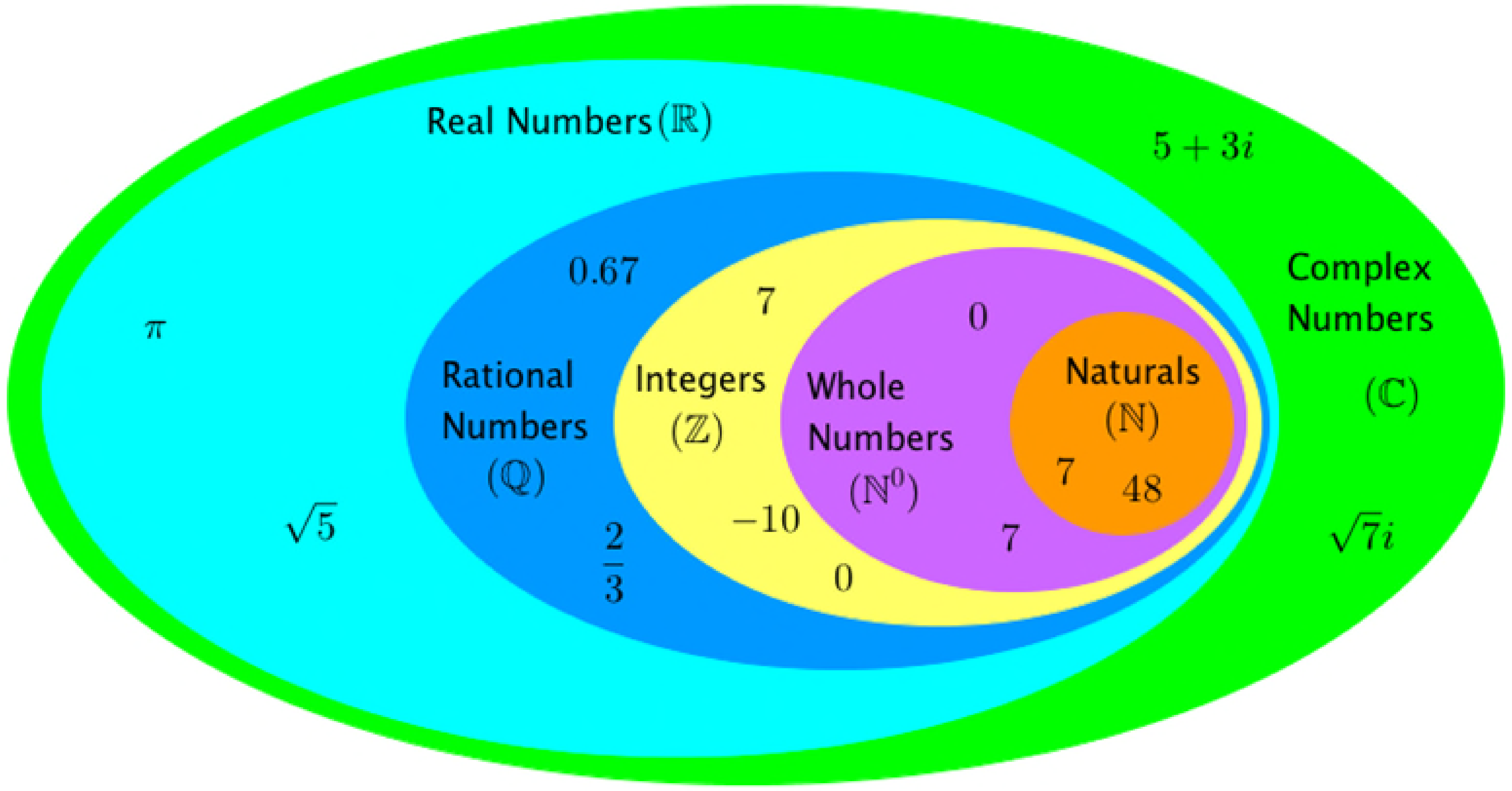
8A.1.4 - Complex conjugation

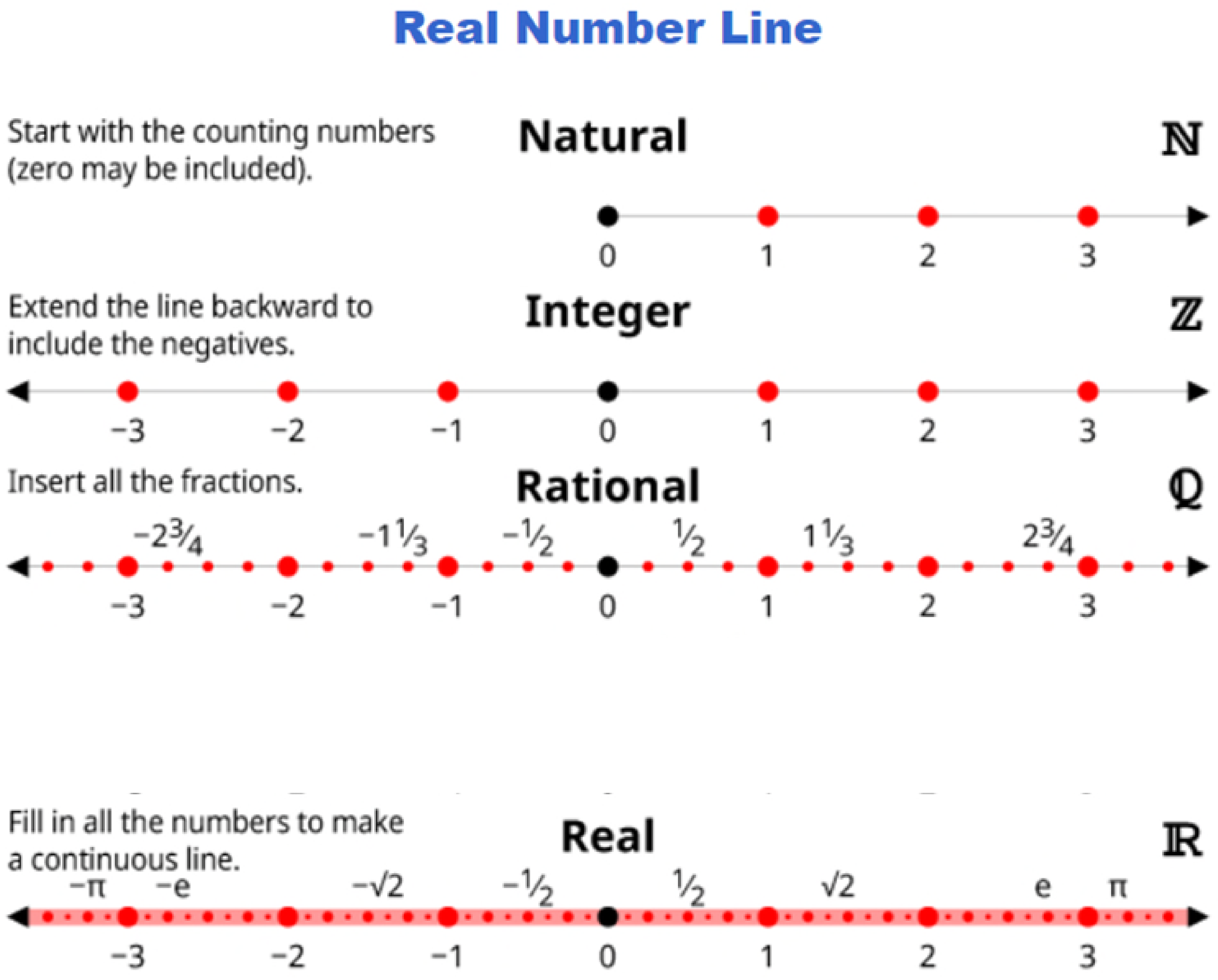
8A.1.1 Complex numbers

- Definitions and Motivations
- number system
- solving quadratic equation for complex roots

8A.1.1 - Definitions and Motivations: Number sets







Ref: https://www.ck12.org/book/ck-12-algebra-ii-with-trigonometry-concepts/r8/section/5.8/
https://www.mathsisfun.com/sets/number-types.html

8A.1.1 - Definitions and Motivations

Define i or $j = \sqrt{-1}$

Define a complex number z to be of the form

$$\underbrace{z = a}_{\text{real part}} + \underbrace{jb}_{\text{imaginary part}},$$

where a is called the **real part** of z (Re(z))and b its **imaginary part** (Im(z)).

Example: $3 + j9$ or $3 + 9j$

If $a = 0$, the complex number is said to be **purely imaginary** and if $b = 0$, it is said to be **purely real**.

In this form, the complex number $z = a + jb$
is called the Rectangular or Cartesian form.

Note: there are 3 forms, the other 2 forms
are called the polar form, and complex exponential form.

8A.1.1 - Definition and Motivation
Example: Why need for complex numbers

There is no real number that solves $x^2 + 4 = 0$.

$$x^2 = -4 \Rightarrow x = \sqrt{(-4)} \Rightarrow x = \sqrt{(-1)(4)} \Rightarrow x = \sqrt{-1}\sqrt{4}$$

Define an imaginary unit j

$$\boxed{j \overset{\text{def}}{=} \sqrt{-1}} \quad \Leftrightarrow \quad j^2 = \sqrt{-1}\sqrt{-1} = -1.$$

More generally, define an imaginary number z to be of the form

$$z = jy, \qquad y \in \mathbb{R}, \text{the set of real numbers}$$

so that

$$z^2 = (jy)^2 = j^2 y^2 = (-1)y^2 = -y^2.$$

In the above example, $x = \sqrt{-1}\sqrt{4} = j(\pm 2) = \pm 2j$ (an imaginary number).

8A.1.1 - Definitions and Motivations: Why need for complex numbers

- Solving quadratic equations: $ax^2+bx+c=0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$D = b^2 - 4ac$

D>0, Two real solutions
D=0, One real solution
D<0, Complex Solutions

Introduce $i \; or \; j \; = \sqrt{-1}$
To have complex solutions!

- Solving cubic equations
  - square roots of negative numbers appear in intermediate steps even when the roots are real
- Fundamental Theorem of Algebra- every polynomial of degree *n* has *n* roots

Ref:https://cyberspacesystem.blogspot.com/2012/03/complex-numbers-quadratic-formula.html

