$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **6.1.3**

Lecture : **Orthogonality**

Topic : **Dot Product**

Concept : **The Dot Product**

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

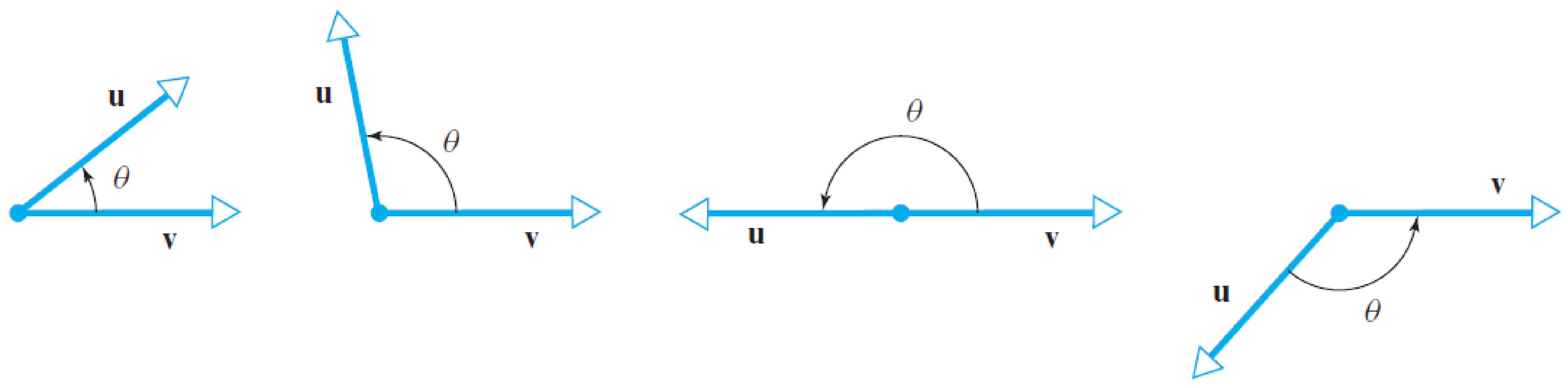
Rev: 25th June 2020

Dot Product

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \le \theta \le \pi$.

How to define "angle" between two vectors in R^2 or R^3? For this purpose, let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in R^2 or R^3 that have been positioned so that their initial points coincide. We define the ***angle between* u *and* v** to be the angle θ determined by $\mathbf{u}$ and $\mathbf{v}$ that satisfies the inequalities $0 \le \theta \le \pi$ (Figure 3.2.4).

The sign of the dot product reveals information about the angle θ that we can obtain by rewriting Formula (12) as

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \tag{13}$$

Since $0 \le \theta \le \pi$, it follows from Formula (13) and properties of the cosine function studied in trigonometry that

- θ is acute if $\mathbf{u} \cdot \mathbf{v} > 0$.
- θ is obtuse if $\mathbf{u} \cdot \mathbf{v} < 0$.
- $\theta = \pi/2$ if $\mathbf{u} \cdot \mathbf{v} = 0$.

