$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.2**

Lecture : **Eigen and Singular Values**

Topic : **Characteristic Equation**

Concept : **How to find eigenvalue and vectors**

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How to find eigenValues ?

Our next objective is to obtain a general procedure for finding eigenvalues and eigenvectors of an $n \times n$ matrix A. We will begin with the problem of finding the eigenvalues of A. Note first that the equation $A\mathbf{x} = \lambda\mathbf{x}$ can be rewritten as $A\mathbf{x} = \lambda I\mathbf{x}$, or equivalently, as

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

For λ to be an eigenvalue of A this equation must have a nonzero solution for $\mathbf{x}$.

$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ (A - \lambda I)x &= 0 \\ (\lambda I - A)x &= 0 \end{aligned}$$

For the above equation to be true, and x not a zero vector (condition for eigen vector),

then $(\lambda I - A)$ must be a singular matrix (not invertible).

See: invertible matrix theorem (corollary of (d))

How to find eigenValues ?

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Lay,4th, pg112 (Ch2.3)

The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices. Each statement in the theorem describes a property of every $n \times n$ invertible matrix. The *negation* of a statement in the theorem describes a property of every $n \times n$ singular matrix. For instance, an $n \times n$ singular matrix is *not* row equivalent to I_n, does *not* have n pivot positions, and has linearly *dependent* columns.

Determinant and Invertibility

THEOREM 4

A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 4 adds the statement "$\det A \neq 0$" to the Invertible Matrix Theorem. A useful corollary is that $\det A = 0$ when the columns of A are linearly dependent. Also, $\det A = 0$ when the *rows* of A are linearly dependent. (Rows of A are columns of A^T, and linearly dependent columns of A^T make A^T singular. When A^T is singular, so is A, by the Invertible Matrix Theorem.) In practice, linear dependence is obvious when two columns or two rows are the same or a column or a row is zero.

Lay, pg171 (Ch3.2)

Clever idea:
The determinant transforms the original problem of $(A - \lambda I)x = 0$,
an equation with two unknowns λ, x into a polynomial with only 1 unknown λ.
Allowing us to solve first for the eigen value λ, the roots of the polynomial.
The polynomial is called the "characteristic equation"of A.

Since $(A - \lambda I)$ is a singular matrix (NOT invertible), therefore

$$\det(A - \lambda I) = 0$$

Characteristic Eqn and Polynomial

EXAMPLE 1 Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

SOLUTION We must find all scalars λ such that the matrix equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has a nontrivial solution. By the Invertible Matrix Theorem in Section 2.3, this problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is *not* invertible, where

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

By Theorem 4 in Section 2.2, this matrix fails to be invertible precisely when its determinant is zero. So the eigenvalues of A are the solutions of the equation

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = 0$$

Determinant of 2x2 matrix

THEOREM 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

$$\det A = ad - bc$$

Theorem 4 says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Lay,4th,pg103,sec 2.2

Characteristic Eqn and Polynomial

Recall that

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

So

$$\begin{aligned} \boxed{\det(A-\lambda I)} &= (2-\lambda)(-6-\lambda)-(3)(3) \\ &= -12+6\lambda-2\lambda+\lambda^2-9 \\ &\boxed{= \lambda^2+4\lambda-21} \\ &= (\lambda-3)(\lambda+7) \end{aligned}$$

If $\det(A-\lambda I)=0$, then $\lambda = 3$ or $\lambda = -7$. So the eigenvalues of A are 3 and -7. ■

Characteristic Eqn

Characteristic Polynomial

Equivalent statements regarding eigenvalue λ

Given $Ax = \lambda x$, the following theorem applies:

Theorem Given a square matrix A and a scalar λ, the following statements are equivalent:

- λ is an eigenvalue of A,
- $N(A - \lambda I) \neq \{\mathbf{0}\}$,
- the matrix $A - \lambda I$ is singular,
- $\det(A - \lambda I) = 0$.

Ref: https://textbooks.math.gatech.edu/ila/characteristic-polynomial.html

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Example: EigenValues of Triangular Matrixes

THEOREM 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

PROOF For simplicity, consider the 3×3 case. If A is upper triangular, then $A - \lambda I$ has the form

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

The scalar λ is an eigenvalue of A if and only if the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, that is, if and only if the equation has a free variable. Because of the zero entries in $A - \lambda I$, it is easy to see that $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable if and only if at least one of the entries on the diagonal of $A - \lambda I$ is zero. This happens if and only if λ equals one of the entries a_{11}, a_{22}, a_{33} in A. For the case in which A is lower triangular, see Exercise 28. ■

THEOREM 3.2 Determinant of a Triangular Matrix

If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal. That is,

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{nn}.$$

Fact 7. The determinant of a lower triangular matrix (or an upper triangular matrix) is the product of the diagonal entries. In particular, the determinant of a diagonal matrix is the product of the diagonal entries.

