Vectors

Overview and Learning Outcomes

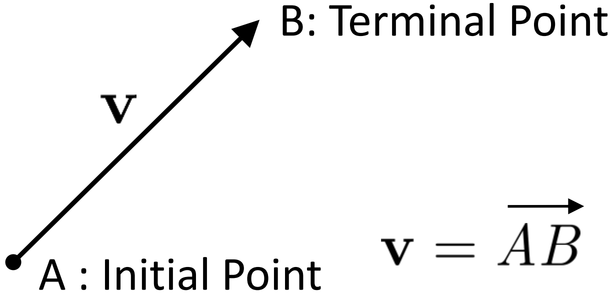
- Vectors in 2-D, 3-D and n-D
  - Perform algebraic and geometric operations on vectors : addition, subtraction, multiplication
  - Understand equivalent and collinear vectors
- Norm, Dot Product, and Distance in R^n
  - Compute norm of a vector in R^n
  - Determine distance between two vectors in R^n
  - Compute the dot product between two vectors in R^n
  - Compute angle between two nonzero vectors in R^n

Overview and Learning Outcomes

- Orthogonality
  - Determine whether two vectors are orthogonal
  - Find equations for lines/planes using a normal vector and a point on the line/plane
- Vector and Parametric equations
  - Express equations of lines in R^2 and R^3 using either vector or parametric equations
  - Express equations of planes in R^n using either vector or parametric equations

I. Vectors in 2-D, 3-D, and n-D

Scalar	:	Only magnitude, e.g., temperature	a, t, x, y	Notation
Vector	:	Both magnitude and direction, e.g. force	$\mathbf{f}, \mathbf{x}, \mathbf{w}, \mathbf{v}$	



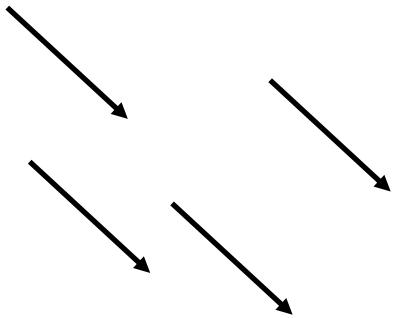
$$\mathbf{v} = \overrightarrow{AB}$$

Length of the arrow $\longrightarrow$ Magnitude of vector

Direction of arrowhead $\longrightarrow$ Direction of vector

<ins>Equivalent or Equal vectors:</ins> Same length and direction

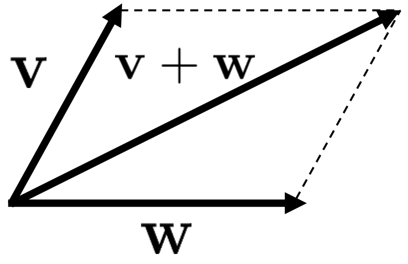
$$\mathbf{v} = \mathbf{w}$$



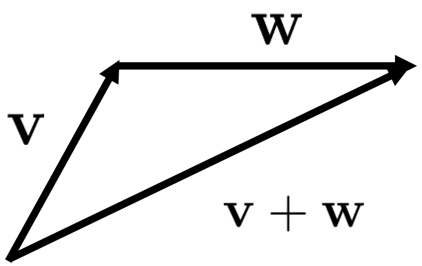
<ins>Zero vector:</ins> Initial and terminal points coincide; length is zero; denoted by **0**.

Vector Addition

Parallelogram Rule for Vector Addition: If $\mathbf{v}$ and $\mathbf{w}$ are vectors in 2-D or 3-D that are positioned so their initial points coincide, then the two vectors form adjacent sides of a parallelogram, and the sum $\mathbf{v}+\mathbf{w}$ is the vector represented by the arrow from the common initial point of $\mathbf{v}$ and $\mathbf{w}$ to the opposite vertex of the parallelogram. [Fig. 1]



