2021 / 5 NOV 2021.

LA / Ch 8 week 12/13.

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Topic: Change of Basis and Representation of Linear Transformation in different basis.

References:

A) Interpreting x, $[x]_B$, $[x]_C$ and their relationship.

a) Lay 5e, pg 241.

B) $T(x)$ linear transformation of Ax ($R^n \to R^n$) and its representation in another basis.

b) pg 292 ~293 Lay 5e.

c) $T(x)$ $R^n \to R^m$ linear transformation in different basis.

c) Lay 5e, pg 291. Matrix for T relative to the bases B and C.

A) Interpreting x, $[x]_B$, $[x]_C$ and their relationship.

a) A basis (set of independent vectors that span a vector space) for vector space V allow us to represent $x \in V$ with a coordinate system, and it will make V act like R^n.

b) different basis allow us to view V differently.

c) Given a basis $B = \{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$ for V, then for each $\underline{x} \in V$, there exist a unique set of scaler $c_1, c_2, \dots c_n$, such that

$$\underset{\in V}{x} \rightarrow \underset{\in R^n}{[x]_{std}} = \beta_1 \underline{b}_1 + \beta_2 \underline{b}_2 + \dots \beta_n \underline{b}_n .$$

represented in std basis

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ b_1 & b_2 & \cdots & b_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$[x]_{std} = B \; [x]_B$$

d) The coordinate of x with respect to the B-basis are the weights $\alpha_1, \alpha_2 \dots \alpha_n$

$$[x]_B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \in R^n$$

and $[x]_{std}$ in std coordinate means wrt std basis

$$\text{std basis} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 1 \end{bmatrix} \Rightarrow \text{each col } e_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R^n$$

$e_1 \quad e_2 \quad e_n$

i^{th} entry.

$$\underset{\in V}{x} \rightarrow \underset{\in R^n}{[x]_{\text{std basis}}}$$

Typically we do not indicate std-basis $[\ ]_{\text{std basis}}$ just

In most cases, we simply write $[x]_{std}$ as x when it is clear from context.

e) $\therefore \; [x]_{std} = \underset{\text{B basis}}{B}[x]_B = \underset{\text{C basis}}{C}[x]_C$

Note B, C are basis of R^n

$\Rightarrow$ these will be square invertible matrix $(n \times n)$ since col of B are independent and span R^n.

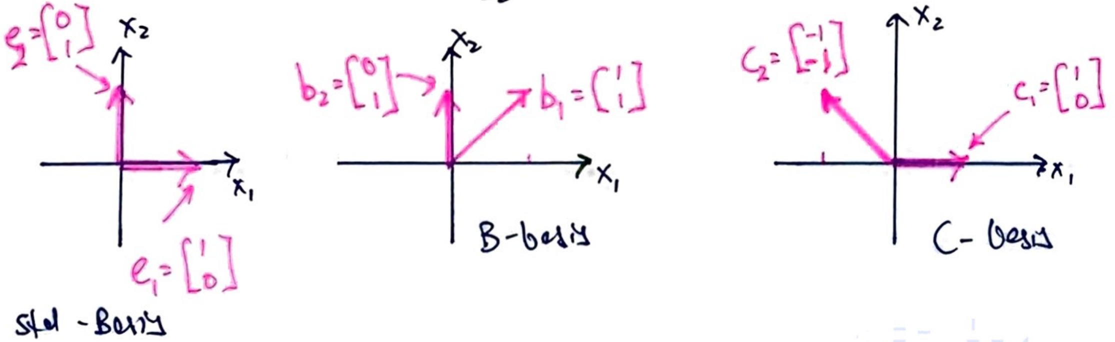
A) Interpreting x, $[x]_B$, $[x]_C$: example 1

From B basis → std basis → C basis.

Q: Given $\begin{bmatrix}2\\3\end{bmatrix}_{\text{std basis}}$, find $[x]_B$, $[x]_C$

given $B = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$, $C = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$

b_1 b_2 c_1 c_2

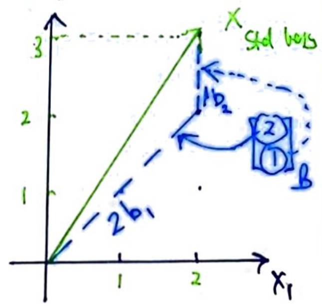


1a) To find $[x]_B$ given $[x]_{std}$ and B, note

$$x = [x]_{std} = B[x]_B$$

$$\therefore [x]_B = B^{-1}[x]_{std}$$

$$\underbrace{\begin{bmatrix}2\\1\end{bmatrix}_B}_{x \text{ in } B \text{ basis}} = \underbrace{\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^{-1}}_{B^{-1}} \underbrace{\begin{bmatrix}2\\3\end{bmatrix}_{std}}_{x \text{ std basis}}$$



