

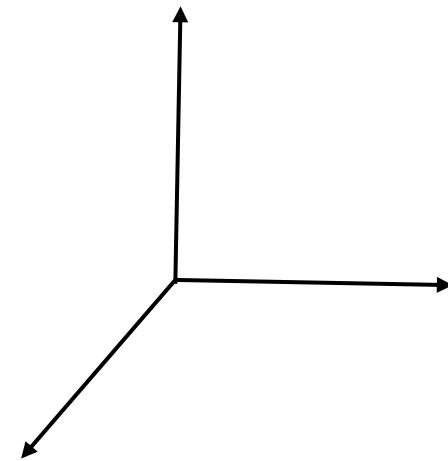
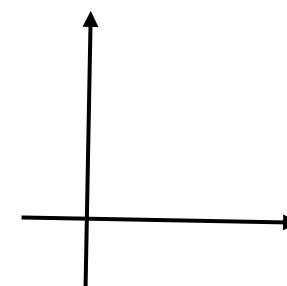
Vector Spaces

Expanding the definition of vectors and the spaces in which they live

Beyond :



$$\begin{bmatrix} 4 \\ 9 \\ 32.5 \end{bmatrix}$$



Overview and Learning Outcomes

- Vector spaces
 - Determine whether a given set with two operations is a vector space
- Subspaces
 - Determine whether a subset of a vector space is a subspace
 - Determine whether a set S of vectors in \mathbb{R}^n span \mathbb{R}^n
- Basis
 - Show that a set of vectors is a basis for a vector space
 - Find the coordinates of a vector relative to a basis

Overview and Learning Outcomes

- Dimension
 - Find a basis for and dimension of null space and column space of a matrix
- Rank
 - Find the rank of a matrix
 - Find the dimension of the row space of a matrix
- Linear Transformation
 - Determine whether a transformation is linear
 - Find the standard matrix for a linear transformation

4.1 Vector spaces

Definition. A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all vectors \mathbf{u} , \mathbf{v} and \mathbf{w} and for all scalars c and d .

1. The sum of \mathbf{u} and \mathbf{v} , denoted by $\mathbf{u} + \mathbf{v}$ is in V .
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each \mathbf{u} in V , there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

6. The scalar multiple of \mathbf{u} by c , denoted by $c\mathbf{u}$ is in V .
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Following simple facts can be proved from the axioms: For each \mathbf{u} in V and scalar c ,

- $0\mathbf{u} = \mathbf{0}$.
- $c\mathbf{0} = \mathbf{0}$.
- $-\mathbf{u} = (-1)\mathbf{u}$.

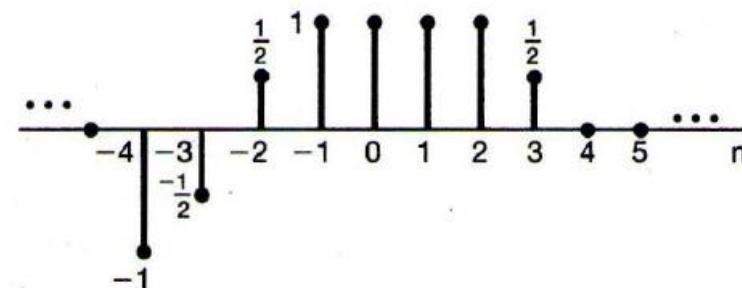
Examples of vector spaces

1. \mathbb{R}^n
2. \mathbb{S} : space of all doubly infinite sequences of numbers

$$\{y_k\} = (\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots)$$

Define addition: If $\{z_k\}$ is another element in \mathbb{S} , then sum $\{y_k\} + \{z_k\}$ is the sequence $\{y_k + z_k\}$ formed by adding corresponding terms of $\{y_k\}$ and $\{z_k\}$.

Define scalar multiplication: The scalar multiple $c\{y_k\}$ is the sequence $\{cy_k\}$.



Examples of vector spaces (contd.)

up to
↓
3. \mathbb{P}_n : polynomials of degree n , ($n \geq 0$)

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

Degree : highest power of t in \mathbf{p}
 If $\mathbf{p}(t) = a_0 \neq 0$, degree of \mathbf{p}
 is zero.

If all the coefficients are zero, \mathbf{p} is called the *zero polynomial*.

Define addition: If $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$, then the sum $\mathbf{p} + \mathbf{q}$ is defined by

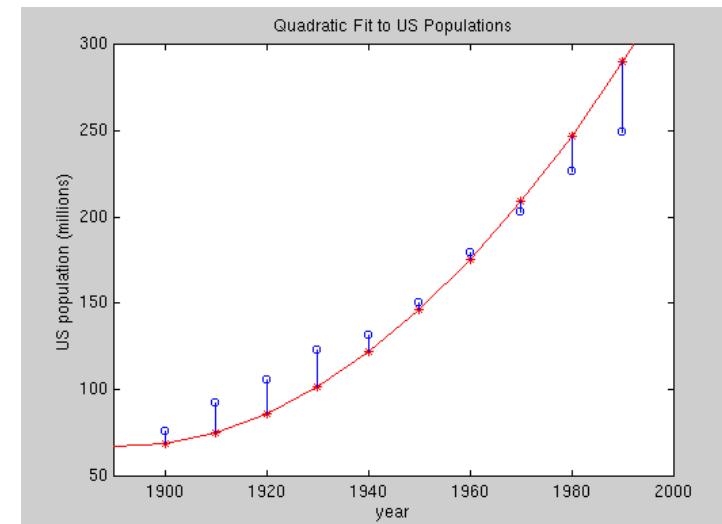
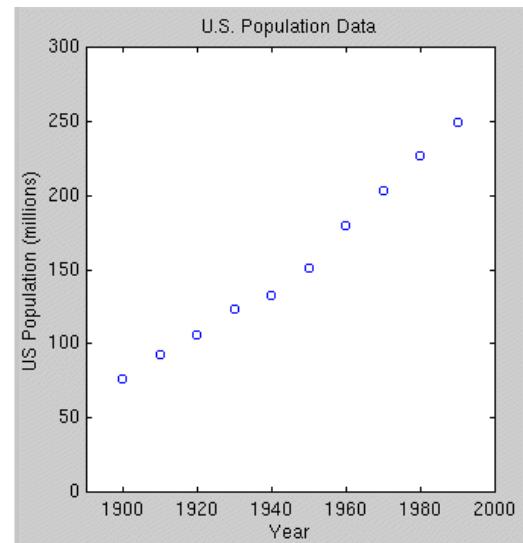
$$\begin{aligned} (\mathbf{p} + \mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n \end{aligned}$$

Define scalar multiplication: The scalar multiple $c\mathbf{p}$ is the polynomial

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = ca_0 + (ca_1)t + \cdots + (ca_n)t^n$$

- Axioms 1 and 6: satisfied because $\mathbf{p} + \mathbf{q}$ and $c\mathbf{p}$ are polynomials of degree less than or equal to n
- Axioms 2, 3, and 7-10: satisfied because of properties of real numbers
- Axiom 4: zero polynomial acts as the zero vector
- Axiom 5: $(-1)\mathbf{p}$ acts as the negative of \mathbf{p}

Non-linear curve fitting



Examples of vector spaces (contd.)

