More References for Complex Numbers

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DFT and Linear Algebra: some nice links

Concepts:

1) An application of compression using DFT

[Can you guess the song? Fourier Music Decomposition – YouTube]

2) Reducible = https://youtu.be/yYEMxqreA10

Good Overview:

1) Simon Xu: Discrete Fourier Transform - Simple Step by Step (youtube.com) (from:3:30)

3) Matlab: https://www.youtube.com/watch?v=QmgJmh2I3Fw

Some Maths:

1) Brunton introduces DFT: The Discrete Fourier Transform (DFT) (youtube.com)
2) Brunton: The DFT Matrix: https://www.youtube.com/watch?v=Xw4voABxU5c
3) SigFy: The Linear Algebra of Fourier Transforms (youtube.com)
4) Van Veen (signal representation using bases): https://www.youtube.com/watch?v=a4Atmssz8-A
5) Jack Gunther: https://www.youtube.com/watch?v=Er-FcErLXrQ

Work examples:

MIT Mysore = https://youtu.be/BQRmJYoFR3M
Exploring technologies (DFT) = https://youtu.be/50_VnwA3LEk
Easy Electronics = https://youtu.be/dE9g72LlPdM
RLC-EEE = https://youtu.be/7VHE3v57XHU
Thansi = https://youtu.be/M0Ez9Pa-6xQ

Ref: Properties of Exponents (for real)

Exponent Rules For $a \neq 0, b \neq 0$	
Product Rule	$a^x \times a^y = a^{x+y}$
Quotient Rule	$a^x \div a^y = a^{x-y}$
Power Rule	$\left(a^x\right)^y = a^{xy}$
Power of a Product Rule	$(ab)^x = a^x b^x$
Power of a Fraction Rule	$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
Zero Exponent	$a^0 = 1$
Negative Exponent	$a^{-x} = \frac{1}{a^x}$
Fractional Exponent	$a^{\frac{x}{y}} = \sqrt[y]{a^x}$

Rules for Roots

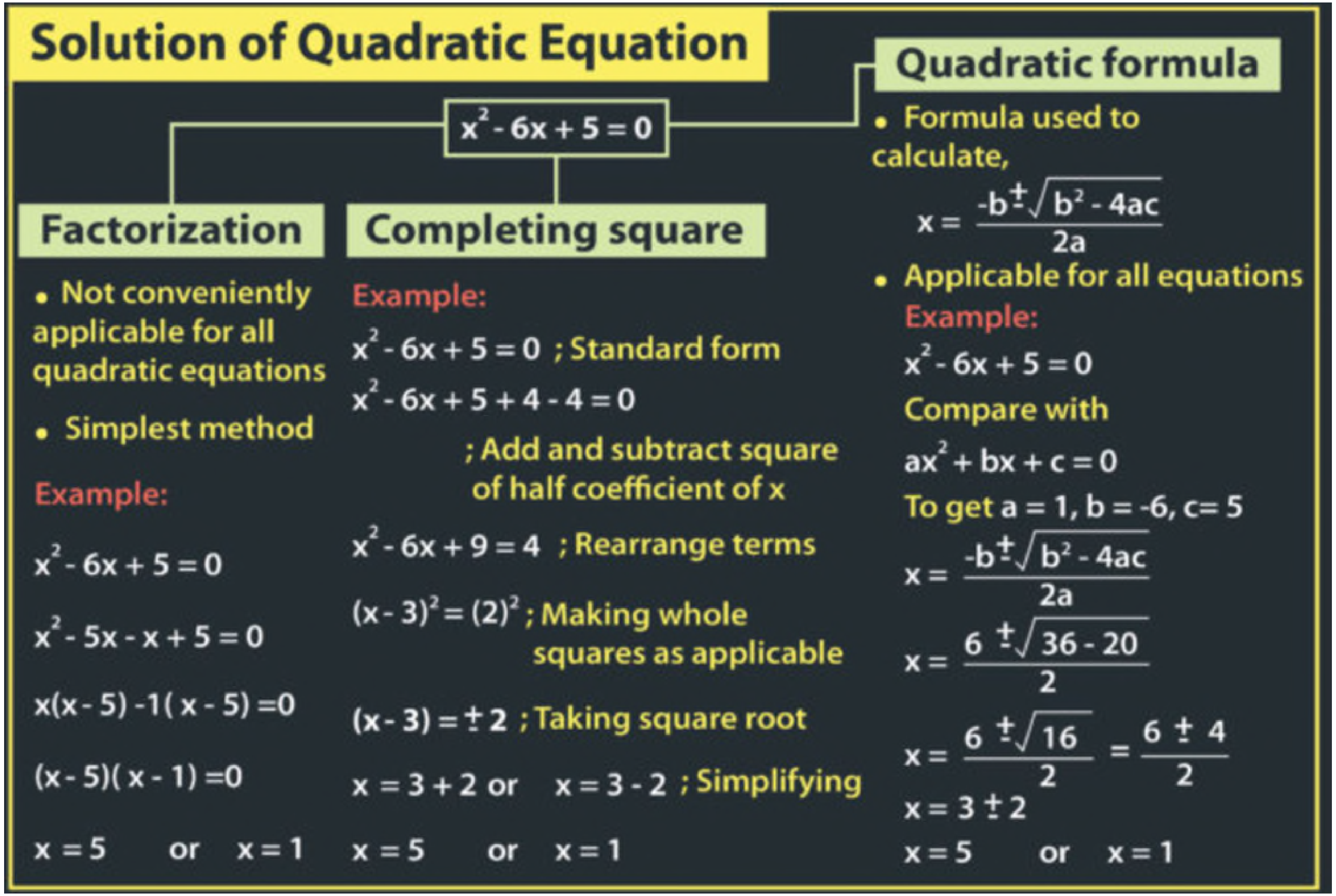
$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

$$\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\left(\sqrt[n]{a}\right)^n = a^{\frac{n}{n}} = a$$