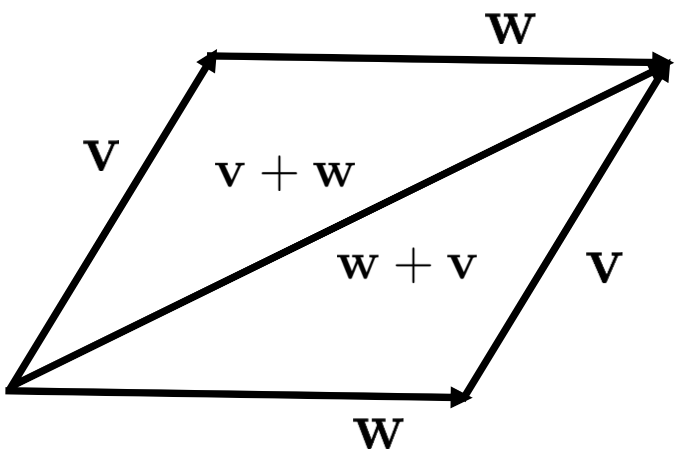
[Fig. 1]



[Fig. 2]

Triangle Rule for Vector Addition: If $\mathbf{v}$ and $\mathbf{w}$ are vectors in 2-D or 3-D that are positioned so that the initial point of $\mathbf{w}$ is at the terminal point of $\mathbf{v}$ then the sum $\mathbf{v}+\mathbf{w}$ is the vector represented by the arrow from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$. [Fig. 2]

Vector Addition(contd.)



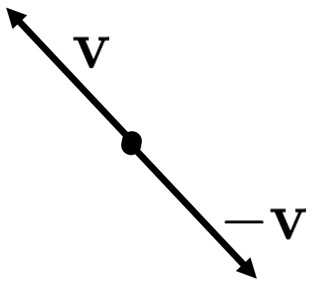
- Construct $\mathbf{v} + \mathbf{w}$ and $\mathbf{w} + \mathbf{v}$ by triangle rule to see that

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

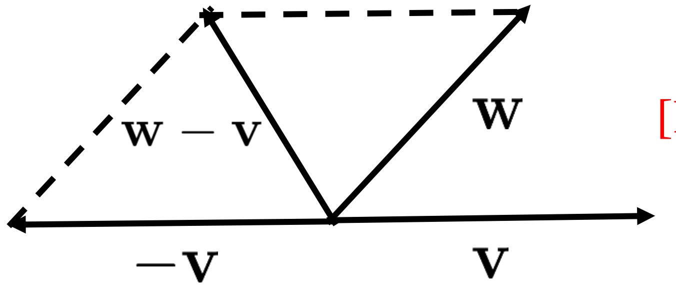
- Sum obtained by triangle rule is the same as sum obtained by the parallelogram rule

Vector Subtraction

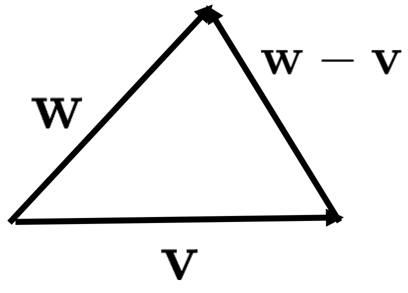
The negative of a vector $\mathbf{v}$ denoted by $-\mathbf{v}$: Same length as $\mathbf{v}$ but of opposite direction



Difference of $\mathbf{v}$ from $\mathbf{w}$ denoted by $\mathbf{w}-\mathbf{v}$: $\mathbf{w}-\mathbf{v}=\mathbf{w}+(-\mathbf{v})$ [Fig. 3]



[Fig. 3]



[Fig. 4]

Position $\mathbf{v}$ and $\mathbf{w}$ so their initial points coincide and draw vector from terminal point of $\mathbf{v}$ to terminal point of $\mathbf{w}$. [Fig. 4]

Scalar Multiplication

$\mathbf{v}$ is a nonzero vector in 2-D or 3-D

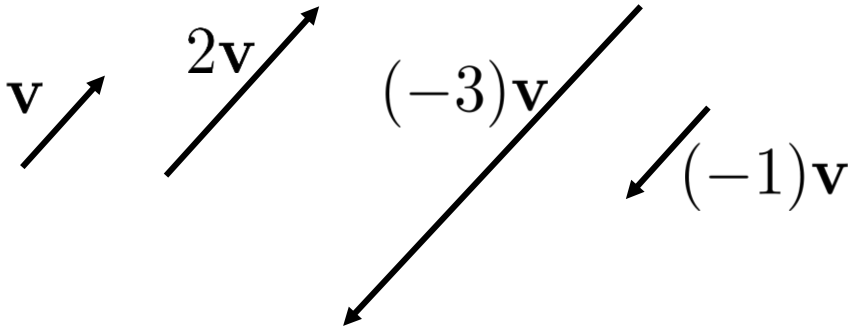
k is a nonzero scalar

Scalar product of $\mathbf{v}$ by k : vector whose length is $|k|$ times the length of $\mathbf{v}$

