$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **6.2.4**

Lecture : **Orthogonality**

Topic : **Orthogonality**

Concept : **Orthonormal Sets & Orthogonal Matrices**

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Orthonormal Set and Orthonormal basis

Orthonormal Sets

A set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors. If W is the subspace spanned by such a set, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal basis** for W, since the set is automatically linearly independent, by Theorem 4.

The simplest example of an orthonormal set is the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$. Any nonempty subset of $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is orthonormal, too. Here is a more complicated example.

THEOREM 4 If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

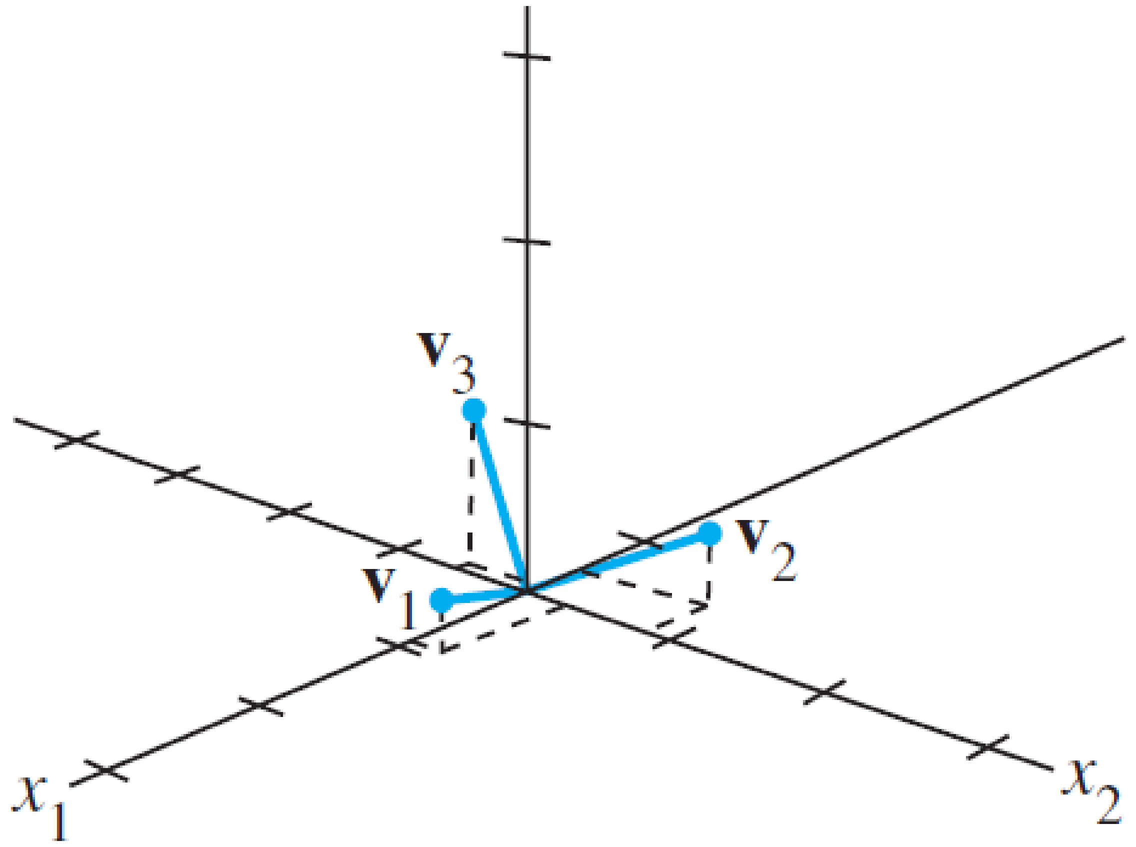

FIGURE 6

Ref: https://www.youtube.com/watch?v=ZJu26chXEiw

Linear Algebra: Orthonormal Basis
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Lay's Linear Algebra and Applications