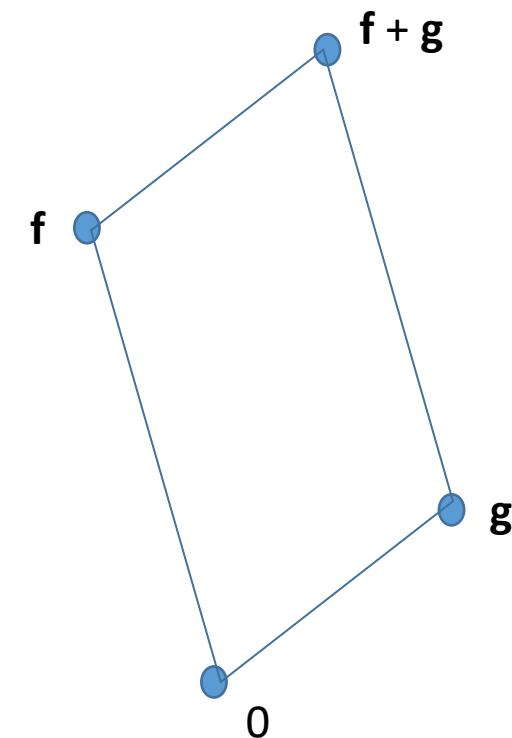
4. Let V be the set of all real-valued functions defined on a set \mathbb{D}

Define addition: $\mathbf{f} + \mathbf{g}$ is the function whose value at t in the domain \mathbb{D} is $\mathbf{f}(t) + \mathbf{g}(t)$.

Define scalar multiplication: The scalar multiple $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$.

- Axioms 1 and 6: obvious
- Axiom 4: zero vector is the function that is identically zero, $\mathbf{f}(t) = 0$
- Axiom 5: Negative of \mathbf{f} is $(-1)\mathbf{f}$
- Rest of the axioms: satisfied because of properties of real numbers

Think of an element of a vector space as one “point” or vector in the vector space



4.2 Subspaces

Definition. A **subspace** of a vector space V is a subset H of V that has three properties:

- a. The zero vector of V is in H .
- b. H is closed under vector addition, i.e., for each \mathbf{u} and \mathbf{v} in H , the sum $\mathbf{u} + \mathbf{v}$ is in H .
- c. H is closed under scalar multiplication, i.e., for each \mathbf{u} in H and each scalar c , the vector $c\mathbf{u}$ is in H .

In applications, subspaces of \mathbb{R}^n arise

- as the set of all solutions to a system of homogeneous linear equations
- as the set of all linear combinations of certain specified vectors

Subspace H of V is itself a *vector space* under vector space operations already defined in V .

- Axioms 1, 4 and 6: same as (a), (b) and (c)
- Axioms 2, 3, and 7-10: automatically true in H because they apply to all elements of V , including those in H
- Axiom 5: if \mathbf{u} is in H , then $(-1)\mathbf{u}$ is in H [by property (c) and by $(-1)\mathbf{u} = -\mathbf{u}$]

Examples of subspaces

1. The set consisting of only the zero vector in a vector space V : **zero subspace** written as $\{\mathbf{0}\}$.

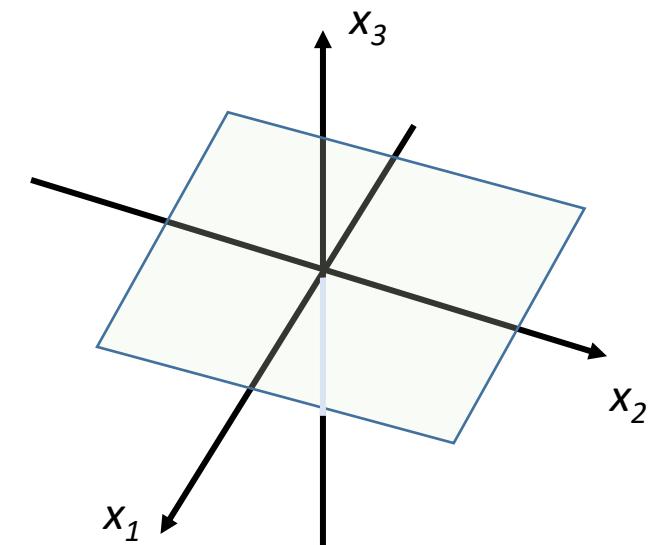
Examples of subspaces (contd.)

2. \mathbb{P} : set of all polynomials with real coefficients, with operations in \mathbb{P} defined as for functions.

For each $n \geq 0$, \mathbb{P}_n is a subspace of \mathbb{P} . [properties (a), (b) and (c)]

3. Is \mathbb{R}^2 a subspace of \mathbb{R}^3 ?

Is the set $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$ a subset of \mathbb{R}^3 ?



4.3 Subspace spanned by a set

Recall: $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the set of all vectors that can be written as linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$.

Exercise 4.3.1

Given \mathbf{v}_1 and \mathbf{v}_2 in a vector space V , let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V .

Theorem 4.1. *If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V , then $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subspace of V .*

$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is called **the subspace spanned** by $\mathbf{v}_1, \dots, \mathbf{v}_p$.

Given any subspace H of V , a **spanning set** for H is a set $\mathbf{v}_1, \dots, \mathbf{v}_p$ in H such that $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

Exercise 4.3.2

Let H be the set of all vectors of the form $(a - 3b, b - a, a, b)$ where a and b are in \mathbb{R} . Show that H is a subspace of \mathbb{R}^4 .

Solution:

$$H = \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}. \quad H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

$\uparrow \qquad \uparrow$
 $\mathbf{v}_1 \qquad \mathbf{v}_2$

$\therefore H$ is a subspace of \mathbb{R}^4 .