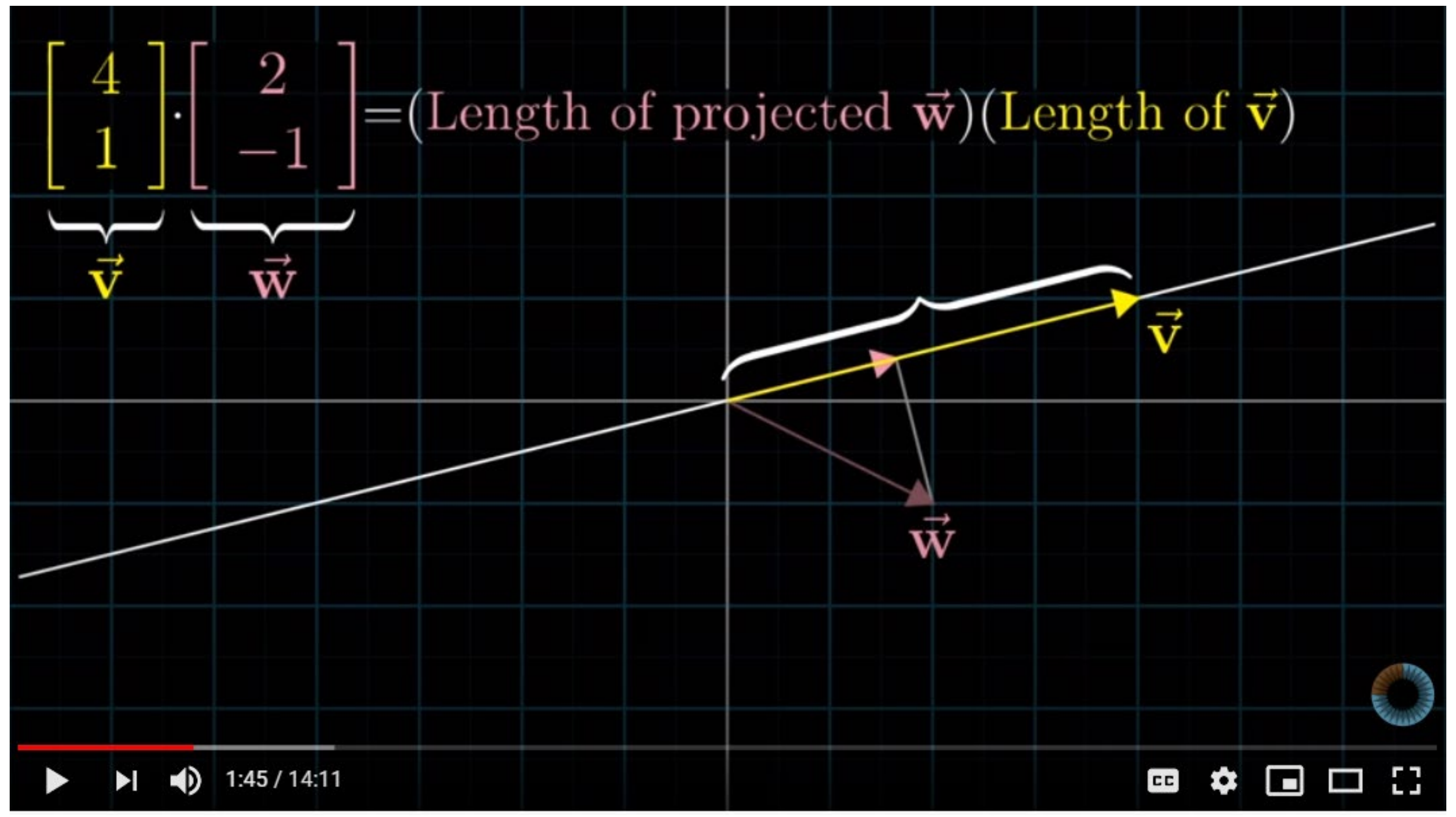
Dot product relates the length of two vectors and the angle (θ) between them.

If vectors u and v are unit vectors,
i.e, $||v|| = ||u|| = 1$,
the dot product is $u.v = \cos\theta$.

Ref:

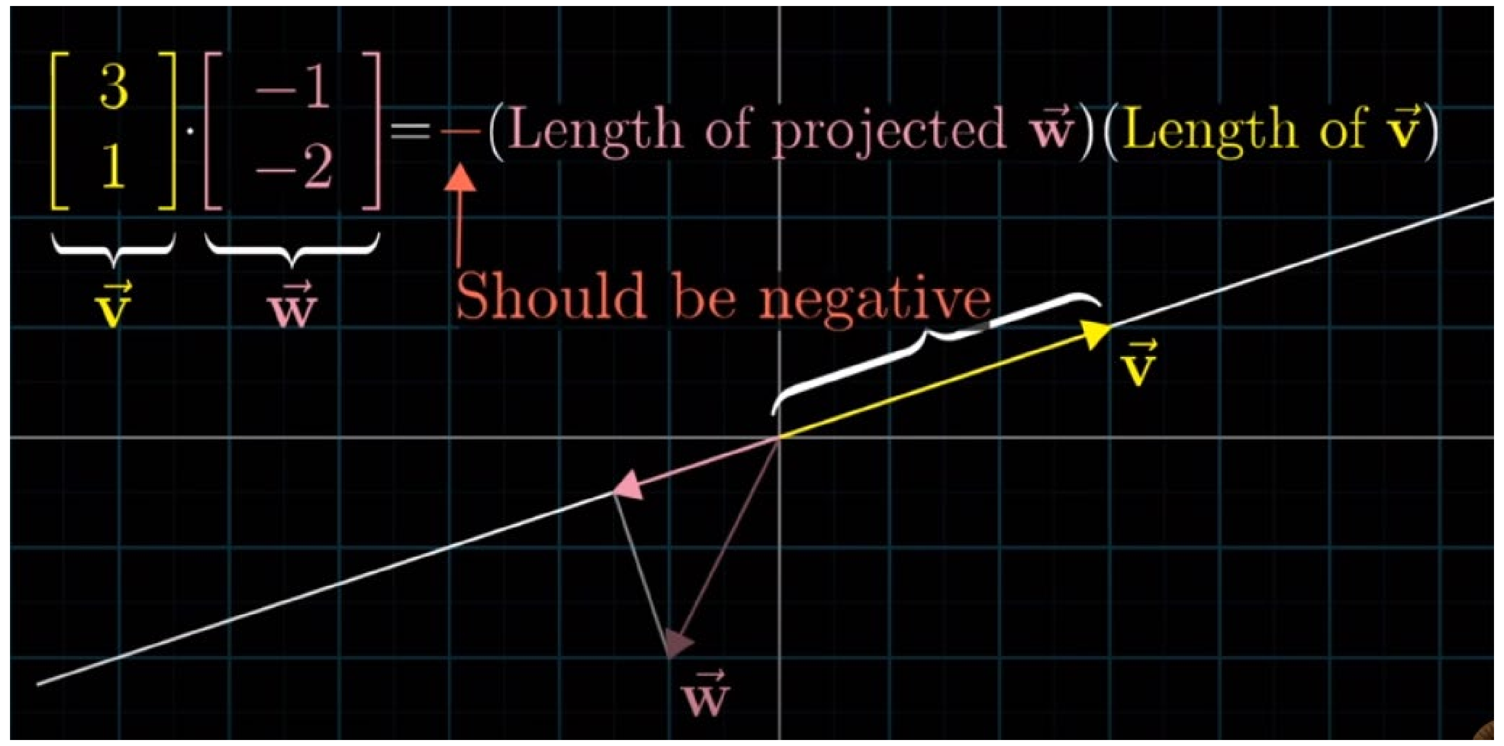
1. Stack Exchange
2. Khan Academy: https://www.youtube.com/watch?v=KDHuWxy53uM
3. 3Blue1Brown, Dot Product and Duality: https://www.youtube.com/watch?v=LyGKycYT2v0
4. MathsTheBeautiful: https://www.youtube.com/watch?v=QPkKWGq_V0U

Dot Product Interpretation



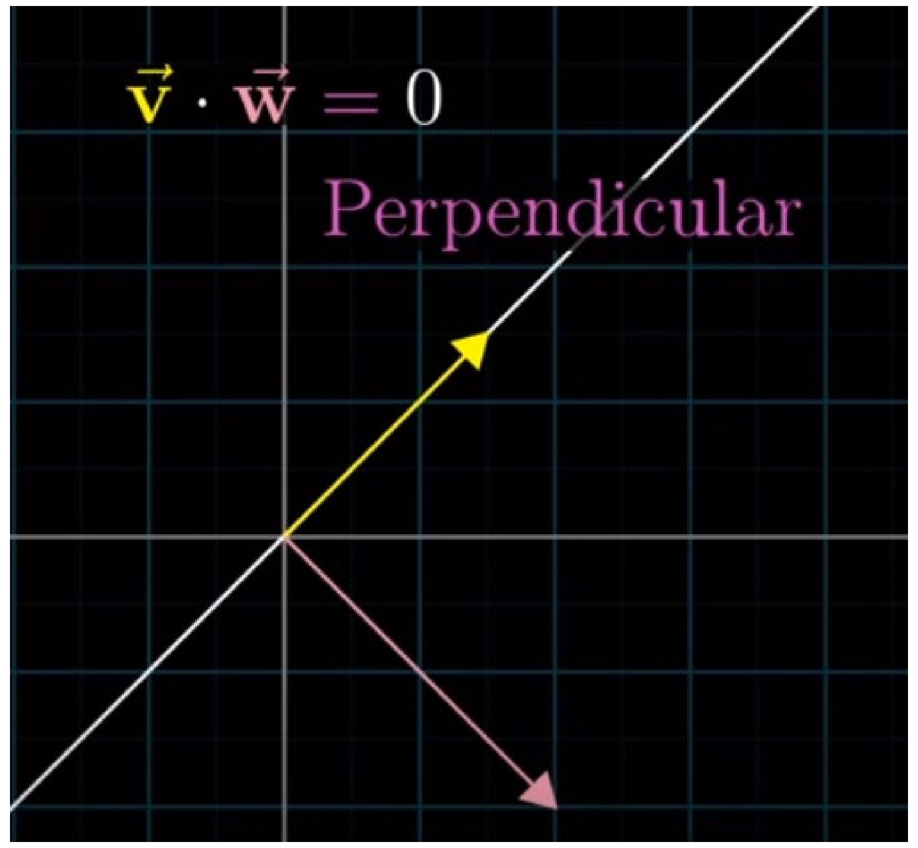
Case 1: When two vectors are pointing nearly towards the same direction, their dot product is +ve.

