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GATE SYLLABUS

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ENGINEERING MATHEMATICS

Linear Algebra: Matrix Algebra, Systems of linear equations, Eigen values and eigen vectors.

Calculus: Mean value theorems, Theorems of integral calculus, Evaluation of definite and improper integrals, Partial Derivatives, Maxima and minima, Multiple integrals, Fourier series. Vector identities, Directional derivatives, Line, Surface and Volume integrals, Stokes, Gauss and Green's theorems.

Differential equations: First order equation (linear and nonlinear), Higher order linear differential equations with constant coefficients, Method of variation of parameters, Cauchy's and Euler's equations, Initial and boundary value problems, Partial Differential Equations and variable separable method.

Complex variables: Analytic functions, Cauchy's integral theorem and integral formula, Taylor's and Laurent' series, Residue theorem, solution integrals.

Probability and Statistics: Sampling theorems, Conditional probability, Mean, median, mode and standard deviation, Random variables, Discrete and continuous distributions, Poisson, Normal and Binomial distribution, Correlation and regression analysis.

Numerical Methods: Solutions of non-linear algebraic equations, single and multi-step methods for differential equations.

Transform Theory: Fourier transform, Laplace transform, Z-transform.

BOOKS RECOMMENDED:

TITLE	AUTHOR(S)
Higher Engineering Mathematics	B.S. Grewal
Advanced Engineering Mathematics	Erwin Kreyszig
Matrices	A.R. Vasishta
Calculus	Shanthi Narayan
Fundamentals of Mathematical Statistics	Gupta & Kapoor
Complex Variables	J.N. Sharma / Shaum's Series, Murrey.R.Speigial
Numerical Methods	S.S. Shastry / Xavier / Balaguruswamy

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Y.V. Gopala Krishna Murthy

Managing Director

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BASIC ENGINEERING MATHEMATICS

(Common to ECE, EEE & EIE)

TOPIC – 1 **LINEAR ALGEBRA**

* Matrix multiplication is associative(if conformability is assured).
  i.e. $A(BC) = (AB)C$
* Matrix multiplication is distributive w. r. t addition of matrices.
  $A.(B + C) = A . B + A . C$
* The matrix multiplication is not always commutative.
  i.e. In general, $AB \neq BA$ (AB need not be equal to BA)
* Whenever $AB = BA$, the matrices A and B are said to commute.
* The equation $AB = O$ does not necessarily imply that atleast one of the matrices A and B is a zero matrix. Eg: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
  i.e.. The product of two non zero matrices can be a zero matrix
* If product of two non zero square matrices A & B is a zero matrix then A and B are singular matrices i.e. $|A| = 0$ and $|B| = 0$
* If A is non singular (i.e. $|A| \neq 0$) and $A.B = 0$ then B is a zero matrix.

UPPER TRIANGULAR MATRIX: A square matrix is said to be upper triangular if all the elements below its principal diagonal are zeros.

LOWER TRIANGULAR MATRIX: A square matrix is lower triangular if all the elements above its principal diagonal are zeros.

DIAGONAL MATRIX: A square matrix is said to be diagonal if all the elements below and above the principal diagonal are zeros.

Note : Product of two diagonal matrices of the same order is a diagonal matrix and follows commutative law (i.e. $AB = BA$).

SCALAR MATRIX: It is a diagonal matrix with same diagonal elements.

UNIT MATRIX(OR IDENTITY MATRIX): A Scalar matrix whose diagonal elements are all equal to 1.

TRACE: Trace of a matrix is the sum of the elements of the principal diagonal.
* $tr(\lambda A) = \lambda.tr(A)$
* $tr(A + B) = tr(A) + tr(B)$
* $tr(AB) = tr(BA)$

TRANSPOSE: Transpose of a matrix A can be obtained by interchanging the rows and columns of A.
It is denoted by A^1 or A^T.
* If A is a matrix of order m x n then A^1 is a matrix of order n x m.
* $(A^1)^1 = A$
* $(A + B)^1 = A^1 + B^1$
* $(kA)^1 = kA^1$
* $(AB)^1 = B^1A^1$

SYMMETRIC AND SKEW SYMMETRIC MATRICES: Let A is square matrix, A is symmetric if $A^1 = A$, and A is skew symmetric if $A^1 = -A$.

* If A is a square matrix then

i) $A + A^1$ is symmetric.

ii) $A - A^1$ is skew symmetric.

iii) $A \cdot A^1$ and $A^1 \cdot A$ are symmetric

* Every square matrix can be written as the sum of a symmetric and a skew symmetric matrices.

i.e. $A = 1/2\,(A + A^1) + 1/2(A - A^1)$

* If A and B are symmetric then AB + BA is symmetric and AB – BA is skew symmetric.

* If A is symmetric then A^n is symmetric($n = 2, 3, 4, \ldots$).

* If A is skew symmetric, then A^n is skew symmetric when n is odd and A^n is symmetric when n is even .

INVERSE OF A MATRIX: If A is a non singular matrix of order n. then a matrix B, if it exists such that

$AB = BA = I_n$ is called inverse of A

* A^{-1} exists $\Leftrightarrow |A| \neq 0$

* $A^{-1} = \text{adj } A / |A|$

* Every non singular matrix possesses a unique inverse

Note :

01. (i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ (iii) $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

02. (a) $(A^{-1})^T = (A^T)^{-1}$

(b) If a non-singular matrix A is symmetric $\Rightarrow$ A^{-1} and A^1 are also symmetric.

Orthogonal Matrix : A square matrix A is said to be orthogonal matrix if $A.A^1 = I = A^1.A$

Note :

01. If A is orthogonal then $A^1 = A^{-1}$

02. (a) If A is orthogonal then A^T & A^{-1} are also orthogonal.

(b) If A & B are orthogonal of the same order then AB is also orthogonal.

PROPERTIES OF DETERMINANTS :-

01. If 'A' is a square matrix then $|A| = |A^T|$

Note: - In a general manner a row or a column is referred as a line.

02. If two parallel lines of a determinant are interchanged, then the determinant retains its numerical value but changes in sign.

Note: - In general, if any line of a determinant be passed over 'm' parallel lines, the resulting determinant $= (-1)^m \Delta$.(where Δ is the initial determinant value)

03. A determinant vanishes if two parallel lines are identical.

04. If each element of a line be multiplied by the same factor, then the whole determinant is multiplied by that factor.

Ex :- 1. $\begin{vmatrix} ka_1 & kb_1 \\ a_2 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

2. $\begin{vmatrix} a_1 & b_1 & lc_1 \\ a_2 & b_2 & lc_2 \\ a_3 & b_3 & lc_3 \end{vmatrix} = l \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Note:- In a determinant, if $R_i = k\,R_j$ (or $C_i = k\,C_j$) then the value of the determinant is zero.

Ex:- $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_1 & kb_1 & kc_1 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = k(0) = 0$ (from property-03)

05. $\begin{vmatrix} a_1 & b_1+c_1 & d_1 \\ a_2 & b_2+c_2 & d_2 \\ a_3 & b_3+c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$

06. If to each elements of a line be added equi- multiples of the corresponding elements of one or more parallel lines the determinant remains unaltered.

07. The determinant of an upper / a lower triangular / diagonal / scalar matrix is equal to the product of leading diagonal elements of the matrix.

08. A, B are square matrices of the same order then $|AB| = |A|\,|B| = |BA|$

09. If A is a non singular matrix (i.e. $|A| \neq 0$) then $|A^{-1}| = \dfrac{1}{|A|}$

10. If A is a square matrix of order-n then (i) $|\text{Adj } A| = |A|^{n-1}$

(ii) $|\text{Adj}(\text{Adj}A)| = |A|^{(n-1)^2}$

11. Determinant of a Skew – symmetric matrix (i.e. $A^T = -A$) of odd order is zero.

12. If A is an orthogonal matrix (i.e. $A^T = A^{-1}$) then $|A| = \pm 1$.

13. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta^1 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

where A,B,C are co-factors of a,b,c then $\Delta^1 = \Delta^2$, which is called **Reciprocal / Adjucate** determinant of Δ.

14. $|I_n| = 1 \quad \forall\ n \in z^+$

15. If 'A' is a square matrix of order 'n' then $|kA| = k^n\,|A|$ where k is a constant.

RANK OF A MATRIX: It is the order of its largest non vanishing minor of the matrix.

* ***Rank of the matrix*** is equal to the number of linearly independent rows(cols) in the matrix.

Note :

1. If a matrix A is of rank-r, then A contains 'r' linearly independent *Vectors* (here vector is either row / column of the matrix).
2. **(i)** $\rho(A+B) \leq \rho(A) + \rho(B)$ **(ii)** $\rho(A-B) \geq \rho(A) - \rho(B)$ **(iii)** $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$

3. Rank of a diagonal matrix is equal to the number of non-zero elements in the diagonal.

* Rank of matrix A is equal to the number of non zero rows(columns) in the row(column) ***Echelon form*** of A.

Note :

01. If A is a non-singular matrix then all the rows /columns (vectors) of A are linearly independent.

02. If A is a singular matrix then vectors of A are linearly dependent.

Non-Homogenous System of Linear Equations :

* The system of linear equations $AX = B$ has a solution (consistent) if and only if Rank of A = Rank of [A | B]

* The system $AX = B$ has
  i) A unique solution if and only if Rank(A) = Rank(A | B) = number of variables.
  ii) Infinitely many solutions $\Leftrightarrow \rho(A) = \rho(A|B) <$ number of variables.
  iii) No solution if $\rho(A) \neq \rho[A \mid B]$ i. e. $\rho(A) < \rho(A \mid B)$

Homogenous System of Linear Equations :

* The system $AX = O$ has
  i) Unique solution (***zero*** solution or ***trivial*** solution) if $\rho(A)$ = number of variables (n).
  ii) Infinitely many number of non-zero (or ***non-trivial***) solutions if $\rho(A) < n$.

Note :

01. In the system of homogenous linear equation $AX = O$
  (i) If A is singular then the system possesses non-trivial solution. (i.e. non-zero solution).
  (ii) If A is non-singular then the system possesses trivial solution (i.e. zero solution).

02. If $\rho(A) = r$, and number of variables = n then, the number of linearly independent solutions of $AX = O$ is $\underline{(n-r)}$.

03. The system of homogeneous linear equations such that the number of unknowns (or variables) exceeds the number of equations necessarily possesses a non-zero solution.

EIGEN VALUES AND EIGEN VECTORS:

Let A be a square matrix of order n and λ be a scalar.
$|A - \lambda I| = 0$ is called the ***characteristic equation*** of A.

The roots of characteristic equation are called ***eigen values*** (***characteristic roots / latent roots***) of A.
Corresponding to each eigen value λ, there exists a <u>*non-zero solution*</u> X such that $(A - \lambda I) X = 0$.
X is called ***eigen vector*** (***characteristic vector or latent vector***) of A.

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS

01. The sum of the eigen values of a matrix is the sum of the principal diagonal elements (i.e. Trace).
02. The Product of the eigen values of a matrix is equal to the determinant of the matrix.

    Note:- In particular, the eigen values of a Diagonal / Scalar / Triangular (either upper or lower) matrix are just the leading diagonal elements and product of the eigen values is just the product of leading diagonal elements.

03. The eigen values of A^T are same as the eigen values of A.
04. If λ is an eigen value of a non-singular matrix A then
    (a) $1/\lambda$ is an eigen value of A^{-1}. $(AA^{-1} = I = A^{-1}A)$
    (b) $|A|/\lambda$ is an eigen value of Adj A . $(A.\text{Adj } A = |A| I)$
05. If λ is an eigen value of an Orthogonal matrix A then $1/\lambda$ is also an eigen value of A $(A^T = A^{-1})$
06. If $\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots \lambda_n$ are eigen values of A, then
    (i) The eigen values of kA are $k\lambda_1, k\lambda_2 \ldots k\lambda_n$. (where k is a scalar)
    (ii) A^m has eigen values $\lambda_1^m, \lambda_2^m \ldots\ldots\lambda_n^m$ (where $m \in z^+$)
    (iii) $A \pm kI$ has eigen values $\lambda_1 \pm k, \lambda_2 \pm k \ldots\ldots\ldots \lambda_n \pm k$
    (iv) $(A \pm kI)^2$ has eigen values $(\lambda_1 \pm k)^2, (\lambda_2 \pm k)^2 \ldots\ldots\ldots (\lambda_n \pm k)^2$
07. The eigen values of an orthogonal matrix have absolute value '1'
08. The eigen values of a symmetric matrix are purely real.
09. The eigen values of skew-symmetric matrix are either purely imaginary or zero.
10. The set of all characteristic roots of a matrix is called ***Spectrum*** of the matrix.
11. Zero is an eigen value of a matrix if and only if the matrix is singular.
12. λ is an eigen value of a non-singular matrix $\Leftrightarrow \lambda \neq 0$.
13. If λ is an eigen value of a matrix A, then the corresponding eigen vector X is not unique.
    i.e. we have infinite number of eigen vectors corresponding to a single eigen value.
14. If $\lambda_1, \lambda_2 \ldots\ldots\ldots \lambda_n$ ~~be distinct~~ eigen values of an n x n matrix A, then the ________ corresponding eigen vectors $X_1, X_2 \ldots\ldots X_n$ form a linearly independent set.
15. For An x n, if some eigen values are repeated, then it may/may not be possible to get 'n' linearly independent eigen vectors for A.

PROBLEMS

1. If AB = BA then which of the following need not be true (n is a +ve integer)
   a) $AB^n = B^nA$ b) $(AB)^n = A^nB^n$
   c) $(A + B)(A - B) = A^2 - B^2$ d) A = I or B = I

2. If a diagonal matrix is commutative with every matrix of the same order then it is necessarily a
   a) a zero matrix   b) a unit matrix   c) a scalar matrix   d) a symmetric matrix

3. If A is any m x n matrix such that AB and BA are both defined then B is a matrix of order
   a) n x n   b) m x m   c) m x n   d) n x m

4. If $A_{m \times n}$ and $B_{n \times p}$ are matrices, then the number of multiplications and additions in computing AB are
a) m p n, mp(n – 1) b) (m – 1) pn, mp(n – 1) c) m p n, (m – 1)(n – 1) d) mn, nm

5. The number of terms in the expansion of the determinant of $A_{n \times n}$ is
a) n^2 b) 2^n c) n! d) n

6. If A = $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & 1 & -3 \end{pmatrix}$ and Adj A = $\begin{pmatrix} -3 & 4 & k \\ -3 & -1 & 4 \\ -3 & 1 & 1 \end{pmatrix}$ then k =
a) –2 b) 4 c) –5 d) 6

7. Rank of unit matrix I_n is
a) one b) zero c) n d) not defined

8. Rank of a non – singular matrix $A_{n \times n}$ is
a) one b) zero c) n d) not defined

9. Rank of a singular matrix $A_{n \times n}$ is
a) n b) zero c) <n d) >n

10. Rank of a diagonal matrix $A_{n \times n}$ is
a) n b) no. of zeros in the diagonal
c) no. of non zero elements in the diagonal d) zero

11. If $A_{m \times 1}$ is non zero column matrix and $B_{1 \times n}$ is a non zero row matrix then ρ(AB) =
a) m b) n c) 1 d) zero

12. If $A_{n \times n}$ is a non singular matrix and $B_{n \times n}$ is a matrix then ρ(AB) =
a) Rank of A b) ρ(B) c) 0 d) 1

13. If $\rho(A_{n \times n})$ is equal to (n – 2) then ρ(Adj A) =
a) (n – 1) b) (n – 2) c) (n – 3) d) zero

14. $\rho(A_{3 \times 3}) = 2$ then | A | =
a) 1 b) 0 c) Any non zero number d) 2

15. If ρ(A) = n then which of the following is false
a) ∃ at least one non zero minor of order 'n'
b) All the minors of A of order greater than n vanish
c) All the minors of A of order n are not zero.
d) If A is nxn matrix then |A| ≠ 0 .

16. If A = $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ then ρ(A) =
a) 2 b) 1 c) 0 d) does not exist

17. If A = $\begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$ then ρ(A) =
a) 2 b) 1 c) 0 d) does not exist

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18. If A = $\begin{pmatrix} 2 & 3 & -1 \\ -4 & 6 & 1 \end{pmatrix}$ then ρ(A) =
a) 0 b) 1 c) 2 d) 3

19. If A = $\begin{pmatrix} 2 & -2 \\ -1 & 1 \\ 3 & -3 \end{pmatrix}$ then ρ(A) =
a) 0 b) 1 c) 2 d) 3

20. If A = $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 4 \\ 2 & 3 & -1 \end{pmatrix}$ then ρ(A) =
a) 0 b) 1 c) 2 d) 3

21. If A = $\begin{pmatrix} 2 & -3 & -1 \\ -4 & 6 & 2 \\ 6 & -9 & -3 \end{pmatrix}$ then ρ(A) =
a) 0 b) 1 c) 2 d) 3

22. If A = $\begin{pmatrix} 2 & -3 & 4 \\ 3 & -2 & 5 \\ 1 & 1 & 1 \end{pmatrix}$ then ρ(A) =
a) 0 b) 1 c) 2 d) 3

23. Which of the statements is false.
Rank of the matrix is equal to
a) The no. of its linearly independent rows b) The no. of its linearly independent columns
c) The no. of non – zero rows - d) The order of its largest non – vanishing minor

24. The system AX = B has no solution if
a) ρ(A) = ρ(A/B) b) ρ(A) < ρ(A/B) c) ρ(A) > ρ(A/B) d) ρ(A) ≥ ρ(A/B)

25. The system AX = 0 in n variables has infinitely many solutions if
a) ρ(A) = n b) ρ(A) > n c) ρ(A) < n d) ρ(A) ≥ n

26. If A is a square matrix then the system AX = 0 has non zero solutions when
a) |A| ≠ 0 b) |A| = 0 c) ρ(A) = no. of variables d) ρ(A) ≥ no. of variables

27. If |A| ≠ 0 then for the system AX = 0, which of the following is false
a) The system has unique solution b) The system has a zero solution
c) The system has a trivial solution d) The system has a non - zero solution

28. The system AX = B has a unique solution if
a) ρ(A) = no. of variables in the system (n) = ρ(A/B) b) ρ(A) < n
c) ρ(A) > n d) ρ(A) = ρ(A/B) < n

29. If ρ(A) = r and number of variables = n, then the number of linearly independent solutions of the system AX = 0 is
a) n b) r c) n – r d) n + r

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30. The system $2X + 3Y + 4Z = 1$
$3Y - Z = 2$
$-6Y + 2Z = 3$ has
a) no solution b) unique solution
c) infinitely many solutions d) two linearly independent solutions

31. If the system $X + Y + Z = 0$
$(\lambda + 1)(Y) + (\lambda + 1)(Z) = 0,$
$(\lambda^2 - 1) Z = 0$ has two linearly independent solutions, then $\lambda =$
a) 1 b) –1 c) 0 d) 3

32. The system given in example 31 has only one independent solution when $\lambda =$
a) 1 b) –1 c) 0 d) 3

33. The system given in example 31 has no linearly independent solutions when $\lambda =$
a) 1 b) –1 c) 0 d) ± 1

34. If $\rho(A) = 1$ and number of variables = 3, then the system $AX = 0$ has
a) Two linearly independent solutions b) 3 linearly independent solutions
c) 1 linearly independent solutions d) no independent solutions

35. The rank of the matrix, every element of which is unity is =
a) one b) zero c) order of the matrix d) >1

36. If A is a skew symmetric matrix then which of the following is false
a) $\rho(A) = 1$ b) $\rho(A) = 0$ c) $\rho(A) = 2$ d) $\rho(A) = 3$

37. Rank of an elementary matrix =
a) order of the matrix b) one c) zero d) two

38. When elementary transformations are applied on a given matrix then its rank
a) will remain same b) reduced by one
c) becomes one d) is equal to the order of matrix

39. $\rho(A)_{m \times n} = r$ then
a) $r \le m, r \le n$ b) $r > m, r \le n$ c) $r > m, r > n$d) d) $r \le m, r > n$

40. Find the rank of the matrix $\begin{pmatrix} 2 & 3 & -1 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 5 \end{pmatrix}$ is =
a) 1 b) 2 c) 3 d) 4

41. If A and B are any two matrices then which of the following is false
a) $\rho(AB) < \rho(A)$ b) $\rho(AB) \le \rho(A)$ c) $\rho(AB) < \rho(B)$ d) $\rho(AB) > \rho(B)$

42. If a square matrix A is orthogonal then
a) $A = A^{1}$ b) $A^{1} = A^{-1}$ c) $A = \overline{A}$ d) $A^{1} = \overline{A}$

43. If $\overline{(A^{1})} = A$ then A is
a) orthogonal b) symmetric c) hermitian d) unitary

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44. If A is skew - hermitian then iA is
a) hermitian b) symmetric c) unitary d) skew – symmetric

45. If a matrix is in row echelon form then its rank is =
a) no. of non – zero rows of the matrix b) no. of zero rows of the matrix
c) order of the matrix d) one

46. If $A_{n \times n}$ is a non singular matrix then which of the following is false
a) $\rho(A) = \rho(A^{1})$ b) $\rho(A) = \rho(A^{-1})$ c) $\rho(A) = \rho(I_n)$ d) $\rho(A) = 1$

47. If $\rho(A) = n$ then $\rho(\text{Adj } A) =$
a) n b) n – 1 c) 1 d) 0

48. If $\rho(A) = n - 1$ then $\rho(\text{Adj } A) =$
a) n b) n – 1 c) 1 d) zero

49. The system $3x + 4y - 2z = 4,$
$6x + 8y - 4z = 10$ has
a) no solution b) infinitely many solutions
c) unique solution d) one linearly independent solution

50. Rank of a non zero matrix is
a) equal to order of the matrix b) equal to the number of rows of the matrix
c) < 1 d) ≥1

51. Eigen values of a matrix $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ are
a) (2, 3) b) (-2, -3) c) (1, 6) d) (-1, -6)

52. Eigen values of the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ are
a) (1, 2, 3) b) (1, 1, 1) c) (2, 2, 2) d) (0, 1, 2)

53. Eigen values of the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ are
a) (3, 2, -1) b) (3, 2, 1) c) (3, 3, -1) d) (4, -1, 0)

54. If '0' and '3' are eigen values of the matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ then the third eigen value is
a) 7 b) 8 c) 15 d) 0

55. If $-1 + \sqrt{3}$ is eigen value of matrix $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$ then the other two eigen values are
a) $(-1-\sqrt{3}, 2)$ b) $(-1+\sqrt{3}, 1)$ c) $(-1-\sqrt{3}, -2)$ d) $(-1+\sqrt{3}, -2)$

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56. If 3 is the eigen value of the singular matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ then the other eigen values of matrix are

a) (0, 15) b) (8, 15) c) (7, 15) d) (0, 3, 2)

57. Which of the following is an eigen vector of the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

a) (1, 0, 0) b) (1, 0, 1) c) (0, 0, 1) d) (1, 1, 1)

58. Which of the following is an eigen vector of the matrix $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$ corresponding to eigen value $\lambda = 6$

a) (4, -1) b) (1, -4) c) (-1, -4) d) (4, 1)

59. If 2, 2, 8 are eigen values of a matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ then the matrix is

a) singular b) non – singular c) skew symmetric d) triangular

60. If 2, 2, 8 are eigen values of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & k \end{pmatrix}$ then k =

a) 0 b) 1 c) 2 d) 3

For the matrix A = $\begin{pmatrix} 3 & -2 & 5 \\ 0 & 5 & 6 \\ 0 & 0 & -4 \end{pmatrix}$ answer the following

61. The eigen values of A^3 are
a) (3, 5, -4) b) (9, 25, 16) c) (27, 125, -64) d) (9, 15, -12)

62. The eigen values of A^{-1} are
a) (1/3, 1/5, -1/4) b) (1/3, 1/5, 1/4) c) (-3, -5, -4) d) (-1/3, -1/5, 1/4)

63. The eigen values of 9A are
a) (27, 45, -36) b) (12, 14, 5) c) (9, 15, -12) d) $(3^9, 5^9, 4^9)$

64. The eigen values of Adj(A) are
a) (3, 5, -4) b) (9, 25, 16) c) (20, 12, -15) d) (-20, -12, 15)

65. The eigen values of A^1 are
a) (3, 5, -4) b) (-3, -5, 4) c) (1/3, 1/5, -1/4) d) does not exist

66. For the matrix $\begin{pmatrix} 2 & -3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ the eigen values of A^{-1} are

a) (2, 1, 0) b) (1/2, 1, ∞) c) (-2, -1, 0) d) does not exist

67. The number of linearly independent eigen vectors corresponding to any distinct eigen value of the matrix $A_{3\times3}$ =
a) 1 b) 2 c) 3 d) cannot be determined

68. If an eigen value λ is repeated two times for a matrix $A_{3\times3}$ then the no. of linearly independent eigen vectors for λ are
a) 2 b) < 2 c) ≤ 2 d) > 2

69. For the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which of the following is an eigen vector

a) (1, 0, 1) b) (1, 1, 0) c) (1, 0, 1) d) (1, 1, 1)

70. For the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ which of the following is not a eigen vector

a) (1, 0, 0) b) (0, 1, 0) c) (0, 0, 1) d) (0, 0, 0)

71. If $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ then A^8 =

a) 5I b) 25I c) 625I d) 3125I

72. If A = $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 5 & 6 \\ 0 & 0 & i & 7 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ then A^4 =

a) I_4 b) $4I_4$ c) $16I_4$ d) $64I_4$

73. If $\lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + \ldots\ldots + K_n = 0$ is the characteristic equation of a matrix A then A^{-1} exists if
a) $K_1 = 0$ b) $K_1 \neq 0$ c) $K_n = 0$ d) $K_n \neq 0$

74. The sum and product of the eigen values of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ are__ and __
a) 7, 12 b) 7, 5
c) 12, 5 d) 7, 9

75. The eigen values of a triangular matrix are
a) diagonal elements of the matrix b) zero c) non – diagonal elements d) ±1

76. The eigen values of a real skew symmetric matrix are
a) real b) ±1 c) purely imaginary or zero d) does not exist

77. The eigen values of an orthogonal matrix are
a) real b) ±1 c) purely imaginary d) of unit modulus

78. For each eigen value of the matrix $A_{3\times3}$, the no. of eigen vectors =
a) 1 b) 2 c) 3 d) ∞

79. If a matrix $A_{3\times3}$ has 3 distinct eigen values then the no. of linearly independent eigen vectors for A =
a) 1 b) 2 c) 3 d) α

80. If zero is eigen value of a square matrix A, then A is
a) singular b) non singular c) orthogonal d) symmetric

81. If 2 is eigen value of a scalar matrix A_{3x3}, then an eigen value of Adj A is
a) 2 b) 3 c) 4 d) 8

82. Which of the following is not an eigen vector of a 2 x 2 unit matrix
a) (0, 0) b) (1, 1) c) (1, 0) d)(0, 1)

Previous Gate Questions :

83. What values of x, y, z satisfy the following system of linear equations.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix}$$ (Gate-2004)

a) x=6, y=3, z=2 b) x=12, y=3, z= -4 c) x=6, y=6, z= -4 d) x=12, y=-3, z=4

84. If matrix $X = \begin{pmatrix} a & 1 \\ -a^2+a-1 & 1-a \end{pmatrix}$ and $X^2 - X + I = 0$. Then the inverse of X is (Gate-2004)

a) $\begin{pmatrix} 1-a & -1 \\ a^2 & a \end{pmatrix}$ b) $\begin{pmatrix} 1-a & -1 \\ a^2-a+1 & a \end{pmatrix}$ c) $\begin{pmatrix} -a & 1 \\ -a^2+a-1 & a-1 \end{pmatrix}$ d) $\begin{pmatrix} a^2-a+1 & a \\ 1 & 1-a \end{pmatrix}$

85. The number of different n×n symmetric matrices with each element being either 0 or 1 is (Gate-2004)
a) 2^n b) 2^{n^2} c) $2^{\frac{n^2+n}{2}}$ d) $2^{\frac{n^2-n}{2}}$

86. Let A, B, C, D be n×n matrices, each with non-zero determinent. ABCD=I then B^{-1} = (Gate-2004)
a) $D^{-1}C^{-1}A^{-1}$ b) CDA c) ADC d) does not exist

87. How many solutions does the following system of linear equations have (Gate-2004)
$-x+5y=-1$ $x-y=2$ $x+3y=3$
a) infinitely many b) two distinct solutions c) unique d) none

88. In an n×n matrix such that non zero entries are covered in a rows and b columns. Then the maximum number of non zero entries, such that no two are on the same row or column is
a) ≤ (a+b) b) ≤ max(a,b) c) ≤ min (M-a, N-b) d) ≤ min (a,b) (Gate-2004)

89. Consider the following system of linear equations

$$\begin{pmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ 5 \\ 7 \end{pmatrix}$$ (Gate-2003)

Notice that the second and third columns of the coefficient matrix are linearly dependent. For how many value of α, does the system of equations have infinitely many solutions.
a) 0 b) 1 c) 2 d) infinitely many

90. The rank of the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is (Gate-2002)

a) 4 b) 2 c) 1 d) 0

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91. Obtain the eigen values of the matrix $A = \begin{pmatrix} 1 & 2 & 34 & 49 \\ 0 & 2 & 43 & 94 \\ 0 & 0 & -2 & 104 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ (Gate-2002)

a) 1, 2, -2, -1 b) -1, -2, -1, -2 c) 1, 2, 2, 1 d) none

92. Consider the following statements
S_1 : The sum of two singular matrices may be singular
S_2 : The sum of two non-singular matrices may be non-singular. (Gate-2001)
Which of the following statements is true.
a) S_1 & S_2 are both true b) S_1 & S_2 are both false
c) S_1 is true and S_2 is false d) S_1 is false and S_2 is true

93. The determinant of the matrix $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 6 & 1 \end{pmatrix}$ is (Gate-2000)

a) 4 b) 0 c) 15 d) 20

94. An n×n array V is defined as follows
V[i, j] = i-j for all i, j 1≤ i≤ n, 1≤j ≤ n. (Gate-2000)
The sum of elements of the array V is
a)0 b) n-1 c) $n^2 - 3n + 2$ d) n(n+1)

KEY

1. d	2. c	3. d	4. a	5. c	6. c	7. c	8. c	9. c	10. c	11. c	12. b
13. d	14. b	15. c & d	16. a	17. b	18. c	19. b	20. d	21. b	22. c	23. c	
24. b	25. c	26. b	27. d	28. a	29. c	30. a	31. b	32. a	33. c	34. a	35. a
36. a	37. a	38. a	39. a	40. d	41. d	42. b	43. c	44. a	45. a	46. d	47. a
48. c	49. a	50. d	51. c	52. c	53. b	54. c	55. a	56. a	57. a	58. d	59. b
60. d	61. c	62. a	63. a	64. d	65. a	66. d	67. a	68. c	69. b	70. d	71. c
72. a	73. d	74. b	75. a	76. c	77. d	78. d	79. c	80. a	81. c	82. a	83. c
84. b	85. c	86. b	87. c	88. d	89. b	90. c	91. a	92. a	93. a	94. a	

MATRIX ALGEBRA (ADDITIONAL PROBLEMS)

1. Consider the following system of equations in three real variables x_1, x_2 and x_3 :
$2x_1 - x_2 + 3x_3 = 1$; $3x_1 + 2x_2 + 5x_3 = 2$; $-x_1 + 4x_2 + x_3 = 3$
This system of equations has *(GATE '05)*
a) no solution b) a unique solution
c) more than one but a finite number of solutions
d) an infinite number of solutions

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2. What are the eigen values of the following 2 × 2 matrix ? *(GATE '05)*

$$\begin{pmatrix} 2 & -1 \\ -4 & 5 \end{pmatrix}$$

a) – 1 and 1 b) 1 and 6 c) 2 and 5 d) 4 and –1

3. Consider the matrices $X_{4\times3}$ $Y_{4\times3}$ and $P_{2\times3}$. The order of $[P\,(X^T Y)^{-1} P^T]^T$ will be
a) 2 × 2 b) 3 × 3 c) 4 × 3 d) 3 × 4 *(GATE '05)*

4. Consider a non homogeneous system of linear equations representing mathematically an over determined system. Such a system will be *(GATE '05)*
a) Consistent having a unique solution b) Consistent having many solutions
c) inconsistent having a unique solution d) inconsistent having no solution

5. Consider the system of equations *(GATE '05)*

$$A_{n\times n} X_{n\times 1} = \lambda\, X_{n\times 1}$$

Where λ is a scalar. Let (λ_i, X_i) be an eigen value and its corresponding eigen vector for real matrix A. Let $I_{n\times n}$ be unit matrix. Which one of the following statement is not correct.
a) For a homogeneous n × n system of linear equations, $(A - \lambda I)\,X = 0$, having a non trivial solution, the rank of $(A - \lambda I)$ is less than n
b) For matrix A^m, m being a positive integer, (λ_i^m, X_i^m) will be the Eigen pair for all i
c) If $A^T = A^{-1}$ then $|\lambda_i| = 1$ for all i
d) If $A^T = A$ then λ_i are real for all i

6. The determinant of the matrix given below is *(GATE '05)*

$$\begin{pmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{pmatrix}$$

a) – 1 b) 0 c) 1 d) 2

7. In the matrix equation PX = Q which of the following is a necessary condition for the existence of atleast one solution for the unknown vector X
a) Augmented matrix [P Q] must have the same rank as matrix P
b) Vector Q must have only non zero elements
c) Matrix P must be singular d) Matrix P must be square

8. For the matrix *(GATE '05)*

$$P = \begin{pmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

One of the eigen values is –2. Which of the following is an eigen vector ?

a) $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ b) $\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$ c) $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ d) $\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$

9. If $R = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{pmatrix}$ the top row of R^{-1} is *(GATE '05)*
a) [5 6 4] b) [5 -3 1] c) [2 0 -1] d) [2 -1 0]

10. The eigen values of the matrix M given below are 15, 3 and 0. *(GATE '05)*

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & n & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

The value of the determinant of the matrix is
a) 20 b) 10 c) 0 d) – 10

11. A is a 3 × 4 matrix and AX = B is an inconsistent system of equations. The highest possible rank of A is *(GATE '05)*
a) 1 b) 2 c) 3 d) 4

12. Which one of the following is an eigen vector of the matrix *(GATE '05)*

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

a) $[1\ \ -2\ \ 0\ \ 0]^T$ b) $[0\ \ 0\ \ 1\ \ 0]^T$ c) $[1\ \ 0\ \ 0\ \ -2]^T$ d) $[1\ \ -1\ \ 2\ \ 1]^T$

13. Let A be a 3 × 3 matrix with rank 2. Then AX = O has *(GATE '05)*
a) only the trivial solution X = 0 b) one independent solution
c) two independent solutions d) three independent solutions

14. Identify which one of the following is an eigen vector of the matrix $A = \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}$
a) $[-1\ \ 1]^T$ b) $[3\ \ -1]^T$ c) $[1\ \ -1]^T$ d) $[-2\ \ 1]^T$ *(GATE '05)*

15. Let $A = \begin{pmatrix} 2 & -0.1 \\ 0 & 3 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} 1/2 & a \\ 0 & b \end{pmatrix}$ then (a + b) = *(GATE '05)*
a) 7 / 20 b) 3 / 20 c) 19/60 d) 11/20

16. Given an orthogonal matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$ *(GATE '05)*

$(A A^T)^{-1}$ is
a) ¼ I_4 b) ½ I_4 c) I d) ¼ I_4

17. If $A = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{pmatrix}$ and $\text{adj } A = \begin{pmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & k & 7 \end{pmatrix}$ then k =
a) – 5 b) 3 c) –3 d) 5 *(GATE '99)*

18. If A and B are real symmetric matrices of order n then which of the following is true
a) $AA^T = I$ b) $A = A^{-1}$ c) $AB = BA$ d) $(AB)^1 = B^1A^1$ *(GATE '94)*

19. The inverse of the matrix
$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$ is *(GATE '94)*

20. The rank of the matrix
$$\begin{pmatrix} 1 & a & a^2 \ldots\ldots\ldots & a^n \\ 1 & a & a^2 \ldots\ldots\ldots & a^n \\ \ldots\ldots\ldots & & & \\ 1 & a & a^2 \ldots\ldots\ldots & a^n \end{pmatrix}$$ of order $(n+1) \times (n+1)$ *(GATE '98)*
Where a is a real number is
a) 1 b) 2 c) n d) depends on a

21. If $\Delta = \begin{pmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{pmatrix}$ then *(GATE '98)*
Which of the following is a factor of Δ
a) a + b b) a – b c) abc d) a + b + c

22. Let $A_{n\times n}$ be a matrix of order n and I_{12} be the matrix obtained by interchanging the first and second rows of I_n. Then A . I_{12} is such that its first *(GATE '97)*
a) row is the same as its second row b) row is the same as second row of A
c) column is the same as the second column of A
d) row is a zero row

23. Let AX = B be a system of linear equations where A is an m × n matrix, B is an m × 1 column matrix. Which of the following is false ? *(GATE '96)*
a) The system has a solution , iff $\rho(A) = \rho[A \mid B]$
b) If m = n and B is a non zero vector then the system has a unique solution
c) If m < n and B is a zero vector then the system has infinitely many solutions
d) The system will have a trivial solution when m = n, B is the zero vector and rank of A is n.

KEY

1. b 2. b 3. a 4. d 5. b 6. a 7. a 8. d 9. b 10. c 11. b 12. a

13. b 14. b 15. a 16. c 17. a 18. d 19. 20. a 21. b 22. c 23. b

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PREVIOUS GATE QUESTIONS

01. Lets A be an n×n real matrix such that $A^2 = I$ and y be an n-dimensional vector. Then the linear system of equations AX = Y has **(IN-2007-1M)**
(a) no solution
(b) a unique solution
(c) more than one but infinitely many dependent solutions.
(d) infinitely many dependent solutions

02. Let A= $\lfloor a_{ij} \rfloor$, $1 \le i, j \le n$ with $n \ge 3$ and $a_{ij} = i \cdot j$. Then the rank of A is **(IN-2007-2M)**
(a) 0 (b) 1
(c) n – 1 (d) n

03. The minimum and maximum eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are –2 and 6 respectively. What is other eigen values? **(CE-2007-1M)**
(a) 5 (b) 3
(c) 1 (d) –1

04. For what values of α and β the following simultaneous equations have an infinite number of solutions? **(CE-2007-2M)**
$x+y+z = 5$, $x+3y+3z = 9$, $x+2y+\alpha z=\beta$
(a) 2, 7 (b) 3, 8
(c) 8, 3 (d) 7, 2

05. The inverse of the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ is **(CE-2007-2M)**
(a) $\frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$ (b) $\frac{1}{3}\begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}$
(c) $\frac{1}{3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$ (d) $\frac{1}{3}\begin{bmatrix} -7 & -2 \\ -5 & -1 \end{bmatrix}$

06. If a square matrix A is real and symmetric then the eigen values **(ME-2007-1M)**
(a) are always real
(b) are always real and positive
(c) are always real and non-negative
(d) occur in complex conjugate pairs.

07. The number of linearly independent eigen vectors of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is **(ME-2007-2M)**
(a) 0 (b) 1
(c) 2 (d) infinite

08. The determinant $\begin{vmatrix} 1+b & b & 1 \\ b & 1+b & 1 \\ 1 & 2b & 1 \end{vmatrix}$ equals to **(PI-2007-1M)**
(a) 0 (b) 2b(b-1)
(c) 2(1-b)(1+2b) (d) 3b(1+b)

09. If A is square symmetric real valued matrix of dimension 2n, the eigen values of A are **(PI-2007-2M)**
(a) 2n distinct real values
(b) 2n real values not necessarily distinct
(c) n distinct pairs of complex conjugate numbers
(d) n pairs of complex conjugate numbers, not necessarily distinct

10. $q_1, q_2 \ldots q_m$ are n-dimensional vectors with m < n. This set of vectors is linearly dependent. Q is the matrix with $q_1, q_2 \ldots q_m$ as the columns. The rank of Q is **(PI-2007-2M)**
(a) less than m (b) m
(c) between m and n (d) n

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11. $X = [x_1, x_2, \ldots. x_n]^T$ is an n-tuple non-zero vector. The n×n matrix $V = X \times T$ **(EE-2007-1M)**
(a) has rank zero (b) has rank 1
(c) is orthogonal (d) has rank n

12. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ then A satisfies the relation **(EE-2007-2M)**
(a) $A+3I+2A^{-1} = 0$ (b) $A^2+2A+2I = 0$
(c) $(A+I)(A+2I) = 0$ (d) $e^A = 0$

13. If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ then A^9 equals **(EE-2007-2M)**
(a) $511\,A+510\,I$ (b) $309\,A + 104\,I$
(c) $154\,A + 155\,I$ (d) e^{9A}

15. The characteristic equation of a (3×3) matrix P is defined as
$\alpha(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0.$
If I denotes identity matrix then the inverse of matrix P will be **(EE-2008-1M)**
(a) $P^2 + P + 2I$ (b) $P^2 + P + I$
(c) $-(P^2 + P + I)$ (d) $-(P^2 + P + 2I)$

16. A is m×n full rank matrix with m > n and I is an identity matrix Let matrix $A^+ = (A^T A)^{-1} A^T$. Then, which one of the following statement is False? **(EE-2008-2M)**
(a) $A A^+ A = A$ (b) $(A A^+)^2 = A A^+$
(c) $A^+ A = I$ (d) $A A^+ A = A^+$

17. If the rank of a (5×6) matrix Q is 4 then which one of the following statement is correct? **(EE-2008-1M)**
(a) Q will have four linearly independent rows and four linearly independent columns
(b) Q will have four linearly independent rows and five linearly independent columns.
(c) $Q Q^T$ will be invertible
(d) $Q^T Q$ will be invertible

19. All the four entries of the 2×2 matrix $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ are non-zero and one of the eigen valves is zero. Which of the following statements is true? **(EC-2008-1M)**
(a) $p_{11}\, p_{22} - p_{12}\, p_{21} = 1$
(b) $p_{11}\, p_{22} - p_{12}\, p_{21} = -1$
(c) $p_{11}\, p_{22} - p_{12}\, p_{21} = 0$
(d) $p_{11}\, p_{22} + p_{12}\, p_{21} = 0$

20. The system of linear equations $\begin{cases} 4x+2y=7 \\ 2x+y=6 \end{cases}$ has **(EC-2008-1M)**
(a) a unique solution
(b) no solution
(c) an infinite no. of solutions
(d) exactly two distinct solutions

21. Consider the matrix $P = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.
The value of e^P is **(EC-2008-2M)**
(a) $\begin{bmatrix} 2e^{-2}-3e^{-1} & e^{-1}-e^{-2} \\ 2e^{-2}-2e^{-1} & 5e^{-2}-e^{-1} \end{bmatrix}$
(b) $\begin{bmatrix} e^{-1}+e^{-2} & 2e^{-2}-e^{-1} \\ 2e^{-1}-4e^{-2} & 3e^{-1}+2e^{-2} \end{bmatrix}$
(c) $\begin{bmatrix} 5e^{-2}-e^{-1} & 3e^{-1}-e^{-2} \\ 2e^{-2}-6e^{-1} & 4e^{-2}+e^{-1} \end{bmatrix}$
(d) $\begin{bmatrix} 2e^{-1}-e^{-2} & e^{-1}-e^{-2} \\ -2e^{-1}+2e^{-2} & -e^{-1}+2e^{-2} \end{bmatrix}$

22. The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{bmatrix}$ has one eigen value to 3. The sum of the other two eigen values is **(ME-2008-1M)**
(a) p (b) p − 1
(c) p − 2 (d) p − 3

23. The eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form of $\begin{bmatrix} 1 \\ a \end{bmatrix}$ & $\begin{bmatrix} 1 \\ b \end{bmatrix}$. What is a + b? **(ME-2008-2M)**
(a) 0 (b) ½
(c) 1 (d) 2

24. For what values of a, if any, will the following system of equations in x, y and z have a solution?
$2x + 3y = 4$
$x + y + z = 4$
$x + 2y - z = a$ **(ME-2008-2M)**
(a) any real number
(b) 0
(c) 1
(d) there is no such value

25. The eigen vector pair of the matrix $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ is **(PI-2008-2M)**
(a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
(c) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

26. Inverse of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is **(PI-2008-2M)**
(a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
(c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$

27. A square matrix B is symmetric if **(CE-2009-1M)**
(a) $B^T = -B$ (b) $B^T = B$
(c) $B^{-1} = B$ (d) $B^{-1} = B^T$

28. In the solution of the following set of linear equations by Gauss-elimination using partial pivoting $5x + y + 2z = 34$, $4y - 3z = 12$ and $10x - 2y + z = -4$. The pivots for elimination of x and y are **(CE-2009-2M)**
(a) 10 and 4 (b) 10 and 2
(c) 5 and 4 (d) 5 and −4

29. The eigen values of the following matrix are $\begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ **(EC-2009-2M)**
(a) 3, 3+5j, 6 − j
(b) − 6+5j, 3+j, 3 − j
(c) 3 + j, 3 − j, 5 + j
(d) 3, −1+3j, −1−3j

30. The eigen values of a (2×2) matrix X are −2 and −3. The eigen values of matrix $(X+I)^{-1}(X + 5I)$ are **(IN-2009-2M)**
(a) −3, −4 (b) −1, −2
(c) −1, −3 (d) −2, −4

31. The matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ rotates a vector about then axis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ by angle of **(IN-2009-2M)**
(a) 30° (b) 60°
(c) 90° (d) 120°

32. For a matrix $[M] = \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix, $[M]^T = [M]^{-1}$. The value of x is given by **(ME-2009-1M)**
(a) $-\frac{4}{5}$ (b) $-\frac{3}{5}$
(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

33. The trace and determinant of a 2×2 matrix are shown to be –2 and –35 respectively. Its eigen values are **(EE-2009-1M)**
(a) –30 and –5 (b) –37 and –1
(c) –7 and 5 (d) 17.5 and –2

34. The value of the determinant $\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$ is **(PI-2009-1M)**
(a) –28 (b) –24
(c) 32 (d) 36

35. The value of x_3 obtained by solving the following system of linear equations is
$x_1 + 2x_2 - 2x_3 = 4$
$2x_1 + x_2 + x_3 = -2$
$-x_1 + 2x_2 - x_3 = 2$ **(PI-2009-2M)**
(a) –12 (b) –2
(c) 0 (d) 12

36. An eigen vector of $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is **(EE-2010-2M)**
(a) $[-1 \quad 1 \quad 1]^T$ (b) $[1 \quad 2 \quad 1]^T$
(c) $[1 \quad -1 \quad 2]^T$ (d) $[2 \quad 1 \quad -1]^T$

37. For the set of equations
$\begin{cases} x_1+2x_2+x_3+4x_4=2 \\ 3x_1+6x_2+3x_3+12x_4=6 \end{cases}$
The following statement is true **(EE-2010-2M)**
(a) only the trivial solution $x_1=x_2=x_3=x_4=0$ exist.
(b) There are no solutions
(c) A unique non-trivial solution exist
(d) Multiple non-trivial solutions exist

38. The eigen values of a skew-symmetric matrix are **(EC-2010-1M)**
(a) always zero
(b) always pure imaginary
(c) either zero or pure imaginary
(d) always real

39. One of the eigen vector of thematrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ is **(ME-2010-2M)**
(a) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (b) $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$
(c) $\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

40. A real n×n matrix $A = [a_{ij}]$ is defind as follows: $\begin{cases} a_{ij} = i, & \text{if } i = j \\ \quad = 0, & \text{otherwise} \end{cases}$
The sum of all n eigen values of A is **(IN-2010-1M)**
(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n-1)}{2}$
(c) $\frac{n(n+1)(2n+1)}{6}$ (d) n^2

41. X and Y are non-zero square matrices of size n×n. If $XY = O_{n\times n}$ then **(IN-2010-2M)**
(a) $|X| = 0$ and $|Y| \neq 0$
(b) $|X| \neq 0$ and $|Y| = 0$
(c) $|X| = 0$ and $|Y| = 0$
(d) $|X| \neq 0$ and $|Y| \neq 0$

42. Consider the following matrix $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$. If the eigen values of A are 4 and 8 then **(CS-2010-2M)**
(a) x = 4, y = 10 (b) x = 5, y = 8
(c) x = -3, y = 9 (d) x = -4, y = 10

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43. The inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ is **(CE-2010-2M)**
(a) $\frac{1}{2}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$
(b) $\frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$
(c) $\frac{1}{14}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$
(d) $\frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

44. The value of q for which the following set of linear algebraic equations $2x+3y=0$, $6x+qy=0$ can have non-trivial solution is **(PI-2010-1M)**
(a) 2 (b) 7
(c) 9 (d) 11

45. If $\{1,0,-1\}^T$ is an eigen vector of the following matrix $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix}$ then the corresponding eigen value is **(PI-2010-1M)**
(a) 1 (b) 2
(c) 3 (d) 5

KEY:

01. b	02. b	03. b
04. a	05. a	06. a
07. b	08. a	09. b
10. a	11. b	12. c
13. a	14.	15. d
16. d	17. a	18.
19. c	20. b	21.
22. c	23. b	24. b
25. a	26. b	27. b
28. c	29. d	30. c
31. c	32. a	33. c
34. b	35. b	36. b
37. d	38. c	39. a
40. a	41. c	42. d
43. b	44. c	45. a

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BASIC ENGINEERING MATHEMATICS

TOPIC – 2 CALCULUS

STANDARD DERIVATES

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$ (Chain Rule)

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\log_e x) = 1/x$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{(x^2-1)}}$

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v(du/dx) - u(dv/dx)}{v^2}$

$\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1} \cdot a$

$\frac{d}{dx}(a^x) = a^x \log_e a$

$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_e a$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{(x^2-1)}}$

$\frac{d}{dx}(\cosh x) = \sinh x$

STANDARD INTEGRALS

$\int x^n\, ds = \frac{x^{n+1}}{n+1} \; (n \neq -1)$

$\int \frac{1}{x} dx = \log_e x$

$\int e^x\, dx = e^x$

$\int a^x\, dx = \frac{a^x}{\log_e x}$

$\int \sin x\, dx = -\cos x$

$\int \cos x\, dx = \sin x$

$\int \tan x\, dx = -\log \cos x$

$\int \cot x\, dx = \log \sin x$

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$\int \sec x\, dx = \log(\sec x + \tan x)$

$\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x)$

$\int \sec^2 x\, dx = \tan x$

$\int \operatorname{cosec}^2 x\, dx = -\cot x$

$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{a-x}{a+x}\right)$

$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right)$

$\int \left(\sqrt{x^2-a^2}\right) dx = \frac{x\sqrt{(a^2-x^2)}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$\int \left(\sqrt{x^2+a^2}\right) dx = \frac{x\sqrt{(a^2+x^2)}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$\int \left(\sqrt{x^2-a^2}\right) dx = \frac{x\sqrt{(x^2-a^2)}}{2} - \frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right)$

$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

$\int \sinh x\, dx = \cosh x$

$\int \cosh x\, dx = \sinh x$

MEAN VALUE THEOREMS

Rolle's Theorem: If f(x) is (i) continuous in [a, b], (ii) differentiable in (a, b) and (iii) f(a) = f(b) then there exists atleast one value $C \in (a, b)$ such that $f^1(C) = 0$.