direction is same as $\mathbf{v}$ if k is $+$

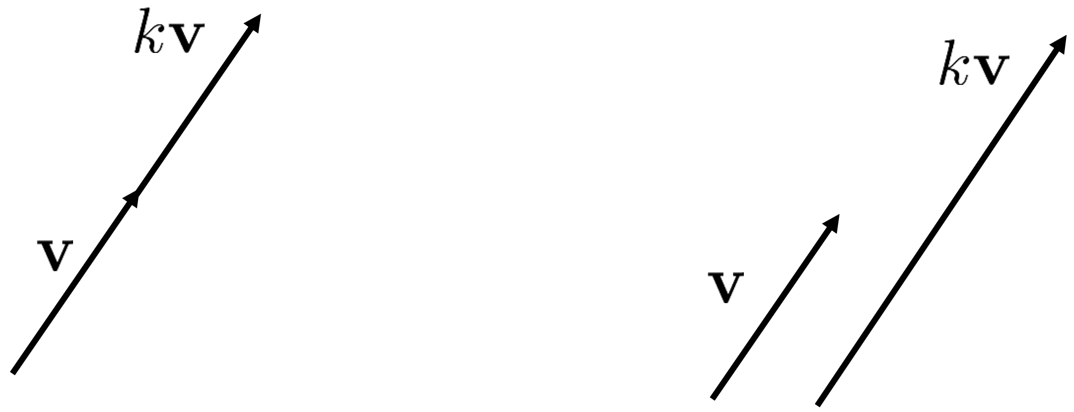
direction is opposite to $\mathbf{v}$ if k is $-$

If $k = 0$ or $\mathbf{v} = \mathbf{0}$, then define $k\mathbf{v}$ to be $\mathbf{0}$.



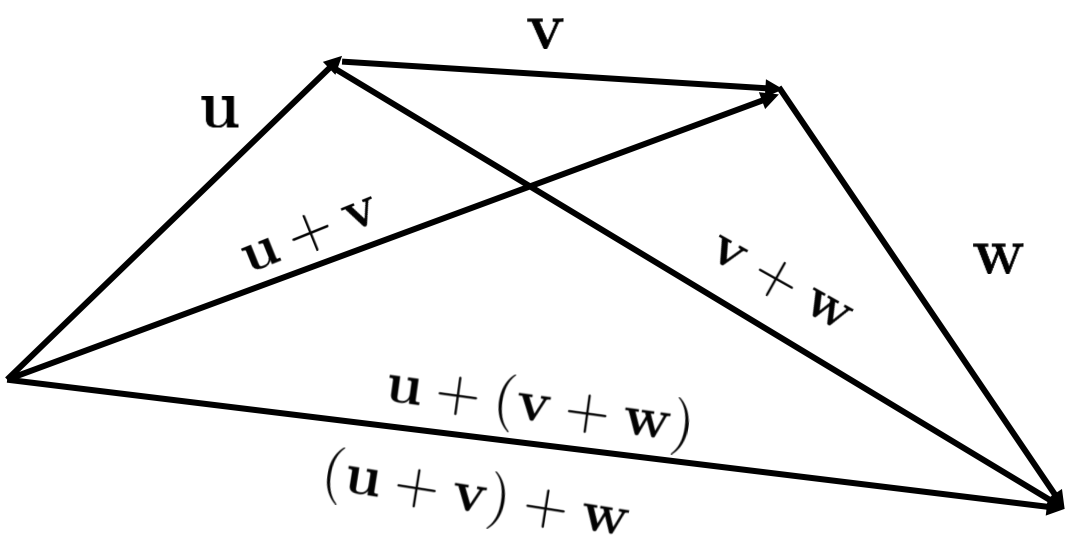
Parallel and Collinear Vectors

They mean the same because translating a vector does not change it!