8A.1.1 - Definitions and Motivations: Fundamental Theorem of Algebra

Fundamental Theorem of Algebra

The polynomial equation of degree n with $a_i \in \mathbb{C}$

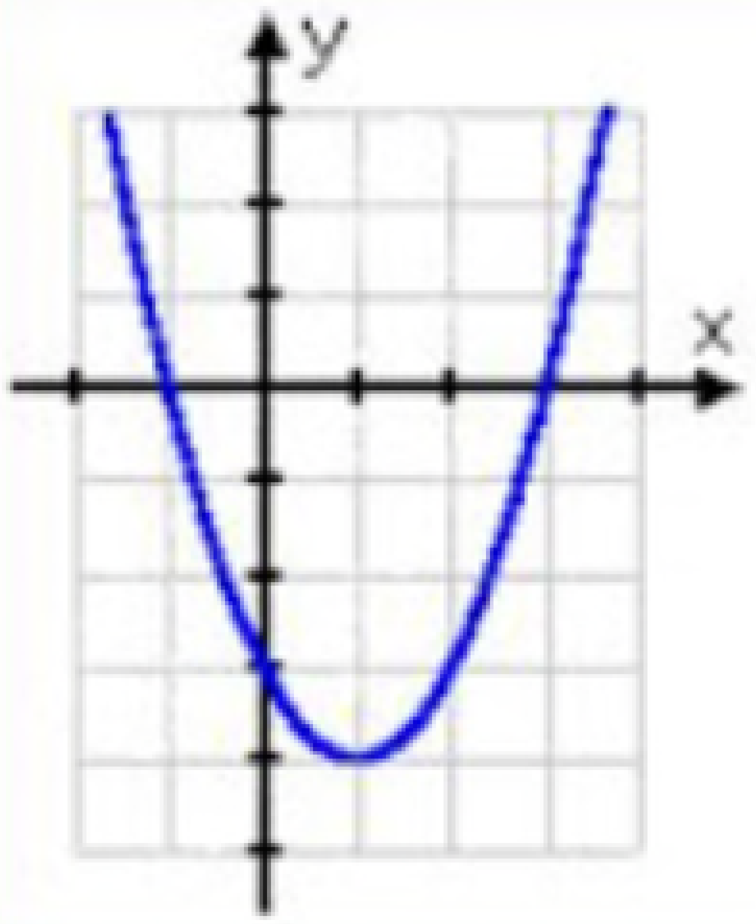
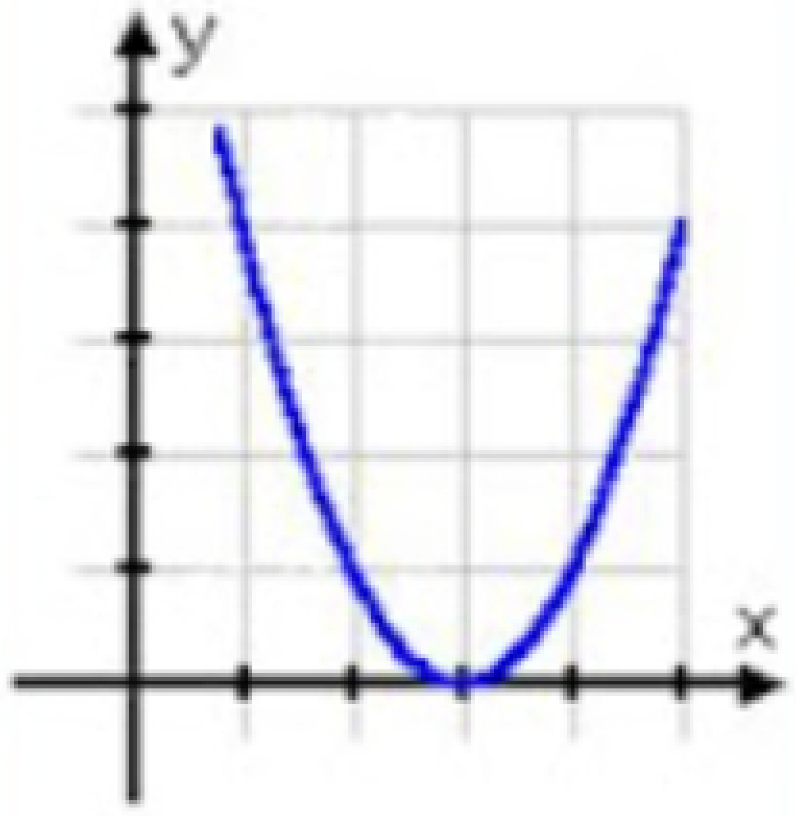
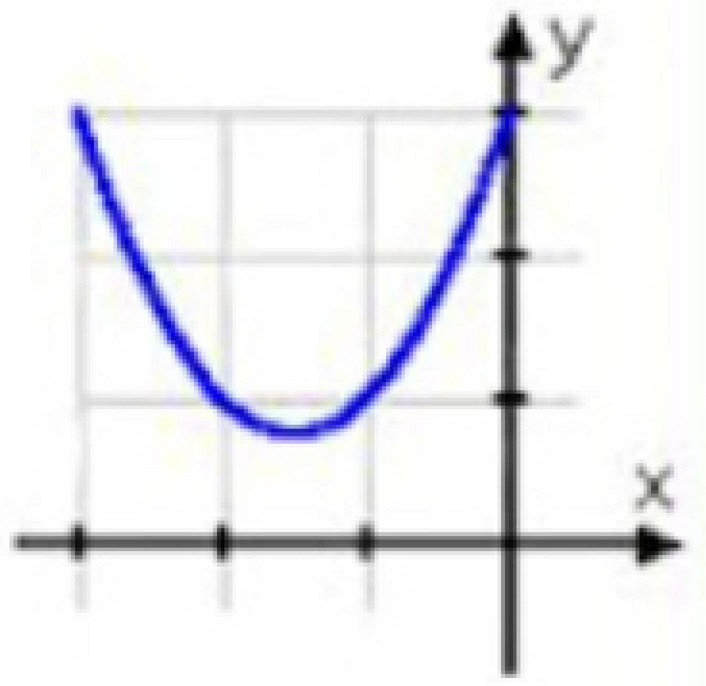
$$P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0$$

has at least 1 solution in $\mathbb{C}$ and the polynomial can be factorised as

$$P(z) = (z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_n)$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are the roots of the polynomial

Ref:
https://cyberspacesystem.blogspot.com/2012/03/complex-numbers-quadratic-formula.html

$x^2 - 2x - 3$	$x^2 - 6x + 9$	$x^2 + 3x + 3$
$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)}}{2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = \frac{-2}{2}, \frac{6}{2} = -1, 3$	$x = \frac{6 \pm \sqrt{(-6)^2 - 4(9)}}{2} = \frac{6 \pm \sqrt{36-36}}{2} = \frac{6 \pm \sqrt{0}}{2} = \frac{6 \pm 0}{2} = 3$	$x = \frac{-3 \pm \sqrt{(3)^2 - 4(3)}}{2} = \frac{-3 \pm \sqrt{9-12}}{2} = \frac{-3 \pm \sqrt{-3}}{2} = -\frac{3}{2} \pm \frac{\sqrt{3}i}{2}$
a positive number inside the square root	zero inside the square root	a negative number inside the square root
two real solutions	one (repeated) real solution	two complex solutions
		
two distinct x-intercepts	one (repeated) x-intercept	no x-intercepts

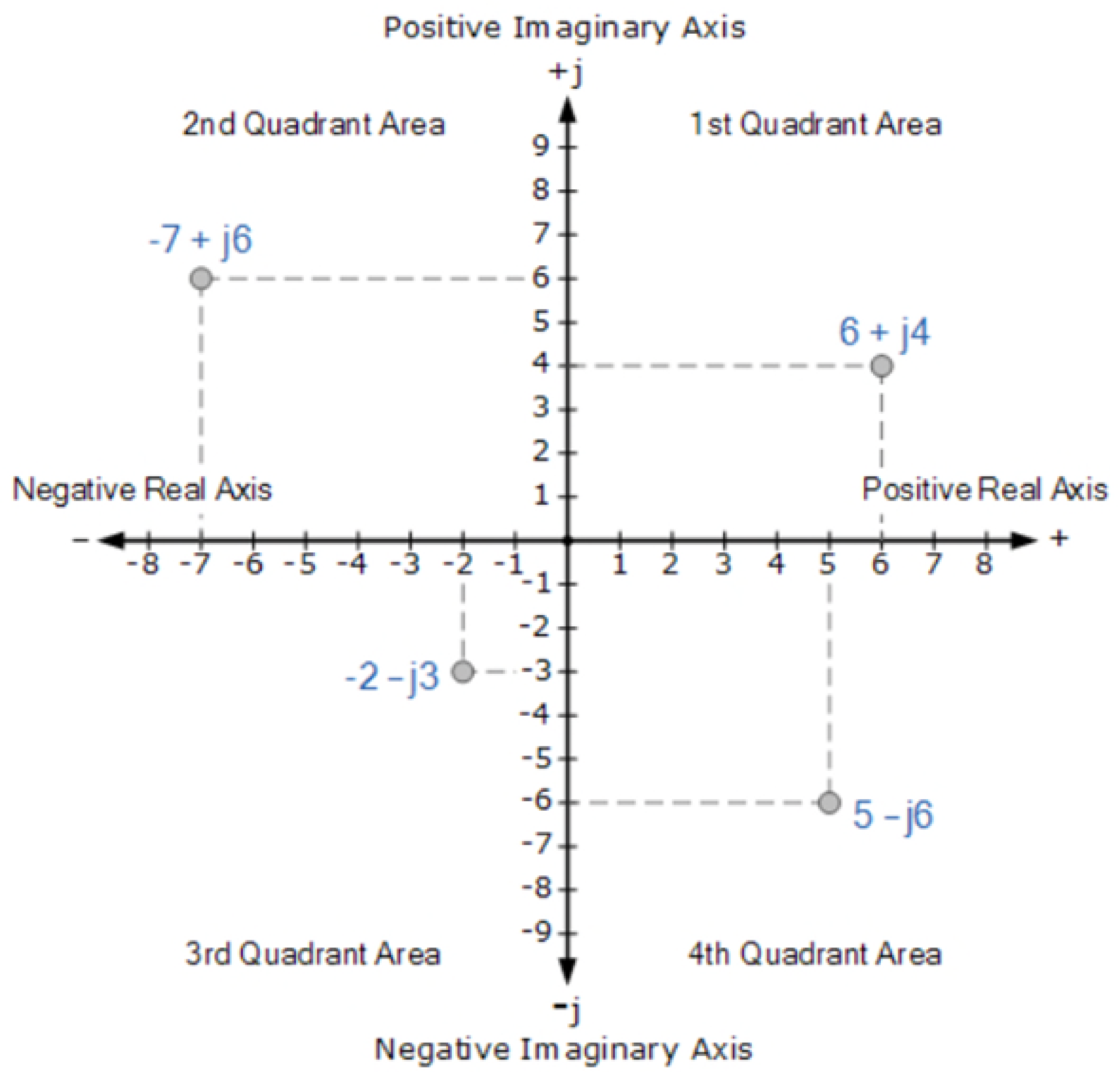
8A.1.2 - The argand diagram, and geometric interpretation
- The polar form, The unit circle

8A.1.2 - Geometric interpretation of the complex number The Argand diagram (complex plane)

An Argand diagram, also known as the complex plane or Argand plane, is a graphical representation use d to visualize complex numbers.

It is named after the Swiss mathematician Jean-Robert Argand.

Four Quadrant Argand Diagram



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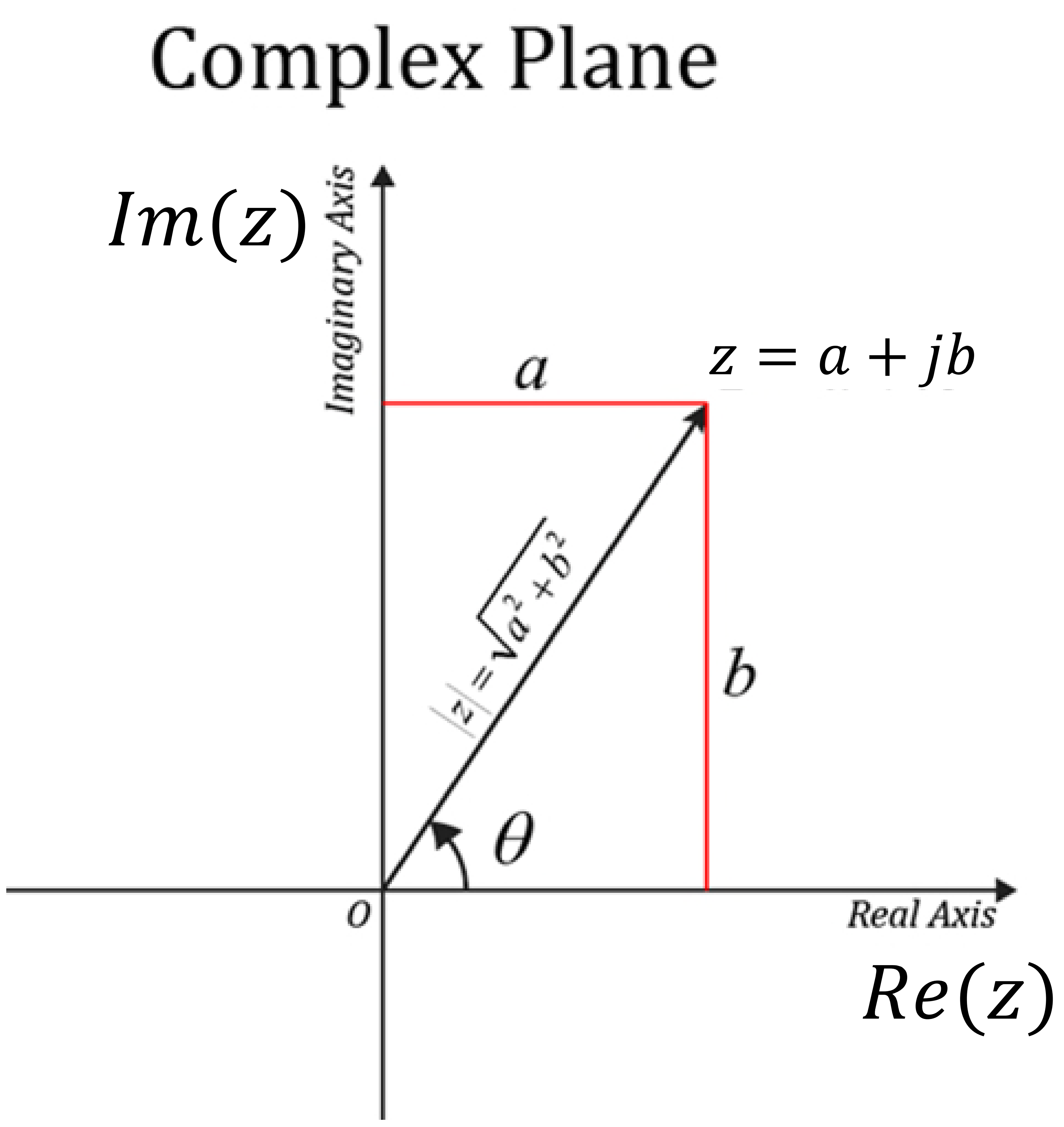
8A.1.2 - Geometric interpretation of the complex number The Argand diagram (complex plane)



The distance of the complex number $z = a + jb$ from the origin $0 + j0$ is termed the **modulus** or **magnitude** of the complex number, denoted as $|z|$, and is computed as $\sqrt{a^2 + b^2}$.

