CX1104: **Linear Algebra for Computing**

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

Chap. No : **6.2.1**

Lecture : **Orthogonality**

Topic : **Orthogonality**

Concept : **Definition of Orthogonality and Orthogonal Complements**

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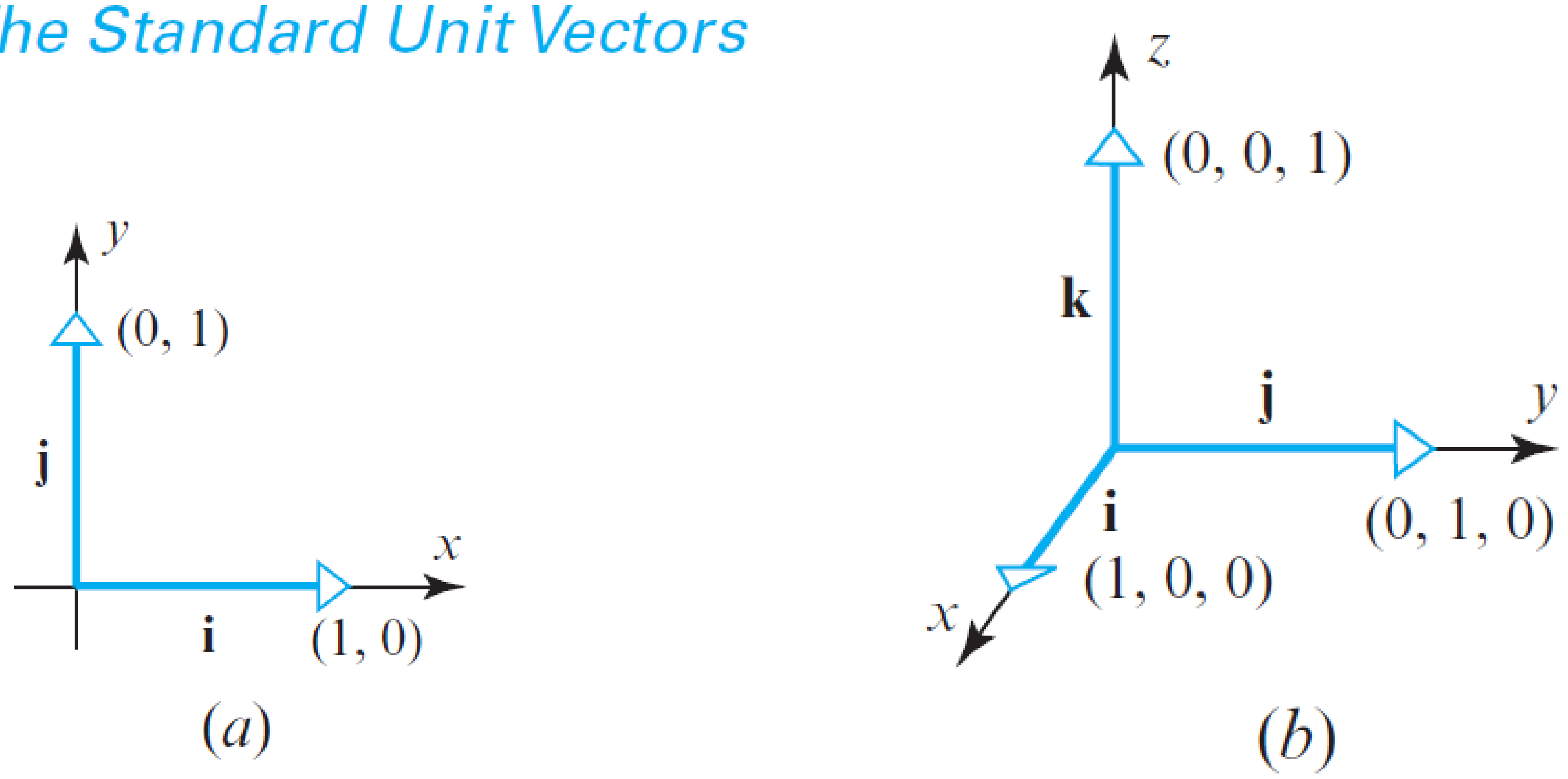
Orthogonality Definition