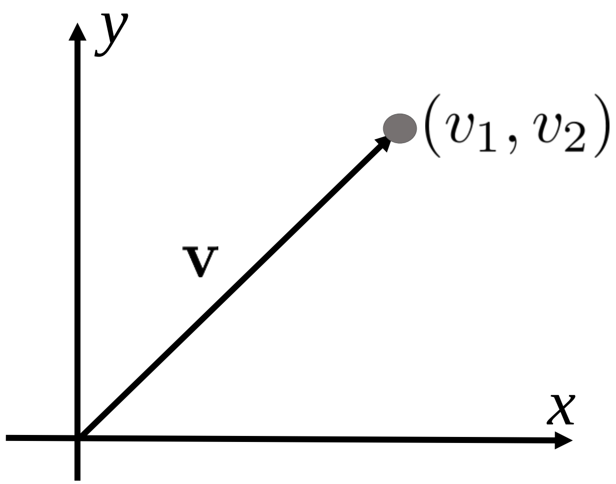


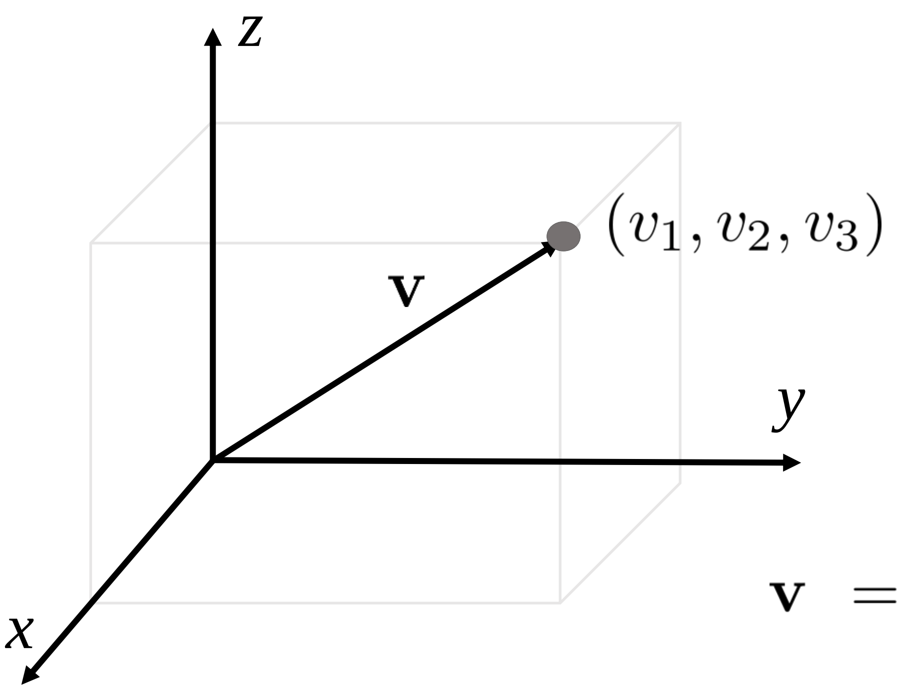
Associative Law for Addition

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$



Vectors in Coordinate Systems





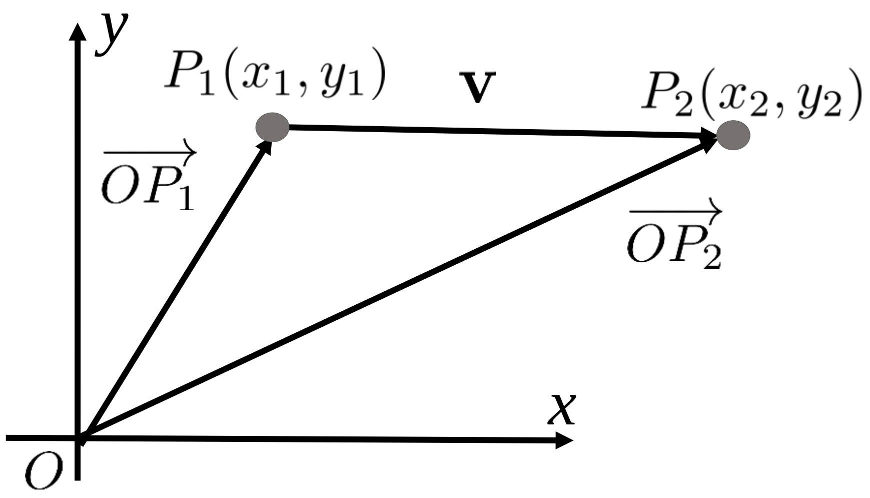
$$\mathbf{v} = \underbrace{(v_1, v_2)}_{\text{components of the vector}} \text{ or } \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\mathbf{v} = \underbrace{(v_1, v_2, v_3)}_{\text{components of the vector}}$$

- Two vectors are equivalent/equal if their components are equal

$\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are equal if and only if $v_1 = w_1$, $v_2 = w_2$ and $v_3 = w_3$.