Proof: Khan's academy determinant of triangular matrix.

https://www.khanacademy.org/math/linear-algebra/matrix-transformations/determinant-depth/v/linear-algebra-upper-triangular-determinant

Example: Meaning of eigenvalue == 0 => not invertible matrix

EXAMPLE 5 Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. The eigenvalues of A are 3, 0, and 2. The eigenvalues of B are 4 and 1. ■

What does it mean for a matrix A to have an eigenvalue of 0, such as in Example 5? This happens if and only if the equation

$$A\mathbf{x} = 0\mathbf{x} \tag{4}$$

has a nontrivial solution. But (4) is equivalent to $A\mathbf{x} = \mathbf{0}$, which has a nontrivial solution if and only if A is not invertible. Thus 0 *is an eigenvalue of A if and only if A is not invertible*. This fact will be added to the Invertible Matrix Theorem in Section 5.2.

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

Example: eigenvalue and algebraic multiplicity

EXAMPLE 3

Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

SOLUTION Form $A - \lambda I$, and use Theorem 3(d):

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$
$$= (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

or

$$(\lambda-5)^2(\lambda-3)(\lambda-1) = 0$$

Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

The eigenvalue 5 in Example 3 is said to have *multiplicity* 2 because $(\lambda - 5)$ occurs two times as a factor of the characteristic polynomial. In general, the (**algebraic**) **multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Lay,4th,pg276

EigenValues and Algebraic multiplicity

EXAMPLE 4 The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplicities.

SOLUTION Factor the polynomial

$$\lambda^6 - 4\lambda^5 - 12\lambda^4 = \lambda^4(\lambda^2 - 4\lambda - 12) = \lambda^4(\lambda - 6)(\lambda + 2)$$

The eigenvalues are 0 (multiplicity 4), 6 (multiplicity 1), and -2 (multiplicity 1). ■

The **set of solutions** $(\lambda_1, \lambda_2, \ldots, \lambda_{N_\lambda})$,that is, the **eigenvalues**, is called the **spectrum** of A. The charaacteristic polynomial can be factored as follows:

$$p(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_{N_\lambda})^{n_{N_\lambda}} = 0.$$

The integer n_i is termed the **algebraic multiplicity** of eigenvalue λ_i .It is the number of times an eigenvalue appears as a root of the characteristic polynomial.

The algebraic multiplicities sum to N (the number of rows in A):