The angle θ, formed by the line connecting the complex number to the origin with the positive x-axis, is known as the argument or amplitude of the complex number. It is symbolized by $\arg(z)$

$$\arg z = \theta, \text{ where } \tan\theta = b/a$$

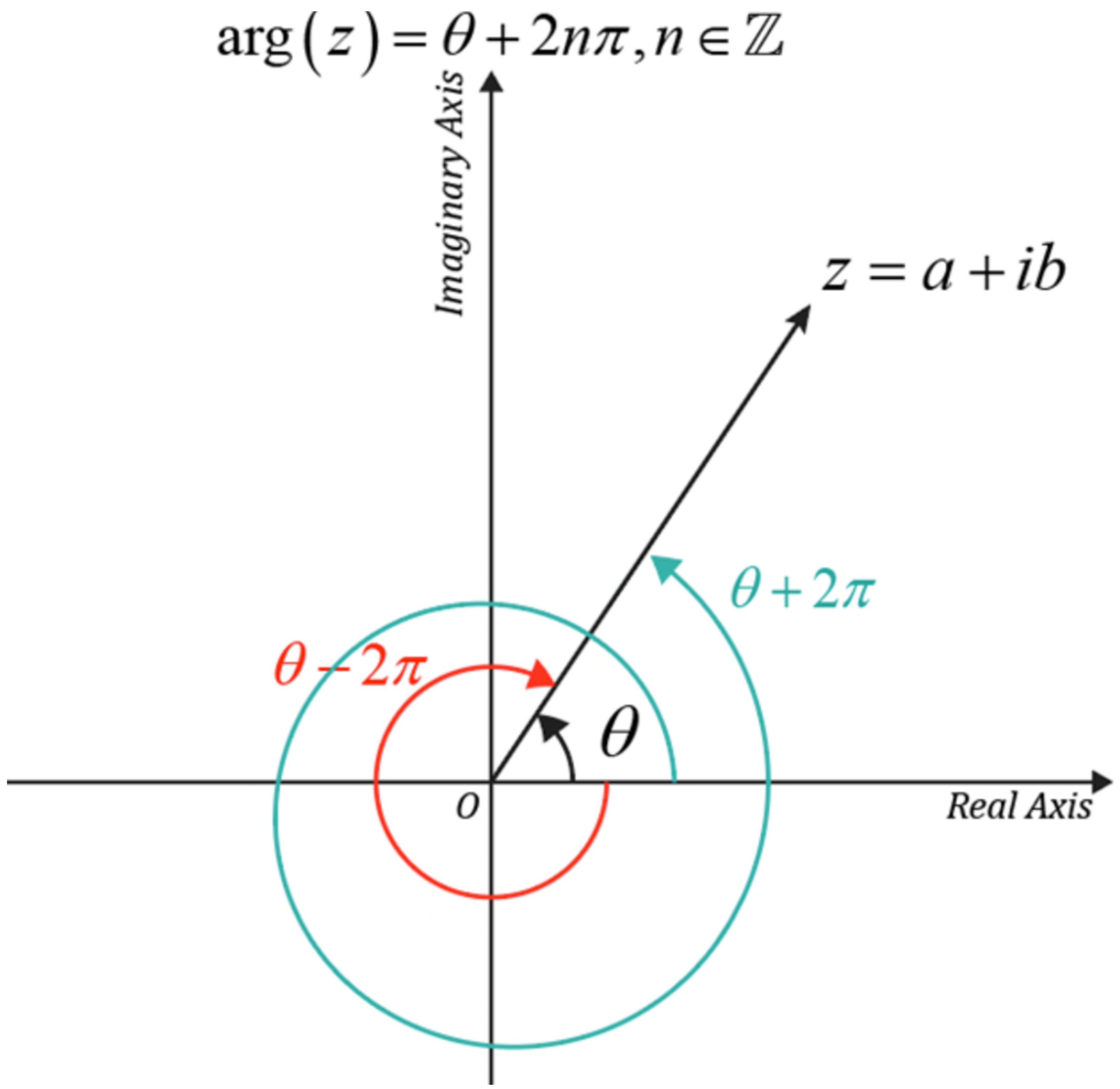
θ is measured from the positive real axis.
$-\pi < \theta \leq \pi$
Important: The value of θ depends on the quadrant of the complex plane that z is in.

Note: angle θ 's unit is in radian and measured against the positive real axis, where rotation of
anti-clockwise for +ve angle
and clockwise for –ve angle

8A.1.2 - The non-uniqueness of θ and Principal Argument

Non-Uniqueness of Argument

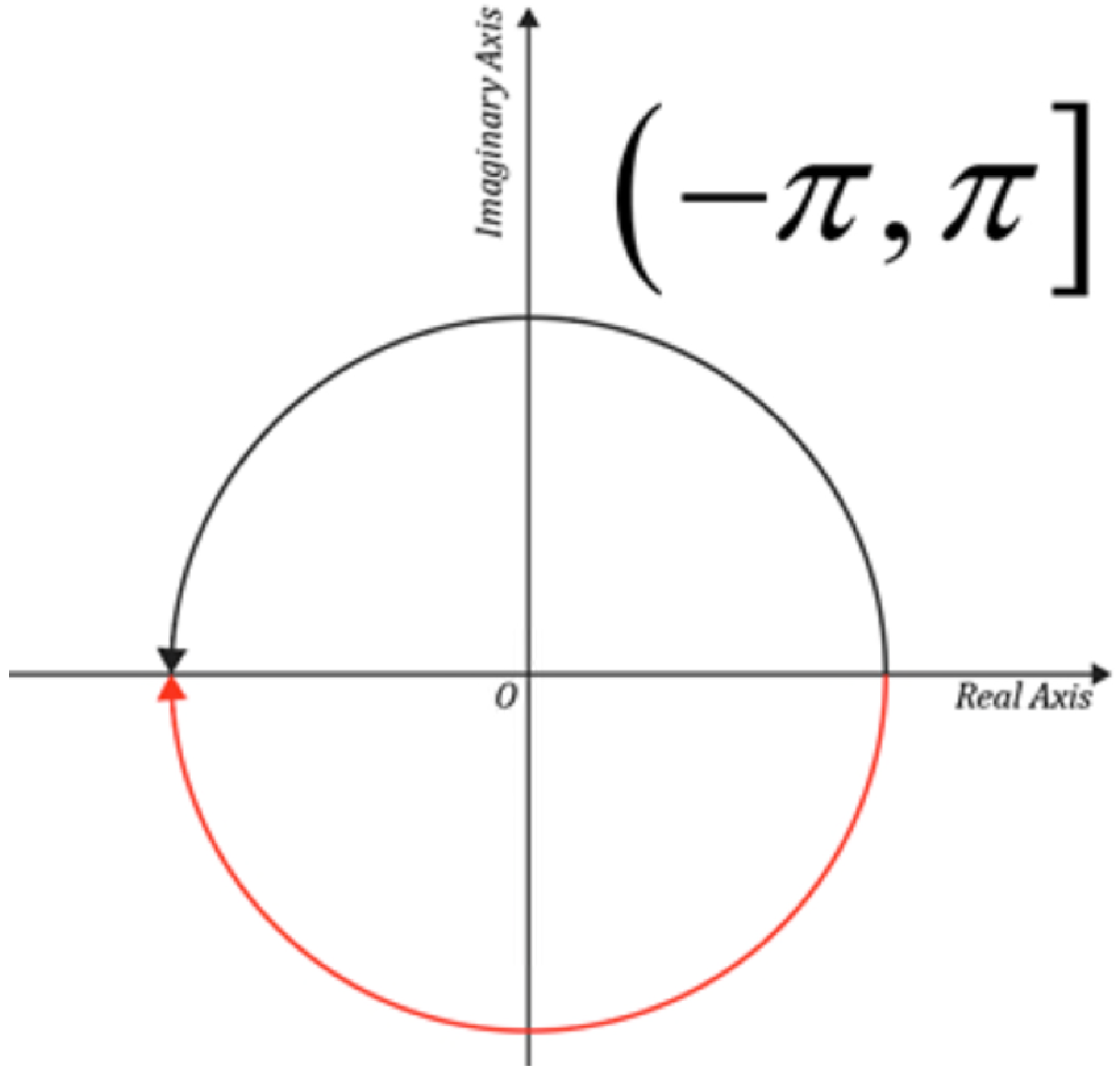
Argument of a complex number is not unique. If θ is the argument of the complex number so can be $\theta + 2\pi$ or $\theta - 2\pi$ or $\theta + 4\pi$ and so on. In general the argument of a complex number is $\theta + 2n\pi$ where θ is any one of the argument.



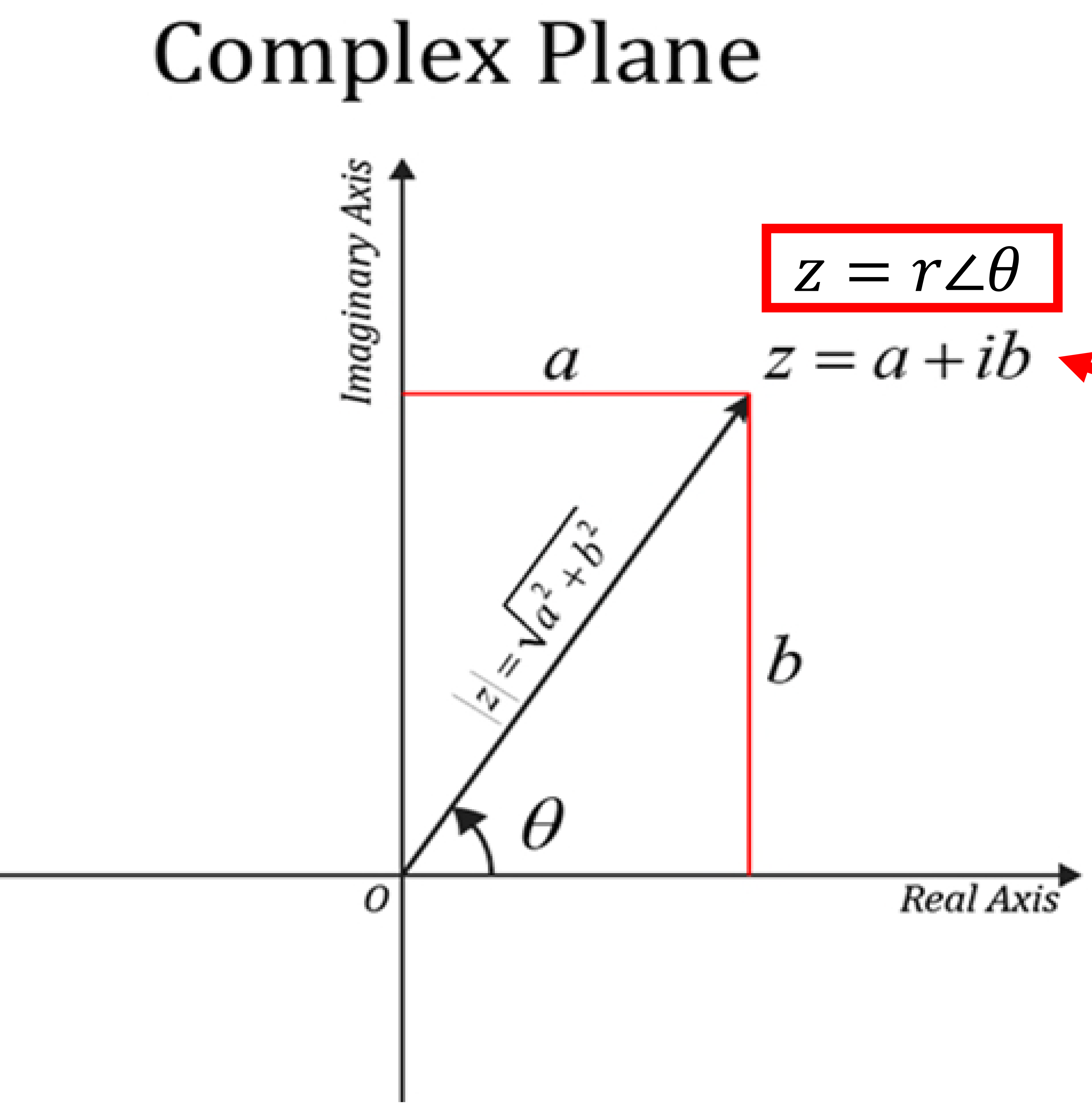
Principal Argument Arg(z) vs arg(z)

To make the argument unique we define principal argument.

Principal Argument of a complex number is that argument of the complex number which lies in the interval $(-\pi, \pi]$. Exactly one such argument out of infinite arguments lie in this interval. $(-\pi, \pi]$ denotes one complete revolution around the circle with centre at origin. An argument lying in $(-\pi, \pi]$ also has the smallest magnitude.