Ref: useful maths background: roots of quadratic functions



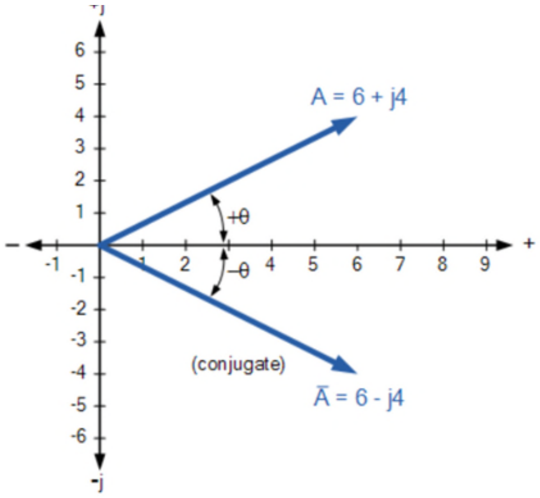
https://www.toppr.com/guides/maths/quadratic-equations/nature-of-roots/

Ref: useful maths background - exponent

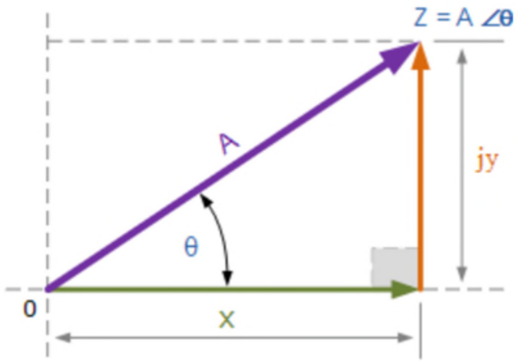
Tip: Easy-to-remember property name	Property expression
Same-base product	$x^a \cdot x^b = x^{a+b}$
Same-base division	$x^a \div x^b = x^{a-b}$
Same-exponent product	$x^a \cdot y^a = (xy)^a$
Same-exponent division	$x^a \div y^a = (x \div y)^a$
Double exponent	$(x^a)^b = x^{a \times b}$
Zero exponent	$x^0 = 1$
Negative exponent	$x^{-a} = \frac{1}{x^a}$
Fractional exponent	$x^{a/b} = \sqrt[b]{x^a}$

3 Different ways to represent complex numbers

Rectangular form



Polar Form Representation of a Complex Number



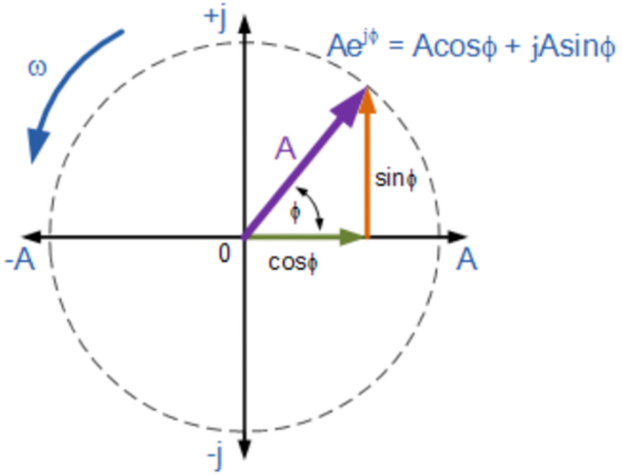
$$A = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$x = A.\cos\theta$$

$$y = A.\sin\theta$$

Complex exponential form



Euler's Formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

$$Z = Ae^{j\phi}$$

$$Z = A\left(\cos\phi + j\sin\phi\right)$$

https://www.electronics-tutorials.ws/accircuits/complex-numbers.html
https://muthu.co/deriving-the-famous-eulers-formula-through-taylor-series/

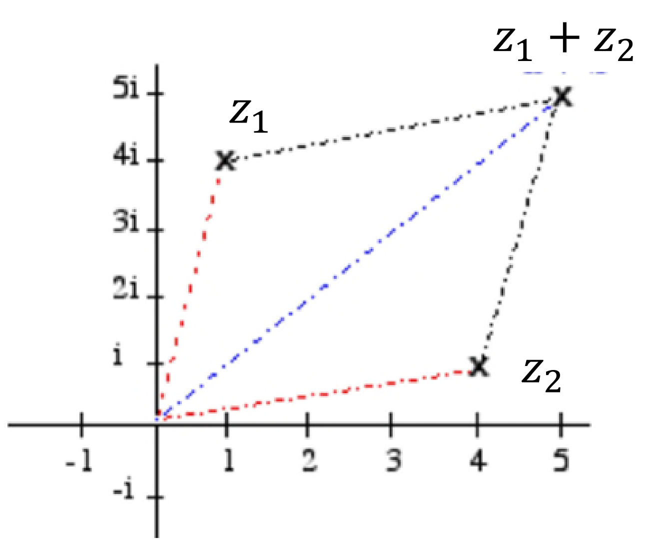
Visualisations: Addition and Subtraction

(a + ib) + (c + id) = (a + c) + i(b + d)