Angle between vectors: acute.



Case 2: When two vectors are pointing away from one another, their dot product is -ve.

Angle between vectors: obtuse.



Case 3: When two vectors are perpendicular, their dot product is zero.

$$v.w = \sum_{i=1}^{n} v_i w_i = v_1 w_1 + v_2 w_2 + \cdots + v_n$$

$$= ||v|| \times ||w|| \times cos\theta$$

Derived in Slide 5

$$v.w = ||v|| \times ||w|| \times cos\theta$$

$$v.w = \underbrace{(||w|| \times cos\theta)}_{\text{Length of projection of } w \text{ onto } v} \times \underbrace{||v||}_{\text{Length of vector } v}$$

$$v.w = \underbrace{(||v|| \times cos\theta)}_{\text{Length of projection of } v \text{ onto } w} \times \underbrace{||w||}_{\text{Length of vector } w}$$

Example

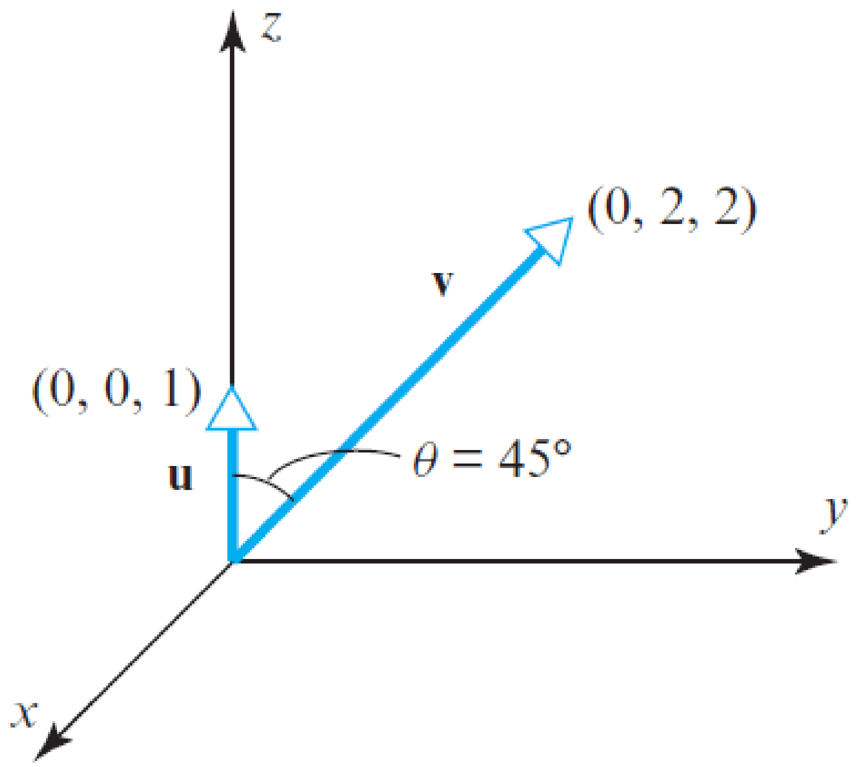


Figure 3.2.5

EXAMPLE 5 Dot Product

Find the dot product of the vectors shown in Figure 3.2.5.

