$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8A.3**

Lecture : **Complex Numbers**

Topic : **DFT**

Concept : **Understanding complex vectors and DFT Analysis and Synthesis equations**

Instructor: **A/P Chng Eng Siong**

TA: Shreyas Gopal

Rev: 23rd July 2024

Content

8A.3.1: Complex vectors: definitions

Given $z \in \mathcal{C}$, let $\boldsymbol{z} \in \mathcal{C}^N$ denote a column vector with N complex numbers.

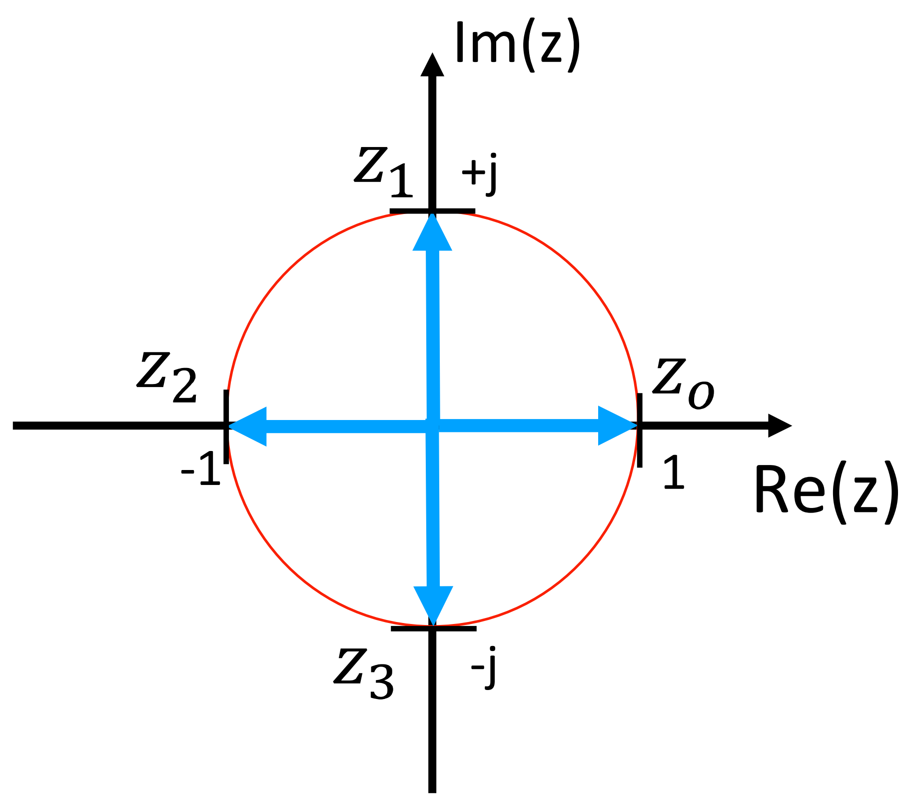
Example: $\boldsymbol{z} \in \mathcal{C}^4$ with elements $z_{n+1} = z_n\left(1e^{\frac{j\pi}{2}}\right)$, and $z_0 = 1$

Alternatively, $z_n = 1e^{\frac{j\pi}{2}n}$, $n = 0..3$

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = (r_1 \cdot r_2) e^{i(\theta_1 + \theta_2)}$$

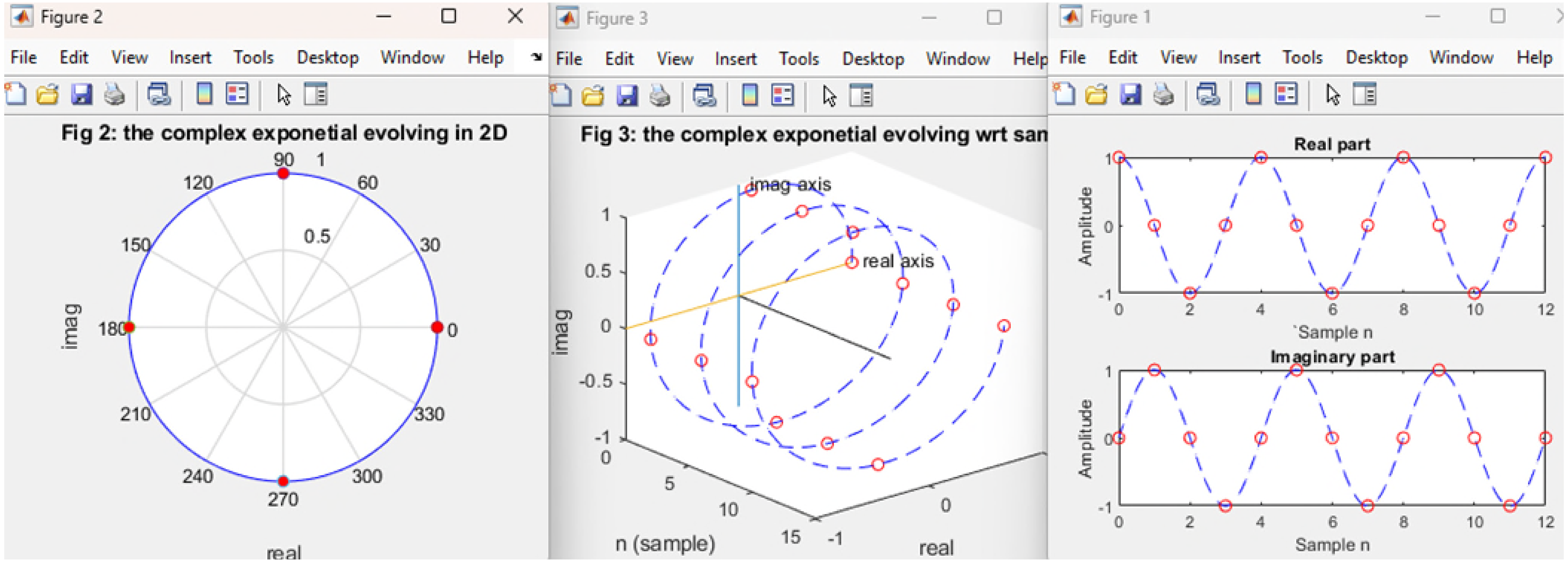
See: 8A.1.2 Multiplication in polar form
Note: multiplication of complex number means length multiplies and angles add.

$$\boldsymbol{z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1e^{\frac{j\pi}{2}0} \\ 1e^{\frac{j\pi}{2}1} \\ 1e^{\frac{j\pi}{2}2} \\ 1e^{\frac{j\pi}{2}3} \end{bmatrix} = \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}$$



Geometrically: z_n is a complex phasor rotating anti-clockwise at each step by $\frac{\pi}{2}$ radian per step.

8A.3.1: complex vectors - visualization



Given $\omega = \frac{\pi}{2}$,

$$z = 1e^{j\omega},$$

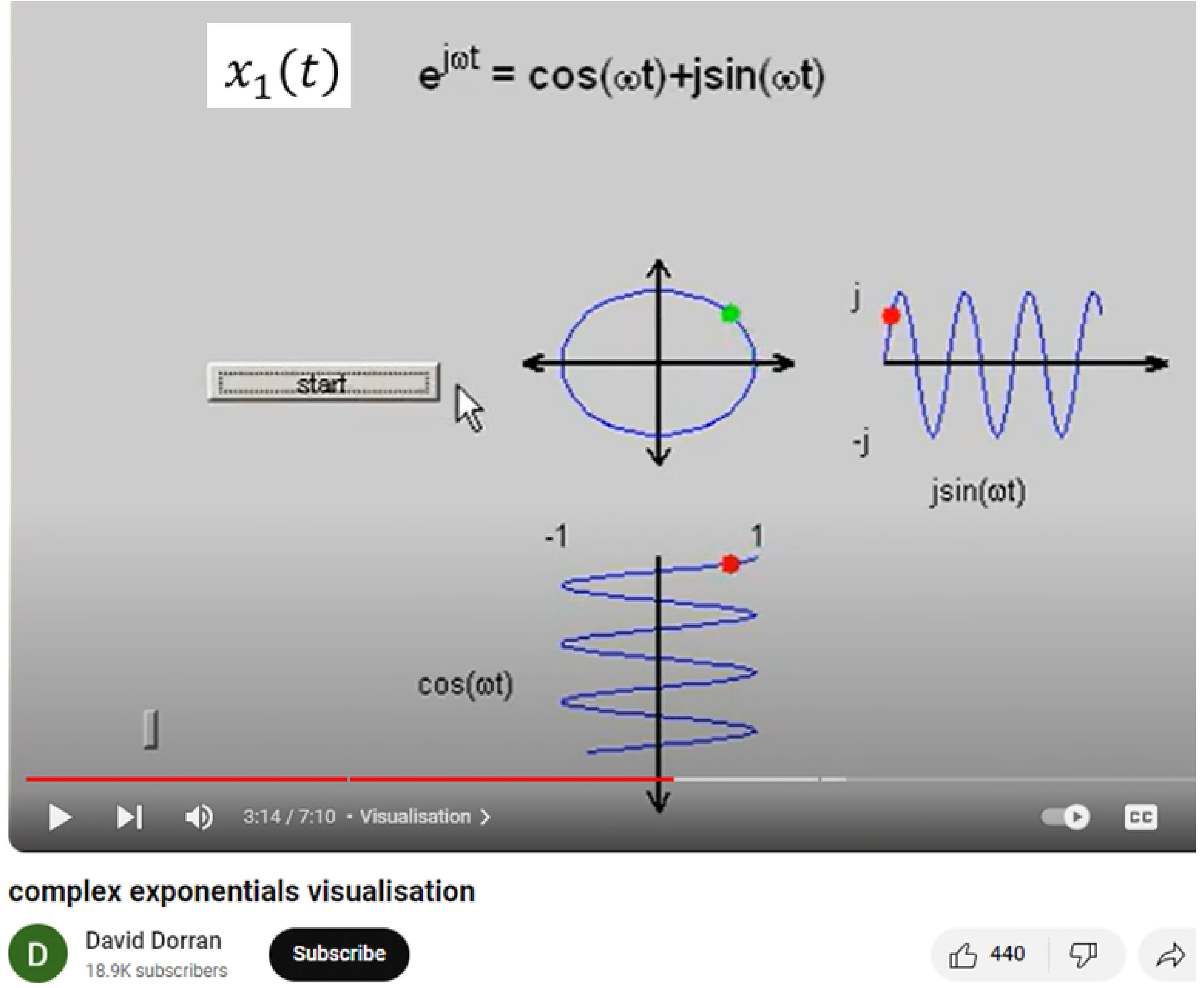
3 views of the complex sequence:

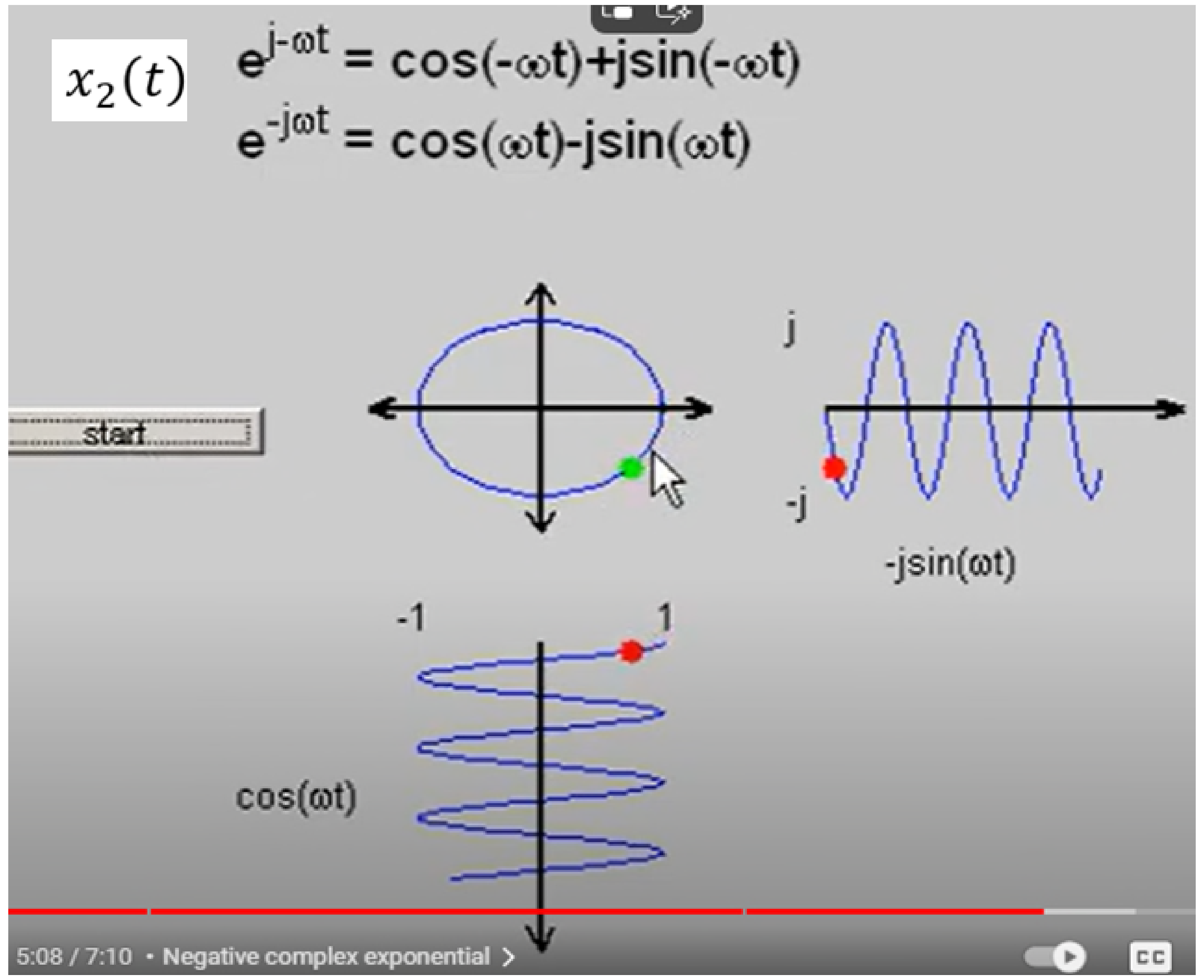
$$z^n = 1e^{j\omega n} = 1e^{j\frac{\pi}{2}n}, \qquad n = 0..12$$