8A.1.2 - Definition and motivation
The polar form of complex number



In this form, the complex number $z = a + jb$ can be represented in polar forms:

i) $z = |z|\angle\theta = r\angle\theta$ is in (abbreviated) polar form

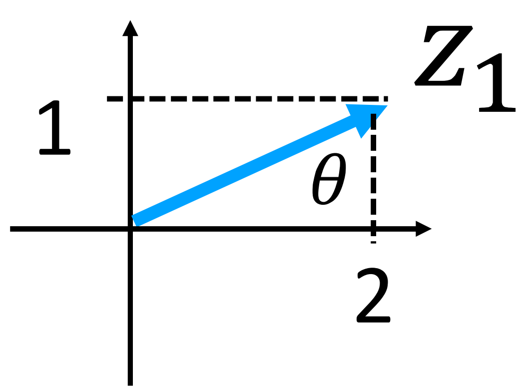
ii) $z = r(\cos(\theta) + j\sin(\theta)) = r\ cis\ \theta$ is in trigonometric polar form

$r = |z|$ (modulus of z)

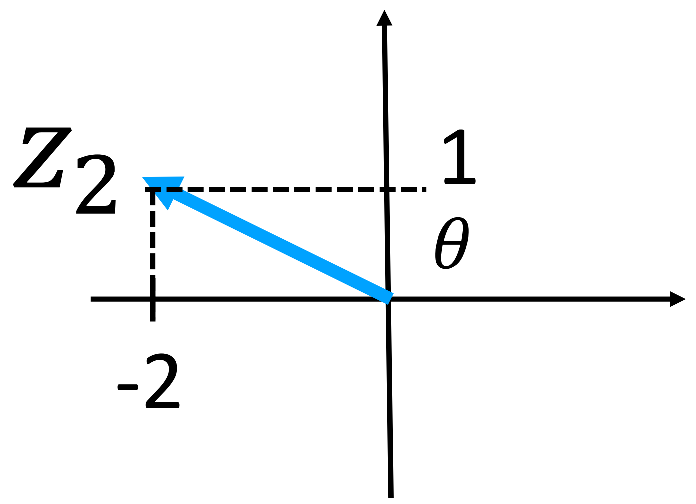
$\theta = arctan(\frac{b}{a})$, $a = |z|\cos(\theta)$, $b = |z|\sin(\theta)$,

Ref: https://math.stackexchange.com/questions/4293260/argument-of-a-complex-number-as-arctan

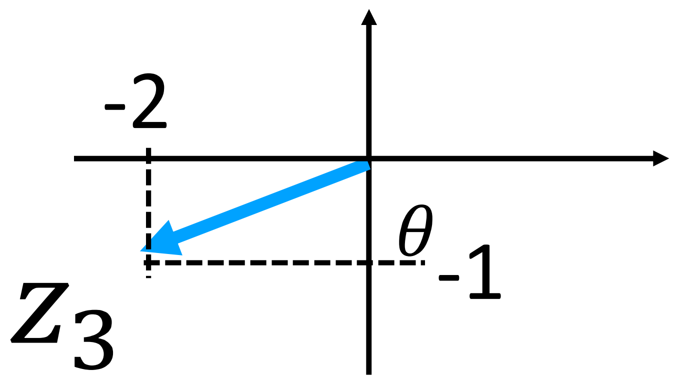
8A.1.2: Example of z in polar form for 4 quadrants



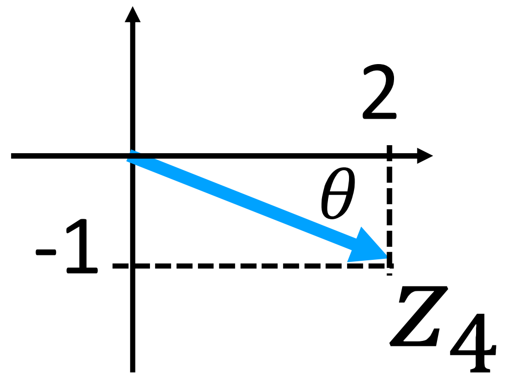
$$z_1 = 2 + 1j$$
$$= \sqrt{5}\angle(0.4636)(rad)$$



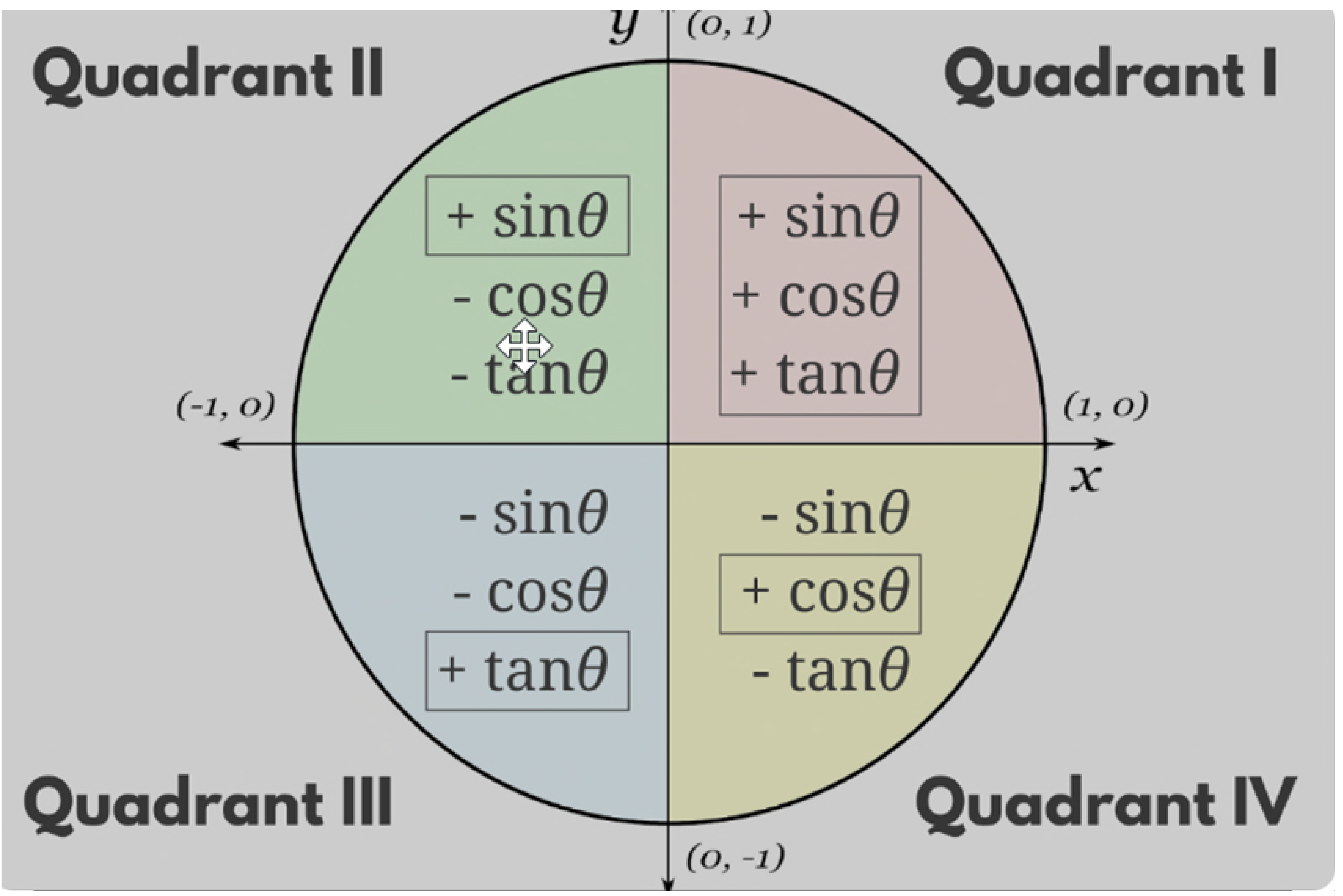
$$z_2 = -2 + 1j$$
$$= \sqrt{5}\angle(\pi - 0.4636)\ (rad)$$
$$= \sqrt{5}\angle(2.678)\ (rad)$$



$$z_3 = -2 - 1j$$
$$= \sqrt{5}\angle(-\pi + 0.4636)\ (rad)$$
$$= \sqrt{5}\angle(-2.678)\ (rad)$$

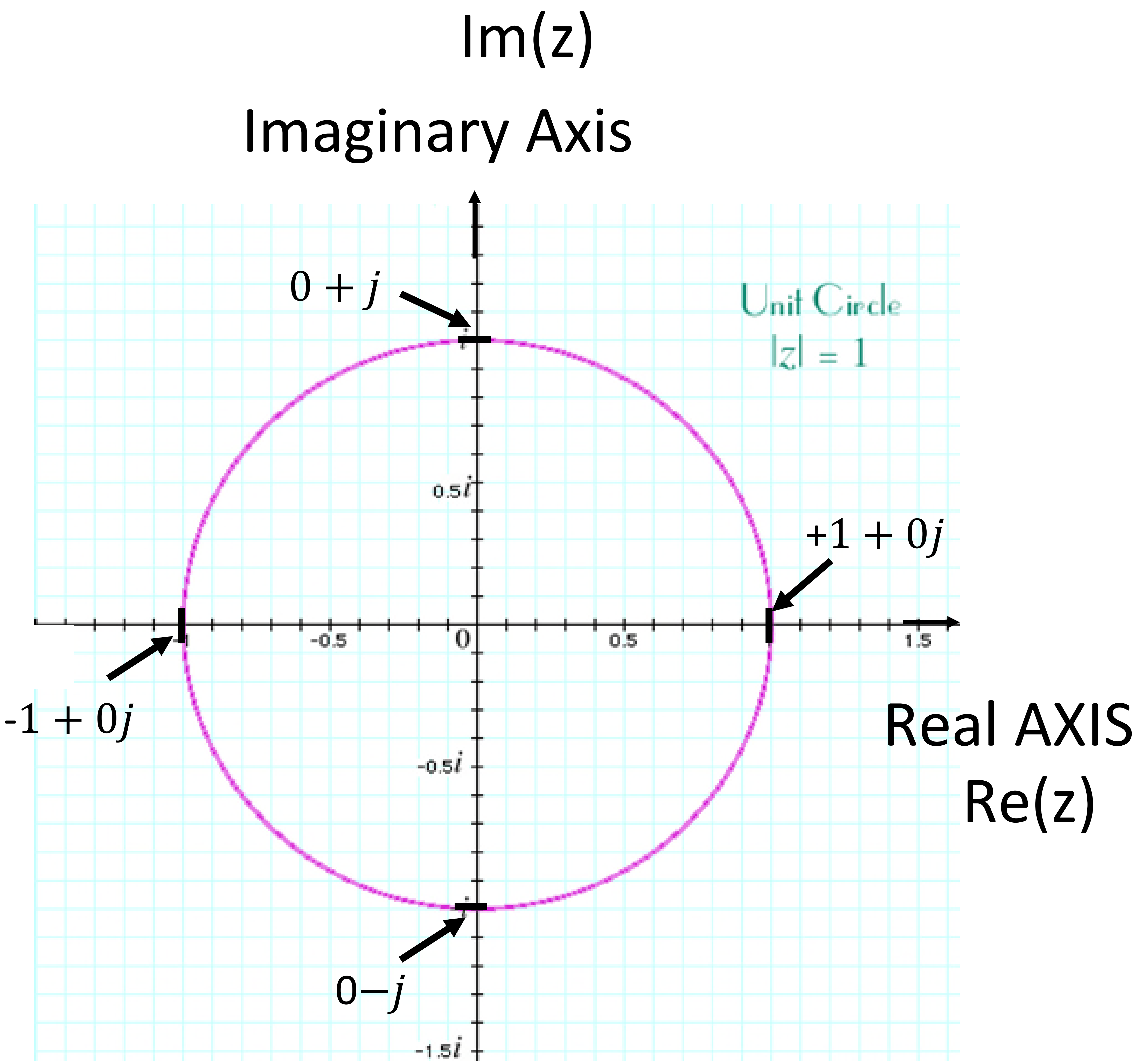


$$z_4 = 2 - 1j$$
$$= \sqrt{5}\angle(-0.4636)(rad)$$



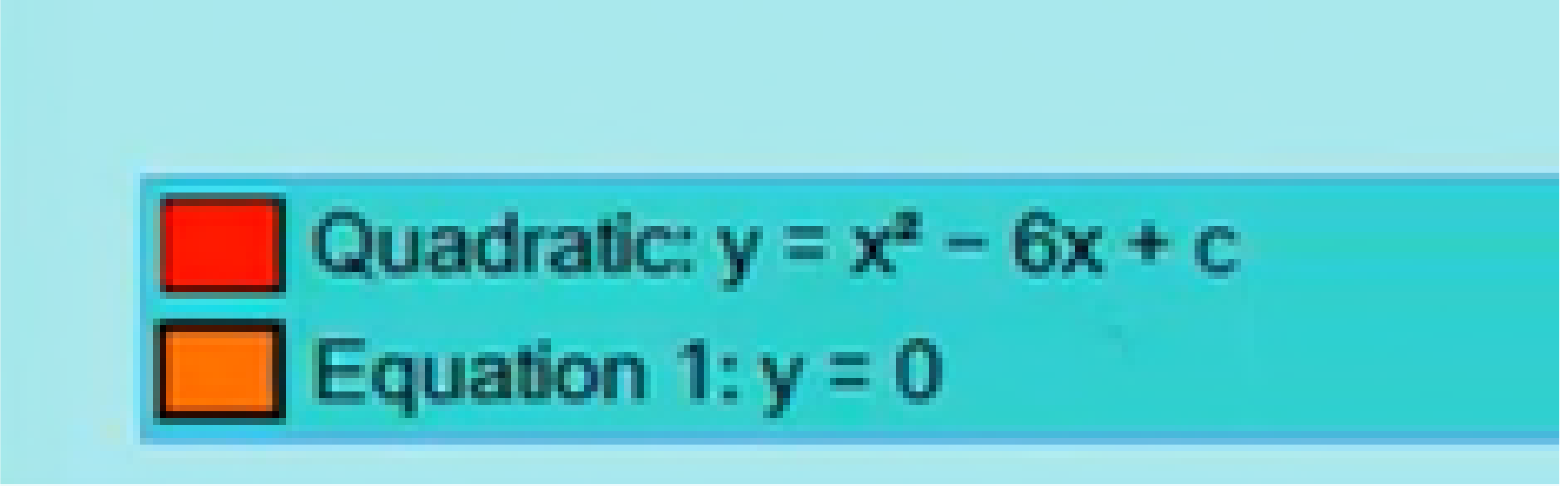
Ref: Reference angles and 4 quadrants
https://www.youtube.com/watch?v=ONPcD6FdoBI

8A.1.2 - definition and motivation
The unit circle in the complex plane.