The Standard Unit Vectors



▲ Figure 3.2.2

When a rectangular coordinate system is introduced in R^2 or R^3, the unit vectors in the positive directions of the coordinate axes are called the ***standard unit vectors***. In R^2 these vectors are denoted by

$$\mathbf{i} = (1, 0) \quad \text{and} \quad \mathbf{j} = (0, 1)$$

and in R^3 by

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \text{and} \quad \mathbf{k} = (0, 0, 1)$$

(Figure 3.2.2). Every vector $\mathbf{v} = (v_1, v_2)$ in R^2 and every vector $\mathbf{v} = (v_1, v_2, v_3)$ in R^3 can be expressed as a linear combination of standard unit vectors by writing

$$\mathbf{v} = (v_1, v_2) = v_1(1, 0) + v_2(0, 1) = v_1\mathbf{i} + v_2\mathbf{j}$$

$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

DEFINITION 1 Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be ***orthogonal*** (or ***perpendicular***) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n.

Recall from Formula (20) in the previous section that the angle θ between two *nonzero* vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is defined by the formula

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

It follows from this that $\theta = \pi/2$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$. Thus, we make the following definition.

EXAMPLE 1 Orthogonal Vectors

(a) Show that $\mathbf{u} = (-2, 3, 1, 4)$ and $\mathbf{v} = (1, 2, 0, -1)$ are orthogonal vectors in R^4.

(b) Let $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be the set of standard unit vectors in R^3. Show that each ordered pair of vectors in S is orthogonal.

Solution (a) The vectors are orthogonal since

$$\mathbf{u} \cdot \mathbf{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$$

Solution (b) It suffices to show that

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$$

because it will follow automatically from the symmetry property of the dot product that

$$\mathbf{j} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{i} = \mathbf{k} \cdot \mathbf{j} = 0$$

Although the orthogonality of the vectors in S is evident geometrically from Figure 3.2.2, it is confirmed algebraically by the computations

$$\mathbf{i} \cdot \mathbf{j} = (1, 0, 0) \cdot (0, 1, 0) = 0$$

$$\mathbf{i} \cdot \mathbf{k} = (1, 0, 0) \cdot (0, 0, 1) = 0$$

$$\mathbf{j} \cdot \mathbf{k} = (0, 1, 0) \cdot (0, 0, 1) = 0$$

Lines and Planes Determined by Points and Normals

One learns in analytic geometry that a line in R^2 is determined uniquely by its slope and one of its points, and that a plane in R^3 is determined uniquely by its "inclination" and one of its points. One way of specifying slope and inclination is to use a *nonzero* vector $\mathbf{n}$, called a ***normal***, that is orthogonal to the line or plane in question. For example, Figure 3.3.1 shows the line through the point $P_0(x_0, y_0)$ that has normal $\mathbf{n} = (a, b)$ and the plane through the point $P_0(x_0, y_0, z_0)$ that has normal $\mathbf{n} = (a, b, c)$. Both the line and the plane are represented by the vector equation

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0 \tag{1}$$

where P is either an arbitrary point (x, y) on the line or an arbitrary point (x, y, z) in the plane. The vector $\overrightarrow{P_0P}$ can be expressed in terms of components as

$$\overrightarrow{P_0P} = (x - x_0,\ y - y_0) \quad \textbf{[line]}$$

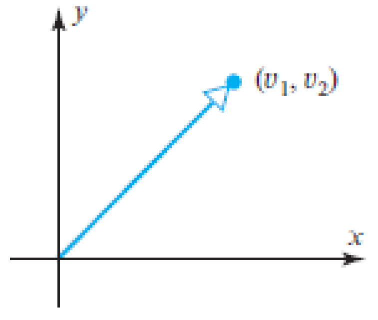
$$\overrightarrow{P_0P} = (x - x_0,\ y - y_0,\ z - z_0) \quad \textbf{[plane]}$$

Thus, Equation (1) can be written as

$$a(x - x_0) + b(y - y_0) = 0 \quad \textbf{[line]} \tag{2}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \textbf{[plane]} \tag{3}$$