Solution The lengths of the vectors are

$$\|\mathbf{u}\| = 1 \quad \text{and} \quad \|\mathbf{v}\| = \sqrt{8} = 2\sqrt{2}$$

and the cosine of the angle θ between them is

$$\cos(45^\circ) = 1/\sqrt{2}$$

Thus, it follows from Formula (12) that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = (1)(2\sqrt{2})(1/\sqrt{2}) = 2$$

Component Form of the Dot Product

Component Form of the Dot Product

For computational purposes it is desirable to have a formula that expresses the dot product of two vectors in terms of components. We will derive such a formula for vectors in 3-space; the derivation for vectors in 2-space is similar.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be two nonzero vectors. If, as shown in Figure 3.2.6, θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the law of cosines yields

$$\|\overrightarrow{PQ}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \tag{14}$$

Since $\overrightarrow{PQ} = \mathbf{v} - \mathbf{u}$, we can rewrite (14) as

$$\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2)$$

or

$$\mathbf{u}\cdot\mathbf{v} = \tfrac{1}{2}(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2)$$

Substituting

$$\|\mathbf{u}\|^2 = u_1^2 + u_2^2 + u_3^2, \qquad \|\mathbf{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

and

$$\|\mathbf{v}-\mathbf{u}\|^2 = (v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2$$

we obtain, after simplifying,

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 \tag{15}$$

The companion formula for vectors in 2-space is

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 \tag{16}$$

Motivated by the pattern in Formulas (15) and (16), we make the following definition.