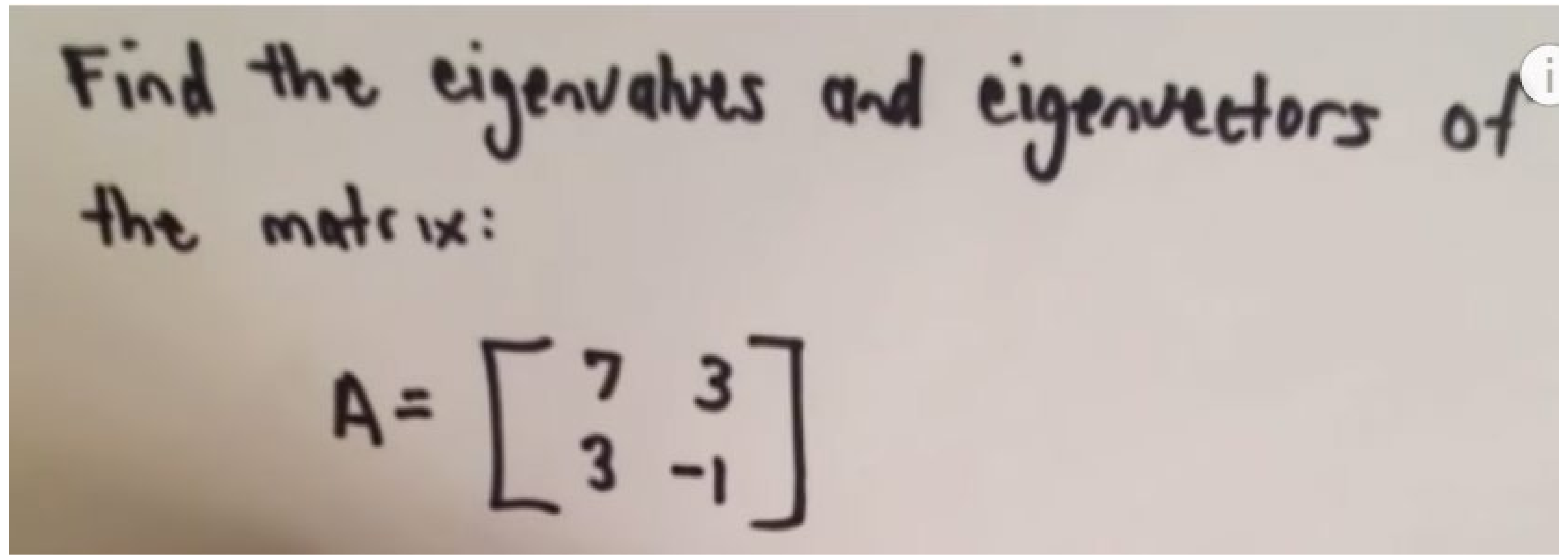
$$\sum_{i=1}^{N_\lambda} n_i = N.$$

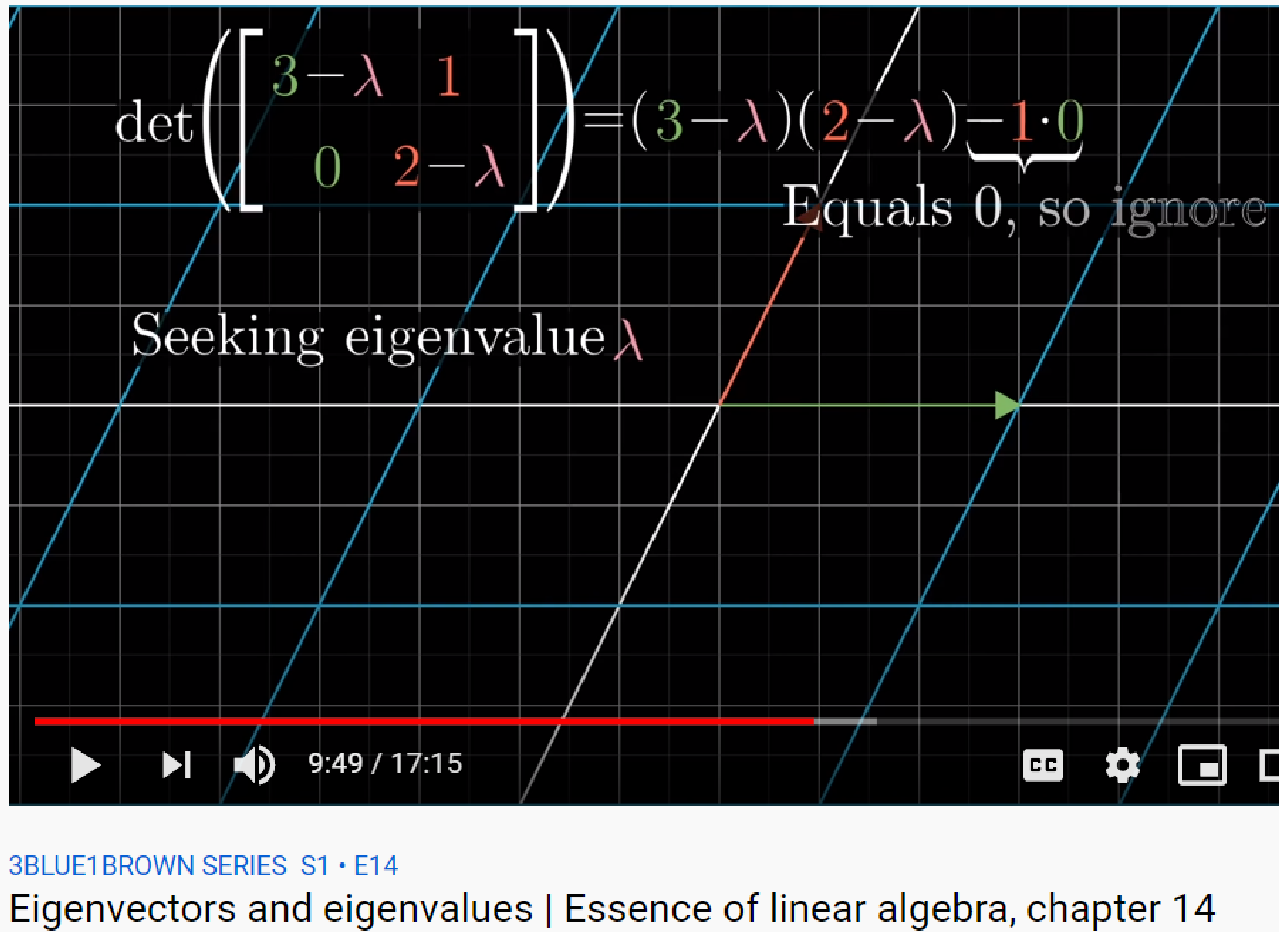
For each eigenvalue λ_i, there is a corresponding EigenSpace $E(\lambda_i)$