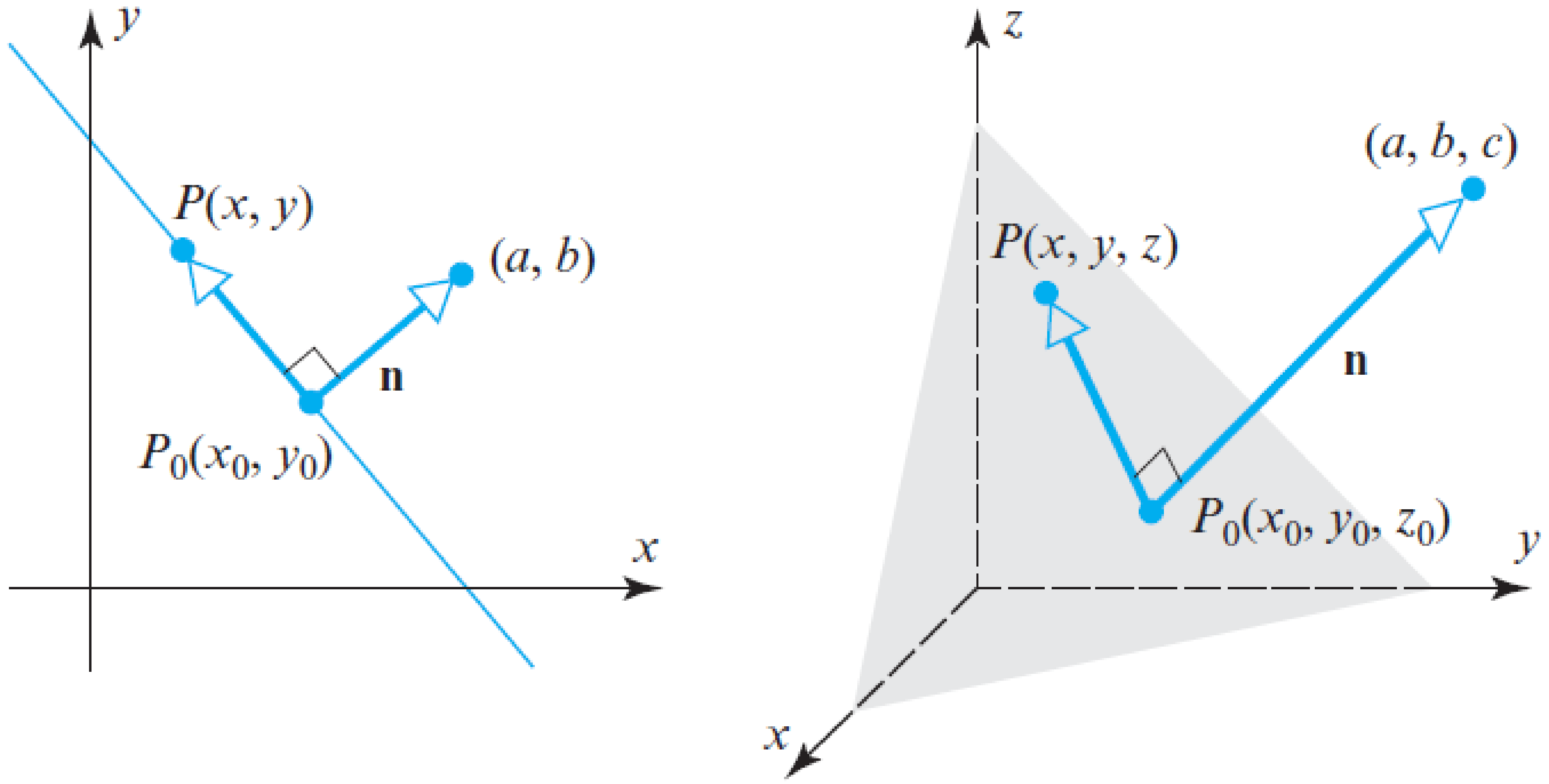
These are called the ***point-normal*** equations of the line and plane.



▲ Figure 3.1.11 The ordered pair (v_1, v_2) can represent a point or a vector.

Note, **n** above represents components of the normal vector and not coordinates.

Remark It may have occurred to you that an ordered pair (v_1, v_2) can represent either a vector with *components* v_1 and v_2 or a point with *coordinates* v_1 and v_2 (and similarly for ordered triples). Both are valid geometric interpretations, so the appropriate choice will depend on the geometric viewpoint that we want to emphasize (Figure 3.1.11).