4.4 The Null Space of a Matrix

System of homogeneous equations

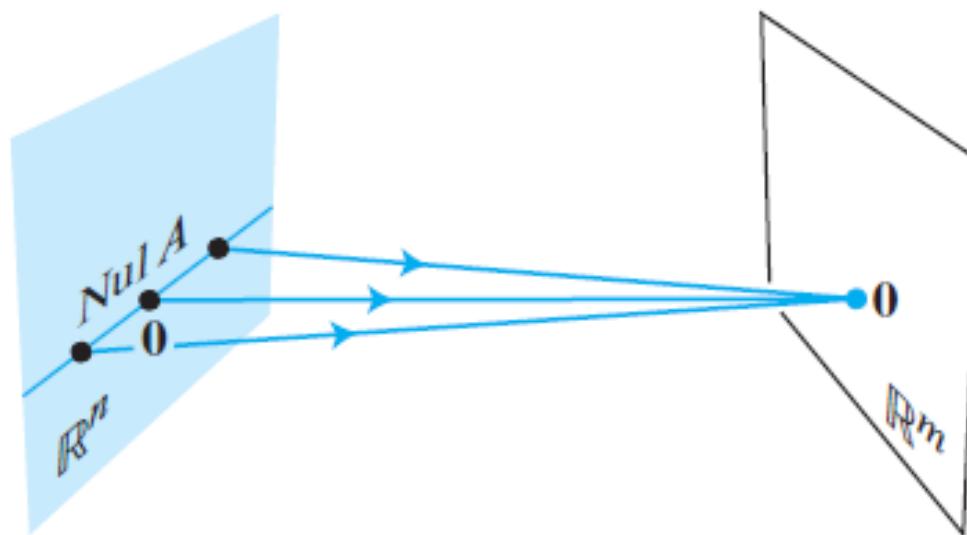
$$\begin{array}{rcl} x_1 & - & 3x_2 & - & 2x_3 = 0 \\ -5x_1 & + & 9x_2 & + & x_3 = 0 \end{array}$$

Rewritten as $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$

Null space: the set of \mathbf{x} that satisfy $A\mathbf{x} = \mathbf{0}$

Definition. The **null space** of an $m \times n$ matrix A , written as $\mathbf{N}(A)$, is the set of all soutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$, i.e.,

$$\mathbf{N}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$



All \mathbf{x} in \mathbb{R}^n mapped into the zero vector in \mathbb{R}^m via the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Exercise 4.4.1

Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if \mathbf{u} belongs to $\mathbf{N}(A)$.

Exercise 4.4.2

Describe the null space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution

Apply elimination on $A\mathbf{x} = \mathbf{0}$.

$$\begin{array}{rcl} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{array} \rightarrow \begin{array}{rcl} x_1 + 2x_2 = 0 \\ 0 = 0 \end{array}$$

The line $x_1 + 2x_2 = 0$ is $\mathbf{N}(A)$. It contains all solutions (x_1, x_2) .

Set free variable x_2 to some value, say, 1. Then $x_1 = -2$.

$\mathbf{N}(A)$ contains all multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Theorem 4.2. *The null space of an $m \times n$ matrix A is a subspace of \mathbb{R}^n . Equivalently, the set of all solutions to the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of \mathbb{R}^n .*

Proof.

□

Exercise 4.4.3

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution

Apply elimination on $A\mathbf{x} = \mathbf{0}$ to obtain reduced row echelon form of augmented matrix $[A \quad \mathbf{0}]$.

$$\left[\begin{array}{cccccc} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{array}{rcl} x_1 - 2x_2 & - & x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 & = & 0 \end{array}$$

General solution : $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + x_5$, with x_2, x_4 , and x_5 as free variables.

Decompose the vector giving the general solution into a linear combination of vectors where *the weights are the free variables*.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}$$

Every linear combination of \mathbf{u} , \mathbf{v} , and \mathbf{w} is an element of $\mathbf{N}(A)$ and vice versa. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for $\mathbf{N}(A)$.

4.5 The Column Space of a Matrix

Definition. The **column space** of an $m \times n$ matrix A , written as $\mathbf{C}(A)$, is the set of all linear combinations of the columns of A . If $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$, then $\mathbf{C}(A) = \text{Span}\{\mathbf{a}_1 \cdots \mathbf{a}_n\}$, i.e.,

$$\mathbf{C}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

Theorem 4.3. *The column space of an $m \times n$ matrix A is a subspace of \mathbb{R}^m .*

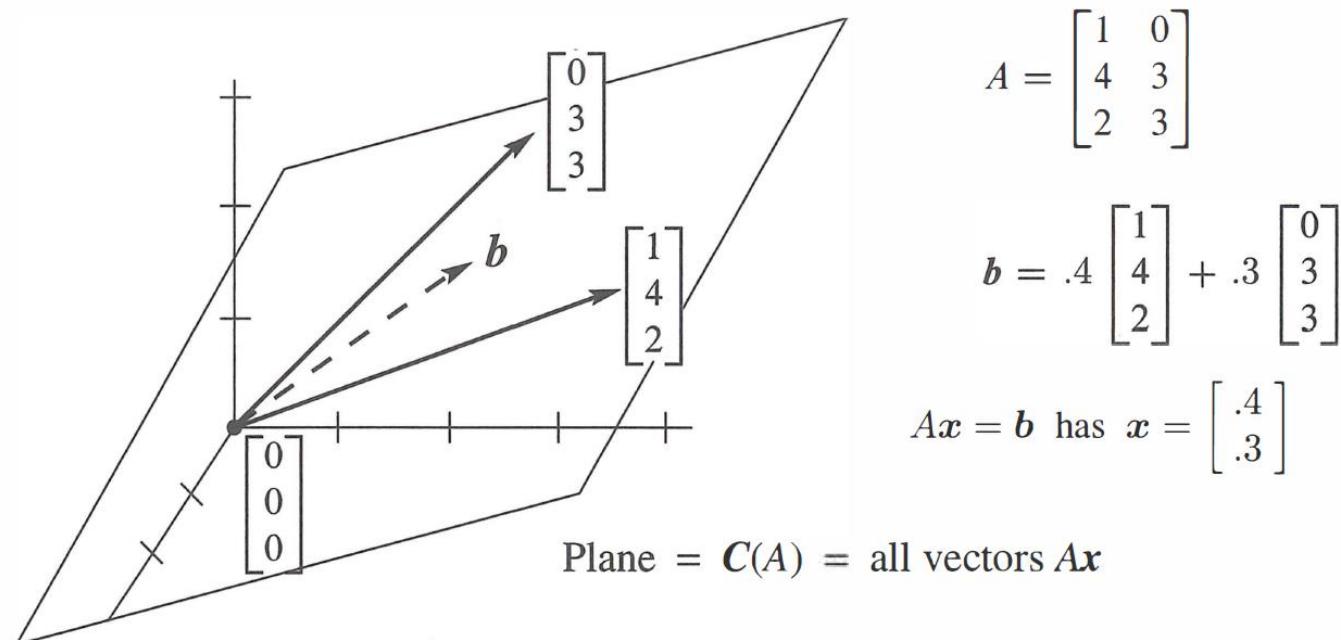
Proof. $\text{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ is a subspace [Theorem 4.1]

Columns of A are in \mathbb{R}^m . □

Recall (from Chapter 1, slide 29)): The columns of A span \mathbb{R}^m if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each \mathbf{b} .

Restating the above:

The column space of an $m \times n$ matrix A is all of \mathbb{R}^m if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each \mathbf{b} in \mathbb{R}^m .



Exercise 4.5.1

Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$.

