

Tutorial 1

Systems of Linear Equations

1. Find the values of k for which the equations

$$\begin{array}{rrcr} x & + & 5y & + & 3z & = & 0 \\ 5x & + & y & - & kz & = & 0 \\ x & + & 2y & + & kz & = & 0 \end{array}$$

have a non-trivial solution.

2. Determine the values of a for which the following linear system has (i) no solutions, (ii) infinite solutions, (iii) exactly one solution:

$$\begin{array}{rcl} x + 2y - 3z & = & 4 \\ 3x - y + 5z & = & 2 \\ 4x + y + (a^2 - 14)z & = & a + 2 \end{array}$$

3. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Is \mathbf{b} in W ? How many vectors are in W ?

4. Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$ and \mathbf{v} be vectors in \mathbb{R}^n . Suppose the vectors \mathbf{u} and \mathbf{v} are in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v}$ is also in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

5. Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

- How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each \mathbf{b} in \mathbb{R}^4 ?
 - Do the columns of B span \mathbb{R}^4 ? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each \mathbf{y} in \mathbb{R}^4 ?
6. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in \mathbb{R}^2 through $(4, 1)$ and the origin.
7. Suppose A is a 3×3 matrix and \mathbf{y} is a vector in \mathbb{R}^3 such that the equation $A\mathbf{x} = \mathbf{y}$ does *not* have a solution. Does there exist a vector \mathbf{z} in \mathbb{R}^3 such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution? Why?

8. Find the value of h for which the vectors $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$ are linearly *dependent*.
9. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation that maps \mathbf{e}_1 into \mathbf{y}_1 and maps \mathbf{e}_2 into \mathbf{y}_2 . Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.
10. Find the standard matrix of the linear transformation
- $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, which first rotates points through $-3\pi/4$ radian (clockwise) and then reflects points through the horizontal x -axis.
 - $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, which first reflects points through the horizontal x -axis and then reflects points through the line $y = x$. Show that the transformation is merely a rotation about the origin. What is the angle of rotation?

Answers

- $k = 1$
- (i) $a = -4$ (ii) $a = +4$ (iii) $a \neq \pm 4$
- Yes, Infinite
-
- a. 3, No b. No, No
- One possibility for $A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$.
- No
- All values of h .
- $\begin{bmatrix} 13 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$
- a. $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, b. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\pi/2$ radians

End