▶ Figure 3.3.1

▶ EXAMPLE 2 Point-Normal Equations

It follows from (2) that in R^2 the equation

$$6(x - 3) + (y + 7) = 0$$

represents the line through the point $(3, -7)$ with normal $\mathbf{n} = (6, 1)$; and it follows from (3) that in R^3 the equation

$$4(x - 3) + 2y - 5(z - 7) = 0$$

represents the plane through the point $(3, 0, 7)$ with normal $\mathbf{n} = (4, 2, -5)$. ◀

Lines and Planes Determined by Points and Normals

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0 \tag{1}$$

Thus, Equation (1) can be written as

$$a(x - x_0) + b(y - y_0) = 0 \quad \textbf{[line]} \tag{2}$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \textbf{[plane]} \tag{3}$$

These are called the ***point-normal*** equations of the line and plane.

THEOREM 3.3.1

(*a*) *If a and b are constants that are not both zero, then an equation of the form*

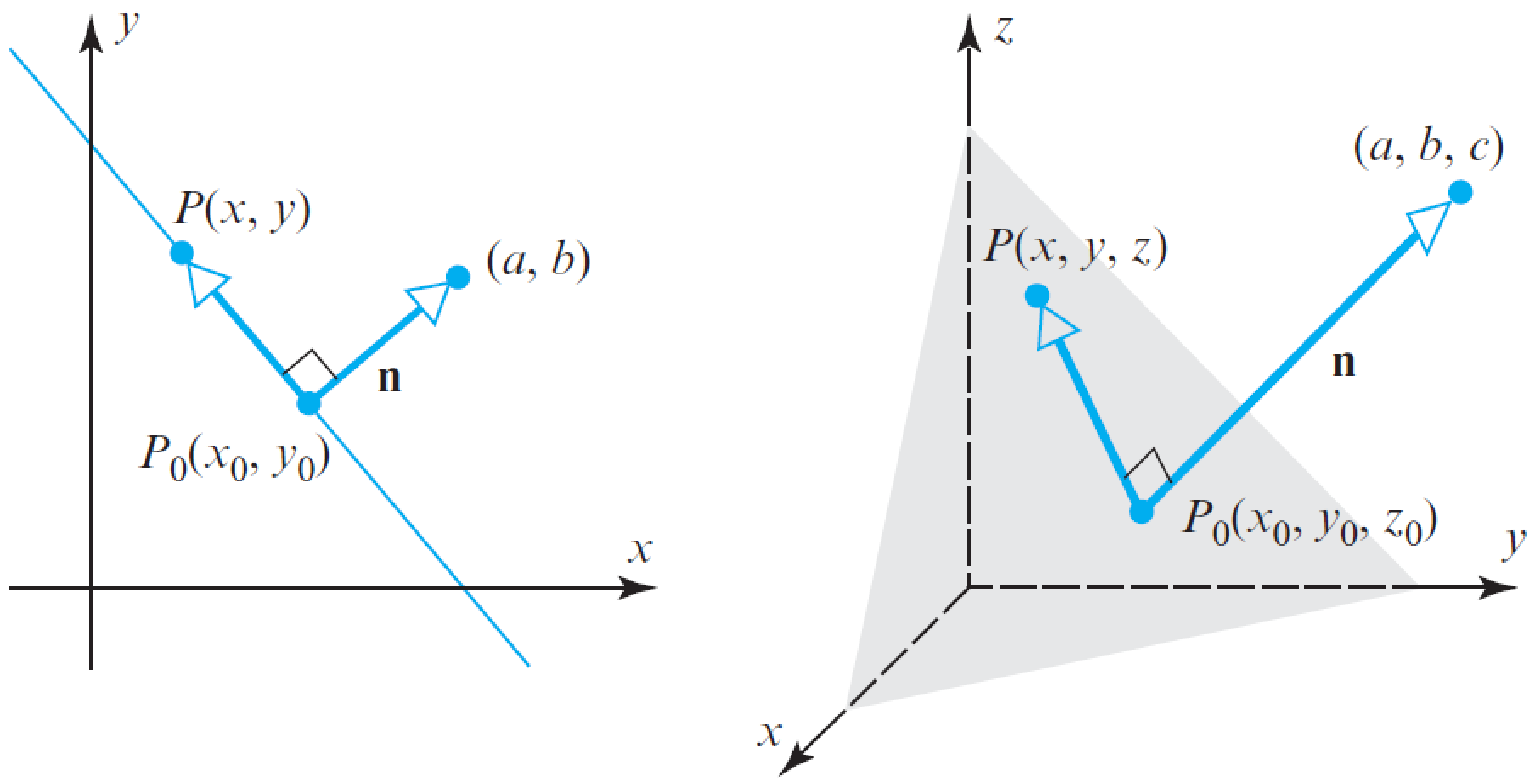
