$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.3**

Lecture : **Eigen and Singular Values**

Topic : **Similarity and Diagonalization**

Concept : **When can we represent $A = PDP^{-1}$**

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Similarity and Diagonalization

Given two nxn matrix A and P, and P is invertible (means it has n-independent column), then we can have a matrix B generated as follows:

$$P^{-1}AP = B$$
$$AP = PB$$
$$A = PBP^{-1}$$

then we say that A and B are similar matrixes,
and the transformation from A to $B = P^{-1}AP$ is called similarity transformation.

In the special case that B is a diagonal matrix, then we also say that A is diagonalizable!

Wiki: insight

Similar matrices represent the same linear map under two (possibly) different bases, with P being the change of basis matrix.[1][2]

See Lecture 8.1.5B

Similarity and Diagonalization

Similar Matrix is an important concept because Similar matrixes share certain characteristics:

In general, any property that is preserved by a similarity transformation is called a ***similarity invariant*** and is said to be ***invariant under similarity***. Table 1 lists the most important similarity invariants. The proofs of some of these are given as exercises.

Table 1 Similarity Invariants

Property	Description
Determinant	A and $P^{-1}AP$ have the same determinant.
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertible.
Rank	A and $P^{-1}AP$ have the same rank.
Nullity	A and $P^{-1}AP$ have the same nullity.
Trace	A and $P^{-1}AP$ have the same trace.
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial.
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues.
Eigenspace dimension	If λ is an eigenvalue of A (and hence of $P^{-1}AP$) then the eigenspace of A corresponding to λ and the eigenspace of $P^{-1}AP$ corresponding to λ have the same dimension.

We like a matrix to be similar to a diagonal matrix

Why we like diagonal matrix?

Diagonal matrixes have nice properties:

1) Eigenvalues of diagonal matrixes are its diagonal element
2) Determinant == product of diagonal entries
3) Rank == number of non-zero entries in the diagonal
4) Multiplication: given A and diagonal matrix D (AD and DA):

   - when we pre-multiply A by a diagonal matrix D, the rows of A are multiplied by the diagonal elements of D;
   - when we post-multiply A by D, the columns of A are multiplied by the diagonal elements of D.

5) A diagonal matrix's inverse is reciprocal of diagonal elements
6) Product of diagonal matrixes are easy to compute.

Ref:
1) https://www.statlect.com/matrix-algebra/diagonal-matrix
2) http://www.robertosmathnotes.com/uploads/8/2/3/9/8239617/la10-3_diagonal_matrices.pdf

When is A Diagonalizable?

Special case of similarity, B is D (a diagonal matrix).

THEOREM 5

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis** of $\mathbb{R}^n$.

Lay, pg 282, Ch 5.3

Ref:https://textbooks.math.gatech.edu/ila/similarity.html

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When is A Diagonalizable? Depends on P (the matrix containing the eigenvectors)

Since P is invertible, its columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ must be linearly independent. Also, since these columns are nonzero, the equations in (4) show that $\lambda_1, \dots, \lambda_n$ are eigenvalues and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are corresponding eigenvectors. This argument proves the "only if" parts of the first and second statements, along with the third statement, of the theorem.

Finally, given any n eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, use them to construct the columns of P and use corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ to construct D. By equations (1)–(3), $AP = PD$. This is true without any condition on the eigenvectors. If, in fact, the eigenvectors are linearly independent, then P is invertible (by the Invertible Matrix Theorem), and $AP = PD$ implies that $A = PDP^{-1}$. ■

Proof: When is A Diagonalizable?

THEOREM 5

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis** of $\mathbb{R}^n$.

PROOF First, observe that if P is any $n \times n$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and if D is any diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, then

$$AP = A[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,] = [\,A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n\,] \tag{1}$$

while

$$PD = P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\,\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n\,] \tag{2}$$

Now suppose A is diagonalizable and $A = PDP^{-1}$. Then right-multiplying this relation by P, we have $AP = PD$. In this case, equations (1) and (2) imply that

$$[\,A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n\,] = [\,\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n\,] \tag{3}$$

Equating columns, we find that

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \quad \ldots, \quad A\mathbf{v}_n = \lambda_n\mathbf{v}_n \tag{4}$$

Examine

$$AP = PD$$

For columns of P being eigenvectors of A, and D the eigenval of A

Practice Problems 2

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Use this information to diagonalize A.