Vectors with Initial Point NOT at Origin



$$\mathbf{v} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$$

- Easily extends to 3-D

n-D space

Real line : R^1

Set of all ordered pairs of real numbers (2-tuples) : R^2 $\mathbf{v} = (v_1, v_2)$

Set of all ordered triples of real numbers (3-tuples) : R^3 $\mathbf{v} = (v_1, v_2, v_3)$

Set of all ordered n-tuples called n-D space: R^n $\mathbf{v} = (v_1, v_2, \ldots, v_n)$

Zero vector in R^n : $\mathbf{0} = (0, 0, \ldots, 0)$

Definition of equivalent/equal vectors carries over from 2-D/3-D.

n-D space (contd.)

Definition. If $\mathbf{v} = (v_1, v_2, \dots, v_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ in R^n, and if k is any scalar, then we define

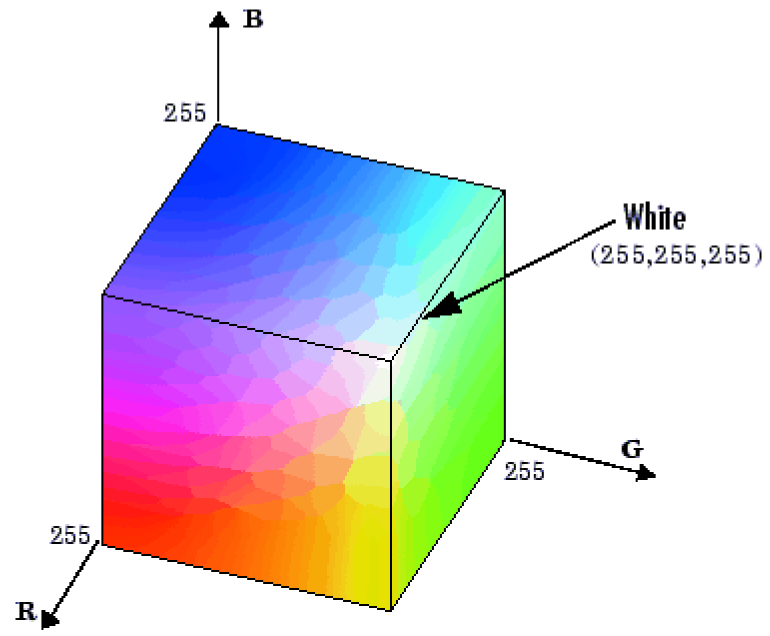
$$\begin{aligned}
\mathbf{v} + \mathbf{w} &= (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n) \\
k\mathbf{v} &= (kv_1, kv_2, \dots, kv - n) \\
-\mathbf{v} &= (-v_1, -v_2, \dots, -v_n) \\
\mathbf{w} - \mathbf{v} &= \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \dots, w_n - v_n)
\end{aligned}$$

n-D space (contd.)

Theorem. *If* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *and* m *are scalars, then*

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
3. $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$
4. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
5. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
6. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
7. $k(m\mathbf{u})=(km)\mathbf{u}$
8. $1\mathbf{u}=\mathbf{u}$

n-D space (contd.)



From: http://www.mathworks.com/help/images/color7.gif

Theorem. *If* $\mathbf{v}$ *is a vector in* R^n *and* k *is a scalar, then*

1. $0\mathbf{v} = \mathbf{0}$
2. $k\mathbf{0} = \mathbf{0}$
3. $-1\mathbf{v} = -\mathbf{v}$

Definition. If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in R^n if it can be expressed in the form

