$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **6.1.4**

Lecture : **Orthogonality**

Topic : **Dot Product**

Concept : **Important Inequalities**

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Cauchy-Schwarz Inequality

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)$$

Formula (20) is not defined unless its argument satisfies the inequalities

$$-1 \le \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \le 1 \tag{21}$$

Fortunately, these inequalities *do* hold for all nonzero vectors in R^n as a result of the following fundamental result known as the ***Cauchy–Schwarz inequality***.

THEOREM 3.2.4 Cauchy–Schwarz Inequality

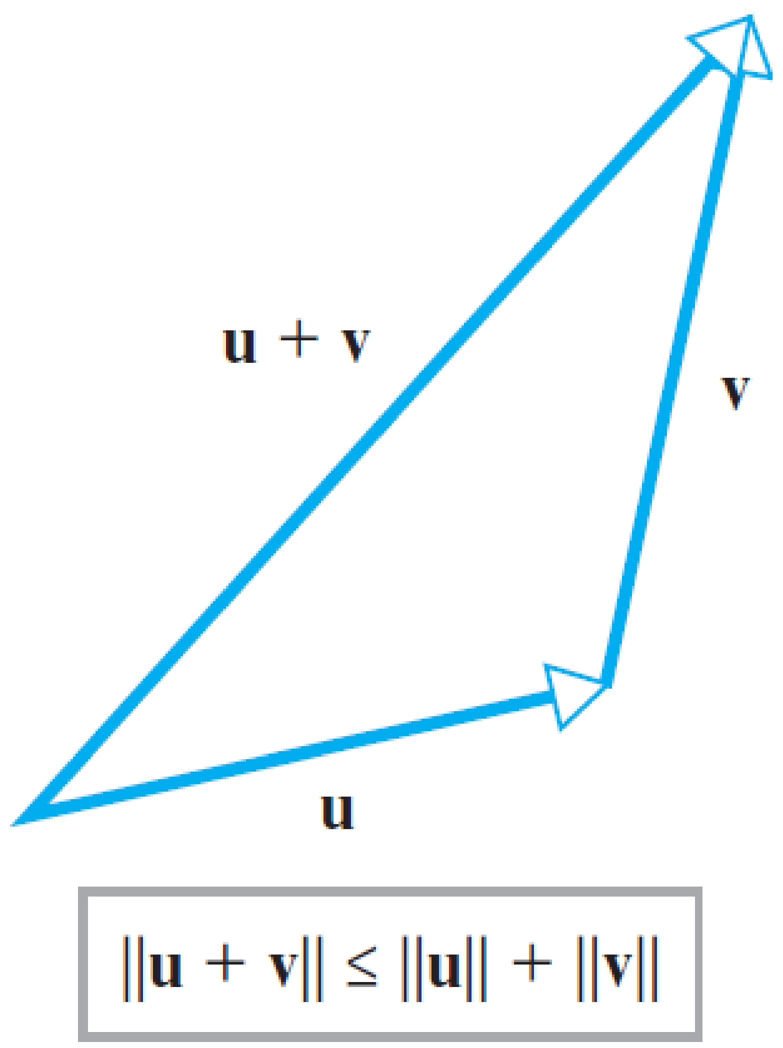
If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n, *then*

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\| \|\mathbf{v}\| \tag{22}$$

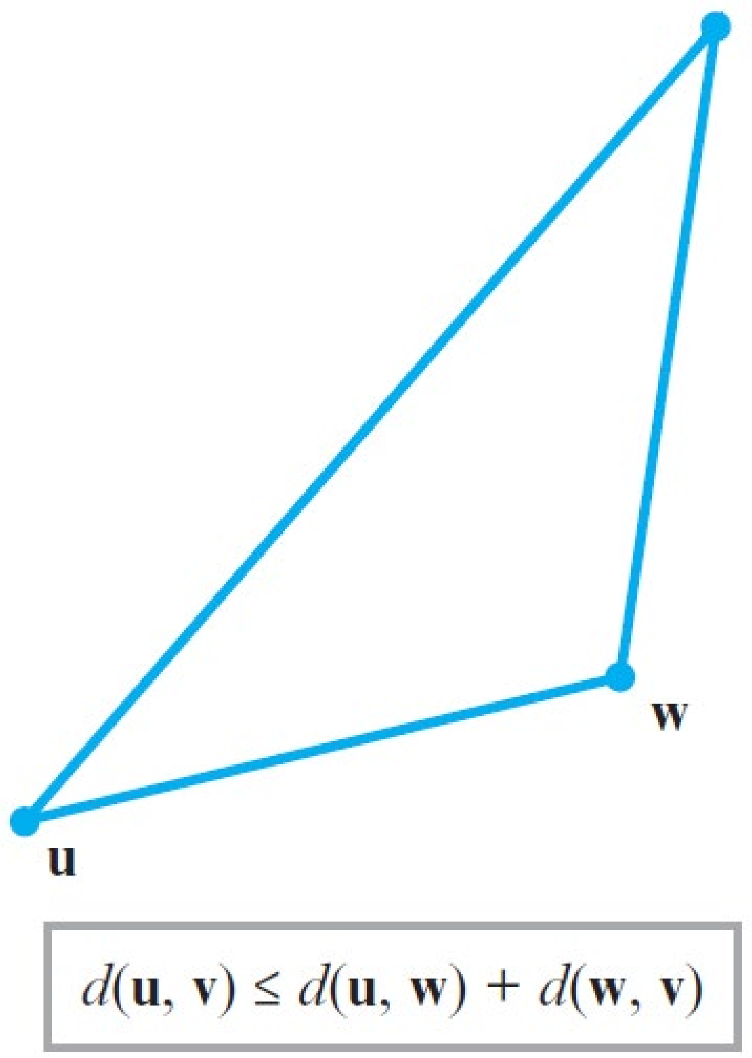
or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \le (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2} \tag{23}$$