Sol:

2. Compute $A\mathbf{v}_1 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \cdot \mathbf{v}_1$, and

$$A\mathbf{v}_2 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \mathbf{v}_2$$

So, $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for the eigenvalues 1 and 3, respectively. Thus

$$A = PDP^{-1}, \quad \text{where} \quad P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

```
% Example in Slide 8.1.3 (diagonalization)
A = [-3 12; -2 7];
[P,D] = eig(A);

P1 = [3 2; 1 1]
D1 = [ 1 0; 0 3]
P1*D1*inv(P1)
```

```
A =

    -3    12
    -2     7


P =

   -0.9487   -0.8944
   -0.3162   -0.4472


D =

    1.0000         0
         0    3.0000
```

```
P1 =

     3     2
     1     1


D1 =

     1     0
     0     3


ans =

   -3.0000   12.0000
   -2.0000    7.0000
```

Recap: Steps to diagonalize a Matrix

Given a matrix A size NxN, to diagonalize it to D, perform:

1) Find the eigenvalues of A.
2) For each eigenvalue, find the eigenvectors of corresponding λ_i
3) If there are N independent eigenvectors v_i, then the matrix A can be represented as:

$$\begin{aligned} AP &= PD \\ A &= PDP^{-1} \\ P^{-1}AP &= D \end{aligned}$$

Where D = diagonal matrix with eigenvalues λ_i
And P is a matrix with columns that are corresponding eigenvectors v_i.

When is A diagonalizable? Sufficient condition: If A has Distinct EigenValues -> Diagonalizable

THEOREM 6 Lay, 4thEd, pg 284, Ch 5.3

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

PROOF Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent, by Theorem 2 in Section 5.1. Hence A is diagonalizable, by Theorem 5. ∎

It is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable. The 3×3 matrix in Example 3 is diagonalizable even though it has only two distinct eigenvalues.

Theorem & proof: distinct eigenvalues means distinct eigenvectors

THEOREM 2 If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Lay 4th, pg 270, Ch 5.1

PROOF Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly dependent. Since $\mathbf{v}_1$ is nonzero, Theorem 7 in Section 1.7 says that one of the vectors in the set is a linear combination of the preceding vectors. Let p be the least index such that $\mathbf{v}_{p+1}$ is a linear combination of the preceding (linearly independent) vectors. Then there exist scalars $c_1, \dots, c_p$ such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{v}_{p+1} \tag{5}$$

Multiplying both sides of (5) by A and using the fact that $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ for each k, we obtain

$$\begin{aligned} c_1A\mathbf{v}_1 + \cdots + c_pA\mathbf{v}_p &= A\mathbf{v}_{p+1} \\ c_1\lambda_1\mathbf{v}_1 + \cdots + c_p\lambda_p\mathbf{v}_p &= \lambda_{p+1}\mathbf{v}_{p+1} \end{aligned} \tag{6}$$

Multiplying both sides of (5) by λ_{p+1} and subtracting the result from (6), we have

$$c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p = \mathbf{0} \tag{7}$$

Since $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent, the weights in (7) are all zero. But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct. Hence $c_i = 0$ for $i = 1, \dots, p$. But then (5) says that $\mathbf{v}_{p+1} = \mathbf{0}$, which is impossible. Hence $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent. ■

Recap: linear independence

Lay 4th Ed, ch1.7) Revision linear independence

THEOREM 7 **Characterization of Linearly Dependent Sets**

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

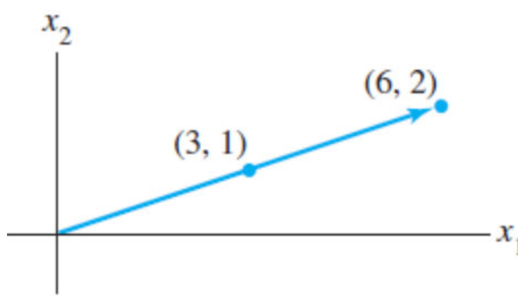
Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = [\, \mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n \,]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

