Developing the Analysis & Synthesis Eqn of DFT // Chng Eng Siong OCT 2024.

Analysis Eqn.
DFT

$$\begin{bmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{bmatrix} = \begin{bmatrix} & & \\ & W & \\ & & \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}$$

DFT coeff $N \times 1$; $N \times N$ DFT matrix. ; $N \times 1$. discrete time sequence.

Motivation: $\underset{N\times 1}{\underline{Y}} = \underset{N\times N}{W} \underset{N\times 1}{\underline{y}}$.

elements of $\underline{Y}$ represents the $\underline{y} \in R^N$ in the freq domain.

// Note : The W matrix is the DFT matrix. in the analysis equation

$$\left\{ \frac{1}{N} W^H = W^{-1} \right\}$$

Synthesis Eqn:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} & & \\ & W^{-1} & \\ & & \end{bmatrix} \begin{bmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{bmatrix}$$

Fourier Basis :

To perform change of coordinate from discrete time to basis $\{ b_k \}_{k=0..N-1}$.

where

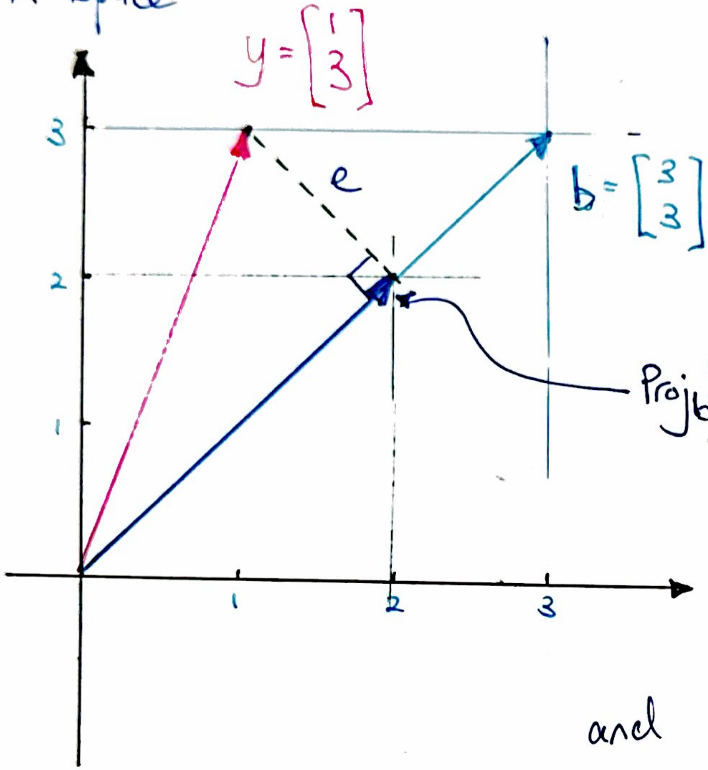
$$b_k = \begin{bmatrix} e^{+j\frac{2\pi}{N}k0} \\ e^{+j\frac{2\pi}{N}k1} \\ \vdots \\ e^{+j\frac{2\pi}{N}k(N-1)} \end{bmatrix} \in C^N \qquad n = 0..(N-1) \qquad \underset{\downarrow}{n=0}$$

i) a vector of N elements, with $n \to 0..(N-1)$ for each element in row n.

ii) change of phase as n increases by angular freq i.e. $\left(\frac{2\pi}{N}k\right)$ rad/sample.

Review: orthogonal projection to a vector

· R^2 space



$$\text{Proj}_b\, y = \left(\frac{b^T y}{\|b\|}\right)\left(\frac{b}{\|b\|}\right) = \langle y, \hat{b} \rangle \hat{b}$$

= Projecting vector v onto line span by b. such that the error $e \perp b$.

and $\hat{b} = \dfrac{b}{\|b\|}$

and $\|b\| = \sqrt{b^T b}$

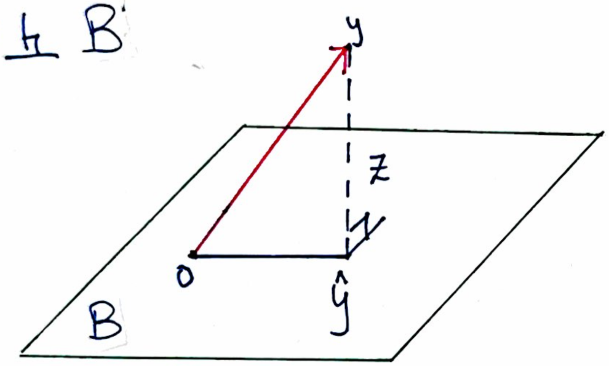
// Ref: 6.2.2 orthogonal projection.