Compressed (zipped) Folder

Matlab code plotting the above

8A.3.1: rotating complex exponentials: more examples





YouTubeLink: https://youtu.be/K_C7htSXORY?si=i_Z5mtR_8aZaHo2V

$$x_1(t) = e^{j(\omega)t}$$

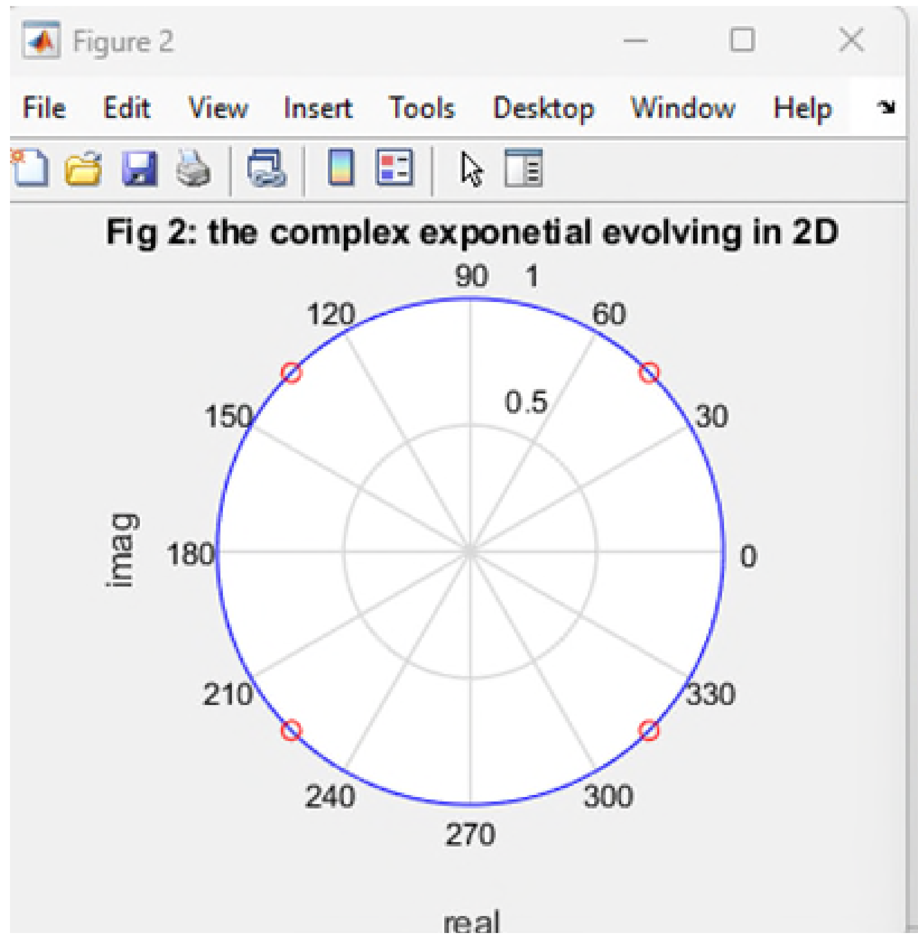
For $\omega = 0..+\pi$, as t increases then the phasor (green dot) rotates **anticlockwise.**

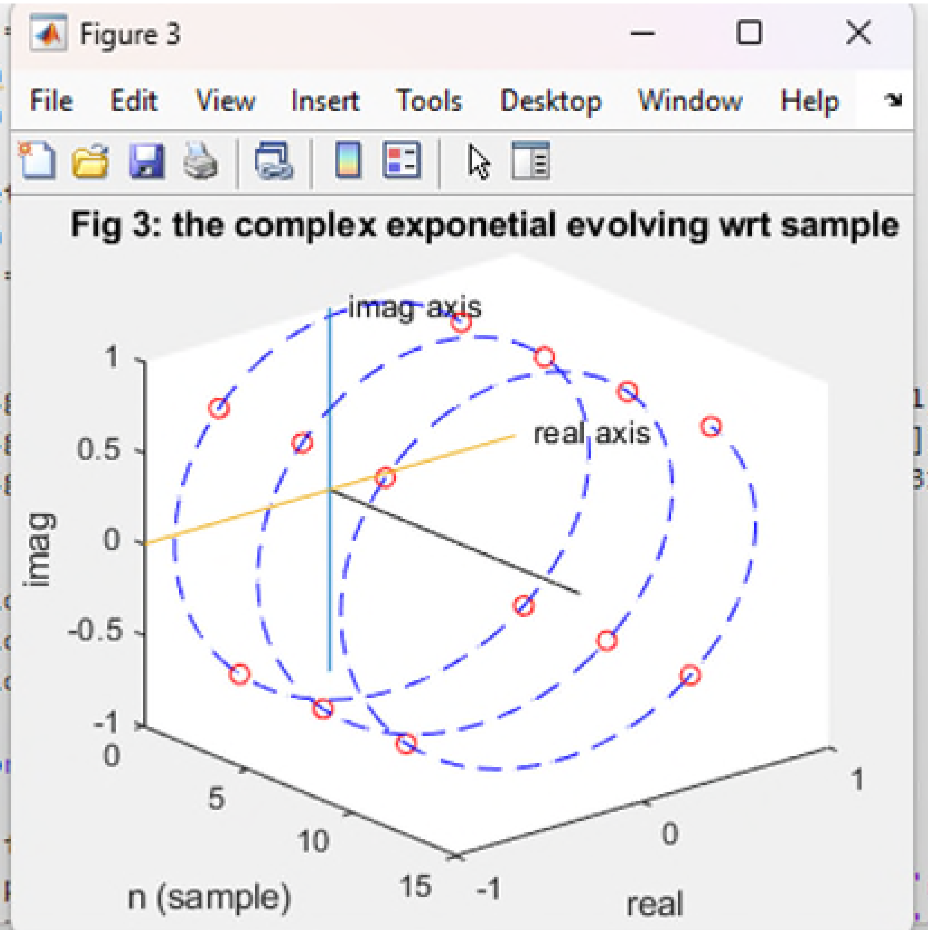
$$x_2(t) = e^{j(-\omega)t}$$

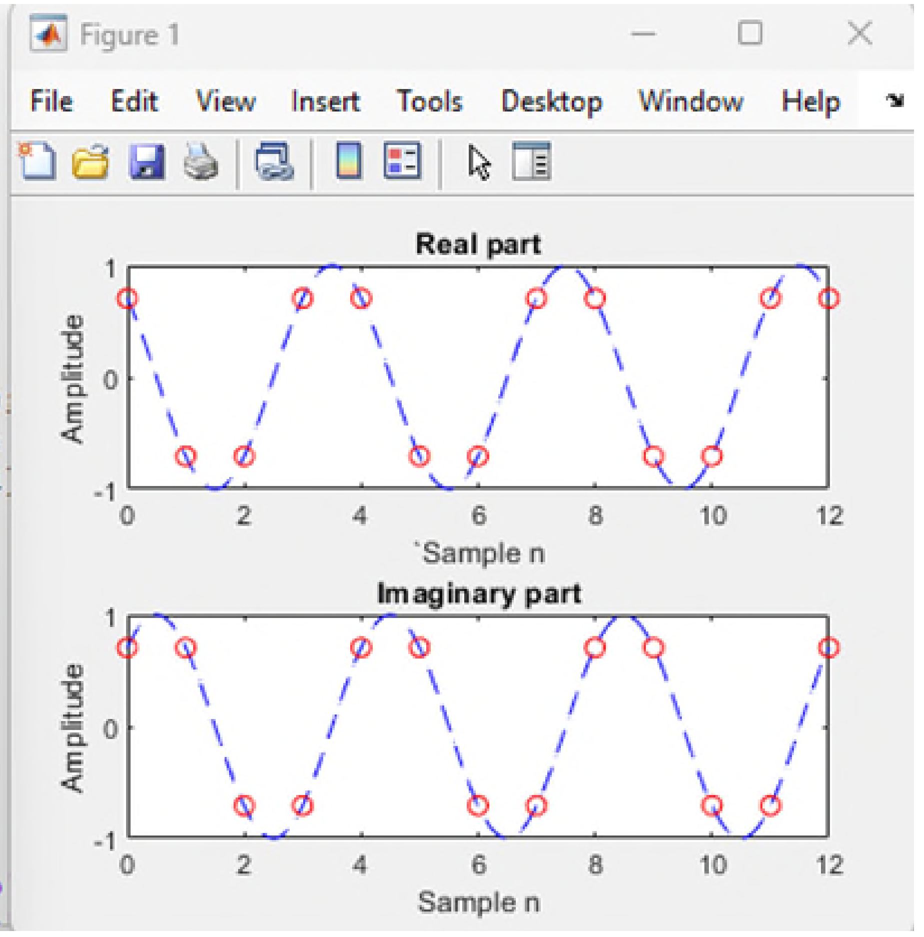
For above, given $(-\omega)$ is between $0..-\pi$, and as t increases, then the phasor (green dot) rotates **clockwise**.

8A.3.1: complex vector scaled by a complex number

$\boldsymbol{b} = \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}$, and $\beta = 0.7071 + j0.7071$, what is $\beta\boldsymbol{b}$?







Ans: $\beta = 0.7071 + j0.7071j = 1e^{\frac{j\pi}{4}}$
the input sequence is scaled by 1,
and phase shifted by $+\frac{j\pi}{4}$.

$$\beta\boldsymbol{b} = (1e^{\frac{j\pi}{4}})\begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix} = (1e^{\frac{j\pi}{4}})\begin{bmatrix} 1e^{\frac{j\pi}{2}0} \\ 1e^{\frac{j\pi}{2}1} \\ 1e^{\frac{j\pi}{2}2} \\ 1e^{\frac{j\pi}{2}3} \end{bmatrix} = \begin{bmatrix} 1e^{\frac{j1\pi}{4}} \\ 1e^{\frac{j3\pi}{4}} \\ 1e^{\frac{j5\pi}{4}} \\ 1e^{\frac{j7\pi}{4}} \end{bmatrix}$$

Note: easier to view multiplication in the complex exponential form.
Conclusion: since $|\beta| = 1$, the operation $\beta\boldsymbol{b}$ simply phase-shifted the original vector $\boldsymbol{b}$ by j$\frac{\pi}{4}$

Content

8A.3.2: Inner products definitions

Real Inner Product

Let V be a vector space over a real field $\mathbb{F}$.

A **(real) inner product** is a mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ that satisfies the real inner product axioms:

$(1')$:	Symmetry		$\forall x, y \in V : \langle x, y \rangle = \langle y, x \rangle$
(2) :	Linearity in first argument	$\forall x, y \in V, \forall a \in \mathbb{F}$:	$\langle ax + y, z \rangle = a \langle x, z \rangle + \langle y, z \rangle$
(3) :	Non-Negative Definiteness	$\forall x \in V$:	$\langle x, x \rangle \in \mathbb{R}_{\ge 0}$
(4) :	Positiveness	$\forall x \in V$:	$\langle x, x \rangle = 0 \implies x = \mathbf{0}_V$

Here, field F is the set of real or complex numbers

If $F = Real$, then the familiar dot product is the special case of inner product.

Complex Inner Product

Let V be a vector space over a complex field $\mathbb{F}$.

A **(complex) inner product** is a mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ that satisfies the complex inner product axioms:

(1) :	Conjugate Symmetry	$\forall x, y \in V$:	$\langle x, y \rangle = \overline{\langle y, x \rangle}$
(2) :	Linearity in first argument	$\forall x, y \in V, \forall a \in \mathbb{F}$:	$\langle ax + y, z \rangle = a \langle x, z \rangle + \langle y, z \rangle$
(3) :	Non-Negative Definiteness	$\forall x \in V$:	$\langle x, x \rangle \in \mathbb{R}_{\ge 0}$
(4) :	Positiveness	$\forall x \in V$:	$\langle x, x \rangle = 0 \implies x = \mathbf{0}_V$

1) https://proofwiki.org/wiki/Definition:Inner_Product
2) https://math.stackexchange.com/questions/476738/difference-between-dot-product-and-inner-product

8A.3.2: What does inner product reveal?
Example 1: orthogonal projection in real vector space

Orthognal projection: (6.2.2)

$$Proj_{\boldsymbol{b}}\boldsymbol{v} = \left(\frac{\boldsymbol{v}.\boldsymbol{b}}{||\boldsymbol{b}||^2}\right)\boldsymbol{b}$$

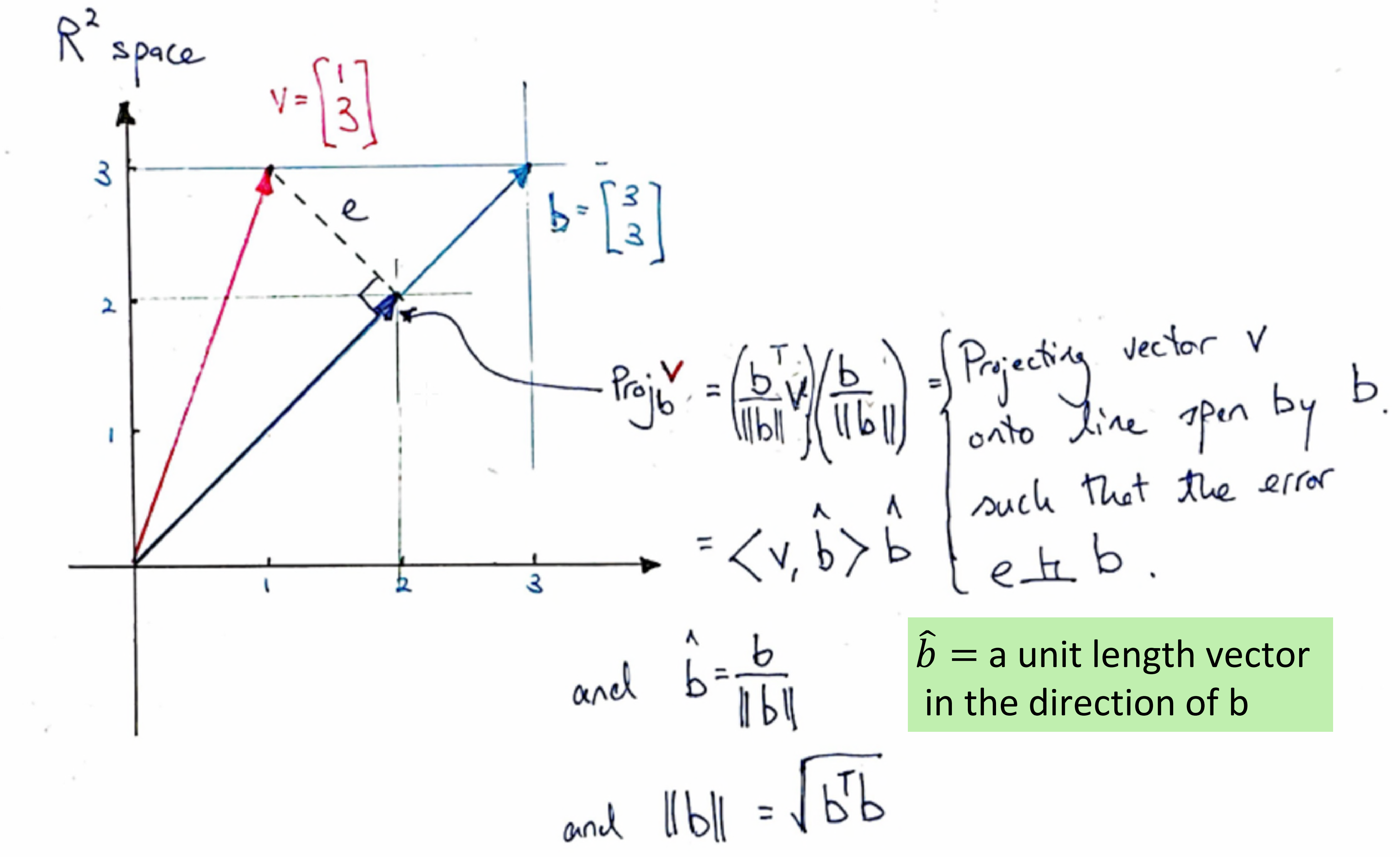
$$Proj_{\boldsymbol{b}}\boldsymbol{v} = < \boldsymbol{v}, \hat{\boldsymbol{b}} > \hat{\boldsymbol{b}}$$

$$< \boldsymbol{v}, \hat{\boldsymbol{b}} > = \hat{\boldsymbol{b}}^T\boldsymbol{v} = \sum_{i=0}^{N-1} v_i \hat{b}_i$$