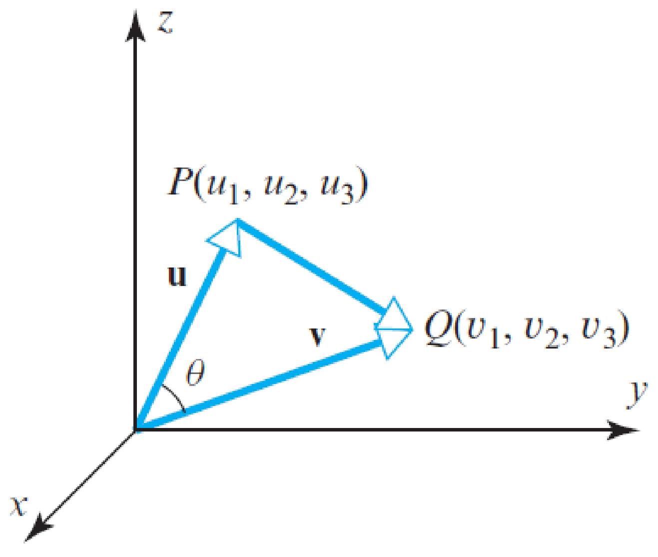


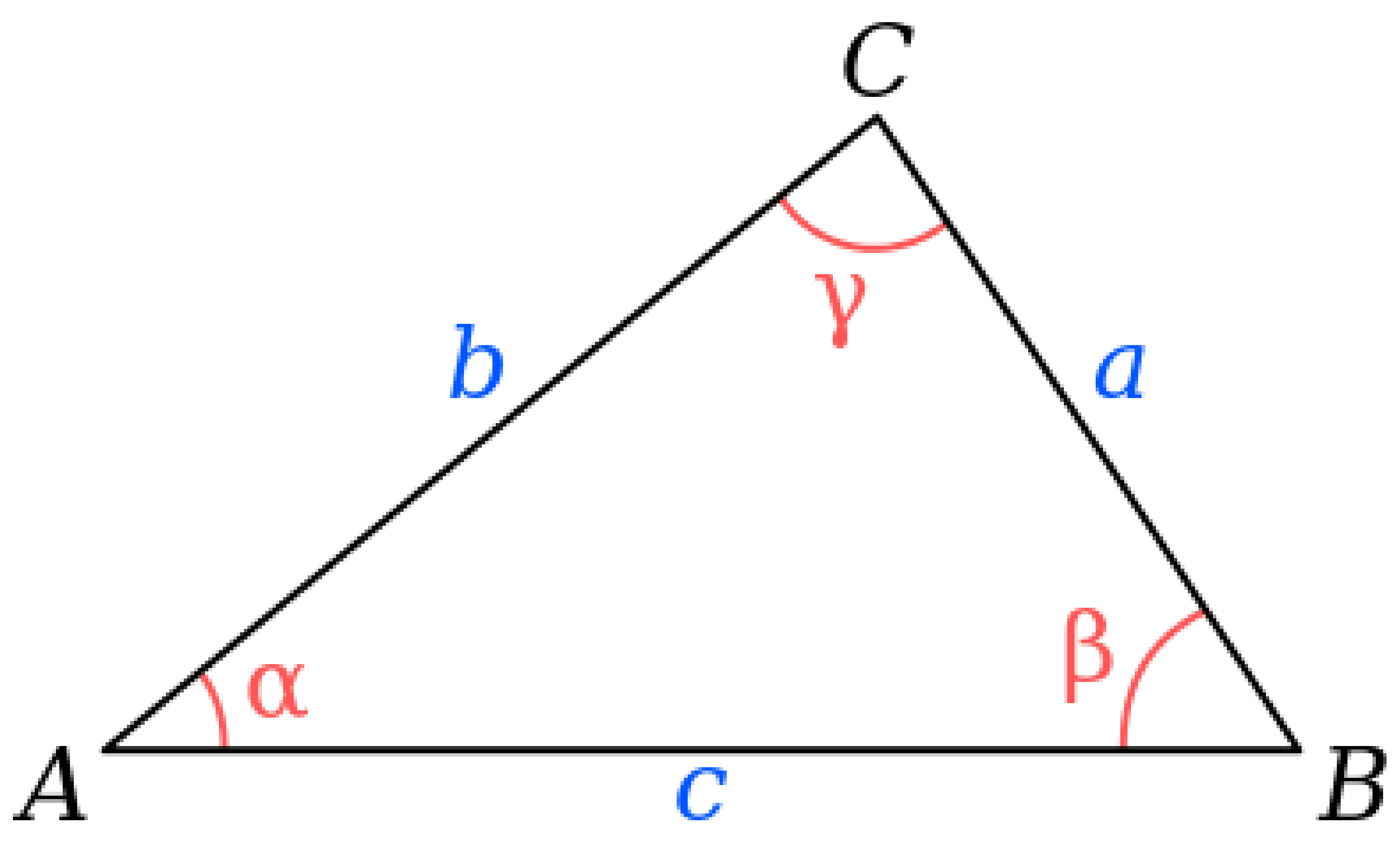
Figure 3.2.6

Although we derived Formula (15) and its 2-space companion under the assumption that $\mathbf{u}$ and $\mathbf{v}$ are nonzero, it turned out that these formulas are also applicable if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ (verify).

DEFINITION 4 If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u}\cdot\mathbf{v}$ and is defined by

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n \tag{17}$$

Reviewing Law of Cosines



$$c^2 = a^2 + b^2 - 2ab\cos\gamma,$$

Example

▶ EXAMPLE 6 **Calculating Dot Products Using Components**

(a) Use Formula (15) to compute the dot product of the vectors **u** and **v** in Example 5.

(b) Calculate $\mathbf{u} \cdot \mathbf{v}$ for the following vectors in R^4:

$$\mathbf{u} = (-1, 3, 5, 7), \quad \mathbf{v} = (-3, -4, 1, 0)$$

Solution (a) The component forms of the vectors are $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (0, 2, 2)$. Thus,

$$\mathbf{u} \cdot \mathbf{v} = (0)(0) + (0)(2) + (1)(2) = 2$$

which agrees with the result obtained geometrically in Example 5.

Solution (b)