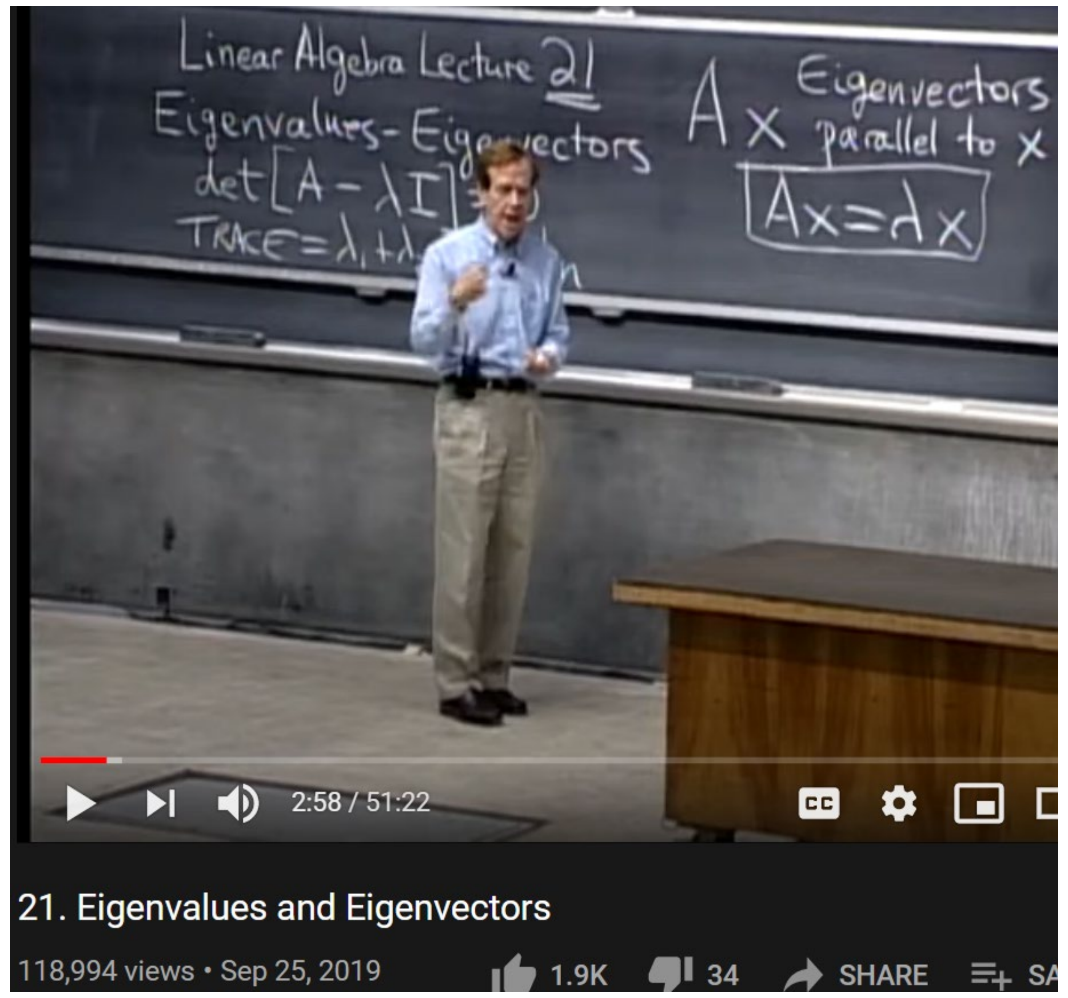
References online

1) PatrickJMT: https://www.youtube.com/watch?v=IdsV0RaC9jM

Work examples:







2) 3Blue1Brown: Ch14, https://www.youtube.com/watch?v=PFDu9oVAE-g

Understanding and perspective

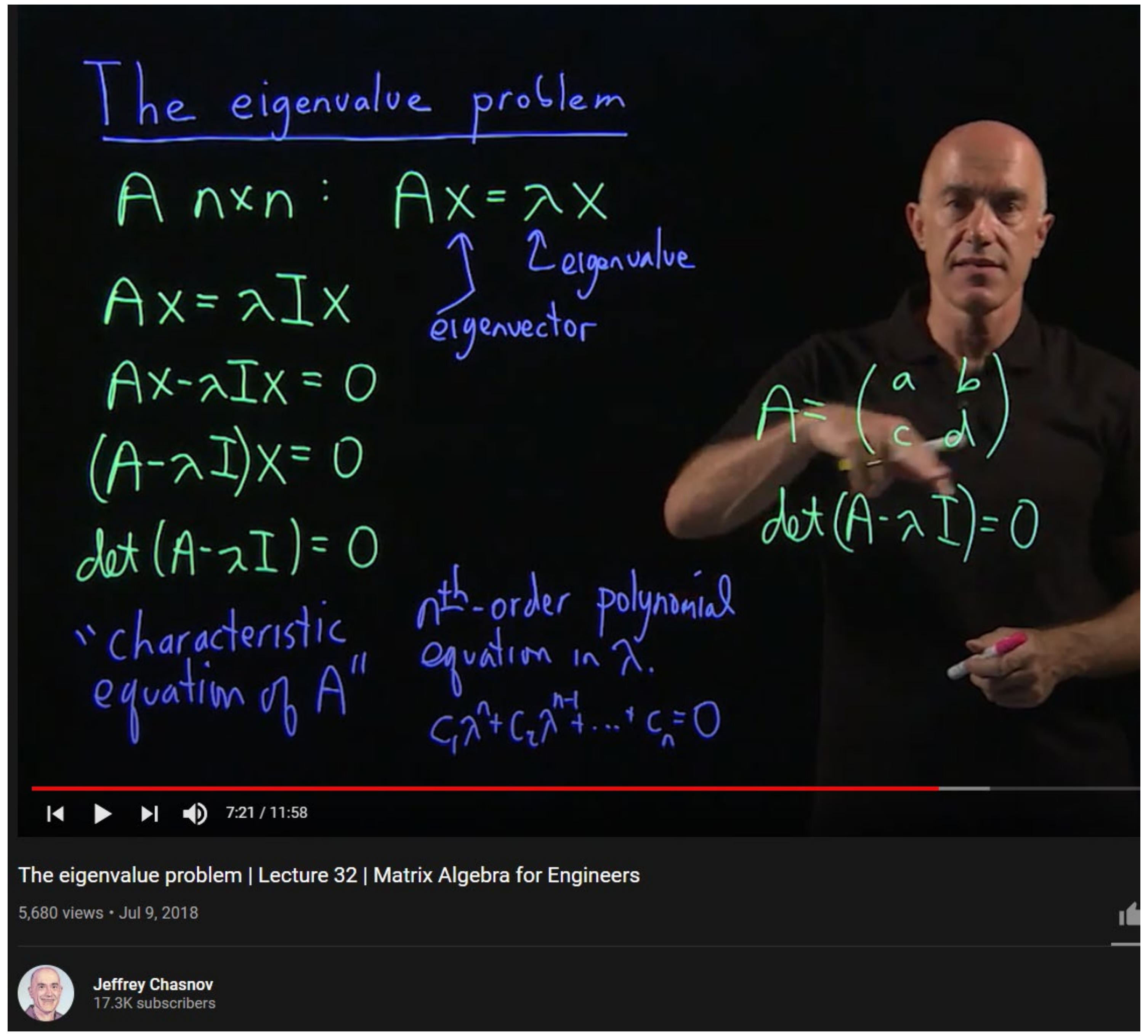
3) Strang introducing EigenValues & vectors

https://www.youtube.com/watch?v=cdZnhQjJu4I

More examples

4. How to find eigenvalues/vectors (process):

a. Chasnov (L33): https://www.youtube.com/watch?v=29keVZGvqME&list=PLkZjai-2Jcxlg-Z1roB0pUwFU-P58tvOx&index=33



5. Steve Brunton's lecture for eigenvalues
https://www.youtube.com/watch?v=OELTJdaU8aA