Where $\hat{\boldsymbol{b}} = \frac{\boldsymbol{b}}{||\boldsymbol{b}||}$, $\hat{\boldsymbol{b}}^T$ = transpose of $\hat{\boldsymbol{b}}$

and

$||\boldsymbol{b}||$ = norm($\boldsymbol{b}$) = length ($\boldsymbol{b}$) = $\sqrt{\boldsymbol{b}^T\boldsymbol{b}}$



$\hat{b}$ = a unit length vector in the direction of b

Inner product tells you how much of one vector is pointing in the direction of another one. If e is a unit vector then $< f, e >$ is the component of f in the direction of e and the vector component of f in the direction e is $< f, e > e$. The vectors f and e are orthogonal when $< f, e >= 0$, in which case f has zero component in the direction e.

https://en.wikibooks.org/wiki/Linear_Algebra/Orthogonal_Projection_Onto_a_Line
https://math.stackexchange.com/questions/988886/what-does-inner-product-actually-mean

8A.3.2: Example 2: Inner product of complex vectors:

What is $< \boldsymbol{a}, \boldsymbol{b} >$ and $||\boldsymbol{b}||$? For complex vectors

Introduction Canonical inner product of complex vectors

$< \boldsymbol{a}, \boldsymbol{b} > = \sum_{i=0}^{N-1} a_i \overline{b_i} = \boldsymbol{b}^H \boldsymbol{a}$

matrix/vector multiplication

Where

$\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{C}^N$

$\boldsymbol{b}^H$ represents conjugate transpose of $\boldsymbol{b}$

$\overline{b_i}$ represents the $\boldsymbol{b}'s\ i^{th}$ element conjugated,

a_i represents the i^{th} element in vectors $\boldsymbol{a}$

Example:

$$\boldsymbol{a} = \begin{bmatrix} 1+j \\ 2-j \\ 3-1j \\ 4+1j \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ +j \\ -1 \\ -j \end{bmatrix}, \quad \overline{\boldsymbol{b}} = \begin{bmatrix} 1 \\ -j \\ -1 \\ +j \end{bmatrix}, \quad \boldsymbol{b}^H = [1, \ -j, \ -1, \ j\ ]$$

$$\begin{aligned} < \boldsymbol{a}, \boldsymbol{b} > &= (1+j)(1) + (2-j)(-j) + (3-1j)(-1) + (4+1j)(+j) \\ &= (1+j) + (-2j + j^2) + (-3+j) + (4j + j^2) \\ &= -4 + 4j \\ &= \boldsymbol{b}^H \boldsymbol{a} \end{aligned}$$

Why 2^{nd} term is conjugated for complex inner product?

Note Norm (b) must be a real number

$= ||\boldsymbol{b}|| = \sqrt{< \boldsymbol{b}, \boldsymbol{b} >} = \sqrt{\boldsymbol{b}^H \boldsymbol{b}} = \sqrt{\sum_{i=0}^{N-1} b_i \overline{b_i}}$

To achieve this for complex vectors, the second entry needs to be conjugated when calculating inner product to ensure that the result is real such that $||\boldsymbol{b}|| = \sqrt{< \boldsymbol{b}, \boldsymbol{b} >} = \sqrt{\boldsymbol{b}^H \boldsymbol{b}}$ will produce a real number.

Example: Find $||\boldsymbol{b}||$

$$< \boldsymbol{b}, \boldsymbol{b} > = \sum_{i=0}^{N-1} b_i \overline{b_i}$$

$$\begin{aligned} &= (1(1)) + (j(-j)) + (-1)(-1) + (-j)(j) \\ &= 1 - j^2 + 1 - j^2 \\ &= 4 \end{aligned}$$

Therefore, $||\boldsymbol{b}|| = \sqrt{< \boldsymbol{b}, \boldsymbol{b} >} = \sqrt{4}$

8A.3.2: Example 2: Inner product of complex vectors:

Find $Proj_{\boldsymbol{b}}\boldsymbol{v} = <\boldsymbol{v}, \widehat{\boldsymbol{b}}> \widehat{\boldsymbol{b}}$

Given $\boldsymbol{b}$ and $\boldsymbol{v} \in \boldsymbol{C}^4$, where

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix}, \; N = 4, \qquad \boldsymbol{b}^H = [1, \; -j, \; -1, \; j\;]$$

$$\mathbf{v} = (1e^{j\frac{\pi}{4}}) \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix} = \left(1e^{j\frac{\pi}{4}}\right) \boldsymbol{b}$$

Find:

$$Proj_{\boldsymbol{b}}\boldsymbol{v} = <\boldsymbol{v}, \widehat{\boldsymbol{b}}> \widehat{\boldsymbol{b}}$$

Lets find $\widehat{\boldsymbol{b}}$, the unit vector pointing in direction of b first.

$$\widehat{\boldsymbol{b}} = \frac{\boldsymbol{b}}{||\boldsymbol{b}||}$$

Because b's element all has length == 1, and it has N elements, then $||\boldsymbol{b}|| = \sqrt{<\boldsymbol{b}, \boldsymbol{b}>} = \sqrt{\boldsymbol{b}^H\boldsymbol{b}} = \sqrt{\boldsymbol{N}}$.

Therefore,

$$\widehat{\boldsymbol{b}} = \frac{\boldsymbol{b}}{||\boldsymbol{b}||} = \frac{\boldsymbol{b}}{\sqrt{\boldsymbol{N}}}$$

$$<\boldsymbol{v}, \widehat{\boldsymbol{b}}> = \widehat{\boldsymbol{b}}^H \boldsymbol{v}$$
$$= \left(\frac{\boldsymbol{b}}{\sqrt{N}}\right)^H \boldsymbol{v}$$
$$= \frac{\boldsymbol{b}^H}{\sqrt{N}} \left(1e^{\frac{j\pi}{4}}\right)\boldsymbol{b},$$

// Note: $\boldsymbol{b}^H\boldsymbol{b} = \boldsymbol{N}$

$$= \frac{N}{\sqrt{N}} \left(1e^{\frac{j\pi}{4}}\right)$$

And $Proj_{\boldsymbol{b}}\boldsymbol{v} = <\boldsymbol{v}, \widehat{\boldsymbol{b}}> \widehat{\boldsymbol{b}}$

$$= \frac{N}{\sqrt{N}} \left(1e^{\frac{j\pi}{4}}\right) \widehat{\boldsymbol{b}}$$
$$= \frac{N}{\sqrt{N}} \left(1e^{\frac{j\pi}{4}}\right) \left(\frac{\boldsymbol{b}}{\sqrt{N}}\right)$$

$$= \left(1e^{\frac{j\pi}{4}}\right) \boldsymbol{b}$$

Conclusions:

Projecting $\boldsymbol{v}$ onto $\boldsymbol{b}$ has no loss of information!

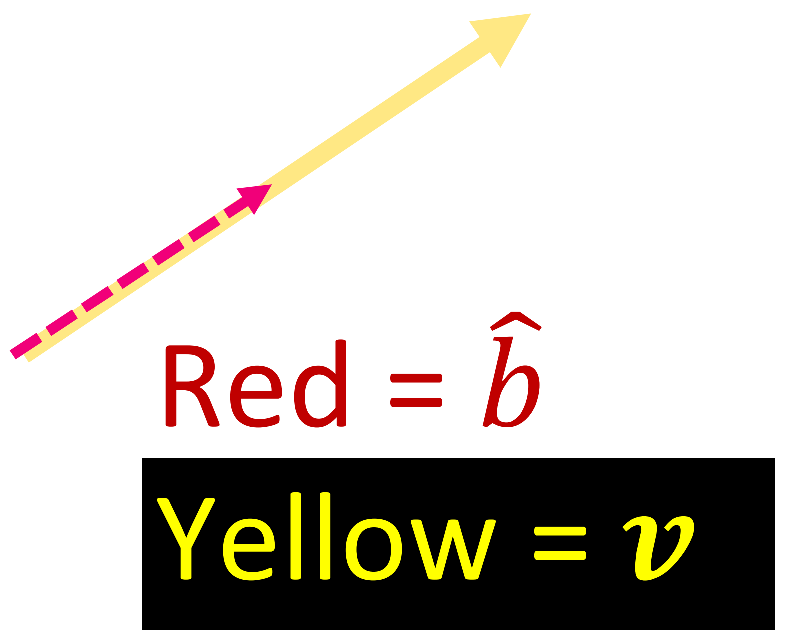
Why: we can think of vectors $\boldsymbol{v}$ and $\boldsymbol{b}$ are vectors in C^N and these 2 vectors have *the same direction but different length!*

Hence, when we use $\boldsymbol{b}$ *to represent* $\boldsymbol{v}$*, we only need to find the correct scaling* $<\boldsymbol{v}, \widehat{\boldsymbol{b}}>$ *to scale* $\widehat{\boldsymbol{b}}$ *to* $\boldsymbol{v}$

8A.3.2 Conclusions:

1) When we project a complex vector $\boldsymbol{v}$ to another complex vector $\boldsymbol{b}$
such that both $\boldsymbol{v}$ and $\boldsymbol{b}$ are rotating at the same frequency (but has different phase),
We still can get back the same vector $\boldsymbol{v}$**.**
Conceptually: vectors $\boldsymbol{v}$ and $\boldsymbol{b}$ have the same direction in C^N vector space.

- The inner product of $\boldsymbol{v}$ with $\widehat{\boldsymbol{b}}$ will reveal the magnitude and phase-shift that's required to construct (synthesize) the vector $\boldsymbol{v}$ from $\widehat{\boldsymbol{b}}$



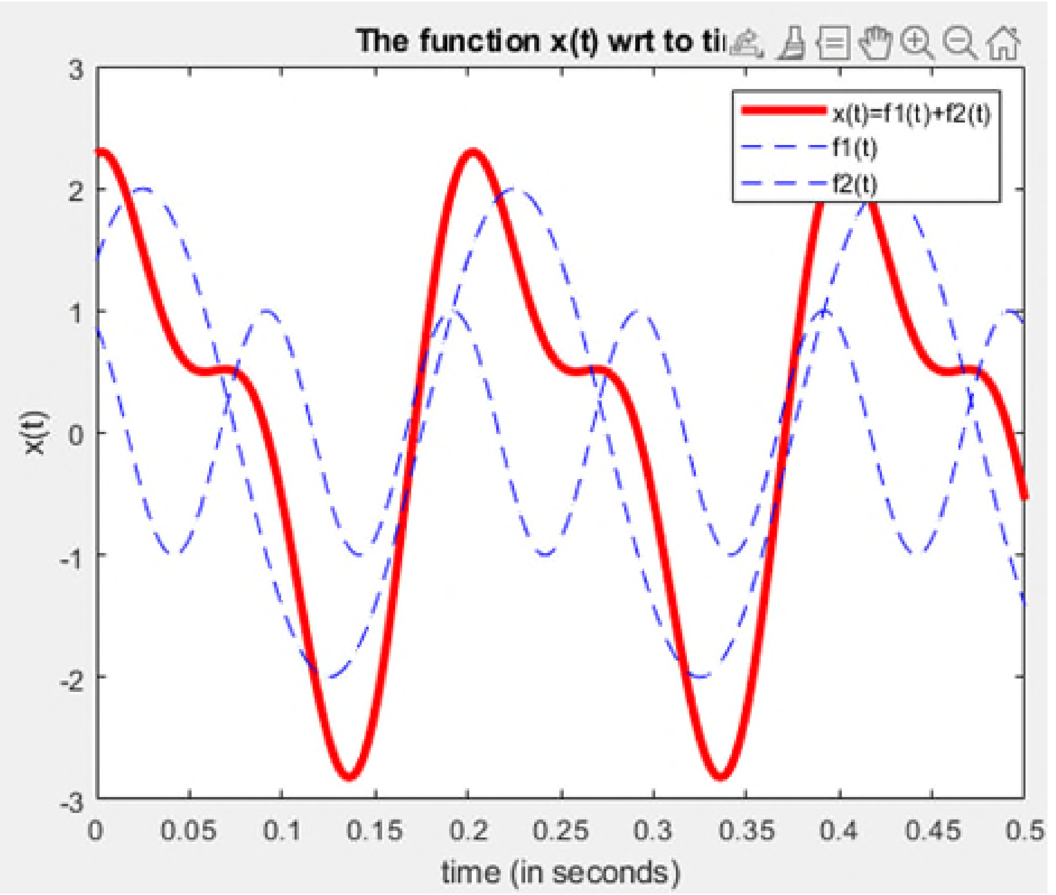
2) Key-insight of Fourier Transform: the set of vectors $\boldsymbol{b}$ **are complex exponentials and not arbitrary basis.** Fourier transform decompose the original signal $\boldsymbol{v}$ **into sum of scaled complex exponentials with different frequencies.**

Content

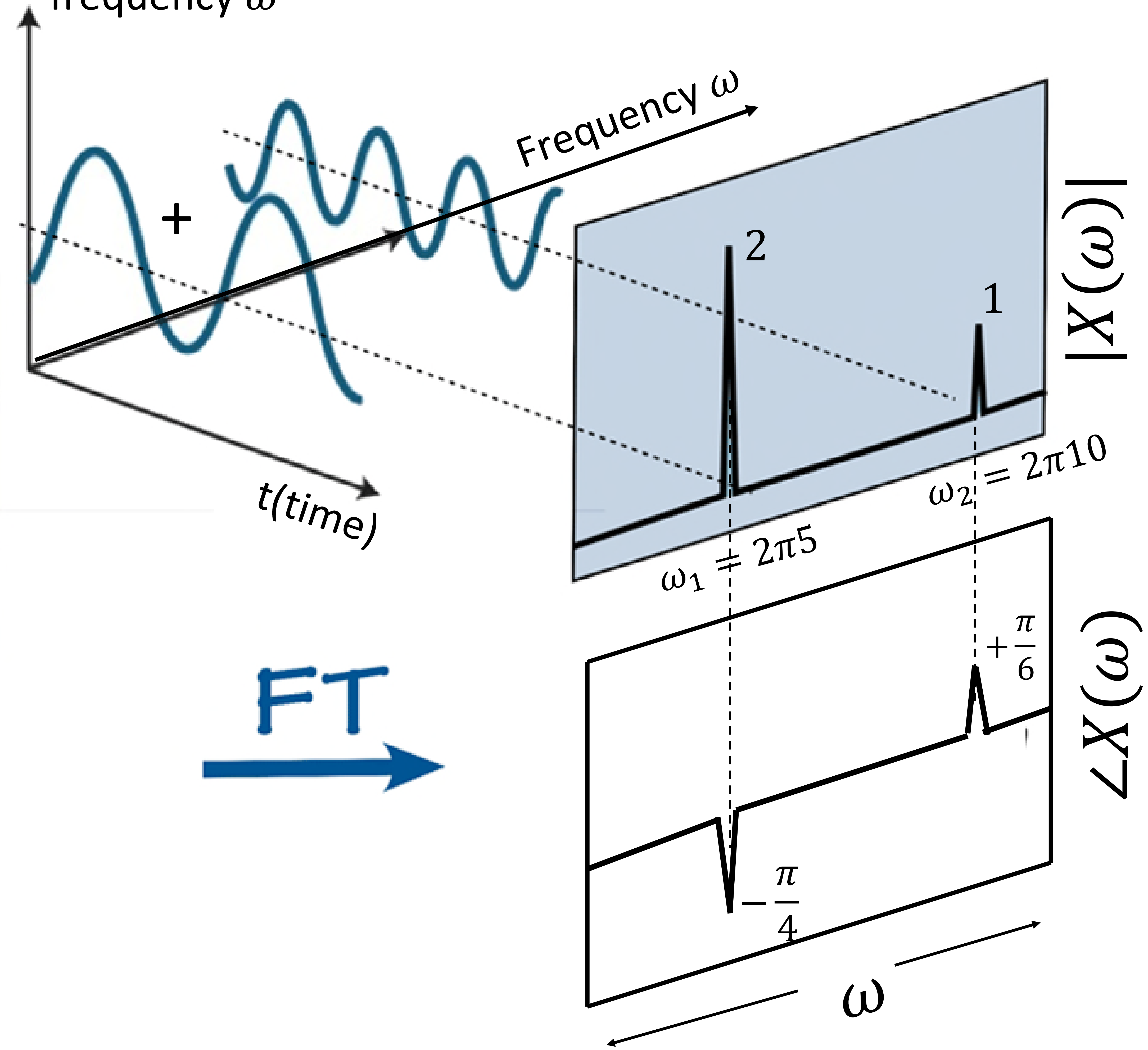
8A3.3: Motivation: Fourier Transform
Decomposing a signal into sum of sinusoids

$$x(t) = \sum_{\omega} |X(\omega)| \cos(\omega t + \angle X(\omega))$$

Sinusoids with frequency ω



=



Warning: the spectrum in this example is based on cosine, but in actual DFT, it is based on complex exponentials