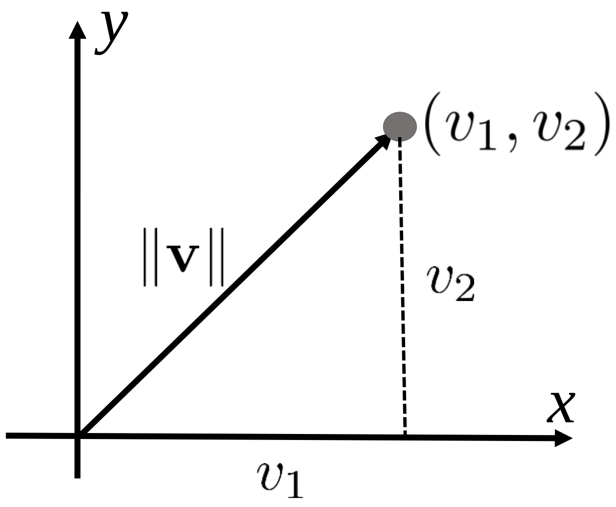
$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

where $k_1, k_2, \ldots, k_n$ are scalars called **coefficients** of the linear combination.

***** End of Vectors in 2-D, 3-D and n-D *****

II. Norm, Dot Product, and Distance in R^n

Norm of a vector: Length of a vector $\mathbf{v}$ denoted by $\|\mathbf{v}\|$.



$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$$

Definition. If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the **norm/length/magnitude** of $\mathbf{v}$ is defined by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{1}$$

Norm of a vector (contd.)

Theorem. *If* $\mathbf{v}$ *is a vector in* R^n *and* k *is a scalar, then*

1. $\|\mathbf{v}\| \geq 0$
2. $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$
3. $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$

Unit vector

- A vector of norm 1.
- Useful for specifying direction when length is not relevant
- If $\mathbf{v}$ is any nonzero vector in R^n, then a unit vector that is in the same direction as $\mathbf{v}$ is

$$\boxed{\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}}$$

called Normalizing $\mathbf{v}$

The Standard Unit Vectors

Unit vectors in the positive directions of the coordinate axes in R^2 and R^3

$\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$

$\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$

Standard unit vectors in R^n :

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

Distance in R^n

Definition. If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then the **distance** between $\mathbf{u}$ and $\mathbf{v}$ denoted by $d(\mathbf{u}, \mathbf{v})$ is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Dot Product

- Angle between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \leq \theta \leq \pi$

Definition. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 and R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the **dot product** (also called **Euclidean inner product**) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \tag{1}$$

If $\mathbf{u} = 0$ or $\mathbf{v} = 0$ then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

- Angle θ can be obtained as $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$
- θ is acute if $\mathbf{u} \cdot \mathbf{v} > 0$
- θ is obtuse if $\mathbf{u} \cdot \mathbf{v} < 0$
- $\theta = \pi/2$ if $\mathbf{u} \cdot \mathbf{v} = 0$

Component Form of Dot Product

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two nonzero vectors, then the component form of their dot product is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{1}$$

Definition. If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the **dot product** (also called the **Euclidean inner product**) of $\mathbf{u}$ and $\mathbf{v}$ denoted by $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Algebraic Properties of the Dot Product

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2$$

Length of a vector in terms of a dot product: $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

Theorem. *If* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *are vectors in* R^n *and* k *is a scalar, then*

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ *[Symmetry]*
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ *[Distributive]*
3. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v}$ *[Homogeneity]*
4. $\mathbf{v} \cdot \mathbf{v} \geq 0$ *and* $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$ *[Positivity]*

Algebraic Properties of the Dot Product (contd.)

Additional properties of dot product:

Theorem. *If* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *are vectors in* R^n *and* k *is a scalar, then*

1. $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$
4. $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$
5. $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

Angles in R^n

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

θ not defined unless its argument satisfies the inequalities

$$-1 \leq \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1$$

Fortunately, the inequalities are indeed satisfied due to Cauchy-Schwarz Inequality

Theorem. *Cauchy-Schwarz Inequality*
If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *are vectors in* R^n,