The *unit circle* (in pink) is the circle of radius 1 centered at 0. It include all complex numbers of absolute value 1, so it has the equation $|z| = 1$

8A.1.2 Example: Visualizing the complex roots



visualising complex roots

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https://youtu.be/5g80j17w0CE?si=HMWc-l8HekEjptxg

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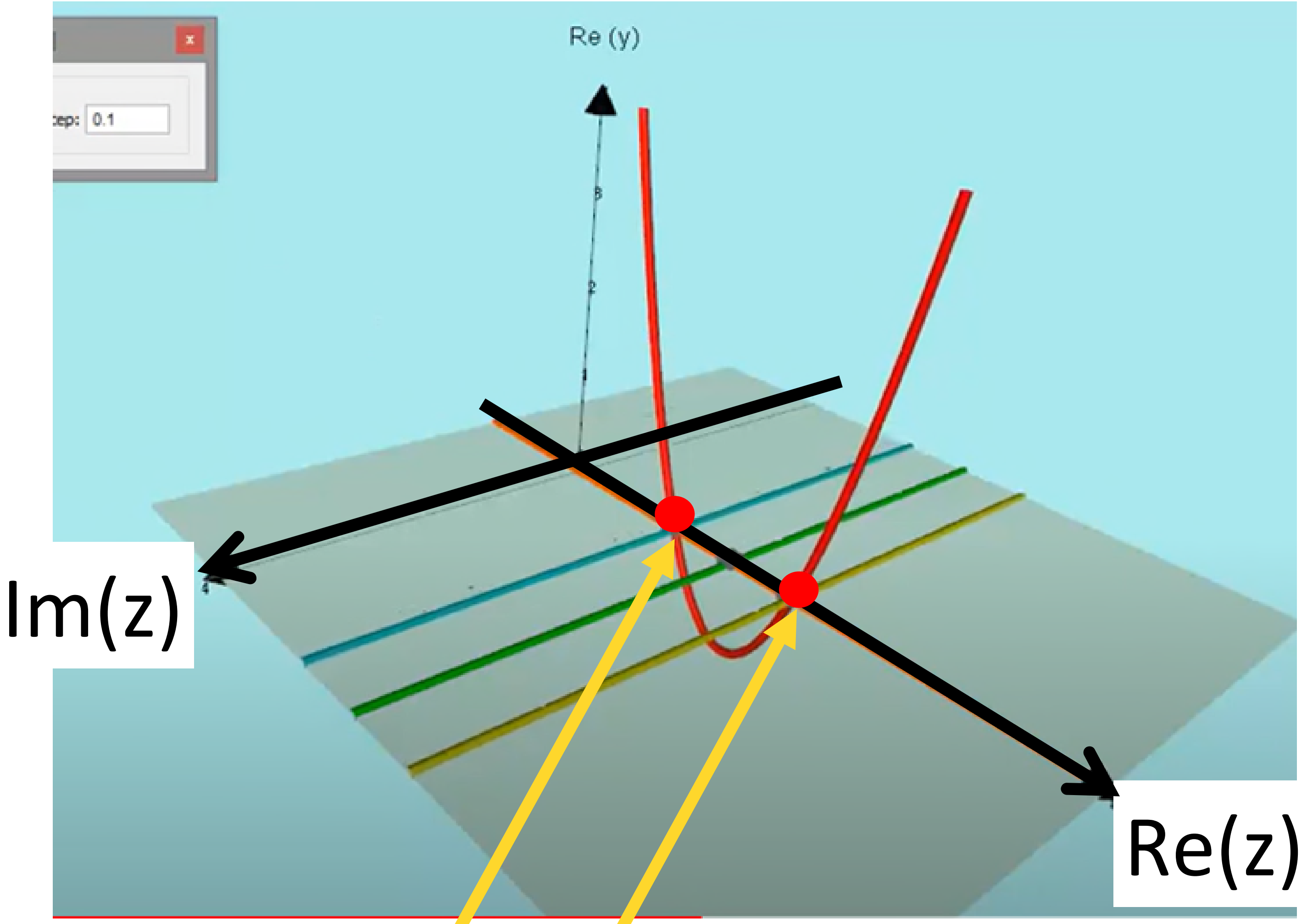


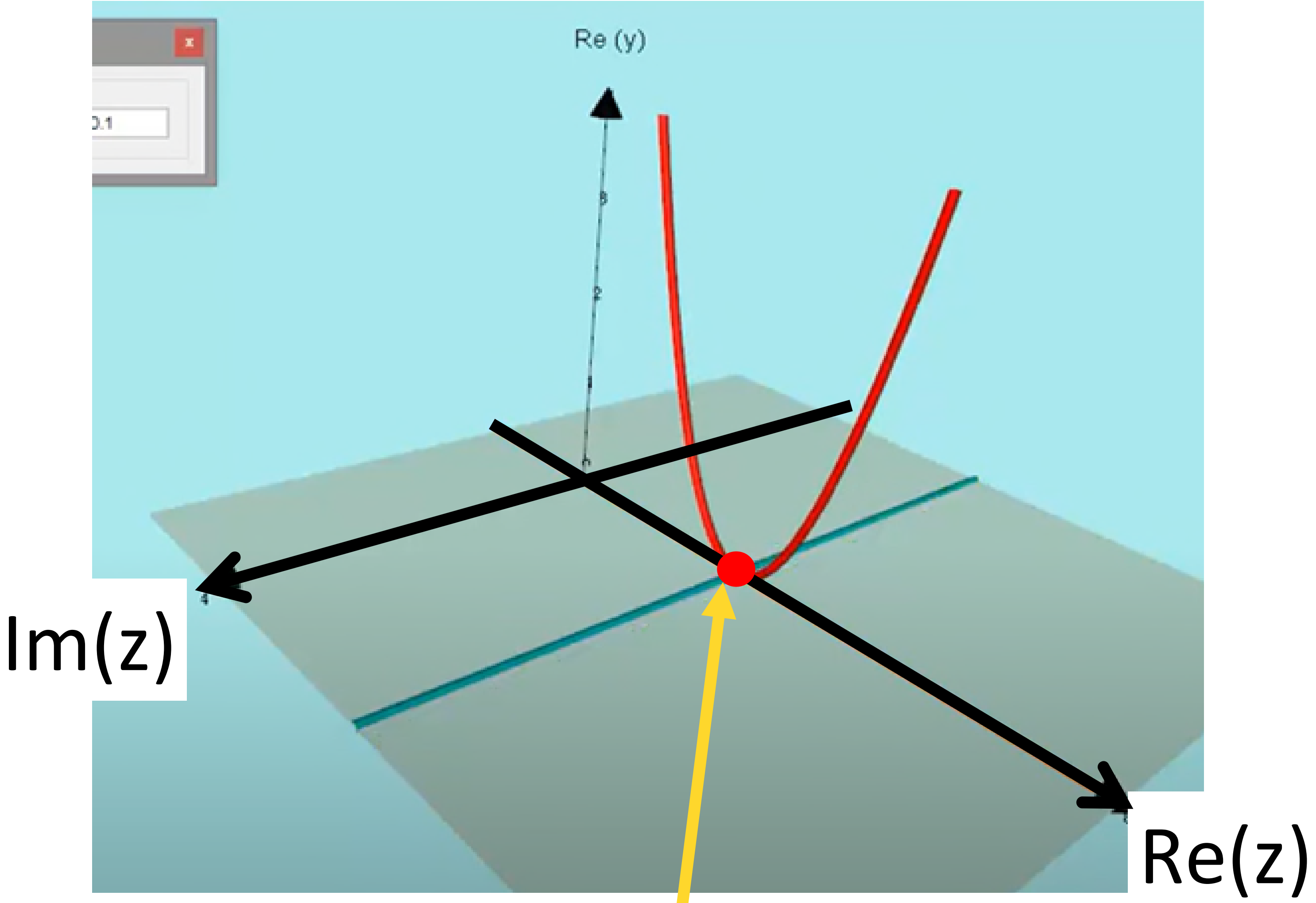
Roots at real values = 2, 4

```
>> roots(([1 -6 8]))

ans =
```

2, 4

c=9



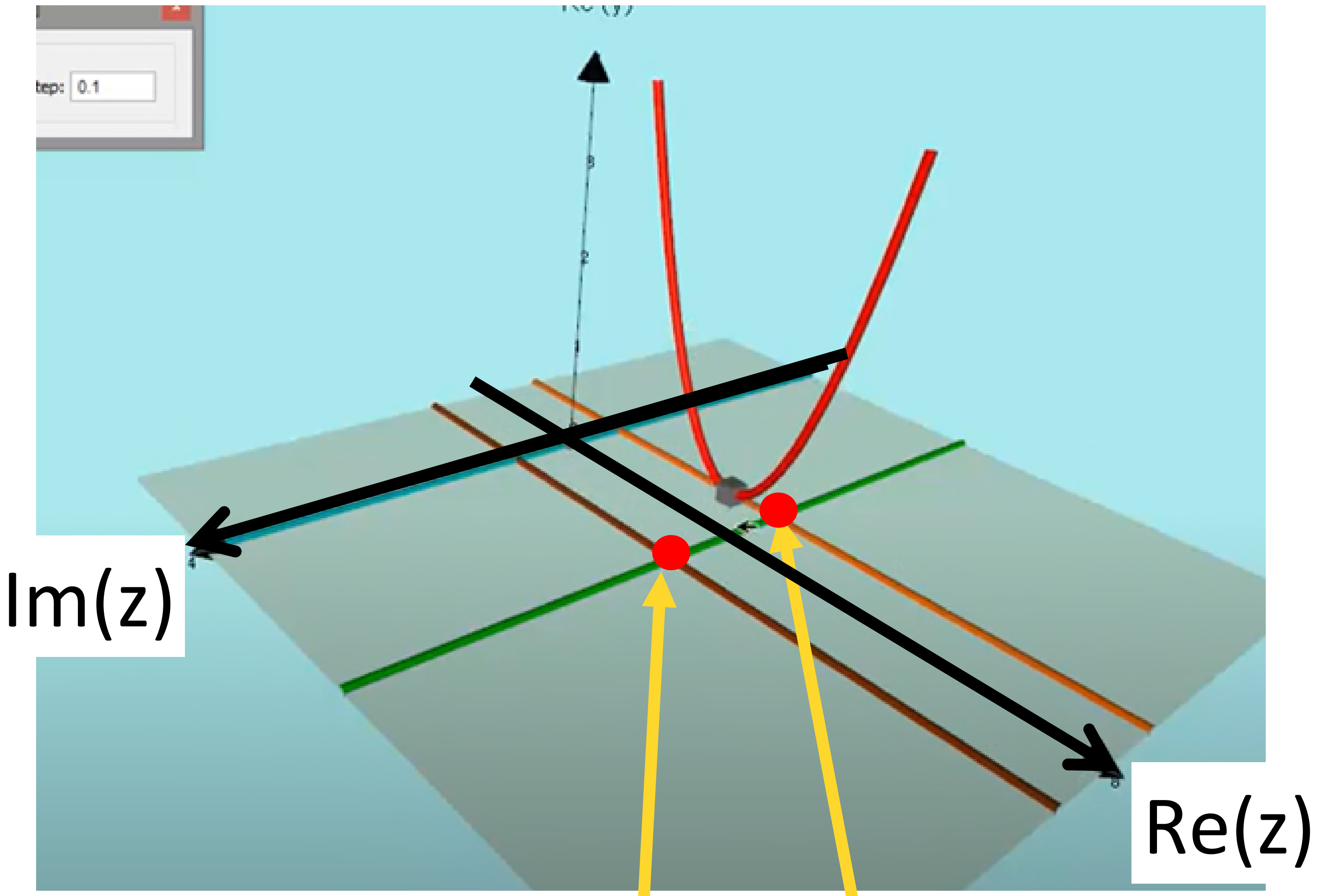
Roots at real value = 3

```
>> roots(([1 -6 9]))

ans =

   3.0000 + 0.0000i
   3.0000 - 0.0000i
```

c=9.5



Roots are complex numbers

```
>> roots([1 -6 9.5])

ans =
```

3+j0.707, 3-j0.707

8A.1.3 - Arithmetic of Complex numbers

- Add, subtract, multiply, and divide,
- Properties
- Multiplication and Division in polar form

8A.1.3 - Arithmetic of Complex numbers (in rectangular form)

Equality: $z_1 = a + ib$, and $z_2 = c + id$ are equal if and only if $a = c$ and $b = d$.