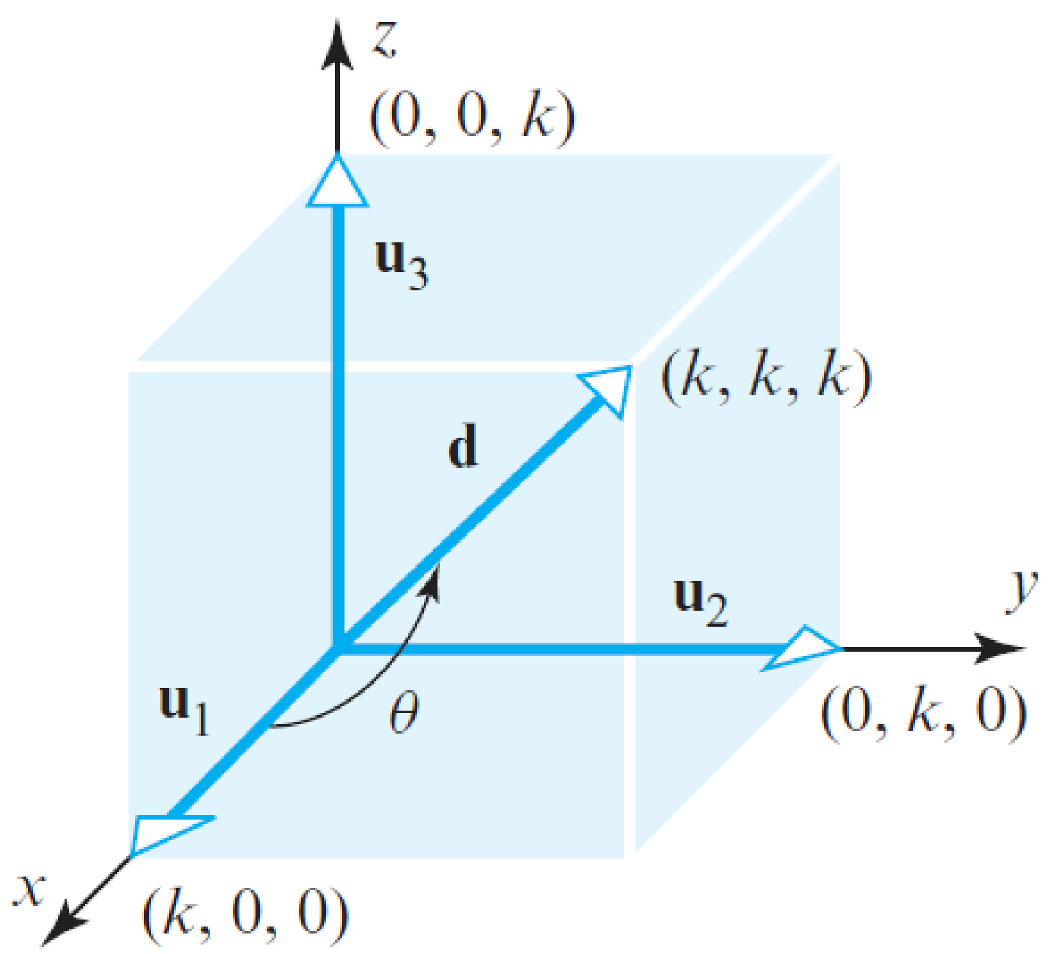
$$\mathbf{u} \cdot \mathbf{v} = (-1)(-3) + (3)(-4) + (5)(1) + (7)(0) = -4$$

In the special case where $\mathbf{u} = \mathbf{v}$ in Definition 4, we obtain the relationship

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2 \tag{18}$$

This yields the following formula for expressing the length of a vector in terms of a dot product:

$$\boxed{\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}} \tag{19}$$



▲ Figure 3.2.7

Note that the angle θ obtained in Example 7 does not involve k. Why was this to be expected?

▶ EXAMPLE 7 **A Geometry Problem Solved Using Dot Product**

Find the angle between a diagonal of a cube and one of its edges.

Solution Let k be the length of an edge and introduce a coordinate system as shown in Figure 3.2.7. If we let $\mathbf{u}_1 = (k, 0, 0)$, $\mathbf{u}_2 = (0, k, 0)$, and $\mathbf{u}_3 = (0, 0, k)$, then the vector

$$\mathbf{d} = (k, k, k) = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

is a diagonal of the cube. It follows from Formula (13) that the angle θ between **d** and the edge $\mathbf{u}_1$ satisfies

$$\cos\theta = \frac{\mathbf{u}_1 \cdot \mathbf{d}}{\|\mathbf{u}_1\|\|\mathbf{d}\|} = \frac{k^2}{(k)(\sqrt{3k^2})} = \frac{1}{\sqrt{3}}$$

With the help of a calculator we obtain

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74° \blacktriangleleft$$

Properties of Dot Product

Dot products have many of the same algebraic properties as products of real numbers.

THEOREM 3.2.2 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ [Symmetry property]

(*b*) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$ [Distributive property]

(*c*) $k(\mathbf{u}\cdot\mathbf{v})=(k\mathbf{u})\cdot\mathbf{v}$ [Homogeneity property]

(*d*) $\mathbf{v}\cdot\mathbf{v}\geq 0$ *and* $\mathbf{v}\cdot\mathbf{v}=0$ *if and only if* $\mathbf{v}=\mathbf{0}$ [Positivity property]

Proof (c) Let $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ and $\mathbf{v}=(v_1,v_2,\ldots,v_n)$. Then

$$\begin{aligned}k(\mathbf{u}\cdot\mathbf{v})&=k(u_1v_1+u_2v_2+\cdots+u_nv_n)\\&=(ku_1)v_1+(ku_2)v_2+\cdots+(ku_n)v_n=(k\mathbf{u})\cdot\mathbf{v}\end{aligned}$$

Proof (d) The result follows from parts (*a*) and (*b*) of Theorem 3.2.1 and the fact that

$$\mathbf{v}\cdot\mathbf{v}=v_1v_1+v_2v_2+\cdots+v_nv_n=v_1^2+v_2^2+\cdots+v_n^2=\|\mathbf{v}\|^2 \blacktriangleleft$$