1) Find C of the Rolle's theorem for f(x) = x (x – 1) (x – 2) in [1, 2]
   a) 1.5 b) 1 – (1/√3) c) 1 + (1/√3) d) 1.25

2) Find C of the Rolle's theorem for $f(x) = e^x$ Sinx in [0, π]
   a) π/4 b) π/2 c) 3π/4 d) does not exist

3) Find C of Rolle's theorem for $f(x) = (x + 2)^3 (x – 3)^4$ in [-2, 3]
   a) 1/7 b) 2/7 c) ½ d) 3/2

4) Find C of Rolle's theorem for $f(x) = e^x$ (Sinx – Cosx) in [π/4, 5π/4]
   a) π/2 b) 3π/4 c) π d) does not exist

5) Find C of Rolle's theorem for $f(x) = x(x+3)\, e^{-x/2}$ in [-3, 0]
a) –1 b) –2 c) –0.5 d) 0.5

6) Find C of Rolle's theorem for $f(x) = \log[(x^2 + ab) / (a+b)x]$
a) (a+b)/2 b) $\sqrt{ab}$ c) 2ab / (a + b) d) (b – a)/2

7) Rolle's theorem cannot be applied for the function f(x) = |x| in [-2, 2] because
a) f(x) is not continuous in [-2, 2] b) f(x) is not differentiable in (-2, 2)
c) $f(-2) \neq f(2)$ d) none of the above

8) Rolle's theorem cannot be applied for the function f(x) = |x+2| in [-2, 0] because
a) f(x) is not continuous in [-2, 0] b) f(x) is not differentiable in (-2, 0)
c) $f(-2) \neq f(0)$ d) none of these

Lagrange's Mean Value Theorem: If f(x) is continuous in [a, b] and differentiable in (a, b) then there exists atleast one value C in (a, b) such that

$$f^1(C) = [f(b) - f(a)] / (b - a)$$

9) Find C of Lagrange's mean value theorem for f(x) = (x – 1) (x – 2) (x – 3) in [1, 2]
a) $2 - 1/\sqrt{3}$ b) $2 + (1/\sqrt{3})$ c) $1 + (1/\sqrt{3})$ d) $1 - (1/\sqrt{3})$

10) Find C of Lagrange's mean value theorem for f(x) = log x in [1, e]
a) e – 2 b) e – 1 c) (e + 1) / 2 d) (e – 1) / 2

11) Find C of Lagrange's mean value theorem for $f(x) = lx^2 + mx + n$ in [a, b]
a) (a + b) / 2 b) $\sqrt{ab}$ c) 2ab / (a + b) d) (b – a)/2

12) Find C of Lagrange's theorem mean value theorem for $f(x) = 7x^2 - 13x - 19$ in [-11/7, 13/7]
a) 1/7 b) 2/7 c) 3/7 d) 4/7

13) Find C of Lagrange's mean value theorem for $f(x) = e^x$ in [0, 1]
a) 0.5 b) log(e – 1) c) log(e + 1) d) log[(e + 1) / (e – 1)]

14) Lagrange's mean value theorem cannot be applied for the function $f(x) = x^{1/3}$ in [-1, 1] because
a) f(x) is not continuous in [-1, 1] b) f(x) is not differentiable in (-1,1)
c) f(x) is neither continuous not differentiable in [-1, 1] d) none of the above

Cauchy's Mean Value Theorem:

If f(x) and g(x) are two functions such that
a) f(x) and g(x) are continuous in [a, b]
b) f(x) and g(x) are differentiable in (a, b)
c) $g^1(x) \neq 0$ for all ~~x~~ in (a, b)

then there exists atleast one value C in (a,b) such that

$$[f^1(C) / g^1(C)] = [f(b) - f(a)] / [g(b) - g(a)]$$

15) Find C of Cauchy's mean value theorem for $f(x) = e^x$ and $g(x) = e^{-x}$ in [a, b]
a) (a + b) /2 b) $\sqrt{ab}$ c) 2ab / (a + b) d) (b – a) / 2

16) Find C of cauchy's mean value theorem for $f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$ in [a, b]
a) (a + b) / 2 b) $\sqrt{ab}$ c) 2ab / (a + b) d) (b – a) / 2

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17) Find C of Cauchy's mean value theorem for the functions 1/x and $1/x^2$ in [a, b]
a) (a + b)/2 b) $\sqrt{ab}$ c) 2ab / (a + b) d) (b – a)/2

18) Find C of Cauchy's mean value theorem for the functions Sinx and Cosx in (-π/2, 0)
a) -π/3 b) -π/4 c) -π/6 d) -π/8

Definite Integrals: The first fundamental theorem of integral calculus:
If f(x) is continuous in [a, b] and F(x) in any antiderivative of f(x) in [a, b] then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The second fundamental theorem of Integral Calculus: If f(x) is continuous on [a, b] then $F(x) = \int_a^x f(t)\,dt$ is differentiable at every point of x in [a,b] and $dF/dx = d/dx \int_a^b f(t)\,dt = f(x)$

Corollary: If f(x) is continuous on [a, b] then there exists a function F(x) whose derivative on [a, b] is f(x).

Theorem:3: If f(x) is continuous on [a, b] and U(x) and V(x) are differentiable functions of x whose values lie in [a, b], then

$$\frac{d}{dx}\int_{U(x)}^{V(x)} f(t)\,dt = f(V(x))\,(dV/dx) - f(U(x))\,(dU/dx)$$

Properties of Definite Integrals:

1) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3) If a < c < b then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if f(x) is even
$= 0$ if f(x) is odd

6) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if f(2a – x) = f(x)
$= 0$ if f(2a – x) = - f(x)

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7) $\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx$ if $f(x + a) = f(x)$

i.e; f(x) is a periodic function with period a

8) $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

9) $\int_0^a x\,f(x)\,dx = a/2 \int_0^a f(x)\,dx$ if $f(a - x) = f(x)$

10) $\int_0^{\pi/2} \mathrm{Sin}^n x\,dx = \int_0^{\pi/2} \mathrm{Cos}^n x\,dx =$

$[(n - 1)/n]\ .\ [(n - 3)/(n - 2)]\ .\ [(n - 5)/(n - 4)]\ \ldots\ldots (2/3)$ if n is odd

$[(n - 1)/n]\ .\ [(n - 3)/(n - 2)]\ .\ [(n - 5)/(n - 4)]\ \ldots\ldots\ 1/2\ .\ (\pi/2)$ if n is even

11) $\int_0^{\pi/2} \mathrm{Sin}^m x\ .\ \mathrm{Cos}^n x\,dx =$

$$\frac{\{(m - 1)(m - 3)(m - 5)\ldots\ldots(2 \text{ or } 1)\}\ .\ [(n - 1)(n - 2)\ldots.(2 \text{ or } 1)]\ .\ k}{(m + n)(m + n - 2)(m + n - 4)\ldots\ldots.(2 \text{ or } 1)}$$

Where $k = \pi/2$ When both m and n are even, otherwise $k = 1$

PROBLEMS

19) $\int_0^2 |1 - x|\,dx =$

a) 1 b) – 1 c) 2 d) 3/2

20) $\int_0^1 x(1 - x)^5\,dx =$

a) 1/42 b) 1/48 c) 1/12 d) 1/56

21) $\int_{-1}^{1} \frac{x^2\ .\ \mathrm{Sinx}\ dx}{x^4 + 1} =$

a) 0 b) π c) 2 d) $\pi/2$

22) $\int_{-\pi/2}^{\pi/2} x^{10}\ .\ \log|(1 + \mathrm{Sinx})/(1 - \mathrm{Sinx})|\,dx =$

a) 0 b) π c) 2 d) $\pi/2$

23) $\int_{-1}^{1} |x|\,dx =$

a) 0 b) 1 c) 2 d) 4

24) $\int_0^{\pi} \mathrm{Sin}^3 x\,dx =$

a) 2/3 b) 4/3 c) 0 d) $\pi/3$

25) $\int_0^{\pi/2} [(\mathrm{Sinx} - \mathrm{Cosx})/(1 + \mathrm{Sinx}\ .\ \mathrm{Cosx})]\,dx =$

a) 0 b) π c) $\pi/2$ d) $\pi/4$

26) $\int_0^{\pi/2} dx/(1 + \sqrt{\cot x}) =$

a) 0 b) π c) $\pi/2$ d) $\pi/4$

27) $\int_0^{\pi/2} (\mathrm{Sin2x}\ .\ \log(\tan x))\,dx =$

a) 0 b) π c) $\pi/2$ d) $\pi/4$

28) $\int_0^{\pi/4} \log(1 + \tan x)\,dx =$

a) 0 b) $\pi/2 \log 2$ c) $\pi/8 \log 2$ d) $-\pi/4 \log 2$

29) $\int_0^{\pi} [(x\ .\ \mathrm{Sinx})/(1 + \mathrm{Cos}^2 x)]\,dx =$

a) π^2 b) $\pi^2/2$ c) $\pi^2/4$ d) $\pi^2/8$

30) $\int_0^{\pi} dx/(a^2\,\mathrm{Cos}^2 x + b^2\,\mathrm{Sin}^2 x) =$

a) 0 b) πab c) π/ab d) $\pi/(a^2 + b^2)$

31) $\int_0^{\pi} x\ .\ \mathrm{Sin}^6 x\ .\ \mathrm{Cos}^4 x\,dx =$

a) $3\pi^2/512$ b) $5\pi^2/256$ c) $3\pi^2/128$ d) $5\pi^2/128$

32) $\int_0^{\pi} [(x\ .\ \tan x)/(\mathrm{Secx} + \tan x)]\,dx =$

a) 0 b) $\pi(\pi - 2)/4$ c) $\pi(\pi - 2)/2$ d) π

33) $\int_0^{\pi^2/4} \mathrm{Sin}\sqrt{x}\,dx =$

a) 0 b) 1 c) 2 d) $\pi/2$

34) $\int_2^3 [\sqrt{x}/(\sqrt{x} + \sqrt{5 - x})]\,dx =$

a) 1 b) 2.5 c) 0.5 d) 1.5

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35) $\int_{-\pi}^{\pi} \sin^4 x\, dx =$

a) $\pi/4$ b) $\pi/2$ c) $3\pi/2$ d) 0

36) $\int_{0}^{\pi} \sin^4 x \cos^5 x\, dx =$

a) 0 b) $3\pi/256$ c) $3\pi/128$ d) $5\pi/128$

37) $\int_{0}^{2\pi} \sin^4 x \cos^6 x\, dx =$

a) $3\pi/128$ b) $3\pi/256$ c) $3\pi/64$ d) 0

38) $\int_{0}^{2\pi} \sin^4 x \cos^5 x\, dx =$

a) 0 b) $3\pi/128$ c) $5\pi/128$ d) $3\pi/256$

IMPROPER INTEGRALS

* $\int_{a}^{b} f(x)\, dx$ is said to be an improper integral of first kind if $a = -\infty$ or $b = \infty$ or both.
* $\int_{a}^{b} f(x)\, dx$ is said to be an improper integral of second kind if $f(x)$ is infinite for one or more values of x in $[a, b]$.
* $\int_{a}^{b} f(x)\, dx$ is said to be convergent if the value of integral is finite.
* If (i) $0 \le f(x) \le g(x)$ for all x and (ii) $\int_{a}^{\infty} g(x)\, dx$ converges then $\int_{a}^{\infty} f(x)\, dx$ also converges.
* If (i) $f(x) \ge g(x) \ge 0$ for all x and (ii) $\int_{a}^{\infty} g(x)\, dx$ diverges then $\int_{a}^{\infty} f(x)\, dx$ also converges.
* If $f(x)$ and $g(x)$ are two functions such that $\underset{n \to \infty}{\text{Lt}} [f(x) / g(x)] = k$ (finite and $\ne 0$) then $\int_{a}^{\infty} f(x)\, dx$ and $\int_{a}^{\infty} g(x)\, dx$ converge or diverge together.
* $\int_{1}^{\infty} (dx / x^p)$ converges when $p > 1$ and diverges when $p \le 1$
* $\int_{a}^{\infty} e^{-px}\, dx$ and $\int_{-\infty}^{b} e^{px}\, dx$ converges for any constant $p > 0$ and diverge for $p \le 0$.

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* The integral $\int_{a}^{b} dx / (b - x)^p$ is convergent iff $p < 1$
* The integral $\int_{a}^{b} dx / (x - a)^p$ converges iff $p < 1$
* Suppose $f(x)$ is continuous in (a, b) and $f(x) \to \infty$ as $x \to b$. If $f(x)$ and $g(x)$ are positive and $\underset{x \to b}{\text{Lt}} [f(x) / g(x)] = l$ (finite and $\ne 0$) then $\int_{a}^{b} f(x)\, dx$ and $\int_{a}^{b} g(x)\, dx$ converge or diverge together.
* Suppose $f(x)$ is continuous in (a, b) and $f(x) \to \infty$ as $x \to a$. If $f(x)$ and $g(x)$ are positive and $\underset{x \to a}{\text{Lt}} [f(x) / g(x)] = l$ (finite and $\ne 0$) then $\int_{a}^{b} f(x)\, dx$ and $\int_{a}^{b} g(x)\, dx$ converge or diverge together.

PROBLEMS

39) Which of the following improper integrals is divergent

a) $\int_{0}^{\infty} [1 / (1 + x^2)]\, dx$ b) $\int_{0}^{1} [x / \sqrt{(1 - x^2)}]\, dx$ c) $\int_{0}^{1} \log x\, dx$ d) $\int_{0}^{\infty} x \,.\, \sin x\, dx$

40) Which of the following improper integrals is divergent

a) $\int_{-\infty}^{\infty} [1 / (1 + x^2)]\, dx$ b) $\int_{0}^{\infty} e^{-x}\, dx$ c) $\int_{1}^{\infty} dx / x^5$ d) $\int_{0}^{2} 1 / x^3\, dx$

41) Which of the following improper integrals is divergent

a) $\int_{0}^{1} [dx / x^{1/3}]\, dx$ b) $\int_{1}^{\infty} [dx / x\sqrt{(x^2 - 1)}]$ c) $\int_{1}^{\infty} [1/x^2]\, dx$ d) $\int_{1}^{\infty} (1/\sqrt{x})\, dx$

42) Which of the following integrals is divergent

a) $\int_{-\infty}^{-1} 1 / x^4\, dx$ b) $\int_{0}^{1} 1 / x^2\, dx$ c) $\int_{0}^{1} [1/\sqrt{(1 - x^2)}]dx$ d) $\int_{0}^{\infty} [1/(x^2 + 2x + 2)]dx$

43) Which of the following improper integrals is divergent.

a) $\int_{0}^{\infty} x^3 \,.\, e^{-x}\, dx$ b) $\int_{0}^{\infty} [\log x / x^3]\, dx$ c) $\int_{0}^{1} x \,.\, \log x\, dx$ d) $\int_{-1}^{1} dx / (x \,.\, x^{1/3})$

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44) Consider the integrals

$I_1 = \int_1^{\infty} dx / [x^2 (1 + e^x)]$ and $I_2 = \int_1^{\infty} \{(x + 1) / x\sqrt{x}\}\, dx$

Which of the following is true

a) I_1 is convergent and I_2 is divergent
b) I_1 is divergent and I_2 is convergent
c) I_1 and I_2 are convergent
d) I_1 and I_2 are divergent

45) Which of the following integrals is divergent

a) $\int_0^1 [1 / \sqrt{(1 - x)}]\, dx$
b) $\int_1^{\infty} [\text{Sin}x / x^3]\, dx$
c) $\int_{-1}^{1} 1/x^2\, dx$
d) $\int_0^2 [1 / \sqrt{|(1 - x^2)|}]\, dx$

46) Which of the following integrals is divergent

a) $\int_0^1 [1 / \sqrt{(x + 4x^3)}]\, dx$
b) $\int_{e^2}^{\infty} dx / [x(\log x)]^3$
c) $\int_0^{\pi/2} \sec x\, dx$
d) $\int_1^{e} [dx / x(\log x)^{1/3}]$

47) Which of the following improper integrals is convergent

a) $\int_1^{\infty} [(x + \sqrt{(x + 1)}) / (x^2 + 2(x^4+1)^{1/5})]\, dx$
b) $\int_2^{\infty} [(3 + 2x^2)^{1/7} / (x^3 - 1)^{1/5}] dx$
c) $\int_3^{\infty} dx / \sqrt{[x(x-1)(x - 2)} \quad (x > 3)]$
d) $\int_1^{\infty} [1 / x^{1/3}(1 + x)^{1/2}] dx$

48) Which of the following improper integrals is divergent

a) $\int_1^{\infty} [x / (1 + x)^3]\, dx$
b) $\int_1^{\infty} dx / [(1 + x)\sqrt{x}]$
c) $\int_0^1 dx / [x^{1/2} + x^3)]$
d) $\int_0^1 dx / [x^2(1+x)^2]$

49) Which of the following improper integrals is / are convergent

a) $\int_0^1 [1 / \{(x - 1)^6 (x - 2)^{1/5}\}]\, dx$
b) $\int_0^1 [x^{a-1} / x + 1]\, dx$ $(a > 0)$
c) $\int_0^{\pi/2} [\sqrt{x} / \text{Sin}x]\, dx$
d) $\int_{1/2}^{1} dx / [x^{1/3} (1 - x)^{1/3}]$

Partial Derivatives and Total Derivatives:

If $u = f(x, y)$ then

$(\partial u / \partial x) = \underset{h \to 0}{\text{Lt}}\ [f(x + h, y) - f(x, y)] / h$ and

$(\partial u / \partial y) = \underset{k \to 0}{\text{Lt}}\ [f(x, y+k) - f(x, y)] / k$

$(\partial/\partial x)\ (\partial u / \partial x) = (\partial^2 u / \partial x^2) = f_{xx}$

$(\partial/\partial u)\ (\partial u / \partial y) = (\partial^2 u / \partial y^2) = f_{yy}$

$(\partial/\partial x)\ (\partial u / \partial y) = (\partial^2 u / \partial u \partial y) = f_{xy}$

$(\partial/\partial y)\ (\partial u / \partial x) = (\partial^2 u / \partial y\, \partial x) = f_{yx}$

In general, $f_{xy} = f_{yx}$

Euler's Theorem: If $u = f(x, y)$ is a homogeneous function of degree n then

$x \cdot u_x + y \cdot u_y = nu$

***Cor.1** If $u = f(x, y, z)$ is a homogeneous function of degree n then

$x \cdot u_x + y \cdot u_y + z \cdot u_z = nu$

$x^2 \cdot u_{xx} + 2xy \cdot u_{xy} + y^2 \cdot u_{yy} = n(n-1)\, u$

* **Cor.2** If $u = f(x, y)$ is not a homogeneous function but $F(u)$ is a homogeneous function of degree n

then

i) $x \cdot u_x + y \cdot u_y = n\, [F(u) / F^1(u)] = G(u)$

ii) $x^2 \cdot u_{xx} + 2xy \cdot u_{xy} + y^2 \cdot u_{yy} = G(u)\, \{G^1(u) - 1\}$

* **Cor.3** If $u = f(x, y) + g(x, y) + h(x, y)$ where f, g, h are homogeneous functions of degrees m, n, p

respectively then

i) $x \cdot u_x + y \cdot u_y = m \cdot f + n \cdot g + p \cdot h$

ii) $x^2 \cdot u_{xx} + 2xy\, u_{xy} + y^2 \cdot u_{yy} = m(m - 1)\, f + n(n - 1)g + p(p - 1)h$

Total derivative

* If $u = f(x, y)$ where x and y are functions of t then the ***total derivative*** of u with respect t is given by

$(du / dt) = (\partial u / \partial x) \cdot (dx / dt) + (\partial u / \partial y) \cdot (dy / dt)$

* Taking $x = t$ in the above equation

$(du / dx) = (\partial u / \partial x) + (\partial u / \partial y) \cdot (dy / dx)$

This formula can be used for finding total derivative of u with respect to x, when x and y are connected by some relation.

* If $f(x, y) = C$ is an implicit function of x and y then

$(dy / dx) = -(f_x / f_y)$

* If $u = f(x, y)$ where $x = g(r, s)$ and $y = h(r, s)$ then

$(\partial u / \partial r) = (\partial u/\partial x) \cdot (\partial x/\partial r) + (\partial u/\partial y) \cdot (\partial y/\partial r)$

$(\partial u / \partial s) = (\partial u/\partial x) \cdot (\partial x/\partial s) + (\partial u/\partial y) \cdot (\partial y/\partial s)$

PROBLEMS

50) The total derivative of x^2y with respect to x, when x and y are connected by the relation $x^2 + xy + y^2 = 1$ is

a) $2xy - x^2\left(\dfrac{2x + y}{x + 2y}\right)$
b) $\dfrac{x^2y + xy^2}{x + 2y}$
c) $\dfrac{x^2y - xy^2}{y + 2x}$
d) $2xy - y^2\left(\dfrac{x + 2y}{2x + y}\right)$

51) If $u = x \log(xy)$ where $x^3 + y^3 + 3xy = 1$ then $(du / dx) =$

a) 0
b) $1 + \log(xy) - [x(x^2 + y) / y(y^2 + x)]$
c) $1 + \log(y/x) - [(x^2 + y) / (x + y^2)$
d) $\log(x/y) + [(x^2 + y) / (x + y^2)$

52) If $u = \text{Sin}(x^2 + y^2)$ where $a^2x^2 + b^2y^2 = C^2$ then $(du / dx) =$

a) $2(1 - a^2/b^2)x \cos(x^2 + y^2)$
b) $2x\, (a^2 + b^2) / b^2\, \text{Cos}(x^2 + y^2)$
c) $2\, x(a^2 - b^2)/a^2\, \text{Cos}(x^2 + y^2)$
d) $2x(a^2 - b^2) / b^2\, \text{Cos}(x^2 + y^2)$

53) If $U = \sin^{-1}(x/y) + \cos(y/x)$ then $U_x / U_y =$
a) x/y b) y/x c) $-x/y$ d) $-y/x$

54) Let $r^2 = x^2 + y^2 + z^2$ and $V = r^n$ then $V_{xx} + V_{yy} + V_{zz} =$
a) 0 b) $n(n+1)\, r^{n-2}$ c) $n(n-1)\, r^{n-2}$ d) $n(n+2)\, r^{n-2}$

55) If $U = (y/z) + (z/x)$ then $x \cdot U_x + y \cdot U_y + z \cdot U_z =$
a) 0 b) xy / z^2 c) yz / x^2 d) zx / y^2

56) If $V = (x^2 + y^2 + z^2)^{-2}$ then $V_{xx} + V_{yy} + V_{zz} =$
a) 0 b) $(x^2 + y^2 + z^2)$ c) $12(x^2 + y^2 + z^2)^{-3}$ d) $(x^2 + y^2 + z^2)^{-2}$

57) If $U = f(r)$ where $r^2 = x^2 + y^2 + z^2$ then $U_{xx} + U_{yy} + U_{zz} =$
a) $f^{11}(r) + (2/r)\, f^1(r)$ b) $f^{11}(r) + (1/r^2)\, f^1(r)$ c) $f^{11}(r) + 3/r\, f^1(r)$ d) $f^{11}(r) - 2/r\, f^1(r)$

58) If $U = f(r)$ where $x = r\cos\theta$ and $y = r\sin\theta$ then $U_{xx} + U_{yy} =$
a) $f^{11}(r) + 1/r\, f^1(r)$ b) $f^{11}(r) + 2/r\, f^1(r)$ c) $f^{11}(r) - 1/r\, f^1(r)$ d) $f^{11}(r) - 2/r\, f^1(r)$

59) If $\sin U = [(x + 2y + 3z) / (x^8 + y^8 + z^8)]$ then $x \cdot U_x + y \cdot U_y + z \cdot U_z =$
a) $(1/7) \tan U$ b) $-7 \tan U$ c) $(1/7) \sec U$ d) $(-1/20) \tan U$

60) If $U = \log[(x^4 + y^4) / (x - y)]$ then $x^2 \cdot U_{xx} + 2xy \cdot U_{xy} + y^2 \cdot U_{yy} =$
a) 0 b) 3 c) -3 d) 1/3

61) If $U = \operatorname{Cosec}^{-1}[(x^{1/4} + y^{1/4}) / (x^{1/5} - y^{1/5})]$ then $x \cdot U_x + y \cdot U_y =$
a) $(1/20) \cot U$ b) $(-1/20) \cot U$ c) $(1/20) \tan U$ d) $(-1/20) \tan U$

62) If $U = [(x^3 + y^3) / (x - y)] + x \sin(x/y)$ then $x^2 \cdot U_{xx} + 2xy \cdot U_{xy} + y^2 \cdot U_{yy} =$
a) 0 b) $2[(x^3 + y^3) / (x - y)]$
c) $[(x^3 + y^3) / (x - y)] - \sin(x/y)$ d) $[(x^3 + y^3) / (x - y)] - \cos(x/y)$

63) If $Z = x^n f_1(y/x) + y^{-n} f_2(x/y)$ then $x(\partial z / \partial x) + y(\partial z/\partial y) + x^2 Z_{xx} + 2xy \cdot Z_{xy} + y^2 \cdot Z_{yy} =$
a) 0 b) $n(n + 1)Z$ c) $n^2 Z$ d) $(n^2 - n) Z$

64) If $U = f(r, s)$ where $r = x + y$ and $s = x - y$ then $U_x + U_y =$
a) $2 U_r$ b) $2 U_s$ c) $-2 U_r$ d) $-2 U_s$

65) If $U = f(x - y, y - z, z - x)$ then $U_x + U_y + U_z =$
a) 0 b) U c) 2U d) 3 U

66) If $Z = f(x, y)$ where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$ then $Z_u - Z_v =$
a) $x \cdot Z_x - y \cdot Z_y$ b) $x \cdot Z_x + y \cdot Z_y$ c) $x \cdot Z_y + y \cdot Z_x$ d) $x \cdot Z_y - y \cdot Z_x$

67) If $U = f(x + Cy) + g(x - Cy)$ then $U_{xx} / U_{yy} =$
a) C^2 b) C^{-2} c) $-C^2$ d) $-C^{-2}$

68) If $U = \log(x^3 + y^3 + z^3 - 3xyz)$ then $U_x + U_y + U_z =$
a) $-3 / (x + y + z)$ b) $3 / (x + y + z)$ c) $9 / (xy + yz + zx)$ d) $-9 / (xy + yz + zx)$

69) If $U = \log(x^3 + y^3 + z^3 - 3xyz)$ then $(\partial/\partial x + \partial/\partial y + \partial/\partial z)^2 u =$
a) $3 / (x + y + z)^2$ b) $-3/(x + y + z)^2$ c) $9 / (x + y + z)^2$ d) $-9 / (x + y + z)^2$

70) If $U = e^{ax + by} f(ax - by)$ then $b.u_x + a.\, u_y$
a) 0 b) 2abU c) $2(a + b)$ U d) $2(a - b)$ U

Maxima and Minima:

Def: A function f(x) has a maximum at x = a if there exists some interval $(a - \delta, a + \delta)$ around 'a' such that $f(a) > f(x)$ for all x in $(a - \delta, a + \delta)$

A function f(x, y) has a minimum at x = a if there exists some interval $(a - \delta, a + \delta)$ around 'a' such that $f(a) < f(x)$ for all values of x in the interval

Extremum: The term used for both for maximum and minimum

A necessary condition for f(a) to be an extreme value of f(x) is $f^1(a) = 0$

The vanishing of $f^1(a) = 0$ is only a necessary but not a sufficient condition for f(a) to be an extreme value of f(x)

Ex: For the function $f(x) = x^3$, f(0) is not an extremum, eventhough $f^1(0) = 0$
Ex: f(0) is a minimum value of f(x) = |x| eventhough $f^1(0)$ does not exist

Stationary Values: A function f(x) is said to be stationary for x = C and f(C) is a stationary value of f(x) if $f^1(C) = 0$

A stationary value may neither be a maximum nor a minimum
Greatest value and least values of a function in the interval [a, b] are f(a) or f(b) or are given by the values of x for which $f^1(x) = 0$

Sufficient conditions for Extrema:
Theorem: f(C) is an extremum of f(x) iff $f^1(x)$ changes sign as x passes through C

Case(i): If $f^1(x)$ changes sign from positive to negative as x passes through C then f(C) is a maximum value of f(x)
Case(ii) : If $f^1(x)$ changes sign from negative to positive as x passes through C then f(C) is a maximum.

Case(iii) : If $f^1(x)$ does not change sign as x passes through C then f(C) is not an extremum.

Theorem: A function f(x) has a maximum at x = a if $f^1(a) = 0$ and $f^{11}(a) < 0$

A function f(x) has a minimum at x = a if $f^1(a) = 0$ and $f^{11}(a) > 0$

PROBLEMS

71) The function $f(x) = 2x^3 - 3x^2 - 36x + 10$ has a maximum at x =
a) 3 b) 2 c) -3 d) -2

72) The minimum value of $f(x) = 2x^3 - 3x^2 - 36x + 10$ is
a) 0 b) -13 c) -17 d) 3

73) A maximum value of f(x) = (logx / x) is
a) e b) e^{-1} c) $e - 1$ d) $e + 1$

74) The function $f(x) = x^x$ has a minimum at x =
a) e b) e^{-1} c) 0 d) $e + 1$

75) The minimum value of $f(x) = x \,.\, \log x$ is
a) e b) e^{-1} c) −   d) $-e^{-1}$

76) The maximum value of $x \,.\, e^{-x}$ is
a) e b) e^{-1} c) 1   d) $-e$

77) The maximum value of $f(x) = \sin x + \cos 2x$ in the interval $[0, \pi]$ is
a) 2   b) 1.5   c) 5/7   d) 9/8

78) $f(x, y) = x^3 + y^3 - 3xy$ has
a) a maximum at (1, 1)   b) a minimum at (1, 1)
c) a saddle point at (1, 1)   d) neither maximum nor minimum

79) At (a, a), $f(x, y) = xy + a^3/x + a^3/y$ has
a) a maximum   b) a minimum
c) a maximum if $a > 0$ d) neither maximum nor minimum

80) At $(\sqrt{2}, -\sqrt{2})$, $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ has
a) a minimum   b) a maximum
c) a saddle point   d) neither maximum nor minimum

81) A rectangular box open at the top is to have a volume 32 C.C. Find the dimensions of the box requiring least material for its construction
a) 4 cm,4 cm,2 cm   b) 2 cm,2 cm,8 cm   c) 16 cm,1 cm,1cm   d) 8 cm,8 cm,½ cm

82) If $f^1(x) = (x + 2)(x - 1)^2(2x - 1)(x - 3)$ then at $x = ½$, f(x) has
a) a maximum   b) a minimum
c) neither maximum nor minimum   d) no stationary point

Constrained Maximum or Minimum:
To find maximum or minimum of $U = f(x, y, z)$ where x, y, z are connected by $\phi(x,y,z) = 0$.

Working Rule: a) Write $F(x, y, z) = f(x, y, z) + \lambda.\phi(x, y, z)$
b) Obtain the equations $F_x = 0$, $F_y = 0$, $F_z = 0$
c) Solve the above equations along with $\phi = 0$ to get stationary point

83) Find the minimum value of $x^2 + y^2 + z^2$ so that $xyz = 8$
a) 8   b) 12   c) 21   d) 27

84) Find the maximum value of $x^2 + y^2 + z^2$ so that $x + y + z = 1$
a) 1   b) ½   c) 1/3   d) ¼

85) Divide 24 into three parts x, y, z so that xy^2z^3 is a maximum
a) 8, 8, 8   b) 4, 8, 12   c) 6, 9, 9   d) 6, 8,10

86) Let $T = 400\, xyz^2$ find maximum value of T so that $x^2 + y^2 + z^2 = 1$
a) 50   b) 100   c) 200   d) 800

87) Find the volume of the greatest rectangular parellelopiped that can be inscribed in the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$
a) $abc / 3\sqrt{3}$ b) $4abc / 3\sqrt{3}$ c) $8abc / 3\sqrt{3}$ d) $2abc / 3\sqrt{3}$

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88) What is the value of $\int_0^{2\pi} (x - \pi)^2 \,.\, \sin x \, dx$ *(GATE '05)*
a) -1   b) 0   c) 1   d) π

KEY

01. c	02. c	03. a	04. c	05. b	06. b	07. b	08. c	09. a	10. b	11. a	12. a
13. b	14. b	15. a	16. b	17. c	18. b	19. a	20. a	21. a	22. a	23. b	24. b
25. a	26. d	27. a	28. c	29. c	30. c	31. a	32. c	33. c	34. c	35. c	36. a
37. a	38. a	39. d	40. d	41. d	42. b	43. d	44. a	45. c	46. c	47. c	48. d
49. b, c	50. a	51. b	52. a	53. d	54. b	55. a	56. a	57. a	58. a	59. b	60. c
61. d	62. b	63. c	64. a	65. a	66. a	67. b	68. b	69. d	70. b	71. d	72. d
73. b	74. b	75. d	76. b	77. d	78. b	79. b	80. a	81. a	82. a	83. b	84. c
85. b	86. a	87. c	88. b.								

MULTIPLE INTEGRALS AND THEIR APPLICATIONS

DOUBLE INTEGRALS

Consider the double Integral $\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x, y)\, dxdy$.

Its value is found as follows:

i) When y_1, y_2 are functions of x and x_1, x_2 are constants, f(x, y) is first integrated with respect to y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated with respect to x within the limits x_1, x_2 i.e.,

$$I_1 = \int_{x_1}^{x_2}\left(\int_{y_1}^{y_2} f(x, y)\, dy\right) dx$$

ii) When x_1, x_2 are functions of y and y_1, y_2 are constants, f(x, y) is first integrated with respect to x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated with respect to y between the limits y_1, y_2 i.e.,

$$I_2 = \int_{y_1}^{y_2}\left(\int_{x_1}^{x_2} f(x, y)\, dx\right) dy$$

iii) When both pairs of limits are constants, it hardly matters whether we first integrate with respect to x and then with respect to y or vice versa.

CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limits of integration. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

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DOUBLE INTEGRALS IN POLAR CO – ORDINATES

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r, \theta)\, dr d\theta$, we first integrate with respect to r between limits $r = r_1$ and $r = r_2$

keeping θ fixed and the resulting expression is integrated w.r.t θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

TRIPLE INTEGRALS

Integral $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\, dxdydz$.

If x_1, x_2 are constants ; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows:

First f(x, y, z) is integrated w.r.t z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t x from x_1 to x_2.

$$\text{Thus } I = \int_{x_1}^{x_2}\left(\int_{y_1(x)}^{y_2(x)}\left(\int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\, dz\right) dy\right) dx$$

CHANGE OF VARIABLES

An appropriate choice of co – ordinates quite facilitates the evaluation of a double or a triple integral. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

1) **In a double integral**, let the variables x, y be changed to the new variables u, v by the transformation

$$x = \phi(u, v), \quad y = \psi(u, v)$$

where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R^1_{uv} in the uv – plane which corresponds to the region R_{xy} in the xy – plane. Then

$$\iint_{R_{xy}} f(x, y)\, dxdy = \iint_{R^1_{uv}} f[\phi(u, v), \psi(u, v)]\, |J|\, dudv$$

Where $J = \dfrac{\partial(x, y)}{\partial(u, v)}\ (\neq 0)$

is the Jacobian of transformation from (x, y) to (u, v) co – ordinates.

2) **For triple integrals**, the formula corresponding to (1) is

$$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R^1_{uvw}} f[x(u, v, w), y(u, v, w), z(u, v, w)]\, |J|\, dudvdw$$

Where $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}\ (\neq 0)$

is the Jacobian of transformation from (x, y, z) to (u, v, w) co – ordinates.

Particular Cases:

i) To change Cartesian co – ordinates (x, y) to polar co –ordinates (r,θ), we have $x = r\cos\theta$, $y = r\sin\theta$ and

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

$$\therefore \iint_{R_{xy}} f(x, y)\, dx\, dy = \iint_{R^1_{r\theta}} f(r\cos\theta, r\sin\theta)\, .\, r\, dr d\theta$$

ii) To change rectangular co – ordinates (x, y, z) to cylindrical co – ordinates (ρ, ϕ, z), we have $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

and $J = \dfrac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$

$$\text{Then} \iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R^1_{\rho\phi z}} f(\rho\cos\phi, \rho\sin\phi, z)\, .\, \rho\, d\rho d\phi dz$$

iii) To change rectangular co – ordinates (x, y, z) to spherical polar co – ordinates (r, θ, ϕ), we have $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

and $J = \dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$

$$\text{Then} \iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R^1_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, .\, r^2\sin\theta\, dr d\theta d\phi$$

PROBLEMS

1. Change the order of integration in the integral $I = \int_{-a}^{a} \int_{0}^{\sqrt{(a^2 - y^2)}} f(x, y)\, dx\, dy$

Ans) $I = \int_{0}^{a} \int_{-\sqrt{(a^2 - x^2)}}^{\sqrt{(a^2 - x^2)}} f(x, y)\, dy dx$

2. Calculate $\int\int r^3\, dr\, d\theta$ over the area included between the circles $r = 2 \sin\theta$ and $r = 4 \sin\theta$

Ans) $2.25\,\pi$

Evaluate the following integrals

3. $\int_{1}^{2} \int_{1}^{3} xy^2\, dx\, dy$ Ans) 13

4. $\int_{0}^{1} \int_{x}^{\sqrt{x}} (x^2 + y^2)\, dx\, dy$ Ans) 3/35

5. $\int_{0}^{1} \int_{4y}^{4} e^{x^2}\, dx\, dy$ Ans) $(e^{16} - 1) / 8$

6. $\int_{0}^{4} \int_{0}^{x^2} e^{y/x}\, dy\, dx$ Ans) $3e^4 - 7$

7. $\int_{0}^{1} \int_{0}^{\sqrt{(1+x^2)}} \left(\frac{1}{1 + x^2 + y^2}\right) dydx$ Ans) $(¼)\,\pi \cdot \log(1 + \sqrt{2})$

8. $\int\int xy\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$ Ans) $a^4/8$

9. $\int\int xy(x + y)\, dx\, dy$ over the area between $y = x^2$ and $y = x$. Ans) 3/56

Evaluate the following integrals by changing the order of integration

10. $\int_{0}^{1} \int_{0}^{\sqrt{(1 - x^2)}} y^2\, dxdy$ Ans) $\pi/16$

11. $\int_{0}^{3} \int_{1}^{\sqrt{(4 - y)}} (x + y)\, dxdy$ Ans) 241/60

12. $\int_{0}^{\infty} \int_{x}^{\infty} (e^{-y} / y)\, dydx$ Ans) 1

13. $\int_{0}^{\infty} \int_{0}^{x} x e^{-x^2/y}\, dy\, dx$ Ans) 1/2

14. Evaluate $\int\int r \sin\theta\, dr\, d\theta$ over the cardioid $r = a(1 - \cos\theta)$ above the initial line. Ans) $4a^3/3$

15. Show that $\int\int_{R} r^2 \sin\theta\, dr\, d\theta = 2a^3/3$, where R is the semi – circle $r = 2a \cos\theta$ above the initial line

16) Evaluate $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)\, dx\, dy\, dz$ Ans) 0

17) Evaluate $\int_{0}^{1} \int_{0}^{\sqrt{(1 - x^2)}} \int_{0}^{\sqrt{(1 - x^2 - y^2)}} xyz\ dxdydz$ Ans) 1/48

18) $\int_{0}^{1} \int_{0}^{2} \int_{1}^{2} x^2yz\, dx\, dy\, dz$ Ans) 7/3

19) $\int_{0}^{4} \int_{0}^{2\sqrt{z}} \int_{0}^{\sqrt{(4z - x^2)}} dy\, dx\, dz$ Ans) 8π

20) $\int_{0}^{a} \int_{0}^{x} \int_{0}^{x+y} e^{x+y+z}\, dz\, dy\, dx$ Ans) $1/8\, e^{4a} - ¾\, e^{2a} + e^{a} - 3/8$

21) Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis. Ans) $8\pi a^3 / 3$

22) Evaluate $\int\int_{R} (x + y)^2\, dx\, dy$, where R is the parallelogram in the XY – plane with vertices (1, 0) (3, 1) (2, 2) (0, 1) using the transformation $u = x + y$ and $v = x - 2y$. Ans) 21

23) Evaluate $\int_{0}^{\infty}\int_{0}^{\infty} e^{-(x^2 + y^2)}\, dx\, dy$ by changing to polar coordinates. Hence show that

$\int_{0}^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{2}}$ Ans) $\sqrt{(\pi/2)}$

PREVIOUS GATE QUESTIONS - "CALCULUS"

1. The integration of $\int \log x \, dx$ has the value *(GATE'94)*
a) $(x \log x - 1)$ b) $\log x - x$ c) $x(\log x - 1)$ d) None of the above

2. The volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and the line $x = 2$ about y – axis is *(GATE'94)*
a) $128\pi / 5$ b) $5 / 128\pi$ c) $127 / 5\pi$ d) None above

3. The function $y = x^2 + (250/x)$ at $x = 5$ attains *(GATE'94)*
a) Maximum b) Minimum c) Neither d) 1

4. The value of ξ in the mean value theorem of $f(b) - f(a) = (b - a) f'(\xi)$ for $f(x) = Ax^2 + Bx + C$ in (a,b) is *(GATE'94)*
a) $b + a$ b) $b-a$ c) $(b + a)/2$ d) $(b - a)/2$

5. The area bounded by the parabola $2y = x^2$ and the lines $x = y - 4$ is equal to *(GATE'95)*
a) 6 b) 18 c) ∞ d) none of the above

6. By reversing the order of integration. $\int_0^2 \int_{x^2}^{2x} f(x, y) dy\, dx$ may be represented as *(GATE'95)*
a) $\int_0^2 \int_{x^2}^{2x} f(x, y)\, dy\, dx$ b) $\int_0^2 \int_y^{\sqrt{y}} (x, y)\, dx\, dy$ c) $\int_0^4 \int_{y/2}^{\sqrt{y}} (x,y)\, dx\, dy$ d) $\int_{x^2}^{2x} \int_0^2 (x,y)\, dy\, dx$

7. The third term in the Taylor's series expansion of e^x about a would be *(GATE'95)*
a) $e^a(x - a)$ b) $e^a/2(x - a)^2$ c) $e^a/2$ d) $e^a/6(x - a)^3$

8. $\lim_{x \to 0} x \sin(1/x)$ is *(GATE'95)*
a) ∞ b) 0 c) 1 d) non – existant

9. The function $f(x) = |x + 1|$ on the interval $[-2,0]$ is *(GATE'95)*
a) continuous and differentiable
b) continuous on the interval but not differentiable at all points
c) neither continuous nor differentiable
d) differentiable but not continuous

10. The function $f(x) = x^3 - 6x^2 + 9x + 25$ has *(GATE'95)*
a) a maxima at $x = 1$ and a minima at $x = 3$ b) a maxima at $x = 3$ and a minima at $x = 1$
c) no maxima, but a minima at $x = 3$ d) a maxima at $x = 1$, but no minima

11. If $f(0) = 2$ and $f(x) = 1 / (5 - x^2)$, the lower and upper bounds of $f(1)$ estimated by the mean value theorem are *(GATE'95)*
a) 1.9, 2.2 b) 2.2, 2.25 c) 2.25, 2.5 d) none of the above

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12. If a function is continuous at a point its first derivative *(GATE'96)*
a) may or may not exist b) exists always
c) will not exist d) has a unique value

13. Area bounded by the curve $y = x^2$ and lines $x = 4$ and $y = 0$ is given by, *(GATE'97)*
a) 64 b) 64/3 c) 128/3 d) 128/4

14. The curve given by the equation $x^2 + y^2 = 3axy$, is *(GATE'97)*
a) symmetrical about x – axis b) symmetrical about y – axis
c) symmetrical about line $y = x$ d) tangential to $x = y = a/3$

15. e^x is periodic, with a period of *(GATE'97)*
a) 2π b) $2i\pi$ c) π d) $i\,\pi$

16. $\lim_{\theta \to 0} \frac{\sin m\theta}{\theta}$, where m is an integer, is one of the following: *(GATE'97)*
a) m b) $m\pi$ c) $m\theta$ d) 1

17. If $y=|x|$ for $x<0$ and $y = x$ for $x \geq 0$, then *(GATE'97)*
a) dy/dx is discontinuous at $x = 0$ b) y is discontinuous at $x = 0$
c) y is not defined at $x = 0$ d) Both y and dy/dx are discontinuous at $x = 0$

18. If $\varnothing(x) = \int_0^{x^2} \sqrt{t}\, dt$, then $(d\varnothing / dx)$ is *(GATE'98)*
a) $2x^2$ b) $\sqrt{x}$ c) 0 d) 1

19. The continuous function $f(x, y)$ is said to have saddle point at (a, b) if *(GATE'98)*
a) $f_x(a, b) = f_y(a, b) = 0$; $f^2_{xy} - f_{xx} f_{yy} < 0$ at (a, b)
b) $f_x(a, b) = 0$; $f_y(a, b) = 0$; $f^2_{xy} - f_{xx} f_{yy} > 0$ at (a, b)
c) $f_x(a, b) = 0$; $f_y(a, b) = 0$; f_{xx} and $f_{yy} < 0$ at (a, b)
d) $f_x(a, b) = f_y(a, b) = 0$; $f^2_{xy} - f_{xx} f_{yy} = 0$ at (a, b)