Arithmetic Operations Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, these operations are defined as follows.

Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

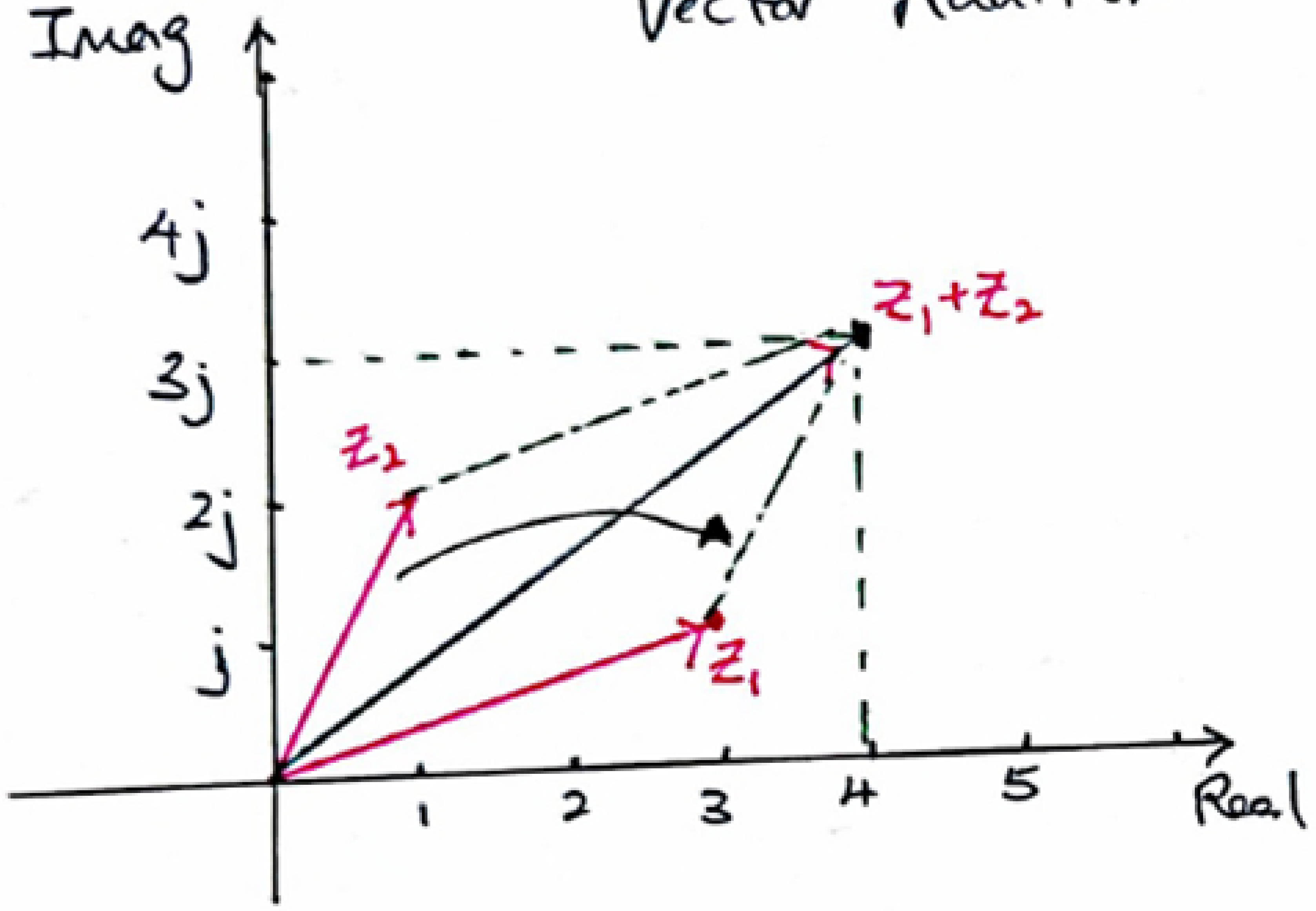
Subtraction: $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

Multiplication:
$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2)$$

Division:
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$
$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

8A.1.3 - Arithmetic of Complex numbers (in Argand plane)

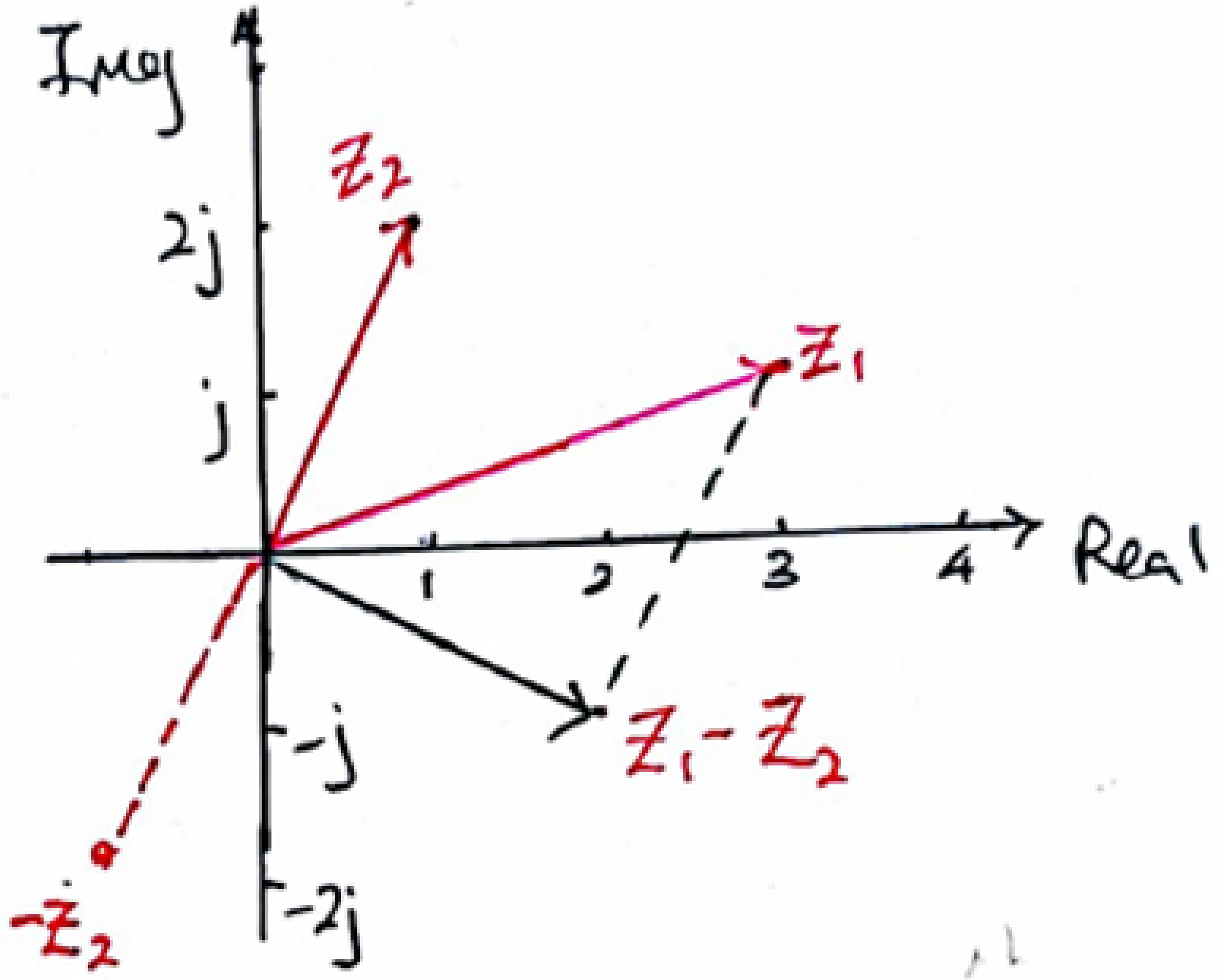
Vector Addition



$z_1 = 3+1j$
$z_2 = 1+2j$
$\therefore z_1 + z_2 = 4+3j$

Note: adding 2 vectors visually is taking z_1's tail and put it at z_1's tip.

Vector Subtraction



$\therefore z_1 - z_2 = 2 - 1j$

$z_1 = 3+1j$
$z_2 = 1+2j$
$\therefore -z_2 = -1-2j$

Note: vector subtraction visually, flip z_2 to $-z_2$, then take $-z_2$ vector and put it at z_1's tip!

8A.1.3 - Other properties of Complex numbers

The familiar commutative, associative, and distributive laws hold for complex numbers.

Commutative laws:

$$\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$$

Associative laws:

$$\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) = (z_1 z_2) z_3 \end{cases}$$

Distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

8A.1.3 - Examples when dealing with $j = \sqrt{-1}$

- As in real number algebra, $-j \overset{\text{def}}{=} (-1)j$

- $j^2 = -1; \quad j^3 = j.j^2 = -j; \quad j^4 = j^2.j^2 = 1; \quad j^5 = j^4.j = j;$
  $j^6 = j^4.j^2 = -1; \quad j^7 = j^4.j^3 = -j;$ and so on.

- $\frac{1}{j} = j^{-1} = -j \qquad (j.j = -1);$
  $\frac{1}{j^2} = j^{-2} = -1; \quad \frac{1}{j^3} = j; \quad \frac{1}{j^4} = 1; \quad \frac{1}{j^5} = -j;$ and so on.

- $x(-j) = (-x)j = -jx$, for all $x \in \mathbb{R}$

- $(\sqrt{-x})^2 = \sqrt{-x}.\sqrt{-x} = \sqrt{(-x).(-x)} = -x.$

CAUTION: $\underbrace{\sqrt{(-x).(-x)}}_{-x} \neq \underbrace{\sqrt{x.x}}_{x}$

and $\sqrt{x^2} = \sqrt{(-x).(-x)}$ OR $\sqrt{(x).(x)} = \pm x$

Important: Always write $\sqrt{-x}$ as $\sqrt{x}.j$ prior to manipulation

8A.1.3 - Example

Now we also saw that if a and b were both positive then $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$. For a second let's forget that restriction and do the following.

$$\sqrt{-9} = \sqrt{(9)\,(-1)} = \sqrt{9}\sqrt{-1} = 3\sqrt{-1}$$

by using the following definition.

$$\boxed{i = \sqrt{-1}}$$

Note that if we square both sides of this we get,

$$\boxed{i^2 = -1}$$

Now, $\sqrt{-1}$ is not a real number, but if you think about it we can do this for any square root of a negative number. For instance,

$$\begin{aligned} \sqrt{-100} &= \sqrt{100}\sqrt{-1} = 10\sqrt{-1} \\ \sqrt{-5} &= \sqrt{5}\,\sqrt{-1} \\ \sqrt{-290} &= \sqrt{290}\,\sqrt{-1} \qquad etc. \end{aligned}$$

Using this definition all the square roots above become,

$$\begin{aligned} \sqrt{-9} &= 3\,i & \qquad \sqrt{-100} &= 10\,i \\ \sqrt{-5} &= \sqrt{5}\,i & \qquad \sqrt{-290} &= \sqrt{290}\,i \end{aligned}$$

In Ch 8A.2_DeMoivre, you will see that there are 2 solutions for $\sqrt{-9}$.