Interpretation

i) $[x]_B = \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}_B \Rightarrow x = \begin{bmatrix}\uparrow & \uparrow\\ b_1 & b_2\\ \downarrow & \downarrow\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}_B$

$= \beta_1 b_1 + \beta_2 b_2$

ii) Graphically, we can see

$x = 2b_1 + 1b_2$ = travel along $\begin{bmatrix}1\\1\end{bmatrix} = b_1$ by 2 units and along $\begin{bmatrix}0\\1\end{bmatrix} = b_2$ by 1 unit.

to reach the same pt x via B-basis

1b) Find $[x]_C$ given $x = \begin{bmatrix}2\\3\end{bmatrix}_{\text{std basis}}$, $C = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$

$$\therefore [x]_{std} = C[x]_C$$

$$\therefore [x]_C = C^{-1}[x]_{std}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}_C = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}2\\3\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}_C = \begin{bmatrix}5\\3\end{bmatrix}_C$$

Sanity check : $x = 5\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}_{std}$

x_1 in C-basis, c_1, x_2 in C-basis, c_2.

A) Interpreting x, $[x]_B$, $[x]_C$: summary & from $[x]_B$ to $[x]_C$.

Summary:

$$[x]_{std} = B[x]_B$$
$$[x]_B = B^{-1}[x]_{std}.$$
$$[x]_{std} = C[x]_C$$
$$[x]_C = C^{-1}[x]_{std}.$$

$$\therefore \; B[x]_B = [x]_{std} = C[x]_C$$

$$\Rightarrow [x]_B = B^{-1}C[x]_C$$

$$\Rightarrow [x]_C = \underbrace{C^{-1}\underbrace{B[x]_B}_{\text{changing to std basis}}}_{\text{changing std to C Basis}} . \blacksquare$$

Note: Pre-multiply with basis B with $[x]_B$

a) $\Rightarrow$ changing to $[x]_{std}$.

b) Pre-multiply with B^{-1} with $[x]_{std}$

$\Rightarrow$ changing to $[x]_B$.

Ex 2: $[x]_C$ from $[x]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_B$ given $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$\therefore [x]_C = C^{-1}B\begin{bmatrix} 2 \\ 1 \end{bmatrix}_B$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \end{bmatrix}_C \; \blacksquare$$

// same results as pg 3

Lay's notation.

let $\underset{C \leftarrow B}{P} = C^{-1}B = C^{-1}[b_1 \ b_2]$ — eqn 1

$\therefore [x]_C = C^{-1}B[x]_B$

$= P[x]_B$.

in Lay's book.

$\underset{C \leftarrow B}{P} = \left[(b_1)_c \ (b_2)_c \right]$ → eqn 2

comparing col of eqn 1 to col of eqn 2.

$\therefore (b_1)_c = C^{-1}b_1$ // i.e col b_1 of B basis change to to C basis.

$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}_C$

$(b_2)_C = C^{-1}b_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_C$

2D

$\therefore \underset{C \leftarrow B}{P} = C^{-1}B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ ☒

Sanity check.

$[x]_C = \underset{C \leftarrow B}{P}[x]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ ☒

Note: Lay's notation $(b_1)_c$

= take col 1 from Basis B and convert to C basis by pre-multiply with C^{-1}. ☒

3) $T(x)$ linear transformation $R^n \to R^n$

represented by Ax.

and $A = PDP^{-1}$

where P = col^n contains a basis.

D = diagonal matrix.

$\therefore AP = PD \Rightarrow$ diagonalisation of A

P = col^n = eigenvectors

D = diagonal elements = eigen values.

then $y = Ax$

$$[y]_{std} = A[x]_{std}$$

$$= (PD\underbrace{P^{-1})[x]_{std}}_{\text{changing to P-basis.}}$$

$$= P\underbrace{D[x]_p}_{\text{transformation using D in P-basis.}}$$

$$= \underbrace{P[y]_p}_{\text{converting to std basis.}}$$

$$= [y]_{std}.$$

$$\begin{tikzcd}
x \arrow[rr, bend left, "T"] \arrow[d] & & y \arrow[d] \\
{[x]_{std}} \arrow[rr, "A"] \arrow[d, "P^{-1}"'] & & {[y]_{std}} \arrow[d, shift right, "P^{-1}"'] \\
{P^{-1}[x]_{std} = [x]_p} \arrow[rr, "D"] & & {[y]_p} \arrow[u, shift right, "P"']
\end{tikzcd}$$

$[y]_{std} = A[x]_{std} = P[y]_p = PDP^{-1}[x]_{std}$.

