

CX1104: Linear Algebra for Computing

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}}_{A \quad m \times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{x \quad n \times 1} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b \quad m \times 1}$$

Chap. No : **8.1.4**

Lecture : **Eigen and Singular Values**

Topic : **Powers of A**

Concept : **Efficiently compute A^K**

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Power of A: Example 1

In many cases, the eigenvalue–eigenvector information contained within a matrix A can be displayed in a useful factorization of the form $A = PDP^{-1}$ where D is a diagonal matrix. In this section, the factorization enables us to compute A^k quickly for large values of k , a fundamental idea in several applications of linear algebra. Later, in Sections 5.6 and 5.7, the factorization will be used to analyze (and *decouple*) dynamical systems.

The following example illustrates that powers of a diagonal matrix are easy to compute.

EXAMPLE 1 If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$
and

$$D^3 = DD^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \quad \text{for } k \geq 1 \quad \blacksquare$$

Example 2: Finding A^k from $A = PDP^{-1}$

EXAMPLE 2 Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k , given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

SOLUTION The standard formula for the inverse of a 2×2 matrix yields

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Then, by associativity of matrix multiplication,

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) = PD \underbrace{(P^{-1}P)}_I DP^{-1} = PDDP^{-1} \\ &= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \end{aligned}$$

Example 2: Finding A^k from $A = PDP^{-1}$

Again,

$$A^3 = (PDP^{-1})A^2 = (PDP^{-1})\underbrace{PD^2P^{-1}}_I = PDD^2P^{-1} = PD^3P^{-1}$$

In general, for $k \geq 1$,

$$\begin{aligned} A^k &= PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix} \quad \blacksquare \end{aligned}$$

A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D .

Impt: when A is diagonalizable to $A = PDP^{-1}$, then

$$A^k = PD^kP^{-1}$$

Practice Problems 1

PRACTICE PROBLEMS

1. Compute A^8 , where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

SOLUTIONS TO PRACTICE PROBLEMS

1. $\det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$. The eigenvalues are 2 and 1, and the corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Next, form

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Since $A = PDP^{-1}$,

$$\begin{aligned} A^8 &= PD^8P^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix} \end{aligned}$$

References

- 1) MIT Strang : Lect 22. Diagonalization and Powers of A
<https://www.youtube.com/watch?v=13r9QY6cmjc>

