MH1812
Extra Exercises of Combinatorics

These are additional combinatorial questions and may not be discussed in the tutorials.

Q1: In how many ways can 5 indistinguishable rooks be placed on an 8-by-8 chessboard so that

(a) no rook can attack each other?

(b) no rook can attack each other and neither the first row nor the first column is empty?

Solution:

(a) We first select 5 rows and place a rook in each row. To determine the positions of the rooks in those rows, we consider all the permutations of 5 columns from 8 columns. Hence the answer is

$$\binom{8}{5}^2 5!.$$

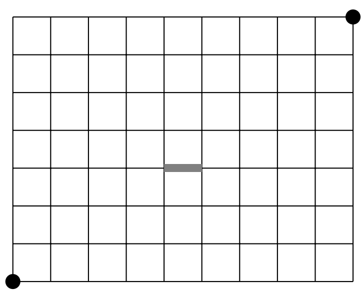
(b) There are two cases: (i) there is a rook at $(1,1)$ and we place 4 rooks in the remaining 7-by-7 chessboard; (ii) there is a rook at $(1,x)$ for some $x \neq 1$ and a rook at $(y,1)$ for some $y \neq 1$, and we place 3 rooks in the remaining 6-by-6 chessboard. Hence the answer is

$$\binom{7}{4}^2 4! + 7^2 \cdot \binom{6}{3}^2 3!. \qquad \square$$

Q2: A secretary works in a building located 9 blocks east and 7 blocks north of his home. Every day he walks 16 blocks to work. (See the map that follows.)

(a) How many different routes are possible for him?

(b) How many different routes if the block (coloured in grey) in the easterly direction, which begins 4 blocks east and 3 blocks north of his home, is under water (and he cannot swim)? (Hint: count the routes that use the block under water.)



Solution:

(a) Each route corresponding to a sequence 16 steps consisting of 9 "eastwards" and 7 "northwards". Hence the answer is (choosing 9 out of 16 positions to fill in "eastwards" and fill the remaining positions "northwards")

$$\binom{16}{7} = 11400$$

(b) A route passing through the grey block must be a route from $(0,0)$ to $(4,3)$, followed by an "eastwards" step and a route from $(5,3)$ to $(9,7)$. Hence there are

$$\binom{7}{4}\binom{8}{4} = 2450$$

routes passing through the grey block and the answer to the question is therefore

$$\binom{16}{7} - \binom{7}{4}\binom{8}{4} = 8990. \qquad \square$$

Q3: Show that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for all integers $n \geq k \geq 1$ by

(a) direct calculation;

(b) a combinatorial argument that relates choosing k items from n items (the left-hand side) to choosing $k-1$ items from $n-1$ items and choosing k items from $n-1$ items (the right-hand side).

Solution:

(a)

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(n-k)!(k-1)!}+\frac{(n-1)!}{(n-k-1)!k!}\\ &= \frac{(n-1)!k}{(n-k)!k!}+\frac{(n-1)!(n-k)}{(n-k)!k!}\\ &= \frac{(n-1)!n}{(n-k)!k!}\\ &= \binom{n}{k}.\end{aligned}$$

(b) The LHS counts the number of ways to select k out of n items. The RHS counts the same thing according to two cases: either the n-th item is selected, or it is not selected. In the first case the remaining $k-1$ items must be selected from the remaining $n-1$ items. The number of ways to do this is $\binom{n-1}{k-1}$. In the second case all k items in the group must be selected from the remaining $n-1$ items. The number of ways to do this is $\binom{n-1}{k}$. Hence the number of selections is $\binom{n-1}{k-1}+\binom{n-1}{k}$, establishing the identity. □

Q4: Show that

$$\binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1}$$

for all integers $n \geq k \geq 1$ by

(a) mathematical induction on n;

(b) a combinatorial argument.

Solution:

(a) Base case: when $n=k$, the LHS is $\binom{k}{k}=1$ and the RHS is $\binom{k+1}{k+1}=1$, hence the identity holds when $n=k$.

Inductive step: Suppose that the identity holds for some n with $n \geq k$, we shall show that the identity holds for $n+1$.

$$\begin{aligned}\text{LHS} &= \binom{k}{k}+\binom{k+1}{k}+\cdots+\binom{n}{k}+\binom{n+1}{k}\\ &= \binom{n+1}{k+1}+\binom{n+1}{k} \quad \text{(by induction hypothesis)}\\ &= \binom{n+2}{k+1} \quad \text{(by Q3)}\end{aligned}$$

This finishes the proof of the inductive step.

Therefore the identity holds for all $n \geq k$.

(b) Select $k+1$ items out of $n+1$ items without replacement and consider the largest item selected. The largest item could be any one of items $k+1$, ..., $n+1$. When the largest item is i, the number of such selections equals to the number of ways to select k items from the remaining $i-1$ items, which is $\binom{i-1}{k}$. Summing up i from $k+1$ to $n+1$ proves the result. $\square$