$$ax + by + c = 0 \tag{4}$$

represents a line in R^2 with normal $\mathbf{n} = (a, b)$.

(*b*) *If a, b, and c are constants that are not all zero, then an equation of the form*

$$ax + by + cz + d = 0 \tag{5}$$

represents a plane in R^3 with normal $\mathbf{n} = (a, b, c)$.



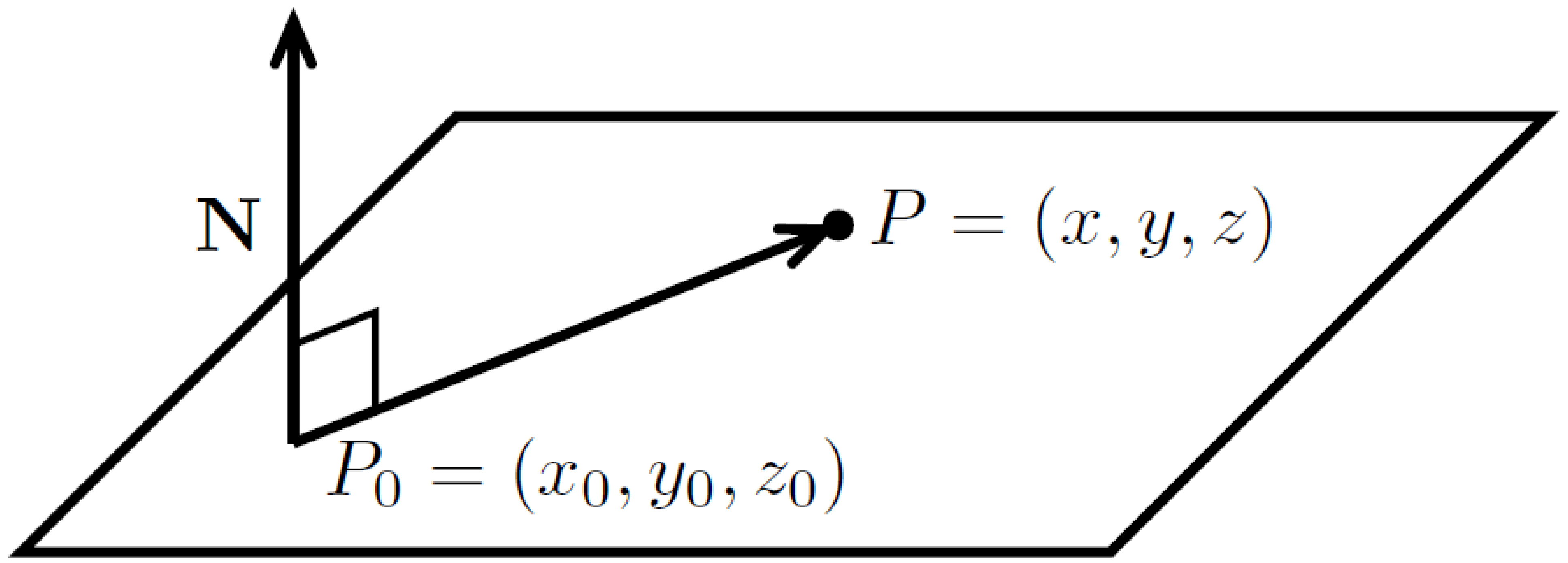
▶ Figure 3.3.1

Lines and Planes Determined by Points and Normals

Let $P = (x, y, z)$ be an arbitrary point in the plane. Then the vector $\overrightarrow{\mathbf{P_0P}}$ is in the plane and therefore orthogonal to $\mathbf{N}$. This means

$$\begin{aligned} & \mathbf{N} \cdot \overrightarrow{\mathbf{P_0P}} = 0 \\ \Leftrightarrow \quad & \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \\ \Leftrightarrow \quad & a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \end{aligned}$$

We call this last equation the point-normal form for the plane.