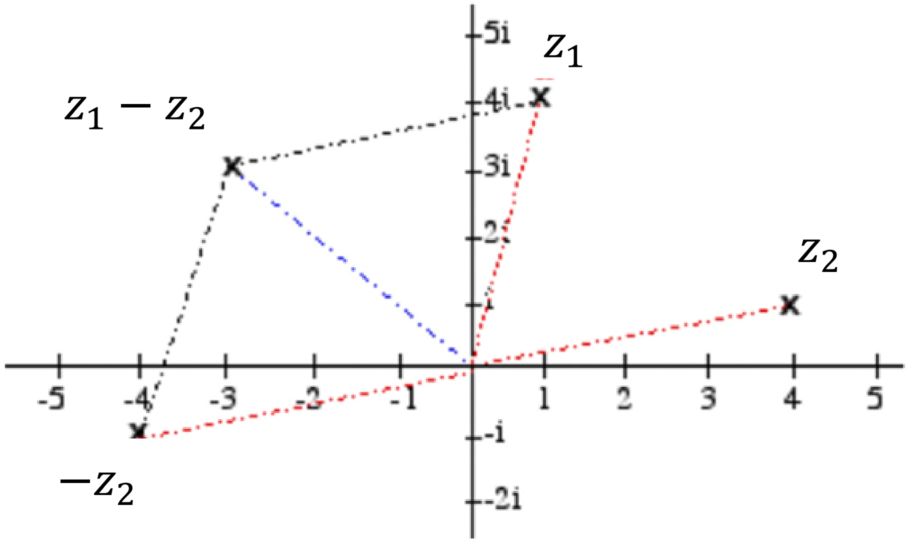
(a + ib) - (c + id) = (a - c) + i(b - d)

$z_1 = 1 + 4i, \qquad z_2 = 4 + i$

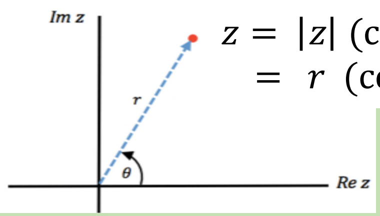
$z_1 + z_2 = (1 + 4i) + (4 + i) = 5 + 5i$



$z_1 - z_2 = (1 + 4i) - (4 + i) = -3 + 3i$



Visualisations: Multiplication of 2 complex number



$$z = |z|\ (\cos(\theta) + i\sin(\theta))$$
$$= \ r\ (\cos(\theta) + i\sin(\theta))$$

modulus $r = |z|$ (depicted as the length of the arrow)

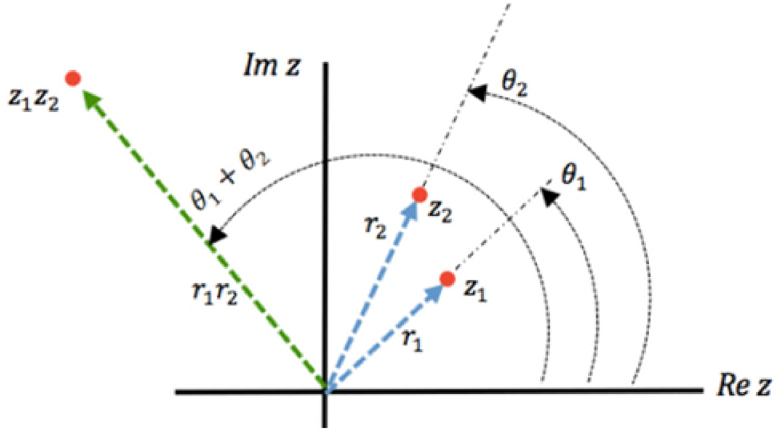
argument θ the angle the arrow makes with the positive real axis

Multiplication in Cartesian Form

Let $z_1 = a + bi$ and $z_2 = c + di$ be complex numbers, then:

$$\begin{aligned} z_1 \cdot z_2 &= (a+bi)\,(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)\,i. \end{aligned}$$

Note: its much easier to perform multiplication of complex numbers in complex exponential form.



$$\begin{aligned} z_1 z_2 &= r_1\,(\cos(\theta_1) + i\sin(\theta_1))\; r_2\,(\cos(\theta_2) + i\sin(\theta_2)) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\ &= r_1 r_2 e^{j(\theta_1 + \theta_2)}) \end{aligned}$$

https://mathspace.co/textbooks/syllabuses/Syllabus-501/topics/Topic-9613/subtopics/Subtopic-127684/?activeTab=theory
https://www.houseofmath.com/encyclopedia/numbers-and-quantities/numbers/complex-numbers/calculations/how-to-multiply-complex-numbers

Examples: multiplication and division of complex numbers in polar form

if $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$ then

Note: r_j is the the modulus (absolute value)of $|z_j|$

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2), \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

Example

If $z_1 = 5\angle(\pi/6)$, and $z_2 = 4\angle(-\pi/4)$ find a) $z_1 z_2$, b) $\frac{z_1}{z_2}$, c) $\frac{z_2}{z_1}$

Solution

a) To multiply the two complex numbers we multiply their moduli and add their arguments. Therefore

$$z_1 z_2 = 20\angle\left(\frac{\pi}{6} + \left(-\frac{\pi}{4}\right)\right) = 20\angle\left(-\frac{\pi}{12}\right)$$

b) To divide the two complex numbers we divide their moduli and subtract their arguments.

$$\frac{z_1}{z_2} = \frac{5}{4}\angle\left(\frac{\pi}{6} - \left(-\frac{\pi}{4}\right)\right) = \frac{5}{4}\angle\frac{5\pi}{12}$$

c)

$$\frac{z_2}{z_1} = \frac{4}{5}\angle\left(-\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{4}{5}\angle\left(-\frac{5\pi}{12}\right)$$

https://harshasnmp.wordpress.com/2017/11/15/multiplication-division-of-complex-numbers-in-polar-form/

DeMoivre Theorem: finding powers of z

$z \quad = |z|e^{j\theta} = |z|(\cos(n\theta) + i\,\sin(n\theta))$

$z^n \; = \left(|z|e^{j\theta}\right) \dots \left(|z|e^{j\theta}\right)$ *; (n of them)*
$\quad = |z|^n e^{jn\theta}$
$\quad = |z|^n(\cos(n\theta) + i\,\sin(n\theta))$

// Fractional power

$z^{1/n} \; = \sqrt[n]{|z|e^{j\theta}}$

$\quad = \sqrt[n]{|z|}\; e^{j\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)} \qquad , k = 0 \cdots (n-1)$

Uses of De Moivre's Theorem

De Moivre's Theorem is used for various purposes. Some of its most important uses are,

- Finding the Roots of Complex Numbers.
- Finding the relationships between Powers of Trigonometric Functions and Trigonometric Angles.
- Solving the Power of Complex Numbers.