8A.1.3 - Multiplication in polar form

Angle sum and difference

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Let $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1+\theta_2] + j[\sin(\theta_1+\theta_2)] \end{aligned}$$

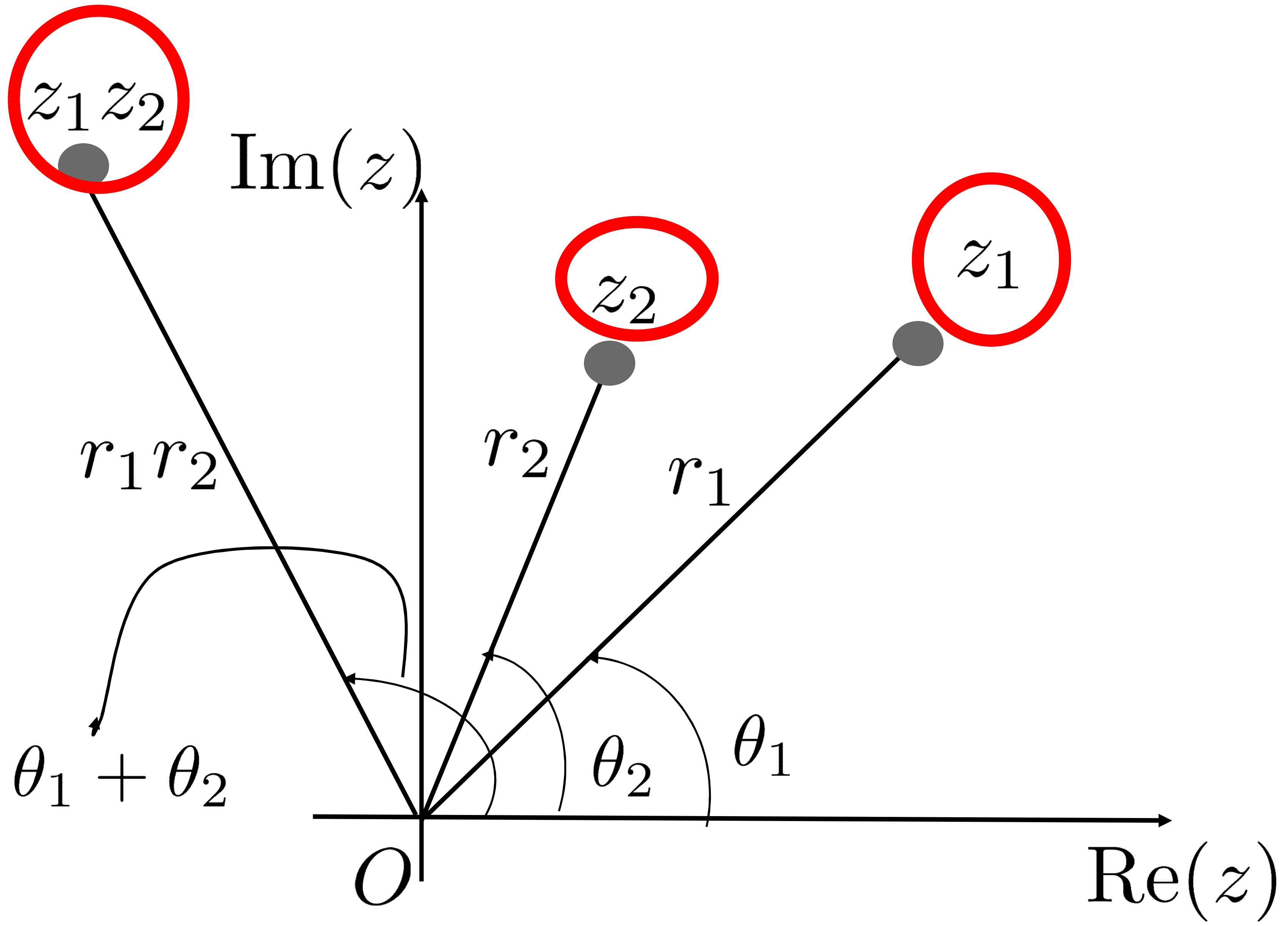
Hence,

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$$

and

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

$$-\pi < \arg(z_1 z_2) \leq \pi$$



In summary: multiplication of complex number can be easily carried out in polar form by:

$$\begin{aligned} z_1 z_2 &= |z_1|\angle(\theta_1)\,|z_2|\angle(\theta_2) \\ &= |z_1||z_2|\ \angle(\theta_1+\theta_2) \end{aligned}$$

8A.1.3 - Division in polar form

Angle sum and difference

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Let $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$. Then

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)} \\
&= \frac{r_1}{r_2}(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 - j\sin\theta_2) \\
&= \frac{r_1}{r_2}[\,(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]
\end{aligned}$$

Therefore,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + j(\sin(\theta_1-\theta_2)]$$

Hence,

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2,$$

$$-\pi < \arg(z_1/z_2) \leq \pi$$

Summary: division of complex number can be easily carried out in polar form by:

$$\frac{z_1}{z_2} = \frac{|z_1|\angle(\theta_1)}{|z_2|\angle(\theta_2)}$$

$$= \frac{|z_1|}{|z_2|}\angle(\theta_1 - \theta_2)$$

8A.1.4 - Complex Conjugation

8A.1.4 - Complex conjugate

Define Complex Conjugate of $z = a + jb$ as $\bar{z} = a - jb$.
The complex conjugate of $z = a - jb$ is $\bar{z} = a + jb$. Thus $\bar{\bar{z}} = z$.

Some results from complex conjugate:

$$z + \bar{z} = 2a = 2\mathrm{Re}(z)$$
$$z - \bar{z} = 2jb = 2j\mathrm{Im}(z)$$

$$z\bar{z} = (a + jb)(a - jb) = a^2 + b^2$$
$$= |r|\angle\theta \, |r|\angle(-\theta)$$
$$= |r|^2\angle 0 = |r|^2$$

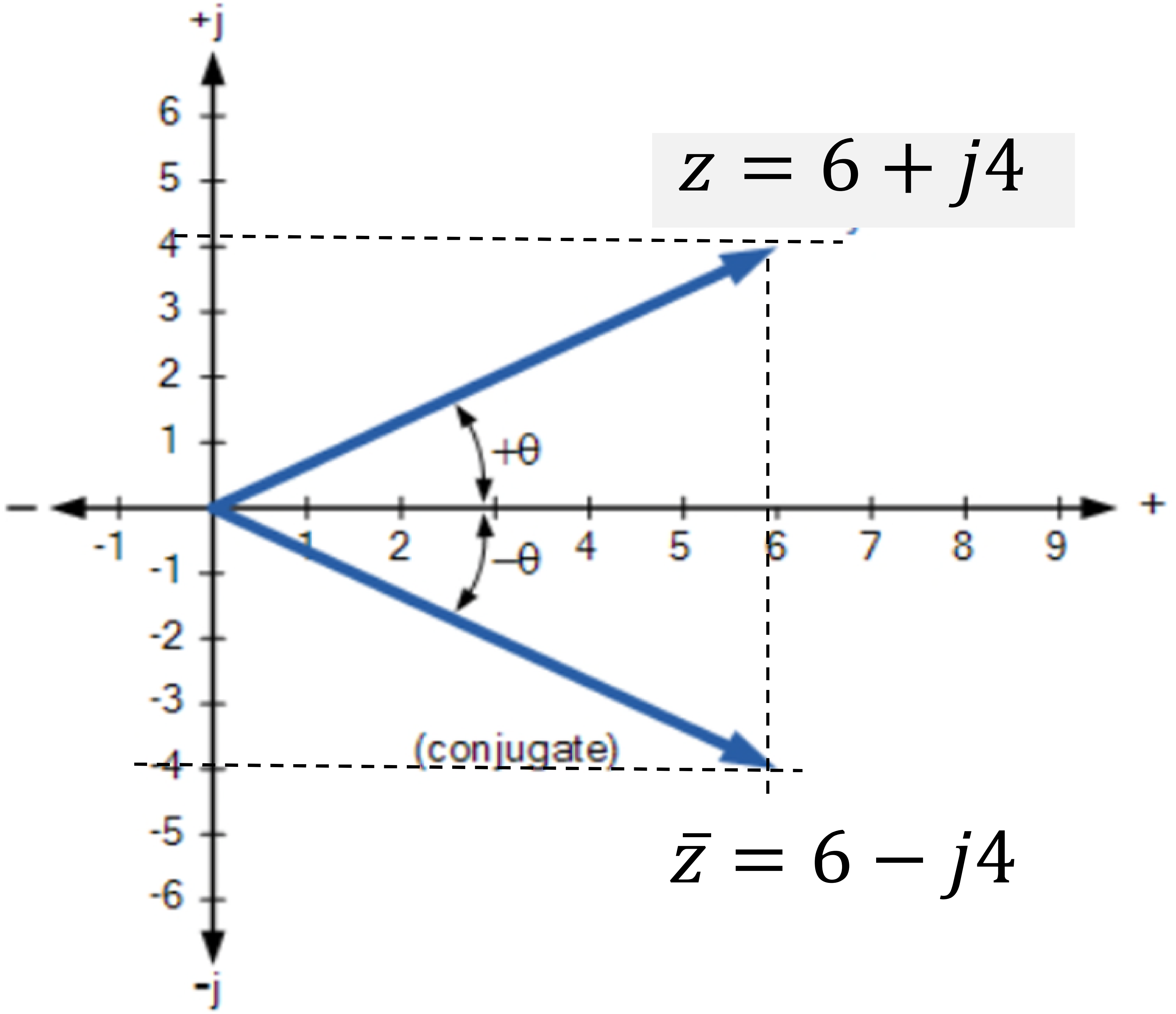
Complex conjugation is important as it allows us to convert a complex value to real value when we multiply by its conjugate

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$= \overline{|r_1|\angle\theta_1 \, |r_2|\angle(\theta_2)}$$
$$= \overline{|r_1||r_2|\angle(\theta_1 + \theta_2)}$$
$$= |r_1||r_2|\angle -(\theta_1 + \theta_2)$$

$$= \overline{|r_1|\angle\theta_1} \; \overline{|r_2|\angle\theta_2}$$
$$= |r_1|\angle(-\theta_1) \, |r_2|\angle(-\theta_2)$$
$$= |r_1||r_2|\angle(-\theta_1 - \theta_2)$$
$$= |r_1||r_2|\angle -(\theta_1 + \theta_2)$$

8A.1.4 - Geometric Interpretation :
Example: Complex conjugate in the Complex plane (Argand Plane)



$z = 6 + j4$

$z = r\angle\theta = 7.211\angle 0.588$

$\bar{z} = 6 - j4$

$\bar{z} = r\angle(-\theta) = 7.211\angle(-0.588)$

Geometric Interpretation: Geometrically, taking the complex conjugate of a complex number corresponds to reflecting the number across the real axis (horizontal axis) in the Argand diagram (complex plane)

8A.1.4 - *More Examples and some identities*

Some examples:

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$.

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i,$$

$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i.$$

Some identities:

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2,$$

$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

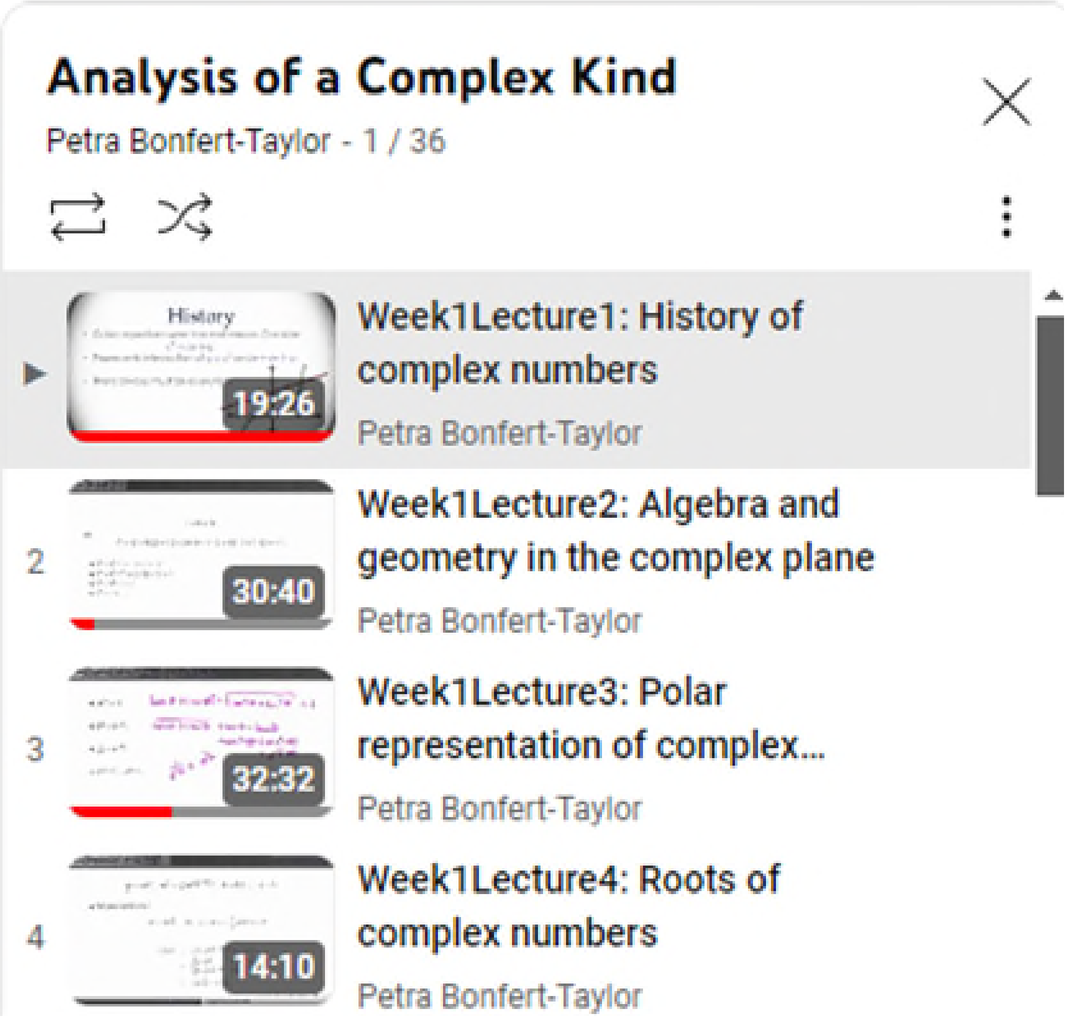
Relevant References for this section

1) U. of surrey course notes on complex numbers
https://www.surrey.ac.uk/sites/default/files/2021-07/10.1-introduction-to-complex-numbers.pdf