20. The Taylor's series expansion of $\sin x$ is *(GATE'98)*
a) $1 - x^2/2! + x^4/4! - \ldots$ b) $1 + x^2/4! + x^4/4! + \ldots$
c) $x + x^3/3! + x^5/5! + \ldots$ d) $x - x^3/3! + x^5/5! - \ldots$

21. A discontinuous real function can be expressed as *(GATE'98)*
a) Taylor's series and Fourier's series
b) Taylor's series and not by Fourier's series
c) neither Taylor's series nor Fourier's series
d) not by Taylor's series, but by Fourier's series

22. Limit of the function $\lim_{n \to \infty} \frac{n}{\sqrt{n^2 + n}}$ is *(GATE'99)*
a) ½ b) 0 c) ∞ d) 1

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23. The function $f(x) = e^x$ is (GATE'99)
a) Even b) Odd c) Neither even nor odd d) None of the above

24. Value of the function $\lim_{x\to a} (x-a)^{(x-a)}$ is (GATE'99)
a) 1 b) 0 c) ∞ d) a

25. Find the maximum and minimum values of the function $f(x) = \sin x + \cos 2x$ over the range $0 < x < 2\pi$. (GATE'99)

26. $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y)\, dx\, dy$ is (GATE - 2000)
(a) 0 (b) π (c) $\pi/2$ (d) 2

27. The limit of the function $f(x) = [(1-a^4)/x^4]$ as $x \to \infty$ is given by (GATE - 2000)
a) 1 b) $\exp[-a^4]$ c) ∞ d) zero

28. The maxima and minima of the function $f(x) = 2x^3 - 15x^2 + 36x + 10$ occur, respectively, at (GATE - 2000)
a) $x = 3$ & $x = 2$ b) $x = 1$& $x = 3$ c) $x = 2$ & $x = 3$ d) $x = 3$ & $x = 4$

29. If $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ (GATE - 2000)
$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ is equal to
a) zero b) 1 c) 2 d) $-3(x^2 + y^2 + z^2)^{-5/2}$

30. Consider the following integral $\lim_{a\to\infty} \int_1^a x^{-4}\, dx$ (GATE - 2000)
a) diverges b) converges to 1/3 c) converges to $-1/a^3$ d) converges to 0

31. The value of the integral is $I = \int_0^{\pi/4} \cos^2 x\, dx$ (GATE'01)
a) $\pi/8 + 1/4$ b) $\pi/8 - 1/4$ c) $-\pi/8 - 1/4$ d) $-\pi/8 + 1/4$

32. Which of the following functions is not differentiable in the domain [-1,1] ? (GATE'02)
a) $f(x) = x^2$ b) $f(x) = x-1$ c) $f(x) = 2$ d) $f(x) = \text{maximum}(x,-x)$

33. The value of the following definite integral is: (GATE'02)
$\int_{-\pi/2}^{\pi/2} (\sin 2x / 1 + \cos x)\, dx$
a) $-2\ln 2$ b) 2 c) 0

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34. The value of the following improper integral is $\int_0^1 x \ln x$ (GATE'02)
a) 1/4 b) 0 c) $-1/4$ d) 1

35. The function $f(x,y) = 2x^2 + 2xy - y^3$ has (GATE'02)
a) only one stationary point at (0,0)
b) two stationary points at (0,0) and (1/6, -1/3)
c) two stationary points at (0,0) and (1, -1)
d) no stationary point

36. $\text{Lt}_{x\to 0} \frac{\sin^2 x}{x}$ is equal to (GATE'03)
a) 0 b) ∞ c) 1 d) -1

37. The area enclosed between the parabola $y = x^2$ and the straight line $y = x$ is (GATE'03)
a) 1/8 b) 1/6 c) 1/3 d) 1/2

38. If $x = a(\theta + \sin\theta)$ and $y = a(1-\cos\theta)$, then (dy/dx) will be equal to (GATE'04)
a) $\sin(\theta/2)$ b) $\cos(\theta/2)$ c) $\tan(\theta/2)$ d) $\cot(\theta/2)$

39. The volume of an object expressed in spherical co-ordinates is given by (GATE'04)
$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 r^2 \sin\phi\, dr\, d\phi\, d\theta$$
The value of the integral
a) $\pi/3$ b) $\pi/6$ c) $2\pi/3$ d) $\pi/4$

40. The value of the function $f(x) = \lim_{x\to 0} \frac{x^3 + x^2}{2x^3 - 7x^2}$ is (GATE'04)
a) 0 b) $-1/7$ c) $1/7$ d) ∞

41. The function $f(x) = 2x^3 - 3x^2 - 36x + 2$ has its maxima at (GATE'04)
a) $x = -2$ only b) $x = 0$ only c) $x = 3$ only d) both $x = -2$ and $x = 3$

42. $\int_{-a}^{a} (\sin^6 x + \sin^7 x)dx$ is equal to (GATE'05)
a) $2\int_0^a \sin^6 x dx$ b) $2\int_0^a \sin^7 x dx$ c) $2\int_0^a (\sin^6 x + \sin^7 x)dx$ d) zero

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43. Changing the order of the integration in the double integral $I = \int_0^8 \int_{x/4}^2 f(x,y)\,dy\,dx$

leads to

$I = \int_r^s \int_p^q f(x,y)\,dy\,dx$. What is q? *(GATE'05)*

a) 4y b) $16y^2$ c) x d) 8

44. By a change of variables $x(u,v) = uv$, $y(u,v) = v/u$ in a double integral, the integrand $f(x,y)$ changes to $f(uv, u/v)\,\phi(u,v)$. Then, $\phi(u,v)$ is *(GATE'05)*

a) 2v/u b) 2uv c) v^2 d) 1

KEY

1. c	2. d	3. b	4. c	5. b	6. c	7. b	8. b	9. b	10. a	11.	12. a
13. b	14. c	15. b	16. a	17. a	18. a	19. a	20. d	21. d	22. d	23. c	24. a
25. Max = 9/8	26. d	27. d	28. c	29. a	30. b	31. a	32. d	33. c	34. c	35. b	
36. a	37. b	38. c	39. a	40. b	41. a	42. a	43. a	44. a			

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PREVIOUS GATE QUESTIONS

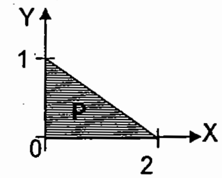
01. The value of $\int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2}\,dx\,dy$ is

(IN-2007-2M)

(a) $\frac{\sqrt{\pi}}{2}$ (b) $\sqrt{\pi}$

(c) π (d) $\frac{\pi}{4}$

02. Consider the shaded triangular region P shown in the figure. What is $\iint_P xy\,dx\,dy$? **(ME-2008-2M)**



(a) $\frac{1}{6}$ (b) $\frac{2}{9}$ (c) $\frac{7}{16}$ (d) 1

03. If f(x, y) is a continuous function defined over $(x, y) \in [0, 1] \times [0, 1]$. Given two constraints, $x > y^2$ and $y > x^2$, the volume under f(x, y) is

(EE-2009-2M)

(a) $\int_{y=0}^{y=1} \int_{x=y^2}^{x=\sqrt{y}} f(x,y)\,dx\,dy$

(b) $\int_{y=x^2}^{y=1} \int_{x=y^2}^{x=1} f(x,y)\,dx\,dy$

(c) $\int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x,y)\,dx\,dy$

(d) $\int_{x=0}^{y=\sqrt{x}} \int_{x=0}^{x=\sqrt{y}} f(x,y)\,dx\,dy$

04. The temperature T(in ^{0}C) at any point (x, y) on a surface is $T = 400\,xy^2$. The highest temperature (in ^{0}C) on the circumference of the circle $x^2 + y^2 = 1$ is

(a) 128 (b) 154 (c) 233 (d) 381

05. A parabolic cable is held between two supports at the same level. The horizontal span between the supports is L.
The sag at the mid-span is h. The equation of the parabola is $y = 4h\frac{x^2}{L^2}$, where x is the horizontal coordinate and y is the vertical coordinate with the origin at the centre of the cable. The expression for the total length of the cable is **(CE-2010-2M)**

(a) $\int_0^L \sqrt{1 + 64\frac{h^2x^2}{L^4}}\,dx$

(b) $2\int_0^{L/2} \sqrt{1 + 64\frac{h^3x^2}{L^4}}\,dx$

(c) $\int_0^{L/2} \sqrt{1 + 64\frac{h^2x^2}{L^4}}\,dx$

(d) $2\int_0^{L/2} \sqrt{1 + 64\frac{h^2x^2}{L^4}}\,dx$

06. The parabolic arc $y = \sqrt{x}$, $1 \le x \le 2$ is revolved around the x-axis. The volume of the solid of revolution is

(ME-2010-1M)

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{4}$ (d) $\frac{3\pi}{2}$

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07. $\int_{-a}^{a}\left[\sin^6 x+\sin^7 x\right]dx$ is equal to

(ME-2005-1M)

(a) $2\int_0^a \sin^6 x\,dx$ (b) $2\int_0^a \sin^7 x\,dx$

(c) $2\int_0^a\left(\sin^6 x+\sin^7 x\right)dx$ (d) zero

08. For the function $f(x)=x^2e^{-x}$, the maximum occurs when x is equal to **(EE-2005-2M)**

(a) 2 (b) 1 (c) 0 (d) –1

09. The value of the integral $\int_{-1}^{1}\frac{1}{x^2}dx$ is

(IN-2005-1M)

(a) 2 (b) does not exists
(c) –2 (d) ∞

10. If $S=\int_1^{\infty}x^{-3}\,dx$ then S has the value

(EE-2005-1M)

(a) $\frac{-1}{3}$ (b) $\frac{1}{4}$

(c) $\frac{1}{2}$ (d) 1

11. For real x, the maximum value of $\frac{e^{\sin x}}{e^{\cos x}}$ is **(IN-2007-2M)**

(a) 1 (b) e (c) $e^{\sqrt{2}}$ (d) ∞

12. Consider the function $f(x)=|x|^3$, where x is real. Then the function f(x) at x = 0 is **(IN-2007-2M)**

(a) continuous but not differentiable
(b) once differentiable but not twice.
(c) twice differentiable but not thrice.
(d) thrice differentiable.

13. The minimum value of function $y=x^2$ in the interval [1, 5] is **(ME-2007-1M)**

(a) 0 (b) 1 (c) 25 (d) undefined

14. $\lim_{x\to 0}\frac{e^x-\left(1+x+\frac{x^2}{2}\right)}{x^3}=$

(ME-2007-2M)

(a) 0 (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) 1

15. If $y=x+\sqrt{x+\sqrt{x+\sqrt{x+\ldots\infty}}}$ then y(2) = ----- **(ME-2007-2M)**

(a) 4 (or) 1 (b) 4 only
(c) 1 only (d) Undefined

16. What is the value of

$\lim_{x\to\pi/4}\frac{\cos x-\sin x}{x-\pi/4}=$ **(PI-2007-1M)**

(a) $\sqrt{2}$ (b) 0
(c) $-\sqrt{2}$ (d) Limit does not exist.

17. For the function $f(x,y)=x^2-y^2$ defined on R^2, the point (0, 0) is **(PI-2007-2M)**

(a) a local mimimum

(b) neither a local minimum (nor) a local maximum.

(c) a local maximum

(d) both a local minimum and a local maximum.

18. $\lim_{x\to 0}\frac{\sin(\theta/2)}{\theta}$ is **(EC-2007-1M)**

(a) 0.5 (b) 1
(c) 2 (d) not defined

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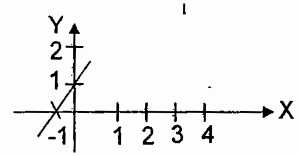
19. The following plot shows a function y which varies linearly with x. The value of the integral $I=\int_1^2 y\,dx$ **(EC-2007-1M)**



(a) 1 (b) 2.5 (c) 4 (d) 5

20. For the function e^{-x}, the linear approximation around x = 2 is

(EC-2007-1M)

(a) $(3-x)e^{-2}$ (b) $1-x$

(c) $\left[3+2\sqrt{2}-(1+\sqrt{2})x\right]e^{-2}$

(d) e^{-2}

21. For | x | << 1, cot h(x) can be approximated as **(EC-2007-1M)**

(a) x (b) x^2 (c) $\frac{1}{x}$ (d) $\frac{1}{x^2}$

22. Consider the function $f(x)=x^2-x-2$. The maximum value of f(x) in the closed interval [–4, 4] is

(EC-2007-2M)

(a) 18 (b) 10
(c) –2.25 (d) indeterminate

23. Consider the function $f(x)=(x^2-4)^2$ where x is a real number. Then the function has **(EE-2008-2M)**

(a) Only one minimum
(b) Only two minima
(c) Three minima
(d) Three maxima

24. Given $y=x^2+2x+10$ the value of $\left.\frac{dy}{dx}\right|_{x=1}$ is equal to **(IN-2008-1M)**

(a) 0 (b) 4 (c) 12 (d) 13

25. $\lim_{x\to 0}\frac{\sin x}{x}$ is **(IN-2008-1M)**

(a) indeterminate (b) 0
(c) 1 (d) ∞

26. The expression $e^{-\ln x}$ for x > 0 is equal to **(IN-2008-2M)**

(a) –x (b) x (c) x^{-1} (d) $-x^{-1}$

27. Consider the function $y=x^2-6x+9$. The maximum value of y obtained when x varies over the internal 2 to 5 is **(IN-2008-2M)**

(a) 1 (b) 3 (c) 4 (d) 9

28. For real values of x, the minimum value of the function $f(x)=e^x+e^{-x}$ is **(EC-2008-2M)**

(a) 2 (b) 1 (c) 0.5 (d) 0

29. Which of the following function would have only odd powers of x in its Taylor series expansion about the point x = 0? **(EC-2008-1M)**

(a) $\sin(x^3)$ (b) $\sin(x^2)$
(c) $\cos(x^3)$ (d) $\cos(x^2)$

30. In the Taylor series expansion of e^x + sin x about the point x = π, the coefficient of $(x-\pi)^2$ is **(EC-2008-2M)**

(a) e^{π} (b) $0.5\,e^{\pi}$
(c) $e^{\pi}+1$ (d) $e^{\pi}-1$

31. The value of the integral of the function $g(x, y)=4x^3+10y^4$ along the straight line segment from the point (0, 0) to the point (1, 2) in the xy-plane is **(EC-2008-2M)**

(a) 33 (b) 35 (c) 40 (d) 56

32. In the Taylor series expansion of e^x about x = 2, the coefficient of $(x-2)^4$ is **(ME-2008-1M)**

(a) $\frac{1}{4!}$ (b) $\frac{2^4}{4!}$

(c) $\frac{e^2}{4!}$ (d) $\frac{e^4}{4!}$

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33. The value of $\lim\limits_{x \to 8} \dfrac{x^{1/3}-2}{x-8}$ is **(ME-2008-1M)**

(a) $\frac{1}{16}$ (b) $\frac{1}{12}$

(c) $\frac{1}{8}$ (d) $\frac{1}{4}$

34. Which of the following integrals is unbounded? **(ME-2008-2M)**

(a) $\int_0^{\pi/4} \tan x\, dx$ (b) $\int_0^{\infty} \frac{1}{1+x^2} dx$

(c) $\int_0^{\infty} x e^{-x}\, dx$ (d) $\int_0^{1} \frac{1}{1-x} dx$

35. The length of the curve $y = \frac{2}{3}x^{3/2}$ between $x = 0$ & $x = 1$ is **(ME-2008-2M)**

(a) 0.27 (b) 0.67 (c) 1 (d) 1.22

36. The value of the integral $\int_{-\pi/2}^{\pi/2} (x\cos x)\, dx$ is **(PI-2008-1M)**

(a) 0 (b) $\pi - 2$ (c) π (d) $\pi + 2$

37. The value of the expression $\lim\limits_{x \to 0} \left[\dfrac{\sin(x)}{e^x x}\right]$ is **(PI-2008-1M)**

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) $\frac{1}{1+e}$

38. The distance between the origin and the point nearest to it on the surface $Z^2 = 1 + xy$ is **(ME-2009-2M)**

(a) 1 (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{3}$ (d) 2

39. The area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$ is **(ME-2009-2M)**

(a) $\frac{16}{3}$ (b) 8 (c) $\frac{32}{3}$ (d) 16

40. The Taylor series expansion of $\dfrac{\sin x}{x-\pi}$ at $x = \pi$ is given by **(CE-2009-2M)**

(a) $1 + \frac{(x-\pi)^2}{3!} + ---$

(b) $-1 - \frac{(x-\pi)^2}{3!} + ---$

(c) $1 - \frac{(x-\pi)^2}{3!} + ---$

(d) $-1 + \frac{(x-\pi)^2}{3!} + ---$

41. The total derivative of the function 'xy' is **(PI-2009-1M)**

(a) x dy + y dx (b) x dx + y dy
(c) dx + dy (d) dx dy

42. At t = 0, the function $f(t) = \dfrac{\sin t}{t}$ has **(EE-2010-2M)**

(a) a minimum
(b) a discontinuity
(c) a point of inflection
(d) a maximum

43. The value of the quantity, where $P = \int_0^1 x e^x\, dx$ is **(EE-2010-1M)**

(a) 0 (b) 1 (c) e (d) 1/e

44. If $e^y = x^{1/x}$ then y has a **(EC-2010-2M)**
(a) maximum at x = e
(b) minimum at x = e
~~(c) masimum at $x = e^{-1}$~~
(d) minimum at $x = e^{-1}$

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45. If $f(x) = \sin|x|$ then the value of $\frac{df}{dx}$ at $x = -\frac{\pi}{4}$ is **(PI-2010-1M)**

(a) 0 (b) $\frac{1}{\sqrt{2}}$ (c) $-\frac{1}{\sqrt{2}}$ (d) 1

46. The integral $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$ is equal to **(PI-2010-1M)**

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) ∞

47. What is the value of $\lim\limits_{n \to \infty}\left(1 - \frac{1}{n}\right)^{2n}$? **(CS-2010-1M)**

(a) 0 (b) e^{-2} (c) $e^{-1/2}$ (d) 1

48. The $\lim\limits_{x \to 0} \dfrac{\sin\left(\frac{2}{3}x\right)}{x}$ is **(CE-2010-1M)**

(a) $\frac{2}{3}$ (b) 1 (c) $\frac{3}{2}$ (d) ∞

49. Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$, the optimal values of f(x, y) is **(CE-2010-1M)**

(a) a minimum equal to $\frac{10}{3}$

(b) a maximum equal to $\frac{10}{3}$

(c) a minimum equal to $\frac{8}{3}$

(d) a maximum equal to $\frac{8}{3}$

50. The infinite series $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + --- \infty$ Converges to **(ME-2010-1M)**
(a) cos(x) (b) sin(x)
(c) sinh(x) (d) e^x

51. The value of the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ **(ME-2010-1M)**

(a) $-\pi$ (b) $\frac{-\pi}{2}$ (c) $\frac{\pi}{2}$ (d) π

52. The function $y = |2 - 3x|$ **(ME-2010-1M)**
(a) is continuous $\forall\, x \in R$ and differential $\forall\, x \in R$
(b) is continuous $\forall\, x \in R$ and differential $\forall\, x \in R$ except at $x = \frac{3}{2}$
(c) is continuous $\forall\, x \in R$ and differential $\forall\, x \in R$ except at $x = \frac{2}{3}$
(d) is continuous $\forall\, x \in R$ and except at x = 3 and differential $\forall\, x \in R$

53. The integral $\int_{-\infty}^{\infty} \delta\left(t - \frac{\pi}{6}\right) 6\sin(t)\, dt$ evaluates to **(IN-2010-1M)**
(a) 6 (b) 3 (c) 1.5 (d) 0

KEY:

01. a	02. a	03. a
04. b	05. d	06. d
07. a	08. a	09. d
10. c	11. c	12. c
13. b	14. b	15. a
16. c	17. b	18. a
19. b	20. a	21.
22. c	23. b	24. b
25. c	26. c	27. c
28. a	29. a	30. b
31. a	32. d	33. b
34. d	35. d	36. a
37. c	38. a	39. a
40. d	41. a	42. d
43. b	44. a	45. c
46. c	47. b	48. a
49. a	50. b	51. d
52. c	53. b	

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TOPIC – 3 VECTOR CALCULUS

* Vector differential operator, Del, written as ∇ is defined by

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

GRADIENT

Let f(x, y, z) be a differentiable function. Gradient of f(x, y, z) written as **∇f** or **gradf** is defined by

$$\nabla f = (\partial f/\partial x)\,\bar{i} + (\partial f/\partial y)\,\bar{j} + (\partial f/\partial z)\,\bar{k}$$

* ∇f defines a vector function
* If f(x, y, z) = 0 is any surface then ∇f is a normal to the surface
* The directional derivative of f(x, y, z) at a point $p(x_1, y_1, z_1)$ along a is $(\nabla f)_p \cdot \frac{\bar{a}}{|a|}$
* The directional derivative of f(x, y, z) is maximum along ∇f and the magnitude of this maximum is $|\nabla f|$

PROBLEMS

1) : Find a unit normal to the level surface $x^2 - y^2 + z = 2$ at (1, -1, 2)
Ans: $\pm\left(\frac{2\bar{i} + 2\bar{j} + \bar{k}}{3}\right)$

2) Find the directional derivative of $f(x, y, z) = x^2yz + 4xz^2$ at (1, -2, -1) along $2\bar{i} - \bar{j} - 2\bar{k}$
Ans: 37/3

3) Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at P(1, 2, 3) in the direction of $\overline{PQ}$ where Q = (5, 0, 4).
Ans: $28/\sqrt{21}$

4) Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at P(1, 2, 3) along Z – axis
Ans: 12

5) What is the directional derivative of $\phi = xy^2 + yz^3$ at P(2, -1, 1) in the direction of a normal to the surface $x \log z - y^2 = -4$ at A(-1, 2, 1).
Ans: $15/\sqrt{17}$

6) Find the directional derivative of $f = y / (x^2 + y^2)$ at P(0, 1) along the line which makes an angle 30° with positive x – axis.
Ans: -1/2

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7) In what direction from the point (1, 1, -1) is the directional derivative of $f = x^2 - 2y^2 + 4z^2$ is a maximum ?
Ans: $2\bar{i} - 4\bar{j} - 8\bar{k}$

8) What is the greatest rate of increase of $K = xyz^2$ at the point (1, 0, 3) ?
Ans: 9

9) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ then

i) $\nabla f(r) = \frac{f'(r)}{r}\,\bar{r}$

ii) $\nabla(r^n) = n\, r^{n-2}\,\bar{r}$

iii) $\nabla(\log r) = (1/r^2)\,.\,\bar{r}$

iv) $\nabla(r) = (\bar{r}/r) = \hat{r}$

v) $\nabla(1/r) = -\bar{r}/r^3$

10) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, -1, 2)
Ans: $\cos^{-1}(8/(3\sqrt{21}))$

11) Find the equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point (1, -1, 2)
Ans: $7x - 3y + 8z = 26$

12) Show that $(\nabla\phi)\,.\,d\bar{r} = d\phi$ where ϕ is a scalar function, and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

13) $\text{grad}(\bar{r}\,.\,\bar{a}) = a$

14) $\text{grad}[\bar{r}\,.\,(\bar{a} \times \bar{b})] = \bar{a} \times \bar{b}$

where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

DIVERGENCE

If $A(x, y, z) = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$, then ***divergence*** of $\bar{A}$ written as $\nabla . \bar{A}$ defined by

$$\nabla . \bar{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

* $\nabla . \bar{A}$ is a scalar function
* $\nabla . \bar{A} \neq \bar{A} . \nabla$
* $\nabla . \bar{A} = 0 \Rightarrow \bar{A}$ is ***solenoidal***
* $\nabla . (\nabla\phi) = \nabla^2\phi$
* $\nabla^2\phi = 0$ is called Laplace's equation
* ∇^2 is ***Laplacian operator***

PROBLEMS

1) If $\bar{r} = x\,i + y\,j + z\,k$ then $\nabla . r =$ Ans: 3

2) If $\bar{A} = x^2y\,\bar{i} - 2xz\,\bar{j} + 2yz\,\bar{k}$ then find $\nabla . \bar{A}$ at (2 , -1, 3)
Ans: -6

3) Determine the constant a so that the vector $\bar{V} = (x + 3y)\,\bar{i} + (y - 2z)\,\bar{j} + (x + az)\,\bar{k}$ is solenoidal
Ans: -2

4) If $\phi = x^2 . y^3 z^4$ then find $\nabla^2\phi$ at (1, -1, 1)
Ans: -20

5) If $\bar{r} = x\,i + y\,j + z\,k$, then $\nabla^2 f(r) =$ Ans: $f^{11}(r) + (2/r)\,f^1(r)$

6) $\nabla^2 (r^n) =$ Ans: $n(n + 1)\, r^{n-2}$

7) $\nabla^2(1/r) =$ Ans: 0

8) $\nabla^2(r^3) =$ Ans: 12r

9) $\nabla^2(\log r) =$ Ans: $1/r^2$

10) $\nabla . (\bar{r} \times \bar{a}) =$ Ans: 0

CURL

If $\bar{A}(x, y, z) = A_1\,\bar{i} + A_2\,\bar{j} + A_3\,\bar{k}$ is a differentiable vector function then ***curl*** of $\bar{A}$, written as, $\nabla \times \bar{A}$ is defined by

$$\nabla \times \bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\bar{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\bar{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\bar{k}$$

* $\nabla \times \bar{A}$ is a vector function.

* If $\nabla \times \bar{A}$ is zero vector then $\bar{A}$ is called irrotational

PROBLEMS

1) $\nabla \times \bar{r} =$ Ans: $\bar{0}$

2) If $\bar{A} = x^2y\,\bar{i} - 2xz\,\bar{j} + 2yz\,\bar{k}$ then find curl A at (-1, 1, 2) and Curl(Curl A)
Ans: $2\bar{i} - 5\bar{k}$ and 4j

3) Find constants a, b, c so that $V = (x + 2y + az)\,\bar{i} + (bx - 3y - z)\,\bar{j} + (4x + cy + 2z)\,\bar{k}$ is irrotational
Ans: a = 4, b = 2, c = -1

4) $\nabla \times (\bar{r} \times \bar{a}) =$ Ans: $-2\bar{a}$

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VECTOR IDENTITIES

If $\bar{A}$ and $\bar{B}$ are differentiable vector functions, and f and g are differentiable scalar functions of position (x, y, z), then

1) $\nabla(f + g) = \nabla f + \nabla g$

2) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

3) $\nabla . (\bar{A} + \bar{B}) = (\nabla . \bar{A}) + (\nabla . \bar{B})$

4) $\nabla \times (\bar{A} + \bar{B}) = (\nabla \times \bar{A}) + (\nabla \times \bar{B})$

5) $\nabla . (f\bar{A}) = (\nabla f) . \bar{A} + f(\nabla . \bar{A})$

6) $\nabla \times (f\bar{A}) = (\nabla f) \times \bar{A} + f(\nabla \times \bar{A})$

7) $\nabla . (\bar{A} \times \bar{B}) = \bar{B} . (\nabla \times \bar{A}) - \bar{A} . (\nabla \times \bar{B})$

8) $\nabla \times (\nabla\phi) = \bar{0}$

9) $\nabla . (\nabla \times \bar{A}) = \bar{0}$

10) $\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla . \bar{A}) - \nabla^2\bar{A}$

PROBLEMS

If $\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$ and $r = |\bar{r}|$ then answer the following .

a) $\nabla . (r^n\,\bar{r}) =$ Ans: $(n + 3)\, r^n$

b) $\nabla . (\bar{r} / r^3) =$ Ans: 0

c) $\nabla . (\bar{r} / r) =$ Ans: 2/r

d) $\nabla \times (r^n\,\bar{r}) =$ Ans: $\bar{0}$

e) $\nabla \times (\bar{r} / r^2) =$ Ans: $\bar{0}$

2) If $\bar{A}$ and $\bar{B}$ are irrotational then $\bar{A} \times \bar{B}$ is ....
Ans: Solenoidal

3) Curl (grad ϕ) = Ans: $\bar{0}$

4) If f and g are differentiable scalar functions then $(\nabla f \times \nabla g)$ is ....
Ans: Solenoidal

5) $\nabla . [\bar{a} \times (\bar{r} \times \bar{a})] =$ Ans: $2a^2$

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6) If $\bar{A} = \bar{r} / r$ then grad(div $\bar{A}$) = ...
Ans: $-2r^{-3}\,\bar{r}$

7) $\nabla \cdot [r\,\nabla(1/r^3)] =$ Ans: $3\,r^{-4}$

8) $\nabla^2\,[\nabla \cdot (\bar{r} / r^2)] =$ Ans: $2\,r^{-4}$

VECTOR INTEGRATION

LINE INTEGRALS: In general, any integral which is to be evaluated along a curve is called a ***line integral***.

Let $\bar{r} = x\,\bar{i} + y\,\bar{j} + z\,\bar{k}$ be the positive vector of any point (x, y, z) on a curve C joining the points P_1 and P_2.

We assume that C is composed of a finite number of curves for each of which $\bar{r}(u)$ has a continuous derivative

Let $\bar{A}(x, y, z) = A_1\,\bar{i} + A_2\,\bar{j} + A_3\,\bar{k}$ be a differentiable vector function. Then the integral of tangential component of A along C from P_1 to P_2 is

$$\int_{P_2}^{P_1} \bar{A} \cdot d\bar{r} = \int_{P_1}^{P_2} A_1\,dx + A_2\,dy + A_3\,dz$$

* If $\bar{A}$ is the force $\bar{F}$ on a particle moving along C, this line integral represents the work done by the force.

* If C is a simple closed curve, then the integral around C is often denoted by

$$\oint_C \bar{A} \cdot d\bar{r}$$

In Aerodynamics and fluid mechanics this integral is called the circulation of A about C, where A represents the fluid velocity.

* If $\bar{A} = \nabla\phi$ (i.e., $\nabla \times \bar{A} = \bar{0}$) in a region R of space, then

1) $\int_{P_1}^{P_2} \bar{A} \cdot d\bar{r}$ is independent of path C joining P_1 and P_2 in R

2) $\int_C \bar{A} \cdot d\bar{r} = 0$ around any closed curve C in R

PROBLEMS

1). If $\bar{F} = 3xy\,\bar{i} - y^2\,\bar{j}$, evaluate $\int \bar{F} \cdot d\bar{r}$ along C where C is the curve in the xy plane, $y = 2x^2$ from (0, 0) to (1, 2).
Ans: -7/6

2) If $\bar{F} = y\bar{i} - x\bar{j}$. Evaluate $\int \bar{F} \cdot d\bar{r}$ from (0, 0) to (1, 1) along the following paths C
i) $y = x^2$
ii) Straight line joining (0, 0) and (1, 1)
iii) The straight lines from (0, 0) to (1, 0) and then to (1, 1)
Ans: -1/3 , 0, -1

3) Find the total work done in moving a particle in a force field given by
$\bar{F} = (3x^2 + 6y)\,\bar{i} - 14yz\,\bar{j} + 20xz^2\,\bar{k}$ along the straight line joining (0, 0, 0) to (1, 1, 1)
Ans: 13/3

4) If $F = (2xy + z^3)\,\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k}$ is irrotational then find the scalar function ϕ, Such that $F = \nabla\phi$.
Ans: $\phi = x^2y + xz^3$

5) In the previous example, find the work done by $\bar{F}$ in moving an object in this field from (1, -2, 1) to (3, 1, 4).
Ans: 202

SURFACE INTEGRALS

Suppose S is a piece wise smooth surface and $\bar{F}(x, y, z)$ is a differentiable vector function over S. Let P be any point on S and let $\bar{n}$ be the unit vector at P in the direction of outward drawn normal to the surface S at P, then

$\iint_S (\bar{F} \cdot \bar{n})\,dS$ is an example of ***surface integral.***

* $\iint_S (\bar{A} \cdot \bar{n})\,dS = \iint_R (\bar{A} \cdot \bar{n}) \dfrac{dx\,dy}{|\bar{n} \cdot \bar{k}|}$ where R is the projection of S in XY - plane.

PROBLEMS

1) Evaluate $\iint_S (\bar{A} \cdot \bar{n})\,dS$ where $\bar{A} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$ and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant.
Ans: 3/8

2) Evaluate $\iint_S (A \cdot n)\,dS$ where $A = z\bar{i} + x\bar{j} - 3y^2z\,\bar{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.
Ans: 90

Note: $\iint_S dS$ = Area of S

VOLUME INTEGRALS

Consider a closed surface S enclosing a volume V then

$\iiint_V \bar{A}\, dV$ and $\iiint_V \phi\, dV$ are examples of ***volume integrals***

GREEN'S THEOREM

If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are differentiable functions of x and y in R, then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

where C is traversed in positive direction

PROBLEMS

1) Evaluate $\int_C (xy + y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.
Ans: -1/20

2) Evaluate $\int_C (3x + 4y)\,dx + (2x - 3y)\,dy$ where C is a circle of radius 2 with center at origin in the xy – plane, is traversed in the positive sense.
Ans: -8π

3) Evaluate $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$.
Ans: 5/3

4) Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = (x^2 + y^2)\,\bar{i} - 2xy\,\bar{j}$ and C is the rectangle in the xy – plane bounded by $x = 0, x = 2, y = 0, y = 3$.
Ans: -36

5) Evaluate $\int_C \bar{A} \cdot d\bar{r}$ where $\bar{A} = (x - y)\,\bar{i} + (x + y)\,\bar{j}$ and C is the boundary of the region bounded by $y = x^2$ and $x = y^2$.
Ans: 2/3

GAUSS DIVERGENCE THEOREM

If V is the volume bounded by a closed surface S and $\bar{A}(x, y, z)$ is a differentiable vector function then

$$\iint_S (\bar{A} \cdot \bar{n})\,dS = \iiint_V (\nabla \cdot \bar{A})\,dV$$

Where $\bar{n}$ is positive normal to S

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PROBLEMS

1) Evaluate $\iint_S (\bar{r} \cdot \bar{n})\,dS$ where S is the surface of the unit sphere $x^2 + y^2 + z^2 = 1$
Ans: 4π

2) Evaluate $\iint_S (\bar{A} \cdot \bar{n})\,dS$ where $\bar{A} = 3x\,\bar{i} - 4y\,\bar{j} + 8z\,\bar{k}$ and S is the surface bounded by $x = 0, x = 3, y = 0, y = 2$ and $z = 0$ and $z = 1$.
Ans: 42

3) Evaluate $\iint_S (\bar{F} \cdot \bar{n})\,dS$ where $\bar{F} = 4xz\,\bar{i} - y^2\,\bar{j} + yz\,\bar{k}$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
Ans: 3/2

4) Evaluate $\iint_S \bar{A} \cdot \bar{n}\,dS$ where $\bar{A} = 4x\,\bar{i} - 2y^2\,\bar{j} + z^2\,\bar{k}$ taken over the region bounded by $x^2+y^2 = 4$, $z = 0$ and $z = 3$.
Ans: 84π

5) Evaluate $\iint_S (\nabla \times \bar{A}) \cdot \bar{n}\,ds$ where $\bar{A} = y\,\bar{i} - z\,\bar{j} + x\,\bar{k}$ and S is the sphere $x^2 + y^2 + z^2 = 1$
Ans: zero

STOKE'S THEOREM

Let S is an open, two – sided surface bounded by a simple closed curve and $\bar{A}(x, y, z)$ is a differentiable vector function then

$$\int_C \bar{A} \cdot d\bar{r} = \iint_S (\nabla \times \bar{A}) \cdot \bar{n}\,dS$$ where C is traversed in the positive direction.

PROBLEMS

1) $\int_C \bar{r} \cdot d\bar{r}$ where C is the curve $x^2 + y^2 = 4$.
Ans: 0

2) $\int_C yz\,dx + zx\,dy + xy\,dz$ where C is the curve. $x^2 + y^2 = 1, z = y^2$.
Ans: 0

3) Evaluate $\int_C \bar{A} \cdot d\bar{r}$ where $\bar{A} = (2x - y)\,\bar{i} - yz^2\,\bar{j} - y^2z\,\bar{k}$ and C is the boundary of the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$
Ans: π

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4) Evaluate $\int_C \bar{A} \cdot d\bar{r}$ where $A = (y - z + 2)\bar{i} + (yz + 4)\bar{j} - xz\bar{k}$ and C is boundary of the surface of

the cube x = 0, y = 0, z = 0, x = 2, y = 2, z = 2 above the xy – plane.

Ans: - 4

PREVIOUS GATE QUESTIONS - "VECTOR CALCULUS"

1. The directional derivative of $f(x, y) = 2x^2 + 3y^2 + z^2$ at point P(2, 1, 3) in the direction of the vector $\vec{a} = \vec{i} - 2\vec{k}$ is *(GATE'94)*

a) $4/\sqrt{5}$ b) $-4/\sqrt{5}$ c) $\sqrt{5}/4$ d) $-\sqrt{5}/4$

2. The derivative of f(x, y) at point (1, 2) in the direction of vector i + j is $2\sqrt{2}$ and in the direction of the vector –2j is –3. Then the derivative of f(x, y) in direction –i – 2j is *(GATE'95)*

a) $2\sqrt{2} + 3/2$ b) $-7/\sqrt{5}$ c) $-2\sqrt{2} - 3/2$ d) $1/\sqrt{5}$

3. The directional derivative of the function f(x,y,z) = x+y at the point P(1,1,0) along the direction $\vec{i} + \vec{j}$ is *(GATE'96)*

a) $1/\sqrt{2}$ b) $\sqrt{2}$ c) $-\sqrt{2}$ d) 2

4. For the function $\varnothing = ax^2y - y^3$ to represent the velocity potential of an ideal fluid, $\nabla^2\varnothing$ should be equal to zero. In that case, the value of 'a' has to be: *(GATE'99)*

a) -1 b) 1 c) -3 d) 3

5. The directional derivative of the following function at (1, 2) in the direction of (4i + 3j) is:

$f(x, y) = x^2 + y^2$ *(GATE'02)*

a) 4/5 b) 4 c) 2/5 d) 1

6. The vector field $\vec{F} = x\vec{i} - y\vec{j}$ (where $\vec{i}$ and $\vec{j}$ are unit vectors) is *(GATE'03)*

a) divergence free, but not irrotational b) irrotational, but not divergence free
c) divergence free and irrotational d) neither divergence free nor irrotational

7. Value of the integral $\oint_c (xydy - y^2dx)$, where, c is the square cut from the first quadrant by the line x = 1 and y = 1 will be (Use Green's theorem to change the line integral into double integral) *(GATE'05)*

a) ½ b) 1 c) 3/2 d) 5/3

8. The line integral $\int \bar{V} \cdot dr$ of the vector function $\bar{V}(r) = 2xyzi + x^2zj + x^2yk$ from the origin to the point P (1,1,1) *(GATE'05)*

a) is 1 b) is zero c) is -1 (d) cannot be determined without specifying the path

KEY

1. b 2. b 3. b 4. d 5. b 6. c 7. c 8. a

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PREVIOUS GATE QUESTIONS

01. Stokes theorem connects **(ME-2005-1M)**

(a) a line integral and a surface integral.

(b) a surface integral and a volume integral.

(c) a line integral and a volume integral

(d) gradient of a function and its surface integral.

02. For the scalar field $u = \frac{x^2}{2} + \frac{y^2}{3}$, the magnitude of the gradient at the point (1, 3) is **(EE-2005-2M)**

(a) $\sqrt{\frac{13}{9}}$ (b) $\sqrt{\frac{9}{2}}$

(c) $\sqrt{5}$ (d) $\frac{9}{2}$

03. If a vector $\bar{R}(t)$ has a constant magnitude than **(IN-2005-1M)**

(a) $\bar{R} \bullet \frac{d\bar{R}}{dt} = 0$ (b) $\bar{R} \times \frac{d\bar{R}}{dt} = 0$

(c) $\bar{R} \cdot \bar{R} = \frac{d\bar{R}}{dt}$ (d) $\bar{R} \times \bar{R} = \frac{d\bar{R}}{dt}$

04. A scalar field is given by $f = x^{2/3} + y^{2/3}$, where x and y are the Cartesian coordinates. The derivative of f along the line y = x directed away from the origin at the point (8, 8) is **(IN-2005-2M)**

(a) $\frac{\sqrt{2}}{3}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{2}{\sqrt{3}}$ (d) $\frac{3}{\sqrt{2}}$

05. Which one of the following is Not associated with vector calculus? **(PI-2005-1M)**

(a) Stoke's theorem
(b) Gauss Divergence theorem
(c) Green's theorem
(d) Kennedy's theorem

06. The surface $\phi(x, y, z)$ is given by $4xz^2 - 3xy^2 + 3x = 5$. The unit vector at the point (1, –2, 2) on the surface is given by **(PI-2005-2M)**

(a) 0.256i + 0.683j + 0.256k
(b) 0.683i + 0.256j + 0.683k
(c) 0.863i + 0.256j + 0.863k
(d) 0.863i + 0.683j + 0.863k

07. The velocity vector is given as $\bar{v} = 5xyi + 2y^2j + 3yz^2k$. The divergence of this velocity vector at (1, 1, 1) is **(CE-2007-2M)**

(a) 9 (b) 10

(c) 14 (d) 15

08. The area of a triangle formed by the tips of vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is **(ME-2007-2M)**

(a) $\frac{1}{2}(\bar{a} - \bar{b}) \bullet (\bar{a} - \bar{c})$

(b) $\frac{1}{2}|(\bar{a} - \bar{b}) \times (\bar{a} - \bar{c})|$

(c) $\frac{1}{2}|\bar{a} \times \bar{b} \times \bar{c}|$

(d) $\frac{1}{2}(\bar{a} \times \bar{b}) \bullet \bar{c}$

09. The angle (in degrees) between two planar vectors $\bar{a} = \frac{\sqrt{3}}{2}i + \frac{1}{2}j$ and $\bar{b} = \frac{-\sqrt{3}}{2}i + \frac{1}{2}j$ is **(PI-2007-1M)**

(a) 30 (b) 60 (c) 90 (d) 120

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10. $f(x) = |x|$ is a function defined for real numbers x. The directional derivative of f at $x = 0$ in the direction $d = -1$ is **(PI-2007-1M)**

(a) 1 (b) 0

(c) $\frac{-1}{2}$ (d) -1

11. Divergence of the vector field $v(x, y, z) = -(x \cos xy + y)i + (y \cos xy)j + [(\sin z^2) + x^2 + y^2]\, k$ is **(EE-2007-1M)**

(a) $2z \cos z^2$
(b) $\sin xy + 2z \cos z^2$
(c) $x \sin xy - \cos z$
(d) none of these

12. Consider points P and Q in xy – plane with P = (1, 0) and Q = (0, 1). The line integral $2\int_P^Q (x\,dx + y\,dy)$ along the semicircle with the line segment PQ as its diameter. **(EC-2008-2M)**

(a) is –1 (b) is 0 (c) 1
(d) depends on the direction (clockwise (or) anti-clickwise) of the semi circle.

13. The divergence of the vector field $(x - y)\bar{i} + (y - x)\bar{j} + (x + y + z)\bar{k}$ is **(ME-2008-1M)**

(a) 0 (b) 1 (c) 2 (d) 3

14. The directional derivative of the scalar function $f(x, y, z) = x^2 + 2y^2 + z$ at the point P = (1, 1, 2) in the direction of the vector $\bar{a} = 3\bar{i} - 4\bar{j}$ is **(ME-2008-2M)**

(a) –4 (b) –2 (c) –1 (d) 1

15. If $\bar{r}$ is the position vector of any point on a closed surface S that encloses the volume V then $\iint_S (\bar{r}\cdot d\bar{s})$ is equal to **(PI-2008-1M)**

(a) $\frac{1}{2}V$ (b) V (c) 2V (d) 3V

16. For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the gradient at the point P(1, 2, –1) is **(CE-2009-1M)**

(a) $2\bar{i} + 6\bar{j} + 4\bar{k}$ (b) $2\bar{i} + 12\bar{j} - 4\bar{k}$
(c) $2\bar{i} + 12\bar{j} + 4\bar{k}$ (d) $\sqrt{56}$

17. For a scalar function $f(x, y, z) = x^2 + 3y^2 + 2z^2$, the directional derivative at the point P(1, 2, –1) in the direction of a vector $\bar{i} - \bar{j} + 2\bar{k}$ is **(CE-2009-2M)**

(a) –18 (b) $-3\sqrt{6}$ (c) $3\sqrt{6}$ (d) 18

18. If a vector field $\bar{V}$ is related to another field $\bar{A}$ through $\bar{V} = \nabla\times\bar{A}$, which of the following is true? **(EC-2009-2M)**

Note: C and S_C refer to any closed contour and any surface whose boundary is C.

(a) $\oint_C \vec{V}\cdot\vec{dl} = \iint_{S_C} \vec{A}\cdot\vec{ds}$

(b) $\oint_C \vec{A}\cdot\vec{dl} = \iint_{S_C} \vec{V}\cdot\vec{ds}$

(c) $\oint_C \nabla\times\vec{V}\cdot\vec{dl} = \iint_{S_C} \nabla\times\vec{A}\cdot\vec{ds}$

(d) $\oint_C \nabla\times\vec{A}\cdot\vec{dl} = \iint_{S_C} \vec{V}\cdot\vec{ds}$

19. A sphere of unit radius is centered at the origin. The unit normal at a point (x, y, z) on the surface of the sphere is the vector. **(IN-2009-1M)**

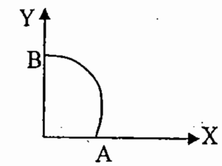
(a) (x, y, z) (b) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(c) $\left(\frac{x}{\sqrt{3}}, \frac{y}{\sqrt{3}}, \frac{z}{\sqrt{3}}\right)$ (d) $\left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}, \frac{z}{\sqrt{2}}\right)$

20. The divergence of the vector field $3xz\hat{i} + 2xy\hat{j} - yz^2\hat{k}$ at a point (1, 1, 1) is equal to **(ME-2009-1M)**

(a) 7 (b) 4 (c) 3 (d) 0

21. A path AB in the form of one quarter of a circle of unit radius is shown in the figure. Integration of $(x+y)^2$ on path AB traversed in a counter – clockwise sense is **(ME-2009-2M)**



(a) $\frac{\pi}{2}-1$ (b) $\frac{\pi}{2}+1$ (c) $\frac{\pi}{2}$ (d) 1

22. $F(x, y) = (x^2 + xy)\,\hat{a}_x + (y^2+xy)\,\hat{a}_y$. Its line integral over the straight line from (x, y) = (0, 2) to (x, y) = (2, 0) evaluates to **(EE-2009-2M)**

(a) –8 (b) 4
(c) 8 (d) 0

23. The line integral of the vector function $\bar{F} = 2x\,\hat{i} + x^2\,\hat{j}$ along the x – axis from x = 1 to x = 2 is **(PI-2009-2M)**

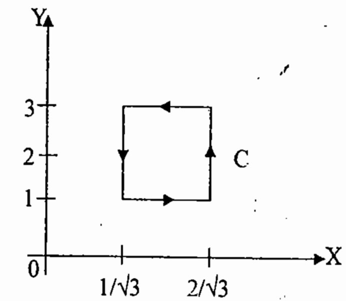
(a) 0 (b) 2.33
(c) 3 (d) 5.33

24. Divergence of the 3 – dimensional radial vector field $\bar{r}$ is **(EE-2010-2M)**

(a) 3 (b) $\frac{1}{r}$

(c) $\hat{i}+\hat{j}+\hat{k}$ (d) $3(\hat{i}+\hat{j}+\hat{k})$

25. If $\vec{A} = xy\,\hat{a}_x + x^2\,\hat{a}_y$ then $\oint \vec{A}\cdot d\vec{l}$ over the path shown in the figure is **(EC-2010-2M)**



(a) 0 (b) $\frac{2}{\sqrt{3}}$

(c) 1 (d) $2\sqrt{3}$

KEY:

01. a	02. c	03. a
04. a	05. d	06.
07. d	08. b	09. d
10.	11. a	12. b
13. d	14. b	15. d
16. b	17. b	18. b
19. a	20. c	21. b
22. c	23. c	24. a
25. c		

BASIC ENGINEERING MATHEMATICS

TOPIC – 4 **PROBABILITY & STATISTICS**

The ***probability*** p of an event A has defined as follows. If A can occur in *s* ways out of a total of *n* equally likely ways then

$$p = p(A) = s/n$$

SAMPLE SPACE: The set S of all possible out comes of some given experiment is called the ***sample space.***

An *event* is a subset of the sample space. The empty set ϕ and S itself are events.