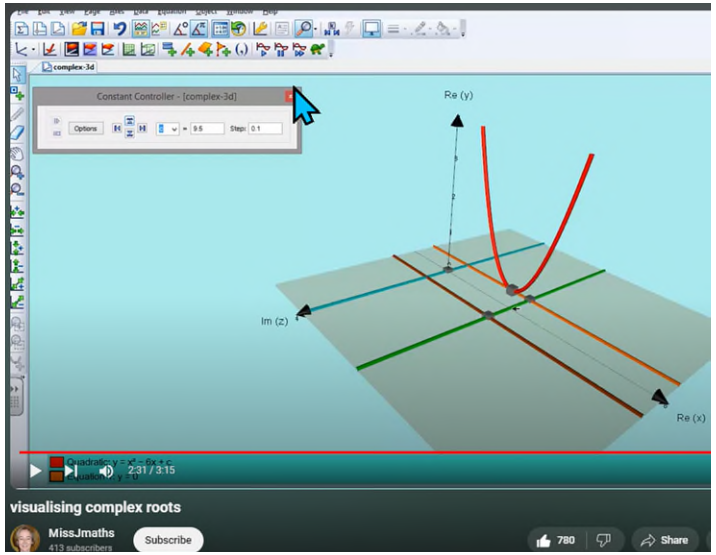
1) Mark Willis: Proof: https://www.youtube.com/watch?v=6UecqjGHR1w
2) https://ccrma.stanford.edu/~jos/st/Direct_Proof_De_Moivre_s.html

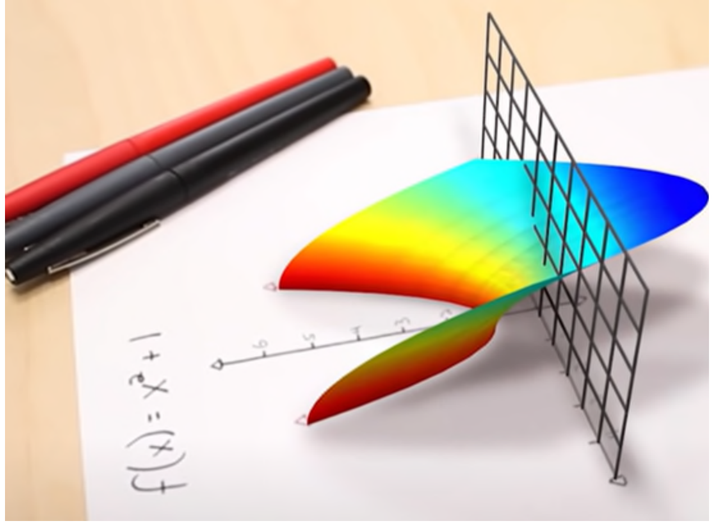
Visualizing complex roots

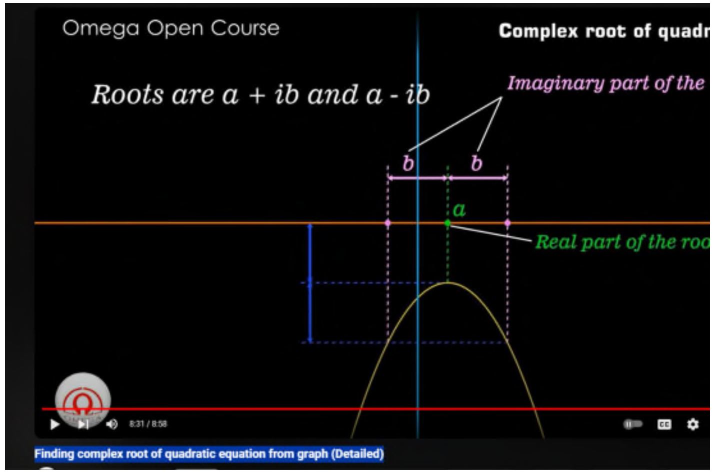
1) https://www.youtube.com/watch?v=5g80j17w0CE

2) Welch Lab: **Imaginary Numbers Are Real [Part 1: Introduction]**
https://youtu.be/T647CGsuOVU?si=dMXaQ1rL7sOoPrcO