- a. If the column space of A is a subspace of \mathbb{R}^k , what is k ?
- b. If the null space of A is a subspace of \mathbb{R}^k , what is k ?
- c. Find a nonzero vector in $\mathbf{C}(A)$ and a nonzero vector in $\mathbf{N}(A)$.
- d. If $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$
 - i. Determine if \mathbf{u} is in $\mathbf{N}(A)$. Could \mathbf{u} be in $\mathbf{C}(A)$?
 - ii. Determine if \mathbf{v} is in $\mathbf{C}(A)$. Could \mathbf{v} be in $\mathbf{N}(A)$?

	$\mathbf{N}(A)$	$\mathbf{C}(A)$
$\text{Nul } A \equiv \mathbf{N}(A)$	<ol style="list-style-type: none"> 1. $\text{Nul } A$ is a subspace of \mathbb{R}^n. 2. $\text{Nul } A$ is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in $\text{Nul } A$ must satisfy. 3. It takes time to find vectors in $\text{Nul } A$. Row operations on $[A \quad \mathbf{0}]$ are required. 4. There is no obvious relation between $\text{Nul } A$ and the entries in A. 5. A typical vector \mathbf{v} in $\text{Nul } A$ has the property that $A\mathbf{v} = \mathbf{0}$. 6. Given a specific vector \mathbf{v}, it is easy to tell if \mathbf{v} is in $\text{Nul } A$. Just compute $A\mathbf{v}$. 7. $\text{Nul } A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. 8. $\text{Nul } A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. 	<ol style="list-style-type: none"> 1. $\text{Col } A$ is a subspace of \mathbb{R}^m. 2. $\text{Col } A$ is explicitly defined; that is, you are told how to build vectors in $\text{Col } A$. 3. It is easy to find vectors in $\text{Col } A$. The columns of A are displayed; others are formed from them. 4. There is an obvious relation between $\text{Col } A$ and the entries in A, since each column of A is in $\text{Col } A$. 5. A typical vector \mathbf{v} in $\text{Col } A$ has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent. 6. Given a specific vector \mathbf{v}, it may take time to tell if \mathbf{v} is in $\text{Col } A$. Row operations on $[A \quad \mathbf{v}]$ are required. 7. $\text{Col } A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every \mathbf{b} in \mathbb{R}^m. 8. $\text{Col } A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps \mathbb{R}^n onto \mathbb{R}^m.

4.6 Kernel and Range of a Linear Transformation

- Generalize definition of linear transformation to include vector spaces

Definition. A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector \mathbf{x} in V a unique vector $T(\mathbf{x})$ in W , such that

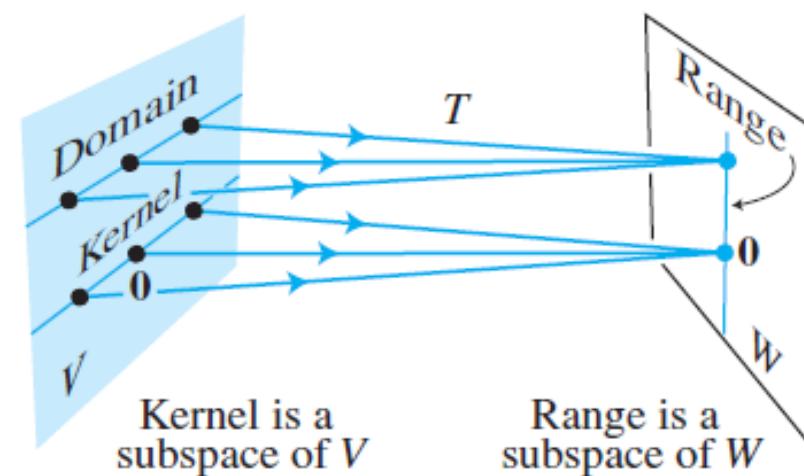
1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all \mathbf{u}, \mathbf{v} in V , and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all \mathbf{u} in V and all scalars c .

Definition. The **kernel**(or **null space**) of such a T is the set of all \mathbf{u} in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector in W).

Definition. The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some \mathbf{x} in V .

If $T(\mathbf{x}) = A\mathbf{x}$, then

- Kernel = $\mathbf{N}(A)$
- Range = $\mathbf{C}(A)$



4.7 Bases

- Linear independence (again! introduced in Sec 1.9 for \mathbb{R}^n)

Definition. An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights c_1, \dots, c_p , not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

- A set containing a single vector \mathbf{v} is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.
- A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of other.
- Any set containing the zero vector is linearly dependent.

Theorem 4.4. *An indexed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some \mathbf{v}_j (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.*

(Skip proof)

Examples: Linearly independent or not?

- $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, and $\mathbf{p}_3(t) = 4 - t$.
- The set $\{\sin t, \cos t\}$ in $C[0, 1]$, the space of all continuous functions on $0 \leq t \leq 1$
- The set $\{\sin t \cos t, \sin 2t\}$ in $C[0, 1]$, the space of all continuous functions on $0 \leq t \leq 1$

Definition. Let H be a subspace of a vector space V . An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** for H if

- (i) \mathcal{B} is a linearly independent set, and
- (ii) the subspace spanned by \mathcal{B} coincides with H , i.e.,

$$H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

- Also true when $H = V$ since every vector space is a subspace of itself.
 \Rightarrow a basis of V is a linearly independent set that spans V .

Examples:

- Invertible matrix $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$

Columns of A form a basis for \mathbb{R}^n

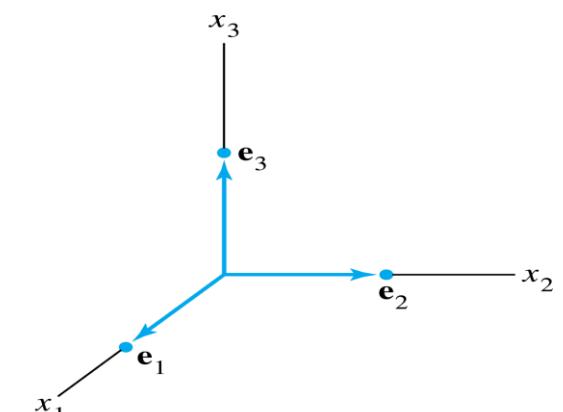
- they are linearly independent
- they span \mathbb{R}^n

- Columns of I_n

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is called the **standard basis** for \mathbb{R}^n

Invertible matrix theorem (Theorem 2.4)



The standard basis for \mathbb{R}^3 .

Exercise 4.7.1:

Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 .

Exercise 4.7.2:

Let $S = \{1, t, t^2, \dots, t^n\}$. Verify that S is a basis for \mathbb{P}_n . This basis is called the **standard basis** for \mathbb{P}_n .

4.8 The Spanning Set Theorem

Exercise 4.8.1:

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$, and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$. Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Then, find a basis for the subspace H .

Theorem 4.5. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V , and let $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

- a. If one of the vectors in S - say, \mathbf{v}_k - is a linear combination of the remaining vectors in S , then the set formed from S by removing \mathbf{v}_k still spans H .
- b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H .