THEOREM 3.2.3 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n, *and if* k *is a scalar, then:*

(*a*) $\mathbf{0}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{0}=0$

(*b*) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$

(*c*) $\mathbf{u}\cdot(\mathbf{v}-\mathbf{w})=\mathbf{u}\cdot\mathbf{v}-\mathbf{u}\cdot\mathbf{w}$

(*d*) $(\mathbf{u}-\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}-\mathbf{v}\cdot\mathbf{w}$

(*e*) $k(\mathbf{u}\cdot\mathbf{v})=\mathbf{u}\cdot(k\mathbf{v})$

Proof (b)

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}&=\mathbf{w}\cdot(\mathbf{u}+\mathbf{v}) && \text{[By symmetry]}\\&=\mathbf{w}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{v} && \text{[By distributivity]}\\&=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w} && \text{[By symmetry]} \blacktriangleleft\end{aligned}$$

▶ **EXAMPLE 8 Calculating with Dot Products**

$$\begin{aligned}(\mathbf{u}-2\mathbf{v})\cdot(3\mathbf{u}+4\mathbf{v})&=\mathbf{u}\cdot(3\mathbf{u}+4\mathbf{v})-2\mathbf{v}\cdot(3\mathbf{u}+4\mathbf{v})\\&=3(\mathbf{u}\cdot\mathbf{u})+4(\mathbf{u}\cdot\mathbf{v})-6(\mathbf{v}\cdot\mathbf{u})-8(\mathbf{v}\cdot\mathbf{v})\\&=3\|\mathbf{u}\|^2-2(\mathbf{u}\cdot\mathbf{v})-8\|\mathbf{v}\|^2 \blacktriangleleft\end{aligned}$$

Property and Example

Table 1

Form	Dot Product	Example	
$\mathbf{u}$ a column matrix and $\mathbf{v}$ a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=\mathbf{v}^T\mathbf{u}$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}^T\mathbf{v}=[1 \quad -3 \quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}=[5 \quad 4 \quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$

If A is an $n \times n$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices, then it follows from the first row in Table 1 and properties of the transpose that

$$A\mathbf{u}\cdot\mathbf{v}=\mathbf{v}^T(A\mathbf{u})=(\mathbf{v}^TA)\mathbf{u}=(A^T\mathbf{v})^T\mathbf{u}=\mathbf{u}\cdot A^T\mathbf{v}$$
$$\mathbf{u}\cdot A\mathbf{v}=(A\mathbf{v})^T\mathbf{u}=(\mathbf{v}^TA^T)\mathbf{u}=\mathbf{v}^T(A^T\mathbf{u})=A^T\mathbf{u}\cdot\mathbf{v}$$

The resulting formulas

$$A\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot A^T\mathbf{v} \tag{26}$$

$$\mathbf{u}\cdot A\mathbf{v}=A^T\mathbf{u}\cdot\mathbf{v} \tag{27}$$

provide an important link between multiplication by an $n \times n$ matrix A and multiplication by A^T.

EXAMPLE 9 Verifying that $A\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot A^T\mathbf{v}$

Suppose that

$$A=\begin{bmatrix}1 & -2 & 3\\2 & 4 & 1\\-1 & 0 & 1\end{bmatrix},\quad \mathbf{u}=\begin{bmatrix}-1\\2\\4\end{bmatrix},\quad \mathbf{v}=\begin{bmatrix}-2\\0\\5\end{bmatrix}$$

Then

$$A\mathbf{u}=\begin{bmatrix}1 & -2 & 3\\2 & 4 & 1\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}-1\\2\\4\end{bmatrix}=\begin{bmatrix}7\\10\\5\end{bmatrix}$$

$$A^T\mathbf{v}=\begin{bmatrix}1 & 2 & -1\\-2 & 4 & 0\\3 & 1 & 1\end{bmatrix}\begin{bmatrix}-2\\0\\5\end{bmatrix}=\begin{bmatrix}-7\\4\\-1\end{bmatrix}$$

from which we obtain

$$A\mathbf{u}\cdot\mathbf{v}=7(-2)+10(0)+5(5)=11$$
$$\mathbf{u}\cdot A^T\mathbf{v}=(-1)(-7)+2(4)+4(-1)=11$$

Thus, $A\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot A^T\mathbf{v}$ as guaranteed by Formula (26). We leave it for you to verify that Formula (27) also holds.