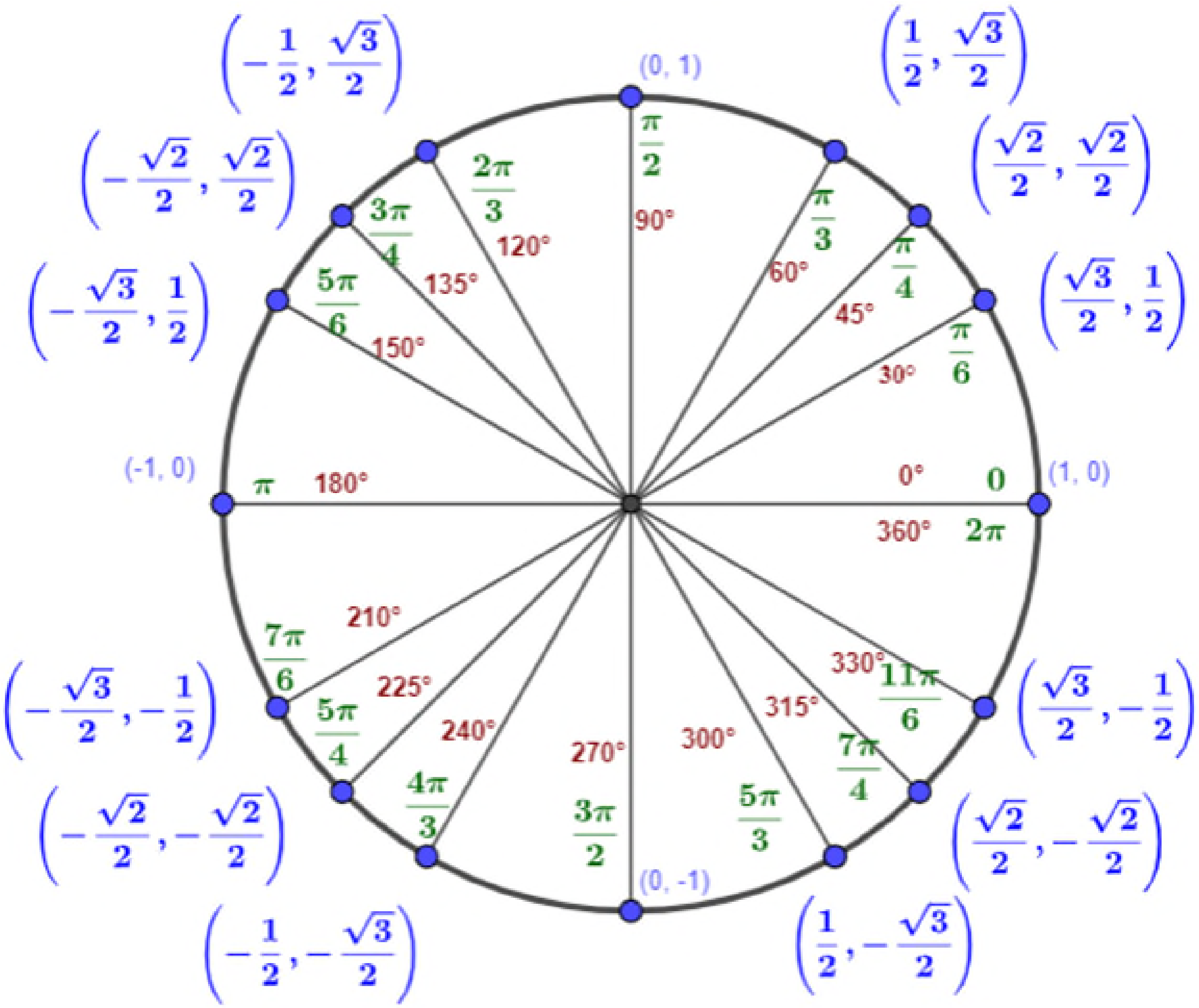
3) Omega Open Course:
https://www.youtube.com/watch?v=i0YkkVrew54&t=322s







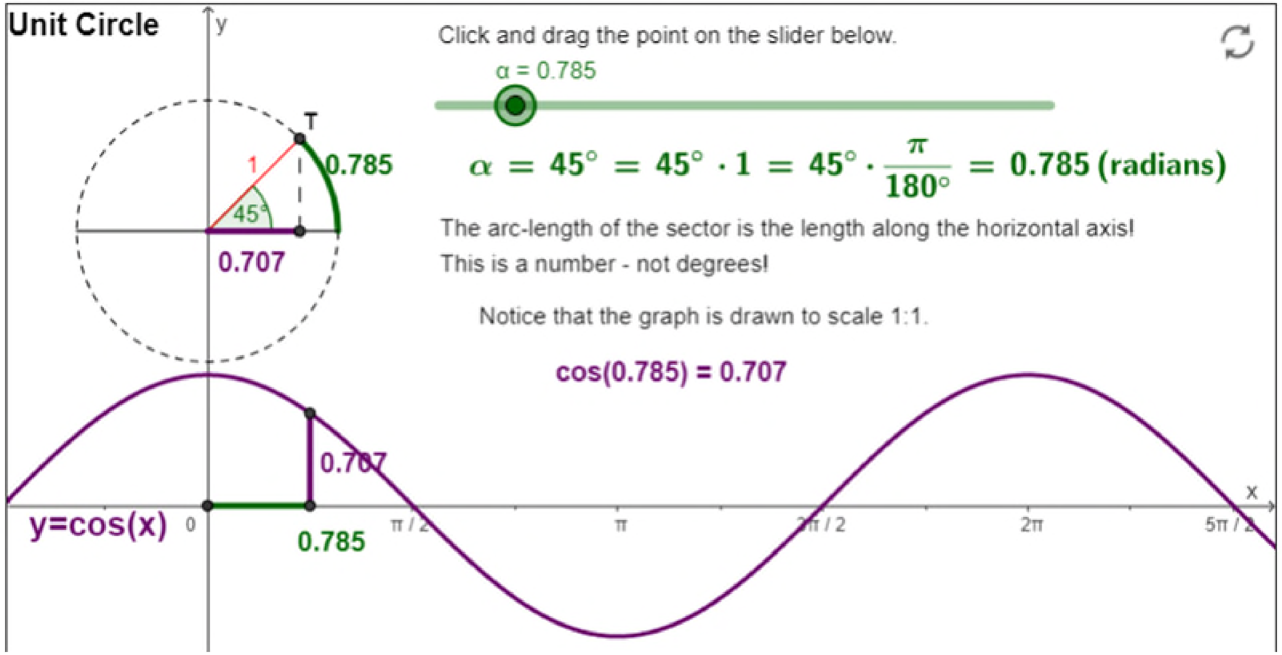
Ref: radian vs degree, and values on the unit circle (cartesian plane)