(Skip proof)

4.9 Bases for $\mathbf{N}(A)$ and $\mathbf{C}(A)$

$\mathbf{N}(A)$

Recall from Exercise 4.4.3:

- found vectors that span $\mathbf{N}(A)$
- vectors were linearly independent (when $\mathbf{N}(A)$ contains non-zero vectors)

So, that method produces a *basis* for $\mathbf{N}(A)$.

$\mathbf{C}(A)$ Exercise 4.9.1

Find a basis for $\mathbf{C}(B)$, where

$$B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution

$$n \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}^p$$

If $p > n$, the columns are linearly dependent.

Each nonpivot column of B is a linear combination of the pivot columns.

$$\mathbf{b}_2 = 4\mathbf{b}_1, \mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$$

By Theorem 4.5, discard \mathbf{b}_2 and \mathbf{b}_4 ; $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span $\mathbf{C}(B)$. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

By Theorem 4.4, since $\mathbf{b}_1 \neq 0$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent.

$\therefore S$ is a basis for $\mathbf{C}(B)$.

- For a matrix A that is *not* in reduced row echelon form

Recall: Any linear dependence relationship among the columns of A can be expressed in the form $A\mathbf{x} = \mathbf{0}$.

If A is row reduced to B , $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have exactly the same set of solutions.

If $A = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n]$, then

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \text{ and } x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n = \mathbf{0}$$

have the same set of solutions.

Columns of A have *exactly the same linear dependence relationships* as the columns of B .

Exercise 4.9.2

It can be shown that

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix B in Exercise 4.9.1. Find a basis for $\mathbf{C}(A)$.

Solution

In Exercise 4.9.1, $\mathbf{b}_2 = 4\mathbf{b}_1$, $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.

So, $\mathbf{a}_2 = 4\mathbf{a}_1$, $\mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_3$ (Check!)

Discard \mathbf{a}_2 and \mathbf{a}_4 . For $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ to be linearly independent, $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ should also be linearly independent, which is true.

Therefore, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ is a basis for $\mathbf{C}(A)$.

Theorem 4.6. *The pivot columns of a matrix A form a basis for $\mathbf{C}(A)$.*

Proof. Let B be the reduced echelon form of A .

The set of pivot columns of B are linearly independent.

Since A is row equivalent to B , the pivot columns of A are linearly independent,
AND

Every nonpivot column of A is a linear combination of the pivot columns of A .

So, the nonpivot columns of A can be discarded from the spanning set for $\mathbf{C}(A)$ [Spanning Set theorem].

This leaves the pivot columns of A as the basis for $\mathbf{C}(A)$. □

4.10 Coordinate Systems

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then for each \mathbf{x} in V , there exists a unique set of scalars c_1, \dots, c_n such that

$$\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n.$$

Definition. Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and \mathbf{x} is in V . The **coordinates of \mathbf{x} relative to the basis \mathcal{B}** (or the **\mathcal{B} -coordinates of \mathbf{x}**) are the weights c_1, \dots, c_n such that $\mathbf{x} = c_1 \mathbf{b}_1 + \dots + c_n \mathbf{b}_n$.

Co-ordinate vector of \mathbf{x} relative to \mathcal{B} , or the **\mathcal{B} -coordinate vector of \mathbf{x}**

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Exercise 4.10.1

Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for \mathbb{R}^2 , where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Suppose an \mathbf{x} in \mathbb{R}^2 has the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find \mathbf{x} .

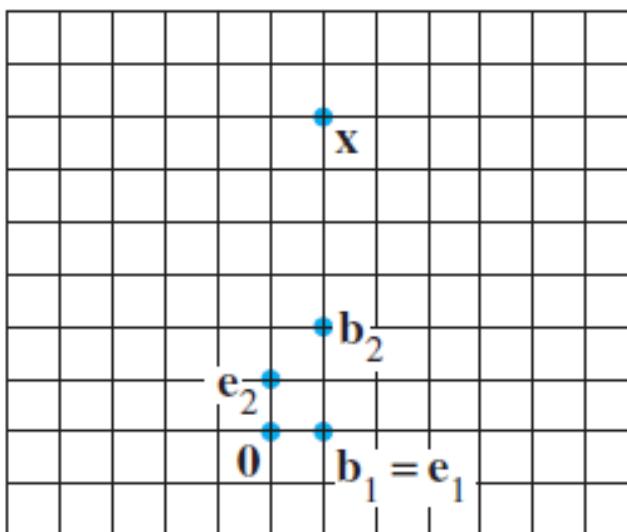
Solution

$$\mathbf{x} = (-2)\mathbf{b}_1 + 3\mathbf{b}_2 = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

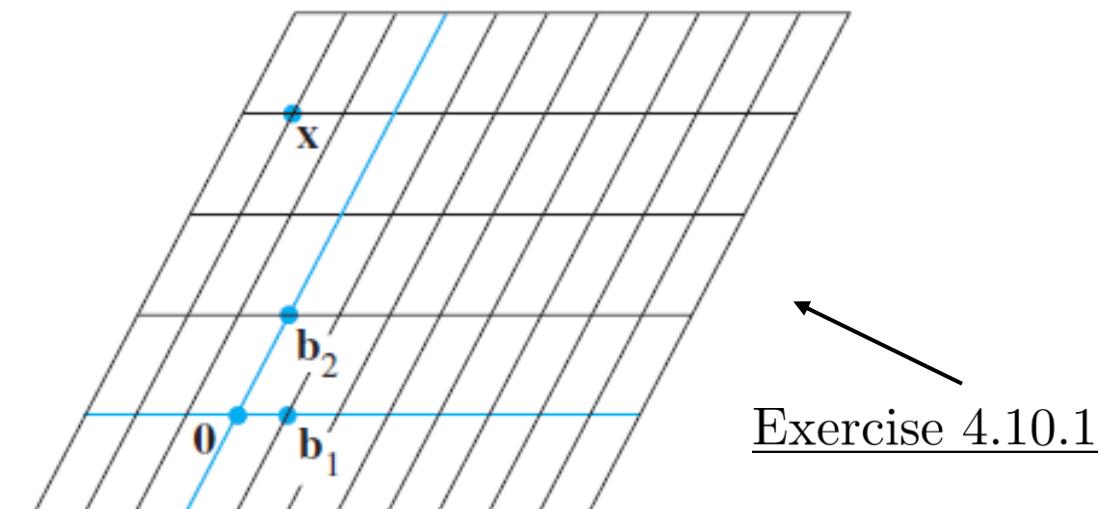
Example

Consider the *standard basis* $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$. If $[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$,

$$\text{then } \mathbf{x} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \text{ i.e., } [\mathbf{x}]_{\mathcal{E}} = \mathbf{x}.$$