Review: Orthogonal Projection
onto any orthogonal basis W.

// Ref: 6.2.5 orthogonal Decomposition
(Theorem 8)
(for complex vectors)

$$y = \hat{y} + z$$

and $z \perp B$



$Proj_B\, y = \hat{y}$ = orthogonal projection of y onto B

B = vector space
with orthogonal basis $\{b_0, b_1, \dots, b_{N-1}\}$

$$\hat{y} = Proj_B\, y = \sum_{k=0}^{N-1} \langle y, \hat{b}_k \rangle \hat{b}_k, \qquad \hat{b}_k = \frac{b_k}{\|b_k\|}$$

$$= \sum_{k=0}^{N-1} \left(\frac{b_k^H y}{\|b_k\|} \right) \frac{b_k}{\|b_k\|}$$; b_k^H = conjugate transpose of col b_k

$$= \sum_{k=0}^{N-1} \frac{b_k^H y}{b_k^H b_k} b_k$$ // complex version of Theorem 8, sec 6.2.5

Note 1: if $y \in B$, then $z = \underline{0}$.
$\hat{y} = y$.

Note 2: if $y \in R^N$ or C^N
and B has an orthogonal basis that spans B,
$\Rightarrow \{b_0, b_1, \dots, b_{N-1}\}$
then $Proj_B\, y = \hat{y} = y$ ∎

Discrete Fourier Transform.

1) Fourier Transform propose to use complex exponential basis B.

$$B = \{b_0, b_1, b_2, \dots, b_{N-1}\}$$

vector space span by

$$B = \left\{ \underbrace{\begin{bmatrix} e^{+j\frac{2\pi}{N}00} \\ e^{+j\frac{2\pi}{N}10} \\ e^{+j\frac{2\pi}{N}20} \\ \vdots \\ e^{+j\frac{2\pi}{N}(N-1)0} \end{bmatrix}}_{b_0}, \underbrace{\begin{bmatrix} e^{+j\frac{2\pi}{N}01} \\ e^{+j\frac{2\pi}{N}11} \\ e^{+j\frac{2\pi}{N}21} \\ \vdots \\ e^{+j\frac{2\pi}{N}(N-1)1} \end{bmatrix}}_{b_1}, \dots \underbrace{\begin{bmatrix} e^{+j\frac{2\pi}{N}0,N-1} \\ e^{+j\frac{2\pi}{N}1,N-1} \\ \vdots \\ e^{+j\frac{2\pi}{N}(N-1)(N-1)} \end{bmatrix}}_{b_{N-1}} \right\}$$

n ↓ sample idx =0..N-1

$b_k \in \mathbb{C}^N$

$k = 0..(N-1)$

$\longrightarrow$ $k = 0..(N-1)$.

angular frequency ω_k at col k $= \left(\frac{2\pi}{N}k\right)$ rad/sample.

2) The Discrete Fourier Transform uses B as the new basis, and B is an orthogonal basis

$$\hat{y} = \mathrm{Proj}_B\, y = \sum_{k=0}^{N-1} \langle y, \hat{b}_k \rangle \hat{b}_k \, ; \quad \hat{b}_k = \frac{b_k}{\|b_k\|}$$

$$\hat{y} = \sum_{k=0}^{N-1} \left(\frac{b_k^H}{\|b_k\|} y \right) \frac{b_k}{\|b_k\|} \, ; \quad \|b_k\|^2 = N$$

$$\hat{y} = \sum_{k=0}^{N-1} \frac{1}{N} (b_k^H y)\, b_k$$

since $y \in B$, then $\hat{y} = y$

$$y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} \in \mathbb{R}^N$$

*. Interpreting the B matrix (N×N)