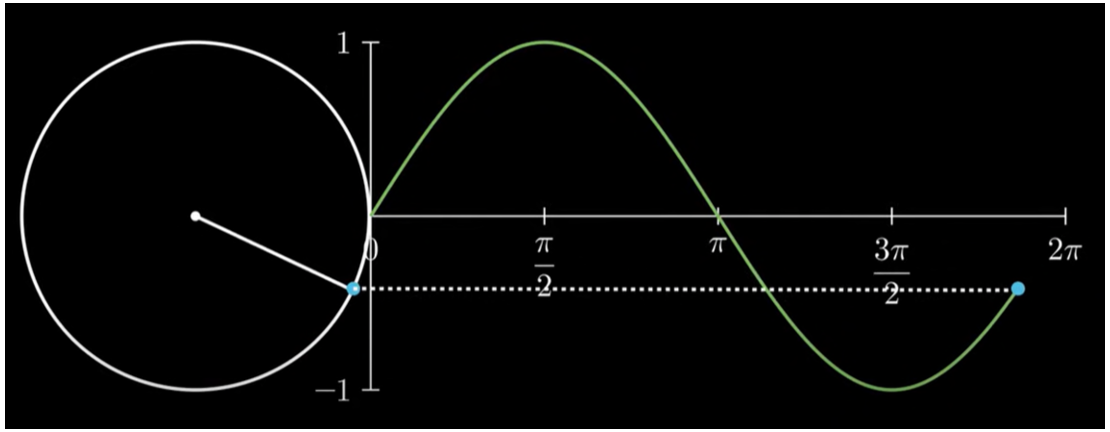


Manocha Academy: https://www.youtube.com/watch?v=LFY5JuQ49Kg

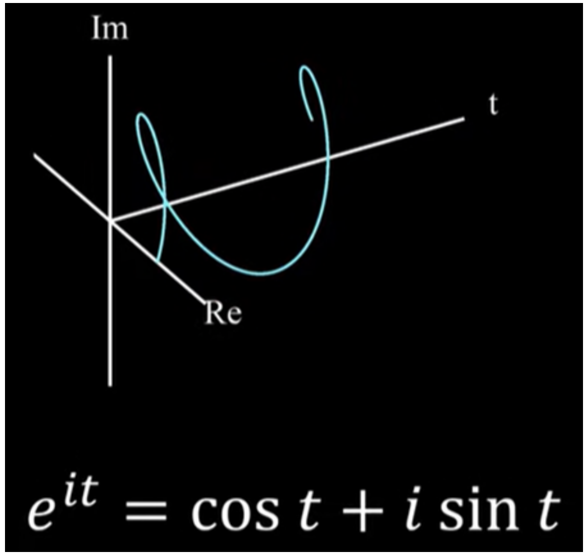
Organic chemistry: https://www.youtube.com/watch?app=desktop&v=V5A

Viewing the rotating phasor e^{it} and sine/cosine





https://www.youtube.com/watch?v=9L57ygCwM4w
https://www.youtube.com/shorts/QfxBGWb96r8