Magnitude spectrum

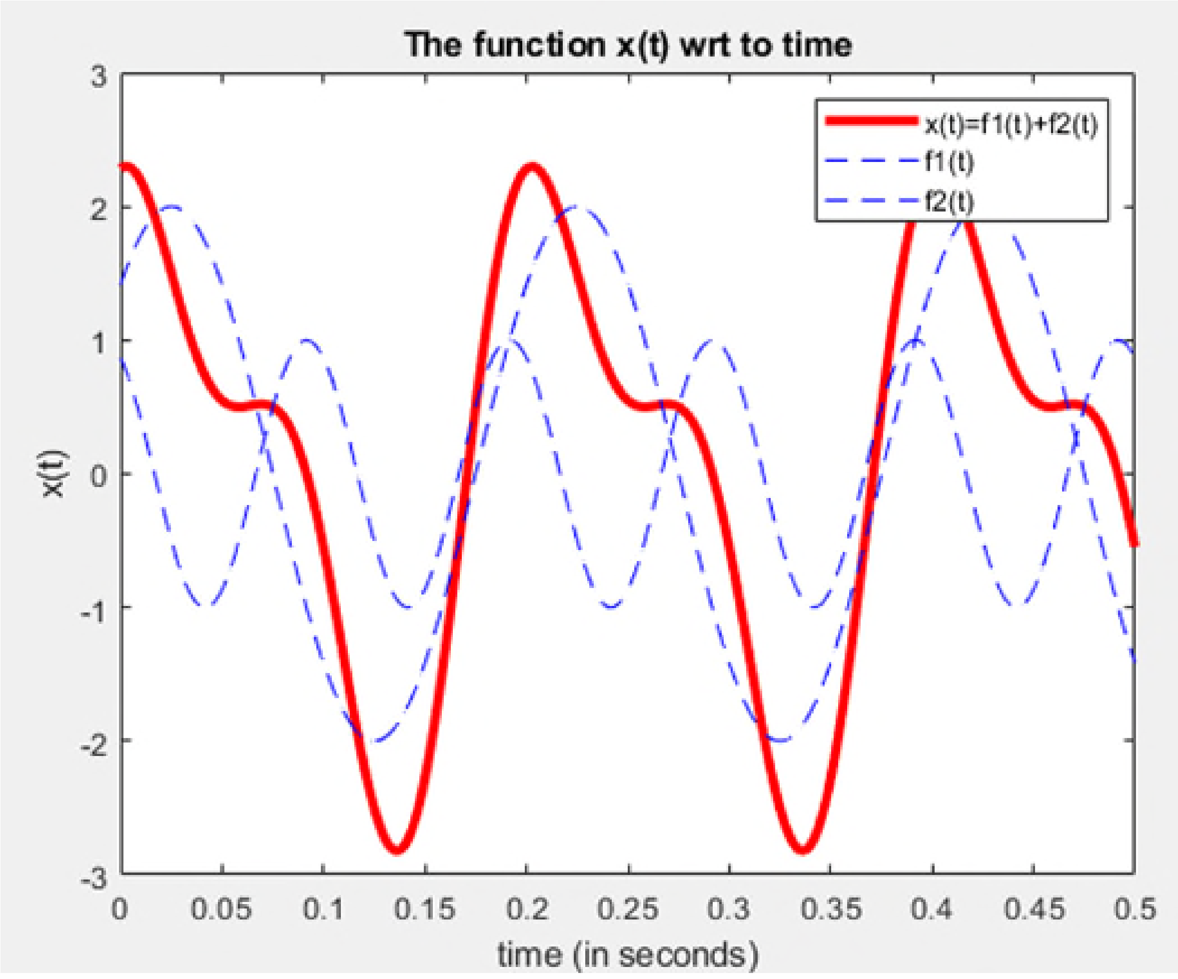
Phase spectrum

Time Domain

Input
$$x(t) = 2\cos\left(2\pi 5t - \frac{\pi}{4}\right) + 1\cos\left(2\pi 10t + \frac{\pi}{6}\right)$$

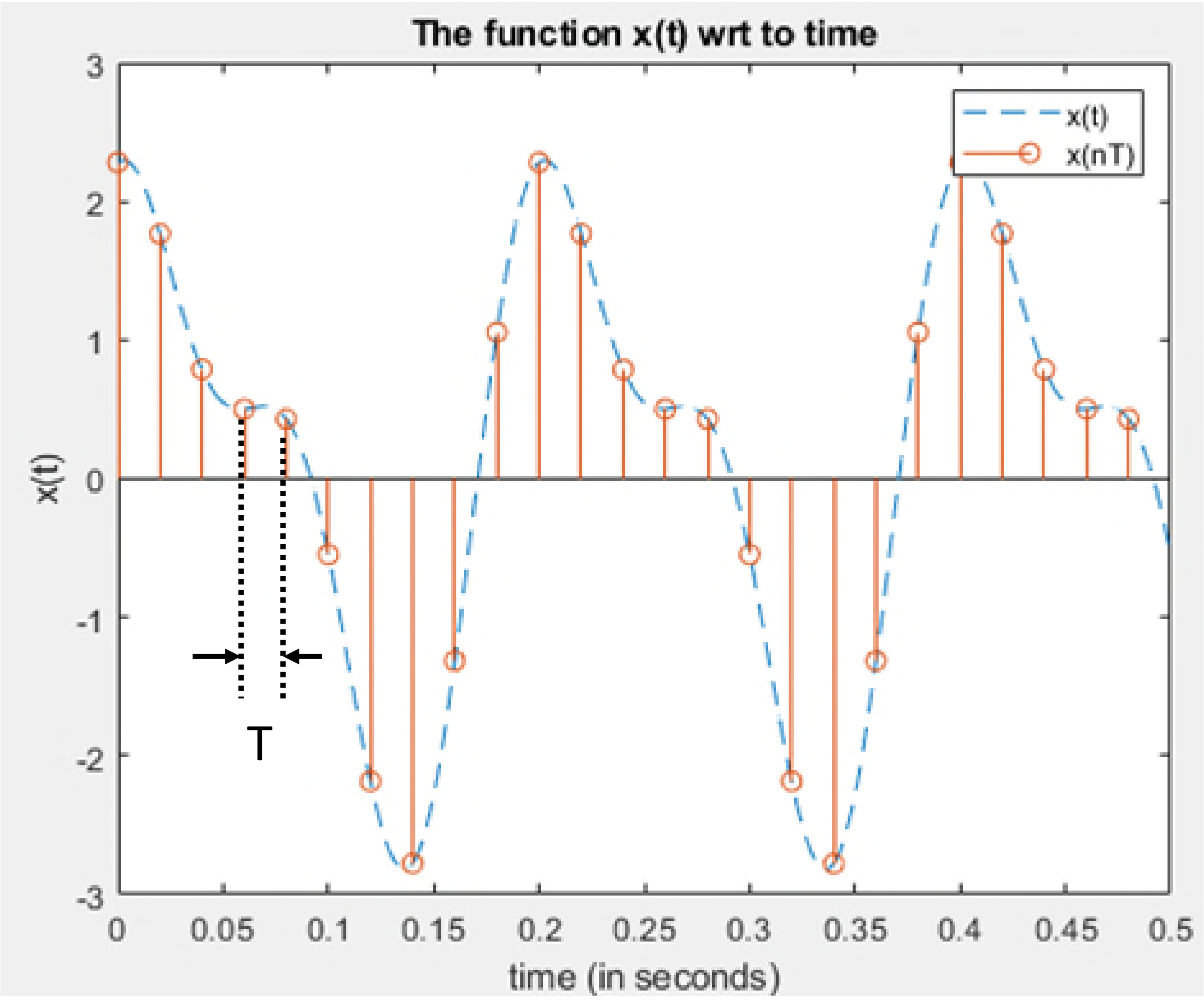
Note: here frequency ω is continuous. Unit = radian/sec

8A3.3: Example – sampling



$$x(t) = 2\cos\left(2\pi 5t - \frac{\pi}{4}\right) + 1\cos\left(2\pi 10t + \frac{\pi}{6}\right)$$

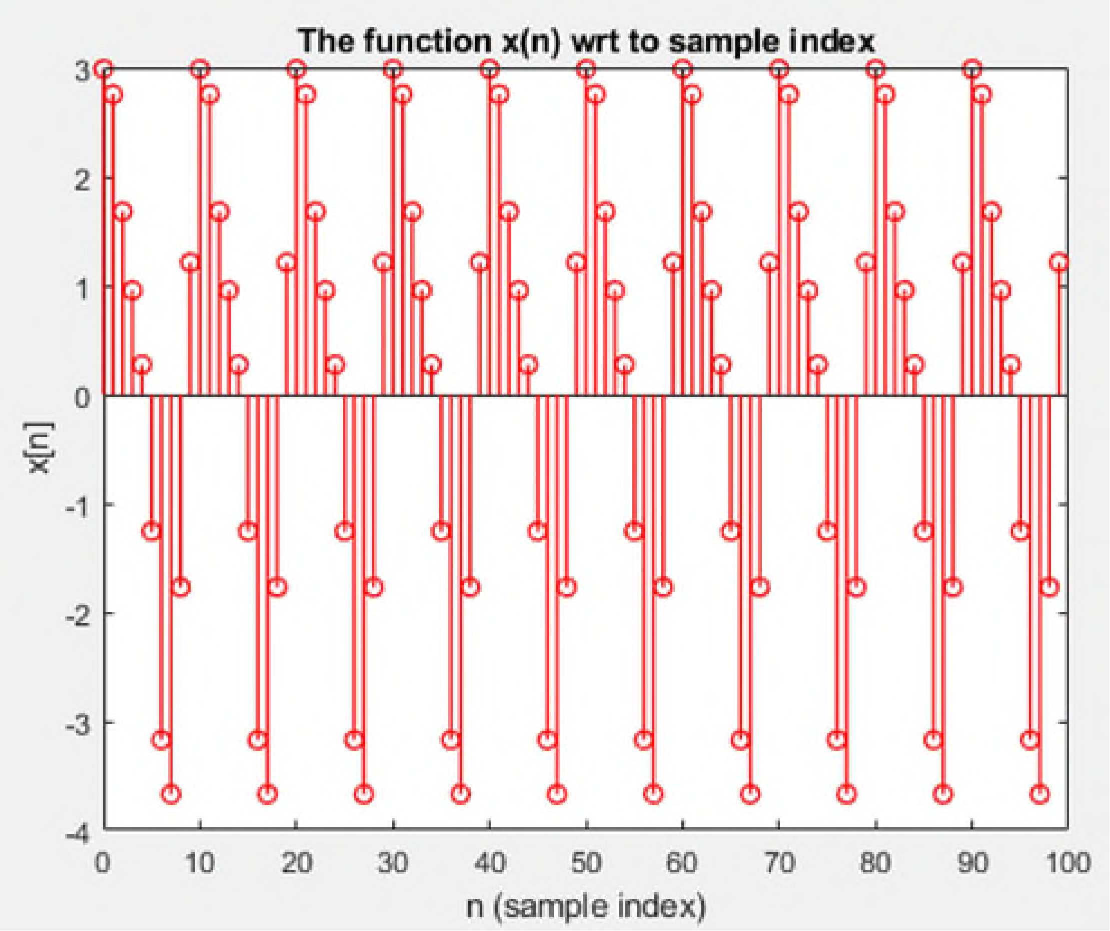
Continuous time



$$x(nT) = 2\cos\left(2\pi 5\left(\frac{n}{50}\right) - \frac{\pi}{4}\right) + 1\cos\left(2\pi 10\left(\frac{n}{50}\right) + \frac{\pi}{6}\right)$$

$$T = \frac{1}{50},\ n \in \mathrm{Z}$$

Sampled version at time step $T = \frac{1}{50}$



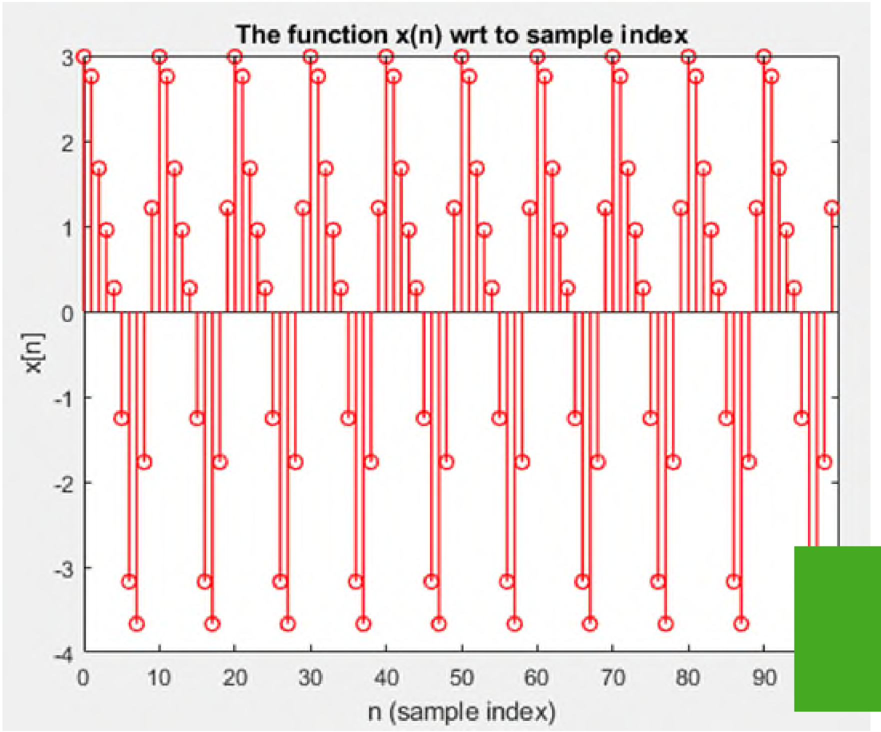
$$x(n) = 2\cos\left(0.2\pi n - \frac{\pi}{4}\right) + 1\cos\left(0.4\pi n + \frac{\pi}{6}\right)$$