EXAMPLE 5 Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

SOLUTION Compute

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= -3/\sqrt{66} + 2/\sqrt{66} + 1/\sqrt{66} = 0 \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= -3/\sqrt{726} - 4/\sqrt{726} + 7/\sqrt{726} = 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= 1/\sqrt{396} - 8/\sqrt{396} + 7/\sqrt{396} = 0 \end{aligned}$$

Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set. Also,

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_1 &= 9/11 + 1/11 + 1/11 = 1 \\ \mathbf{v}_2 \cdot \mathbf{v}_2 &= 1/6 + 4/6 + 1/6 = 1 \\ \mathbf{v}_3 \cdot \mathbf{v}_3 &= 1/66 + 16/66 + 49/66 = 1 \end{aligned}$$

which shows that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are unit vectors. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal set. Since the set is linearly independent, its three vectors form a basis for $\mathbb{R}^3$. See Fig. 6. ■

```
S3 = [ 3/sqrt(11)   -1/sqrt(6)   -1/sqrt(66);
       1/sqrt(11)   2/sqrt(6)    -4/sqrt(66);
       1/sqrt(11)   1/sqrt(6)     7/sqrt(66)];

S3
checkOrthognality = S3'*S3
```

```
S3 =

    0.9045   -0.4082   -0.1231
    0.3015    0.8165   -0.4924
    0.3015    0.4082    0.8616


checkOrthognality =

    1.0000    0.0000    0.0000
    0.0000    1.0000    0.0000
    0.0000    0.0000    1.0000
```

NOTE: The 0's correspond to dot products of orthogonal vectors. See next slide for explanation of result!

2

Orthonormal Sets and U^TU

THEOREM 6

An $m \times n$ matrix U has orthonormal columns if and only if $U^TU = I$.

PROOF To simplify notation, we suppose that U has only three columns, each a vector in $\mathbb{R}^m$. The proof of the general case is essentially the same. Let $U = [\mathbf{u}_1 \ \ \mathbf{u}_2 \ \ \mathbf{u}_3]$ and compute

$$U^TU = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} [\mathbf{u}_1 \ \ \mathbf{u}_2 \ \ \mathbf{u}_3] = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 & \mathbf{u}_1^T\mathbf{u}_3 \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 & \mathbf{u}_2^T\mathbf{u}_3 \\ \mathbf{u}_3^T\mathbf{u}_1 & \mathbf{u}_3^T\mathbf{u}_2 & \mathbf{u}_3^T\mathbf{u}_3 \end{bmatrix} \tag{4}$$

The entries in the matrix at the right are inner products, using transpose notation. The columns of U are orthogonal if and only if

$$\mathbf{u}_1^T\mathbf{u}_2 = \mathbf{u}_2^T\mathbf{u}_1 = 0, \quad \mathbf{u}_1^T\mathbf{u}_3 = \mathbf{u}_3^T\mathbf{u}_1 = 0, \quad \mathbf{u}_2^T\mathbf{u}_3 = \mathbf{u}_3^T\mathbf{u}_2 = 0 \tag{5}$$

The columns of U all have unit length if and only if

$$\mathbf{u}_1^T\mathbf{u}_1 = 1, \quad \mathbf{u}_2^T\mathbf{u}_2 = 1, \quad \mathbf{u}_3^T\mathbf{u}_3 = 1 \tag{6}$$