$$B = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ b_0 & b_1 & \cdots & b_{N-1} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} b_{0,0} & b_{0,1} & \cdots & b_{0,N-1} \\ b_{1,0} & b_{1,1} & & \vdots \\ b_{2,0} & \vdots & \ddots & \vdots \\ \vdots & & & \\ b_{N-1,0} & b_{N-1,1} & \cdots & b_{N-1,N-1} \end{bmatrix}$$

where $b_{n,k} = e^{+j\frac{2\pi}{N}nk}$

∴ each col^n of b_k is a sequence of complex exponential length = 1, (angular frequency) rotating anti-clockwise at $\frac{2\pi}{N}k$ rad/sample.

Note: compare to $\underbrace{W}_{DFT}$ matrix which is rotating clockwise at $\frac{2\pi}{N}k$ rad/sample

for col^n k } since
or row k } symmetric matrix.

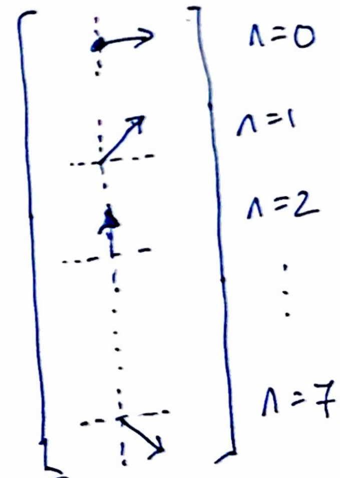
n idx, K idx

$$\underset{k\ idx}{b_0} = \begin{bmatrix} e^{+j\frac{2\pi}{N}0,0} \\ e^{+j\frac{2\pi}{N}1,0} \\ \vdots \\ e^{+j\frac{2\pi}{N}(N-1),0} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

a constant value = 1 when $n = 0..(N-1)$

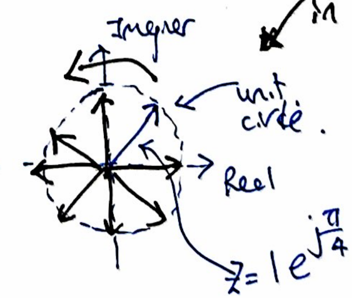
n idx, k idx

$$\underset{k\ idx}{b_{k=1}} = \begin{bmatrix} e^{+j\frac{2\pi}{N}0,1} \\ e^{+j\frac{2\pi}{N}1,1} \\ e^{+j\frac{2\pi}{N},2,1} \\ \vdots \\ e^{+j\frac{2\pi}{N}\cdot((N-1)(1))} \end{bmatrix} =$$



e.g. N = 8

rotating anti-clockwise at $\frac{2\pi}{8} = \frac{\pi}{4}$ rad/sample.



* in 2D plot of phasor $e^{j\frac{2\pi}{8}(1)n}$

for k = 1 since $\frac{2\pi}{N}(1) = \frac{2\pi}{8} = \pi/4$ rad/sample

*

* Show that the cols of B are orthogonal.

Let b_l, b_k represents the cols of B.

To show orthogonality, since $b_k \in C^N$, we need inner product $\langle -, - \rangle$.

$$\langle b_l, b_k \rangle = \begin{cases} 0 & \text{if } l \neq k \\ N & \text{if } l == k. \end{cases}$$

$$\langle b_l, b_k \rangle = \sum_{n=0}^{N-1} b_{n,l} \overline{b_{n,k}}$$

Note: 2nd element is conjugated!