Discrete time signal – time is now represented as sample index

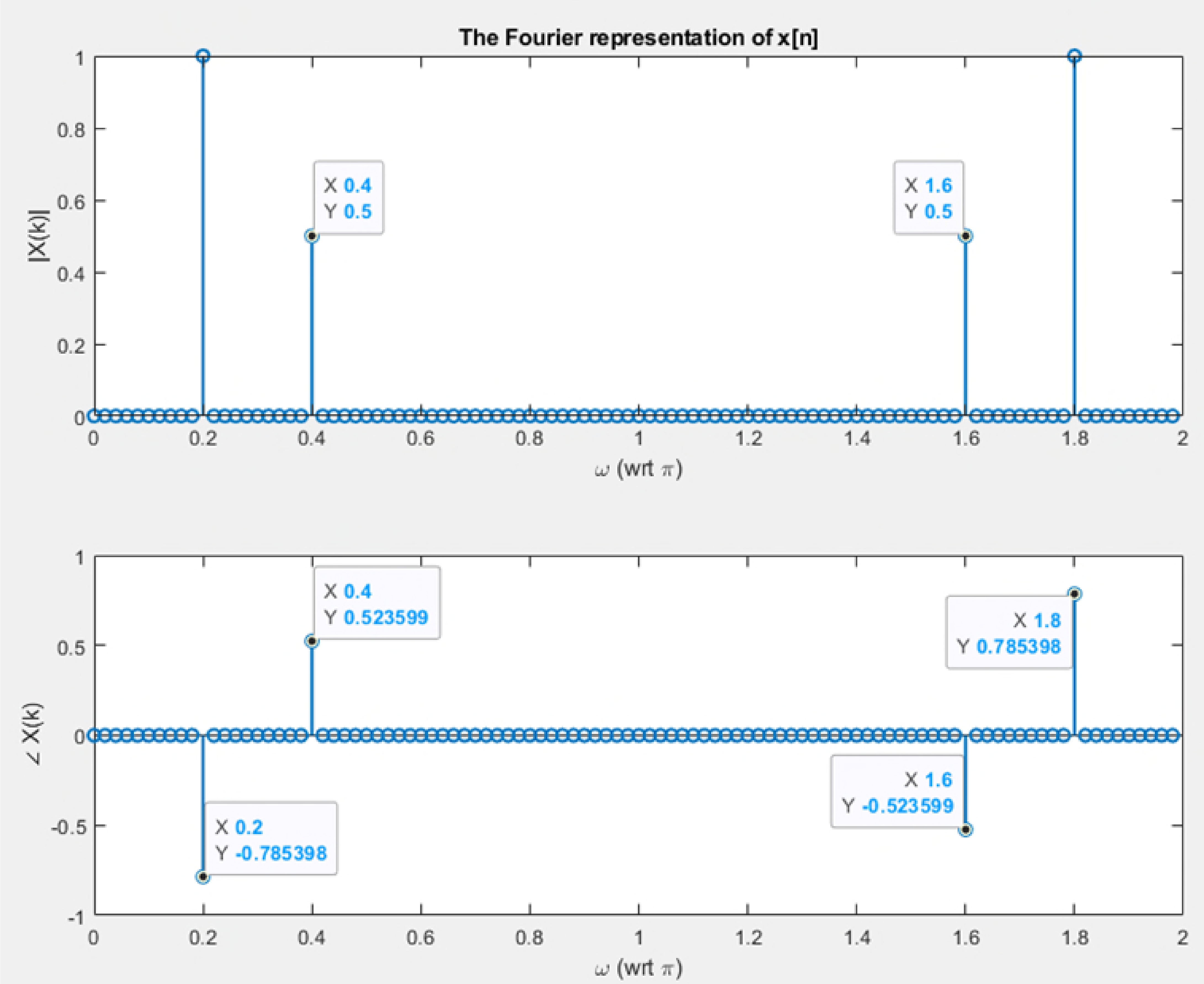
x[n] = Extracted 100 samples

Code= plot_DFT_example3.m

8A3.3: Discrete Time Fourier Transform reveals how x[n] is decomposed to complex exponentials



FT



100 samples of Discrete Time sequence

$4\ complex\ exponentials$

(1) $1e^{j(0.2\pi n-\frac{\pi}{4})}$

(2) $0.5e^{j(0.4\pi n+\frac{\pi}{6})}$

(3) $0.5e^{j(1.6\pi n-\frac{\pi}{6})}$

$= 0.5e^{j(-0.4\pi n-\frac{\pi}{6})}$

$= 0.5e^{-j(0.4\pi n+\frac{\pi}{6})}$

(4) $1e^{j(1.8\pi n+\frac{\pi}{4})}$

$= 1e^{j(-0.2\pi n+\frac{\pi}{4})}$

$= 1e^{-j(0.2\pi n-\frac{\pi}{4})}$

Note: $2\cos\left(0.2\pi n-\frac{\pi}{4}\right) = 1e^{j(0.2\pi n-\frac{\pi}{4})} + 1e^{-j(0.2\pi n-\frac{\pi}{4})}$

$And\ :1\cos\left(0.4\pi n+\frac{\pi}{6}\right) = 0.5e^{j(0.4\pi n+\frac{\pi}{6})} + 0.5e^{-j(0.4\pi n+\frac{\pi}{6})}$

Note: here frequency ω is for discrete time,
unit = radian/sample

$8A.3.3$ DFT is orthogonal decompositon

Review:

Chap. No : **6.2.3**
Lecture : **Orthogonality**
Topic : **Orthogonality**
Concept : **Orthogonal Sets & Orthogonal Basis**

THEOREM 5

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j} \qquad (j = 1, \ldots, p)$$

PROOF As in the preceding proof, the orthogonality of $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ shows that

$$\mathbf{y}\cdot\mathbf{u}_1 = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p)\cdot\mathbf{u}_1 = c_1(\mathbf{u}_1\cdot\mathbf{u}_1)$$

Since $\mathbf{u}_1\cdot\mathbf{u}_1$ is not zero, the equation above can be solved for c_1. To find c_j for $j = 2, \ldots, p$, compute $\mathbf{y}\cdot\mathbf{u}_j$ and solve for c_j. ■

Interpretations:

1) The basis is orthogonal, hence the contribution of each $\boldsymbol{u}_{j=1..p}$ is independent of other $\boldsymbol{u}_{j=1..p}$.
2) Therefore, $|c_j|$ tells us how much $\boldsymbol{u}_j$ contributes to $\boldsymbol{y}$ in terms of magnitude.
3) In the DFT representation, we will have DFT coefficients $c_j \in C$ and Fourier basis $\boldsymbol{u}_j \in C^N$, there the value of $\angle c_j$ tells us how much phase shift is required of $\boldsymbol{u}_j$ to phase-align to y.

$8A.3.3$ Introducing DFT Synthesis equation, $x = \left(\frac{1}{N}\right) \boldsymbol{W}^H \boldsymbol{X}$

DFT Synthesis equation: representing $x[n]$ as sum of scaled+phase-shifted orthogonal complex exponentials vectors

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad W_N^{-kn} = e^{j\frac{2\pi}{N}kn} \quad n, k = 0..(N-1)}$$

Inverse DFT

Signal vector $\boldsymbol{x} = \left(\frac{1}{N}\right) \boldsymbol{W}^H \boldsymbol{X}$

$X[k]$ represents the required $|.|$ and $\angle$ to scaled columns of W^H to represent $\boldsymbol{x}$

$n = 0..(N-1)$ $\quad$ $k = 0..(N-1)$

$$\underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\text{Signal vector } \boldsymbol{x}} = \frac{1}{N} \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j(2\pi/N)} & e^{j(4\pi/N)} & \cdots & e^{j(2(N-1)\pi/N)} \\ 1 & e^{j(4\pi/N)} & e^{j(8\pi/N)} & \cdots & e^{j(4(N-1)\pi/N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j(2(N-1)\pi/N)} & e^{j(4(N-1)\pi/N)} & \cdots & e^{j(2(N-1)(N-1)\pi/N)} \end{bmatrix}}_{\text{IDFT Matrix } \boldsymbol{W}^H = W_N^{-kn}} \underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\text{DFT coeff } \boldsymbol{X}},$$

$n = 0..(N-1)$ $\quad$ $k = 0..(N-1)$

This scaling makes $\left(\frac{1}{N}\boldsymbol{W}^H\right)\boldsymbol{W} = \boldsymbol{I}$, hence

$$\boldsymbol{x} = \left(\frac{1}{N}\boldsymbol{W}^H\right)\boldsymbol{X}$$
$$= \left(\frac{1}{N}\boldsymbol{W}^H\right)\boldsymbol{W}\boldsymbol{x}$$
$$= \boldsymbol{I}\boldsymbol{x} = \boldsymbol{x}$$

Warning: the place of scaling (1/N) is not uniform. But in DFT, 1/N is placed in the inverse DFT equation

Let $x \in R^N$ be signal vector
Let $X \in C^N$ be DFT coefficients vector

Let $W \in C^{NxN}$ be DFT matrix, and
$W^H \in C^{NxN}$ be IDFT matrix

where W^H is conjugate transpose of W

8A.3.3 The Fourier Basis: columns of W^H = W_N^{-kn} : Example N = 8

The columns of the Synthesis Equation of W^H

The complex exponential at digital angular frequency $\Omega = \frac{2\pi}{N} k \left(\frac{rad}{sample}\right)$

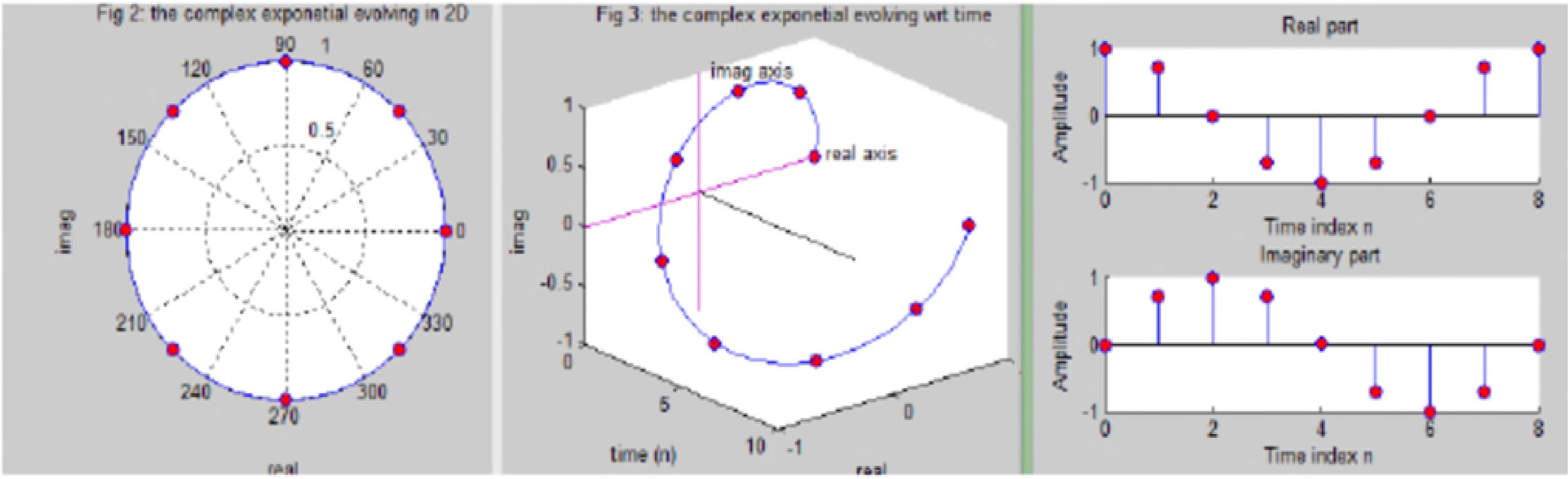
Note:

$$W_N^{-kn} = e^{j\frac{2\pi}{N}kn}, \qquad n, k = 0..(N-1)$$

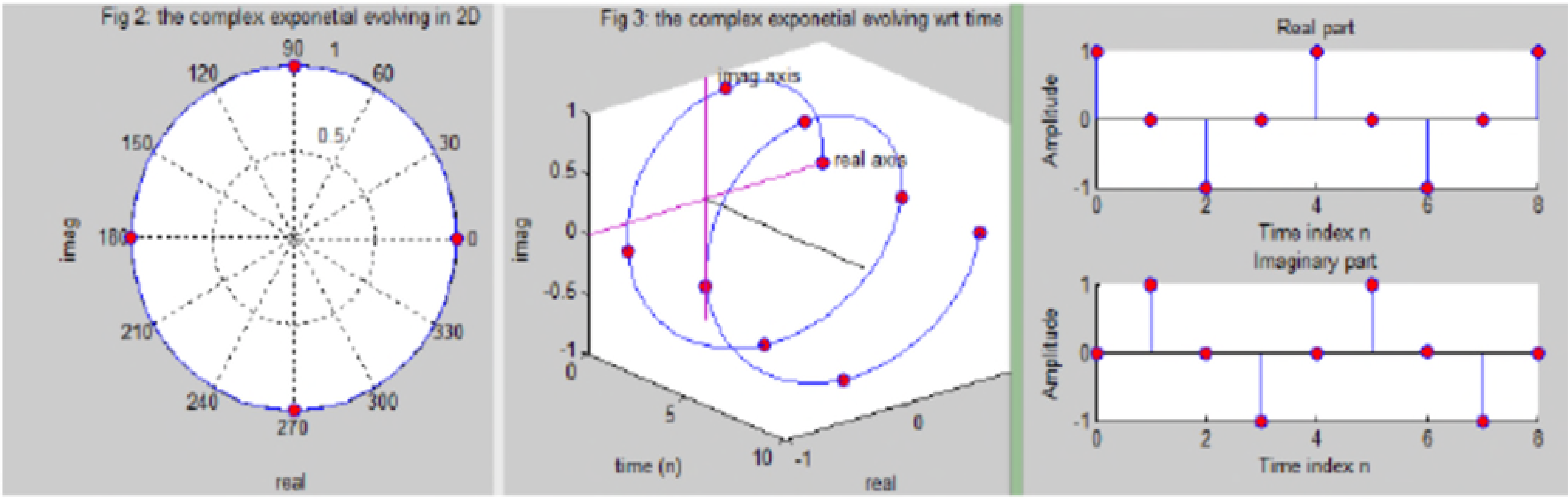
$$W_N^{kn} = e^{-j\frac{2\pi}{N}kn}, \qquad n, k = 0..(N-1)$$