Standard graph paper

 \mathcal{B} -graph paperExercise 4.10.1

Exercise 4.10.2

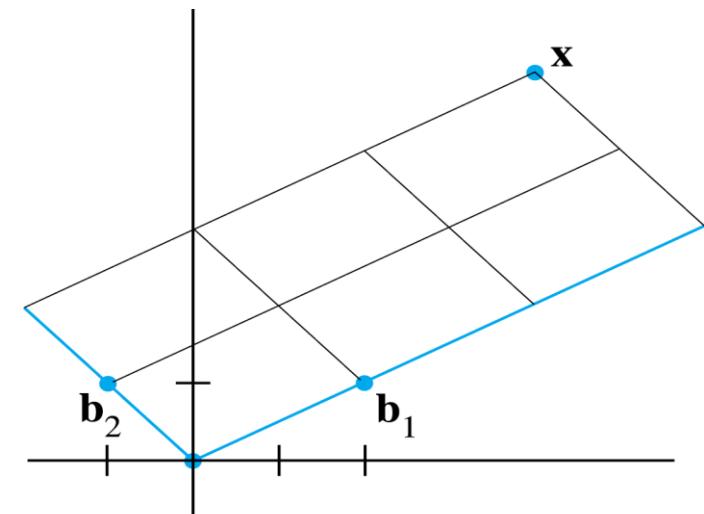
Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of \mathbf{x} relative to \mathcal{B} .

Solution

$$\text{Let } [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \text{ or } [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



The \mathcal{B} -coordinate vector of \mathbf{x} is $(3, 2)$.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ P_{\mathcal{B}} & [\mathbf{x}]_{\mathcal{B}} & \mathbf{x} \end{array}$$

The matrix $P_{\mathcal{B}} = [\mathbf{b}_1 \ \mathbf{b}_2]$ changes the \mathcal{B} -coordinates of a vector \mathbf{x} into the standard coordinates of \mathbf{x} .

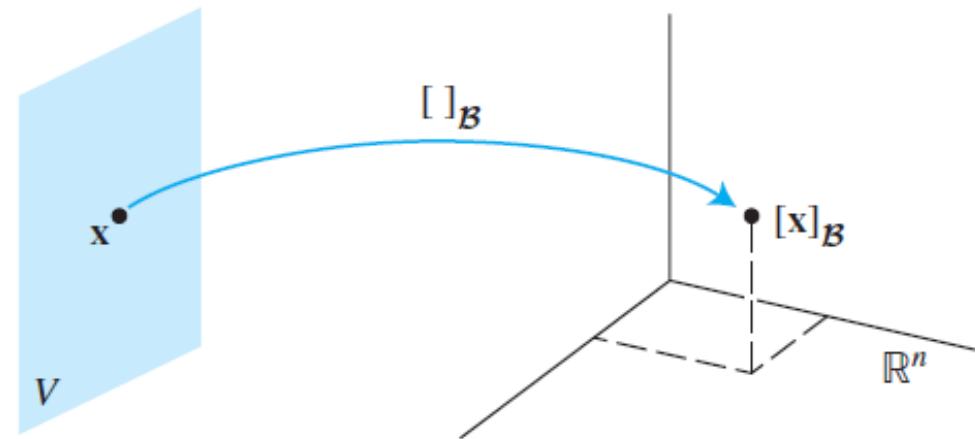
$$\mathbf{x} = P_{\mathcal{B}} [\mathbf{x}]_{\mathcal{B}}$$

Since the columns of $P_{\mathcal{B}}$ form a basis for \mathbb{R}^2 , $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem). Therefore,

$$P_{\mathcal{B}}^{-1} \mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$ can be seen as a one-to-one linear transformation $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$

Extending to \mathbb{R}^n :



If $\mathbf{u}_1, \dots, \mathbf{u}_p$ are in V and if c_1, \dots, c_p are scalars, then

$$[c_1 \mathbf{u}_1 + \cdots + c_p \mathbf{u}_p]_{\mathcal{B}} = c_1 [\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p [\mathbf{u}_p]_{\mathcal{B}}$$

The \mathcal{B} -coordinate vector of a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_p$ is the *same* linear combination of their coordinate vectors.

4.11 Dimension of a Vector Space

Definition. If V is a vector space, the **dimension** of V , written as $\dim V$, is the number of vectors in a basis for V . The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.

Examples

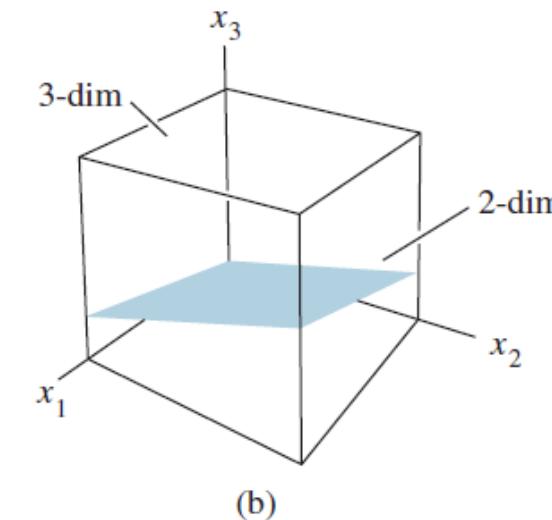
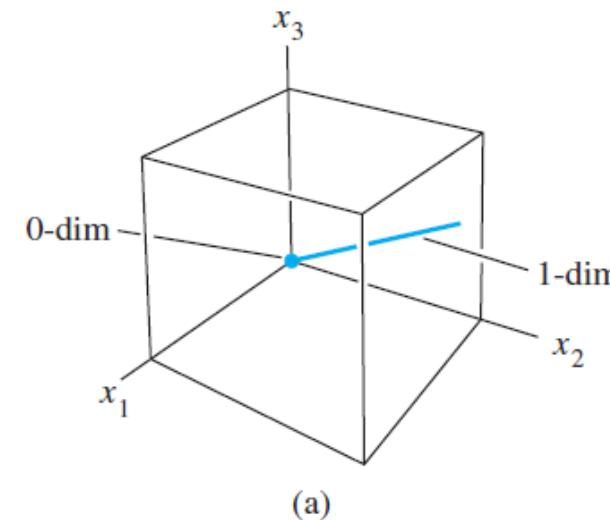
- Standard basis for \mathbb{R}^n contains n vectors $\Rightarrow \dim \mathbb{R}^n = n$
- Standard polynomial basis $\{1, t, t^2\} \Rightarrow \dim \mathbb{P}_2 = 3$

Exercise 4.11.1 Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

Example

- Subspaces of \mathbb{R}^3



4.12 Dimensions of $\mathbf{C}(A)$ and $\mathbf{N}(A)$

$\mathbf{C}(A)$

Pivot columns of A form a basis for $\mathbf{C}(A)$.

So, $\dim \mathbf{C}(A) = \text{number of pivot columns of } A$.

$\mathbf{N}(A)$

For $m \times n$ matrix A , suppose $A\mathbf{x} = \mathbf{0}$ has k free variables.

Spanning set for $\mathbf{N}(A)$ will have k linearly independent vectors - one for each free variable.

So, $\dim \mathbf{N}(A) = \text{number of free variables, } k$.