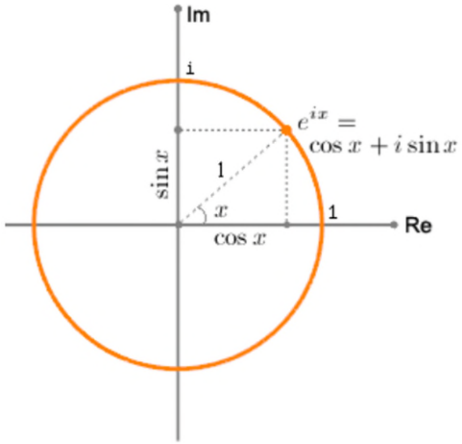


https://www.geogebra.org/m/wARRxDYf

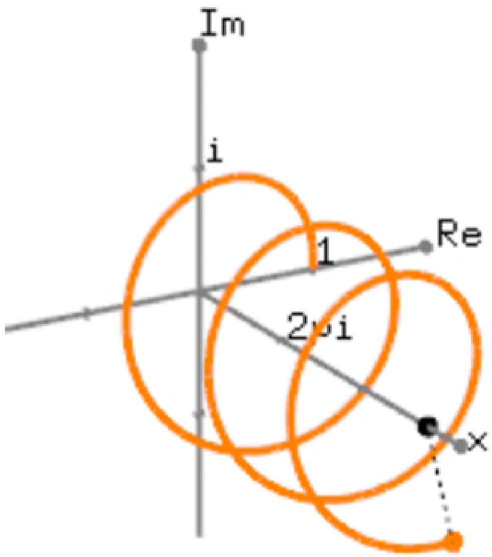
Euler Equation

$$e^{ix} = \cos(x) + i\ sin(x)$$

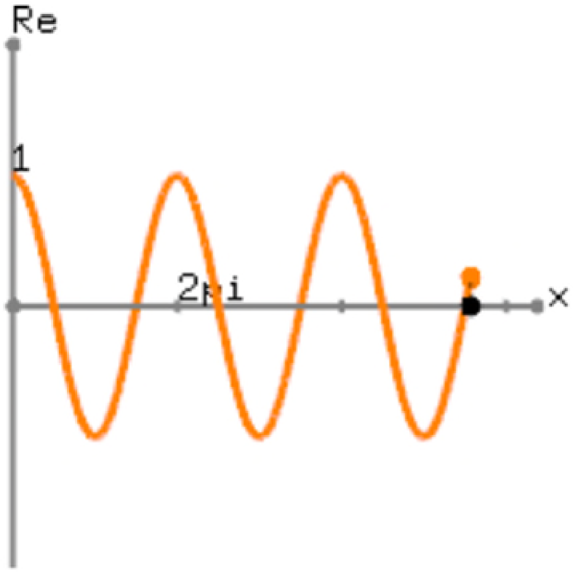
$$e^{-ix} = \cos(x) - i\ sin(x)$$



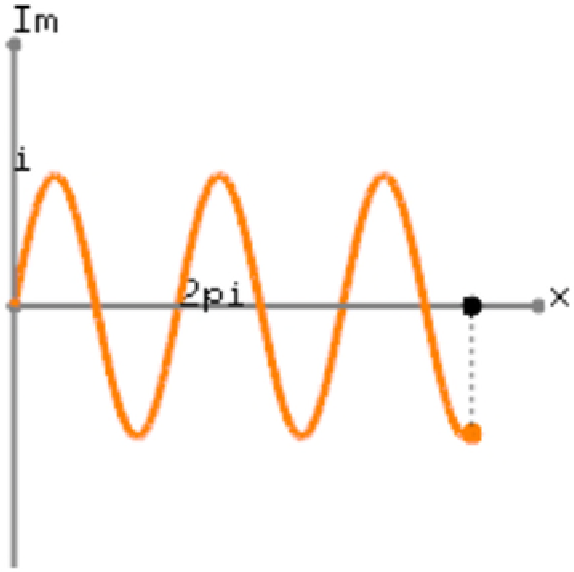
Graph of e^{ix} on the complex plane



Graph of e^{ix}



Real number part of e^{ix}



Imaginary number part of e^{ix}

Graph of e^{ix}*, as x increases, the phasor is rotating anti-clockwise.*

https://www.songho.ca/math/euler/euler.html

Time vs Frequency Representations: what is $e^{j(\frac{2\pi}{N}k)n}$,

k^{th} harmonic: $b_{k,n} = e^{j\frac{2\pi}{N}kn}$

The complex exponential at digital angular frequency $\Omega = \frac{2\pi}{N}k\ (\frac{rad}{sample})$

$$e^{j(\frac{2\pi}{N}k)n}, \qquad n = 0..(N-1)$$

To appreciate, represent $c_k = |c_k|e^{j\theta}$ (complex exponential form)

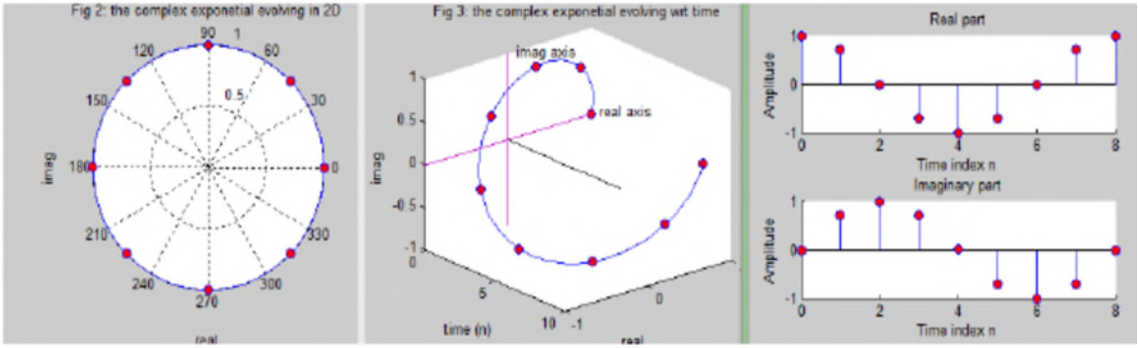
See that to reconstruct $x[n]$, the complex exponential at $\Omega = \frac{2\pi}{N}k$ is modified by c_k in its magnitude and phase

$$c_k e^{j(\frac{2\pi}{N}k)n} = |c_k|e^{j\theta}e^{j(\frac{2\pi}{N}k)n}$$

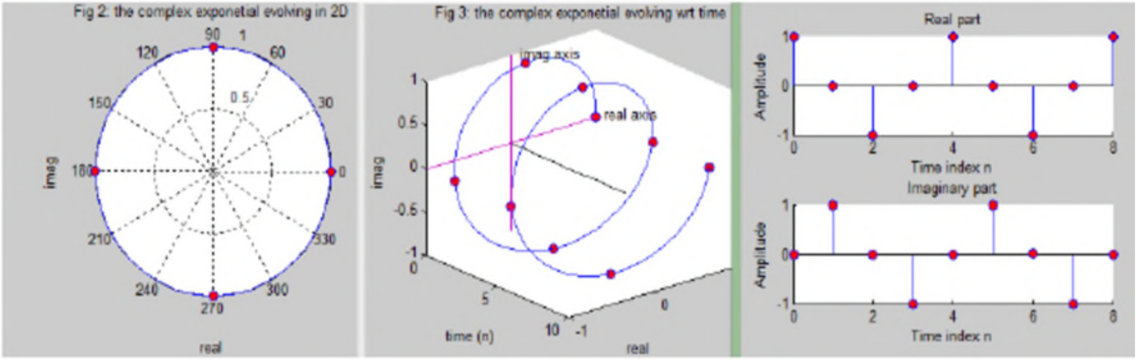
$$= |c_k|e^{j\left(\left(\frac{2\pi}{N}k\right)n+\theta\right)}$$

$N = 8,$
$n = 0..(N-1)$

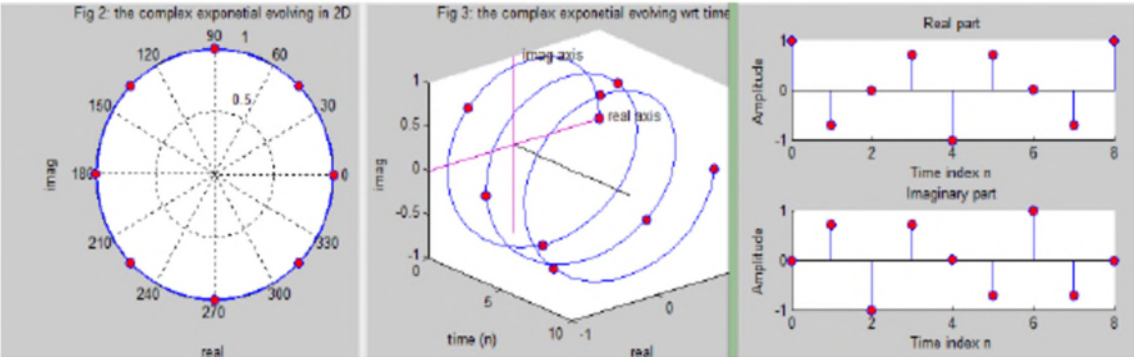
$$b_{1,n} = e^{j\frac{2\pi}{N}(1)n}$$



$$b_{2,n} = e^{j\frac{2\pi}{N}(2)n}$$



$$b_{3,n} = e^{j\frac{2\pi}{N}(3)n}$$



Prof Fowler: Fourier Basis

- Matrix and Vectors – good to understand basis for DFT, DTFS http://www.ws.binghamton.edu/fowler/fowler%20personal%20page/EE523_files/Ch_11_1%20Matrices%20and%20Vectors.pdf