ϕ = Impossible event
S = Certain event

$A \vee B$ is the event that occurs if A occurs or B occurs (or both)
$A \wedge B$ is the event that occurs if A occurs and B occurs

A^C, the *complement* of A , is the event that occurs if A does not occur

MUTUALLY EXCLUSIVE EVENTS: Two events A and B are called mutually exclusive, if they are disjoint i.e.. $A \wedge B = \phi$. In other words, A and B are mutually exclusive if they cannot occur simultaneously.

Axioms Of Probability:

1. For every event A, $0 \le P(A) \le 1$
2. $P(S) = 1$
3. If A and B are mutually exclusive events, then
   $P(A \vee B) = P(\text{ A or B }) = P(A) + P(B)$
4. If $A_1, A_2, \ldots\ldots A_n$ are mutually exclusive then
   $P(A_1 \vee A_2 \vee \ldots\ldots \vee A_n) = P(A_1) + P(A_2) + \ldots\ldots + P(A_n)$
5. If ϕ is the empty set, $P(\phi) = 0$
6. If A^C is the complement of an event A, then
   $P(A^C) = 1 - P(A)$
7. If $A \subset B$, then $P(A) \le P(B)$
8. If A and B are any two events then
   $P(A - B) = P(A) - P(A \wedge B)$
9. ***Addition theorem of probability:***
   If A and B are any two events, then
   $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$
10. For any three events A, B, C
    $P(A \vee B \vee C) = P(A) + P(B) + P(C) - P(A \wedge B) - P(B \wedge C) - P(C \wedge A) + P(A \wedge B \wedge C)$

NOTE: A finite or countably infinite probability space is said to be *discrete* and an uncountable space is said to be ***non – discrete*** (continuous).

OBJECTIVES:

1. Three horses A, Band C are in a race. A is twice as likely to win as B and B is twice as likely
   to win as C. What is the probability that B or C wins?
   a) 2/7 b) 3/7 c) 4/7 d) 6/7

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2. A card is selected at random from an ordinary pack of 52 cards. Probability of selecting a spade card or a face card is
   a) 3/32 b) 23/52 c) 22/52 d) 25/52
3. Let two items be choosen from a lot containing 12 items of which 4 are defective. What is the probability that atleast one item is defective?
   a) 19/33 b) 14/33 c) 1/11 d) 13/33
4. A number is selected at random from first 200 natural numbers. Find the probability that the number is divisible by 6 or 8?
   a) 1/3 b) 1/4 c) 1/5 d) 2/3
5. A point is selected at random inside a circle. Find the probability p that the point is closer to the centre of the circle than to its circumference?
   a) 1/3 b) 1/4 c) 1/5 d) 2/3
6. Let A and B be events with $p(A) = 3/8$, $P(B) = 1/2$ and $P(A \wedge B) = 1/4$ then which of the following is false.
   a) $P(A^C \vee B^C) = 3/4$ b) $P(A^C \wedge B^C) = 3/8$ c) $P(A \wedge B^C) = 1/8$ d) $P(B \wedge A^C) = 5/8$
7. Of 120 students, 60 are studying French, 50 are studying Spanish and 20 are studying French and Spanish. If a student is selected at random then which of the following is not correct.
   a) Probability that the student is studying French or Spanish is 0.75.
   b) Probability that the student is studying neither French nor Spanish is 0.25.
   c) Probability that the student is studying Spanish but not French is 0.25.
   d) Probability that the student is studying French but not Spanish is 0.3.
8. In a class of 100 students, 40 failed in mathematics, 30 failed in physics, 25 failed in chemistry, 20 failed in maths and physics, 15 failed in physics and chemistry, 10 failed in chemistry and maths, 5 failed in maths, physics and chemistry. If a student is selected at random then the probability that he passed in all three subjects is
   a) 0.4 b) 0.45 c) 0.55 d) 0.65

Conditional Probability:

Let E be an arbitrary event in a sample space S with $P(E) > 0$. the Probability that an event A occurs once E has occurred or, in other words, ***Conditional Probability of 'A' given E,*** written P(A/E), is defined as follows

$$P(A/E) = \frac{P(A \wedge E)}{P(E)}$$

NOTE:
Let S be a finite equiprobable space with events a and E. Then

$$P(A/E) = \frac{\text{number of elements in } (A \wedge E)}{\text{number of elements in E}}$$

Multiplication Theorem:
If A and B are any two events then

$$P(A \wedge B) = P(A).\ P(B/A) = P(B).\ P(A/B)$$

NOTE:
If A and B are ***Independent events*** then
$P(A/B) = P(A)$ and $P(B/A) = P(B)$
$P(A \wedge B) = P(A).\ P(B)$

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9. Let a pair of dice be tossed. If the sum is 6, find the probability that one of the dice is a 2.
a) 1/5 b) 2/5 c) 3/5 d) 4/5

10. A man visits a couple who have two children. One of the children, a boy, comes in to the room . Find the probability p that the other is also a boy
a) 1/3 b) 2/3 c) 1/2 d) 3/4

11. Let A and B be events with $P(A) = 3/8$, $P(B) = 5/8$ and $P(A \vee B) = 3/4$. Find the conditional probability $P(A/B)$
a) 1/3 b) 2/5 c)3/4 d)1/2

12. In certain college, 25% of the students failed mathematics, 15% of the students failed in Chemistry, and 10% of the students failed in both maths and chemistry. A student is Selected at random. If he failed chemistry, what is the probability that he failed in maths?
a) 2/3 b)2/5 c) 3/5 d)1/5

13. A die is tossed. If the number appeared is odd, what is the probability that it is prime?
a) 1/3 b) 2/3 c) 3/4 d)1

14. In a certain college, 4% of the men and 1% of the women are taller than 1.8m. Further more, 60% of the students are women. Now if a student is selected at random and is taller than 1.8m, what is the probability that the student is a woman ?
a)3/11 b) 4/11 c) 5/11 d) 6/11

15. We are given three urns as follows. Urn A contains 3 red and 5 white marbles, Urn b contains 2 red and 1 white marble, Urn C contains 2 red and 3 white marbles. An urn is selected at and a marble is drawn from the urn. If the marble is red, what is the probability that it came from urn A?
a) 45/173 b) 37/165 c) 27/109 d) 39/185

16. A coin, weighted so that $P(H) = 2/3$ and $P(T) = 1/3$ is tossed. If heads appears, then a number is selected at random from the numbers 1 through 9. If tails appears, then a number is selected at random from the numbers 1 through 5. Find the probability P that an even number is selected.
a) 67/145 b) 58/135 c)74/157 d)43/142

17. A box contains three coins, two of them fair and one two headed. A coin is selected at random and tossed twice. If heads appears both times, what is the probability that the coin istwo headed?
a) 2/3 b)1/3 c)3/4 d)1/2

18. Ann urn contains 3 red marbles and 7 white marbles. A marble is drawn from the urn and a marble of the colour is then put in to the urn. A second marble is drawn from the urn. If both marbles where of the same colour . What is the probability P that they were both white?
a) 5/6 b)7/8 c)8/9 d)9/10

Random Variable And Expectation:

Suppose that to each point of a sample space we assign a number. We then have a function defined on the sample space. This function is called a ***random variable*** or more precisely a ***random function***. It is usually denoted by a capital letter such as X or Y.

A random variable which takes on a finite or countably infinite number of values is called a ***discrete random variable.*** While one which takes on non countably infinite number of values is called a ***non – discrete or continuous random variable.***

Discrete Probability Distribution:

Let X be a discrete random variable and suppose that the possible values which it can assume are given by $x_1, x_2, \ldots\ldots$ are arranged in increasing order of magnitude. Suppose also that these values are assumed with probabilities given by

$$P(X = x_i) = f(x_i) \quad i = 1, 2, \ldots\ldots$$

Or $$P(X = x) = f(x)$$

In general, f(x) is a probability function

If

1, $f(x) \geq 0$

2, $\sum_{x} f(x) = 1$ where sum is taken over all possible values of x.

Continuous Probability Distribution

If X is a continuous distribution random variable then a function which satisfies the following requirements is called ***probability distribution or probability density function*** of X

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Mathematical Expectation:-

1. For a discrete random variable X having the possible values $x_1, x_2, \ldots\ldots, x_n$ the expectation of x is defined as

$$E(X) = x_1 . P(x_1) + x_2 . P(x_2) + \ldots + x_n P(x_n) = \sum_{i=1}^{n} x_i.P(x_i)$$

2. For a continuous random variable X having density function f(x) the expectation of x is defined as

$$E(x) = \int_{-\infty}^{\infty} x. f(x)\,dx$$

3. If X is a discrete random variable having probability density function f(x), they

$$E\{g(x)\} = \sum_{x} [g(x). f(x)]$$

4. If X is a continuous random variable having p.d.f. g(x) they $E[g(x)] = \int_{-\infty}^{\infty} [g(x). f(x)]\,dx$

5. If c is any constant, then $E(c X) = c. E(X)$

6. If X and Y are any random variables, then $E(x + y) = E(x) + E(y)$

7. If X and Y are independent random variables then $E(XY) = E(X). E(Y)$

8. Variance & standard deviation

$Var(x) = E[(x - \mu)^2]$

S. D. of $X = \sigma x = \sqrt{var(x)}$

$Var(cx) = C^2\ Var(x)$

$\sigma^2 = E[(x - \mu)^2] = E(x^2) - [E(x)]^2$

$Var(x \pm y) = var(x) + var(y)$

19. Find the constant C such that the function

$$f(x) = Cx^2, \; 0 < x < 3.$$
$$= 0. \text{ otherwise}$$

is a probability density function. Hence find $P(1 < x < 2)$

20. A random variable x has density function $f(x) = C/(x^2 + 1)$ where $-\infty < x < \infty$.
a) find the value of the constant C b) $P(1/3 \le x^2 \le 1) = ?$

21. The density function of a random variable X is given by $f(x) = x/2 \quad 0 < x < 2$
$= 0$ otherwise

Then the mean and variance of X are
a) 4/3, 2/9 b) 2/3, 4/9 c) 4/3, 4/9 d) 2/3, 2/9

22. The expectation of discrete random variable X whose probability function is given by
$f(x) = (1/2)^x, \; x = 1, 2, 3, \ldots\ldots$ is
a) 1 b) 2 c) 3 d) 4

23. A continuous random variable X has probability density given by $f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \le 0 \end{cases}$
then expectation of X

a) 0.1 b) 0.25 c) 0.5 d) 0.75

24. On a rainy day an umbrella salesman can earn Rs. 300, and on a fair day (no rain) he loses Rs. 60. What is his expected income per day, if the probability for a rainy day is 0.3
a) Rs 24 b) Rs 36 c) Rs 48 d) Rs 64

25. In a lottery there are 200 prizes of Rs 5, 20 prizes of Rs. 25 and 5 prizes of Rs. 100. Assuming that 10,000 prizes tickets are to be issued and sold what is the fair price to pay for the ticket?(Or if some one purchases a lottery ticket his expectation is--------)
a) Rs 0.2 b) Rs. 0.4 c) Rs. 0.5 d) Rs. 0.6

26. A random variable X has the following probability function

x:	0	1	2	3	4	5	6
P (x):	k	3k	5k	7k	9k	11k	13k

Then $P(3 < x \le 6) =$
a) 11/109 b) 22/49 c) 33/49 d) 44/49

27. Find the expectation of the sum of points in tossing three fair dice
a) 10 b) 10.5 c) 11 d) 11.5

28. A player tosses a fair die. If a prime number occurs he wins that number of rupees, but if a non prime number occurs he loses that number of rupees. His expectation in rupees for each tossing is
a) 1/6 b) 1/2 c) −1/2 d) −1/6

29. The joint density function of two random variables X and Y is given by

$$f(x, y) = xy/96, \; 0 < x < 4, \; 1 < y < 5$$
$$= 0, \quad \text{otherwise.}$$

Which of the following is false
a) E (X) = 8/3 b) E(Y) = 31/9 c) E(XY) = 248/27 d) E(2X + 3Y) = 47/3 e) none

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30. Three machines A, B and C produce respectively 50%, 30%,and 20% of total number of items of a factory. The percentages of defective output of these machines are 3%, 4% and 5% respectively. If an item is selected at random and is found to be defective then the probability that it is produced by machine B is
a) 15/37 b) 16/37 c) 14/34 d) 12/37

BINOMIAL DISTRIBUTION (BERNOULLI'S DISTRIBUTION)

Bernoulli trial: In an experiment if the probability will not change from one trial to the next (tossing a coin or die), such trials are called Bernoulli's trial

If p is the probability that an event will happen in any single Bernoulli's trial (called the probability of success). Then q = 1 - p is the probability that the event will fail to happen (called the probability of failure)

The probability that the event will happen exactly x times in n trials (i.e., x successes and n-x failures will occur) is given by the probability function

$$f(x) = P(X = x) = C(n, x) \; p^x \; q^{n-x}$$

where the random variable X denotes the number of successes in n trials and $x = 0, 1, 2, \ldots\ldots n$.

For Binomial Distribution:

Mean $= \mu = n \cdot p$

Variance $= \sigma^2 = n\,p\,q$

S. D. $= \sigma = \sqrt{n\,p\,q}$

coefficient of skewnen $= \dfrac{q - p}{\sqrt{npq}}$

kurtosis $= 3 + \dfrac{6 - 6\,pq}{n\,p\,q}$

31. Ten coins are thrown simultaneously. Which of the following is wrong
a) probability of getting atleast one head is 1023/1024
b) probability of getting atmost 8 heads is 1013/1024
c) probability of getting atleast 2 heads is 1013/1024
d) probability of getting exactly 8 heads is 11/1024

32. Out of 2000 families with 4 children each, how many families would you expect to have atleast one boy?
a) 1250 b) 1875 c) 1500 d) 1825

33. If 20% of the bolts produced by a machine are defective, determine the probability that out of 4 bolts chosen, the number of defective bolts is less than 2
a) 27/64 b) 81/256 c) 27/256 d) 512/625

34. The probability of getting a total of 7 atleast once in three tosses of a pair of fair dice is
a) 125/216 b) 91/216 c) 117/216 d) 99/216

35. If the probability of a defective bolt is 0.1, then the mean and standard deviation for the number of defective bolts in a total of 400 bolts are------ and -------
a) 40, 6 b) 36, 9 c) 36, 6 d) 40, 9

36. How many dice must be thrown so that there is better than even chances of getting a 6
a) 4 b) 5 c) 6 d) 7

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37. Which of the following statements is true
   a) The mean of the Binomial distribution is 5 and standard deviation is 3
   b) For a Binomial distribution, mean is 6 and variance is 9
   c) For a Binomial distribution mean is 3 and variance is 2
   d) None of the above

Poisson distribution:

Let X be a discrete random variable which can take on the values 0, 1, 2.....such that the probability of X is given by

$$f(x) = P(X = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!} \quad x = 0, 1, 2, \ldots\ldots$$

Where λ is a given positive constant called the parameter of the distribution.

For Poisson Distribution:

Mean $= \mu = \lambda$ Variance $= \sigma^2 = \lambda$

S.D $= \sigma = \sqrt{\lambda}$

Coefficient of Skewness $= 1/\sqrt{\lambda}$

Coefficient of Kurtossis $= 3 + (1/\lambda)$

NOTE:

→ When n is large and p is small then Binomial distribution is very closely approximated by poisson distribution.

→ Poisson distribution is a limiting case of binomial distribution as $n \to \infty$ and $p \to 0$

38. If the probability that an individual suffers a bad reaction from injection of a serum is 0.001. Determine the probability that out of 2000 individuals, exactly 3 individuals suffer a bad reaction.
   a) 0.12 b) 0.08 c) 0.18 d) 0.003

39. In the above problem, find the probability that more than 2 individuals will suffer a bad reaction
   a) 0.823 b) 0.632 c) 0.523 d) 0.323

40. In problem (38), find the probability that atleast one individual suffer a bad reaction
   a) 0.87 b) 0.64 c) 0.92 d) 0.47

41. If X follows poisson distribution such that $P(X = 1) = P(X = 2)$ then $P(X = 0) =$
   a) e^{-1} b) e^{-2} c) e^{-3} d) e^{-4}

42. In a certain factory of turning razor blades, there is a small chance (1/500) for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing atleast one defective blade in a consignment of 10,000 packets.
   a) 9802 b) 198 c) 2 d) 196

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Normal Distribution:

Normal distribution is another limiting form of the binomial distribution under the following conditions

i) n, the number of trials is indefinitely large. ii) Neither p nor q is very small.

The normal probability (curve) density function with mean μ and standard deviation σ is given by the equation $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp[-(x-\mu)^2/2\sigma^2]$, $-\infty < x < \infty$

PROPERTIES:

1. The curve is bell shaped and symmetrical about the line $x = \mu$
2. Mean, Median and Mode of the distribution coincide.
3. As x increases numerically f(x) decreases rapidly.
4. The maximum probability occurs at the point $x = \mu$, and given by
   $$[P(x)]_{max} = \frac{1}{\sigma\sqrt{2\pi}}$$
5. x- axis is an asymptote to the curve.
6. Coefficient of skewness is zero.
7. Coefficient of kurtosis is 3.
8. Total area under the normal curve is unity.
9. Area property $P(\mu - \sigma < X < \mu + \sigma) = 0.6826$
   $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$
   $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$
10. Mean deviation (about mean) $\approx (4/5)\,\sigma$
11. Standard normal distribution. If we let $Z = \frac{X - \mu}{\sigma}$ then the mean of Z is 0 and variance is 1.
    The probability density function for Z is
    $$f(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$$

$$P(a \le X \le b) = \int_a^b f(x)\,dx = \int_{z_1}^{z_2} f(z)\,dz$$

= Area under the standard normal curve between z_1 and z_2

$P(\cancel{-1 \le Z \le 1}) = 0.6827$, $P(-2 \le Z \le 2) = 0.9545$, $\cancel{P(-3 \le Z \le 3) = 0.9973}$

$P(-a \le Z \le a) = 2\,P(0 \le Z \le a) = 2$ (Area under the normal curve between $Z = 0$ and $Z = a$)

$P(-a \le Z \le b) = P(-a \le Z \le 0) + P(0 \le \cancel{Z \le b})$
$= P(0 \le Z \le a) + P(0 \le Z \le b)$
= (Area under the normal curve between $Z = 0$ and $Z = a$) + (Area between $Z = 0$ and $Z = b$)

$P(Z \ge a) = 0.5 - P(0 \le Z \le a)$
$= 0.5 -$ (Area between $Z = 0$ and $Z = a$)

$P(Z \le -a) = P(Z \ge a)$

$P(Z_1 \le Z \le Z_2) = P(0 \le Z \le Z_2) - P(0 \le Z \le Z_1)$

$P(-Z_1 \le Z \le -Z_2) = P(0 \le Z \le Z_1) - P(0 \le Z \le Z_2)$

$P(Z \le a) = 0.5 + P(0 \le Z \le a)$

$P(Z \ge -a) = P(Z \le a)$

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43. Area under normal curve between Z = 0and Z = 1.2 is 0. 3849. Which of the following statements is false.
a) P(Z >1.2) = 0.1151 b) P(Z <1.2) = 0.8849
c) P(-1.2 <Z <1.2) = 0.7698 d) P(Z > -1.2) = 0.1151

44. The mean inside diameter of a sample of 200 washers produced by a machine is 12mm and the standard deviation is 0.02mm. The purpose for which these washers are intended allows a maximum tolerance in the diameter of 11.97 to 12.03mm. Other wise the washers are considered to be defective. Determine the percentage of non defective washers produced by the machine, assuming the diameters are normally distributed.(Area under the normal curve between Z = 0 and Z = 1.5 is 0.4332)
a) 43.32% b) 86.64% c) 93.32% d) 54.68%

45. Suppose that the temperature during june is normally distributed with mean 20^0 c and standard deviation 3.33^0. Find the probability P that the temperature is between 21.11^0c and 26.66^0c (Area under the normal curve between Z = 0 and Z = 2 is 0.4772 and between Z = 0 and Z = 0.33 is 0. 1293)
a) 0.3479 b) 0.6065 c) 0.8479 d) 0.1065

46. A die is tossed 180 times. Using normal distribution find the probability that the face 4 will turn up atleast 35 times
(Area under the normal curve between Z = 0 and Z = 1 is 0. 3413)
a) 0.1587 b)0.8413 c) 0.6587 d)0.3413

47. Suppose the waist measurements of 500 boys are normally distributed with mean 66cm and standard deviation 5cm. Find the number of boys with waists ≤ 70cm (Area under the normal curve between z = 0 and Z = 0.8 is 0.2881)
a) 394 b) 288 c) 788 d) 112

48. Among 10,000random digits, find the probability P that the digit 3 appears at most 950 times. (Area under normal between Z = 0 and Z = 1.67 is 0.4525)
a) 0.4525 b) 0.9525 c) 0.91 d) 0.0475

KEY

1. b 2. c 3. a 4. b 5. b 6. d 7. d 8. b 9. b 10. a 11. b 12. a
13. b 14. a 15. a 16. b 17. a 18. b 19. C = 1/9, P(1 < X < 2) =(7/27)
20. C = 1/π, P((1/3) ≤ X^2 ≤ 1) = 1/6 21. a 22. b 23.c 24. c 25. a 26. c 27. b
28. d 29. e 30. d 31. d 32. b 33. d 34. b 35. a 36. a 37. c 38. c 39. d
40. a 41. b 42. b 43. d 44. b 45. a 46. a 47. a 48. d

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PROBABILITY (ADDITIONAL PROBLEMS)

1) Let f(x) be the continuous probability function of a random variable X. The probability that a < X < b is
a) f(b – a) b) f(b) – f(a) c) $\int_a^b f(x)\,dx$ d) $\int_a^b x \,.\, f(x)\,dx$

2) Which one of the following statements is not true
a) The measure of skewness depends upon the amount of dispersion
b) In a symmetric distribution the values of mean, mode and median are the same
c) In a positively skewed distribution: mean > median > mode
d) In a negatively skewed distribution: mode > mean > median

3) A bag contains 10 blue marbles and 30 red marbles. A marble is drawn from the bag, its color recorded and it is put back in the bag. This process is repeated 3 times. The probability that no two of the marbles drawn have the same color is *(GATE'05[IT])*
a) 1/36 b) 1/6 c) ¼ d) 1/3

4) If P and Q are two random events, then the following is true *(GATE'05[EE])*
a) Independence of P and Q implies that Probability(P ∩ Q) = 0
b) Probability (P ∩ Q) ≥ Probability(P) + Probability(Q)
c) If P and Q are mutually exclusive then they must be independent
d) Probability(P ∩ Q) ≤ Probability(P)

5) A fair coin is tossed 3 times in succession. If the first toss produces a head, then the probability of getting exactly two heads in three tosses is *(GATE'05[EE])*
a) 1/8 b) ½ c) 3/8 d) ¾

6) Two dice are thrown simultaneously. The probability that the sum of numbers on both exceeds 8 I *(GATE'05[PI])*
a) 4/36 b) 7/36 c) 9/36 d) 10/36

7) A lot has 10% defective items. Ten items are chosen randomly from this lot. The probability that exactly 2 of the chosen items are defective is *(GATE'05[ME])*
a) 0.0036 b) 0.1937 c) 0.2234 d) 0.3874

8) A single die is thrown two times. What is the probability that the sum is neither 8 nor 9? *(GATE'05[ME])*
a) 1/9 b) 5/36 c) ¼ d) ¾

9) The probability that there are 53 Sundays in a randomly chosen leap year is *(GATE'05[IN])*
a) 1/7 b) 1/14 c) 1/28 d) 2/7

10) Find the probability of not getting a total of 7 or 11 on either of two tosses of a pair of fair dice?
a) 25/36 b) 21/49 c) 35/64 d) 49/81

11) Find the probability of a 4 turning up atleast once in two tosses of a fair dice?
a) 8/32 b) 10/36 c) 11/36 d) 13/36

12) A fair dice is rolled twice. The probability that an odd number will follow an even number is *(GATE'05[EC])*
a) ½ b) 1/6 c) 1/3 d) 1/4

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13) In a population of N families, 50% of the families have three children, 30% of families have two children and the remaining families have one child. What is the probability that a randomly picked child belongs to a family with two children? *(GATE'04[IT])*
a) 3/23 b) 6/23 c) 3/10 d) 3/5

14) If a fair coin is tossed 4 times, what is the probability that two heads and two tails will result? *(GATE'04[CS])*
a) 3/8 b) ½ c) 5/8 d) ¾

15) An exam paper has 150 multiple choice questions of 1 mark each, with each question having four choices. Each incorrect answer fetches -0.25 marks. Suppose 1000 students choose all their answers randomly with uniform probability. The sum total of the expected marks obtained by all the students is *(GATE'04[CS])*
a) 0 b) 2550 c) 7525 d) 9375

16) In a class of 200 students, 125 students have taken programming language course, 85 students have taken data structures course, 65 students have taken computer organization course, 50 students have taken both programming languages and data structures, 35 students have taken both programming languages and computer organization, 30 students have taken both data structures and computer organization, 15 students have taken all the three courses.
How many students have not taken any of the three courses? *(GATE'04(IT))*
a) 15 b) 20 c) 25 d) 35

17) Let P(E) denote the probability of an event E. Given P(A) =1, P(B) = ½ the values of P(A/B) and P(B/A) respectively are *(GATE'03[CSE])*
a) ¼, ½ b) ½, ¼ c) ½, 1 d) 1, ½

18) Four fair coins are tossed simultaneously. The probability that atleast one heads and atleast one tails turn up is *(GATE'02[CS])*
a) 1/16 b) 1/8 c) 7/8 d) 15/16

19) Seven car accidents occurred in a week, what is the probability that they all occurred on the same day ? *(GATE'01[CS])*
a) $1/7^7$ b) $1/7^6$ c) $1/2^7$ d) $7/2^7$

20) E_1 and E_2 are events in a probability space satisfying the following constraints
$P(E_1) = P(E_2)$; $P(E_1 \cup E_2) = 1$; E_1 & E_2 are independent
Then $P(E_1)$ = *(GATE'2000[CS])*
a) 0 b) ¼ c) ½ d) 1

21) Suppose that the expectation of a random variable X is 5. Which of the following statements is true? *(GATE'99[CS])*
a) There is a sample point at which X has the value 5
b) There is a sample point at which X has the value > 5
c) There is a sample point at which X has a value ≥ 5 d) none of the above

22) Consider two events E_1 and E_2 such that $P(E_1) = ½$, $P(E_2) = 1/3$ and $P(E_1 \cap E_2) = 1/5$. Which of the following statements is true? *(GATE'99[CS])*
a) $P(E_1 \cup E_2) = 2/3$ b) E_1 and E_2 are independent
c) E_1 and E_2 are not independent d) $P(E_1 / E_2) = 4/5$

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23) A die is rolled three times. The probability that exactly one odd number turns up among the three outcomes is *(GATE'98[CS])*
a) 1/6 b) 3/8 c) 1/8 d) ½

24) The probability that it will rain today is 0.5. The probability that it will rain tomorrow is 0.6. The probability that it will rain either today or tomorrow is 0.7. What is the probability that it will rain today and tomorrow? *(GATE'97[CS])*
a) 0.3 b) 0.25 c) 0.35 d) 0.4

25) The probability that a number selected at random between 100 and 999 (both inclusive) will not contain the digit 7 is *(GATE'95[CS])*
a) 16/25 b) $(9/10)^3$ c) 27/75 d) 18/25

26) The probability of an event B is $\bar{p}_1$. The probability of events A and B occur together is p_2. While the probability that A and B occur together is p_3. The probability of event A in terms of p_1, p_2 and p_3 is
a) $p_1 + p_2$ b) $p_2 + p_3$ c) $p_3 + p_1$ d) $p_1 + p_3 - p_2$

27) Let A and B be any two arbitrary events then which one of the following is true?
a) $P(A \cap B) = P(A). P(B)$ b) $P(A \cup B) = P(A) + P(B)$
c) $P(A / B) = P(A \cap B) . P(B)$ d) $P(A \cup B) \le P(A) + P(B)$

28) The probability that a new Airport will get an award for its design is 0.16. The probability that it will get an award for its efficient use of materials is 0.24 and probability that it will get both the awards is 0.11. What is the probability that it will get only one of the two awards?
a) 0.29 b) 0.18 c) 0.21 d) 0.19

29) There are 27 students in a class. What is the probability that atleast 3 of them have their birthday in the same month?
a) 1/9 b) 1/12 c) ¼ d) 1

30) A jar has 5 marbles, one of each of the colors, red, white, blue, green and yellow. If 4 marbles are removed from the jar, what is the probability that the yellow one is removed?
a) 1/5 b) ½ c) 4/5 d) ¾

31) If the probability that a communication system has a high fidelity is 0.81 and the probability that it will have high fidelity and high selectivity is 0.18. What is the probability that a system with high fidelity will also have high selectivity?
a) 2/9 b) 7/9 c) 0.63 d) 0.37

32) A jar contains 4 marbles. 2 red and 2 white. Two marbles are chosen at random. If p_1 is the probability that the marbles chosen are of same color and p_2 is the probability that the marbles chosen be of different colors, then which of the following is true?
a) $p_1 = p_2$ b) $p_1 = 2p_2$ c) $p_2 = 2p_1$ d) $2p_1 = 3p_2$

33) The average grade for an examination is 74 and the standard deviation is 7. If 12% of the class are given A's and the grades are curved to follow normal distribution then what is the lowest possible A?
[The area under the standard normal curve to the left of Z = 1.175 is 0.88]
a) 79 b) 81 c) 83 d) 85

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34) A and B play a game in which they toss a fair coin three times. The one obtaining heads first wins the game. If A tosses the coin first and if the total value of the stakes is Rs. 20. How much should be contributed by B in order that the game is fair?
a) Rs. 6.66 b) Rs. 7.50 c) Rs. 8 d) Rs. 8.25

35) An inefficient secretary places 5 different letters into 5 different addressed envelopes at random.
Find the probability that atleast one of the letters will arrive at the proper destination.
a) 4/5 b) 1/120 c) 19/30 d) 119/120

36) Determine the probability p, that a non defective bolt will be found next, if out of 600 bolts already examined, 12 were defective?
a) 0.02 b) 0.04 c) 0.96 d) 0.98

37) Each of the three identical jewelry boxes has two shelves. In each shelf of the first box there is a gold watch. In each shelf of the second box there is a silver watch. In one shelf of the third box there is a gold watch while in the other there is a silver watch. If we select a box at random, open one of the shelves and find it to contain a silver watch. What is the probability that the other shelf of the box has the gold watch?
a) ½ b) 1/3 c) 1/4 d) 3/4

38) Box P has 2 red balls and 3 blue balls and box Q has 3 red balls and 1 blue ball. A box is selected as follows.
i) Select a box
ii) Choose a ball from the selected box such that each ball in the box is equally likely to be chosen. The probabilities of selecting boxes P and Q are 1/3 and 2/3 respectively.
Given that a ball selected in the above process is a red ball, the probability that it came from the box P is
a) 4/19 b) 5/19 c) 2/9 d) 19/30

39) For a probability distribution, If 'a' is the mean, 'b' is the mode and 'c' is the standard deviation then the coefficient of skewness = ...
a) (b-a) / c b) (a-c) / b c) (b-c) / a d) (a-b) / c

KEY:

1. c 2. d 3. b 4. d 5. b 6. d 7. b 8. d 9. d 10. d 11. c 12. d
13. b 14. a 15. d 16. c 17. d 18. c 19. b 20. d 21. c 22. c 23. b 24. d
25. d 26. b 27. d 28. b 29. d 30. c 31. a 32. c 33. c 34. b 35. c 36. d
37. b 38. a 39. d

MEAN / MEDIAN / MODE / SD

01. Calculate the arithmetic mean for following data 1600, 1560, 1440, 1530, 1670, 1860, 1750, 1910, 1490, 1800
(a) 1660 (b) 1661 (c) 1670 (d) 1560

02. Arithmetic mean of 24, 28, 29, 34, 18, 22, 26, 30, 32, 24, 20 is
(a) 26.09 (b) 26 (c) 24 (d) 34

03. Arithmetic mean of the natural numbers from 1 to n is
(a) $\frac{n(n+1)}{2}$ (b) $\frac{n+1}{2}$ (c) n/2 (d) none

04. If 10 is added to each and every item of a data, then the arithmetic mean
(a) Is increased by 10 times (b) is not increased
(c) is greater by 10 (d) none

05. The man of 25 values was calculated as 78.4, but while taking them an item 69 was misread as 96, the correct mean is
(a) 77.32 (b) 78.4 (c) 76 (d) 69

06. The median of 55, 100, 75, 80, 90, 85, 95, 45, 70, 70, 55
(a) 75 (b) 85 (c) 90 (d) none

07. The simplest measure of dispersion is
(a) S. D (b) Range (c) MD (d) QD

08. The measure of dispersion which is used to find more consistent data is
(a) SD (b) MD (c) QD (d) Range

09. Standard deviation of 27, 35, 40, 35, 36, 36, 29 is
(a) 17.14 (b) 4.14 (c) 34 (d) none

10. For a symmetrical distribution QD is
(a) 2/3 SD (b) SD (c) MD (d) 6/5 MD

11. Standard deviation of 3, 5, 7, 9, 11, 13 is
(a) 12 (b) 11 (c) 11.66 (d) 3.4

12. If the first and third quartiles of a data are 5, 10 then QD is
(a) 5/2 (b) 3 (c) 2 (d) 1

13. If the least and greatest values of a data are 5.95 then the coefficient of range is
(a) 10/9 (b) 9/10 (c) 1/10 (d) none

14. If standard deviation of a data is 3, mean is 20 then coefficient of variation is
(a) 156 (b) 3/20 (c) 20/3 (d) none

15.

Classes	0-10	10-20	20-30
Frequency	5	14	6

mean is
(a) 8.55 (b) 8 (c) 9 (d) 10

16. Range of 1, 4, 90, 100, 4 is
(a) 99 (b) 73 (c) 72 (d) 11

17. The mean of a set of numbers is '$\overline{X}$', if each number is increased by λ, the mean of new set is
(a) $\overline{X}$ (b) $\overline{X}+\lambda$ (c) $\lambda\overline{X}$ (d) λ

18. If mean = (3 Md – Mode) X, then the value 'X' is
(a) 1 (b) 2 (c) ½ (d) 3/2

KEY

1. b	2. a	3. b	4. c	5. a	6. a	7. b	8. a	9. b
10. a	11. d	12. a	13. b	14. a	15. a	16. a	17. b	18. c

REGRESSION ANALYSIS

Regression is the estimation or prediction of unknown values of one variable from known values of another variable.

Regression measures the nature and extent of correlation.

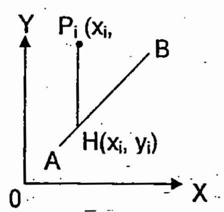
LINEAR REGRESSION

If two variables x and y are correlated i.e., there exists an association or relationship between them, then the scatter diagram will be more or less concentrated round a curve. This curve is called the curve of regression and the relationship is said to be expressed by means of *curvilinear regress-ion*. In the particular case, when the curve is a straight line, it is called a *line of regression* and the regression is said to be linear.

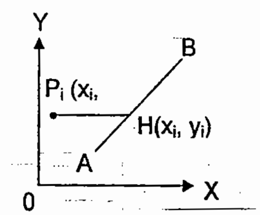
A line of regression is the straight line which gives the best fit in the least squares sense to the given frequency.

If the line of regression is so chosen that the sum of squares of deviation parallel to the axis of y is minimized [see Fig(A)], it is called the line of regression of y on x and it gives the best estimate of y for any given value of x.

If the line of regression is so chosen that the sum of squares of deviation parallel to the axis of x is minimized [see Fig(B)], it is called the line of regression of x on y and it gives the best estimate of x for any given value of y.



Fig(A)



Fig(B)

Simple regression establishes the relationship between two variables (one is dependent and other is independent). But in multiple regression the number of variables is more than two (one dependent and two or more independent).

In linear regression, the relationship between the variables is linear and is represented by a straight line called as regression line.

Ex(1): $y = a + bx$ is known as regression line of y on x.
Ex(2): $x = a_0 + a_1y$ is known as regression line of x on y
Ex(3): Reciprocal curve $y = 1 / (a + bx)$.

This can be transformed to simple linear regression equation, as follows if $1/y = z \Rightarrow y = 1/z$

$$\Rightarrow 1/z = 1/(a + bx) \Rightarrow z = a + bx$$

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CURVILINEAR (OR) (NON – LINEAR) REGRESSION

In this regression, the regression equation $y = f(x)$ is non – linear.
Ex: polynomial, exponential, power, reciprocal functions.

POLYNOMIAL REGRESSION

Let $y = a + bx + cx^2 + \dots\dots\dots + kx^{m-1}$ be a polynomial in x of degree (m – 1)
And let (x_i, y_i) ; i = 1, 2, ……,n be the given set of n – observations.

Now, the constants a, b, c, …k ('m' constants) can be estimated with the help of 'm' – normal equations given by

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 + \dots\dots + k\,\Sigma x^{m-1}$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\,\Sigma x^3 + \dots\dots + k\,\Sigma x^m$$
$$\text{- - - - - - - - - -}$$
$$\text{- - - - - - - - - - -}$$
$$\Sigma x^{m-1}y = a\Sigma x^{m-1} + b\Sigma x^m + c\Sigma x^{m+1} + \dots\dots k\Sigma x^{2(m-1)}$$

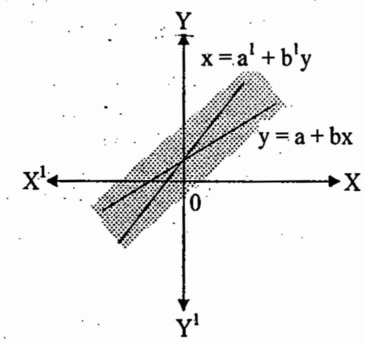
CORRELATION

In some contexts, the changes in one variable are related to the changes in the other variable. This phenomenon is called correlation and such a data connecting the two variables is called bivariate population.

Ex(1) : The yield of a crop varies with the amount of rainfall. If an increase (or decrease) in the values of one variable corresponds to an increase (or decrease) in the other, then that correlation is said to be positive. If the increase (or decrease) in one variable corresponds to the decrease (or increase) in the other, then that correlation is said to be negative. If there is no relationship indicated between the two variables, then they are said to be independent or uncorrelated.

Ex(2) : The price of a commodity increases with the reduction in its supply.

Scatter Diagram: To obtain a measure of relationship between the two variables, we plot their corresponding values on the graph taking one of the variables along the X – axis and the other along the Y – axis. The resulting diagram showing a collection of dots is called a scatter diagram.



Let the origin be shifted to (x, y) where x , y are the means of x's and y's.

Then the new co –ordinates are given by

$$X = x - \bar{x} \quad ; \quad Y = y - \bar{y}$$

Now the points (X, Y) are distributed over the four quadrants of XY – plane. The product XY is positive in the I and II quadrants but negative in the II and IV quadrants. The algebraic sum of the products can be taken as the trend of the dots in all the four quadrants.

∴ (i) If ΣXY is positive, then the trend of the dots is through the first and third quadrants.

(ii) If ΣXY is negative, then the trend of the dots is in the second and fourth quadrants.

(iii) If ΣXY is zero, then the points indicate no trend i.e., the points are evenly distributed over all the four quadrants.

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COEFFICIENT OF CORRELATION

The numerical measure of correlation is called the coefficient of correlation and is defined by the relation $r = \sum XY / n\sigma_x \sigma_y$ --- (1)

Where $X = x - \bar{x}$ $Y = y - \bar{y}$

σ_x = Standard deviation of x's σ_y = Standard deviation of y's

and n = number of points

$$\therefore \quad r = \frac{\sum XY}{\sqrt{\sum X^2}\sqrt{\sum Y^2}} \quad \text{--- (2)}$$

LINES OF REGRESSION

The line of regression of y on x is $y - \bar{y} = r \cdot (\sigma_y/\sigma_x)(x - \bar{x})$

and the line of regression of x on y is $x - \bar{x} = r \cdot (\sigma_x/\sigma_y)(y - \bar{y})$

$r \cdot (\sigma_y/\sigma_x)$ is called the regression co-efficient of y on x and is denoted by b_{yx}.

$r \cdot (\sigma_x/\sigma_y)$ is called the regression co-efficient of x on y and is denoted by b_{xy}.

Note:

If r = 0, the two lines of regression become $y = \bar{y}$ and $x = \bar{x}$ which are two straight lines parallel to X and Y axes respectively and passing through their means $\bar{y}$ and $\bar{x}$. They are mutually perpendicular.

If $r = \pm 1$, the two lines of regression will coincide.

PROPERTIES OF REGRESSION CO-EFFICIENTS

1. Correlation co-efficient is the geometric mean between the regression co-efficients.
2. If one of the regression co-efficients is greater than unity, the other must be less than unity.
3. Arithmetic mean of regression co-efficients is greater than the correlation co-efficient.
4. Regression co-efficients are independent of the origin but not of scale.
5. The correlation co-efficient and the two regression co-efficients have same sign.

ANGLE BETWEEN TWO LINES OF REGRESSION

If θ is the acute angle between the two regression lines in the case of two variables x and y, then

$$\tan\theta = \frac{1 - r^2}{|r|} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$ where r, σ_x, σ_y have their usual meanings.

Note:

1. When r = 0, θ = π/2

The two lines of regression are perpendicular to each other.

2. When $r = \pm 1$, $\tan\theta = 0$ so that, θ = 0 or π.

Hence the lines of regression coincide and there is perfect correlation between the two variables x and y.

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OBJECTIVES

1. Which of the following is a curvilinear regression

a) straight line b) $ax + by + cz + d = 0$ c) exponential function d) none of these

2. The normal equations of the geometric curve $y = ax^b$

a) (i) $\sum \log y = \sum \log a + b \sum \log x$
(ii) $\sum \log x \log y = \log a \sum \log x + b \sum (\log x)^2$

b) (i) $\sum \log y = \log a + b \sum \log x$
(ii) $\sum \log xy = \log a \sum x + b \sum \log x^2$

c) (i) $y = a + bx$
(ii) $\sum xy = na + b\sum x^2$

d) none of these

3(a). If a polynomial $y = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_kx^k$ is to be fitted for a given data of 'n' observations, then the number of normal equations required are ....

a) k b) k + 1 c) n + 1 d) n

3(b). The first normal equation of the above Q.No3(a) is given by

a) $y = na_0 + a_1\sum x + a_2\sum x^2 + \ldots + a_k\sum x^k$
b) $\sum y = a_0 + a_1\sum x + a_2\sum x^2 + \ldots + a_k\sum x^k$
c) $\sum y = na_0 + a_1\sum x + a_2\sum x^2 + \ldots + a_k\sum x^k$
d) $a_0\sum y = \sum a_0 + a_1\sum x + a_2\sum x^2 + \ldots + a_k\sum x^k$

4. The normal equations of the reciprocal curve $y = \dfrac{1}{A + Bx}$ are given by (if z = 1/y) ....

a) $y = \sum A + B\sum x$, $\sum xy = A\sum x + B\sum x^2$
b) $z = \sum A + B\sum x$, $\sum zx = A\sum x + B\sum x^2$
c) $z = \sum A + B\sum y$, $\sum zy = A\sum y + B\sum y^2$
d) none of these

5. The equation $y = a_0 + a_1x_1 + a_2x_2 + \ldots\ldots + a_kx_k$ indicates ....

a) simple linear regression b) simple nonlinear regression
c) curvilinear multiple regression d) linear multiple regression

6. The last normal equation of the curve $y = a + bx + cx^2 + dx^3$ is given by

a) $\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 + d\sum x^5$
b) $\sum x^4y = a\sum x^4 + b\sum x^5 + c\sum x^6 + d\sum x^7$
c) $\sum x^3y = a\sum x^3 + b\sum x^4 + c\sum x^5 + d\sum x^6$
d) none of these

7. The regression coefficient of y = a + bx is given by

a) $(r\sigma_x)/\sigma_y$ b) $(r\sigma_y)/\sigma_x$ c) σ_y/σ_x d) σ_x/σ_y

8. If the two regression lines are known, then r = ...

a) A.M of the two regression coefficients b) G.M of the two regression coefficients
c) H.M of the two regression coefficients d) product of the two regression coefficients

9(a). If n = 25, $\sum x = 125$, $\sum x^2 = 650$, $\sum y = 100$, $\sum y^2 = 460$ and $\sum xy = 508$ and (8, 12); (6, 8) are considered as (6, 14); (8, 6) then change occurs in finding the correct value of r is......

a) $\sum x^2 = 636$, $\sum xy = 520$ b) $\sum y^2 = 426$, $\sum xy = 520$
c) $\sum y^2 = 436$, $\sum xy = 520$ d) $\sum x^2 = 646$, $\sum xy = 510$

9(b). From the data given above in Q. No 9(a) the correct value of r is given by ....

a) 0.76 b) 0.56 c) 0.77 d) 0.67

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10. The equations of regression of lines are $y = 0.5x + a$ and $x = 0.4y + b$, then $r =$ ....
a) 0.2 b) -0.2 c) $-\sqrt{0.2}$ d) $\sqrt{0.2}$

11(a). If $y = x+1$ and $x = 3y - 7$ are the two lines of regression, then $\overline{y} =$ ...., $\overline{x} =$ ....
a) 3, 2 b) 2, 3 c) -3, -2 d) -2, -3

11(b). In the above Q. No 11(a), $r =$ ....
a) $-\sqrt{3}$ b) $\sqrt{3}$ c) $\sqrt{2}$ d) $-\sqrt{2}$

12. If $x - \overline{x} = b(y - \overline{y})$, then $b =$ ....
a) $(r\sigma_x)/\sigma_y$ b) $(r\sigma_y)/\sigma_x$ c) σ_y/σ_x d) σ_x/σ_y

KEY

1. c	2. a	3(a). b	3(b). c	4. b	5. d	6. c	7. b
8. b	9(a). c	9(b). d	10. d	11(a). a	11(b). b	12. a	

PROBLEMS

1. If $r = 0$, then the two lines of regression are ....

2. If the two regression co-efficients are 0.8 and 0.2, what would be the value of co-efficient of correlation.

3. The two regression equations of the variables x and y are $x = 19.13 - 0.87$ and $y = 11.64 - 0.50x$. Find (i) mean of x's, (ii) mean of y's and (iii) the correlation co-efficient between x and y.

4. Two random variables have the regression lines with equations $3x + 2y = 26$ and $6x + y = 31$. Find the mean values and the correlation coefficient between x and y.

5. Compute the co-efficient of correlation between x and y from the following data.

x	65	66	67	68	69	70	71
y	67	68	66	69	72	72	69

KEY

1. parallel to the axes 2. 0.4 3. (i) 15.79 (ii) 3.74 (iii) -0.66

4. $\overline{x} = 9.06$; $\overline{y} = 5.52$; $r = 0.46$ 5. 0.67

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PROBLEMS

1. The regression lines are $3x + 12y = 19$ and $3y + 9x = 46$.
(i) The value of correlation coefficient is (ii) Mean values of x and y are
Ans: (i) $r = 0.0833$ (ii) $\overline{x} = 5$; $\overline{y} = 0.33$

2. Find the correlation coefficient (r) if $\sigma_x = 3, \sigma_y = 1.4, b_{yx} = 0.28$
Ans: $r = 0.6$

3. Find correlation coefficient (r)

X	-1	0	3	1
Y	1	2	3	2

Ans: $r = 0.8520$

4. $(\overline{X}, \overline{Y}) = (9.2, 16.5)$; $(\sigma_X, \sigma_Y) = (2.1, 4.2)$ then two lines of regression are:
Ans: $Y = 1.68X + 1.044$; $X = 0.42Y + 2.27$

5. The regression line, y on x and x on y are $y = ax + b$ and $x = cy + d$.
(i) Find $\overline{x} =$ ..... and $\overline{y} =$ .....
(ii) Correlation coefficient between x and y is
Ans: (i) $\left(\dfrac{bc+d}{1-ac}, \dfrac{ad+b}{1-ac}\right)$ (ii) $\sqrt{ac}$

6. In a partially destroyed laboratory record, only the lines of regression of y on x and x on y are available as $4x - 5y + 33 = 0$ and $20x - 9y = 107$ respectively. Calculate x, y and the coefficient of correlation between x and y.
Ans: 13, 17, 0.6

7. While calculating correlation co – efficient between two variables x and y from 25 pairs of observations, the following results were obtained. $n = 25$, $\Sigma x = 125$, $\Sigma x^2 = 650$, $\Sigma y = 100$, $\Sigma y^2 = 460$, $\Sigma xy = 125$. Later it was discovered at the time of checking that the paths of values

x	y
8	12
6	8

were copied down as

x	y
6	14
8	6

Obtain the correct value of correlation co – efficient.
Ans: 2/3

8. Following tables gives the data on rainfall and discharge in a certain river. Obtain the line of regression of y on x.