$$|\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2}$$

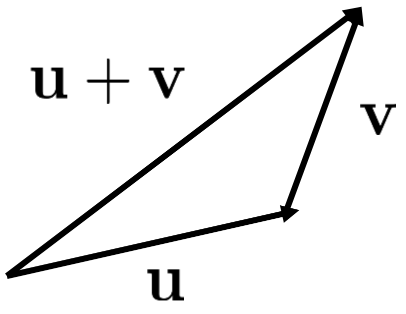
Geometry in R^n

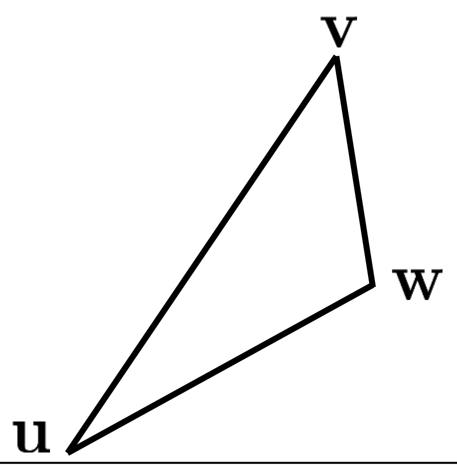
Concepts from geometry extend to R^n

Theorem. *If* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *are vectors in* R^n *and* k *is a scalar, then*

1. $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ *[Triangle inequality for vectors]*

2. $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$ *[Triangle inequality for distances]*





***** End of Norm, Dot Product and Distance in R^n *****

III. Orthogonality

Orthogonal Vectors: Angle θ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

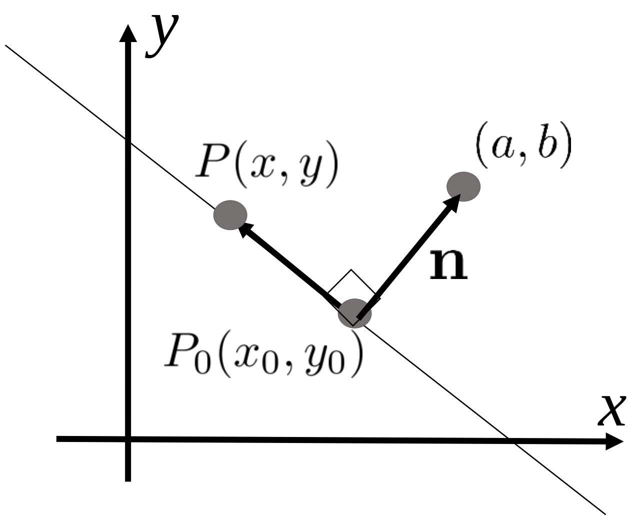
$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

Definition. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be **orthogonal** if $\mathbf{u}\cdot\mathbf{v} = 0$. A nonempty set of vectors in R^n is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal.

Lines and Planes Determined by Points and Normals

Line in R^2 determined uniquely by its slope and one of its points
Plane in R^2 determined uniquely by its "inclination" and one of its points

Use a nonzero vector $\mathbf{n}$, called a **normal**, that is orthogonal to the line or plane to specify slope or inclination.



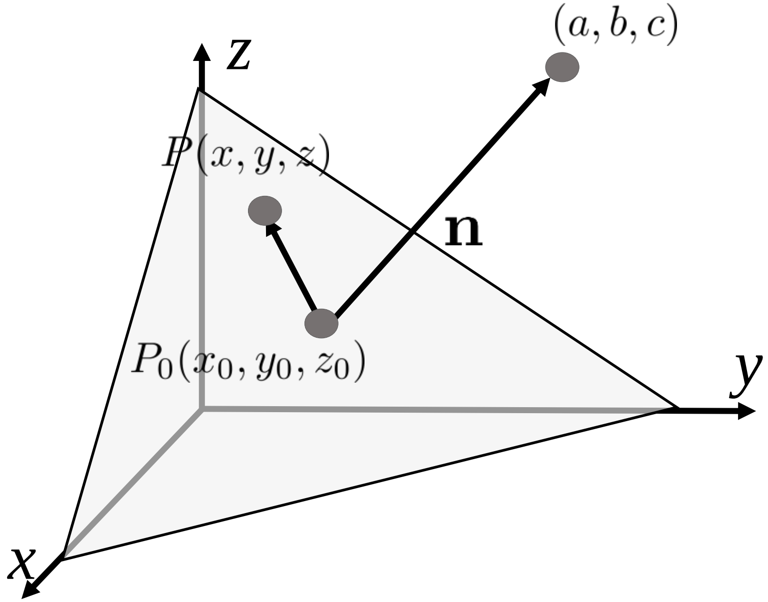
$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