$$\boxed{\text{where } \begin{aligned} b_{n,k} &= e^{+j\frac{2\pi}{N}nk} \\ b_{n,k} &= \text{element } n \text{ of col } k \text{ of } B \end{aligned}}$$

$$\langle b_l, b_k \rangle = b_k^H b_l \quad // \; b^H = \text{Hermition} = \text{conjugate transpose!}$$

$$= \overline{b_k}^T b_l$$

Note: when $l = k$, $\overline{b_k}^T b_k = N$ ◼

Let $\omega = e^{+j\frac{2\pi}{N}}$, $\omega^{-nk} = e^{-j\frac{2\pi}{N}nk}$

$$\langle b_l, b_k \rangle = \sum_{n=0}^{N-1} \omega^{+nl} \omega^{-nk}$$

$$= \sum_{n=0}^{N-1} \omega^{(l-k)n}$$

if $(l-k) \neq 0$ then

$$\langle b_l, b_k \rangle = \frac{1-\omega_0^N}{1-\omega_0} = 0$$

bcos:

$$\omega_0 = \omega^{(l-k)} \quad \neq 0$$

$$= \left(1 e^{j\frac{2\pi}{N}}\right)^{(l-k)} \neq 0.$$

$$\omega_0^N = 1 e^{j\frac{2\pi}{N}(l-k)N}$$

$$= 1 e^{j 2\pi (l-k) \neq 0}$$

$$= 1 \; ◼$$

$$S_n = ar^0 + ar^1 + \dots + ar^{n-1}$$

$$= \sum_{k=0}^{N-1} ar^k$$

$$= \frac{a - ar^n}{1-r}$$

Proof:

$$S_n = ar^0 + ar^1 + \dots + ar^{n-1}$$

$$rS_n = ar^1 + ar^2 + \dots + ar^n$$

$$S_n - rS_n = ar^0 - ar^n$$

$$S_n(1-r) = a - ar^n$$

$$\therefore S_n = \frac{a-ar^n}{1-r}$$

$$= a\left(\frac{1-r^n}{1-r}\right).$$

∴ Synthesis Eqn

$$
\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}
= \frac{1}{N}
\underbrace{\begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ b_0 & b_1 & \cdots & b_{N-1} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}}_{\substack{(N \times N) \\ B \text{ matrix.}}}
\underbrace{\overset{\substack{W \text{ matrix} \\ (N \times N)}}{\begin{bmatrix} \leftarrow \bar{b}_0 \rightarrow \\ \leftarrow \bar{b}_1 \rightarrow \\ \vdots \\ \leftarrow \bar{b}_{N-1} \rightarrow \end{bmatrix}}
\overset{(N \times 1)}{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}}}_{b_k^H y = Y(k)}
$$

Note
b_k^H = conjugate transpose
$\bar{b}_k$ = conjugate

and the DFT coeff is

$$
\begin{bmatrix} Y(0) \\ Y(1) \\ \vdots \\ Y(N-1) \end{bmatrix}
=
\underbrace{\begin{bmatrix} \leftarrow \bar{b}_0 \rightarrow \\ \leftarrow \bar{b}_1 \rightarrow \\ \vdots \\ \leftarrow \bar{b}_{N-1} \rightarrow \end{bmatrix}}
\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}}
$$

analysis eqn.

$$\underset{\text{DFT coeff}}{Y} = \underset{W.}{\text{DFT matrix}} \quad \underset{\text{Discrete Time seq.}}{\underline{y}}$$

Synthesis Eqn.

$$y = \left(\frac{1}{N} W^{H}\right) Y$$

Note: is symmetric.

W^H = conjugate transpose

each row is complex exponential rotating clockwise at $\omega = \frac{2\pi}{N} k$ rad/sample

$$Y = W y$$

$$
\begin{array}{r}
\text{row } 0 \rightarrow \\
\text{row } 1 \rightarrow \\
\\
\bar{b}_k \rightarrow \\
\\
\text{row}(N-1) \rightarrow
\end{array}
\begin{bmatrix}
e^{-j\frac{2\pi}{N}0,0}, & e^{-j\frac{2\pi}{N}1,0}, & e^{-j\frac{2\pi}{N}2,0}, & \cdots, & e^{-j\frac{2\pi}{N}(N-1),0} \\
e^{-j\frac{2\pi}{N}0,1}, & e^{-j\frac{2\pi}{N}1,1}, & & \cdots, & e^{-j\frac{2\pi}{N}N-1,1} \\
\vdots & \vdots & \vdots & & \vdots \\
e^{-j\frac{2\pi}{N}0,N-1}, & e^{-j\frac{2\pi}{N}1,N-1} & \cdots & & e^{-j\frac{2\pi}{N}(N-1)(N-1)}
\end{bmatrix}
$$