$[y]_p = D[x]_p = DP^{-1}[x]_{std}$.

$\therefore$ A is T in std basis = std matrix of T.

D is T in P basis.

// see Lay 5e section 1.9 (pg 71) Matrix of linear transformation. (Theorem 10).

$T: R^n \to R^m$ be linear transformation.

$T(x) = Ax$; $A \in R^{m \times n}$, $x \in R^n$

where $A = [T(e_1), T(e_2), .., T(e_n)]$

where e_j is col^n j of identity matrix.

3B

Example : $A = \begin{bmatrix} 2 & 0 \\ 2.5 & -0.5 \end{bmatrix}$.

can be diagonalize to

$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$

eigen vector   eigenvalue

a) Write the close form sol^n of

$x(n)$, given $x(n) = A\,x(n-1)$; $n \geq 1$

$x(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\therefore$ $x(1) = A\,x(0)$

$x(2) = A\,x(1) = AA\,x(0) = A^2 x(0)$

$\vdots$

$x(n) = A^n x(0)$.

$= (PDP^{-1})^n x(0)$.

// since A can be diagonalize to P, D

means $AP = PD$

$\therefore A = PDP^{-1}$.

$\therefore$ $x(n) = \underbrace{(PDP^{-1})(PDP^{-1})\ldots}_{n \text{ times}} x(0)$

$= PD^n \underline{P^{-1} x(0)}$

→ convert to P basis

→ transformation in P basis.

→ convert to std basis

$\therefore$ $x(n) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & -\frac{1}{2}^n \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 4 \end{bmatrix}}$

$[x(0)]_P = \begin{bmatrix} 1 \\ 3 \end{bmatrix}_P$.

$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-\frac{1}{2})^n \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left(2^n + 3(-\frac{1}{2})^n \right)$.

$x(n) = 2^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3(-\frac{1}{2})^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ⋇

ex:

$\therefore x(10) = 2^{10} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3(-\frac{1}{2})^{10} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ⋇

4A

$R^2 \to R^3$: Lay 5e, pg 291

$$[T(x)]_C = M[x]_B.$$

$$M = \Big[[T(b_1)]_C \ \ [T(b_2)]_C \cdots [T(b_n)]_C \Big]$$

↳ matrix for T relative to B and C.

interpreting col of M.

$$[T(b_1)]_C = C^{-1}[\ \ ]_{\text{std basis}} \quad = \text{converting to C-basis.}$$

$$T(b_1) = Ab_1$$

$$\underbrace{C^{-1}\underbrace{AB}_{T \text{ acting on } B \text{ basis}}}_{\text{converting to C-basis.}}$$

$$\begin{tikzcd}
x \arrow[r, "T"] \arrow[d] & y = T(x) \\
{[x]_{std}} \arrow[r, "A"] \arrow[d, bend right, "B^{-1}"'] & {[y]_{std} = A[x]_{std}.} \arrow[d, bend left, "C^{-1}"] \\
{[x]_B} \arrow[r, "M"] \arrow[u, bend right, "B"'] & {[y]_C = C^{-1}[y]_{std}} \arrow[u, bend left, "C"]
\end{tikzcd}$$

Graphically.

$$[y]_{std} = A[x]_{std}$$

$$[y]_C = M[x]_B.$$

C basis to std

$$[y]_{std} = C[y]_C = CM[x]_B$$

B basis to std.

$$[y]_{std} = \underbrace{CMB^{-1}}_{=A}[x]_{std}.$$

$$\therefore A = CMB^{-1}$$

$$\Rightarrow C^{-1}AB = M.$$

converting to C-basis

A acting on col of B

Transformation from $R^2 \to R^3$. / Lay 5e pg 291

$y = Ax \quad ; x \in R^2, y \in R^3$

$A \in 3\times 2$ matrix.

Given B basis matrix representing a $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

C matrix representing a basis for R^3 $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

find $[y]_c = M[x]_B$; M represent A from B basis to C basis.

given $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $[x]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Ans:

$\therefore [x]_{std} = B[x]_B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

$\therefore [y]_{std} = A[x]_{std} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}_{std}$.

4B

$\therefore [y]_{std} = A[x]_{std}$.

$= A \underbrace{B[x]_B}$. $\longrightarrow$ from $[x]_B \to [x]_{std}$

convert to C basis from std basis.

$[y]_c = C^{-1}[y]_{std} = \underbrace{C^{-1}AB}_{M}[x]_B$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}_B$

C^{-1} A B $[X]_B$

$[y]_c = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$ $\underset{A:C\leftarrow B}{M} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \\ 1 & 1 \end{bmatrix}$

#

$[y]_{std} = C[y]_c = AB[x]_B$

$= \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}_{std}$.