Rainfall x(inches)	1.53	1.78	2.60	2.95	3.42
Discharge y(1000 CC)	33.5	36.3	40.0	48.8	53.5

Ans: $y = 9.67x + 18.03$

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RANK CORRELATION

Let there be n individuals having different characteristics. By considering two particular characteristics A, B of the individuals, ranking is given in an order of merit. Now, the correlation between these n – pairs of ranks is called rank correlation in the characteristics A & B for that group of individuals.

Let x_i, y_i be the ranks of the i^{th} individual in A and B respectively. Assuming that no two individuals are equal in either case and each of the variables taking the rank value 1,2,3, ……n.

Let $d_i = x_i - y_i = (x_i - x) - (y_i - y)$
$= x_i - y_i$

Rank correlation coefficient is denoted by ρ.

$$\therefore \boxed{\rho = 1 - \frac{6\,\Sigma d_i^2}{n(n^2-1)}}$$

Note:

1. This formula is called Spearmann's formula for rank correlation
2. $\Sigma d_i = \Sigma(x_i - y_i) = \Sigma x_i - \Sigma y_i = 0$ always.

PROBLEMS

1. Ten participants in a contest are ranked by two judges as follows.

x	1	6	5	10	3	2	4	9	7	8
y	6	4	9	8	1	2	3	10	5	7

Calculate the rank correlation ~~coefficient~~ ρ.

Ans: 0.6 nearly

2. Calculate ρ from the following data showing ranks of 10 students in two subjects.

Maths	3	8	9	2	7	10	4	6	1	5
Physics	5	9	10	1	8	7	3	4	2	6

Ans: 0.8545

REPEATED RANKS

In this case we add the factor $[m(m^2 - 1)]/12$ to Σd^2 where 'm' is the number of times an item is repeated. And this correlation factor is to be added for each repeated value.

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PROBLEMS

1. Obtain the rank correlation co – efficient for the following data

x	68	64	75	50	64	80	75	40	55	64
y	62	58	68	45	81	60	68	48	50	70

Ans: 0.5454

2. A sample of 12 fathers and their eldest sons have the following data about their heights in inches. Calculate ρ.

Fathers	65	63	67	64	68	62	70	66	68	67	69	71
Sons	68	66	68	65	69	66	68	65	71	67	68	70

Ans: 0.722

PREVIOUS GATE QUESTIONS - "PROBABILITY"

1. The probability that two friends share the same birth-month is *(GATE'98)*
   a) 1/6 b) 1/12 c) 1/144 d) 1/24
2. In a manufacturing plant, the probability of making a defective bolt is 0.1. The mean and standard deviation of defective bolts in a total of 900 bolts are respectively
   a) 90 and 9 b) 9 and 90 c) 81 and 9 d) 9 and 81
   (GATE - 2000)
3. Manish has to travel from A to D changing buses at stops B and C enroute. The maximum waiting time at either stop can be 8 minutes each, but any time of waiting up to 8 minutes is equally likely at both places. He can afford up to 13 minutes of total waiting time if he is to arrive at D on time, What is the probability that Manish will arrive late at D? *(GATE'02)*
   a) 8/13 b) 13/64 c) 119/128 d) 9/128
4. A box contains 10 screws, 3 of which are defective. Two screws are drawn at random with replacement. The probability that none of the two screws is defective will be *(GATE'03)*
   a) 100% b) 50% c) 49% d) none of these
5. A hydraulic structure has four gates which operate independently. The probability of failure of each gate is 0.2. Given that gate 1 has failed, the probability that both gates 2 and 3 will fail is
   a) 0.240 b) 0.200 c) 0.040 d) 0.008
   (GATE'04)
6. From a pack of regular playing cards, two cards are drawn at random. What is the probability that both cards will be kings, if the first card is NOT replaced? *(GATE'04)*
   a) 1/26 b) 1/52 c) 1/169 d) 1/221

7. A lot has 10% defective items. Ten items are chosen randomly from this lot. The probability that exactly 2 of the chosen items are defective is *(GATE'05)*
(a) 0.0036 (b) 0.1937 (c) 0.2234 (d) 0.3874

PREVIOUS GATE QUESTIONS - "STATISTICS"

8. Four arbitrary points (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4), are given in the xy – plane using the method of least squares, if, regressing y upon x gives the fitted line y = ax + b; and regressing x upon y gives the fitted line x = cy + d, then *(GATE'99)*
a) the two fitted lines must coincide (b) the two fitted lines need not coincide
c) it is possible that ac = 0 (d) a must be 1/c

9. A regression model is used to express a variable Y as a function of another variable X. This implies that *(GATE'02)*
a) there is a causal relationship between Y and X
b) a value of X may be used to estimate a value of Y
c) values of X exactly determine values of Y
d) there is no causal relationship between Y and X

10. The following data about the flow of liquid was observed in a continuous chemical process plant

Flow rate (litres/sec)	7.5 to 7.7	7.7 to 7.9	7.9 to 8.1	8.1 to 8.3	8.3 to 8.5	8.5 to 8.7
Frequency	1	5	35	17	12	10

Mean flow rate of the liquid is *(GATE'04)*
a) 8.00 litres/sec b) 8.06 litres/sec c) 8.16 litres/sec d) 8.26 litres/sec

KEY

1. b 2. a 3. 4. d 5. c 6. d 7. b 8. b 9. b 10. c

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PREVIOUS GATE QUESTIONS

01. The life of a bulb (in hours) is a random variable with an exponential distribution $f(t) = \alpha e^{-\alpha t}$, $0 \le t \le \infty$. The probability that its value lies b/w 100 and 200 hours is **PI – 2005 – 2M**
(a) $e^{-100\alpha} - e^{-200\alpha}$
(b) $e^{-100} - e^{-200}$
(c) $e^{-100\alpha} + e^{-200\alpha}$
(d) $e^{-200\alpha} - e^{-100\alpha}$

02. Assume that the duration in minutes of a telephone conversation follows the exponential distribution $f(x)=\frac{1}{5}e^{-x/5}$, $x \ge 0$. The probability that the conversation will exceed five minutes is **IN – 2007 – 2M**
(a) $\frac{1}{e}$ (b) $1-\frac{1}{e}$
(c) $\frac{1}{e^2}$ (d) $1-\frac{1}{e^2}$

03. If the standard deviation of the spot speed of vehicles in a highway is 8.8 kmph and the mean speed of the vehicles is 33 kmph, the coefficient of variation in speed is **CE – 2007 – 2M**
(a) 0.1517 (b) 0.1867
(c) 0.2666 (d) 0.3646

04. Let X and Y be two independent random variables. Which one of the relations b/w expectation (E), variance (V_{ar}) and covariance (C_{ov}) given below is FALSE? **ME – 2007 – 2M**
(a) E(XY) = E(X) E(Y)
(b) $C_{ov}(X, Y) = 0$
(c) $V_{ar}(X+Y) = V_{ar}(X) + V_{ar}(Y)$
(d) $E(X^2Y^2) = (E(X))^2\ (E(Y))^2$

05. Two cards are drawn at random in succession with replacement from a deck of 52 well shuffled cards Probability of getting both 'Aces' is **PI – 2007 – 1M**
(a) $\frac{1}{169}$ (b) $\frac{2}{169}$
(c) $\frac{1}{13}$ (d) $\frac{2}{13}$

06. The random variable X taken on the values 1, 2 (or) 3 with probabilities $\frac{2+5P}{5}$, $\frac{1+3P}{5}$ and $\frac{1.5+2P}{5}$ respectively The values of P and E(X) are respectively **PI – 2007 – 2M**
(a) 0.05, 1.87 (b) 1.90, 5.87
(c) 0.05, 1.10 (d) 0.25, 1.40

07. If X is a continuous random variable whose probability density function is given by
$$f(x) = \begin{cases} k(5x-2x^2), & 0 \le x \le 2 \\ 0 & , \text{otherwise} \end{cases}$$
Then P(x > 1) is **PI – 2007 – 2M**
(a) 3/14 (b) 4/5
(c) 14/17 (d) 17/28

08. If E denotes expectation, the variance of a random variable X is given by **EC – 2007- 1M**
(a) $E(X^2) - E^2(X)$
(b) $E(X^2) + E^2(X)$
(c) $E(X^2)$
(d) $E^2(X)$

09. An examination consists of two papers, paper 1 and paper 2. The probability of failing in paper 1 is 0.3 and that in paper 2 is 0.2. Given that a student has failed in paper 2, the probability of failing in paper 1 is 0.6. The probability of a student failing in both the papers is **EC – 2007 – 2M**
(a) 0.5 (b) 0.18
(c) 0.12 (d) 0.06

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10. X is uniformly distributed random variable that take values between 0 and 1. The value of $E(X^3)$ will be **EE – 2008 - 1M**
(a) 0 (b) 1/8
(c) 1/4 (d) 1/2

11. A random variable is uniformly distributed over the interval 2 to 10. Its variance will be **IN-2008-2M**
(a) 16/3 (b) 6
(c) 256/9 (d) 36

12. Consider a Gaussian distributed random variable with zero mean and standard deviation σ. The value of its cumulative distribution function at the origin will be **IN-2008-2M**
(a) 0 (b) 0.5
(c) 1 (d) 10σ

13. $P_x(x) = Me^{(-2|x|)} + Ne^{(-3|x|)}$ is the probability density function for the real random variable X, over the entire x-axis M and N are both positive real numbers. The equation relating M and N is **EC-2008-2M**
(a) $M+\frac{2}{3}N=1$ (b) $2M+\frac{1}{3}N=1$
(c) $M+N=1$ (d) $M+N=3$

14. A coin is tossed 4 times. What is the probability of getting heads exactly 3 times? **ME-2008-1M**
(a) 1/4 (b) 3/8
(c) 1/2 (d) ¾

15. For a random variable $x(-\alpha < x < \alpha)$ following normal distribution, the mean is $\mu = 100$. If the probability is $P = \alpha$ for $x \geq 110$. then the probability of x lying b/w 90 and 110 i.e. $P(90 \leq x \leq 110)$ will be equal to **PI-2008-1M**
(a) $1-2\alpha$ (b) $1-\alpha$
(c) $1-\alpha/2$ (d) 2α

16. In a game, two players X and Y toss a coin alternately. Whosever gets a 'head' first, wins the game and the game is terminated. Assuming that player X starts the game the probability of player X winning the game is ______ **PI-2008-2M**
(a) 1/3 (b) 1/3
(c) 2/3 (d) ¾

17. The standard normal probability function can be approximated as
$$F(x_N) = \frac{1}{1+\exp\left(-1.7255\,x_N |x_N|^{0.12}\right)}$$
where x_N = standard normal deviate. If mean and standard deviation of annual precipitation are 102 cm and 27 cm respectively, the probability that the annual precipitation will be b/w 90 cm and 102 cm is **CE-2009-2M**
(a) 66.7 % (b) 50.0 %
(c) 33.3 % (d) 16.7 %

18. A fair coin is tossed 10 times. What is the probability that only the first two tosses will yield heads? **EC-2009-1M**
(a) $\left(\frac{1}{2}\right)^2$ (b) $10_{C_2}\left(\frac{1}{2}\right)^2$
(c) $\left(\frac{1}{2}\right)^{10}$ (d) $10_{C_2}\left(\frac{1}{2}\right)^{10}$

19. Consider two independent random variables X and Y with identical distributions. The variables X and Y take values 0, 1 and 2 with probabilities ½, ¼ and ¼ respectively. What is the conditional probability $P(X+Y = 2/X - Y = 0)$? **EC-2009-2M**
(a) 0 (b) 1/16
(c) 1/6 (d) 1

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20. A discrete random variable X takes value from 1 to 5 with probabilities as shown in the table. A student calculates the mean of X as 3.5 and her teacher calculates the variance to X as 1.5. Which of the following statements is true?

k	1	2	3	4	5
P(X=k)	0.1	0.2	0.4	0.2	0.1

EC-2009-2M
(a) Both the student and the teacher are right
(b) Both the student and the teacher are wrong
(c) The student is wrong but the teacher is right
(d) The student is right but the teacher is wrong

21. A screening test is carried out to detect a certain disease. It is found that 12% of the positive reports and 15% of the negative reports are incorrect. Assuming that the probability of a person getting a positive report is 0.01, the probability that a person tested gets an incorrect report is **IN-2009-2M**
(a) 0.0027 (b) 0.0173
(c) 0.1497 (d) 0.2100

22. If three coins are tossed simultaneously, the probability of getting at least one head is **ME-2009-1M**
(a) 1/8 (b) 3/8
(c) ½ (d) 7/8

23. The standard deviation of a uniformly distributed random variable b/w 0 and 1 is **ME-2009-2M**
(a) $\frac{1}{\sqrt{12}}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\frac{5}{\sqrt{12}}$ (d) $\frac{7}{\sqrt{12}}$

24. Assume for simplicity that N people, all born in April (a month of 30 days) are collected a room, consider the event of at least two people in the room being born on the same date of the month even if in different years e.g.1980 and 1985. What is the smallest N so that the probability of this event exceeds 0.5 is? **EE-2009-2M**
(a) 20 (b) 7
(c) 15 (d) 16

25. A box contains 4 white balls and 3 red balls. In succession, two balls are randomly selected and removed from the box. Given that first removed ball is white, the probability that the 2nd removed ball is red is **EE-2010-2M**
(a) 1/3 (b) 3/7
(c) ½ (d) 4/7

26. A fair coin is tossed independently four times. The probability of the event "The number of times heads show up is more than the number of times tails show up" is **EC-2010-2M**
(a) 1/16 (b) 1/8
(c) 1/4 (d) 5/16

27. What is the probability that a divisor of 10^{99} is a multiple of 10^{96}? **CS-2010-2M**
(a) 1/625 (b) 4/625
(c) 12/625 (d) 16/625

28. The diameters of 10,000 ball bearings were measured the mean diameter and standard deviation were found to be 10 mm and 0.05 mm respectively. Assuming Gaussian distribution of measurements, it can be expected that the number of measurements more than 10.15 mm will be **IN-2010-1M**
(a) 230 (b) 115
(c) 15 (d) 2

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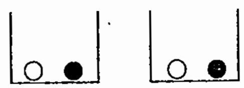
29. Consider a company that assembles computers. The probability of a faulty assembly of any computer is p. The company therefore subjects each computer to a testing process. This testing process gives the correct result for any computer with a probability of q. What is the probability of a computer being declared faulty?
CS-2010-2M
(a) $pq + (1-p)(1-q)$ (b) $(1-q)\,p$
(c) $(1-p)\,q$ (d) pq

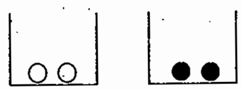
30. A box contains 2 washers, 3 nuts and 4 bolts. Items are drawn from the box at random one at a time without replacement. The probability of drawing 2 washers first followed by 3 nuts and subsequently the 4 bolts is
ME-2010-2M
(a) 2/315 (b) 1/630
(c) 1/1260 (d) 1/2520

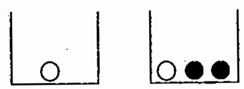
31. Two coins are simultaneously tossed. The probability of two heads simultaneously appearing is
CE-2010-1M
(a) 1/8 (b) 1/6
(c) ¼ (d) ½

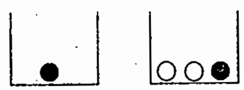
32. If a random variable X satisfies the poission's distribution with a mean value of 2-then the probability that $X \geq 2$ is
PI-2010-1M
(a) $2e^{-2}$ (b) $1-2e^{-2}$
(c) $3e^{-2}$ (d) $1-3e^{-2}$

33. Two white and two black balls, kept in two bins, are arranged in four ways as shown below. In each arrangement, a bin has to be chosen randomly and only one ball needs to be picked randomly fron the chosen bin. Which one of the following arrangements has the highest probability for getting a white ball picked?
PI-2010-2M

(a)


(b)


(c)


(d)


KEY:

01. a	02. a	03. c
04. d	05. a	06. a
07. d	08. a	09. c
10. c	11. a	12. b
13. a	14. b	15. a
16. c	17. d	18. a
19. b	20. b	21. c
22. b	23. a	24. b
25. c	26. d	27. a
28. d	29. a	30. c
31. c	32. d	33. c

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BASIC ENGINEERING MATHEMATICS

TOPIC – 5 DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATIONS OF FIRST ORDER

- A **differential equation** is an equation which involves differential coefficients or differentials.

Ex: i) $e^x\,dx + e^y\,dy = 0$ ii) $(d^2x / dt^2) + n^2x = 0$
iii) $[1 + (dy/dx)^2]^{3/2} / (d^2y / dx^2) = c$ iv) $x\,(\partial u/\partial x) + y\,(\partial u / \partial y) = 2u$
v) $(\partial^2 y / \partial t^2) = c^2\,(\partial^2 y / \partial x^2)$ are all examples of differential equations.

- An **ordinary differential equation** is that in which all the differential coefficients have reference to a single independent variable.
- A **partial differential equation** is that in which there are two or more independent variables and partial differential coefficients with respect to any of them.
- The **order** of a differential equation is the order of the highest derivative appearing in it.
- The **degree** of a differential equation is the degree of the highest derivative occurring in it, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

PRACTICAL APPROACH TO DIFFERENTIAL EQUATIONS

The study of a differential equation consists of three phases:

- Formulation of differential equation from the given physical situation, called **modelling**
- Solutions of this differential equation, evaluating the arbitrary constants from the given conditions, and
- Physical interpretation of the solution.

FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed in an attempt to eliminate certain arbitrary constant from a relation in the variables and constants.

Ex: Form the differential equation of simple harmonic motion given by $x = A\cos(nt + \alpha)$.
Ans: $(d^2x / dt^2) + n^2x = 0$

Ex: Obtain the differential equation of all circles of radius a and centre (h, k)
Ans: $[1 + (dy/dx)^2]^3 = a^2(d^2y/dx^2)^2$

Solution Of a Differential Equation:

- A ***solution (or integral)*** of a differential equation is a relation between the variables which satisfies the given differential equation.
- The ***general (or complete) solution*** of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.
- A ***particular solution*** is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

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SOLUTION OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

Method – I (Variables – Separable) : If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the variables are said to be separable. Thus the general form of such an equation is $f(y)\,dy = \phi(x)\,dx$

Integrating both sides, we get $\int f(y)\,dy = \int \phi(x)\,dx + c$ as its solution.

NOTE : An equation of the form $(dy/dx) = f(ax+by+c)$ can be reduced to variable separable form by substituting $ax + by + c = z$.

Method – II (Homogeneous Equations) : The differential equation is of the form

$$(dy/dx) = [f(x, y) / \phi(x, y)].$$

Where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y.

To solve a homogeneous equation i) Put $y = vx$, then

$(dy/dx) = v + x\,(dv/dx)$,

ii) Separate the variables v and x, and integrate.

Method – III (Equations Reducible to Homogeneous Form) :

The equations of the form $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a^1x + b^1y + c^1}$ --- (1)

can be reduced to the homogeneous form as follows:

Case(i): When $(a/a^1) \neq (b/b^1)$

Putting $x = X + h$, $y = Y + k$ (h, k being constants)

Where $h = \dfrac{bc^1 - b^1c}{ab^1 - b^1a}$ $k = \dfrac{ca^1 - c^1a}{ab^1 - ba^1}$ (when $ab^1 - ba^1 \neq 0$)

Case(ii): When $(a/a^1) = (b/b^1)$

Let $(a/a^1) = (b/b^1) = (1/m)$ (say)

Then $a^1 = am$, $b^1 = bm$

And (1) becomes

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c^1}$$

Put $ax + by = t$ and solve by variable separable method

Method – IV (Linear Equation) : A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

The standard form of a linear equation of the first order, commonly known as Leibnitz's linear equation, is

$(dy/dx) + Py = Q$ where P, Q are the functions of x.

and $ye^{\int Pdx} = \int Q\, e^{\int Pdx}\, dx + c$ is the required solution.

Note: It is important to remember that $I.F = e^{\int Pdx}$

And the solution is $y(I.F) = \int Q\,(I.F)\,dx + c$.

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Method – V (Bernoulli's Equation) :

The equation $(dy/dx) + Py = Qy^n$ --- (1)

Where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation.

To solve (1) , divide both sides by y^n, so that

$$y^{-n}(dy/dx) + Py^{1-n} = Q \quad --- \quad (2)$$

Put $y^{1-n} = z$

Then (2) becomes $[1/(1-n)]\,(dz/dx) + Pz = Q$ (or) $(dz/dx) + P(1-n)z = Q(1-n)$

Which is Leibnitz's linear in z and can be solved easily.

NOTE : An equation of the form $f'(y)\,(dy/dx) + f(y)\,P(x) = Q(x)$ is the another form of Bernoulli's differential equation. And this euation can be reduced to linear differential equation by substituting $f(y) = Z$.

Method – VI (Exact Differential Equations) :

- **Def:** A differential equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be exact if its left hand member is the exact differential of some function $u(x, y)$ i.e., $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

- **Theorem:** The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The solution of $Mdx + Ndy = 0$ is

$\int M\,dx + \int$ (Terms of N not containing x) $dy = c$

(Treating y as constant)

Method – VII (Equations Reducible To Exact Equations):

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an integrating factor. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows:

Rule - I (I.F found by inspection): In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as a part of an exact differential. In this connection the following integrable combinations prove quite useful:

$$xdy + ydx = d(xy)$$

$$\frac{xdy - ydx}{x^2} = d(y/x)\,; \qquad \frac{xdy - ydx}{xy} = d[\log(y/x)]$$

$$\frac{xdy - ydx}{y^2} = -d(x/y)\;; \qquad \frac{xdy - ydx}{x^2 + y^2} = d[\tan^{-1}(y/x)]$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left((1/2)\log\left(\frac{x + y}{x - y}\right)\right)$$

Rule – II (I.F of a homogeneous equation): If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $1/(Mx + Ny)$ is an integrating factor ($Mx + Ny \neq 0$).

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Rule – III (I.F for an equation of the type $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$):

If the equation $M dx + N dy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor $(Mx - Ny \neq 0)$.

Rule – IV (In the equation $M dx + N dy = 0$):

If $\dfrac{(\partial M/\partial y - \partial N/\partial x)}{N}$ be a function of x only = f(x) say, then $e^{\int f(x)\,dx}$ is an integrating factor.

Rule – V (In the equation $M dx + N dy = 0$):

If $\dfrac{(\partial N/\partial x - \partial M/\partial y)}{M}$ be a function of y only = F(y) say, then $e^{\int F(y)dy}$ is an integrating factor.

PROBLEMS

01. $x\,(dy/dx) + \cot y = 0$ if $y = \pi/4$ when $x = \sqrt{2}$. **Ans:** $x = 2\cos y$.

02. $(x+1)(dy/dx) + 1 = 2e^{-y}$ **Ans:** $(x+1)(2 - e^{y}) = c$

03. $(dy/dx) = \cos(x + y + 1)$ **Ans:** $x = \operatorname{cosec}(x+y+1) - \cot(x+y+1) + c$

04. $x^4\,(dy/dx) + x^3y + \operatorname{cosec}(xy) = 0$ **Ans:** $\cos(xy) + 1/2x^2 = c$

05. $y^2 + x^2\,(dy/dx) = xy\,(dy/dx)$ **Ans:** $y/x - \log y = c$

06. $xy\log(x/y)\,dx + [y^2 - x^2\log(x/y)]\,dy = 0$ **Ans:** $\log y - \dfrac{x^2}{4y^2}\left(2\log\left(\dfrac{y}{x}\right) - 1\right) = c$

07. $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0$ **Ans:** $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

08. $(4x - 6y - 1)\,dx + (3y - 2x - 2)\,dy = 0$ **Ans:** $x - y + (3/4)\log(8x - 12y - 5) = c$

09. $(x + 2y)(dx - dy) = dx + dy$ **Ans:** $\log(x + y + 1/3) + (3/2)(y - x) = c$

10. $(dy/dx) + y\cot x = 4x\operatorname{cosec} x$, Given that $y = 0$, when $x = \pi/2$ **Ans:** $y\sin x = 2x^2 - \pi^2/2$

11. $\dfrac{dy}{dx} = -\left(\dfrac{x + y\cos x}{1 + \sin x}\right)$ **Ans:** $y(1 + \sin x) = c - x^2/2$

12. $y\,e^{y}\,dx = (y^3 + 2xe^{y})\,dy$ **Ans:** $xy^{-2} = c - e^{-y}$

13. $y\log y\,dx + (x - \log y)\,dy = 0$ **Ans:** $x = \tfrac{1}{2}\log y + c(\log y)^{-1}$

14. $r\sin\theta\,d\theta + (r^3 - 2r^2\cos\theta + \cos\theta)\,dr = 0$ **Ans:** $r(1 + ce^{-r^2}) = 2\cos\theta$

15. $r\sin\theta - \cos\theta\,(dr/d\theta) = r^2$ **Ans:** $1/r = \sin\theta + c\cos\theta$

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16. $2xy' = 10x^3y^5 + y$ **Ans:** $x^2 + (4x^5 + c)y^4 = 0$

17. $(x^3y^2 + xy)\,dx = dy$ **Ans:** $1/y = -x^2 + 2 + ce^{-x^2/2}$

18. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + 1}{2xy}$ **Ans:** $y^2 = x^2 - cx - 1$

19. $x(x - y)\,dy + y^2\,dx = 0$ **Ans:** $y/x = \log y + c$

20. $\tan y\left(\dfrac{dy}{dx}\right) + \tan x = \cos y\cos^2 x$ **Ans:** $\cos y = \cos x(\sin x + c)$

21. $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{(xy)}}$ **Ans:** $\sqrt{x} = 2\sqrt{y}(\log y + c)$

25. $(x^2 + y^2 - a^2)\,x\,dx + (x^2 - y^2 - b^2)\,y\,dy = 0.$ **Ans:** $x^4 + 2x^2y^2 - y^4 - 2a^2x^2 - 2b^2y^2 = c$

26. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$ **Ans:** $\dfrac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$

27. $ye^{xy}dx + (xe^{xy} + 2y)\,dy = 0$ **Ans:** $e^{xy} + y^2 = c$

28. $y\sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$ **Ans:** $3y\cos 2x + 6y + 2y^3 = c$

29. $(\sec x\tan x\tan y - e^x)\,dx + \sec x\sec^2 y\,dy = 0$ **Ans:** $e^x = \sec x\tan y + c$

30. $[y(1 + 1/x) + \cos y]\,dx + (x + \log x - x\sin y)\,dy = 0$ **Ans:** $y(x + \log x) + x\cos y = c$

31. $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}$ **Ans:** $3\log x - (y/x)^3 = c$

32. $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$ **Ans:** $xy + \log(x/y) - (1/xy) = c$

33. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ **Ans:** $(y + 2/y^2)\,x + y^2 = c$

34. $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$ **Ans:** $x^2(ay^2 - xy) = c$

35. $(x^4e^x - 2mxy^2)\,dx + 2mx^2y\,dy = 0$ **Ans:** $e^x + m(y/x)^2 = c$

36. $y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$ **Ans:** $(x/y) + e^{x^3} = c$

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HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together. Thus the general linear differential equation of the n^{th} order is of the form

$$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + p_ny = X,$$

where $p_1, p_2, \ldots\ldots p_n$ and X are functions of x only.

Linear differential equations with constant co - efficients are of the form

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + k_ny = X,$$

where $k_1, k_2, \ldots\ldots k_n$ are constants.

Denoting d/dx, d^2/dx^2, d^3/dx^3, etc. by D, D^2, D^3 etc. so that

$dy/dx = Dy$, $d^2y/dx^2 = D^2y$, $d^3x/dx^3 = D^3y$ etc, the equation above can be written in the symbolic form

$(D^n + k_1D^{n-1} + \ldots\ldots + k_n)y = X$, i.e., $f(D)y = X$,

where $f(D) = D^n + k_1D^{n-1} + \ldots\ldots + k_n$, is a polynomial in D.

Thus the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e, f(D) can be factorised by ordinary rules of algebra and the factors may be taken in any order.

Working Procedure To Solve The Equation:

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + k_ny = X,$$

of which the symbolic form is

$(D^n + k_1D^{n-1} + \ldots\ldots + k_{n-1}D + k_n)y = X.$

Step I : To find the complementary function

i) Write the A.E

$D^n + k_1D^{n-1} + \ldots\ldots + k_{n-1}D + k_n = 0$ and solve it for D.

ii) Write the C.F as follows:

Roots of A.E	C.F
1. $m_1, m_2, m_3 \ldots\ldots$ (real and different roots)	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \ldots\ldots$
2. $m_1, m_1, m_3 \ldots\ldots$ (two real and equal roots)	$(c_1 + c_2x)e^{m_1x} + c_3e^{m_3x} + \ldots\ldots$
3. $m_1, m_1, m_1, m_4 \ldots$(three real and equal roots)	$(c_1 + c_2x + c_3x^2)e^{m_1x} + c_4e^{m_4x} + \ldots\ldots$
4. $\alpha + i\beta, \alpha - i\beta, m_3\ldots$(a pair of imaginary roots)	$e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} +$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5 \ldots$(two pairs of equal imaginary roots)	$e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x + c_5e^{m_5x} + \ldots\ldots$
6. $\alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}, m_3$ (a pair of irrational roots)	$e^{\alpha x}(c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x) + c_3e^{m_3x} +$

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Step II : To find the particular integral

From symbolic form P. I = $\dfrac{1}{D^n + k_1D^{n-1} + \ldots\ldots + k_{n-1}D + k_n}$

X = $[1/f(D)]X$ or $[1/\phi(D^2)]X$

i) When $X = e^{ax}$

P. I = $[1/f(D)]\, e^{ax}$, put D = a, $[f(a) \neq 0]$

P.I = $[x / f'(D)]\, e^{ax}$, put D = a, $[f(a) = 0\ \&\ f'(a) \neq 0]$

P.I = $[x^2 / f''(D)]\, e^{ax}$, put D = a, $[f'(a) = 0\ \&\ f''(a) \neq 0]$ and So on.....

ii) When X = sin(ax + b) or cos(ax + b)

P. I = $[1/\phi(D^2)]\sin(ax + b)$ [or $\cos(ax + b)$], put $D^2 = -a^2$ $[\phi(-a^2) \neq 0]$

P. I = $[1/\phi'(D^2)]\sin(ax + b)$ [or $\cos(ax + b)$], put $D^2 = -a^2$ $[\phi(-a^2) = 0\ \&\ \phi'(-a^2) \neq 0]$

P. I = $[1/\phi''(D^2)]\sin(ax + b)$ [or $\cos(ax + b)$], put $D^2 = -a^2$ $[\phi'(-a^2) = 0\ \&\ \phi''(-a^2) \neq 0]$

and so on ....

iii) When $X = x^m$, m being a positive integer (or) $a_0 + a_1x + a_2x^2 + \ldots.+ a_mx^m$ $(a_m \neq 0)$

P. I = $[1/f(D)]\, x^m = [f(D)]^{-1} x^m$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

iv) When $X = e^{ax}V$, where V is a function of x i.e sin(bx+c) or cos(bx+c) or x^m

P. I = $[1/f(D)]\, e^{ax}V = e^{ax}\,[1/f(D+a)]\, V$

And then evaluate $[1/f(D+a)]V$ as in (i), (ii) and (iii)

v) When X is any function of x

P. I = $[1/f(D)]X$

Resolve $[1/f(D)]$ into partial fractions and operate each partial fraction on X by using

$[1/(D - a)]X = e^{ax}\int X e^{-ax}\, dx$

Step III: To find the complete solution

Then the C.S is y = C.F + P.I

METHOD OF VARIATION OF PARAMETERS

This method is quite general and applies to equations of the form.

$y^{11} + py^1 + qy = X$

Where p, q and X are functions of x. It gives P.I = $-y_1\int (y_2X / W)\, dx + y_2\int (y_1X / W)\, dx$

Where y_1 and y_2 are the solutions of $y^{11} + py^1 + qy = 0$

And W = $\begin{vmatrix} y_1 & y_2 \\ y_1^1 & y_2^1 \end{vmatrix}$ is called the Wronskian of y_1, y_2.

EQUATIONS REDUCIBLE TO LINEAR EQUATIONS WITH CONSTANT CO–EFFICIENTS

➢ **Cauchy's homogeneous linear equation:** An equation of the form

$x^n(d^ny/dx^n) + k_1x^{n-1}(d^{n-1}y/dx^{n-1}) + \ldots\ldots + k_{n-1}x(dy/dx) + k_ny = X$

Where X is a function of x, is called Cauchy's homogeneous linear equation.

Such equations can be reduced to linear differential equations with constant coefficients, by putting

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$x = e^t$ or $t = \log x$. and $D = d/dt$

Then $x\,(dy/dx) = Dy$, $x^2(d^2y/dx^2) = D(D-1)y$. Similarly, $x^3(d^3y/dx^3) = D(D-1)(D-2)y$ and so on.

➢ **Legendre's Linear Equation:**

An equation of the form

$$(ax+b)^n\,(d^ny/dx^n) + k_1(ax+b)^{n-1}\,(d^{n-1}y/dx^{n-1}) + \dots\dots\dots + k_ny = X$$

Where k's are constants and X is a function of x, is called Legendre's linear equation.

Such equations can be reduced to linear equations with constant coefficients by the substitution $ax + b = e^t$, i.e., $t = \log(ax+b)$ and $D = d/dt$

Then $(ax+b)\,dy/dx = a\,Dy$, $(ax+b)^2\,(d^2y/dx^2) = a^2\,D(D-1)y$

Similarly $(ax+b)^3\,(d^3y/dx^3) = a^3\,D(D-1)(D-2)y$ and so on.

PROBLEMS

38. $\dfrac{d^2x}{dt^2} + 3a\dfrac{dx}{dt} - 4a^2x = 0$ **Ans:** $x = c_1e^{-4at} + c_2e^{at}$

39. $y'' - 2y' + 10y = 0$, $y(0) = 4$, $y'(0) = 1$ **Ans:** $y = e^x(4\cos 3x - \sin 3x)$

40. $\dfrac{d^3y}{dx^3} + y = 0$ **Ans:** $y = c_1e^{-x} + e^{x/2}\left(c_2\cos\dfrac{\sqrt{3}x}{2} + c_3\sin\dfrac{\sqrt{3}x}{2}\right)$

41. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$ **Ans:** $y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$

42. Solve $\dfrac{d^4y}{dx^4} + a^4y = 0$ **Ans :** $e^{ax/\sqrt{2}}\,[C_1\cos(ax/\sqrt{2}) + C_2\sin(ax/\sqrt{2})] + e^{-ax/\sqrt{2}}\,[C_3\cos(ax/\sqrt{2}) + C_4\sin(ax/\sqrt{2})]$

43. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x$. Also find y when $y = 0$, $\dfrac{dy}{dx} = 1$ at $x = 0$

Ans: $y = \dfrac{3}{5}e^{-2x}(\cos x + 3\sin x) - \dfrac{e^x}{10} - \dfrac{e^{-x}}{2}$

44. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 4\cos^2 x$. **Ans:** $y = c_1e^{-x} + c_2e^{-2x} + \dfrac{1}{10}(3\sin 2x - \cos 2x) + 1$

45. $(D^2 - 4D + 3)\,y = \sin 3x\cos 2x$.

Ans: $y = c_1e^x + c_2e^{3x} + \dfrac{1}{884}(10\cos 5x - 11\sin 5x) + \dfrac{1}{20}(\sin x + 2\cos x)$

46. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^{2x} - \cos^2 x$. **Ans:** $y = (c_1 + c_2x)\,e^{-x} - \dfrac{1}{2} + \dfrac{1}{50}(-4\sin 2x + 3\cos 2x) + \dfrac{e^{2x}}{9}$

47. $(D^3 - D)\,y = 2x + 1 + 4\cos x + 2e^x$ **Ans:** $y = c_1 + c_2e^x + c_3e^{-x} + xe^x - (x^2 + x) - 2\sin x$.

48. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^{-2x}\sin 2x$ **Ans:** $y = c_1e^{-2x} + c_2e^{-3x} - \dfrac{e^{-2x}}{[illegible]}(\cos 2x + 2\sin 2x)$

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49. $(D^2 + 4)\,y = e^x\sin^2 x$.

Ans: $y = c_1\cos 2x + c_2\sin 2x + \dfrac{e^x}{2}\left(\dfrac{1}{5} - \dfrac{1}{17}(4\sin 2x + \cos 2x)\right)$

50. $(D^2 + 4D + 3)y = e^{-x}\sin x + x$

Ans: $y = c_1e^{x} + c_2e^{-3x} - \dfrac{1}{5}e^{-x}(2\cos x + \sin x) + \dfrac{1}{3}(x - 4/3)$

51. $\dfrac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$

Ans: $y = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \dfrac{e^{3x}}{11}\left(x^2 - \dfrac{12}{11}x + \dfrac{50}{121}\right) + \dfrac{e^x}{17}(4\sin 2x - \cos 2x)$

52. $\dfrac{d^2y}{dx^2} + 4y = x\sin x$.

Ans: $y = c_1\cos 2x + c_2\sin 2x + \dfrac{1}{9}(3x\sin x - 2\cos x)$

53. $(D^2 - 1)\,y = x\sin x + (1 + x^2)\,e^x$.

Ans: $y = c_1e^x + c_2e^{-x} - ½\,(x\sin x + \cos x) + (xe^x/12)(2x^2 - 3x + 9)$

54. $(D^2 + a^2)\,y = \tan ax$.

Ans: $y = c_1\cos ax + c_2\sin ax - \dfrac{1}{a^2}(\cos ax)\,[\log(\sec ax + \tan ax)]$

55. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$ **Ans:** $y = c_1x^4 + c_2x^{-1} + \dfrac{x^4}{5}\log x$

56. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = (1+x)^2$ **Ans:** $y = (c_1 + c_2\log x)\,x^2 + (1/4) + 2x + (1/2)x^2(\log x)^2$

57. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$ **Ans:** $y = 2(\log x)^3 + c_1\log x + c_2$

58. $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2\log x$ **Ans:** $y = \dfrac{c_1}{x} + c_2x^4 - \dfrac{x^2}{6} - \dfrac{1}{2}(\log x + 3/8)$

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59. $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$

Ans: $y = c_1x^{-1} + \{c_2 \cos(\log x) + c_3 \sin(\log x)\}x + 5x + 2(\log x)/x$

60. $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$

Ans: $y = \{c_1 \cos(\log x) + c_2 \sin(\log x)\} + \dfrac{1}{4}(\log x)\ [\sin(\log x) - \log x \cos(\log x)]$

61. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4 \cos \log(1+x)$

Ans: $y = c_1 \cos t + c_2 \sin t + 2t \sin t$, where $t = \log(1+x)$

62. $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Ans: $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \dfrac{1}{108}[(3x+2)^2 \log(3x+3)]$

PARTIAL DIFFERENTIAL EQUATIONS

Definition: A Partial Differential Equation is that in which there are two or more independent variables and partial derivatives with respect to any of them.

Order And Degree: The order of a partial differential equation is the order of the highest derivative occurring in it and the degree of a partial differential equation is the degree of the highest derivative appearing in it.

Ex(1): $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$ is a first order and first degree partial differential equation.

Ex(2): $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$ is a second order and first degree partial differential equation.

Ex(3): $\left(\dfrac{\partial T}{\partial x}\right)^3 + \dfrac{\partial T}{\partial t} = 0$ of order 1 and degree 3.

Standard Notation: We consider x, y as independent variables and z as dependent variable.

$p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\partial y$, $t = \partial^2 z/\partial y^2$

Formation of Partial Differential Equations: Partial differential equations can be formed by eliminating the arbitrary constants (or) some specific functions.

Elimination of arbitrary constants: In this process we can observe the following.

1. If the number of arbitrary constants is equal to the number of independent variables, then we obtain a first order partial differential equation.
2. If the number of arbitrary constants is more than the number of independent variables, then we obtain a higher order partial differential equation.

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PROBLEMS

1. $2z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$	(a, b)	**Ans:** $xp + yq = 2z$
2. $(x^2+a)(y^2+b) = z$	(a, b)	**Ans:** $pq = 4xyz$
3. $(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$	(a, b)	**Ans:** $p^2 + q^2 = \tan^2\alpha$
4. $z = xy + y(\sqrt{x^2 - a^2}) + b^2$	(a, b)	**Ans:** $px + qy = pq$
5. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	(a, b, c)	**Ans:** $zs + pq = 0$

Elimination of arbitrary functions: In this case order of the partial differential equation is same as the number of arbitrary functions.

PROBLEMS

1. $z = f(x^2 - y^2)$	**Ans:** $xq + yp = 0$
2. $z = (x+y)\phi(x^2 - y^2)$	**Ans:** $xq + yp = z$
3. $xyz = f(x+y+z)$	**Ans:** $x(y-z)p + y(z-x)q = z(x-y)$
4. $z = y^2 + 2f(1/x + \log y)$	**Ans:** $px^2 + qy = 2y^2$
5. $f(x^2 + y^2, z - xy) = 0$	**Ans:** $xq - py = x^2 - y^2$
6. $z = x f_1(x+y) + f_2(x+y)$	**Ans:** $r - 2s + t = 0$
7. $z = y f(x) + x g(y)$	**Ans:** $px + qy = xy s + z$

SOLUTION OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Linear partial differential equation: A partial differential equation is said to be linear if the dependent variable and its differential co-efficients occur in first degree only but not multiplied together.

The general form of a linear partial differential equation of the first order is given by

$$Pp + Qq = R \quad \text{---} \quad (1)$$

Where P, Q, R are functions of x, y, z. It is also called Lagrange's linear equation.

Solution Procedure:

1. Write the Lagrange's auxillary equations or subsidiary equations of (1) as follows
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{---} \quad (2)$$
2. Solve any two simultaneous equations of (2). Let the solutions be $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$
3. The complete solution of (1) is given by $\phi(u, v) = 0$ --- (3) or $v = \Psi(u)$ --- (4)

PROBLEMS

1. $yzp - xzq = xy$ — **Ans:** $f(x^2 + y^2, x^2 - z^2) = 0$

2. $p - q = \log(x + y)$ — **Ans:** $\phi[x + y, x - z/\log(x+y)] = 0$

3. $z(z^2 + xy)(px - qy) = x^4$ — **Ans:** $f(xy, x^4 - z^4 - 2xyz^2) = 0$

4. $p \sin x + q \cos y = \operatorname{Tan} z$

Ans: $\phi\left\{\dfrac{\operatorname{Cosec} x - \operatorname{Cot} x}{\operatorname{Sec} y + \operatorname{Tan} y}, \dfrac{\operatorname{Sec} y + \operatorname{Tan} y}{\operatorname{Sin} z}\right\} = 0$

LANGRANGE'S METHOD OF MULTIPLIERS

Consider the partial differential equation $Pp + Qq = R$ --- (1) then auxillary equations are given by $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$ --- (2).

Now choose the multipliers P_1, Q_1, R_1 and

P_2, Q_2, R_2 in such a way that

each ratio $= \dfrac{P_1\,dx + Q_1\,dy + R_1\,dz}{P_1P + Q_1Q + R_1R} = \dfrac{P_2\,dx + Q_2\,dy + R_2\,dz}{P_2P + Q_2Q + R_2R}$

and denominators vanish.

$\Rightarrow P_1dx + Q_1dy + R_1dz = 0$

By integration, solution is given by $u(x, y, z) = c_1$

and similarly $P_2dx + Q_2dy + R_2dz = 0$

$\therefore$ By integration, solution is given by $v(x, y) = c_2$

$\therefore$ Complete solution of (1) is given by $f(u, v) = 0$ (or) $\phi(u) = v$

PROBLEMS

1. $xzp + yzq = xy$ — **Ans:** $f(x/y, xy - z^2) = 0$

2. $(z - y)p + (x - z)q = y - x$ — **Ans:** $f(x + y + z, x^2 + y^2 + z^2) = 0$

3. $(y + zx)p - (x + yz)q = x^2 - y^2$ — **Ans:** $f(x^2 + y^2 - z^2, xy + z) = 0$

4. $px(x + y) = qy(x + y) - (2x + 2y + z)(x - y)$ — **Ans:** $f[xy, (x + y)(x + y + z)] = 0$

5. $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$ — **Ans:** $f[x - y - z, (x^2 - y^2)/z^2] = 0$

6. $\dfrac{(y - z)}{yz}p + \dfrac{(z - x)}{zx}q = \dfrac{x - y}{xy}$ — **Ans:** $\phi(x + y + z, xyz) = 0$

7. $(mz - ny)\dfrac{\partial z}{\partial x} + (nx - lz)\dfrac{\partial z}{\partial y} = ly - mx$ — **Ans:** $x^2 + y^2 + z^2 = f(lx + my + nz)$

8. $(x^2 - y^2 - z^2)p + 2xy\,q = 2xz$ — **Ans:** $f[y/z, (x^2 + y^2 + z^2)/z] = 0$

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NON – LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

Definition: The equations in which the degrees of p and q are higher than *one* are called non – linear partial differential equations of the first order. To solve such equations follow the types given below.

Type – I: Equation contains p, q only.

Let the given partial differential equation be $f(p, q) = 0$ --- (1)

Then complete solution is given by $z = ax + by + c$ --- (2)

Where $f(a, b) = 0$ --- (3)

From (2) $\partial z/\partial x = p = a$ and $\partial z/\partial y = q = b$

$\therefore$ From (3) equation (1) is satisfied.

From (3) we can write $b = \phi(a)$

$\therefore$ Complete solution of (1) is given by $z = ax + \phi(a)y + c$ where a, c are arbitrary constants.

PROBLEMS

1. $p^2 + q^2 = npq$ — **Ans:** $z = ax + (ay/2)[n \pm \sqrt{(n^2 - 4)}] + c$

2. $x^2p^2 + y^2q^2 = z^2$ — **Ans:** $\log z = a \log x \pm (\sqrt{1 - a^2})\log y + c$

3. $\sqrt{p} + \sqrt{q} = 1$ — **Ans:** $z = ax + (1 - \sqrt{a})^2y + c$

Type – II: Equation containing p, q, z only.

Let it be $f(z, p, q) = 0$ --- (1)

Consider a trial solution $z = g(t)$ --- (2)

Where $t = x + ay$ --- (3)

Then from (2) $\dfrac{\partial z}{\partial x} = p = g^1(t)\dfrac{\partial t}{\partial x} = g^1(t) = \dfrac{dz}{dt}$

and $\dfrac{\partial z}{\partial y} = q = g^1(t)\dfrac{\partial t}{\partial y} = a\,g^1(t) = a\dfrac{dz}{dt}$

$\therefore$ (1) becomes $f(z, dz/dt, a\,dz/dt) = 0$ --- (4)

Now (4) is an ordinary differential equation in z with the independent variable t. Let the solution of (4) be $\phi(z, t, b) = 0$ i.e., $\phi(z, x + ay, b) = 0$ --- (5) is the complete solution of (1) ; where a, b are arbitrary constants.

PROBLEMS

1. $zpq = p + q$ — **Ans:** $az^2 = 2(1 + a)(x + ay) + b$

2. $p(1 + q) = qz$ — **Ans:** $\log(az - 1) = x + ay + b$

3. $q^2 = z^2p^2(1 - p^2)$ — **Ans:** $z^2 = (x + ay + b)^2 + a^2$

4. $z^2(p^2x^2 + q^2) = 1$ — **Ans:** $z^2\sqrt{(1 + a^2)} = \pm 2(\log x + ay) + b$

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Type – III: Separable equation of the form $f(x, p) = g(y, q)$.

Let $f(x, p) = g(y, q) = a$ (constant)
Solve $f(x, p) = a$ and $g(y, q) = a$ for p and q, we get $p = \phi(x, a)$ and $q = \Psi(y, a)$

But $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

$\Rightarrow dz = p\,dx + q\,dy$

By integration, we get the required solution as follows.