Example 1: Find the plane through the point (1,4,9) with normal $\langle 2, 3, 4 \rangle$.

Answer: Point-normal form of the plane is $2(x-1) + 3(y-4) + 4(z-9) = 0$. We can also write this as $2x + 3y + 4z = 50$.

Orthogonal Complements

[redacted] If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be **orthogonal to** W. The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^{\perp}$ (and read as "W perpendicular" or simply "W perp").

1. A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

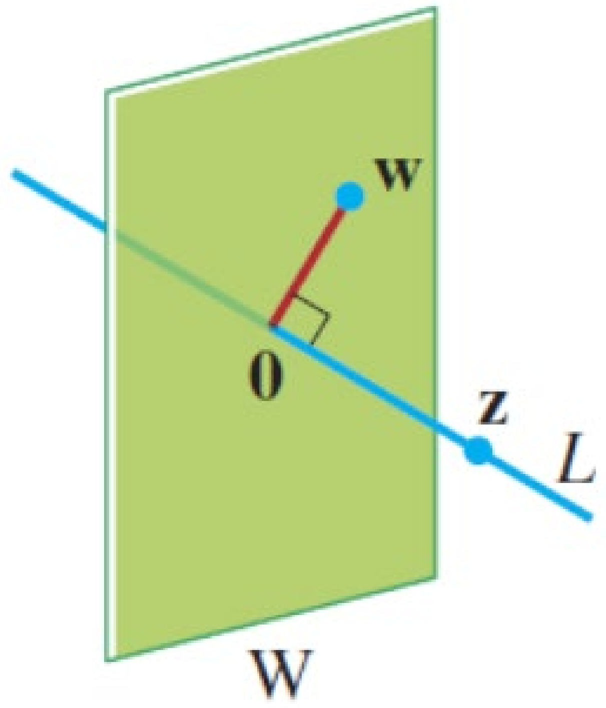


FIGURE 7

A plane and line through $\mathbf{0}$ as orthogonal complements.

EXAMPLE 6 Let W be a plane through the origin in $\mathbb{R}^3$, and let L be the line through the origin and perpendicular to W. If $\mathbf{z}$ and $\mathbf{w}$ are nonzero, $\mathbf{z}$ is on L, and $\mathbf{w}$ is in W, then the line segment from $\mathbf{0}$ to $\mathbf{z}$ is perpendicular to the line segment from $\mathbf{0}$ to $\mathbf{w}$; that is, $\mathbf{z} \cdot \mathbf{w} = 0$. See Figure 7. So each vector on L is orthogonal to every $\mathbf{w}$ in W. In fact, L consists of *all* vectors that are orthogonal to the $\mathbf{w}$'s in W, and W consists of all vectors orthogonal to the $\mathbf{z}$'s in L. That is,