$N = 8,$
$n = 0..(N-1)$

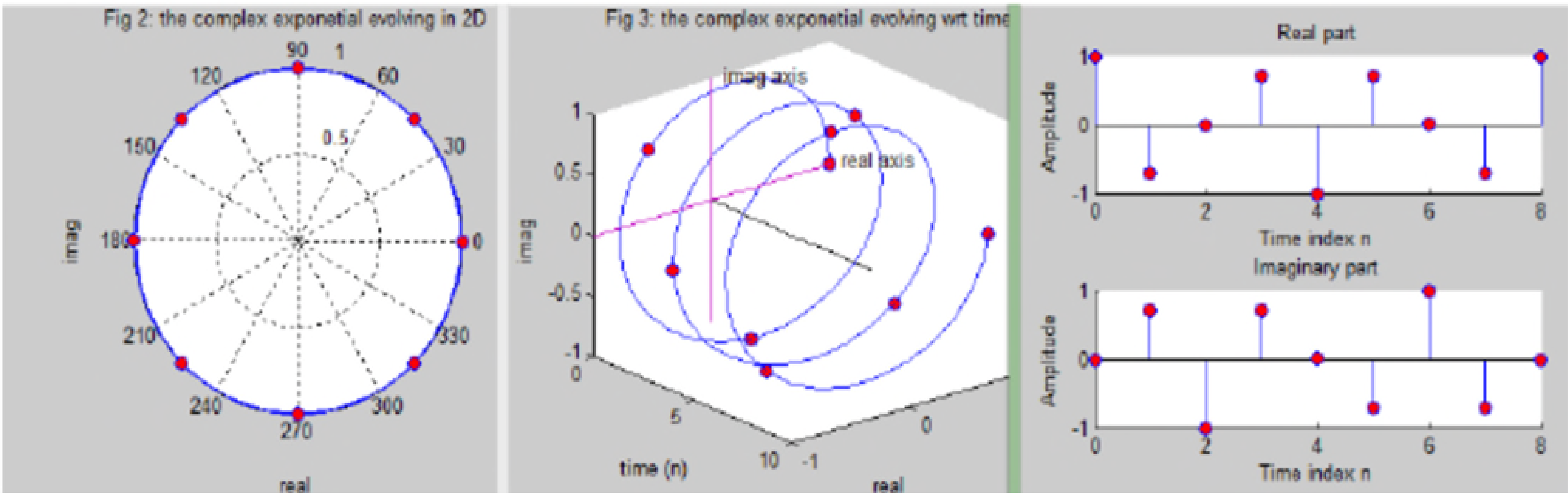
$$W_8^{-1n} = e^{j\frac{2\pi}{8}(1)n}$$



$$W_8^{-2n} = e^{j\frac{2\pi}{8}(2)n}$$



$$W_8^{-3n} = e^{j\frac{2\pi}{8}(3)n}$$



$$W_8^{-1n} = \begin{array}{cc} \begin{bmatrix} 1\angle 0.25\pi\ 0 \\ 1\angle 0.25\pi\ 1 \\ 1\angle 0.25\pi\ 2 \\ 1\angle 0.25\pi\ 3 \\ 1\angle 0.25\pi\ 4 \\ 1\angle 0.25\pi\ 5 \\ 1\angle 0.25\pi\ 6 \\ 1\angle 0.25\pi\ 7 \end{bmatrix} & n = 0..(N-1) \end{array} \qquad W_8^{-2n} = \begin{bmatrix} 1\angle 0.5\pi\ 0 \\ 1\angle 0.5\pi\ 1 \\ 1\angle 0.5\pi\ 2 \\ 1\angle 0.5\pi\ 3 \\ 1\angle 0.5\pi\ 4 \\ 1\angle 0.5\pi\ 5 \\ 1\angle 0.5\pi\ 6 \\ 1\angle 0.5\pi\ 7 \end{bmatrix}$$

$$n = 0..(N-1)$$

8A.3.3 The DFT Analysis Equation and DFT Matrix W

DFT Analysis equation: Finding $X[k]$, the DFT coefficients.

DFT coeff $\boldsymbol{X} = \boldsymbol{W}\,\boldsymbol{x}$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad W_N^{kn} = e^{-j\frac{2\pi}{N}kn}}$$

Discrete Fourier Transform

$\boldsymbol{W}$ is the DFT matrix

Note:

$$\boldsymbol{W}^H\boldsymbol{W} = N\boldsymbol{I}$$

$\boldsymbol{I}$ **= identity matrix**

Because:

1) Columns of W are orthogonal
2) Each element of W has |.| = 1
3) Therefore $w_j^H w_k$ = N (when $j == k$)
   and $w_j^H w_k$ = 0 (when $j \neq k$)

Note:

$\boldsymbol{W}^H$ is the conjugate transpose of $\boldsymbol{W}$

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\text{DFT coeff } \boldsymbol{X}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j(2\pi/N)} & e^{-j(4\pi/N)} & \cdots & e^{-j(2(N-1)\pi/N)} \\ 1 & e^{-j(4\pi/N)} & e^{-j(8\pi/N)} & \cdots & e^{-j(4(N-1)\pi/N)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j(2(N-1)\pi/N)} & e^{-j(4(N-1)\pi/N)} & \cdots & e^{-j(2(N-1)(N-1)\pi/N)} \end{bmatrix}}_{\text{DFT Matrix } \boldsymbol{W} = W_N^{kn}} \cdot \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\text{Signal vector } \boldsymbol{x}}$$

$k = 0..(N-1)$

$n = 0..(N-1)$

$k = 0..(N-1)$

$n = 0..(N-1)$

8A.3.3) Visualizing the W matrix as rotating phasors

W is the Analysis matrix, also commonly known as the Fourier matrix or the DFT matrix.
Each row of W represents the k^{th} harmonic complex exponential sequence.

Visually, the rows of W are rotating clockwise (when no aliasing occurs)
because its digital frequency at the k-th column is $\frac{-2\pi}{N}k$ (rad/sample). Example below is for N = 8

Ref: https://en.wikipedia.org/wiki/DFT_matrix

Fig on bottom left representing the W matrix in polar coordinates to visualize the complex exponential rotation.
The circle is the unit circle.

Note the –ve sign in the complex exponential, this makes the sequence rotates clockwise.
This is different to the synthesis matrix W^H whose columns are rotating anti-clockwise.

$n = 0..(N-1)$

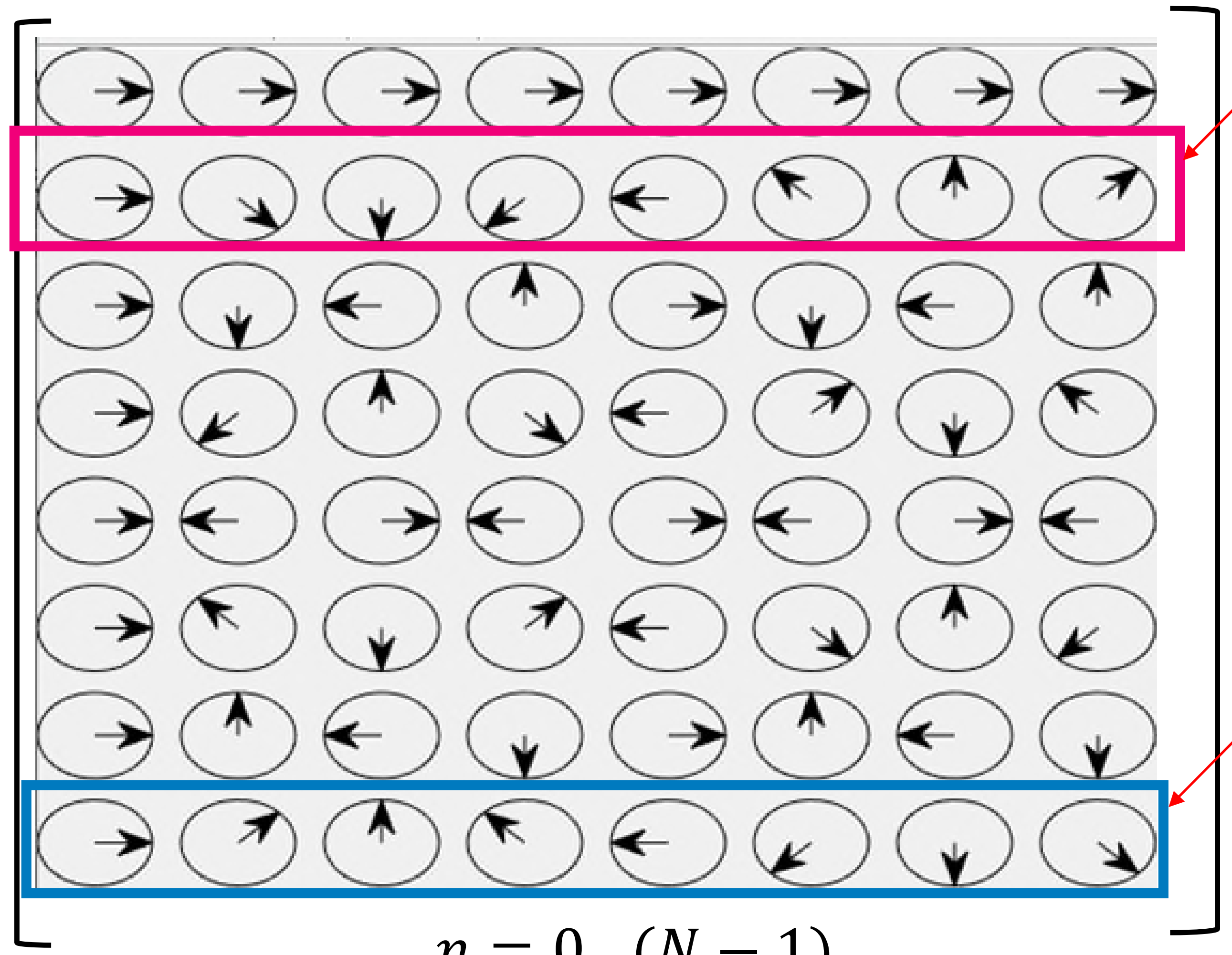
$$W = \begin{bmatrix}
\omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\
\omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\
\omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\
\omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\
\omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\
\omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\
\omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\
\omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49}
\end{bmatrix}$$

$k = 0..(N-1)$

$$w^n = \left(e^{-j\frac{2\pi}{N}}\right)^n$$

-ve radian/sample rotating clockwise

$W =$



$n = 0..(N-1)$

$k = 0..(N-1)$

Row 1 of W

$$(W_8^{1n}) = [w^0, \quad w^1, \quad w^2, \quad w^3, \quad \ldots \;, w^7\ ]$$
$$= [\,1\angle\left(-\frac{\pi}{4}0\right), 1\angle\left(-\frac{\pi}{4}1\right), 1\angle\left(-\frac{\pi}{4}2\right), 1\angle\left(-\frac{\pi}{4}3\right), \ldots, 1\angle\left(-\frac{\pi}{4}7\right)]$$

Row 7 of W (Angle $-\frac{7}{4}\pi$ *same as* $\frac{\pi}{4}$), see that row 1 and row 7 are rotating same frequency BUT opposite directions

$$(W_8^{7n}) = \left[1\angle\left(-\frac{\pi}{4}7.0\right),\ 1\angle\left(-\frac{\pi}{4}7.1\right),\ 1\angle\left(-\frac{\pi}{4}7.2\right),\ 1\angle\left(-\frac{\pi}{4}7.3\right), \ldots, 1\angle(-\frac{\pi}{4}7.7)\right]$$
$$= \left[1\angle\left(\frac{\pi}{4}1.0\right),\quad 1\angle\left(\frac{\pi}{4}1.1\right),\quad 1\angle\left(\frac{\pi}{4}2\right),\quad 1\angle\left(\frac{\pi}{4}3\right),\quad \ldots,\quad 1\angle(\frac{\pi}{4}7)\right]$$

8A.3.3) Example: calculating DFT and IDFT for 4-pt sequence

Calculate the four-point DFT of the aperiodic signal $x[k]$ considered in Example one.

Solution

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j(2\pi/N)} & e^{-j(4\pi/N)} & e^{-j(6\pi/N)} \\ 1 & e^{-j(4\pi/N)} & e^{-j(8\pi/N)} & e^{-j(12\pi/N)} \\ 1 & e^{-j(6\pi/N)} & e^{-j(12\pi/N)} & e^{-j(18\pi/N)} \end{bmatrix}}_{\text{DFT matrix: W}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j(2\pi/4)} & e^{-j(4\pi/4)} & e^{-j(6\pi/4)} \\ 1 & e^{-j(4\pi/4)} & e^{-j(8\pi/4)} & e^{-j(12\pi/4)} \\ 1 & e^{-j(6\pi/4)} & e^{-j(12\pi/4)} & e^{-j(18\pi/4)} \end{bmatrix}}_{\text{DFT matrix: W}} \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3-j2 \\ -3 \\ 3+j2 \end{bmatrix}.$$

Calculate the inverse DFT of $X[r]$ considered in Example two.

Solution

Arranging the values of the DFT coefficients in the DFT vector X, we obtain $X = [5\ 3 - j2\ -3\ \ 3 + j2]$.

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j(2\pi/N)} & e^{j(4\pi/N)} & e^{j(6\pi/N)} \\ 1 & e^{j(4\pi/N)} & e^{j(8\pi/N)} & e^{j(12\pi/N)} \\ 1 & e^{j(6\pi/N)} & e^{j(12\pi/N)} & e^{j(18\pi/N)} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}$$

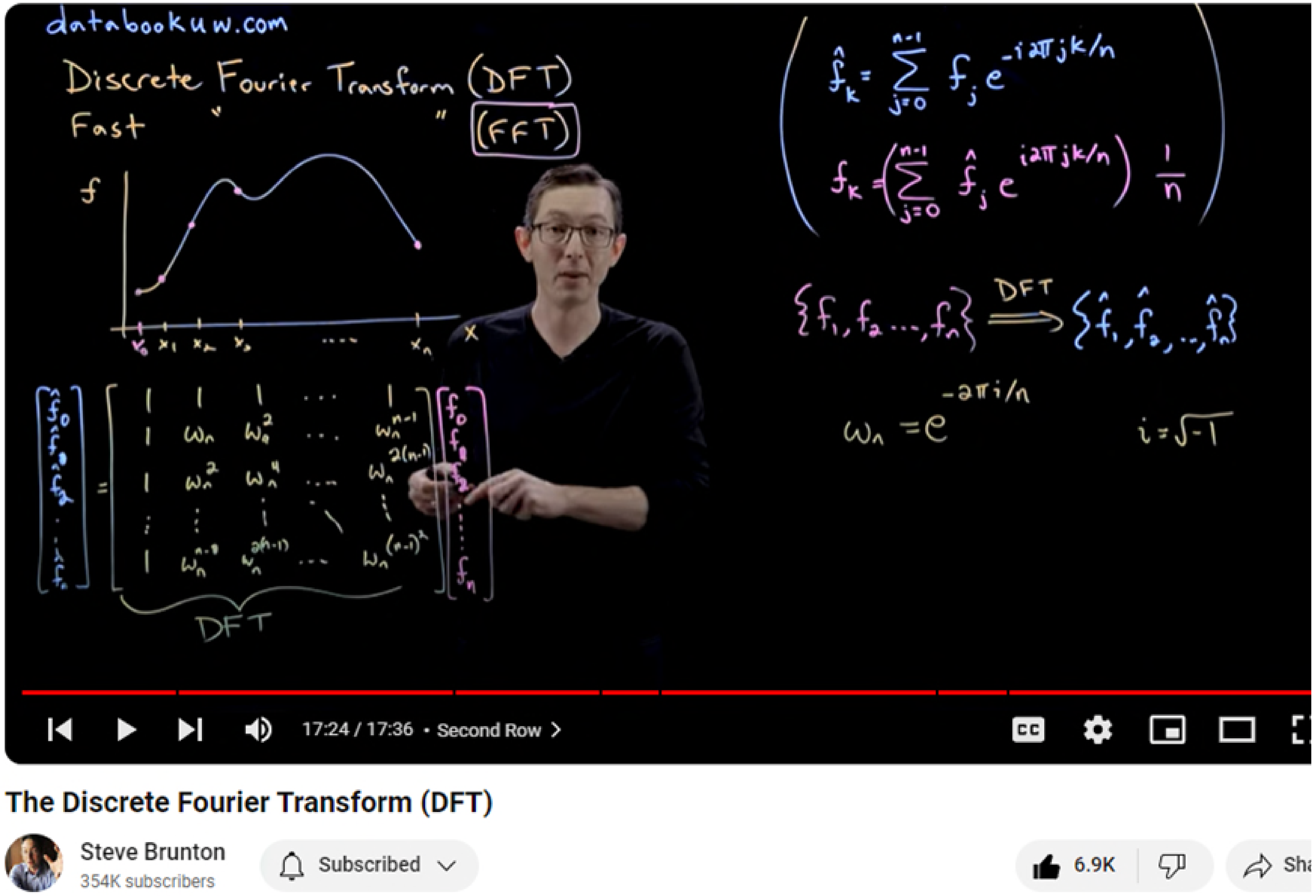
$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j(2\pi/4)} & e^{j(4\pi/4)} & e^{j(6\pi/4)} \\ 1 & e^{j(4\pi/4)} & e^{j(8\pi/4)} & e^{j(12\pi/4)} \\ 1 & e^{j(6\pi/4)} & e^{j(12\pi/4)} & e^{j(18\pi/4)} \end{bmatrix} \begin{bmatrix} 5 \\ 3-j2 \\ -3 \\ 3+j2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ 12 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix}.$$

The above values for the DT sequence x[k] are the same as the ones obtained in Example On the left.