DFT Coefficients as Inner Product Results:

Now lets see how these ideas relate to the DFT. We've already seen that we can interpret the N-pt. IDFT as an expansion of the signal vector in terms of the linear combination of N ON vectors $\mathbf{d}_k$ defined above. (Note that for the IDFT case this is just saying that we are building the signal out of a sum of complex sinusoids.) From the above theory, we now know how to compute the required coefficients for any ON expansion so we should see what this idea gives for the IDFT case. Our main general result above was that the expansion coefficients are found from the inner products of the

vector to be expanded and the various ON basis vectors: so for the IDFT case the coefficients are $\langle \mathbf{x}, \mathbf{d}_k \rangle$. But, we know from standard DFT theory that the coefficients of the IDFT expansion are just the DFT values $X[k]$. Combining these two points of view gives

$$\boxed{X[k] = \langle \mathbf{x}, \mathbf{d}_k \rangle}.$$

Let's take a look at this an verify that the right-hand side of this is consistent with what we know from standard DFT theory. From vector inner product theory we know that the right side of this is

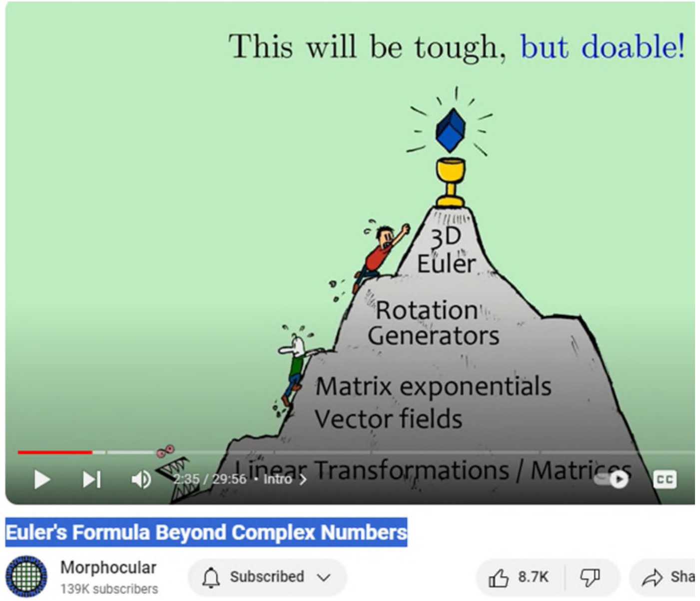
$$\begin{aligned} \langle \mathbf{x}, \mathbf{d}_k \rangle &= \sum_{n=0}^{N-1} x[n] d_k^*[n] \\ &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \end{aligned}$$

which is the DFT of the signal, so we see that things work as stated. Actually, we have been a little bit loose here – the $\mathbf{d}_k$ vectors as given above are orthogonal but not ortho<u>normal</u> so there is a 1/N term in the IDFT equation used back in the "DFT from Basis Viewpoint" section. To really do this using ON basis vectors we would have to put a $1/\sqrt{N}$ term in front of the $\mathbf{d}_k$ vectors, but that would lead to forms that aren't exactly the same as the conventional DFT/IDFT expressions; try as an exercise redoing this development with <u>true</u> ON basis vectors.

Ref Advance: Interesting videos

1) Euler's Formula Beyond Complex Numbers

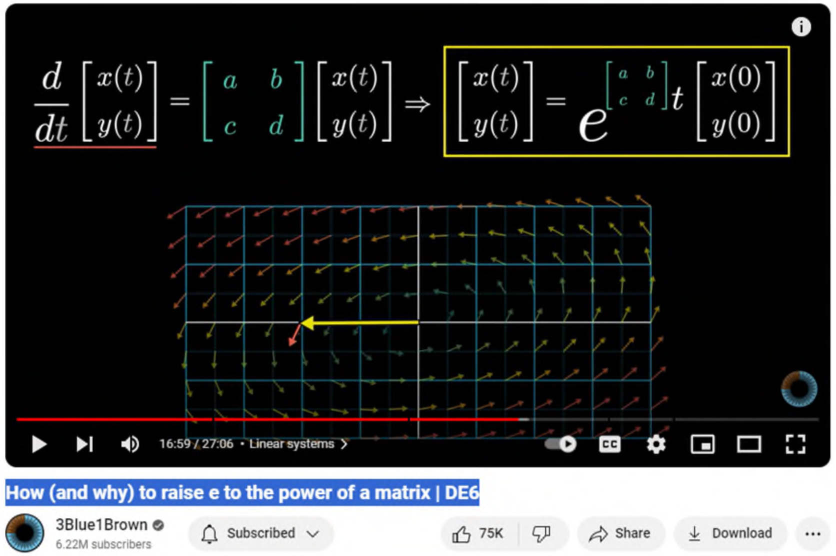
https://www.youtube.com/watch?v=Y1gOYtQYRXo



for matrix A

$$e^A := A^0 + \frac{A^1}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

2) How (and why) to raise e to the power of a matrix | DE6



Some good references: DFT

1) https://ubcmath.github.io/MATH307/dft/dft.html
2) https://math.stackexchange.com/questions/2413218/proof-of-orthonormality-of-basis-of-dft
3) https://en.wikipedia.org/wiki/DFT_matrix
4) https://www2.seas.gwu.edu/~simhaweb/quantum/modules/appendix/dft.html#:~:text=The%20DFT%20is%20a%20linear,could%20be%20considered%20an%20algorithm.
5) https://www.youtube.com/watch?v=W5iXRA4ro1g
6) https://www.statlect.com/matrix-algebra/discrete-Fourier-transform#:~:text=they%20are%20orthogonal%3B,function%20taken%20at%20different%20frequencies.