$$L = W^{\perp} \quad \text{and} \quad W = L^{\perp}$$

■

Ref: Lay 5e, pg 336

Orthogonal Complements

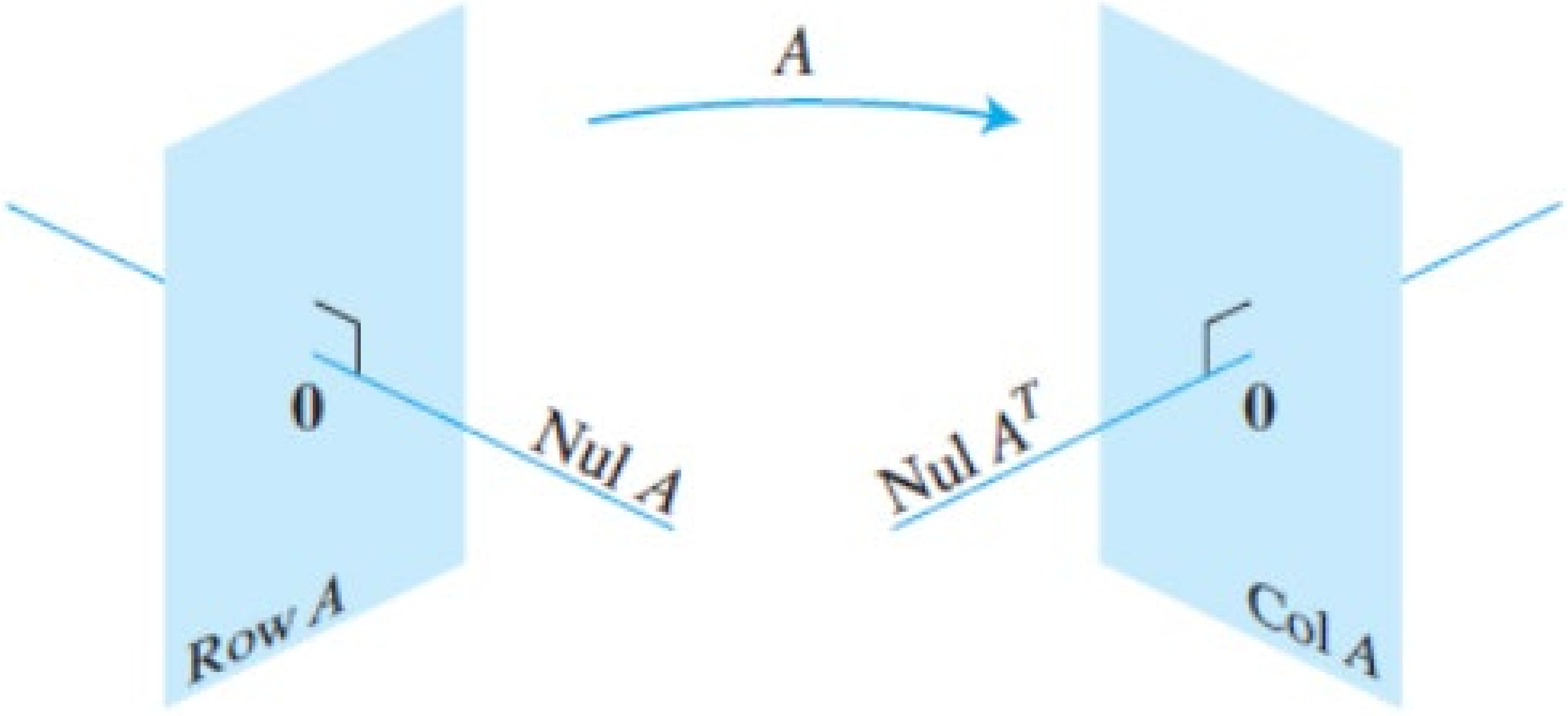


FIGURE 8 The fundamental subspaces determined by an $m \times n$ matrix A.

THEOREM 3

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\text{Row } A)^\perp = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^\perp = \text{Nul } A^T$$

PROOF The row–column rule for computing $A\mathbf{x}$ shows that if $\mathbf{x}$ is in Nul A, then $\mathbf{x}$ is orthogonal to each row of A (with the rows treated as vectors in $\mathbb{R}^n$). Since the rows of A span the row space, $\mathbf{x}$ is orthogonal to Row A. Conversely, if $\mathbf{x}$ is orthogonal to Row A, then $\mathbf{x}$ is certainly orthogonal to each row of A, and hence $A\mathbf{x} = \mathbf{0}$. This proves the first statement of the theorem. Since this statement is true for any matrix, it is true for A^T. That is, the orthogonal complement of the row space of A^T is the null space of A^T. This proves the second statement, because Row A^T = Col A. ■

Remark: A common way to prove that two sets, say S and T, are equal is to show that S is a subset of T and T is a subset of S. The proof of the next theorem that Nul A = (Row $A)^\perp$ is established by showing that Nul A is a subset of (Row $A)^\perp$ and (Row $A)^\perp$ is a subset of Nul A. That is, an arbitrary element $\mathbf{x}$ in Nul A is shown to be in (Row $A)^\perp$, and then an arbitrary element $\mathbf{x}$ in (Row $A)^\perp$ is shown to be in Nul A.

Ref: Lay 5e, pg 337