$\int dz = z = \int \phi(x, a)\, dx + \int \Psi(y, a)\, dy + b$

PROBLEMS

1. $p^2q^2 + x^2y^2 = x^2q^2 (x^2 + y^2)$ **Ans:** $z = (1/3)(a^2 + x^2)^{3/2} + \sqrt{(y^2 - a^2)} + b$

2. $z^2(p^2 + q^2) = x^2 + y^2$ **Ans:** $z^2 = x\sqrt{(x^2 + a^2)} + y\sqrt{(y^2 - a^2)} + a^2 \log x\left(\frac{+\sqrt{(x^2 + a^2)}}{y + \sqrt{(y^2 - a^2)}}\right) + 2b$

3. $p^2 + q^2 = x + y$ **Ans:** $z = (2/3)(a + x)^{3/2} + (2/3)(y - a)^{3/2} + b$

Type – IV: The Clairaut's equation $z = px + qy + f(p, q)$.

The solution is given by $z = ax + by + f(a, b)$ Since $\partial z / \partial x = a = p$ & $\partial z / \partial y = b = q$

PROBLEMS

1. $pqz = p^2(xq + p^2) + q^2(yp + q^2)$ **Ans:** $z = ax + by + (a^3/b) + (b^3/a)$

2. $(p - q)(z - px - qy) = 1$ Ans: $z = ax + by + \frac{1}{(a - b)}$

PREVIOUS GATE QUESTIONS - "DIFFERENTIAL EQUATIONS"

1. The necessary and sufficient for the differential equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ to be exact is *(GATE'94)*
   a) $M = N$ b) $\partial M/\partial x = \partial N/\partial y$ c) $\partial M/\partial y = \partial N/\partial x$ d) $\partial^2 M/\partial x^2 = \partial^2 N/\partial y^2$

2. The differential equation $EI\,(d^4y / dx^4) + P\,(d^2y/dx^2) + ky = 0$ is *(GATE'94)*
   a) Linear of Fourth order b) Non – Linear of fourth order
   c) Non – Homogeneous d) Linear and Fourth degree

3. For the differential equation $(dy/dt) + 5y = 0$ with $y(0) = 1$, the general solution is *(GATE'94)*
   (a) e^{5t} (b) e^{-5t} (c) $5e^{-5t}$ (d) $e^{\sqrt{-5t}}$

4. Solve for y, if $(d^2y/dt^2) + 2(dy/dt) + y = 0$; with $y(0) = 1$ and $y'(0) = -2$ *(GATE'94)*
   Ans : $y = (1-t)\,e^{-t}$

5. The differential equation $y^{11} + (s^3 \sin x)^5\, y^1 + y = \cos x^3$ is *(GATE'95)*
   a) homogeneous b) non linear
   c) second order linear d) non homogeneous with constant coefficients

6. Solve $(d^4v/dx^4) + 4\lambda^4 v = 1 + x + x^2$ *(GATE'96)*
   Ans: $y = e^{\lambda x}(c_1\cos\lambda x + c_2\sin\lambda x) + e^{-\lambda x}(c_3\cos\lambda x + c_4\sin\lambda x) + (1 + x + x^2)/(4\lambda^4)$

7. The particular solution for the differential equation $(d^2y/dx^2) + 3\,(dy/dx) + 2y = 5\cos x$ is
   a) $0.5\cos x + 1.5\sin x$ b) $1.5\cos x + 0.5\sin x$ c) $1.5\sin x$ d) $0.5\cos x$
   (GATE'96)

8. The one dimensional heat conduction partial differential equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ is *(GATE'96)*
   a) parabolic b) hyperbolic c) elliptic d) mixed

9. For the differential equation, $f(x, y)(dy/dx) + g(x, y) = 0$ to be exact, *(GATE'97)*
   a) $\partial f / \partial y = \partial g / \partial x$ b) $\partial f / \partial x = \partial g / \partial y$ c) $f = g$ d) $\partial^2 f / \partial x^2 = \partial^2 g / \partial y^2$

10. The differential equation, $(dy/dx) + Py = Q$, is a linear equation of first order only if, *(GATE'97)*
    a) P is a constant but Q is a function of y b) P and Q are functions of y or constants
    c) P is a function of y but Q is a constant d) P and Q are functions of x or constants

11. Solve $(d^4y / dx^4) - y = 15\cos 2x$ *(GATE'98)*
    Ans: $y = c_1e^x + c_2e^{-x} + (c_3\cos x + c_4\sin x) + \cos 2x$

12. The general solution of the differential equation $x^2(d^2y/dx^2) - x\,(dy/dx) + y = 0$ is *(GATE'98)*
    a) $Ax + Bx^2$ (A,B are constants) b) $Ax + B\log x$ (A,B are constants)
    c) $Ax + Bx^2\log x$ (A,B are constants) d) $Ax + Bx\log x$ (A,B are constants)

13. The radial displacement in a rotating disc is governed by the differential equation
    $\frac{d^2u}{dx^2} \quad \frac{1}{x}\frac{du}{dx} \quad \frac{u}{x^2} = 8x$ where u is the displacement and x is the radius.
    If $u = 0$ at $x = 0$, and $u = 2$ at $x = 1$, calculate the displacement at $x = ½$
    Ans: 1/8 *(GATE'98)*

14. $(d^2y/dx^2) + (x^2 + 4x)(dy/dx) + y = x^8 - 8$ *(GATE'99)*
    The above equation is a
    a) partial differential equation b) nonlinear differential equation
    c) non - homogeneous differential equation d) ordinary differential equation

15. If c is a constant, solution of the equation $dy/dx = 1 + y^2$ is *(GATE'99)*
    a) $y = \sin(x + c)$ b) $y = \cos(x + c)$ c) $y = \tan(x + c)$ d) $y = e^x + c$

16. Find the solution of the differential equation $d^2y/dt^2 + \lambda^2 y = \cos(\omega t + k)$ with initial conditions $y(0) = 0$, $dy/dt(0) = 0$. Here λ, ω and k are constants. Use either the method of undetermined coefficients or the operator $(D = d/dt)$ based method. *(GATE - 2000)*
    Ans :$y = c_1\cos\lambda t + c_2\sin\lambda t + [1/(\lambda^2 - w^2)]\cos(\omega t + k)$ where $c_1 = \cos k/(w^2 - \lambda^2)$ and $c_2 = w\sin t/\lambda(\lambda^2 - w^2)$

17. The number of boundary conditions required to solve the following differential equation is $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ *(GATE'01)*
    a) 2 b) 0 c) 4 d) 1

18. The solution for the following differential equation with boundary conditions $y(0) = 2$ and $y^1(1) = -3$ is *(GATE'01)*

$$(d^2y/dx^2) = 3x - 2$$

a) $y = x^3/3 - x^2/2 + 3x - 6$ b) $y = 3x^3 - x^2/2 - 5x + 2$
c) $y = x^3/2 - x^2 - 5x/2 + 2$ d) $y = x^3 - x^2/2 + 5x + 3/2$

19. Solve the differential equation $\frac{d^2y}{dx^2} + y = x$ with the following conditions: *(GATE'01)*

(i) at $x = 0, y = 1$ (ii) at $x = \text{——}, y = \text{——}$ **Ans:** $y = x + \cos x$

20. The solution of the differential equation $(dy/dx) + y^2 = 0$ is *(GATE'03)*
a) $y = 1/(x + c)$ b) $y = (-x^3/3) + c$ c) ce^x d) unsolvable as equation is non-linear

21. Biotransformation of an organic compound having concentration (x) can be modeled using an ordinary differential equation $(dx / dt) + kx^2 = 0$, where k is the reaction rate constant. If $x = a$ at $t = 0$, the solution of the equation is *(GATE'04)*
a) $x = ae^{-kt}$ b) $1/x = 1/a + kt$ c) $x = a(1 - e^{-kt})$ d) $x = a + kt$

22. Transformation to linear form by substituting $v = y^{1-n}$ of the equation *(GATE'05)*
$(dy / dt) + p(t)y = q(t)y^n$; $n > 0$ will be
a) $(dv / dt) + (1 - n) pv = (1 - n)q$ b) $(dv / dt) + (1 - n) pv = (1 + n)q$
c) $(dv / dt) + (1 + n) pv = (1 - n)q$ d) $(dv / dt) + (1 + n) pv = (1 + n)q$

23. The solution of $(d^2y / dx^2) + 2 (dy / dx) + 17y = 0$; $y(0) = 1$, $(dy/dx)_{(\pi/4)} = 0$ in the range $0 < x < \pi/4$ is given by *(GATE'05)*
a) $e^{-x}(\cos 4x + ¼ \sin 4x)$ b) $e^{x}(\cos 4x - ¼ \sin 4x)$
c) $e^{-4x}(\cos x - ¼ \sin x)$ d) $e^{-4x}(\cos 4x - ¼ \sin 4x)$

24. If $x^2 (dy/dx) + 2xy = (2\ln x / x)$ and $y(1) = 0$, then what is $y(e)$? *(GATE'05)*
a) e b) 1 c) 1/e d) $1/e^2$

25(a). The complete solution of the ordinary differential equation $(d^2y/dx^2) + p (dy/dx) + q y = 0$ is $y = c_1 e^{-x} + c_2e^{-3x}$ then p and q are *(GATE'05)*
a) p = 3, q = 3 b) p=3,q= 4 c) p= 4,q=3 d) p= 4,q= 4

25(b). Which of the following is a solution of the differential equation *(GATE'05)*
$(d^2y /dx^2) + p(dy /dx) + (q+1) y=0$?
a) e^{-3x} b) xe^{-x} c) xe^{-2x} d) x^2e^{-2x}

KEY

1. c 2. a 3. b 5. c 7. a 8. a 9. b 10. d 12. d 14. c & d 15. c
17. c 18. c 20. a 21. b 22. a 23. a 24. d 25(a). c 25(b). C

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PREVIOUS GATE QUESTIONS

01. The solution of the first order differential equation $\dot{x}(t) = -3\,x(t)$, $x(0) = x_0$ is **EE – 2005 – 1M**
(a) $x(t) = x_0\, e^{-3t}$ (b) $x(t) = x_0\, e^{-3}$
(c) $x(t) = x_0\, e^{-t/3}$ (d) $x(t) = x_0\, e^{-1}$

02. For the equation $\ddot{x}(t)+3\dot{x}(t)+2x(t)=5$, the solution x(t) approaches the following values as $t \to \infty$ **EE – 2005 – 2M**
(a) 0 (b) 5/2
(c) 5 (d) 10

03. The general solution of the differential equation $(D^2 - 4D + 4)y = 0$, is of the form (given $D=\frac{d}{dx}$ and c_1, c_2 are constants) **IN – 2005 – 2M**
(a) $c_1 e^{2x}$ (b) $c_1 e^{2x} + c_2 e^{-2x}$
(c) $c_1 e^{2x} + c_2 x e^{2x}$ (d) $c_1 e^{2x} + c_2 x e^{-2x}$

04. The differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = c^2\left[\frac{d^2y}{dx^2}\right]^2 \text{ is of}$$

PI – 2005 – 1M
(a) 2nd order and 3rd degree
(b) 3rd order and 2nd degree
(c) 2nd order and 2nd degree
(d) 3rd order and 3rd degree

05. The degree of the differential equation $\frac{d^2x}{dt^2}+2x^3=0$ is **CE – 2007 – 1M**
(a) 0 (b) 1
(c) 2 (d) 3

06. The solution for the differential equation $\frac{dy}{dx}=x^2 y$ with the condition that y = 1 at x = 0 is **CE – 2007 – 1M**
(a) $y=e^{1/2x}$ (b) $\ln(y)=\frac{x^3}{3}+4$
(c) $\ln(y)=\frac{x^2}{2}$ (d) $y=e^{x^3/3}$

07. A body originally at 60^0 cools down to 40^0c in 15 minutes when kept in air at a temperature of 25^0 c. What will be the temperature of the body at the end of 30 minutes? **CE – 2007 – 2M**
(a) 35.2^0c (b) 31.5^0c
(c) 28.7^0c (d) 15^0c

08. The solution of $\frac{dy}{dx}=y^2$ with initial value $y(0) = 1$ is bounded in the interval **ME – 2007 – 2M**
(a) $-\infty \le x \le \infty$ (b) $-\infty \le x \le 1$
(c) $x < 1, x > 1$ (d0 $-2 \le x \le 2$

09. The solution of the differential equation $k^2\frac{d^2y}{dx^2}=y-y_2$ under the boundary conditions (i) $y = y_1$ at $x = 0$ and (ii) $y = y_2$ at $x = \infty$, where k, y_1 and y_2 are constants is **EC – 2007 – 2M**
(a) $y=(y_1-y_2)e^{\frac{-x}{k^2}}+y_2$
(b) $y=(y_2-y_1)\,e^{\frac{-x}{k}}+y_1$
(c) $y=(y_1-y_2)\sinh\left(\frac{x}{k}\right)+y_1$
(d) $y=(y_1-y_2)\,e^{\frac{-x}{k}}+y_2$

10. A function y(t) satisfies the following differential equation $\frac{d}{dt}y(t)+y(t)=\delta(t)$ where $\delta(t)$ is the delta function. Assuming zero initial condition and denoting the unit step function by u(t), y(t) can be of the form. **EE – 2008 – 1M**
(a) e^t (b) e^{-t}
(c) $e^t\, u(t)$ (d) $e^{-t}\, u(t)$

11. Consider the differential equation $\frac{dy}{dx}=1+y^2$. Which one of the following can be particular solution of this differential equation?

IN – 2008 – 2M

(a) $y = \tan(x + 3)$
(b) $y = \tan x + 3$
(c) $x = \tan(y + 3)$
(d) $x = \tan y + 3$

12. Which of the following is a solution to the differential equation

$$\frac{d}{dt}x(t)+3x(t)=0,\ x(0)=2$$

EC – 2008 – 2M

(a) $x(t)=3e^{-t}$ (b) $x(t)=2e^{-3t}$
(c) $x(t)=-3/2\,t^2$ (d) $x(t)=3\,t^2$

13. Given that $\ddot{x}+3x=0$ and $x(0) = 1$, $\dot{x}(0)=0$, what is $x(1)$?

MC – 2008 – 1M

(a) – 0.99 (b) – 0.16
(c) 0.16 (d) 0.99

14. It is given that $y'' + 2y' + y = 0$, $y(0) = 0$, $y(1) = 0$. What is $y(0.5)$?

MC – 2008 – 2M

(a) 0 (b) 0.37
(c) 0.62 (d) 1.13

15. The solutions of the differential equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0$ are

PI – 2008 – 2M

(a) $e^{-(1+i)x}, e^{-(1-i)x}$ (b) $e^{(1+i)x}, e^{(1-i)x}$
(c) $e^{-(1+i)x}, e^{(1+i)x}$ (d) $e^{(1+i)x}, e^{-(1+i)x}$

16. Solution of the differential equation $3y\frac{dy}{dx}+2x=0$ represents a family of

CE – 2009 – 2M

(a) ellipses (b) circles
(c) parabolas (d) hyperbolas

17. The order of the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+y^4=e^{-t}$ is

EC – 2009 – 1M

(a) 1 (b) 2
(c) 3 (d) 4

18. Match each differential equation in group – I to its family of solution curves from Group – II

CE – 2009 – 2M

Group – I	Group – II
P. $\frac{dy}{dx}=\frac{y}{x}$	1. Circles
Q. $\frac{dy}{dx}=-\frac{y}{x}$	2. Straight lines
R. $\frac{dy}{dx}=\frac{x}{y}$	3. Hyperbolas
S. $\frac{dy}{dx}=-\frac{x}{y}$	

(a) P – 2, Q – 3, R – 3, S – 1
(b) P – 1, Q – 3, R – 2, S – 1
(c) P – 2, Q – 1, R – 3, S – 3
(d) P – 3, Q – 2, R – 1, S – 2

19. The solution of $x\frac{dy}{dx}+y=x^4$ with condition $y(1) = 6/5$ is

ME – 2009 – 2M

(a) $y=\frac{x^4}{5}+\frac{1}{x}$ (b) $y=\frac{4x^4}{5}+\frac{4}{5x}$
(c) $y=\frac{x^4}{5}+1$ (d) $y=\frac{x^5}{5}+1$

20. The solution of the differential equation $\frac{d^2y}{dx^2}=0$ with boundary conditions (i) $\frac{dy}{dx}=1$ at $x = 0$ and (ii) $\frac{dy}{dx}=1$ at $x = 1$ is

PI – 2009 – 2M

(a) $y = 1$
(b) $y = x$
(c) $y = x + c$, when c is an arbitrary constant
(d) $y = c_1 x + c_2$, where c_1 & c_2 are arbitrary constants

21. The homogeneous part of the differential equation $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=r$ (p, q, r are constants) has real distinct roots if

PI – 2009 – 1M

(a) $p^2 - 4q > 0$ (b) $p^2 - 4q < 0$
(c) $p^2 - 4q = 0$ (d) $p^2 - 4q = r$

22. For the differential equation $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+8x=0$ with initial conditions $x(0) = 1$ and $\left(\frac{dx}{dt}\right)_{t=0}=0$, the solution is

EE – 2010 – 2M

(a) $x(t) = 2e^{-6t} - e^{-2t}$
(b) $x(t) = 2e^{-2t} - e^{-4t}$
(c) $x(t) = -e^{-6t} + 2e^{-4t}$
(d) $x(t) = e^{-2t} + 2e^{-4t}$

23. A function n(x) satisfies the differential equation $\frac{d^2n(x)}{dx^2}-\frac{n(x)}{L^2}=0$ where L is a constant. The boundary conditions are : $n(0) = k$ and $n(\alpha) = 0$. The solution to this equation is

EC – 2010 – 1M

(a) $n(x) = k\exp(x/L)$
(b) $n(x) = k\exp(-x/\sqrt{L})$
(c) $n(x) = k^2\exp(-x/L)$
(d) $n(x) = k\exp(-x/L)$

24. The solution of the differential equation $\frac{dy}{dx}-y^2=1$ satisfying the condition $y(0) = 1$ is

PI – 2010 – 2M

(a) $y=e^{x^2}$ (b) $y = \sqrt{x}$
(c) $y = \cot(x + \pi/4)$
(d) $y = \tan(x + \pi/4)$

25. Which one of the following differential equations has a solution given by the function $y = 5\sin(3x + \pi/3)$

PI – 2010 – 1M

(a) $\frac{dy}{dx}-\frac{5}{3}Cos(3x)=0$
(b) $\frac{dy}{dx}+\frac{5}{3}Cos(3x)=0$
(c) $\frac{d^2y}{dx^2}+9y=0$
(d) $\frac{d^2y}{dx^2}-9y=0$

26. The order an degree of the differential equation $\frac{d^3y}{dx^3}+4\sqrt{\left(\frac{dy}{dx}\right)^3+y^2}=0$ are respectively

CE-2010-1M

(a) 3 and 2 (b) 2 and 3
(c) 3 and 3 (d) 3 and 1

27. The solution to the ordinary differential equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$ is

CE – 2010 – 2M

(a) $y = c_1e^{3x} + c_2e^{-2x}$
(b) $y = c_1e^{3x} + c_2e^{2x}$
(c) $y = c_1e^{-3x} + c_2e^{2x}$
(d) $y = c_1e^{-3x} + c_2e^{-2x}$

28. Consider the differential equation $\frac{dy}{dx}+y=e^x$ with $y(0) = 1$. The value of y(1) is

IN – 2010 – 2M

(a) $e + e^{-1}$ (b) $\frac{1}{2}[e-e^{-1}]$
(c) $\frac{1}{2}[e+e^{-1}]$ (d) $2[e-e^{-1}]$

29. If $f = a_0x^n + a_1x^{n-1}y + \ldots\ldots + a_{n-1}xy^{n-1} + a_ny^n$, where a_i (i = 0 to n) are constants then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$ is

IN – 2005 – 01M

(a) $\frac{f}{n}$ (b) $\frac{n}{f}$
(c) nf (d) $n\sqrt{f}$

30. The partial differential equation $\frac{\partial^2\varphi}{dx^2}+\frac{\partial^2\varphi}{dy^2}+\left(\frac{\partial\varphi}{\partial x}\right)+\left(\frac{\partial\varphi}{\partial y}\right)=0$ has

ME – 2007 – 1M

(a) degree 1 and order 2
(b) degree 1 and order 1
(c) degree 2 and order 1
(d) degree 2 and order 2

31. Let $f = y^n$. What is $\frac{\partial^2 f}{\partial x \partial y}$ at x = 2, y = 1?

ME – 2008 – 2M

(a) 0
(b) $ln\ 2$
(c) 1
(d) $\frac{1}{\ln 2}$

32. For the partial differential equation $\frac{\partial^2 u}{\partial x^2}=\pi^2\frac{\partial u}{\partial t}$ in the domain $0 \le x \le 1$ with boundary conditions u(0, t) = 0 and u(1, t) = 0 and initial condition $u(x,0) = \sin(\pi x)$, the solution of the differential equation is

ME – 2008 – 2M

(a) $e^{-t}\sin(\pi x)$
(b) $e^{t}\sin(\pi x)$
(c) $e^{-\pi t}\sin(\pi x)$
(d) $e^{\pi t}\sin(\pi x)$

33. The blasius equation $\frac{d^3f}{d\eta^3}+\frac{f}{2}\frac{d^2f}{d\eta^2}=0$ is a

(a) 2^{nd} order non-linear ordinary differential equation
(b) 3^{rd} order non-linear ordinary differential equation
(c) 3^{rd} order linear ordinary differential equation
(d) mixed order non-linear ordinary differential equation

34. The partial differential equation that can be formed from z = ax + by + ab has the form (with $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$)

CE – 2010 – 2M

(a) z = px + qy
(b) z = px + pq
(c) z = px + qy + pq
(d) z = qy + py

KEY:

01. a	02. b	03. c
04. c	05. b	06. b
07. b	08. c	09. b
10. b	11. a	12. b
13. b	14. a	15. a
16. a	17. b	18. a
19. c	20. c	21. a
22. b	23. d	24. d
25. c	26. a	27. c
28. c	29. c	30. a
31. c	32. a	33. b
34. c		

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BASIC ENGINEERING MATHEMATICS

TOPIC – 6 FOURIER SERIES

PERIODIC FUNCTIONS

If at equal intervals of abscissa x, the value of each ordinate f(x) repeats itself i.e. $f(x) = f(x+\alpha)$ for all x, then y=f(x) is called a periodic function having period α.

Eg: Sin x & Cos x are periodic functions having a period 2π.

To analyze some periodic functions it is necessary to express a function in a series of sines and cosines; Such a series is called ***Fourier series***, which may be written in a general interval $(c, c+2l)$ as follows:

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{l}x\right)+\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{l}x\right)$$

Where $a_0=\frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$a_n=\frac{1}{l}\int_c^{c+2l} f(x)\,\mathrm{Cos}\left(\frac{n\pi}{l}x\right)dx$$

$$b_n=\frac{1}{l}\int_c^{c+2l} f(x)\,\mathrm{Sin}\left(\frac{n\pi}{l}x\right)dx$$

* Fourier series for the ***even function*** f(x) in the interval $(-l, l)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\,\mathrm{Cos}\left(\frac{n\pi}{l}x\right)$$

Where $a_0=\frac{2}{l}\int_0^{l} f(x)\,dx$,

$a_n=\frac{2}{l}\int_0^{l} f(x)\,\mathrm{Cos}\left(\frac{n\pi}{l}x\right)dx$ and $b_n=0$

Note: A function f(x) is said to be even if f(-x) = f(x) and odd if f(-x) = -f(x)

* Fourier series for an ***odd function*** f(x) in the interval $(-l, l)$ *is*

$$f(x)=\sum_{n=1}^{\infty} b_n\,\mathrm{Sin}\left(\frac{n\pi}{l}x\right)$$

Where $b_n=\frac{2}{l}\int_0^{l} f(x)\,\mathrm{Sin}\left(\frac{n\pi}{l}x\right)dx$

DIRCHLET'S CONDITIONS FOR A FOURIER EXPANSION

Any function f(x) can be developed as a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$$; where a_0, a_n, b_n are constants provided

1) f(x) is periodic, single valued and finite.
2) f(x) has a finite number of discontinuities in any one period.
3) f(x) has at the most a finite number of maxima and minima.

* The above conditions are sufficient for the expansion of a function in Fourier series
* Fourier series of even functions in the interval $(-l, l)$ contains only cosine terms
* Fourier series of odd functions in the interval $(-l, l)$ contains only sine terms

HALF RANGE SERIES

* ***Half range sine series*** of f(x) in the range $(0,l)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$$

Where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx$

* ***Half range cosine series*** of f(x) in the range $(0,l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right)$$

Where $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi}{l}x\right)dx$

$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

IMPORTANT RELATIONS

1. $\int_\alpha^{\alpha+2\pi} \cos nx\,dx = \int_\alpha^{\alpha+2\pi} \sin nx\,dx = 0$ $(n \neq 0)$
2. $\int_\alpha^{\alpha+2\pi} \cos^2 nx\,dx = \int_\alpha^{\alpha+2\pi} \sin^2 nx\,dx = \pi$ $(n \neq 0)$
3. $\int_\alpha^{\alpha+2\pi} \cos mx \cos nx\,dx = \int_\alpha^{\alpha+2\pi} \sin mx \sin nx\,dx = 0$ $(m \neq n)$
4. $\int_\alpha^{\alpha+2\pi} \sin mx \cos nx\,dx = 0$ $(m \neq n)$

5. For an even function $\int_{-l}^{l} f(x)\,dx = 2\int_0^l f(x)\,dx$ and for an odd function $\int_{-l}^{l} f(x)\,dx = 0$
6. $\sin n\pi = \cos(n+½)\pi = 0$ & $\sin(n+½)\pi = \cos n\pi = (-1)^n$ (where n = 0,1,2,3…)

PROBLEMS

01. The coefficient of sin x in the expansion of $f(x) = x^2$ in the internal $(-\pi, \pi)$ is

a) $\sum_{n=1}^{\alpha} \frac{(-1)^n}{n^2}$ b) $\sum_{n=1}^{\alpha} \frac{1}{n^2}$ c) $\frac{\pi^2}{6}$ d) 0

02. The term a_0 in the fourier series of the function f(x) = 0 for $-\pi < x < 0$
= Sin x for $0 < x < \pi$ is

a) $2/\pi$ b) $1/\pi$ c) 0 d) 1/2

03. The function $f(x) = -x + 1 \quad -\pi \le x \le 0$
$= x + 1 \quad 0 \le x \le \pi$ has the following terms in its expansion

a) Cosine b) Sine c) Both Cosine & Sine d) Cannot be predicted

04. In the Fourier series for $f(x) = \pi x$ in $0 \le x \le 2$ the coefficient of $\cos \pi x$ =

(a) $1/n$ (b) $1/n^2$ (c) $(-1)^n/n^2$ (d) 0

05. If f(x) = 0, $-2 < x < 0$
= 1, $0 < x < 2$
then the term independent of x in the Fourier Series of f(x) =

(a) 0 (b) 1 (c) 1 / 2 (d) 2

06. For the function given in Q.No.05, the coefficient of $\cos(n\pi x/2)$ =

(a) 0 (b) $1/n$ (c) $1/n^2$ (d) $-1/n$

07. For the function given in Q.No.05, the coefficient of $\sin(n\pi x/2)$ =

(a) $[1+(-1)^n]/n\pi$ (b) $[1-(-1)^n]/n\pi$ (c) 0 (d) $1/n\pi$

08. A function can be expanded in Fourier Series if it satisfies

(a) Drichlet's condition (b) Euler's conditions
(c) Fourier's conditions (d) Bernoulli's formula

09. Which of the following functions cannot be expanded in Fourier Series in the interval $(-\pi, \pi)$

(a) e^x (b) x^2 (c) Cosec x (d) | x |

10. In the range $0 < x < \pi$, if $C = (4C/\pi)\{\sin x + (\sin 3x)/3 + (\sin 5x)/5 + \ldots\ldots\}$
then $1 - (1/3) + (1/5) - (1/7) + \ldots\ldots\ldots \infty$ =

(a) π (b) $\pi/2$ (c) $\pi/4$ (d) $\pi/6$

11. The Fourier Series of $x - x^2$ in the interval $(-\pi, \pi)$ contains

(a) only sine terms (b) only cosine terms
(c) both sine & cosine terms (d) neither sine terms nor cosine terms

12. The Fourier Series of 'x Sinx' in the interval $(0, 2\pi)$ contains

(a) only sine terms (b) only cosine terms
(c) both sine & cosine terms (d) neither sine terms nor cosine terms

13. Which of the following is not a Dirchlet's condition
(a) f(x) is periodic (b) f(x) is single valued
(c) f(x) is finite (d) f(x) is even function

14. In the Fourier Series of the function
$f(x) = 0$ when $-2 < x < -1$
$= k$ when $-1 < x < 1$
$= 0$ when $1 < x < 2$
then constant term =
(a) k (b) k /2 (c) 2k (d) 4k

15. If f(x) is an odd function in (0,2) then its Fourier Series is
(a) $\Sigma\,[b_n \operatorname{Sin}(n\pi x/2)]$ (b) $(a_0/2) + \Sigma\,[a_n \operatorname{Cos}(n\pi x)]$
(c) $(a_0/2) + \Sigma\,[b_n \operatorname{Sin}(n\pi x)]$ (d) $(a_0/2) + \Sigma\,(a_n \operatorname{Cos} n\pi x + b_n \operatorname{Sin} n\pi x)$

16. The Fourier Series of $f(x) = 1 - x^2$ in (-1, 1) contains
(a) only sine terms (b) only cosine terms
(c) both sine & cosine terms (d) neither sine terms nor cosine terms

17. In the interval $(0, \pi)$, the constant term in the Cosine series of $f(x) = x$ is
(a) $\pi/2$ (b) π (c) 0 (d) $\pi/4$

18. In the interval $(0,\pi)$, the constant term in the Sine series of $f(x) = x$
(a) $\pi/2$ (b) π (c) 0 (d) $\pi/4$

19. If f(x) = x is expressed as a half range Cosine series in (0,2) then the constant term in the Series is
(a) 1 (b) 2 (c) 0 (d) –1

20. If f(x) = x is expressed as a half range Sine series in (0,2), then the coefficient of $\operatorname{Sin} \pi x$ in the series is
(a) $2/\pi$ (b) $-2/\pi$ (c) $1/\pi$ (d) $-1/\pi$

21. In $(0,\pi)$, If a constant 'c' is expanded as a half range Sine series, then the coefficient of Sin 5x is
(a) $4c/5\pi$ (b) $2c/5\pi$ (c) 0 (d) $c/5\pi$

22. If $f(x) = x^2$ is expanded as a Cosine series in $(0,\pi)$ then the constant term =
(a) $\pi^2/3$ (b) $2\pi^2/3$ (c) $\pi^2/3$ (d) $3\pi^2/2$

23. In $(0,\pi)$, If a constant 'c' is expanded as a half range Sine series, then the coefficient of Sin 2x is
(a) $4c/5\pi$ (b) $2c/5\pi$ (c) 0 (d) $c/5\pi$

24. To expand f(x) as a Sine series in (0,2)
(a) f(x) should be an even function . (b) f(x) should be an odd function
(c) f (x) should be neither even nor odd function
(d) It hardly matters whether f(x) is even or odd or neither

25. If $u(t) = 0$ when $(-T/2) < t < 0$
$= E \operatorname{Sin} \omega t$ when $0 < t < (T/2)$
and $T = 2\pi/\omega$ then the constant term in the Fourier Series =
(a) E/ω (b) E/π (c) $2E/\omega$ (d) $2E/\pi$

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KEY

01.d 02.a 03.a 04. d 05. c 06. a 07. b 08. a 09. c 10. c 11. c 12. c

13. d 14.b 15.d 16. b 17. a 18. c 19. a 20. b 21. a 22. a 23. c 24. d 25. b

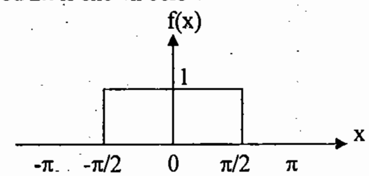
PREVIOUS GATE QUESTIONS - "FOURIER SERIES"

1. Fourier expansion of the periodic function $f(t) = t^3$ with a period $(= 2\pi/\omega)$ has the form

a) $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$ b) $\frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin n\omega t$ *(GATE'96)*

c) $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t$ d) $\sum_{n=1}^{\infty} b_n \sin n\omega t$

2. A function with a period 2π is shown below *(GATE - 2000)*



The Fourier series for this function is given by

a) $f(x) = (½) + \sum_{n=1}^{\infty} (2/n\pi)\ \operatorname{Sin}(n\pi/2) \cos nx$ b) $f(x) = \sum_{n=1}^{\infty} (2/n\pi)\ \sin(n\pi/2) \cos nx$

c) $f(x) = (½) + \sum_{n=1}^{\infty} (2/n\pi) \operatorname{Sin}(n\pi/2) \sin nx$ d) $f(x) = \sum_{n=1}^{\infty} (2/n\pi) \operatorname{Sin}(n\pi/2) \sin nx$

3. The Fourier series expansion of a symmetric and even function, f(x) where *(GATE'03)*
$f(x) = 1 + (2x/\pi)$, $-\pi \le x \le 0$ and
$= 1 - (2x/\pi)$, $0 \le x \le \pi$
will be

a) $\sum_{n=1}^{\infty} (4/\pi^2 n^2)(1 + \cos n\pi) \cos nx$ b) $\sum_{n=1}^{\infty} (4/\pi^2 n^2)(1 - \cos n\pi) \cos nx$

c) $\sum_{n=1}^{\infty} (4/\pi^2 n^2)(1 - \sin n\pi) \sin nx$ d) $\sum_{n=1}^{\infty} (4/\pi^2 n^2)(1 + \sin n\pi) \sin nx$

KEY

1. a 2. a 3. b

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PREVIOUS GATE QUESTIONS

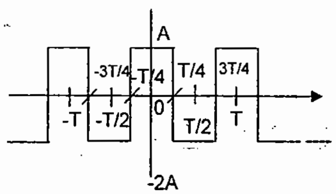
01. The Fourier series for the function $f(x) = \sin^2 x$ is (EE-2005-2M)

(a) $\sin x + \sin 2x$ (b) $1 - \cos 2x$
(c) $\sin 2x + \cos 2x$ (d) $0.5 - 0.5 \cos 2x$

02. The Fourier series of a real periodic function has only (EC-2009-1M)
P. cosine terms if it is even
Q. sine terms if it is even
R. cosine terms if it is odd
S. sine terms if it is odd.
Which of the above statements are correct?
(a) P and S (b) P and R
(c) Q and S (d) Q and R

03. The period of signal $x(t) = 8 \sin[0.8 \pi t + \pi/4]$ is (EE-2010-1M)
(a) $0.4 \pi s$ (b) $0.8 \pi s$
(c) 125 s (d) 2.5 s

04. The trigonometric Fourier series for the wave form f(t) shown below contains

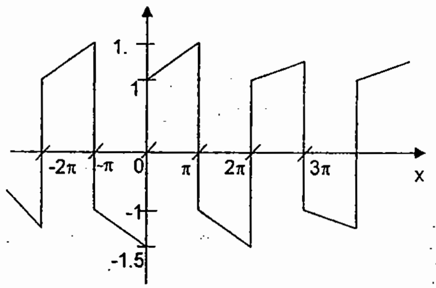


(a) Only cosine terms and zero value for dc component.
(b) Only cosine terms and a +ve value for the dc component.
(c) Only cosine terms and a –ve value for the dc component.
(d) Only sine terms and a –ve value for the dc component.

05. f(x), shown in the adjoining figure is represented by

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)].$$

The value of a_0 is (IN-2010-1M)



(a) 0 (b) $\pi/2$ (c) π (d) 2π

KEY:

01. d	02. a	03. d
04. c	05. a	

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BASIC ENGINEERING MATHEMATICS

TOPIC – 7 TRANSFORM THEORY

LAPLACE TRANSFORMS

Definition : Let f(t) be a function of t defined for all positive values of t. Then the Laplace transform of f(t), denoted by L{f (t)} and is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

provided that the integral exists. 's' is a parameter which may be real or complex number L{f(t)} being clearly a function of 's' is briefly written as $\overline{f}(s)$ i.e. $L\{f(t)\} = \overline{f}(s)$,

which can also be written as $f(t) = L^{-1} \{\overline{f}(s)\}$.

Then f(t) is called the **Inverse Laplace transform** of $\overline{f}(s)$. The symbol L, which transforms $\overline{f}(t)$ into f(s), is called the Laplace transformation operator.

Existence Conditions :

$\int_0^\infty e^{-st} f(t)\, dt$ exists if $\int_0^\lambda e^{-st} f(t)\, dt$ can actually be evaluated and its limit as $\lambda \to \infty$ exists. Otherwise we may use the following theorem :

If f (t) is continuous and $\underset{t\to\infty}{Lt} \{e^{-at} f(t)\}$ is finite ; then the Laplace transform of f(t), i.e. $\int_0^\infty e^{-st} f(t)\, dt$ exists for s > a.

It should however, be noted that the above conditions are sufficient rather than necessary. For example, $L(1/\sqrt{t})$ exists, though $1/\sqrt{t}$ is infinite at t = 0. Similarly a function f(t) for which $\underset{t\to\infty}{Lt} \{e^{-at} f(t)\}$ is finite and having a finite discontinuity will have a Laplace transform for s > a.

TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

The direct application of the definition gives the following formulae:

S.No.	f(t)	L[f(t)]		$L[e^{at}f(t)]$	
1.	1	$\frac{1}{s}$	(s > 0)	$\frac{1}{s-a}$	(s-a > 0)
2.	k	$\frac{k}{s}$	(s > 0)	$\frac{k}{s-a}$	(s-a > 0)

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3.	t^n	$\frac{n!}{s^{n+1}}$	$(s>0)$	$\frac{n!}{(s-a)^{n+1}}$ where $n \in N$
		$\frac{\Gamma n+1}{s^{n+1}}$	$(s>0)$	$\frac{\Gamma n+1}{(s-a)^{n+1}}$ where $n \notin Z^-$
4.	sinat	$\frac{a}{s^2+a^2}$	$(s>0)$	$\frac{a}{(s-a)^2+a^2}$
5.	cosat	$\frac{s}{s^2+a^2}$	$(s>0)$	$\frac{s}{(s-a)^2+a^2}$
6.	sinhat	$\frac{a}{s^2-a^2}$	$(s>\lvert a\rvert)$	$\frac{a}{(s-a)^2-a^2}$
7.	coshat	$\frac{s}{s^2-a^2}$	$(s>\lvert a\rvert)$	$\frac{s}{(s-a)^2-a^2}$

PROPERTIES OF LAPLACE TRANSFORMS

I. Linearity property. If a, b, c be any constants and f, g, h any functions of t, then

$$L\{a\,f(t)+b\,g(t)-c\,h(t)\}=a\,L\{f(t)\}+b\,L\{g(t)\}-c\,L\{h(t)\}$$

II. First shifting property. If $L\{f(t)\}=\overline{f}(s)$, then $L\{e^{at}f(t)\}=\overline{f}(s-a)$.

III. Change of scale property. If $L\{f(t)\}=\overline{f}(s)$, then $L\{f(at)\}=\frac{1}{a}\overline{f}\left(\frac{s}{a}\right)$, $a>0$

IV. Transforms of Derivatives

(1) If $f^1(t)$ is continuous and $L\{f(t)\}=\overline{f}(s)$, then $L\{f^1(t)\}=s\overline{f}(s)-f(0)$

(2) If $f^1(t)$ and its first $(n-1)$ derivatives are continuous, then
$L\{f^n(t)\}=s^n\overline{f}(s)-s^{n-1}f(0)-s^{n-2}f^1(0)-\ldots\ldots-f^{(n-1)}(0)$.

V. Transforms of Integrals

If $L\{f(t)\}=\overline{f}(s)$, then $L\left\{\int_0^t f(u)\,du\right\}=\frac{1}{s}\overline{f}(s)$.

VI. Multiplication by t^n

If $L\{f(t)\}=\overline{f}(s)$, then $L\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}[\overline{f}(s)]$, where $n=1,2,3\ldots$

VII. Division by t

If $L\{f(t)\}=\overline{f}(s)$, then $L\left\{\frac{1}{t}f(t)\right\}=\int_s^\infty \overline{f}(s)\,ds$.

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INVERSE LAPLACE TRANSFORMS

(1) $L^{-1}\left\{\frac{1}{s}\right\}=1$

(2) $L^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$

(3) $L^{-1}\left\{\frac{1}{s^n}\right\}=\frac{t^{n-1}}{(n-1)!}$, $n=1,2,3,\ldots$

(4) $L^{-1}\left\{\frac{1}{(s-a)^n}\right\}=\frac{e^{at}t^{n-1}}{(n-1)!}$

(5) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at.$

(6) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at.$

(7) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\}=\frac{1}{a}\sinh at.$

(8) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cosh at.$

(9) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{1}{2a}t\sin at.$

(10) $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=\frac{1}{2a^3}(\sin at-at\cos at).$

(11) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}=\frac{1}{2a}(\sin at+at\cos at)$

PROPERTIES OF INVERSE LAPALACE TRANSFORMS

I. Shifting property for inverse Laplace transforms.

If $L^{-1}\{\overline{f}(s)\}=f(t)$, then
(i) $L^{-1}\{\overline{f}(s-a)\}=e^{at}f(t)=e^{at}L^{-1}\{\overline{f}(s)\}$.

(ii) $L^{-1}\{\overline{f}(s+a)\}=e^{-at}f(t)=e^{-at}L^{-1}\{\overline{f}(s)\}$.

II. If $L^{-1}\{\overline{f}(s)\}=f(t)$ and $f(0)=0$, then $L^{-1}\{s\overline{f}(s)\}=\frac{d}{dt}\{f(t)\}$

In general, $L^{-1}\{s^n\overline{f}(s)\}=\frac{d^n}{dt^n}\{f(t)\}$ provided $f(0)=f^1(0)=\ldots=f^{(n-1)}(0)=0$

III. If $L^{-1}\{f(s)\}=f(t)$, then $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\}=\int_0^t f(t)\,dt$

IV. If $L^{-1}\{\overline{f}(s)\}=f(t)$, then $L^{-1}\left\{\frac{d^n}{ds^n}[\overline{f}(s)]\right\}=(-1)^n t^n f(t)$

V. If $L^{-1}\{\overline{f}(s)\}=f(t)$, then $L^{-1}\left\{\int_s^\infty \overline{f}(s)\,ds\right\}=f(t)/t$

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CONVOUTION THEOREM

If $L^{-1}\{f(s)\} = f(t)$ and $L^{-1}\{g(s)\} = g(t)$,

then $L^{-1}\{\overline{f}(s)\,\overline{g}(s)\} = \int_0^t f(u)\, g(t-u)du = f * g$

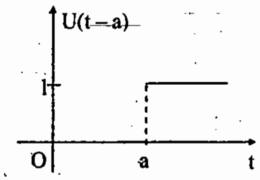
[f * g is called the **convolution** or **falting** of f and g.]

UNIT STEP FUNCTION

Def:- The unit step function $u(t-a)$ is defined as follows :

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t \geq a \end{cases}$$

where 'a' is always positive.



Transform of Unit Function : $L\{u(t-a)\} = e^{-as}/s$.

Second shifting property. If $L\{f(t)\} = \overline{f}(s)$, then

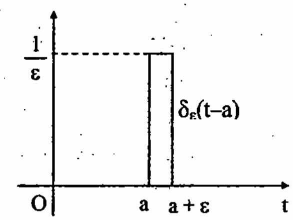
$$L\{f(t-a)\, u(t-a)\} = e^{-as}\,\overline{f}(s)$$

UNIT IMPULSE FUNCTION

Unit impulse function is considered as the limiting form of the function.

$\delta_\varepsilon(t-a) = 1/\varepsilon,$ $\quad a \leq t \leq a+\varepsilon$

$= 0,$ $\quad$ otherwise.



Thus the unit impulse function $\delta(t-a)$ is defined as follows :

$\delta(t-a) = \infty$ for $t = a$; $= 0$ for $t \neq a$,

such that $\int_0^\infty \delta(t-a)dt = 1 \quad (a \geq 0)$

Transform of Unit Impulse Function : $L\{\delta(t-a)\} = e^{-as}$

PERIODIC FUNCTION

If $f(t)$ is a periodic function with period T, i.e. $f(t+T) = f(t)$, then $L\{f(t)\} = \dfrac{\int_0^T e^{-st} f(t)\,dt}{(1-e^{-sT})}$

PROBLEMS

Find the Laplace Transforms of the following

01. $\cos(at+b)$. **Ans :** $\dfrac{s\cos b - a\sin b}{s^2+a^2}$

02. $\sin at \sin bt$. **Ans :** $\dfrac{2abs}{[s^2+(a+b)^2][s^2+(a-b)^2]}$

03. $\cos^3 2t$ **Ans :** $\dfrac{s(s^2+28)}{(s^2+4)(s^2+36)}$

04. $e^{-at}\sinh bt$ **Ans :** $\dfrac{b}{(s+a)^2-b^2}$

05. $t^3 e^{-3t}$ **Ans :** $\dfrac{6}{(s+3)^4}$

06. $e^{-t}\sin^2 t$ **Ans :** $\dfrac{2}{(s+1)(s^2+2s+5)}$

07. $\cosh at \sin at$ **Ans :** $\dfrac{a(s^2+2a^2)}{s^4+4a^4}$

08. $\sinh 3t \cos^2 t$ **Ans :** $\dfrac{3}{2}\left(\dfrac{1}{s^2-9} + \dfrac{s^2-13}{s^4-10s^2+169}\right)$

09. $f(t) = \begin{cases} e^t, 0 < t < 1 \\ 0, t > 1 \end{cases}$ **Ans :** $\dfrac{1}{1-s}\,[e^{1-s}-1]$.

10. $f(t) = \begin{cases} \sin t, 0 < t < \pi \\ 0, t > \pi \end{cases}$ **Ans :** $\dfrac{1+e^{-\pi s}}{s^2+1}$

11. $f(t) = \begin{cases} t^2, 0 < t < 2 \\ t-1, 2 < t < 3 \\ 7, t > 3 \end{cases}$ **Ans :** $\dfrac{2}{s^3} - \dfrac{e^{-2s}}{s^3}(2+3s+3s^2) + \dfrac{e^{-3s}}{s^2}(5s-1)$.

12. $f(t) = |t-1| + |t+1|, t \geq 0$. **Ans :** $\dfrac{2}{s}\left(1 + \dfrac{e^{-s}}{s}\right)$

13. Find $L\left\{\int_0^t e^{-t}\cos t\,dt\right\}$ **Ans :** $\dfrac{s+1}{s(s^2+2s+2)}$

14. Given $L\{2\sqrt{(t/\pi)}\} = 1/s^{3/2}$, show that $L\{1/\sqrt{(\pi t)}\} = 1/\sqrt{s}$.

15. $t\sin^2 t$ **Ans :** $\dfrac{2(3s^2+4)}{s^2(s^2+4)^2}$

16. $t^2\cos at$. **Ans :** $\dfrac{2s^3-6a^2s}{(s^2+a^2)^3}$

17. $te^{-2t}\sin 2t$. **Ans :** $\dfrac{4(s+2)}{(s^2+4s+8)^2}$

18. $te^{-t}\cosh t$. **Ans :** $\dfrac{s^2+2s+2}{(s^2+2s)^2}$

19. (sin at)/t. **Ans :** $\cot^{-1}\left(\frac{s}{a}\right)$

20. (cos 2t – cos 3t)/t. **Ans :** $(1/2)\ \log\ \{(s^2+9)/(s^2+4)\}$.

21. $(e^{-t}\sin t)/t$. **Ans :** $\cot^{-1}(s+1)$

22. (1 – cos 2t) / t. **Ans :** $\frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$

23. $(1-\cos t)/t^2$ **Ans :** $\cot^{-1}s - (½)s\ \log\ (1+s^{-2})$.