$$\overrightarrow{P_0P} = (x - x_0, y - y_0)$$

$$\mathbf{n} = (a, b)$$

$$a(x - x_0) + b(y - y_0) = 0$$

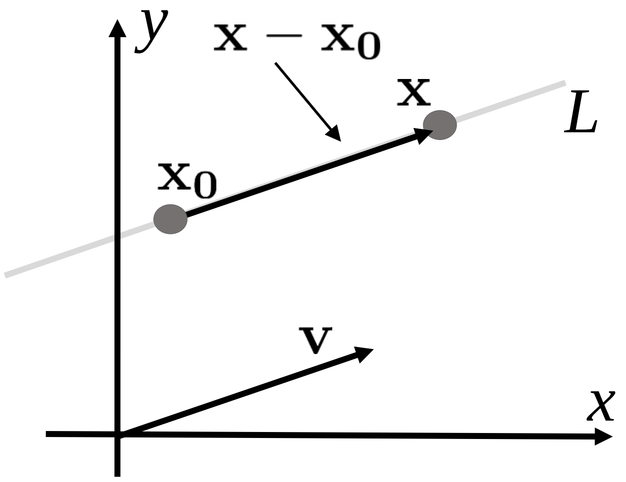
Point-normal equation of line



Point-normal equation of plane?

Vector and Parametric Equations of Lines in R^2 and R^3

- Previously, equations of lines and planes determined in point-normal form
- Here, look at other ways of specifying lines and planes
- Equation for a line that contains point $\mathbf{x_0}$ and is parallel to vector $\mathbf{v}$



For any point $\mathbf{x}$ on line L in R^2 or R^3, <u>vector form of line</u>

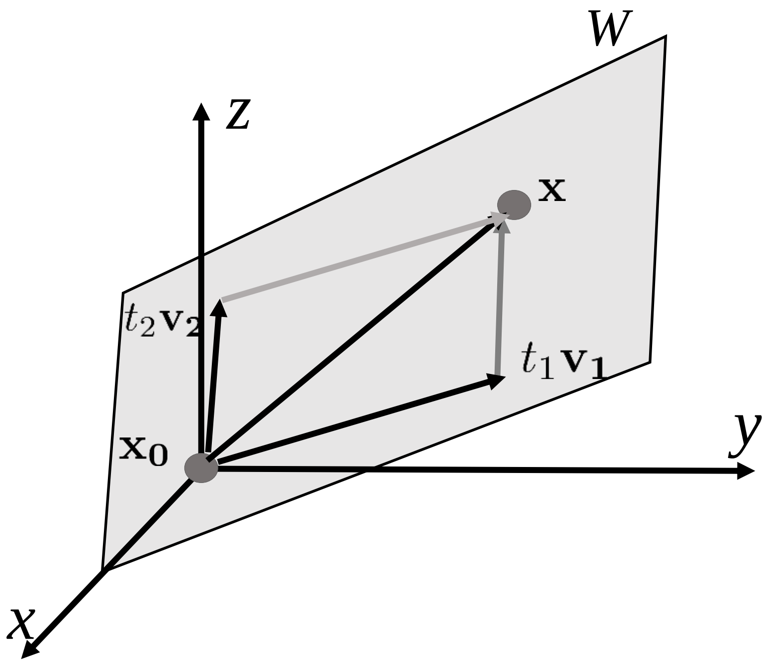
$$\mathbf{x} - \mathbf{x_0} = t\mathbf{v} \quad \text{or} \quad \mathbf{x} = \mathbf{x_0} + t\mathbf{v}$$

If $\mathbf{x}_0 = \mathbf{0}$, line passes through origin.

Variable t called a ***parameter***;
$\mathbf{x}$ traces out the line L as t varies from $-\infty$ to ∞;
Extends to R^n

Vector and Parametric Equations of Planes in R^3

Equation for the plane W that contains point $\mathbf{x_0}$ and is parallel to the noncollinear vectors $\mathbf{v_1}$ and $\mathbf{v_2}$



<u>Vector form of plane</u>

$\mathbf{x} - \mathbf{x_0} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ or $\mathbf{x} = \mathbf{x_0} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$

If $\mathbf{x}_0 = \mathbf{0}$, plane passes through the origin.

Variable t_1 and t_2 called ***parameters***;
$\mathbf{x}$ traces out the plane W as t_1 and t_2 are independently varied from $-\infty$ to ∞ ;
Extends to R^n
Parametric equations are formed when vectors are expressed in terms of their components

***** End of Chapter 1*****