More work examples: Easy Electronics: N=4, Worked example
https://www.youtube.com/watch?v=dE9g72LlPdM

References

Ref: DFT lecture by Brunton



https://youtu.be/nl9TZanwbBk?si=nnJ99MHY64U8NMeV

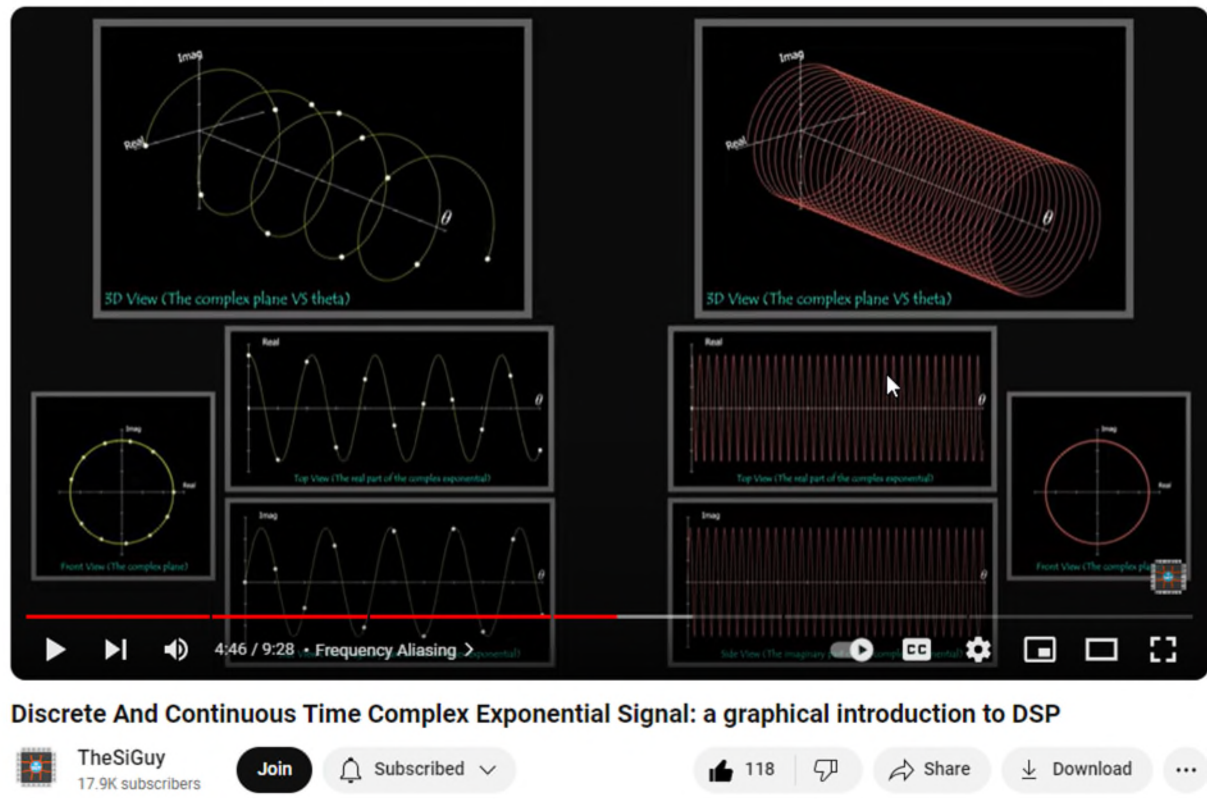
References:

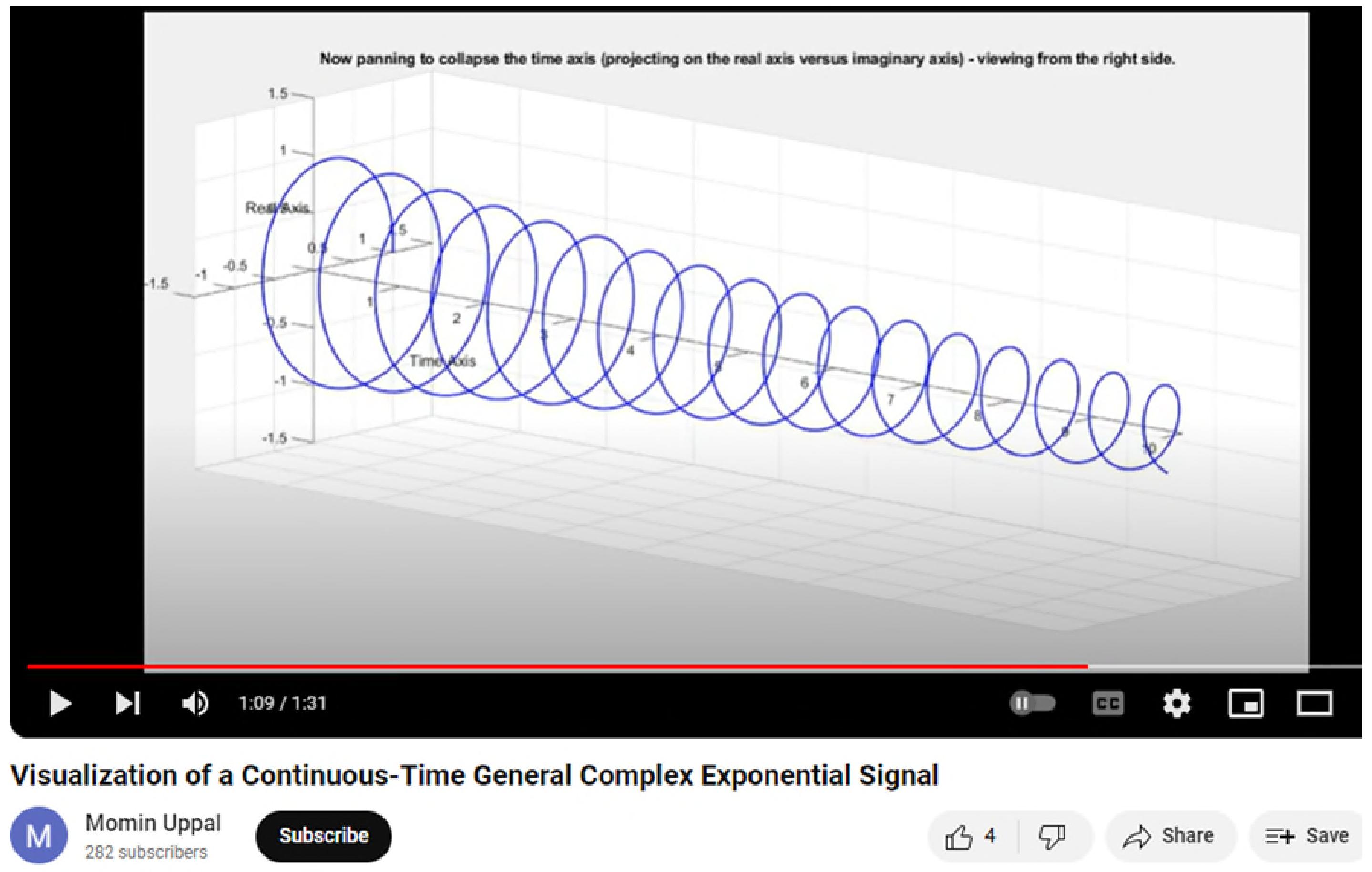
1) DFT Wikipedia: https://en.wikipedia.org/wiki/DFT_matrix
2) UBC Maths: https://ubcmath.github.io/MATH307/dft/dft.html
3) StatLect: https://www.statlect.com/matrix-algebra/discrete-Fourier-transform

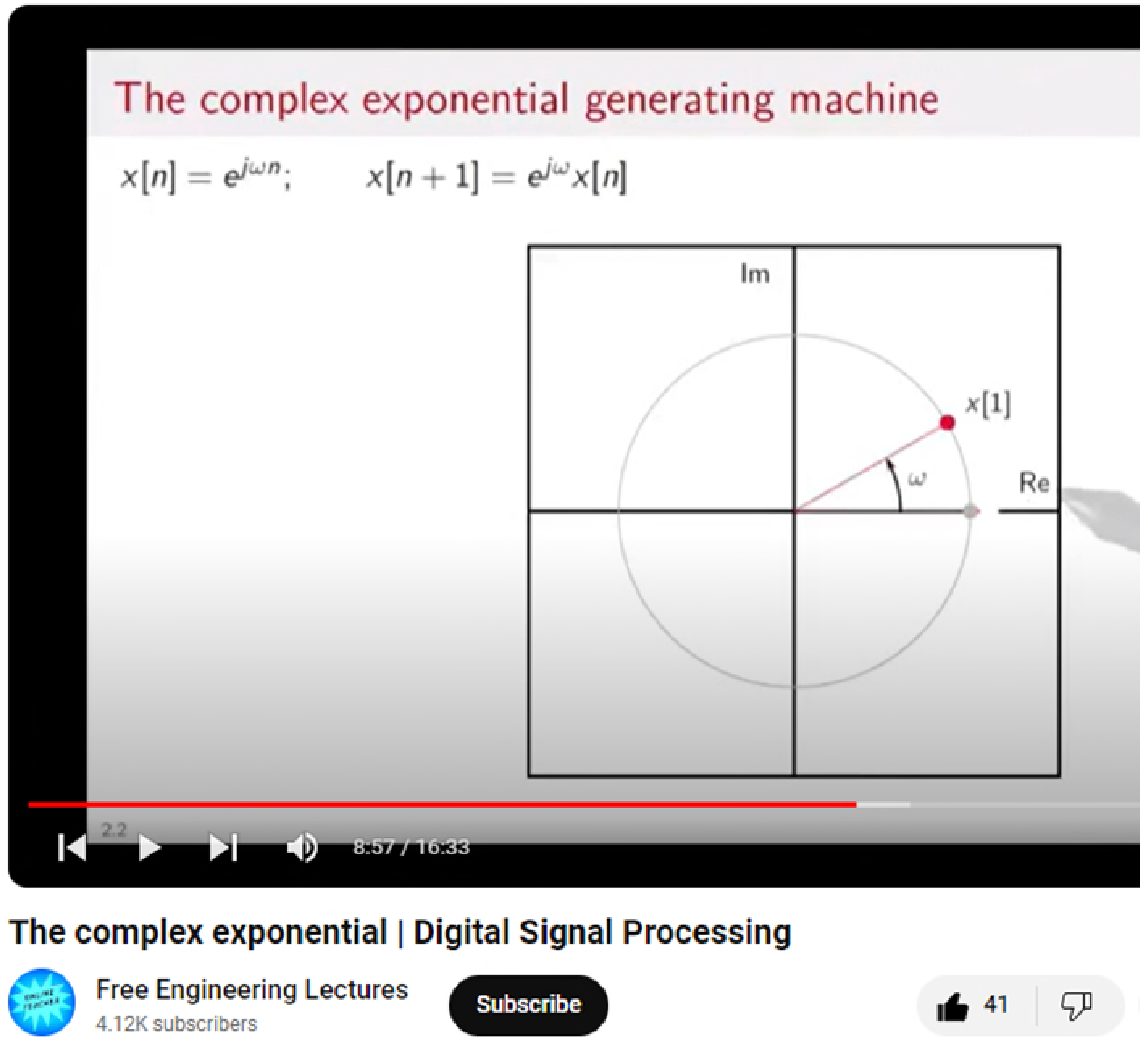
4) Lecture 8 DFT by Lecturer: Dr.Manal Khadhim. University of Iraq
https://uotechnology.edu.iq/dep-electromechanic/typicall/lecture%20interface/lecture/system3-4/4rth_class/SignalsandSystems/lec8.pdf

5)What is DFT matrix : https://www.statlect.com/matrix-algebra/discrete-Fourier-transform
6)What is unitary : https://www.statlect.com/matrix-algebra/unitary-matrix
7) Lecture 8 DFT by Lecturer: Dr.Manal Khadhim. University of Iraq.
https://uotechnology.edu.iq/dep-electromechanic/typicall/lecture%20interface/lecture/system3-4/4rth_class/SignalsandSystems/lec8.pdf

Ref: visualizing complex exponentials







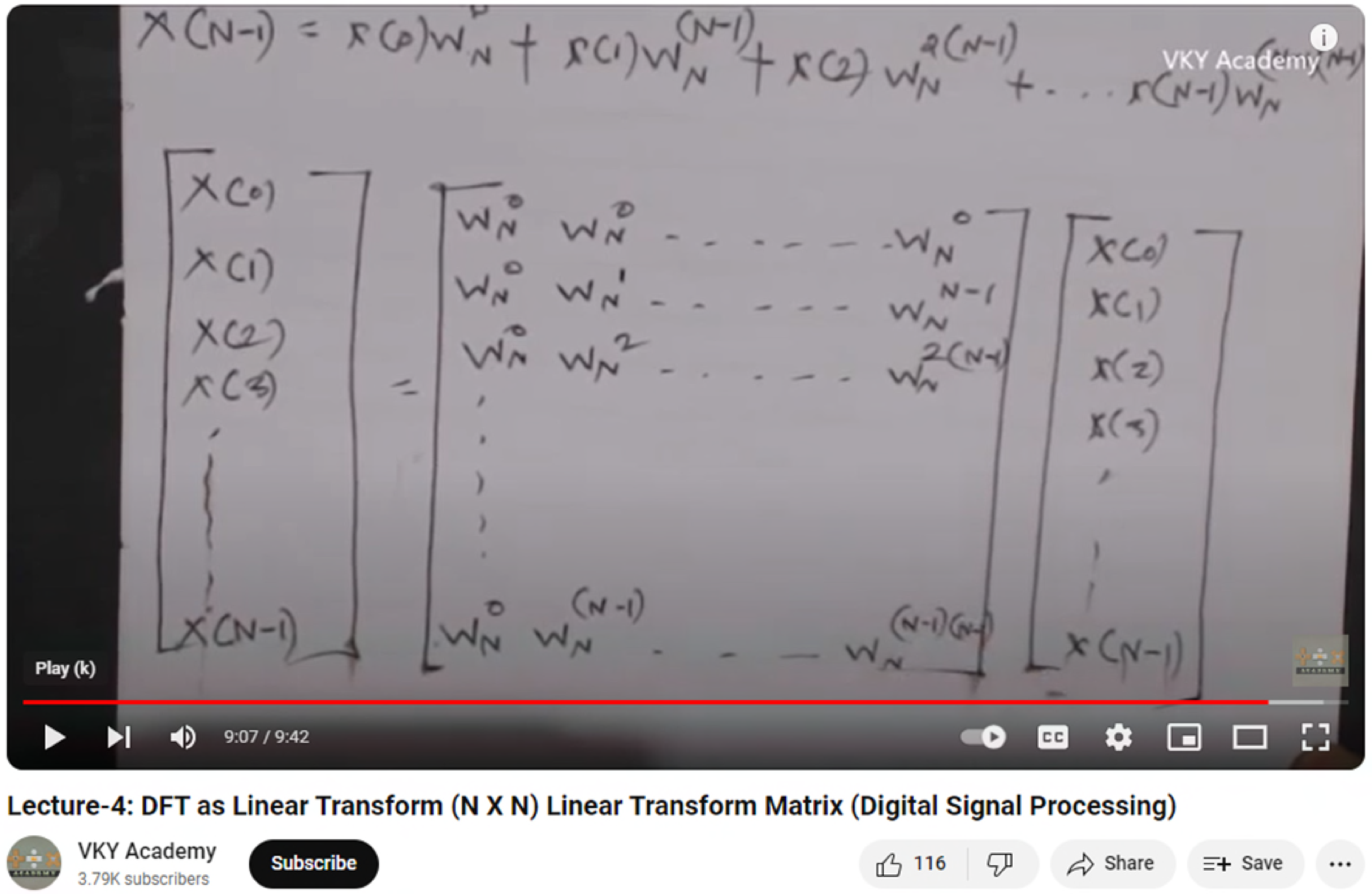
https://youtu.be/gD4Gh_SUUQ4?si=ZO7NgvU4ThgwT88r