$\mathbf{C}(A)$ Exercise 4.9.1

Find a basis for $\mathbf{C}(B)$, where

$$B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_5] = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution

$$n \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}^p$$

If $p > n$, the columns are linearly dependent.

Each nonpivot column of B is a linear combination of the pivot columns.

$$\mathbf{b}_2 = 4\mathbf{b}_1, \mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$$

By Theorem 4.5, discard \mathbf{b}_2 and \mathbf{b}_4 ; $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span $\mathbf{C}(B)$. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

By Theorem 4.4, since $\mathbf{b}_1 \neq 0$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent.

$\therefore S$ is a basis for $\mathbf{C}(B)$.

Exercise 4.4.3

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution

Apply elimination on $A\mathbf{x} = \mathbf{0}$ to obtain reduced row echelon form of augmented matrix $[A \quad \mathbf{0}]$.

$$\left[\begin{array}{cccccc} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{array}{rcl} x_1 - 2x_2 & - & x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 & = & 0 \end{array}$$

General solution : $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + x_5$, with x_2, x_4 , and x_5 as free variables.

Decompose the vector giving the general solution into a linear combination of vectors where *the weights are the free variables*.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_2 \mathbf{u} + x_4 \mathbf{v} + x_5 \mathbf{w}$$

Every linear combination of \mathbf{u} , \mathbf{v} , and \mathbf{w} is an element of $\mathbf{N}(A)$ and vice versa. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for $\mathbf{N}(A)$.

Exercise 4.12.1

Find the dimension of the null space and column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution

Echelon form of augmented matrix $[A \quad \mathbf{0}]$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- 3 free variables (x_2, x_4, x_5) $\Rightarrow \dim \mathbf{N}(A) = 3$
- A has two pivot columns $\Rightarrow \dim \mathbf{C}(A) = 2$

4.13 The Row Space

The set of all linear combinations of the rows (row vectors) of a matrix A is called the **row space** of A .

Since rows of A are columns of A^T , row space is denoted by $\mathbf{C}(A^T)$.

For an $m \times n$ matrix, each row has n entries $\Rightarrow \mathbf{C}(A^T)$ is a subspace of \mathbb{R}^n .

Theorem 4.7. *If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B .*

(Skip proof)

Exercise 4.13.1

Find bases for the row space, column space and null space of

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Solution

Reduce A to echelon form:

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row space $\mathbf{C}(A^T)$

From Theorem 4.7,

$$\text{Basis for } \mathbf{C}(A^T) : \{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$$

Column space $\mathbf{C}(A)$

Pivots are in columns 1, 2 and 4. Hence, columns 1, 2, and 4 of A (not B !) form the basis:

$$\text{Basis for } \mathbf{C}(A) : \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

Null space $\mathbf{N}(A)$

Need reduced row echelon form of A :

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{0}$ is equivalent to $C\mathbf{x} = \mathbf{0}$

$$\begin{array}{rcl} x_1 + x_3 & & + x_5 = 0 \\ x_2 - 2x_3 & & + 3x_5 = 0 \\ & & x_4 - 5x_5 = 0 \end{array}$$

Similar to Exercise 4.4.3 (slide 21),

$$\text{Basis for } \mathbf{N}(A) : \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

4.14 Rank

Definition. The **rank** of A is the dimension of the column space of A .

Recall: $A\mathbf{x} = \mathbf{b}$ is consistent when \mathbf{b} is in $\mathbf{C}(A)$.

Rank of A is the dimension of the set of \mathbf{b} such that $A\mathbf{x} = \mathbf{b}$ is consistent.

Suppose $A \in \mathbb{R}^{n \times n}$ has rank p . This means that if we take all vectors $\mathbf{x} \in \mathbb{R}^{n \times 1}$, then $A\mathbf{x}$ spans p dimensional space \Rightarrow no unique solution.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Rank of $A = 1 \Rightarrow A\mathbf{x}$ spans a 1 dimensional space,
i.e., a line $y_2 = 2y_1$ passing through the origin.

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$$

Points on $x_1 - x_2$ plane are mapped onto a line $y_2 = 2y_1$.
(Also see Exercise 4.13.1)

- The dimensions of the column space and the row space of an $m \times n$ matrix A are equal.

Example

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Echelon form of A : $A \sim B$ =

$$\begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 4.6 (slide 42), rank of A is the number of pivot columns in A , i.e., the number of pivot positions in B .

Rows with pivots in B form a basis for the row space of A .

Therefore, rank of A is also the dimension of the row space of A .

- Rank of $A = \dim \mathbf{C}(A)$ = the number of columns with pivots
- $\dim \mathbf{N}(A) =$ the number of free variables (in $A\mathbf{x} = \mathbf{0}$)
= the number of columns without pivots
- Obviously,

$$\#(\text{columns with pivots}) + \#(\text{columns without pivots}) = \#(\text{columns})$$

Theorem 4.8. *The Rank Theorem*

If A is a matrix with n columns then,

$$\text{rank of } A + \dim \mathbf{N}(A) = n.$$

- Gives a relationship between the solution set of $A\mathbf{x} = \mathbf{0}$, i.e., $\mathbf{N}(A)$ and the set of vectors \mathbf{b} that make $A\mathbf{x} = \mathbf{b}$ consistent, i.e., $\mathbf{C}(A)$

Exercise 4.14.1

A scientist has found two solutions to a homogeneous system of 40 equations in 42 variables. The solutions are linearly independent and span $\mathbf{N}(A)$.

Can the scientist be *certain* that an associated nonhomogeneous system (with the same coefficients) has a solution?

Solution

A : 40×42 coefficient matrix.

The two solutions are a basis for $\mathbf{N}(A)$.

Therefore, the $\dim \mathbf{N}(A) = 2 \Rightarrow \dim \mathbf{C}(A) = 42 - 2 = 40$.

Since \mathbb{R}^{40} is the only subspace of \mathbb{R}^{40} whose dimension is 40, $\mathbf{C}(A)$ must be all of \mathbb{R}^{40} .

\Rightarrow Every non-homogeneous equation $A\mathbf{x} = \mathbf{b}$ has a solution.