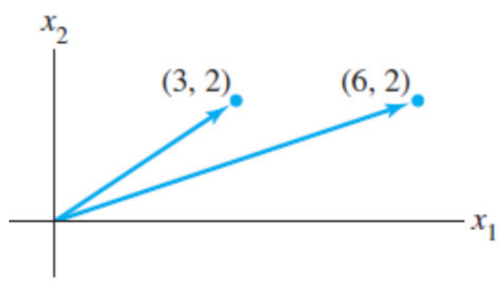
$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following important fact.

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)



Linearly dependent



Linearly independent

Example: Distinct EigenValues -> Diagonalizable

EXAMPLE 5 Determine if the following matrix is diagonalizable.

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

SOLUTION This is easy! Since the matrix is triangular, its eigenvalues are obviously 5, 0, and -2. Since A is a 3×3 matrix with three distinct eigenvalues, A is diagonalizable. ■

See Slide 8.1.2 (pg 8) To see slide “EigenValues of Triangular Matrixes”

THEOREM 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

Distinct eigenvalue is a sufficient BUT not necessary condition to have linearly independent eigenvectors.

Example 3 (later) shows that eigenvalues are repeated, but it is still diagonalizable.
And Example 4 shows counter-example.

When is a matrix with repeated roots diagonalizable? Introducing algebraic and geometrix multiplicity.

Algebraic
multiplicity = multiplicity of eigenvalues λ_k

Geometric
multiplicity = Dimension of eigenspace corresponding to eigenvalue λ_k

Ref: https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk10/10_algebraic_and_geometric_multiplicities.html

Introducing terminology: Algebraic and Geometric Multiplicity

Eigenspaces

Let λ be an eigenvalue of A. Recall that the eigenvectors of A for λ are the nonzero vectors in the nullspace of $A - \lambda I$. We call the nullspace $A - \lambda I$ the **eigenspace** of A for λ denoted by $\mathcal{E}_A(\lambda)$. In other words, $\mathcal{E}_A(\lambda)$ consists of all the eigenvectors of A for λ and the zero vector.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Note that -1 is an eigenvalue of A. Then $A - (-1)I_2 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. The nullspace of this matrix is spanned by the single vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence, $\mathcal{E}_A(-1)$ is the span of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Sanity check:

```
% example in sli
A = [1 2; 1 0]
[P,D] = eig(A)
```

```
A =

     1     2
     1     0


P =

    0.8944   -0.7071
    0.4472    0.7071


D =

     2     0
     0    -1
```

Ref:https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk10/10_algebraic_and_geometric_multiplicities.html

Geometric Multiplicity (is the dimension of Eigen Space)
Algebraic Multiplicity (is the number of repeated roots)

Algebraic multiplicity vs geometric multiplicity

The **geometric multiplicity** of an eigenvalue λ of A is the dimension of $\mathcal{E}_A(\lambda)$. In the example above, the geometric multiplicity of -1 is 1 as the eigenspace is spanned by one nonzero vector.

In general, determining the geometric multiplicity of an eigenvalue requires no new technique because one is simply looking for the dimension of the nullspace of $A - \lambda I$.

The **algebraic multiplicity** of an eigenvalue λ of A is the number of times λ appears as a root of p_A. For the example above, one can check that -1 appears only once as a root. Let us now look at an example in which an eigenvalue has multiplicity higher than 1.

In mathematics, the **dimension** of a vector space *V* is the cardinality (i.e. the number of vectors) of a basis of *V* over its base field.[1]

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

```
% Example:
% repeated roots lambda=1,
% BUT eigenspace ==1
A = [1 2; 0 1]
[P,D] = eig(A)
```

```
A =

     1     2
     0     1


P =

    1.0000   -1.0000
         0    0.0000


D =

     1     0
     0     1
```

Ref:https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk10/10_algebraic_and_geometric_multiplicities.html

When is a matrix with repeated roots diagonalizable? Introducing algebraic and geometrix multiplicity.