The theorem follows immediately from (4)–(6). ■

Lay's Linear Algebra and Applications

Matlab Example: When U is square (3x3)

```
>> U = [3/sqrt(11) -1/sqrt(6) -1/sqrt(66);
1/sqrt(11) 2/sqrt(6) -4/sqrt(66);
1/sqrt(11) 1/sqrt(6) 7/sqrt(66)]

U =

    0.9045   -0.4082   -0.1231
    0.3015    0.8165   -0.4924
    0.3015    0.4082    0.8616

>> U'*U

ans =

    1.0000    0.0000    0.0000
    0.0000    1.0000    0.0000
    0.0000    0.0000    1.0000

>> U*U'

ans =

    1.0000   -0.0000    0.0000
   -0.0000    1.0000    0.0000
    0.0000    0.0000    1.0000
```

vs U is rectangle (3x2)

```
>> U2=[U(:,1) U(:,2)]

U2 =

    0.9045   -0.4082
    0.3015    0.8165
    0.3015    0.4082

>> U2'*U2

ans =

    1.0000    0.0000
    0.0000    1.0000

>> U2*U2'

ans =

    0.9848   -0.0606    0.1061
   -0.0606    0.7576    0.4242
    0.1061    0.4242    0.2576
```

```
>> P=U2*U2'

P =

    0.9848   -0.0606    0.1061
   -0.0606    0.7576    0.4242
    0.1061    0.4242    0.2576

>> P*P

ans =

    0.9848   -0.0606    0.1061
   -0.0606    0.7576    0.4242
    0.1061    0.4242    0.2576
```

```
>> rank(P*P)

ans =

     2
```

Question: If U (a mx2 matrix) ONLY has 2 **orthonormal** columns, (m>2), what is the characteristics of matrixes: U^TU and UU^T?

Ans: U^TU = identity (2x2) matrix

$P = UU^T$ = (mxm)matrix but not identity (only rank 2)

It spans the column space of U, lets call it *W*.

(later we show) it is a projection matrix P

P can be used to projecting a given vector y onto W by

$\hat{y} = UU^Ty = Py$ (see Ch 6.2.5, pg 7 Theorem 10)

Orthonormal Sets and Orthogonal matrix

THEOREM 7

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x})\cdot(U\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$

c. $(U\mathbf{x})\cdot(U\mathbf{y}) = 0$ if and only if $\mathbf{x}\cdot\mathbf{y} = 0$

Properties (a) and (c) say that the linear mapping $\mathbf{x} \mapsto U\mathbf{x}$ preserves lengths and orthogonality. These properties are crucial for many computer algorithms.

Lay's Linear Algebra and Applications

6.2 Orthogonal Sets 343

EXAMPLE 6 Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$. Notice that U has orthonormal columns and

$$U^TU = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Verify that $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

SOLUTION

$$U\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\|U\mathbf{x}\| = \sqrt{9+1+1} = \sqrt{11}$$

$$\|\mathbf{x}\| = \sqrt{2+9} = \sqrt{11}$$

Theorems 6 and 7 are particularly useful when applied to *square* matrices. An **orthogonal matrix** is a square invertible matrix U such that $U^{-1} = U^T$. By Theorem 6, such a matrix has orthonormal columns.[1] It is easy to see that any *square* matrix with orthonormal columns is an orthogonal matrix. Surprisingly, such a matrix must have orthonormal *rows*, too.

Orthogonal Matrix definitions

Ref: https://en.wikipedia.org/wiki/Orthogonal_matrix **Important!**

In linear algebra, an **orthogonal matrix**, or **orthonormal matrix**, is a real square matrix whose columns and rows are orthonormal vectors.

One way to express this is

$$Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I,$$

where Q^{T} is the transpose of Q and I is the identity matrix.

This leads to the equivalent characterization: a matrix Q is orthogonal if its transpose is equal to its inverse:

$$Q^{\mathrm{T}} = Q^{-1},$$

where Q^{-1} is the inverse of Q.

Orthogonal Matrix
MUST be SQUARE!

Note: confusion
If you have a mxn matrix called U *with* its column ortho-normal, and m>n (tall matrix)

1) IT IS NOT an orthogonal matrix since it satisfy ONLY $U^TU = I$ (nxn)
2) its UU^T is mxm matrix BUT it is not equals to I) .Instead UU^T is a projection matrix and has rank n.

Note: There is no standard name for "rectangular matrix with orthonormal columns"