24. Evaluate

(a) $\int_0^\infty te^{-2t}\cos t\,dt$. **Ans :** $\frac{3}{25}$

(b) $\int_0^\infty te^{-t}\sin^4 t\,dt$. **Ans :** $\frac{8(s+1)}{s(s^2+2s+17)}$

25. Prove that $\int_0^\infty (e^{-t}\sin^2 t)/t\ dt = (1/4)\log 5$.

Find the Inverse Laplace Transforms of the following

26. $\frac{2s-5}{4s^2+25}+\frac{4s-18}{9-s^2}$ **Ans :** $\frac{1}{2}\left(\cos\frac{5t}{2}-\sin\frac{5t}{2}\right)$ - 4cosh3t + 6 sinh3t

27. $\frac{3s}{s^2+2s-8}$ **Ans :** $e^{2t}+2e^{-4t}$

28. $\frac{3s+7}{s^2-2s-3}$ **Ans :** $4e^{3t}-e^{-t}$

29. $\frac{s^2+s-2}{s(s+3)(s-2)}$ **Ans :** $\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$

30. $\frac{s^2-10s+13}{(s-7)(s^2-5s+6)}$ **Ans :** $2e^{3t}-\frac{3}{5}e^{2t}-\frac{2}{5}e^{7t}$

31. $\frac{s}{(s^2-1)^2}$ **Ans :** (½)t sinh t

32. $\frac{1+2s}{(s+2)^2(s-1)^2}$ **Ans :** $(1/3)t(e^t-e^{-2t})$

33. $\frac{s}{(s-3)(s^2+4)}$ **Ans :** $(1/13)(3e^{3t}-3\cos 2t+2\sin 2t)$

34. $\frac{s^3}{s^4-a^4}$ **Ans :** (½) [cos at + cosh at]

35. $\frac{1}{s^3-a^3}$ **Ans :** $\left(\frac{1}{3}a^2\right)[e^{at}-e^{-at/2}\{\cos(\sqrt{3}at/2)+(\sqrt{3}/2)\sin(\sqrt{3}at/2)\}]$

36. $\frac{s^2+s}{(s^2+1)(s^2+2s+2)}$ **Ans :** $\frac{1}{5}(1+e^{-t})\sin t+\frac{3}{5}(1-e^{-t})\cos t$

37. $\frac{s+2}{(s^2+4s+5)^2}$ **Ans :** $(1/2)\,te^{-2t}\sin t$

38. $\frac{s}{s^4+s^2+1}$ **Ans :** $(2/\sqrt{3})\sinh(½t)\sin(½\sqrt{3}t)$

39. $\frac{1}{s^2(s+2)}$ **Ans :** $\frac{1}{4}(e^{-2t}+2t-1)$.

40. $\frac{1}{s(s+2)^3}$ **Ans :** $\frac{1}{8}-\frac{1}{4}(t^2+t+\frac{1}{2})e^{-2t}$

41. $\frac{s}{a^2s^2+b^2}$ **Ans :** $\frac{1}{a^2}\cos\left(\frac{bt}{a}\right)$

42. $\frac{s}{(s+a)^2}$ **Ans :** $(1-at)\,e^{-at}$

43. $\frac{2as}{(s^2+a^2)^2}$ **Ans :** t sin at.

44. $\frac{s^2}{(s+a)^3}$ **Ans :** $(1/2)(a^2t^2-4at+2)\,e^{-at}$.

45. $\log\left(\frac{1+s}{s}\right)$ **Ans :** $(1-e^t)/t$

46. $\log\left(\frac{s+a}{s+b}\right)$ **Ans :** $(1/t)(e^{-bt}-e^{-at})$

47. $\log\left\{\frac{s+1}{(s+2)(s+3)}\right\}$ **Ans :** $e^{-t}-e^{-2t}-e^{-3t}$

48. $\frac{1}{2}\log\left\{\frac{s^2+b^2}{s^2+a^2}\right\}$ **Ans :** (1/t) (cos at – cos bt)

49. $\log\left\{1-\left(\frac{a^2}{s^2}\right)\right\}$ **Ans :** (2/t) (1 – cosh at)

50. $\tan^{-1}\left(\frac{2}{s}\right)$ **Ans :** (sin 2t) / t

51. $\cot^{-1}(s+1)$ Ans : $(e^{-t}\sin t)/t$.

52. $s\log\left(\frac{s-1}{s+1}\right)$ Ans : $\frac{2(\sinh t - t\cosh t)}{t^2}$

53. $\log\left(\frac{s+1}{s-1}\right)$ Ans: $2\,\frac{\text{Sinh}t}{t}$

54. $\text{Cot}^{-1}\left(\frac{s}{2}\right)$ Ans: $\frac{\sin 2t}{t}$

Using Convolution theorem, evaluate :

55. $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$ Ans : $(1/a^3)(at - \sin at)$.

56. $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$ Ans : $t(e^{-t}+1) + 2(e^{-t}-1)$

57. $L^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\}$ Ans : $(1/16)(e^{2t} - e^{-2t} - 4te^{-2t})$

58. $L^{-1}\left\{\frac{s}{(s^2+2)(s^2+4)}\right\}$ Ans : $(1/3)(\cos t - \cos 2t)$

Solve the following Equations by the Laplace Transform Method

59. $(dx/dt) + x = \sin wt,\ x(0) = 2.$ Ans : $x = \left(2 + \frac{\omega}{\omega^2+1}\right)e^{-t} + \frac{\sin\omega t - \omega\cos\omega t}{\omega^2+1}$

60. $y^{11} + 4y^1 + 3y = e^{-t},\ y(0) = y^1(0) = 1.$ Ans : $y = \frac{7}{4}e^{-t} - \frac{3}{4}e^{-3t} + \frac{1}{2}te^{-t}$

61. $(D^2-1)x = a\cosh t,\ x(0) = x^1(0) = 0.$ Ans : $x = (at/2)\sinh t.$

62. $y^{11} + y = t,\ y(0) = 1,\ y^1(0) = -2.$ Ans : $y = t - 3\sin t + \cos t$

63. $y^{11} - 3y^1 + 2y = 4t + e^{3t}$, when $y(0) = 1$ and $y^1(0) = -1.$ Ans : $y = 2t + 3 + (1/2)(e^{3t} - e^t) - 2e^{2t}.$

64. **Evaluate :**

(i) $L\{e^{t-1}u(t-1)\}$ Ans : $e^{-s}/(s-1)s$

(ii) $L\{(t-1)^2u(t-1)\}$ Ans : $2e^{-s}/s^3$

(iii) $L\{t^2u(t-3)\}$. Ans : $e^{-3s}(2 + 6s + 9s^2)/s^3$

(iv) $L\{t^2u(t-1) + \delta(t-1)\}$ Ans : $e^{-s}(2 + 2s + s^2 + s^3)/s^3$

(v) $L\ \{e^{t-1}\ u(t-1)\}$ Ans : $e^{-s}/(s-1)$

(vi) $L\{(t-1)^2\ u(t-1)\}$ Ans : $2e^{-s}/s^3$

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65. **Find the Inverse Laplace transforms of the following**

(i) $\frac{e^{-\pi s}}{s^2+1}$ Ans : $-\sin t\,.\,u(t-\pi)$

(ii) $\frac{se^{-as}}{s^2+w^2}$, $a > 0$. Ans : $\cosh w(t-a)\,.\,u(t-a)$

(iii) $\frac{e^{-s}}{(s-1)(s-2)}$ Ans : $[e^{2(t-1)} - e^{t-1}]\,u(t-1)$

(iv) $\frac{e^{-s}}{(s+1)^3}$ Ans : $(1/2)\,e^{-(t-1)}(t-1)^2\,u(t-1)$

(v) $\frac{e^{-\pi s}}{s^2+1}$ Ans: $-\sin t\ \ u(t-\pi)$

66. Find the Laplace transform of the square-wave (or meander) function of period 'a' defined as

$f(t) = 1,$ when $0 < t < a/2,$
$\quad = -1,$ when $a/2 < t < a.$ Ans : $(1/s)\tanh(sa/4).$

67. Find the Laplace transform of the saw-toothed wave of period T, given $f(t) = t/T$ for $0 < t < T$. Ans : $(1/s^2T) - e^{-sT}/s(1-e^{-sT}).$

68. Find the Laplace transform of the triangular wave of period '2a' given by

$f(t) = t,$ $0 < t < a$
$\quad = 2a - t,$ $a < t < 2a.$ Ans : $(1/s^2)\tanh(1/2)as.$

69. Find $L\{F(t)\}$ Where $F(t) = \sin t,\ 0<t<\pi$
$\quad = 0\ ,\ \pi<t<2\pi$ Ans: $\frac{1}{(s^2+1)(1-e^{-\pi s})}$

PREVIOUS GATE QUESTIONS - "LAPLACE TRANSFORMS"

1. The inverse Laplace transform of $(s+9)/(s^2+6s+13)$ is---- *(GATE'95)*
a) $\cos 2t + 9\sin 2t$ b) $e^{-3t}\cos 2t - 3e^{-3t}\sin 2t$
c) $e^{-3t}\sin 2t + 3e^{-3t}\cos 2t$ d) $e^{-3t}\cos 2t + 3e^{-3t}\sin 2t$

2. Using Laplace transform, solve the initial value problem $9y^{11} - 6y^1 + y = 0$ *(GATE'96)* $y(0) = 3$ and $y^1(0) = 1$, where prime denotes derivative with respect to t.

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3. The Laplace Transform of a unit step function $u_a(t)$, defined as *(GATE'98)*

$$u_a(t) = 0 \quad \text{for } t < a \quad \text{is}$$
$$= 1 \quad \text{for } t > a,$$

a) e^{-as}/s b) se^{-as} c) $s - u(0)$ d) $se^{-as} - 1$

4. $(s+1)^{-2}$ is the Laplace transform of *(GATE'98)*
a) t^2 b) t^3 c) e^{-2t} d) te^{-t}

5. The Laplace transform of the function *(GATE'99)*
$f(t) = k,\ 0 < t < c$
$= 0,\ c < t < \infty$, is
a) $(k/s)\,e^{-sc}$ b) $(k/s)e^{sc}$ c) $k\,e^{-sc}$ d) $(k/s)(1 - e^{-sc})$

6. Laplace transform of $(a + bt)^2$ where 'a' and 'b' are constants is given by: *(GATE'99)*
a) $(a + bs)^2$ b) $1/(a + bs)^2$
c) $(a^2/s) + (2ab/s^2) + (2b^2/s^3)$ d) $(a^2/s) + (2ab/s^2) + (b^2/s^3)$

7. Let $F(s) = £[f(t)]$ denote the Laplace transform of the function f(t). Which of the following statements is correct? *(GATE - 2000)*

a) $£[df/dt] = 1/s\,F(s)$; $£\left\{\int_0^t f(\tau)\,d\tau\right\} = sF(s) - f(0)$

b) $£[df/dt] = s\,F(s) - F(0)$; $£\left\{\int_0^t f(\tau)\,d\tau\right\} = -dF/ds$

c) $£[df/dt] = s\,F(s) - F(0)$; $£\left\{\int_0^t f(\tau)\,d\tau\right\} = F(s-a)$

d) $£[df/dt] = s\,F(s) - F(0)$; $£\left\{\int_0^t f(\tau)\,d\tau\right\} = 1/s\,F(s)$

8. The inverse Laplace transform of $1/(s^2 + 2s)$ is *(GATE'01)*
a) $(1 - e^{-2t})$ b) $(1 + e^{+2t})/2$ c) $(1 - e^{+2t})/2$ d) $(1 - e^{-2t})/2$

9. The Laplace transform of the following function is *(GATE'02)*

$$f(t) = \begin{cases} \text{Sin}t & \text{for } 0 \le t \le \pi \\ 0 & \text{for } t > \pi \end{cases}$$

a) $1/(1 + s^2)$ for all $s > 0$ b) $1/(1 + s^2)$ for all $s < \pi$
c) $(1 + e^{-\pi s})/(1 + s^2)$ for all $s > 0$ d) $e^{-\pi s}/(1 + s^2)$ for all $s > 0$

10. Using Laplace transforms, solve $(d^2y/dt^2) + 4y = 12t$. *(GATE'02)*
given that $y = 0$ and $dy/dt = 9$ at $t=0$.

11. A delayed unit step function is defined as $u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \ge a \end{cases}$, Its Laplace transform is *(GATE'04)*

a) $a\,e^{-as}$ b) e^{-as}/s c) e^{as}/s d) e^{as}/a

KEY

1. d 2. $y = 3e^{x/3}$ 3. a 4. d 5. d 6. c 7. d 8. d 9. c
10. $y = 3\sin 2x + 3x$ 11. b

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FOURIER TRANSFORMS

Fourier Transform(F.T): Let f(x) be a function defined on R. Then the Fourier transform of f(x) is defined by

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx = F(s) \quad \text{----} \quad (1)$$

$$\text{and } F^{-1}\{F(s)\} = (1/2\pi)\int_{-\infty}^{\infty} e^{-isx} F(s)\,ds = f(x) \quad \text{----} \quad (2)$$

is the inverse Fourier transform of F(s). It is also called inversion formulae of (1)

Fourier Sine Transform (F.S.T): The Fourier Sine transform of the function f(x) in $(0, \infty)$ is defined by

$$F_s\{f(x)\} = F_s(s) = \int_0^{\infty} f(x)\,\text{Sin}\,sx\,dx \quad \text{----} \quad (3)$$

$$\text{and} \quad f(x) = (2/\pi)\int_0^{\infty} F_s(s)\,\text{Sin}\,sx\,ds \quad \text{----} \quad (4)$$

is called the inverse Fourier Sine Transform of $F_s(s)$.

Fourier Cosine Transform(F.C.T): The Fourier Cosine transform of the function f(x) is $(0, \infty)$ is defined by

$$F_c\{f(x)\} = F_c(s) = \int_0^{\infty} f(x)\,\text{Cos}\,sx\,dx \quad \text{----} \quad (5)$$

$$\text{and } f(x) = (2/\pi)\int_0^{\infty} F_c(s)\,\text{Cos}\,sx\,ds \quad \text{----} \quad (6)$$

is called the inverse Fourier Cosine Transform of $F_C(s)$.

PROPERTIES OF FOURIER TRANSFORMS

1. **Linear property:** If F(s) and G(s) are Fourier transforms of f(x) and g(x) then
$F\{C_1 f(x) + C_2 g(x)\} = C_1 F(s) + C_2 G(s)$

2. **Change of scale property(C.S.P):** If F(s) is the Fourier transform of f(x) then
$F\{f(ax)\} = (1/a) F(s/a),\ a \ne 0.$

Note: Similarly we can show that $F_s\{f(ax)\} = (1/a) F_s(s/a)$ and $F_c\{f(ax)\} = (1/a) F_c(s/a)$

3. **Shifting property:** If F(s) is the Fourier transform of f(x) then
$F\{f(x-a)\} = e^{ias} F(s)$

Note: $F\{f(x + a)\} = e^{-ias} F(s)$

4. **Modulation Theorem:** If $F\{f(x)\} = F(s)$ then $F\{f(x)\,\text{Cos}\,ax\} = (½)[F(s+a) + F(s-a)]$
$F\{f(x)\,\text{Cos}\,ax\} = (½)[F(s+a) + F(s-a)]$

NOTE: If $F\{f(x)\} = F(s)$ then $F\{e^{iat} f(x)\} = F(s+a)$

Remarks: Similarly if $F_s(s)$ and $F_c(s)$ are the Fourier Sine Transform and Fourier Cosine Transform of f(x) then

i) $F_s\{f(x) \cos ax\} = (½)\,[F_s(s+a) + F_s(s-a)]$
ii) $F_c\{f(x) \sin ax\} = (½)\,[F_s(s+a) - F_s(s-a)]$
iii) $F_s\{f(x) \sin ax\} = (½)\,[F_c(s-a) - F_c(s+a)]$
iv) $F_c\{f(x) \cos ax\} = (½)\,[F_c(s+a) + F_c(s-a)]$

5. **Fourier Transform of derivatives**

i) $F_s\{f^1(x)\} = -s\,F_c(s)$
ii) $F_c\{f^1(x)\} = f(0) + s\,F_s(s)$
iii) $F\{\frac{d^n}{dx^n} f(x)\} = (-is)^n F(s)$

6. **Multiplication with 'x':**

i) $F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} [F(s)]$

ii) $F_c\{x\, f(x)\} = (d/ds)\,\{F_s[f(x)]\}$

iii) $F_s\{x\, f(x)\} = -(d/ds)\,\{F_c[f(x)]\}$

7. $F\{f(-x)\} = F(-s)$

8. $F\{\overline{f(x)}\} = \overline{F(-s)}$

9. $F\{\overline{f(-x)}\} = \overline{F(s)}$

10. $\int_0^\infty F_c[f(x)]\,.\,F_c[g(x)]\,ds = (\pi/2)\int_0^\infty f(x)\,g(x)\,dx$

11. $\int_0^\infty F_s[f(x)]\,.\,F_s[g(x)]\,ds = (\pi/2)\int_0^\infty f(x)\,g(x)\,dx$

PROBLEMS

01. Find the Fourier Cosine transform of $f(x) = \begin{cases} x\,, & 0 < x < 1 \\ 2 - x\,, & 1 < x < 2 \\ 0\,, & x > 2 \end{cases}$

Ans) $\frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}$

02. Find the Fourier transform of $e^{-x^2/2}$, $-\infty < x < \infty$

Ans) $\sqrt{2\pi}\; e^{-s^2/2}$

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03. Find the Fourier Sine transform and Fourier Cosine transform of e^{-ax}/x
Ans) $\tan^{-1}(s/a)$ and $(-½)\,\log(s^2 + a^2)$

04. Find the Fourier Sine transform of xe^{-ax}
Ans) $\frac{2as}{(S^2 + a^2)^2}$

05. Find the Fourier Cosine transform of $f(x) = 1/(1+x^2)$ and hence derive Fourier Sine transform of $\phi(x) = x/(1+x^2)$
Ans) $\frac{\pi}{2} e^{-s}$ and $\frac{\pi}{2} e^{-s}$

06. Find the Fourier Sine transform and Fourier Cosine transform of e^{-ax}, $a>0$
Ans) $\frac{s}{s^2 + a^2}$ and $\frac{a}{s^2 + a^2}$

Convolution: The convolution of two functions f(x) and g(x) over $(-\infty, \infty)$ is defined as

$$f * g = \int_{-\infty}^{\infty} f(t)\; g(x-t)\,dt.$$

It is also called **Falting.**

Convolution theorem for Fourier Transforms:

$$F\{f(x) * g(x)\} = F\{f(x)\}\,.\,F\{g(x)\}$$

Parseval's Identity for Fourier Transforms:

If the fourier transforms of f(x) and g(x) are F(s) and G(s) respectively, then

i) $(1/2\pi)\int_{-\infty}^{\infty} F(s)\,\overline{G(s)}\,ds = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx$

ii) $(1/2\pi)\int_{-\infty}^{\infty} [F(s)]^2\,ds = \int_{-\infty}^{\infty} [f(x)]^2\,dx$

Parseval's Identities for Fourier Sine transform and Fourier Cosine transform:

i) $(2/\pi)\int_0^\infty F_s(s)\; G_s(s)\,ds = \int_0^\infty f(x)\,g(x)\,dx$

ii) $(2/\pi)\int_0^\infty [F_s(s)]^2\,ds = \int_0^\infty [f(x)]^2\,dx$

iii) $(2/\pi)\int_0^\infty F_c(s)\,.\,G_c(s)\,ds = \int_0^\infty f(x)\,g(x)\,dx$

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iv) $(2/\pi) \int_0^{\infty} [F_c(s)]^2 \, ds = \int_0^{\infty} [f(x)]^2 \, dx$

PROBLEMS

1. Using Parseval's identities prove that

$$\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

2. Using Parseval's identities calculate

i) $\int_0^{\infty} dx / (a^2+x^2)^2$ ii) $\int_0^{\infty} x^2 / (a^2+x^2)^2 \, dx$ if $a > 0$

Ans) i) $\pi/4a^3$ ii) $\pi/4a$

3. Find Fourier transform of f(x) if $f(x) = \begin{cases} 1 - |x| , & |x| < 1 \\ 0 , & |x| > 1 \end{cases}$

hence deduce that $\int_0^{\infty} (\text{Sin}t / t)^4 \, dt = \pi/3$

Ans) $(2/s^2)(1 - \text{Cos } s)$

4. Show that the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 , & |x| < a \\ 0 , & |x| > a > 0 \end{cases}$

is $4\left[\frac{\text{Sin } as - as \text{ Cos } as}{s^3}\right]$ hence deduce that $\int_0^{\infty} (\text{Sin}t - t\,\text{Cos}t) / t^3 \, dt = \pi/4$. Using Parseval's

identity show that $\int_0^{\infty} [(\text{Sin}t - t\,\text{Cos}t)/t^3]^2 \, dt = \pi/15$

FINITE FOURIER TRANSFORMS

Finite Fourier Sine transform:

Let f(x) be a function defined on $(0, l)$ then the Finite Fourier Sine transform of f(x) is defined as

$$F_s\{f(x)\} = \int_0^{l} f(x) \, \text{Sin}(n\pi x / l) \, dx = F_s(n) \qquad \text{where 'n' is an integer}$$

The inverse Finite Fourier Sine transform of $F_s\{f(x)\}$ is given by

$$F(x) = (2/l) \sum_{n=1}^{\infty} F_s\{f(x)\} \, \text{Sin}(n\pi x/l)$$

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Finite Fourier Cosine transform:

The Finite Fourier Cosine transform of f(x) in $(0, l)$ is

$F_c\{f(x)\} = \int_0^{l} f(x) \, \text{Cos}(n\pi x / l) \, dx = F_c(n)$ and the **inverse Finite Fourier Cosine transform** of

$F_c\{f(x)\}$ is given by $F(x) = (1/l) F_c(0) + (2/l) \sum_{n=1}^{\infty} F_c\{f(x)\} \, \text{Cos}(n\pi x / l)$

PROBLEMS

1. Find the finite Fourier Sine transform and Fourier Cosine transform of $f(x) = x^2$ in $0<x<l$

Ans) $F_S\{f(x)\} = \frac{l^2}{n\pi}(-1)^{n+1} + \frac{2l^3}{n^3\pi^3}[(-1)^n - 1]$ and $F_C\{f(x)\} = 2l^3 \, (-1)^n / n^2\pi^2$

2. Find the Fourier Sine transform and Fourier Cosine transform of $f(x) = e^{ax}$ in $(0, l)$

Ans) $\frac{n\pi l}{a^2l^2 + n^2\pi^2}[(-1)^{n+1} e^{al} + 1]$ and $\frac{al^2}{a^2l^2 + n^2\pi^2}[e^{al}(-1)^n - 1]$

3. Find the Fourier Cosine transform of $f(x) = \text{Sin } ax$ in $(0, \pi)$

Ans) 0 , if $n + a$ is even

$\frac{2n}{a^2 - n^2}$, if $n + a$ is odd

4. Find f(x), if its Sine transform as

$F_s(s) = \frac{1 - \text{Cos } s\pi}{s^2\pi^2}$, $0 < x < \pi$, $s = 1, 2, 3, \ldots\ldots\ldots\ldots$

Ans) $f(x) = (2/\pi^3) \sum [(1 - \text{Cos } n\pi) / n^2] \, \sin nx$

5. Find f(x), if its Cosine transform is $F_c(s) = \frac{\text{Cos}(2s\pi/3)}{(2s+1)^2}$, $0 < x < 1$

Ans) $f(x) = 1 + 2 \sum_{s=1}^{\infty} \frac{\text{Cos}(2s\pi/3)}{(2s+1)^2} \text{Cos}(s\pi x)$

ADDITIONAL PROBLEMS

1. Find the Fourier Transform of $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

Hence evaluate $\int_0^{\infty} f(x) \, e^{isx} \, dx$

Ans: $\begin{cases} 2[(\text{Sin } s) / s] & s \neq 0 \\ 2 & s = 0 \end{cases}$

2. Find the Fourier Transform of $f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0 , & |x| > 1 \end{cases}$

Ans: $4/s^3 (\text{Sin } s - s \cos s)$

3. Find Fourier Sine Transform of $e^{-|x|}$. Hence show that $\int_0^{\infty} \frac{x \sin mx}{1 + x^2} \, dx = \pi e^{-m} / 2 \; (m > 0)$

Ans: $s / (s^2 + 1)$

4. Find the Finite Fourier Sine Transform of $f(x) = 1$ in $0 < x < \pi$
Ans: $\frac{1-(-1)^s}{s}$

5. Find the Finite Fourier Cosine Transform of $f(x) = 1$ in $0 < x < \pi$
Ans: 0

6. Find the Finite Fourier Sine Transform and Finite Fourier Cosine Transform of $f(x) = x$ in $(0, \pi)$
Ans: Finite Fourier Sine Transform = $[\pi(-1)^{s+1}]/s$

Finite Fourier Cosine Transform = $\begin{cases} \left(\frac{(-1)^s - 1}{s^2}\right), & s = 1, 2, 3, \ldots\ldots \\ 0, & s = 0 \end{cases}$

7. Find Finite Fourier Sine Transform of $f(x) = 2x$ in $0 < x < 4$.
Ans: $(32 / n\pi)(-1)^{n+1}$

8. Find Fourier Cosine Transform of $f(x) = [1 - (x/\pi)]^2$ in $(0, \pi)$
Ans: $(2/\pi)(1/s^2)$

9. Find the Fourier Cosine Transform of f(x) if $f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ -1 & \pi/2 < x < \pi \end{cases}$

Ans: $\begin{cases} 2/r \operatorname{Sin}(r\pi/2) & r > 0 \\ 0 & r = 0 \end{cases}$

10. If F(s) is the Fourier transform of f(x) then F.T of [f(ax)] =
(A) F (s/a) (B) F(as) (C) (1/a) F (s/a) (D) a F (s/a)

11. If F(s) is the Fourier transform of f(x) then F{f(x-a)} =
(A) $e^{isa} f(s)$ (B) $e^{-ias} f(s)$ (C) $e^{ias} f(as)$ (D) $e^{ias} f(s/a)$

12. The finite fourier sine transform of f(x) in the interval $0 < x < l$ is given by

(A) $(1/l)\int_0^l f(x)\sin(n\pi x/l)\,dx$ (B) $\int_0^l f(x)\sin(n\pi x/l)\,dx$

(C) $(2/l)\int_0^l f(x)\sin(n\pi x/l)\,dx$ (D) $(2/l)\int_0^l f(x)\sin(n\pi x)\,dx$

KEY

10. c 11. a 12. b

Z – TRANSFORMS

Def:- If the function u_n is defined for discrete values (n = 0, 1, 2,….) and $u_n = 0$ for $n<0$, then its Z-transform is defined to be

$$Z(u_n) = U(z) = \Sigma u_n z^{-n} \text{ whenever the infinite series converges.}$$

The inverse Z – transform is written as $Z^{-1}[U(z)] = u_n$.

Linearity Property :

If a, b, c be any constants and u_n, v_n, w_n be any discrete functions, then
$$Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$$

Damping Rule

If $Z(u_n) = U(z)$, then $Z(a^{-n} u_n) = U(az)$

Cor. $Z(a^n u_n) = U(z/a)$

Shifting u_n to the Right

If $Z(u_n) = U(z)$, then $Z(u_{n-k}) = z^{-k} U(z)$ for $k > 0$

Shifting u_n to the Left

If $Z(u_n) = U(z)$, then

$$Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \ldots - u_{k-1} z^{-(k-1)}]$$

TWO BASIC THEOREMS

(1) Initial value theorem: If $Z(u_n) = U(z)$, then $u_0 = \underset{z\to\infty}{\text{Lt}}\, U(z)$

(2) Final value theorem : If $Z(u_n) = U(z)$, then $\underset{n\to\infty}{\text{Lt}}\, U(n) = \underset{z\to 1}{\text{Lt}}\, (z-1)\,U(z)$

SOME USEFUL Z-TRANSFORMS

S. No	Sequence u_n $(n \geq 0)$	Z – transform : U (z)
1.	n	$z/(z-1)^2$
2.	n^2	$(z^2+z)/(z-1)^3$
3.	n^p	$-z\,(d/dz)\,[Z\,(n^{p-1})]$, p + ve integer.
4.	$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$	1
5.	$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$	$z/(z-1)$
6.	a^n	$z/(z-a)$
7.	na^n	$az/(z-a)^2$
8.	n^2a^n	$(az^2 + a^2z)/(z-a)^3$
9.	$\sin n\theta$	$\dfrac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$
10.	$\cos n\theta$	$\dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$
11.	$a^n \sin n\theta$	$\dfrac{a z \sin\theta}{z^2 - 2az\cos\theta + a^2}$
12.	$a^n \cos n\theta$	$\dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$
13.	$\sinh n\theta$	$\dfrac{z \sinh\theta}{z^2 - 2z\cosh\theta + 1}$
14.	$\cosh n\theta$	$\dfrac{z(z - \cosh\theta)}{z^2 - 2z\cosh\theta + 1}$
15.	$a^n \sinh n\theta$	$\dfrac{a z \sinh\theta}{z^2 - 2az\cosh\theta + a^2}$
16.	$a^n \cosh n\theta$	$\dfrac{z(z - a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}$
17.	$a^n u_n$	$U(z/a)$
18.	u_{n+1}	$z\,[U(z) - u_0]$
	u_{n+2}	$z^2\,[U(z) - u_0 - u_1z^{-1}]$
	u_{n+3}	$z^3\,[U(z) - u_0 - u_1z^{-1} - u_2z^{-2}]$
19.	$u_n - k$	$z^{-k}\,U(z)$
20.	$n\,u_n$	$-z\,(d/dz)\,[U(z)]$

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21.	$(1/n)\,u_n$	$-\int_0^z z^{-1}\,U(z)\,dz$
22.	u_0	$\underset{z\to\infty}{\mathrm{Lt}}\,U(z)$
23.	$\underset{n\to\infty}{\mathrm{Lt}}\,(u_n)$	$\underset{z\to 1}{\mathrm{Lt}}\,[(z-1)\,U(z)]$

SOME USEFUL INVERSE Z-TRANSFORMS

S. No	U(z)	Inverse z-transform u_n
1.	$\dfrac{z}{z-a}$	a^n
2.	$\dfrac{z^2}{(z-a)^2}$	$(n+1)\,a^n$
3.	$\dfrac{z^3}{(z-a)^3}$	$(1/2!)\,(n+1)\,(n+2)\,a^n$
4.	$\dfrac{1}{z-a}$	$a^{n-1}\,u(n-1)$
5.	$\dfrac{1}{(z-a)^2}$	$(n-1)\,a^{n-2}\,u(n-2)$
6.	$\dfrac{1}{(z-a)^3}$	$1/2\,(n-1)\,(n-2)\,a^{n-3}\,u(n-3)$

CONVOLUTION THEOREM

If $Z^{-1}\,[U(z)] = u_n$ and $Z^{-1}\,[V(z)] = v_n$, then

$$Z^{-1}\,[U(z)\,.\,V(z)] = \sum_{m=0}^{n} u_m\,.\,v_{n-m} = u_n * v_n$$

PROBLEMS

1. Find $Z\{a^n/n!\}$ Ans: $e^{a/z}$

2. Find $Z\{1/n!\}$ Ans: $e^{1/z}$

3. Find $Z\{na^{n-1}\}$ Ans: $\dfrac{z}{(z-a)^2}$

4. Find $Z\{1/(n+1)\}$ Ans: $-z\log(1 - 1/z)$

5. Find $Z\{\sin(n\pi/2)\}$ and $Z\{\cos(n\pi/2)\}$ Ans: $\dfrac{z}{z^2+1}$ & $\dfrac{z^2}{z^2+1}$

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6. $Z\{\text{Cosh}n\theta\}$ Ans: $\dfrac{z(z-\text{Cosh}\theta)}{z^2-2z\cosh\theta+1}$

7. $Z\{\text{Sinh}n\theta\}$ Ans: $\dfrac{z\,\text{Sinh}\theta}{z^2-2z\text{Cosh}\theta+1}$

8(a). $Z\left\{\dfrac{a^n e^{-n}}{n!}\right\}$ Ans: $e^{a/(ez)}$

8(b). $Z\left\{\dfrac{a^n e^{-a}}{n!}\right\}$ Ans: $e^{a(1-z)/z}$

9. $Z\{\text{Cos}(n+1)\theta\}$ Ans: $\dfrac{(z\text{Cos}\theta-1)z}{z^2-2z\text{Cos}\theta+1}$

10. If $Z\{f_n\}=\dfrac{z^2}{z^2-5z+6}$. Find $Z\{f_{n+2}\}$. Ans: $\dfrac{19z^2-30z}{z^2-5z+6}$

11. If $Z\{U_n\}=\dfrac{2z^2+4z+12}{(z-1)^3}$. Find U_2 and U_3 Ans: 2 and 12

12. $Z\{(n+1)^2\}$ Ans: $\dfrac{z^3+z^2}{(z-1)^3}$

13. $Z\{n\,\text{Cos}n\theta\}$ Ans: $\dfrac{z^3\text{Cos}\theta-2z^2+z\text{Cos}\theta}{(z^2-2z\text{Cos}\theta+1)^2}$

14. $Z\{(n-1)^2\}$ Ans: $\dfrac{z^3-3z^2+4z}{(z-1)^3}$

15. If $Z\{U(z)\}=\dfrac{2z^2+5z+14}{(z-1)^4}$, prove that $U_0=0$, $U_1=0$, $U_2=2$ and $U_3=13$.

16. $Z\{\text{Cos}^2(n\pi/4)\}$ Ans: $\dfrac{1}{2}\left\{\dfrac{z}{z-1}+\dfrac{z^2}{z^2+1}\right\}$

17. $Z\{\text{Sin}^3(n\pi/6)\}$ Ans: $\dfrac{3}{4}\left\{\dfrac{z\text{Sin}\pi/6}{z^2-2z\text{Cos}(\pi/6)+1}\right\}-\dfrac{1}{4}\,\dfrac{z}{z^2+1}$

18. $Z\{\text{Sin}[(n\pi/2)+\alpha]\}$ Ans: $\dfrac{z^2\text{Sin}\alpha+z\text{Cos}\alpha}{z^2+1}$

19. $Z\{\text{Cosh}(an)\,\text{Sin}bn\}$

Ans: $\dfrac{z\text{Sin}b}{2}\left\{\dfrac{e^{-a}}{z^2e^{-2a}+2ze^{-a}\text{Cos}b+1}+\dfrac{e^{a}}{z^2e^{2a}-2ze^{a}\text{Cos}b+1}\right\}$

20. $Z\left\{\dfrac{1}{(n+1)(n+2)}\right\}$ Ans: $(z^2-z)\log(1-1/z)+z$

21. $Z\{1/n(n-1)\}$ Ans: $(1/z)\log[z/(z-1)]$

22. $Z\left\{\dfrac{2n+3}{(n-1)(n+2)}\right\}$ Ans: $z(z+1)\log\{z/(z-1)\}-z$

Find the inverse Z – transforms of the following.

1. $\dfrac{z-4}{z^2+5z+6}$ Ans: $-6(-2)^{n-1}+7(-3)^{n-1}$

2. $\dfrac{z^2}{z^2-4z+3}$ Ans: $\frac{1}{2}(3^{n+1}-1)$

3. $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$ Ans: $2-2.\,2^n+3n.\,2^n$

4. $\dfrac{z-4}{(z-1)(z-2)^2}$ Ans: $3(2)^{n-1}-(n-1)2^{n-1}-3$

5. Use Convolution theorem, to evaluate $Z^{-1}\left\{\dfrac{z^2}{(z-a)(z-b)}\right\}$ Ans: $\dfrac{a^{n+1}-b^{n+1}}{(a-b)}$

6. Show that $\dfrac{1}{n!}*\dfrac{1}{n!}=\dfrac{2^n}{n!}$ and $\dfrac{1}{n!}*\dfrac{1}{n!}*\dfrac{1}{n!}=\dfrac{3^n}{n!}$

7. Using Convolution theorem, show that $Z^{-1}\left\{\left(\dfrac{z}{z-a}\right)^2\right\}=(n+1)\,a^n$

8. Find $Z^{-1}\left\{\left(\dfrac{z}{z-1}\right)^3\right\}$ (using Convolution theorem) Ans: $(\frac{1}{2})(n+1)(n+2)$

9. Find $Z^{-1}\left\{\dfrac{8z-z^3}{(4-z)^3}\right\}$ Ans: $(n^2+7n+4)\,4^{n-1}$

10. $Z^{-1}\left\{\dfrac{z}{(z-1)(z^2+1)}\right\}$ Ans: $(\frac{1}{2})-(\frac{1}{2})\text{Cos}(n\pi/2)-(\frac{1}{2})\text{Sin}(n\pi/2)$

11. $Z^{-1}\left\{\log\left(\dfrac{z}{z+1}\right)\right\}$ Ans: $f_n=\begin{cases}0, & n\le 0\\ (-1)^n/n, & n\ge 1\end{cases}$

12. $Z^{-1}\left\{\dfrac{z}{(z+1)^2}\right\}$ Ans: $f_n=(-1)^{n+1}.\,n$

13. $Z^{-1}\left\{\dfrac{z}{(z-1)^2}\right\}$ Ans: $f_n=n$

14. $Z^{-1}\{\log[1+(a/z)]\}$ Ans: $f_n=\begin{cases}(-1)^{n+1}a^n/n, & n\ge 1\\ 0, & n\le 0\end{cases}$

OBJECTIVES

01. $Z\{1/n!\} =$
(A) e^z (B) $e^{1/z}$ (C) $e^{1/(z-1)}$ (D) $e^{1/z} - 1$

02. $Z\{\sin(n\pi/2)\}$
(A) $z/(z^2+1)$ (B) $z^2/(z^2+1)$ (C) $z/(z^2-1)$ (D) $z^2/(z^2-1)$

03. $Z(n^2 a^n)$
(A) $\frac{(az)^2+az}{(az-1)^3}$ (B) $\frac{a(z^2+az)}{(z-a)^3}$ (C) $\frac{(az)^2-az}{(az+1)^3}$ (D) $\frac{a(z^2+az)}{(z+a)^3}$

04. $Z^{-1}[z/(z+1)] =$
(A) 1 (B) $(-1)^n$ (C) $n.(-1)^n$ (D) n

05. Z – transform of unit impulse function is
(A) 1 (B) 0 (C) z (D) z^2

06. Z – transform of discrete unit step function is
(A) $z/(z-1)$ (B) $z/(z-1)^2$ (C) $z^2/(z-1)$ (D) $z^2/(z-1)^2$

07. If $z(u_n) = \overline{u}(z)$ then $u_0 =$
(A) $\underset{z\to\infty}{Lt}\ \overline{u}(z)$ (B) $\underset{z\to 1}{Lt}\ \overline{u}(z)$ (C) $\underset{z\to\infty}{Lt}\ z\,\overline{u}(z)$ (D) $\underset{z\to 1}{Lt}\ z\,\overline{u}(z)$

08. Which one of the following is incorrect
(A) $\underset{z\to\infty}{Lt}\ z^2\,[\overline{u}(z) - u_0 - u_1/z] = u_2$ (B) $\underset{z\to 1}{Lt}\ u_n = \underset{z\to\infty}{Lt}\ (z-1)\overline{u}(z)$
(C) $z(u_{n+2}) = z^2\,[\overline{u}(z) - u_0 - u_1/z]$ (D) $z(u_{n+1}) = z[\overline{u}(z) - u_0]$

9. Z – transforms of (3^n) is
(A) $z/(z+3)$ (B) $z/(z-3)$ (C) $3z/(z-1)$ (D) $3/(z-3)$

10. Z – transform of $(-4)^n$ is
(A) $z/(z-4)$ (B) $z/(4-z)$ (C) $z/(z+4)$ (D) $(z-4)/z$

11. Z – transform of $(n.2^n) =$
(A) $2z/(z-1)^2$ (B) $2z/(z-2)^2$ (C) $z/(2z-1)^2$ (D) $2z/(z+2)^2$

12. Inverse z – transform of $2z/(z-1)$ is
(A) 2 (B) 1/2 (C) –2 (D) –1/2

13. If z – transform of $u_n = f(z)$ then z transform of U_{n-2} is
(A) $z^2 f(z)$ (B) $z^{-2} f(z)$ (C) $2z\ f(z)$ (D) $(z/2)\ f(z)$

14. If z – transform of $u_n = (2z^2 - 3z)/(3z^2+4)$ then $u_0 =$
(A) 2/3 (B) 3/2 (C) 0 (D) –3/4

KEY

1. B 2. A 3. B 4. B 5. A 6. A 7. A 8. B 9. B 10. C 11. B 12. A
13. B 14. A

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PREVIOUS GATE QUESTIONS

01. The Laplace transform of a function f(t) is $F(s) = \frac{5s^2+23s+6}{s(s^2+2s+2)}$. as $t \to \infty$, f(t) approaches **(EE-2005-2M)**
(a) 3 (b) 5 (c) 17/2 (d) ∞

02. Laplace transform of $f(t) = \cos(pt+q)$ is **(PI-2005-2M)**
(a) $\frac{s\cos q - p\sin q}{s^2+p^2}$ (b) $\frac{s\cos q + p\sin q}{s^2+p^2}$
(c) $\frac{s\sin q - p\cos q}{s^2+p^2}$ (d) $\frac{s\sin q + p\cos q}{s^2+p^2}$

03. If F(s) is the Laplace transform of the function f(t) than Laplace transform of $\int_0^t f(x)\,dx$ is **(ME-2007-2M)**
(a) $\frac{1}{s}F(s)$ (b) $\frac{1}{s}F(s) - f(0)$
(c) $s\,F(s) - f(0)$ (d) $\int F(s)ds$

04. Laplace transform of $8t^3$ is **(PI-2008-1M)**
(a) $\frac{8}{s^4}$ (b) $\frac{16}{s^4}$
(c) $\frac{24}{s^4}$ (d) $\frac{48}{s^4}$

05. Laplace transform of sin ht is **(PI-2008-2M)**
(a) $\frac{1}{s^2-1}$ (b) $\frac{1}{1-s^2}$
(c) $\frac{s}{s^2-1}$ (d) $\frac{s}{1-s^2}$

06. Laplace transform of $f(x) = \cosh(ax)$ is **(CE-2009-2M)**
(a) $\frac{a}{s^2-a^2}$ (b) $\frac{s}{s^2-a^2}$
(c) $\frac{a}{s^2+a^2}$ (d) $\frac{s}{s^2+a^2}$

07. Given that F(s) is the one-sided Laplace transorm of f(t), the Laplace transform of $\int_0^t f(\tau)\,d\tau$ is **(EC-2009-2M)**
(a) $s\,F(s) - f(0)$ (b) $\frac{1}{s}F(s)$
(c) $\int_0^s f(\tau)\,d\tau$ (d) $\frac{1}{s}[F(s) - f(0)]$

08. The inverse Laplace transform of $\frac{1}{(s^2+s)}$ is **(ME-2009-1M)**
(a) $1 + e^t$ (b) $1 - e^t$
(c) $1 - e^{-t}$ (d) $1 + e^{-t}$

09. Given $f(t) = L^{-1}\left[\frac{3s+1}{s^3+4s^2+(k-3)s}\right]$.
If $\underset{t\to\infty}{Lt}\ f(t) = 1$ then value of k is **(EE-2010-2M)**
(a) 1 (b) 2 (c) 3 (d) 4

10. The Laplace trnaform of f(t) is $\frac{1}{s^2(s+1)}$. The function **(ME-2010-2M)**
(a) $t - 1 + e^{-t}$ (b) $t + 1 + e^{-t}$
(c) $-1 + e^{-t}$ (d) $2t + e^t$

11. u(t) represents the unit step function. The Laplace transform of $u(t-\tau)$ is **(IN-2010-1M)**
(a) $\frac{1}{s\tau}$ (b) $\frac{1}{s-\tau}$
(c) $\frac{e^{-s\tau}}{s}$ (d) $e^{-s\tau}$

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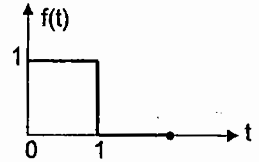
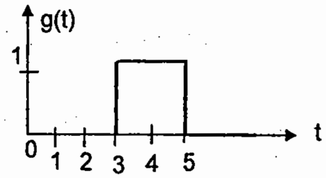
12. Given f(t) and g(t) as shown below
(EE-2010-(2M+2M))





(i) g(t) can be expressed as

(a) $g(t) = f(2t-3)$
(b) $g(t) = f(t/2-3)$
(c) $g(t) = f(2t-3/2)$
(d) $g(t) = f(t/2-3/2)$

(ii) The Laplace transform of g(t) is

(a) $\frac{1}{s}(e^{3s}-e^{5s})$ (b) $\frac{1}{s}[e^{-5s}-e^{-3s}]$

(c) $\frac{e^{-3s}}{s}[1-e^{-2s}]$ (d) $\frac{1}{s}[e^{5s}-e^{3s}]$

13. If u(t) is the unit step and δ(t) is the unit impulse function, the inverse z-transform of $F(z) = \frac{1}{Z+1}$ for $k \geq 0$ is
(EE-2005-2M)

(a) $(-1)^k\,\delta(k)$ (b) $\delta(k)-(-1)^k$
(c) $(-1)^k\,u(k)$ (d) $u(k)-(-1)^k$

KEY:

01. a	02. a	03. a
04. d	05. a	06. b
07. b	08. c	09. d
10. a	11. c	12. d
13. c		

TOPIC – 8 COMPLEX VARIABLES

Complex Variables (Complex Numbers):

A Complex number may be defined as an ordered pair (x, y), where x and y are real numbers. Which are called real and imaginary parts of Z respectively.

Thus the numbers of the form $Z = x+iy$ where $x, y \in R$(Real set) are called complex numbers. The conjugate of a complex number $Z= x+iy$ is $\bar{Z} = x - iy$.

Polar Form of Complex Numbers:

Put $x = r\cos\theta$, $y = r\sin\theta$ in $Z = x + iy$. Then $Z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.
Thus $Z = re^{i\theta}$ represents the polar form of the complex number $Z = x+iy$ and $\bar{Z} = re^{-i\theta}$. Where $r = \sqrt{x^2+y^2}$ is called the modulus and $\theta = \tan^{-1}(y/x)$ is called the amplitude of the complex number.

The particular value of 'θ' satisfying the equation $\tan\theta = y/x$ and which lies between $-\pi$ and π is called the principal value of the amplitude.

Properties of Moduli and amplitude of complex numbers:

a) Modulus of sum or difference of two complex numbers is always less than or equal to their sum of their moduli.
$|Z_1+Z_2| \leq |Z_1| + |Z_2|$
$|Z_1 - Z_2| \leq |Z_1| + |Z_2|$

b) The modulus of difference or sum of two complex numbers is always greater than or equal to the difference of their moduli.
$|Z_1 - Z_2| \geq |Z_1| - |Z_2|$
$|Z_1+Z_2| \geq |Z_1| - |Z_2|$

c) The modulus of the product of two complex numbers is equal to the product of their moduli is
$|Z_1Z_2Z_3\ldots\ldots\ldots Z_n| = |Z_1|\,|Z_2|\,|Z_3|\,\ldots\ldots\ldots\,|Z_n|$

d) The modulus of the quotient of two complex numbers is equal to the quotient of their moduli i.e., $|Z_1/Z_2| = |Z_1| / |Z_2|$

e) The amplitude of the product of two complex numbers is equal to their sum of the amplitudes \ is if $Z_1 = r_1e^{i\theta}$ and $Z_2 = r_2e^{i\alpha}$
Then $|Z_1Z_2| = r_1r_2\,e^{i(\theta+\alpha)}$

f) The amplitude of the quotient of two complex numbers is equal to the difference of their amplitudes i.e., $|Z_1/Z_2| = r_1/r_2\,e^{i(\theta-\alpha)}$

g) If the amplitude is π/2. Then the complex number is purely imaginary, but if the amplitude is 0 or π. Then the complex number is purely real.

Neighbourhood of a point Z_0: The set of all points within the circle whose center is Z_0, is called neighbourhood of a point.

Analytical Functions:

A complex valued function f(Z) is said to be analytic at a point Z_0 if f(Z) is differentiable not only at Z_0, but at every point of some neighbourhood of Z_0.

A complex valued function f(Z) is analytic in a region R, if $f^1(Z)$ exists at every point of the region R.

Note: An analytic function is also known as regular, holomorphic, monogenic function.

Entire Function: A function f(Z) which is analytic everywhere in the complex plane (argand plane) is called an entire function.

Cauchy – Riemann(C - R) Equations: The Cauchy – Riemann equations are applied to determine whether the given complex valued function f(Z) = u + iv is analytic or not.