In general, **the algebraic multiplicity and geometric multiplicity of an eigenvalue can differ. However, the geometric multiplicity can never exceed the algebraic multiplicity**.

It is a fact that summing up the algebraic multiplicities of all the eigenvalues of an $n \times n$ matrix A gives exactly n. **If for every eigenvalue of A, the geometric multiplicity equals the algebraic multiplicity, then A is said to be *diagonalizable***.

See also (Theorem 7) in Lay, 4thEd, pg 285, Ch 5.3

Ref: https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk10/10_algebraic_and_geometric_multiplicities.html

Example 3: Diagonalizable A with repeated eigenValue

EXAMPLE 3 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

SOLUTION There are four steps to implement the description in Theorem 5.

Step 1. ***Find the eigenvalues of A.*** As mentioned in Section 5.2, the mechanics of this step are appropriate for a computer when the matrix is larger than 2×2. To avoid unnecessary distractions, the text will usually supply information needed for this step. In the present case, the characteristic equation turns out to involve a cubic polynomial that can be factored:

$$\begin{aligned} 0 = \det(A - \lambda I) &= -\lambda^3 - 3\lambda^2 + 4 \\ &= -(\lambda - 1)(\lambda + 2)^2 \end{aligned}$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$.

Note: In this example, the eigenvalues are NOT distinct ($\lambda = -2$), i.e repeated, But this matrix is diagonalizable.

Algebraic multiplicity of eigenvalue = -2 is 2
AND
Geometric multiplicity of eigenvalue = -2 is ALSO 2.

HENCE there is a complete set of linearly independent eigenvectors for A, allowing A to be diagonalizable.

Lay4th, pg 283, Ch 5.2

Example 3: Diagonalizable A with repeated eigenValue

Step 2. ***Find three linearly independent eigenvectors of A.*** *Three* vectors are needed because A is a 3×3 matrix. This is the critical step. If it fails, then Theorem 5 says that A cannot be diagonalized. The method in Section 5.1 produces a basis for each eigenspace:

$$\text{Basis for } \lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -2: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

Step 3. ***Construct P from the vectors in step 2.*** The order of the vectors is unimportant. Using the order chosen in step 2, form

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Sanity check:

```
% Example: slide 5.1.3, pg 17
A = [1 3 3; -3 -5 -3; 3 3 1]
[P,D] = eig(A)
```

```
A =

     1     3     3
    -3    -5    -3
     3     3     1


P =

   -0.5774   -0.7876    0.4206
    0.5774    0.2074   -0.8164
   -0.5774    0.5802    0.3957


D =

    1.0000         0         0
         0   -2.0000         0
         0         0   -2.0000
```

```
P23 = P(:,2:3);
v2  = [-1 1 0]'
c2 = pinv(P23)*v2
v2_est = P23*c2
```

```
c2 =

    0.7121
   -1.0440


v2_est =

   -1.0000
    1.0000
   -0.0000
```

Note that matlab produces col 2,3 of P that does not resemble v_2 *and* v_3.
But you can check that v_2 *and* v_3 can be formed by appropriate linear combinations of col2,3, of P!

Example 3: Diagonalizable A with repeated eigenValue

Step 4. Construct D from the corresponding eigenvalues. In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P. Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

It is a good idea to check that P and D really work. To avoid computing P^{-1}, simply verify that $AP = PD$. This is equivalent to $A = PDP^{-1}$ when P is invertible. (However, be sure that P is invertible!) Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \quad \blacksquare$$

Example 4: NOT Diagonalizable A with repeated eigenValue

EXAMPLE 4 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

SOLUTION The characteristic equation of A turns out to be exactly the same as that in Example 3:

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$. However, it is easy to verify that each eigenspace is only one-dimensional:

$$\text{Basis for } \lambda = 1: \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -2: \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

There are no other eigenvalues, and every eigenvector of A is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$. Hence it is impossible to construct a basis of $\mathbb{R}^3$ using eigenvectors of A. By Theorem 5, A is *not* diagonalizable. ■

Note: In this example, the eigenvalues are NOT distinct ($\lambda = -2$), i.e repeated,
But this matrix is NOT diagonalizable.