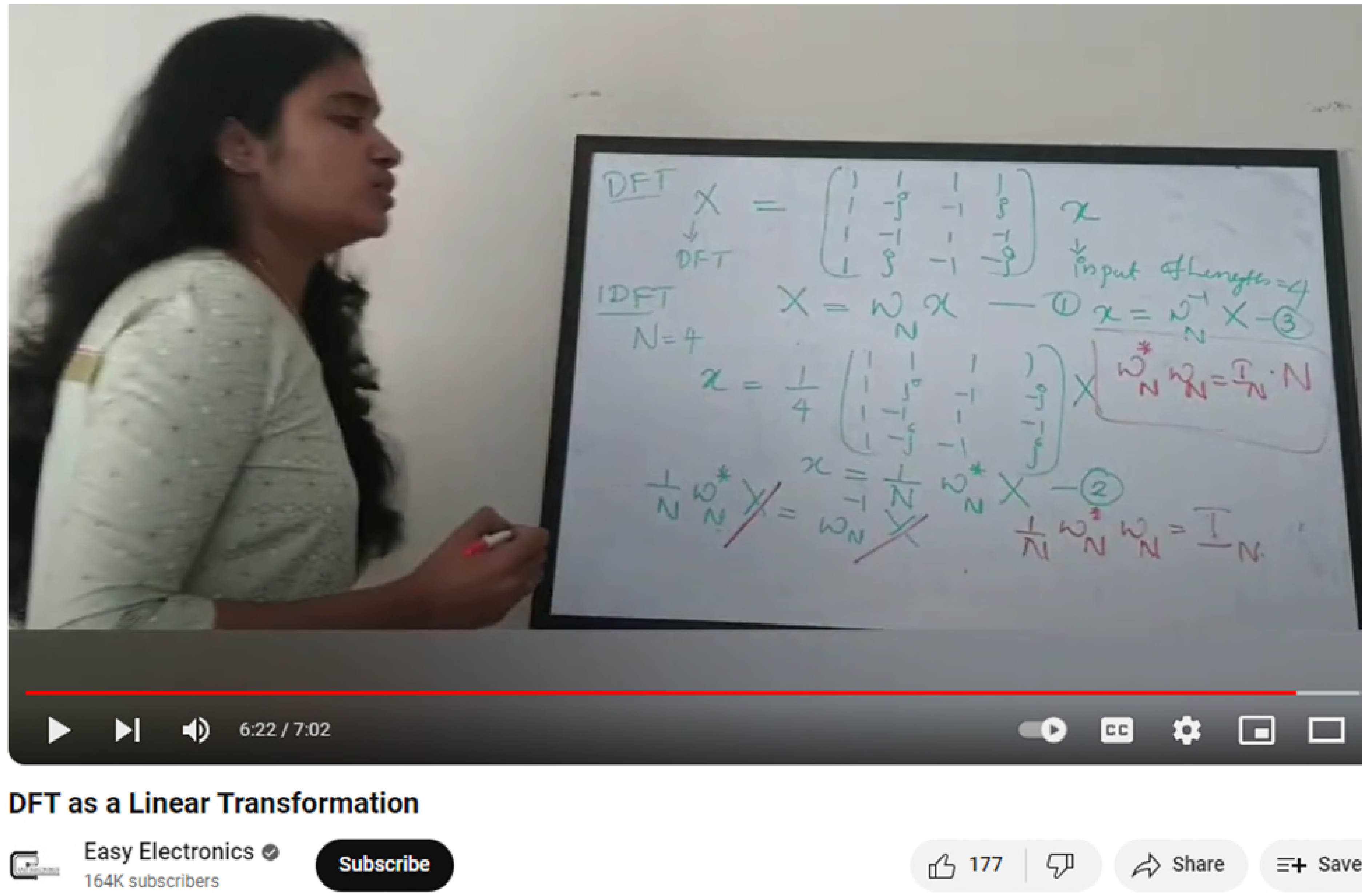
https://www.youtube.com/watch?v=geBSEZJ84Lg
The intuition is great!
But warning: did he draw the complex exponential rotating in wrong direction (?)

Classic explanation by Martin Vetterli

https://www.youtube.com/watch?v=ZzP63nHToa4&list=PLZBpE9maMtF92DepgJDIWCrAhwCFJ_ZPe&index=3

Ref: Video explaining DFT Analysis $X = Wx$





VKY Academy: Deriving matrix form of $X = Wx$
https://www.youtube.com/watch?v=5MRDBYm-Qsw

Easy Electronics: N=4, Worked example
https://www.youtube.com/watch?v=dE9g72LlPdM

Ref: Proof that cols of DFT matrix are orthogonal (1)

Proof that the Fourier basis is orthogonal.

Let b_l, b_k represents the col^s of W

$$b_k = \begin{bmatrix} \uparrow \\ e^{-j\frac{2\pi}{N}kn} \\ \downarrow \end{bmatrix}_{n=0..(N-1)} \in \mathbb{C}^N ; \quad \omega = e^{-j\frac{2\pi}{N}}$$

to show orthogonality,

$$\langle b_l, b_k \rangle = \begin{cases} 0 & \text{if } l \neq k \\ N & \text{if } l == k \end{cases}$$

$$= \sum_{n=0}^{N-1} b_{l,n} \overline{b_{k,n}} = b_k^H b_l$$

where $b_{k,n}$ = element n of col vector b_l.

$\overline{b_{k,n}}$ = conjugate of element n of b_l.

Let $\omega = e^{-j\frac{2\pi}{N}}$, $\omega^{ln} = e^{-j\frac{2\pi}{N}ln}$

$\therefore$

$$\langle b_l, b_k \rangle = \sum_{n=0}^{N-1} b_{l,n} \overline{b_{k,n}} = b_k^H b_l$$

$$= \sum_{n=0}^{N-1} \omega^{ln} \omega^{-kn}$$

$$= \sum_{n=0}^{N-1} \omega^{(l-k)n}$$

(A) if $l == k$, $\therefore$

$$\langle b_l, b_l \rangle = b_l^H b_l = \sum_{n=0}^{N-1} \omega^{0n}$$

$$= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}0}$$

$$= N \quad \#$$

Ref: https://math.stackexchange.com/questions/2413218/proof-of-orthonormality-of-basis-of-dft

Ref: Proof that cols of DFT matrix are orthogonal (2)

3

(B) If $(l-k) \neq 0$

then $\sum_{n=0}^{N-1} \omega^{(l-k)n}$ is a finite geometric series

Proof: let $r = \omega^{(l-k)}$

$$\therefore \sum_{n=0}^{N-1} r^n$$

$$S_{N-1} = r^0 + r^1 + r^2 + \dots + r^{N-1} \quad ; \quad r^0 = 1$$

$$r S_{N-1} = r^1 \quad r^2 + r^3 + \dots + r^N$$

$$\therefore S_{N-1} - r S_{N-1} = 1 - r^N$$

$$\therefore S_n = \frac{1-r^N}{1-r} = \frac{1-\omega^{(l-k)N}}{1-\omega^{(l-k)}}$$

$= 0$ // see proof next page.

4

lets work out $\omega^{(l-k)N}$; $l \neq N$

$$\omega^{(l-k)N} = 1e^{-j\frac{2\pi}{N}(l-k)N} \quad ; \text{ N cancels!}$$

$$= 1e^{-j\underbrace{2\pi(l-k)}_{\neq 0}}$$

$$= 1e^{-j2\pi m} \quad ; \quad m = l-k \neq 0$$

$$= 1 \quad \boxtimes$$

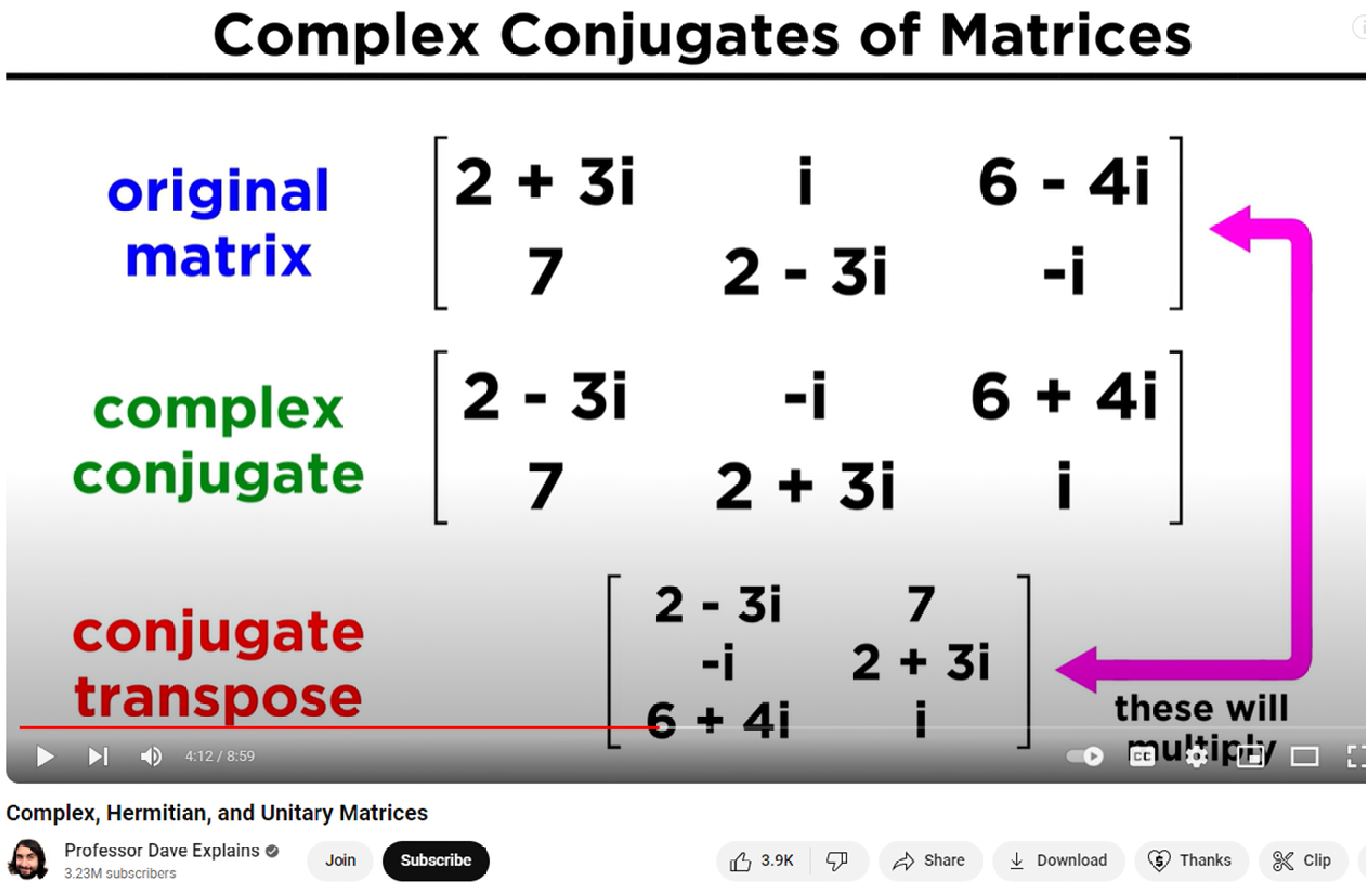
$$\therefore S_n = \frac{1-\omega^{(l-k)N}}{1-\omega^{(l-k)}}$$

$$= \frac{1-1}{1-\underbrace{\omega^{(l-k)}}_{\neq 0}} = \frac{0}{1-\underbrace{1e^{-j\frac{2\pi}{N}(l-k)}}_{\neq 0}}$$

$$= 0. \quad \boxtimes$$

References: conjugate transpose U^H

1) Prof Dave Explains: Complex Matrixes
https://www.youtube.com/watch?v=DUuTx2nbizM



unitary (complex)

- columns form orthonormal vectors

$U^H = U^{-1}$

(conjugate transpose = inverse)

Ref: Basic properties of inner products

Basic properties [edit]

In the following properties, which result almost immediately from the definition of an inner product, x, y and z are arbitrary vectors, and a and b are arbitrary scalars.

- $\langle \mathbf{0}, x\rangle = \langle x, \mathbf{0}\rangle = 0.$
- $\langle x, x\rangle$ is real and nonnegative.
- $\langle x, x\rangle = 0$ if and only if $x = \mathbf{0}.$
- $\langle x, ay + bz\rangle = \overline{a}\langle x, y\rangle + \overline{b}\langle x, z\rangle.$
  This implies that an inner product is a sesquilinear form.
- $\langle x + y, x + y\rangle = \langle x, x\rangle + 2\,\mathrm{Re}(\langle x, y\rangle) + \langle y, y\rangle$, where Re denotes the real part of its argument.

From: Wikipedia
https://en.wikipedia.org/wiki/Inner_product_space#
https://proofwiki.org/wiki/Inner_Product_is_Sesquilinear

Convention variant [edit]

Some authors, especially in physics and matrix algebra, prefer to define inner products and sesquilinear forms with linearity in the second argument rather than the first. Then the first argument becomes conjugate linear, rather than the second. Bra-ket notation in quantum mechanics also uses slightly different notation, i.e. $\langle\cdot|\cdot\rangle$, where $\langle x|y\rangle := (y, x)$.

Example: projecting $y = \alpha x$ into x (proof for pg11)

If $\boldsymbol{y} = \alpha\, \boldsymbol{x}$, what is $Proj_x \boldsymbol{y}$?

$where$: $\alpha \in C,\ x \in C^N$

$$\hat{x} = \frac{x}{||x||}$$

$\hat{x}^H$ = conjugate transpose of $\hat{x}$

$$Proj_x y = ?$$

$$= \alpha \underline{x}$$

Proof

$$= \langle y, \hat{x} \rangle \hat{x}$$

$$= (\hat{x}^H y)\hat{x}$$

$$= (\hat{x}^H \alpha x)\hat{x}$$

$$= \left(\alpha \frac{1}{||x||} x^H x\right)\hat{x}$$

$$= \alpha \frac{||x||^2}{||x||} \frac{x}{||x||}$$

$$= \alpha x.$$