Theorem 4.9. *The Invertible Matrix Theorem [Theorem 2.4] (continued)*

Let A by an $n \times n$ matrix. Then the statement ‘ A is invertible’ is equivalent to the following statements:

11. The columns of A form a basis for \mathbb{R}^n

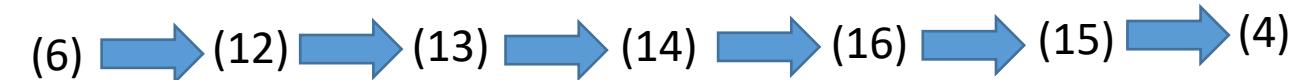
12. $\mathbf{C}(A) = \mathbb{R}^n$

13. $\dim \mathbf{C}(A) = n$

14. Rank of $A = n$

15. $\mathbf{N}(A) = \{\mathbf{0}\}$

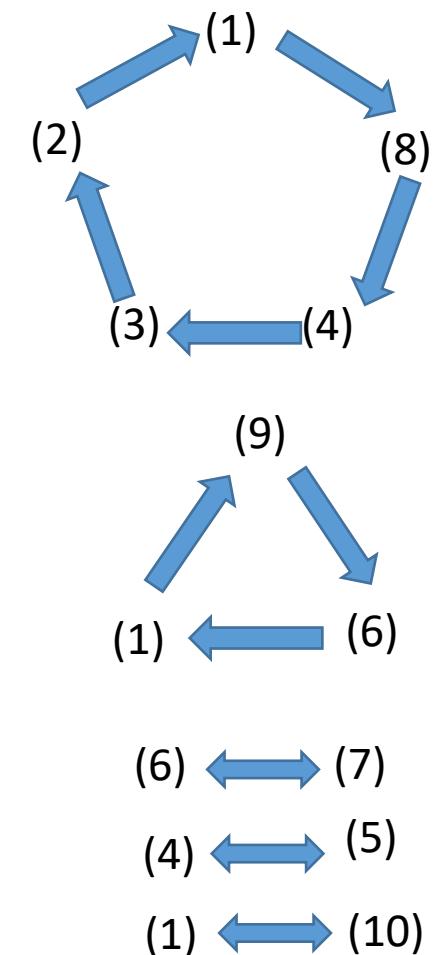
16. $\dim \mathbf{N}(A) = 0$



Theorem 2.4. *The Invertible Matrix Theorem*

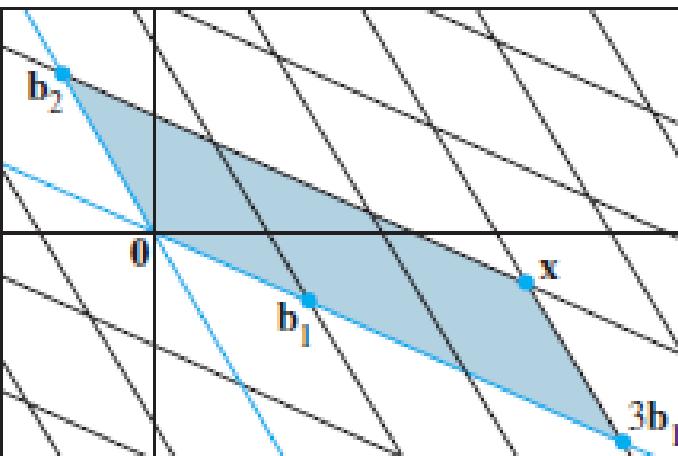
Let A be a square $n \times n$ matrix. Then the following statements are equivalent, i.e., for a given A , the statements are either all true or all false.

1. A is an invertible matrix.
2. A is row equivalent to I_n .
3. A has n pivot positions.
4. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbb{R}^n .
7. The columns of A span \mathbb{R}^n .
8. There is an $n \times n$ matrix C such that $CA = I$.
9. There is an $n \times n$ matrix D such that $AD = I$.
10. A^T is an invertible matrix.

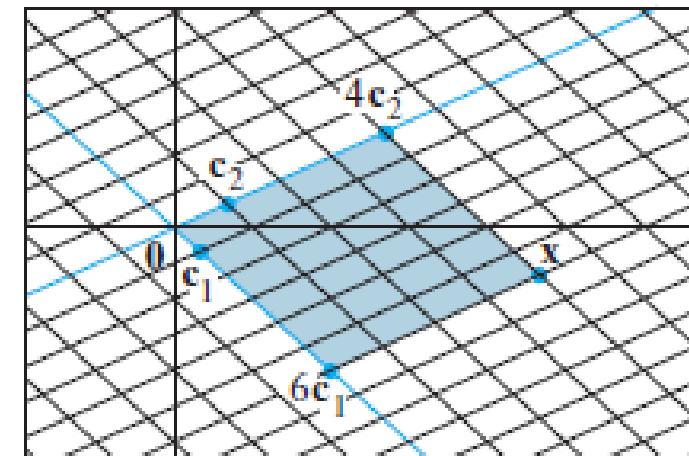


4.15 Change of basis

- A problem could be more easily solved by changing the basis from \mathcal{B} to \mathcal{C}
- How $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$ are related for each \mathbf{x} in V .



(a)



(b)

Two co-ordinate systems for the same vector space

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \xleftarrow{\text{Relation ?}} \quad [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Exercise 4.15.1

Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V , such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \text{ and } \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

If $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$, i.e., $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{C}}$.

Solution

Coordinate mapping is a linear transformation (slide 48)

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} \\ &= 3[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} \end{aligned}$$

$$[\mathbf{x}]_{\mathcal{C}} = [[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}}] \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

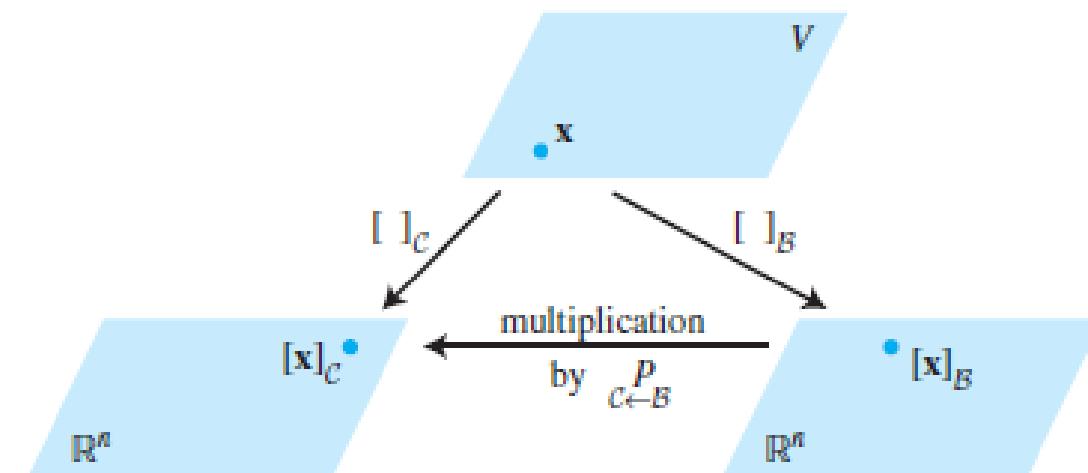
$P_{\mathcal{C} \leftarrow \mathcal{B}}$

Theorem 4.9. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V . Then there is a unique $n \times n$ matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{x}]_{\mathcal{B}}$$

The columns of $P_{\mathcal{C} \leftarrow \mathcal{B}}$ are the \mathcal{C} -coordinate vectors of the vectors in the basis B , i.e.,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}} \quad \cdots \quad [\mathbf{b}_n]_{\mathcal{C}}]$$



Two coordinate systems for V

***** END OF CHAPTER *****