Algebraic multiplicity of eigenvalue = -2 is 2
BUT
Geometric multiplicity of eigenvalue = -2 is ONLY 1.

Hence incomplete basis of eigenvectors for A =>
A is NOT diagonalizable.

Example 4: NOT Diagonalizable A with repeated eigenValue

It is NOT possible to diagonalize as A does
NOT have a full set of independent eigenvectors.
Sanity check: what does Matlab produce?

```
A =

     2     4     3
    -4    -6    -3
     3     3     1

>> [P,D] = eig(A)

P =

   0.5774 + 0.0000i   0.7071 + 0.0000i   0.7071 - 0.0000i
  -0.5774 + 0.0000i  -0.7071 + 0.0000i  -0.7071 + 0.0000i
   0.5774 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i


D =

   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -2.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -2.0000 - 0.0000i
```

Note that columns of P are the eigen vectors.
Column 2 == Column 3.

Compare Column 2 to v_2 .
We see Direction is the same, BUT scaled differently.
Matlab gives eigenVectors with norm == 1

References

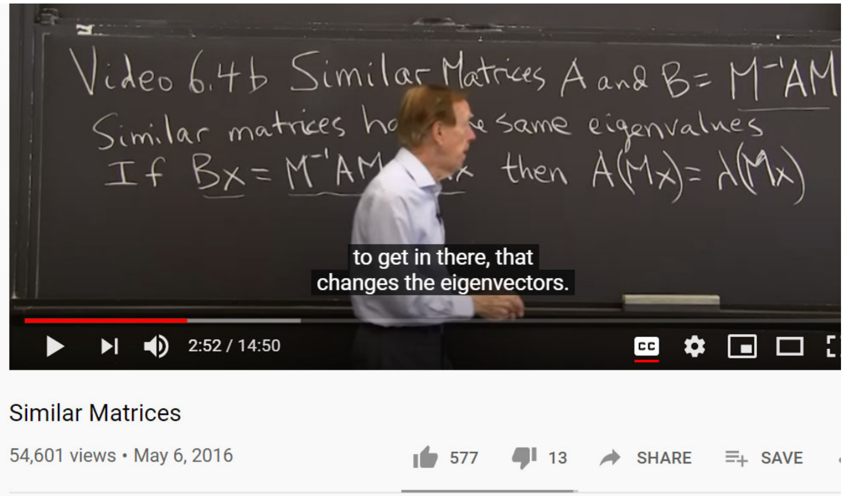
Similar Matrixes
A) Trefor Bazett: Similar matrices have similar properties
Link: https://www.youtube.com/watch?v=jNtiENbAcFM

B) MIT Strang: “Similar Matrixes”
https://www.youtube.com/watch?v=TSdXJw83kyA
https://www.youtube.com/watch?v=LKMGo8G7-vk
https://www.youtube.com/watch?v=KUuxdk_V7To

C) Technion 1M :
https://youtu.be/MJic8o5ph5M,
https://youtu.be/l2BEIOnG54k

D) Dan Yasaki https://youtu.be/ObTVoBQDBx8

E) Mathaholic:
https://www.youtube.com/watch?v=H2pv1Rug0RQ



Prof. Strang shows similar matrices A and B have the same eigenvalues in the first 3 minutes and works out some examples of similar matrices later!

Diagonalization:
A) MIT Strang: “Diagonalizing a matrix”
https://www.youtube.com/watch?v=U8R54zOTVLw

B) MIT Strang: “Diagonalization and Powers of A”
https://www.youtube.com/watch?v=13r9QY6cmjc
Example: Time == 22:10 (triangular matrix and eigenvalues)
Time == 24:00 (algebraic multiplicity==2, geometric multiplicity=1) -> not diagonalizable.