Triangle Inequality



▲ Figure 3.2.8



▲ Figure 3.2.9

THEOREM 3.2.5 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *then:*

(*a*) $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ **[Triangle inequality for vectors]**

(*b*) $d(\mathbf{u},\mathbf{v}) \le d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$ **[Triangle inequality for distances]**

Proof:

Proof (a)

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{u}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{v})\\
&= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2\\
&\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 && \leftarrow \text{Property of absolute value}\\
&\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \leftarrow \text{Cauchy–Schwarz inequality}\\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2
\end{aligned}$$

This completes the proof since both sides of the inequality in part (*a*) are nonnegative.

Proof (b) It follows from part (*a*) and Formula (11) that

$$\begin{aligned}
d(\mathbf{u},\mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| = \|(\mathbf{u}-\mathbf{w}) + (\mathbf{w}-\mathbf{v})\|\\
&\le \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\| = d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v}) \quad ◀
\end{aligned}$$

Parallelogram Equation for Vectors

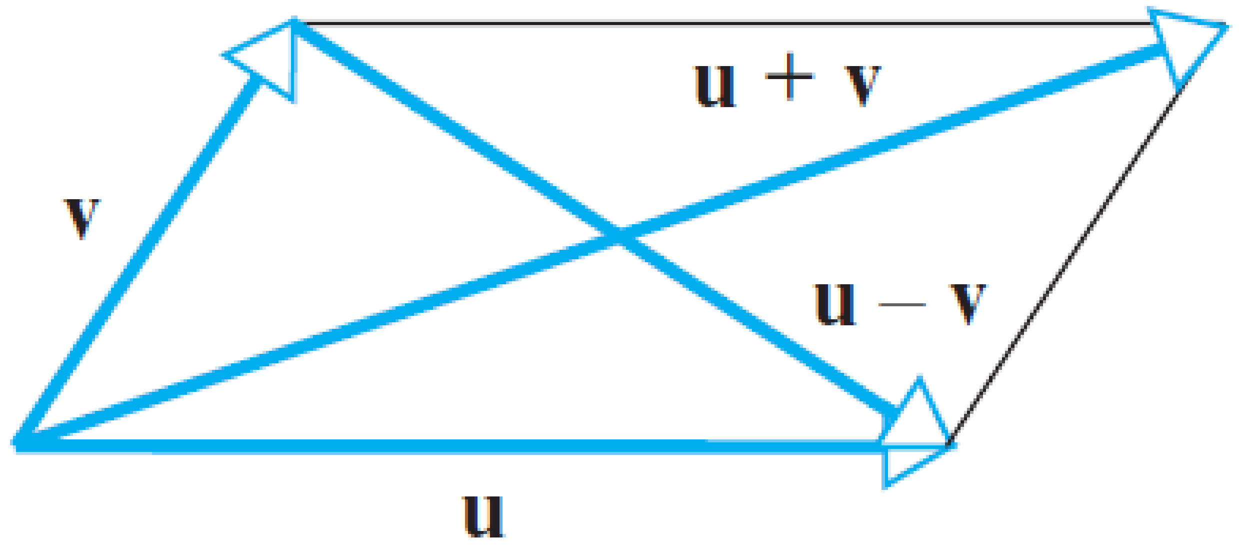


Figure 3.2.10

THEOREM 3.2.6 **Parallelogram Equation for Vectors**

If **u** *and* **v** *are vectors in* R^n, *then*

$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2=2\left(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2\right) \tag{24}$$

Proof:

THEOREM 3.2.6 **Parallelogram Equation for Vectors**

If **u** *and* **v** *are vectors in* R^n, *then*

$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2=2\left(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2\right) \tag{24}$$

Proof

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})+(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= 2(\mathbf{u}\cdot\mathbf{u})+2(\mathbf{v}\cdot\mathbf{v})\\ &= 2\left(\|\mathbf{u}\|^2+\|\mathbf{v}\|^2\right) \quad ◀\end{aligned}$$

THEOREM 3.2.7 *If* **u** *and* **v** *are vectors in* R^n *with the Euclidean inner product, then*

$$\mathbf{u}\cdot\mathbf{v}=\tfrac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2-\tfrac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 \tag{25}$$

Proof

$$\|\mathbf{u}+\mathbf{v}\|^2=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})=\|\mathbf{u}\|^2+2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2$$
$$\|\mathbf{u}-\mathbf{v}\|^2=(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})+\|\mathbf{v}\|^2$$

from which (25) follows by simple algebra. ◀

Reference

Materials in these slides have been taken from:

Anton and Rorres, “Linear Algebra”, 11th edition , Wiley.

Chapter: 3.1, 3.2

Euclidean Vector Spaces