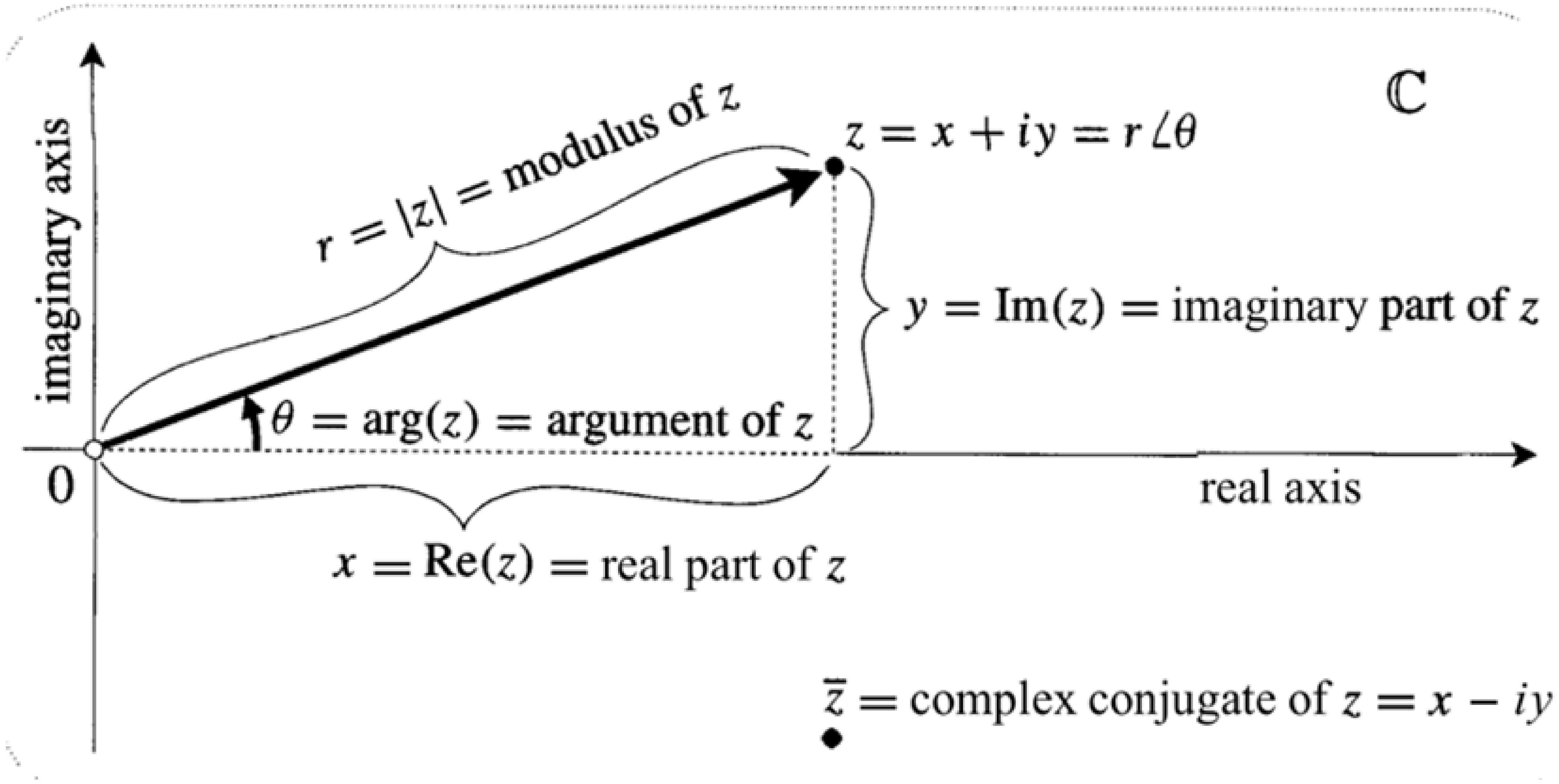
2) Beginner's guide:
https://www.matrix.edu.au/beginners-guide-maths-extension-2/introduction-to-complex-numbers/

3) Online book: UC Davis
https://math.libretexts.org/Bookshelves/Linear_Algebra/Book:_Linear_Algebra_(Schilling_Nachtergaele_and_Lankham)/02:_Introduction_to_Complex_Numbers

4) Petra's Bonfert Taylor complex analysis videos: Lect1~4
https://www.youtube.com/watch?v=CVpMpZpd-5s&list=PLi7yHjesblV0sSfZzWdSUXGO683n_nJdQ&index=1



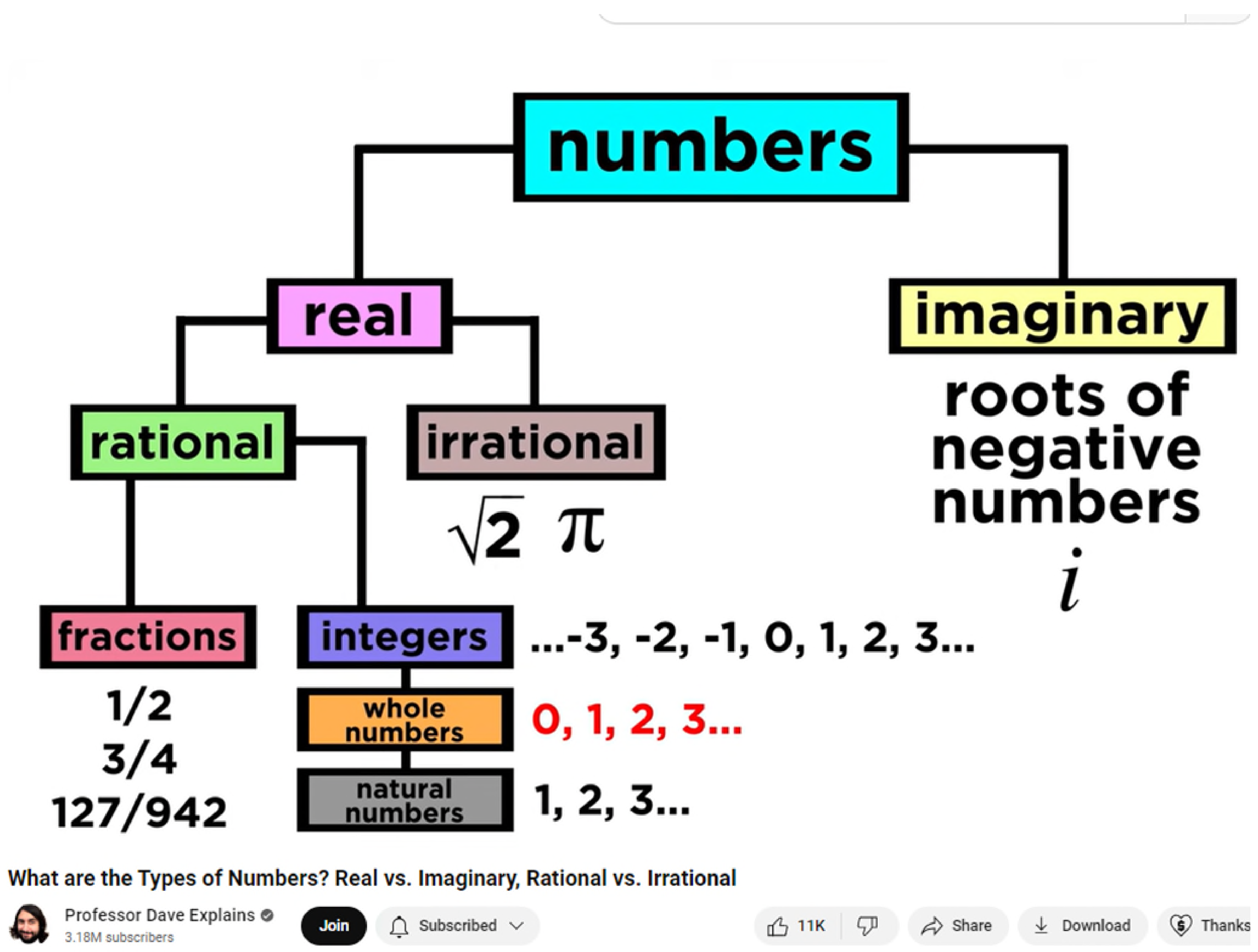
Ref: Terminologies



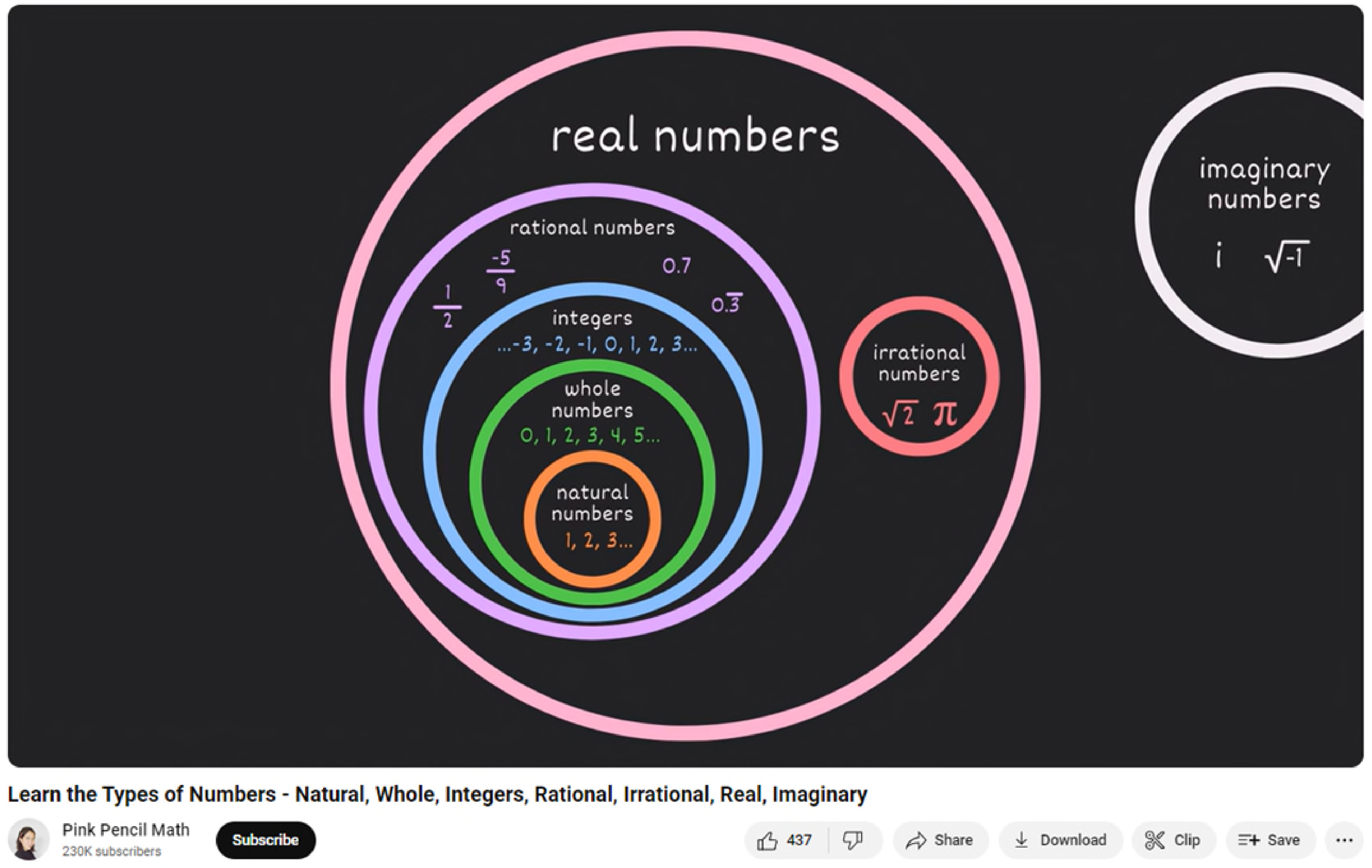
Name	Meaning	Notation
modulus of z	length r of z	$\lvert z\rvert$
argument of z	angle θ of z	$\arg(z)$
real part of z	x coordinate of z	$\mathrm{Re}(z)$
imaginary part of z	y coordinate of z	$\mathrm{Im}(z)$
imaginary number	real multiple of i	
real axis	set of real numbers	
imaginary axis	set of imaginary numbers	
complex conjugate of z	reflection of z in the real axis	$\bar{z}$

Other References

1) Professor Dave: Types of numbers
https://www.youtube.com/watch?v=QUGmwPwtbpg



Hoser: The story of (almost) All Numbers
https://www.youtube.com/watch?v=ggOPJ8gafPo



2) Pink Pencil Math: Types of numbers
https://www.youtube.com/watch?v=WxXZaP8Y8pI