1. The C.R equations are given by
   $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$
   or $u_x = v_y$ and $u_y = -v_x$

2. If f(Z) is defined in polar form then the C.R equations in polar form are given by
   $\partial u/\partial r = 1/r\ \partial v/\partial\theta$ and $\partial v/\partial r = -1/r\ \partial u/\partial\theta$
   or $v_\theta = ru_r$ and $u_\theta = -r\,v_r$

3) **Harmonic functions:** Any function $\phi(x, y)$ satisfying the Laplace's equation $\nabla^2\phi = 0$.
   i.e., $(\partial^2\phi / \partial x^2) + (\partial^2\phi/\partial y^2) = 0$ is called an harmonic function.

Theorem: If f(Z) = u+iv is an analytic function. Then the real and imaginary parts u and v satisfy Laplace's equation. (i.e, u and v are harmonic).
Thus $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$ and $(\partial^2 v/\partial x^2) + (\partial^2 v/\partial y^2) = 0$.

Note: The polar form of laplace's equations are given by $r^2u_{rr} + ru_r + u_{\theta\theta} = 0$ and
$r^2v_{rr} + rv_r + v_{\theta\theta} = 0$.

Properties of Analytic Function:

1. If f(Z) and g(Z) are analytic, then $f(Z) \pm g(Z)$; f(Z) g(Z); f(Z)/g(Z) $[g(Z) \neq 0]$; are also analytic.

2. If f(Z) is analytic, then it is differentiable and continuous,
   i.e., (analyticity $\Rightarrow$ differentiability $\Rightarrow$ continuity).

3. The derivative of an analytic function is also an analytic function.

4. If f(Z) = u+iv is analytic, then the family of curves defined by $u(x,y) = C_1$ and $v(x,y) = C_2$ form an orthogonal system. i.e., $u(x,y) = C_1$ are orthogonal trajectories of $v(x,y) = C_2$ and vice versa.

5. The real part u(x,y) of an analytic function f(Z) = u(x,y) + iv(x,y) is known as the conjugate harmonic function and is uniquely determined upto an arbitrary real additive constant.

Note: The harmonic conjugate here is not to be confused with the conjugate $Z = \bar{x} - iy$.

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CONSTRUCTION OF ANALYTIC FUNCTIONS / DETERMINATION OF CONJUGATE FUNCTION

I) Method (Total Derivative Method):

a) Let the real part u(x,y) of an analytic function f(Z) = u+iv be given.
Then to find v(x,y) we proceed as follows : Since v is a function of x and y. The total derivative of v is given by

$$dv = \partial v/\partial x \,.\, dx + \partial v/\partial y \,.\, dy \qquad \ldots(1)$$

$\therefore$ By using C.R equations $u_x = v_y$ & $u_y = -v_x$ equation (1) can be written as

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad \ldots\ldots\ldots(2)$$

The R.H.S of the above equation is of the form M dx + N dy where $M = -\partial u/\partial y$ & $N = \partial u/\partial x$
$\therefore\ \partial M/\partial y = -\partial^2 u/\partial y^2$ and $\partial N/\partial x = \partial^2 u/\partial x^2$
Since u is harmonic, it satisfies Laplace equation. i.e., $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0 \Rightarrow$
$\partial M/\partial y = \partial N/\partial x$
Hence equation (2) is an exact differential equation. Therefore integrating (2) both sides we get $V = \int(-u_y)dx + \int u_x\,dy + C$
Note: While integrating $\int(-u_y)dx$ keep 'y' as constant and while integrating $\int u_x\,dy$. Integrate only those terms which are independent of 'x' (i.e., only those terms of u_x which do not contain x). If equation (2) is not exact, it may be reducible to exact.

b) Similarly when v(x,y) is given. Then to find u(x,y). We use the total derivative of u

$$du = \frac{\partial u}{\partial x}du + \frac{\partial u}{\partial y}dy$$

$$du = v_y\,dx - v_x\,dy$$

$$\Rightarrow u = \int v_y\,dx - \int v_x\,dy + C$$

Thus the analytic function is f(Z) = u+iv

Examples:

1) If the real part of an analytic function is given by $u(x,y) = x^2 - y^2 - y$. Then find v(x,y)
Given $u(x,y) = x^2 - y^2 - y$
$\partial u/\partial x = 2x \quad \partial u/\partial y = 2y - 1$
$\therefore\ dv = \int(-\partial u/\partial y)\ dx + \int \partial u/\partial x\ dy$
$dv = \int(2y+1)dx + \int 2x\,dy$ (Exact equation)
$dv = 2xy + x$
Thus $f(Z) = u + iv = (x^2-y^2-y) + i(2xy + x)$.

2. If the imaginary part $v(x, y) = y^2 - x^2$. Find u(x, y)
$du = (\partial v / \partial y)\ dx - (\partial v / \partial x)\,dy$
$\int du = \int 2y\,dx + \int 2x\,dy$ (exact equation)
$u = 2xy$ $\quad\therefore\ f(Z) = u+iv = 2xy + i(y^2 - x^2) = iZ^2$

3. If $u = e^x\,Cosy$ find v(x,y)
$dv = \int(-\partial u / \partial y)\ dx + \int(\partial u / \partial x)\,dy$
$\int dv = \int e^x Siny\,dx + \int e^x\,Cosy\,dy$ (Exact equation)
$v = e^x\,Siny$
$\therefore f(Z) = e^x Cosy + ie^x\,Siny$
$= e^x\,(Cosy + i\,Siny) = e^Z$.
Thus $f(Z) = e^Z$.

Milne - Thompson Method:

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In this method, we get f(Z) in terms of 'Z', irrespective of whether u(x,y) or v(x,y) is given.

I. If u(x,y) is given

Take $f^1(Z) = u_x - i\, u_y$

Replace x by Z and y by 0 in $f^1(Z)$.

Then integrate $f^1(Z)$ with respect to Z.

II. If v(x,y) is given

Take $f^1(Z) = v_y + iv_x$

Replace x by Z and y by 0 in $f^1(Z)$.

Then integrate $f^1(Z)$ with respect to 'Z'.

Examples:

1. If $u = (1/2)\log(x^2+y^2)$. Find the analytic function f(Z).

Let $f^1(Z) = u_x - i\, u_y$

$u_x = x/x^2+y^2 \quad u_y = y/x^2+y^2$

$\therefore f^1(Z) = x/x^2+y^2 - i\, y/x^2+y^2$

put x = Z and y = 0. Then we get

$f^1(Z) = 1/Z + 0$

$\therefore \int f^1(Z) = \int (1/Z)\, dZ$

$f(Z) = \log Z$

2. If the real part u(x,y) of an analytic function f(Z) is $e^x(x\,\text{Cos}y - y\,\text{Sin}y)$. Find f(Z).

$u_x = e^x\,\text{Cos}y + e^x(x\,\text{Cos}y - y\,\text{Sin}y)$

$u_y = e^x\,(-x\,\text{Sin}y - \text{Sin}y - y\,\text{Cos}y)$

Put x = Z and y = 0 in the above equations Then

$u_x = Z\,e^Z + e^Z = e^Z(Z+1) \quad ; \quad v_x = 0$

$\therefore f^1(Z) = e^Z(Z+1)$

$f(Z) = \int e^Z(Z+1)\, dZ = Ze^Z$

EXERCISE – I

1. For what values of a , b and c, the function defined by $f(Z) = (x+ay) + i\,(bx+cy)$ is analytic.
   a) a = 1, b = 2, c = 3 b) a = -b, c = 1 c) a = b = c = 1 d) none of the above

2. For what values of a,b,c and d, the function defined by $f(Z) = (x^2+axy+by^2) + i\,(cx^2+dxy+y^2)$ is analytic.
   a) a = d = 2,b = c = -1 b) a = c = -1,b = d = 3
   c) a = 2,b = 1,c = 2,d = 1 d) none of the above

3. The value of 'P' for which the function defined in polar form as $f(Z) = r^2\,\text{Cos}2\theta + ir^2\,\text{Sin}P\theta$ is analytic.
   a) P = 0 b) P = 2 c) P = -1 d) none

4. The value of 'P' for which the function is defined by $f(x) = (1/2)\log(x^2+y^2) + i\tan^{-1}(Px/y)$ is analytic
   a) P = 1 b) P = 0 c) P = -1 d) none

5. An electrostatic field in the xy – plane is given by the potential function $\phi = x^2 - y^2$. Then the stream function 'ψ' is
   a) x^2 b) 2xy c) –2xy d) x^3

6. If the potential function is $\log(x^2+y^2)$. Then the stream function is
   a) $\tan^{-1}(y/x)$ b) $2\tan^{-1}(y/x)$ c) $½\tan^{-1}(y/x)$ d) none

7. In the complex potential function $f(Z) = \phi + i\,\psi$. If $\psi = e^x\,\text{Sin}y$. Then ϕ is
   a) $e^{-x}\,\text{Sin}y$ b) $e^x\,\text{Cos}y$ c) $e^{-x}\,\text{Cos}y$ d) none

8. The analytic function f(Z) whose real part is $u = x^3 - 3xy^2$ is
   a) iZ^3 b) Z^3 c) Z^4 d) none

9. The analytic function f(Z) corresponding to its real part $u = 2x - x^3 + 3xy^2$ is
   a) $f(Z) = 2Z - Z^3$ b) $f(Z) = Z^3$ c) 2Z+2 d) none

10. If $u(x,y) = (x-1)^3 - 3xy^2+3y^2$. Then f(Z) in terms of 'Z' is
    a) Z^3 b) $(Z-1)^3$ c) Z^3+3Z^2 d) none

11. If $u(x,y) = e^x\,(x\text{Cos}y - y\text{Sin}y)$. Then f(Z) is
    a) Z^2e^Z b) Ze^Z c) e^Z d) none

12. If $u - v = e^x(\text{Cos}y - \text{Sin}y)$ and f(Z) is analytic then
    a) $f(Z) = e^Z$ b) e^{Z2} c) e^{2Z} d) none

13. The function $W = Z^2$ is
    a) Analytic everywhere b) analytic only at (0,0) c) no where analytic d) none

14. The function $f(Z) = e^Z$ is an
    a) entire function b) no where analytic c) analytic only at (0,0) d) none

15. The value of 'b' for which $u(x,y) = e^{bx}\text{Cos}5y$ is harmonic.
    a) b = ±2 b) b = ±5 c) b = 0 d) none

16. The orthogonal trajectory corresponding to the family of curves $v(r,\theta) = r^2\,\text{Cos}2\theta - r\,\text{Cos}\theta + 2$ is
    a) $r\text{Sin}\theta + C$ b) $-r^2\,\text{Sin}2\theta + r\,\text{Sin}\theta + C$ c) $re^{\text{Sin}\theta}$ d) none

17. If $u = e^{-\theta}\,\text{Cos}(\log r)$ is harmonic. Then $v(r,\theta)$ is
    a) $e^{\theta}\,\text{Sin}(\log r)$ b) $e^{-\theta}\log(\text{Cos}\, r)$ c) $e^{-\theta}\,\text{Sin}(\log r)$ d) none

18. If $u = -r^3\,\text{Sin}3\theta$ is harmonic. Then its harmonic conjugate $v(r,\theta)$ is
    a) $r^3\,\text{Sec}\theta$ b) $r^3\,\text{Cos}3\theta$ c) $r^2\,\text{Sin}3\theta$ d) none of the above

19. The analytic function f(Z) is such that $\text{Re}\{f^1(Z)\} = 3x^2 - 4y - 3y^2$ and $f(1+i) = 0$. Then f(Z) is
    a) Z^3+2iZ^2+6-2i b) Z^2+2iZ c) Z^3+2iZ^2-6 d) none

20. If $\text{Im}\{f^1(Z)\} = 6x(2y-1)$ and $f(0) = 3 - 2i$; $f(1) = 6 - 5i$. Then $f(1+i)$ is
    a) 6+3i b) 3i – 6 c) 0 d) none

KEY

1. b	2. a	3. b	4. c	5. b	6. b	7. b	8. b	9. a	10. b	11. b	12. a
13. a	14. a	15. b	16. b	17. c	18. b	19. a	20. a				

Complex Line Integrals:

Let f(Z) be a complex valued function. Then the complex line integral of f(Z) along the given path 'C' is denoted by $\int_C f(Z)\,dZ$. Where 'C' is the given path of integration.

Note: Every complex line integral can be expressed as a real line integral.

If $f(Z) = u+iv$ and $dZ = dx + i\,dy$.

Then $\int_C f(Z)\,dZ = \int_C (u+iv)(dx+i\,dy) = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$

Properties of Line Integrals

1. Linearity property:

$$\int_C [k_1 f(Z) + k_2 f(Z)]\,dZ = k_1\int_C f(Z)\,dZ + k_2\int_C f(Z)\,dZ$$

2. $\int_a^b f(Z)\,dZ = -\int_b^a f(Z)\,dZ$

3. Partitioning of paths: If C is expressed as $C_1 + C_2$

$$\int_C f(Z)\,dZ = \int_{C_1} f(Z)\,dZ + \int_{C_2} f(Z)\,dZ$$

4. ML – Inequality:

$\left|\int_C f(Z)\,dZ\right| \le ML$ where $|f(Z)| \le M$ every where on 'C' and L is the length of the curve 'C'.

Note: If the integrand f(Z) is analytic. Then the line integral is independent of the path of integration.

Examples:

1. Evaluate $\int_0^{(1+i)} (x^2 - iy)\,dZ$ along a) $y = x$ & b) $y = x^2$

a) Along $y = x$; $dy = dx$. $dz = dx + i\,dx = (1+i)\,dx$

Also $x^2 - iy = (x^2 - ix)$

$$\therefore \int_0^{(1+i)} (x^2 - iy)\,dZ = \int_0^1 (x^2 - ix)(1+i)\,dx = (1+i)\int_0^1 (x^2 - ix)\,dx$$

$$= (1+i)\left[x^3/3 - ix^2/2\right]_0^1 = (1+i)[1/3 - i/2]$$

$$= (1+i)(2-3i)/6$$

$$= (5-i)/6$$

b) Along $y = x^2 \Rightarrow dy = 2x\,dx$.

$\therefore dZ = dx + i2x\,dx = (1+2ix)\,dx$

$$\int_0^{(1+i)} (x^2 - iy)\,dZ = \int_0^1 (x^2 - ix^2)(1+2ix)\,dx = (1-i)\int_0^1 x^2(1+2i\,x)\,dx$$

$$= (1-i)\left[x^3/3 + 2i(x^4/4)\right]_0^1 = (1-i)(2+3i)/6 = (5+i)/6$$

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EXERCISE – II

1. $\int_{3i}^{(1-i)} 4Z\,dZ =$ ----

a) $16 - 4i$ b) $18 - 4i$ c) 0 d) none

2. The value of the integral $\int_0^{(3+i)} Z^2\,dZ$ along $y = x/3$ is

a) $(6 + 26i)/3$ b) $(6 + 21i)/3$ c) 0 d) none

3. The integral $\int_0^{(4+2i)} Z\,dZ$ is

a) Is independent of the path joining 0& 4+2i

b) not independent of the path joining 0 & 4+2i

c) The value of the integral is zero d) none of the above

4. $\oint_C (Z+1)\,dZ$ where 'C' is the boundary of the square with the vertices at the points Z=0, Z=1, Z=1+i & Z=i is

a) $2\pi i$ b) 0 c) 3i d) none

5. $\int_C dZ/(Z-a)$ where 'C' is the circle $|Z - a| = r$ is

a) 0 b) πi c) $2\pi i$ d) none

6. $\int_C (Z-a)^n\,dZ$ where 'c' is $|Z - a| = r$ is

a) 0 b) $2\pi i$ c) $2^n(\pi i)^n$ d) none

7. The value of the integral $\int_C (Z - Z^2)\,dZ$ where 'C' is the upper half of the circle $|Z - 2| = 3$ Is

a) 29 b) 15 c) 30 d) none

8. The value of the integral $\int \mathrm{Re}(Z)\,dZ$, if C is the shortest path joining the points 1+i to 3+2i is

a) 3+2i b) 4+2i c) $-4 - 2i$ d) none

9. The upper bound for the absolute value of the integral $\int (e^Z - Z)\,dZ$. Where 'C' is the boundary of the triangle with the vertices at the points Z = 0, -4 and 3i is

a) 3 b) 2 c) 4 d) none

10. The value of $\int_C (Z^2 - Z + 2)\,dZ$. Where 'C' is the upper half of the circle $|Z| = 1$ taken in the positive direction is

a) $-1/3$ b) $-14/3$ c) 18/13 d) none

11. $\int_0^i Z\,e^{Z^2}\,dZ$ is

a) $\frac{1}{2}[1 - 1/e]$ b) $\frac{1}{2}[e^{-1} - 1]$ c) 0 d) none

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12. $\int_{0}^{\pi i} \cos Z \, dZ$

a) $\sin\pi$ b) $i \sin \pi$ c) $i \sin h\pi$ d) none

13. $\int_{0}^{2i} \sin hZ \, dZ$ is

a) $\cos h2 - 1$ b) $\cos 2 - 1$ c) $\cos 2$ d) none

14. $\int_{0}^{2\pi} Z^2 \sin 4Z \, dZ$ is ______

a) $2i$ b) 2π c) $-\pi^2$ d) 0

15. $\int_{-i}^{+i} Z \cos hZ^2 \, dZ$ is

a) π b) πi c) 0 d) none

16. $\int_{1}^{1+\pi i} e^Z \, dZ$ is

a) e b) $-2e$ c) $2\pi ie$ d) none

17. The upper bound for the absolute value of the integral $\int_C Z^2 \, dZ$, 'C' is the straight line from 0 to 1+i.

a) $2\sqrt{2}$, M = 2, L = $\sqrt{2}$ b) $2\sqrt{3}$, M=3, L=1

c) $-2\sqrt{2}$, M=1, L=3 d) none of the above

18. $\int_C dZ/Z^2+1$ where 'C' is $|Z| = 2$ in the first quadrant. Then the value of M_L is

a) $\pi/3$ b) $\pi/2$ c) 0 d) none

KEY:

1. b	2. a	3. a	4. b	5. c	6. a	7. c	8. b	9. c	10. b	11. b	12. c
13. b	14. c	15. c	16. b	17. a	18. a						

Cauchy's Integral Theorem:

If f(Z) is analytic in a simply connected region 'R' bounded by a closed curve 'C'.

$\oint_C f(Z) \, dZ = 0$

Example:

1. Evaluate $\oint (5Z^4 - Z^3 + 2) \, dZ$ around

a) $|Z| = 1$ b) square with vertices (0,0) (1,0) (1,1) (0,1)

Solution: The function $f(Z) = 5Z^4 - Z^3 + 2$ is analytic everywhere. Therefore by Cauchy's integral theorem $\oint_C f(Z) \, dZ = 0$. Since the given regions are closed regions.

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Cauchy's Integral Formula:

If f(Z) is analytic inside and on a simple closed curve 'C' and 'a' is any point inside 'C' then $f(a) = 1/2\pi i \int_C f(Z)/(Z-a) \, dZ$

Note: The cauchy's integral formula expresses the value of an analytic function at an interior point of a region 'R' in terms of its values on the boundary of the region.

Cauchy's inequality: If f(Z) is analytic inside and on a circle 'C' of radius 'r' and centre at Z= a. Then $|f^n(a)| \le M \cdot n! / r^n$ for n = 0, 1, 2, ... etc.

Where M is a constant such that $|f(Z)| < M$ on 'C', i.e., M is an upper bound of $|f(Z)|$ on C.

Note: The Cauchy's integral formula for the n^{th} derivative at Z = a is given by

$$f^n(a) = (n! / 2\pi i) \int_C f(Z) / (Z - a)^{n+1} \, dZ$$

Problems on Cauchy's integral formula:

1) Evaluate $\int_C e^{2Z} / (Z+1)^4$ where 'C' is the circle $|Z| = 3$.

Clearly the point Z = -1 lies inside $|Z| = 3$.

By the general formula $\int_C f(Z) / (Z-a)^{n+1} \, dZ = 2\pi i/n! \; f^n(a)$

Comparing with the given integral we have n = 3, a = -1

$f(Z) = e^{2Z}$

$f^{1}(Z) = 2e^{2Z}$; $f^{11}(Z) = 4e^{2Z}$

$f^{111}(Z) = 8 e^{2Z}$

$$\text{Hence } \int_C \frac{e^{2Z}}{(Z+1)^4} dZ = 2\pi i / 6 \times f^{111}(-1)$$

$$\int_C \frac{e^{2Z}}{(Z+1)^4} dZ = (8/3) \pi i e^{-2}$$

2. $\int_C \dfrac{[\sin\pi Z^2 + \cos\pi Z^2]}{(Z-1)(Z-2)} dZ$ where 'C' is the circle $|Z| = 3$.

$$\text{Since } \frac{1}{(Z-1)(Z-2)} = \frac{1}{Z-2} - \frac{1}{Z-1}$$

$$\therefore \int_C \frac{\sin\pi Z^2 + \cos\pi Z^2}{(Z-1)(Z-2)} dZ = \int \frac{\sin\pi Z^2 + \cos\pi Z^2}{(Z-2)} dZ - \int \frac{\sin\pi Z^2 + \cos\pi Z^2}{(Z-1)} dZ$$

$$= 2\pi i \times f(2) - 2\pi i \times f(1)$$

$$= 2\pi i [\sin 4\pi + \cos 4\pi] - 2\pi i [\sin\pi + \cos\pi]$$

$$= 2\pi i - (-2\pi i) = 4\pi i$$

EXERCISE – III

Evaluate by Cauchy's integral formula/Residue Theorem.

1) $\int_c dz / (z^2 + 9)$ where 'c' is the circle $|Z - 3i| = 4$ is

a) π b) $\pi/2$ c) $\pi/3$ d) None

2) $\int_c \dfrac{e^Z}{(Z^2 + 1)} dZ$ where 'c' is the circle $|Z - i| = 4$ is

a) $\pi(\cos 1 + i \sin 1)$ b) $2\pi i \sin 1$ c) 0 d) None

3) $\int_c \frac{\tan Z}{(Z^2-1)} dZ$ where 'c' is $|Z| = 3/2$
a) $2\pi i$ b) 0 c) 2π i tan 1 d) none

4) $\int_c \frac{\cos \pi Z}{(Z^2-1)} dZ$ where 'c' is the rectangle bounded by the vertices $\pm 2 \pm i$
a) 0 b) $2\pi i$ c) π d) none

5) $\int_c \frac{e^{2Z}}{(Z-1)(Z-2)} dZ$ where 'c' is $|Z| = 3$
a) $2\pi i (e^2 - e)$ b) $2\pi i (e^4 - e^2)$ c) ½ e^2 d) none

6) $\int_c \frac{\sin Z}{(Z^2 - iZ + 2)} dZ$ where 'c' is $|Z+2| = 2$
a) 0 b) $2\pi i$ c) $2\pi^2$ d) none

7) Evaluate problem (6) when 'C' is rectangle with the vertices (1,0) ; (1,3) ; (-1,3) ;(-1,0).
a) $(2\pi i/3)$ Sinh2 b) $(2\pi i/3)$ Sinh2i c) 0 d) none of the above

8. $\int_c \frac{e^{2Z}}{(Z-1)^4} dz$ where 'c' is the circle $|Z| = 2$
a) $4\pi i/3$ b) $8\pi i e^2/3$ c) $8\pi/3e^2$ d) none

9. $\int_c \frac{\cos Z \, dZ}{(Z-\pi)}$ where 'c' is $|Z-1| = 3$
a) $-2\pi i$ b) $2\pi i$ c) πi d) none of the above

10. $\int_c \frac{\sin^2 Z}{(Z-\pi/6)^3} dZ$ where 'c' is $|Z| = 2$
a) 0 b) $-\pi i$ c) πi d) none

11. $\int_c \frac{e^{-2Z}}{(Z+1)^3} dZ$ where 'c' is $|Z| = 3/2$
a) $8\pi i e^2$ b) $4\pi i e^2$ c) $2\pi i$ d) none

12. $\int_c \frac{Z+1}{Z^2} dZ$ where 'c' is $|Z| = 3$
a) $2\pi i$ b) πi c) 0 d) none

13. $\int_c \frac{\sin 3Z}{(Z+\pi/2)} dZ$ where 'c' is $|Z| = 1$
a) 0 b) $2\pi i$ c) $-2\pi i$ d) none

14. $\int_c \frac{Z+4}{Z^2+2Z+5} dZ$ where 'c' is $|Z| = 1$
a) 0 b) $2\pi i$ c) $-2\pi i$ d) none

15. $\int_c \frac{e^{\pi Z}}{(Z^2+1)^2} dZ$ where 'c' is $|Z - i| = 1$
a) $\pi(\pi i - 1)/2$ b) $(2\pi - i)/6$ c) 0 d) none of the above

16. $\int_c \frac{e^Z + Z \sinh Z}{(Z-\pi i)^2} dz$ where 'c' is $|Z| = 4$
a) $-(1+\pi i)2\pi i$ b) $(1+\pi i)\, 2\pi i$ c) 0 d) none

17. $\int_c \frac{Z+1}{Z^3 - 2Z^2} dz$ where 'c' is $|Z| = 1$
a) $-3\pi i$ b) $-3\pi i/2$ c) $3\pi i/2$ d) none

18. $\int_c \frac{5Z^2 - 3Z + 2}{(Z-1)^3} dz$ where 'c' is a circle containing the point Z = 1
a) $5\pi i$ b) $10\pi i$ c) $4\pi i$ d) none

KEY

1. c 2. b 3. c 4. a 5. b 6. a 7. a 8. b 9. a 10. c 11. b 12. a 13. a
14. a 15. a 16. a 17. b 18. b

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Taylor's Theorem: If f(z) is analytic inside a circle 'C' with center at 'a'. Then for all 'z' inside 'c'.

$$f(z) = f(a) + (z-a) f^1(a) + \frac{(z-a)^2}{2!} f^{11}(a) + \ldots = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Where $a_n = \frac{1}{2\pi i} \int_c \frac{f(w)}{(w-a)^{n+1}} dw$

Laurent's Series: If f(z) is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles C_1 and C_2 with center at 'a' and respective radii r_1 and r_2 $(r_1 > r_2)$. Then for all 'z' in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $a_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(w)}{(w-a)^{n+1}} dw$ n = 0, 1, 2…….

where $b_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(w)}{(w-a)^{-n+1}} dw$ n = 1, 2, 3…….

Note (1): Maclaurin's series is a particular case of Taylor's Series, i.e., when a = 0.

Thus $f(z) = f(0) + z f^1(0) + \frac{z^2}{2!} f^{11}(0) + \ldots$ is the required Machaurin's series.

Note (2): In practice, the Laurents Series is obtained by rearrangement, manipulation and using
standard series expansions (both Maclaurin's & Taylors) and not by using the formulae for
a_n and b_n.

Note (3): The region of convergence (validity) of Laurents series is the annulus region
$r_2 < | z - a | < r_1$.

TYPES OF SINGULARITIES

1. Zeros of f(z): The values of z for which f(z) = 0 are called zero's of f(z).
If f(z) is analytic in the neighborhood of z = a. then the Taylor's series of f(z) is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

If $a_0 = a_1 = a_2 = \ldots = a_{n-1} = 0$. Then z = a is a zero of order 'n'. $(a_n \neq 0)$

2. Isolated Singularity: If z = a is a singularity of f(z) and if there exists no other singularity within a small neighborhood surrounding the point z = a then z = a is said to be isolated singularity.

eg: $f(z) = \frac{z+1}{z(z-4)}$

Here z = 0 and z = 4 are isolated singularities since there exists no other singularity around the points z = 0 and z = 4.

Note: In the Laurent's series, the term

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$\sum_{n=0}^{\infty} a_n (z-a)^n$ is called analytical part and $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ is called the principal part

3. Removable singularity: When all the coefficients of b_n are zero, i.e if the principal does not exist then z = a is a removable singularity. (i.e. If $\lim_{z\to a} \frac{f(z)}{(z-a)}$ exists finitely)

Eg: $f(z) = \frac{\sin (z-a)}{z-a)}$, z = a is a singularity.

$= \frac{1}{z-a}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!}\ldots\ldots\right] = \frac{1}{3!} - \frac{(z-a)^2}{5!} + \frac{(z-a)^4}{7!}\ldots\ldots$

Hence z = a is a removable singularity also $\lim_{z\to a.} \frac{\sin (z-a)}{(z-a)} = 1$ finite

4. Poles of f(z): In the Laurent's series if the principal part contains a single term say $(b_1/(z-a))$ then z = a is called a simple pole or pole of order one, similarly if the principal part contains terms upto $b_2/(z-a)^2$. Then z = a is a double pole. In general if the principal part contains a finite number of terms say 'm' terms i.e. upto $(b_m/(z-a)^m)$. Then z = a is called a pole of order 'm'.

eg: (1) $f(z) = \frac{e^z}{z-3}$; z = 3 is a simple pole

(2) $f(z) = \frac{e^{2z}}{(z-1)^2 (z^2+9)}$ z = 1 a double pole and z = ± 3i are simple poles.

5: Essential Singularity: If the principal part contains infinite number of terms i.e., infinite number of negative powers of (z – a) then z = a is called an essential singularity of f(z).

eg: $f(z) = e^{1/(z-2)} = 1 + \frac{1}{z-2} + \frac{1}{2!(z-2)^2} + \frac{1}{3!(z-2)^3} + \ldots..$

Here z = 2 is an essential singularity.

6. Isolated essential singularity: The limit point of zero's is an isolated essential singularity.

7. Non – Isolated Essential Singularity: The limit point of poles is an Non – Isolated essential singularity.

1. Expand $f(z) = e^z$ about z = a (i.e in powers of z – a).

$f(z) = e^{z-a+a} = e^a.e^{(z-a)}$

$= e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!}$

2. Expand $f(z) = \frac{a}{bz+c}$ about z_0.

$f(z) = \frac{a}{bz+c} = \frac{a}{bz - bz_0 + bz_0 + c}$

$= \frac{1}{(bz_0 + c)} \left(\frac{a}{1 + \frac{b(z-z_0)}{bz_0 + c}}\right)$

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Let $bz_0 + c = d$ and $b/d = e$. Then we get

$f(z) = \frac{a}{d}\left(\frac{1}{1+e(z-z_0)}\right) = \frac{a}{d}\,[1+e(z-z_0)]^{-1}$

Then by binomial theorem we get

$f(z) = \frac{a}{d}\left(\sum_{n=0}^{\infty} (1)^n e^n (z-z_0)^n\right)$ Hence

$= \frac{a}{bz_0+c} \sum (-1)^n \left(\frac{b}{bz_0+c}\right)^n (z-z_0)^n$ (where $|z-z_0| < \frac{1}{e}$)

EXERCISE – IV

Find the Laurent's series about the indicated singularity for each of the following functions.

1. $\frac{e^{2z}}{(z-1)^3}$ at z = 1
2. $(z-3)\sin \frac{1}{z+2}$ at z = -2
3. $\frac{z - \sin z}{z^3}$ at z = 0
4. $\frac{z}{(z+1)(z+2)}$ at z = – 2
5. $\frac{1}{z^2 (z-3)^2}$ at z = 3
6. $\frac{\sin z}{(z-\pi)}$ at z = π
7. $\frac{e^{-z}}{(z-1)^2}$ at z = 1
8. $\frac{1}{1+z^2}$ at z = 0
9. $\frac{3}{3z - z^2}$ at z = 1
10. $\frac{1}{(z-1)(z-2)}$ at z = 0
11. $\frac{z-1}{z^2}$ at z = 1
12. $\frac{e^z}{z(z+1)}$ at z = 2

ANSWERS

1. $e^2/(Z-1)^3 + 2e^2/(Z-1)^2 + 2e^2/(Z-1) + (4e^2/3) + (2e^2/3)(Z-1) + \ldots\ldots.$
2. $1 - 5/(Z+2) - (1/6)(Z+2)^2 + (5/6)(Z+2)^3 + \ldots\ldots$
3. $(1/3!) - (Z^2/5!) + (Z^4/7!) + \ldots\ldots\ldots.$
4. $2/(Z+2) + 1 + (Z+2) + (Z+2)^2 + \ldots\ldots..$
5. $(1/9)(Z-3)^2 - (2/27)(Z-3) + (1/27) - 4(Z-3)/243 \ldots\ldots\ldots..$
6. $-1 + (Z-\pi)^2/3! - (Z-\pi)^4/5! \ldots\ldots..$
7. $e^{-1}[1/(Z-1)^2 - [1/(Z-1) + 1/(2!) - (Z-1)/3! \ldots\ldots\ldots..]$
8. $\Sigma (-1)^n Z^{2n}$
9. $(3/2) - (3/4)(Z-1) + (9/8)(Z-1)^2$

10. $(1/2) + (3/4)Z + (7/8)Z^2 + (15/16)Z^3 + \ldots\ldots\ldots\ldots$

11. $\sum_{n=1}^{\infty} (-1)^{n+1} n(Z-1)^n$

12. $(e^2/6)\,(1 + (Z-2)/6 + (7/36)(Z-2)^2 + \ldots\ldots\ldots$

EXERCISE – V

What kind of singularities the following functions have at the indicated points.

1. $1/(1-e^Z)$ at $Z = 2\pi i$
   a) a simple pole b) pole of order 3 c) removable singularity d) none

2. $1/(\text{Sin}Z - \text{Cos}Z)$ at $Z = \pi/4$
   a) essential b) removable c) simple pole d) none

3. $\text{Cot}\,\pi Z/(Z-a)^2$ at $Z = a$
   a) simple pole b) a double pole c) essential d) none

4. $(Z^2+4)/e^Z$ at $Z = \infty$.
   a) essential singularity b) isolated essential singularity c) simple pole d) none

5. $f(Z) = \text{Sin}(1/1\text{-}Z)$ at $Z = 1$
   a) essential singularity b) isolated essential c) both a & b d) none

6. $(1-e^{2Z})/Z^4$ at $Z = 0$
   a) simple pole b) pole of order 4 c) pole of order 3 d) none

7. $(1-\text{Cos}Z)/Z$ at $Z = 0$
   a) essential singularity b) isolated singularity c) removable singularity d) none

8) $f(Z) = e^{Z/(Z-2)}$ at $Z = 2$
   a) simple pole b) essential singularity c) zero of f(Z) d) none

9) $f(Z) = 1/Z(e^Z - 1)$ at $Z = 0$
   a) simple pole b) pole of order 2 c) pole of order 3 d) none

10) $f(Z) = Z\,e^{1/Z^2}$ the point $Z = 0$ is
   a) an essential singularity b) simple pole c) pole of order '2' d) none

11) $f(Z) = Z^2 - 1/(Z+1)(Z-1)^3$ the point $Z = 1$ is
   a) a simple pole b) pole of order '3' c) pole of order '2' d) none

12) For the function $f(Z) = 1/e^Z$ which of the following statement is true
   a) the function has no zeros b) the function has no singularities
   c) The function has infinite singularities d) none of the above

KEY:

1. a 2. c 3. b 4. b 5. c 6. c 7. c 8. b 9. b 10. a 11. c 12. b

Evaluation of Residues at the Poles / Singularities

1. If $Z = a$ is a simple pole of f(Z) then Residue of f(z) at $z = a$ is $\lim_{Z\to a} (Z-a)\,f(Z)$

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2. If $Z = a$ is a double pole then Residue of f(z) at $Z = a$ is $\lim_{Z\to a} \{d/dZ\,(Z-a)^2\, f(Z)\}$

3. In general if $Z=a$ is a pole of order 'm'
Then Res of f(z) at $Z=a$ is $(1/(m-1)!)\ \lim_{Z\to a} \{d^{m-1}/dZ^{m-1}\,[(Z-a)^m \cdot f(Z)]\}$

Note: If $f(Z)=\phi(Z)/\psi(Z)$ and $Z=a$ has a simple pole at $z = a$ where $\phi(a) \neq 0$, $\psi(a)= 0$ and $\psi^1(a) \neq 0$ then the residue of f(z) at $Z=a$ is given by $\phi(a)/\psi^1(a)$

EXERCISE – VI

Find the poles, singularities and residues at each pole for the following

1. $e^{2Z}/(Z-1)^3$
2. $(Z-3)\,\text{Sin}(1/Z+2)$
3. $(Z-\text{Sin}Z)/Z^3$
4. $1/Z^2(Z-3)^2$
5. $e^{Z/(Z-2)}$
6. $(Z^2-2Z)/(Z+1)^2(Z^2+4)$
7. $e^Z\,\text{Cosec}^2 Z$
8. $(\text{Cot}Z \cdot \text{Coth}Z)/Z^3$
9. $(9Z+i)/(1-\text{Cos}Z)$
10. $Z^2/(Z-1)^2(Z+2)$
11. $50Z/(Z+4)(Z-1)^2$
12. $(1+Z)/(1-\text{Cos}Z)$
13. $3Z/Z^2+2Z+5$
14. $(4Z-3)/Z(Z-1)(Z-2)$
15. $1-e^{2Z}/Z^4$
16. $1+e^Z/(\text{Sin}Z + Z\,\text{Cos}Z)$
17. Tan Z between $0\&\pi$
18. $Z^2/(Z^4-1)$
19. $Ze^{iZ}/(Z^2+a^2)$
20. Find Residue at $Z=0$ for $\text{Cosec}^2 Z/Z^3$.

ANSWERS

1. $Z = 1$ a pole of order 3 ; $\text{Res} = 2e^2$
2. $Z = -2$ essential singularity ; Res = -5
3. $Z=0$ is a removable singularity ; Res = 0
4. $Z = 0, 3$ are poles of order '2' ; Residues are 2/27 and –2/27
5. $Z=2$ essential singularity ; Res = 2e
6. $Z = -1$ double pole Res = -14/25 ; $Z = \pm 2i$ simple poles Res : (7+i)/25, (7–i)/25
7. $Z = n\pi$ for n = 0,1, ……… etc ; $\text{Res} = e^{n\pi}$
8. $Z = 0$; Res = -7/45
9. Pole $Z = 0$ Res = i ; $Z = \pm i$, Res = -5i, 4i
10. $Z = -2$ Simple pole Res = 4/9 ; $Z=1$ double pole Res = 5/9
11. Pole $Z = -4i$ Res = -8 ; Pole $Z = 1$ double pole Res = 8
12. $Z=0$ Simple pole ; Res = 2
13. $Z = -1\pm 2i$ simple poles ; Residues are (3/4)(2 – i) and (3/4)(2+i)
14. Simple poles at $Z=0, 1, 2$; Residues are –3/2, -1, 5/2.
15. $Z=0$ is a pole of order 3 ; Res = -4/3
16. $Z=0$ is a singular point ; Res = 1
17. $Z=\pi/2$ is a pole in $0<Z<\pi$; Res = -1
18. $Z = \pm i$ & $Z=\pm 1$ are simple poles ; Residues are i/4, -i/4, 1/4 and –1/4 respectively
19. $Z = \pm ai$ are simple poles ; Residues are $e^{-a}/2$ and $e^{a}/2$.
20. 1/6

Cauchy's Residue Theorem:

If f(Z) is analytic within and on a simple closed curve 'C' except at a finite number of poles inside 'C'. Then

$$\int_C f(Z)\,dZ = 2\pi i \times [\text{Sum of the residues at the poles inside 'C'}]$$

Note: All the problems of Exercise – III can be solved by using Cauchy's Residue Theorem.

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IMPORTANT FORMULAE

1. Relation between Circular and Hyperbolic functions:

a) Sin iZ = i SinhZ

b) Cos iZ = CoshZ

c) Tan iZ = i tan hZ

d) Cos h iZ = Cos Z

e) Sin h iZ = i SinZ

f) tan h iZ = i tanZ

g) tan Z = (Sin2x + i Sinh2y) / (Cos2x + Cosh2y)

h) Cot Z = (Sin2x – i Sinh2y) / (Cosh2y - Cos2x)

i) SechZ = (2CoshxCosy – i 2SinhxSiny) / (Cosh2x + Cos2y)

j) tanhZ = (Sinh2x + i Sin2y) / (Cos2x + Cosh2y)

k) SecZ = (2CosxCoshy + i 2SinxSinhy) / (Cos2x+Cosh2y)

l) CosecZ = (2SinxCoshy – i 2CosxSinhy) / (Cosh2y – Cos2x)

2. Zero's of f(Z)

1. f(Z) = SinZ ; Z = ±nπ for n = 0,1,2,........

2. f(Z) = CosZ ; Z = ±(2n+1)π/2 for n = 0,1,2,........

3. f(Z) = SinhZ ; Z = ±nπi for n = 0,1,2,........

4. f(Z) = CoshZ ; Z = ±(2n+1/2)πi/2 for n = 0,1,2,........

5. f(Z) = tanZ ; Z = ±nπ for n = 0,1,2,........

6. f(Z) = CotZ ; Z = ± (2n+1)π/2 for n = 0,1,2,.......

7. f(Z) = tanhZ ; Z = ±nπi for n = 0,1,2,........

8. f(Z) = cothZ ; Z = ±(2n+1/2)πi for n = 0,1,2,........

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3. Different Values of the following terms

1. $1^{\sqrt{2}} = \text{Cos}2\sqrt{2}\,k\pi + i\,\text{Sin}2\sqrt{k}\,\pi$ for k = 0,1,2,.......

2. $\text{Re}\{(1-i)^{1+i}\} = e^{\log\sqrt{2} - 7\pi/4 - 2k\pi} \cdot \text{Cos}(7\pi/4 + \log\sqrt{2})$

3. The modulus of $(-i)^{-i} = e^{3\pi/2 + 2k\pi}$, k = 0,1,2,3,....... (All values are the principal values)

4. $i^i = e^{-\pi/2 \pm 2n\pi}$; Principal Value $e^{-\pi/2}$

5. $(1+i)^{2-i} = 2e^{\pi/4 + 2n\pi}$ [Sin log√2 + i Cos log √2)]
   n = 0; principal value is $2e^{\pi/4}$[Sinlog√2+iCoslog√2]

6. $1^i = e^{-2k\pi}$, k =0 ; Principal value = 1

7. $i^{-2i} = e^{(4n+1)\pi}$ for n = 0,1,2,......
   for n = 0 principal value is e^{π}

8. $(1-i)^{1+i} = \sqrt{2}\,(1-i)\,e^{(2n+1/4)\pi}\,e^{i\log\sqrt{2}}$ n =0,1,2,........... etc
   principal value : $(1-i)\,e^{\pi/4}\,e^{i\log\sqrt{2}}$

9. $\log(i^i) = -(2n+1/2\pi)$, n = 0, ±1, ±2

10. The real part of the principal value of $i^{\log(1+i)}$ is $e^{-\pi^2/8}$ Cos(π/4 log2)

11. $(1+i)^i = e^{i\log(1+i)} = e^{i[\log\sqrt{2} + i(\pi/4 + 2n\pi)]}$

12. $\log Z = e^{i\log\sqrt{2}}\; e^{-(\pi/4 \pm 2n\pi)} = e^{-(\pi/4 \pm 2n\pi)} = e^{-(\pi/4 \pm 2k\pi)}$[Cos log √2 + i Sin log √2]

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PREVIOUS GATE QUESTIONS

01. Which one of the following is Not true for complex numbers Z_1 and Z_2 ? **(CE-2005-1M)**

(a) $\frac{Z_1}{Z_2} = \frac{Z_1 \overline{Z_2}}{|Z_2|^2}$

(b) $|Z_1 + Z_2| \le |Z_1| + |Z_2|$

(c) $|Z_1 - Z_2| \le |Z_1| - |Z_2|$

(d) $|Z_1 + Z_2|^2 + |Z_1 - Z_2|^2 = |Z_1|^2 + 2|Z_2|^2$

02. Consider likely applicability of cauchy's Integral theorem to evaluate the following integral counter clockwise around the unit circle C.
$I = \int_c \sec z\, dz$, z being a complex variable. The value of I will be **(CE-2005-2M)**

(a) $I = 0$: singularities set $= \phi$

(b) $I = 0$: singularities

set $= \left\{\pm \frac{2n+1}{2}\pi / n = 0,1,2....\right\}$

(c) $I = \pi/2$: singularities set $= \{\pm n\pi : n = 0,1,2....\}$

(d) None of above

03. Consider the circle $|Z - 5 - 5i| = 2$ in the complex plane (x, y) with $Z = x + iy$. The minimum distance from the origin to the circle is **(IN-2005-2M)**

(a) $5\sqrt{2} - 2$ (b) $\sqrt{54}$

(c) $\sqrt{34}$ (d) $5\sqrt{2}$

04. Let $Z^3 = \overline{Z}$, where Z is a complex number not equal to zero. Then Z is a solution of **(IN-2005-2M)**

(a) $Z^2 = 1$ (b) $Z^3 = 3$

(c) $Z^4 = 1$ (d) $Z^9 = 1$

05. The function

$w = u + iv$

$= \frac{1}{2}\log(x^2 + y^2) + i \tan^{-1}(y/x)$

is not analytic at the point **(PI-2005-2M)**

(a) (0, 0) (b) (0, 1)

(c) (1, 0) (d) 2, 0)

06. Let $j = \sqrt{-1}$ then one value of j^j is **(IN-2007-1M)**

(a) $\sqrt{j}$ (b) -1 (c) $\pi/2$ (d) $e^{-\pi/2}$

07. For the function $\frac{\sin z}{z^3}$ of a complex variable z, the point $z = 0$ is **(IN-2007-2M)**

(a) a pole of order 3

(b) a pole of order 2

(c) a pole of order 1

(d) not a singularity

08. Potential function ϕ is given as $\phi = x^2 - y^2$, what will be the stream function ϕ with the condition $\phi = 0$ at $x = y = 0$? **(CE-2007-2M)**

(a) $2xy$ (b) $x^2 + y^2$

(c) $x^2 - y^2$ (d) $2x^2y^2$

09. If $\phi(x, y)$ and $\phi(x, y)$ are functions with continuous 2^{nd} order derivations then $\phi(x, y) + i\,\phi(x, y)$ can be expressed as an analytic function of $x + iy$ ($i = \sqrt{-1}$) when **(ME-2007-1M)**

(a) $\frac{\partial \phi}{\partial x} = \frac{-\partial \psi}{\partial x}, \frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial y}$

(b) $\frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x}, \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$

(c) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \psi}{\partial y^2} = 1$

(d) $\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0$

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10. If a complex variable $z = \frac{\sqrt{3}}{2} + i\frac{1}{2}$ then z^4 is **(PI-2007-1M)**

(a) $2\sqrt{3} + 2i$ (b) $\frac{-1}{2} + i\frac{\sqrt{3}}{2}$

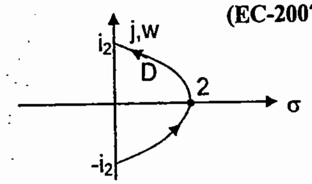
(c) $\frac{\sqrt{3}}{2} - i\frac{1}{2}$ (d) $\frac{\sqrt{3}}{8} + i\frac{1}{8}$

11. The value of $\oint_C \frac{dz}{(1+z^2)}$ where C is the contour $|z - i/2| = 1$ is **(EE-2007-2M)**

(a) $2\pi i$ (b) π

(c) $\tan^{-1} z$ (d) $\pi\, i \tan^{-1} z$

12. If the semi-circular contour D of radius 2 is as shown in the figure, then the value of the integral $\oint_D \frac{1}{(s^2 - 1)} ds$ is **(EC-2007-2M)**



(a) $i\pi$ (b) $-i\pi$ (c) $-\pi$ (d) π

13. The equation $\sin(z) = 10$ has **(EC-2008-1M)**

(a) no real (or) complex solution

(b) exactly two distinct complex solutions.

(c) a unique solution.

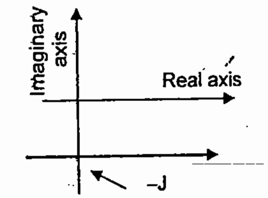
(d) an infinite number of complex solutions.

14. The residue of the function $f(z) = \frac{1}{(z+2)^2(z-2)^2}$ at $z = 2$ is **(EC-2008-2M)**

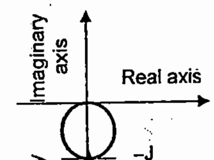
(a) $\frac{-1}{32}$ (b) $\frac{-1}{16}$ (c) $\frac{1}{16}$ (d) $\frac{1}{32}$

15. A complex variable $z = x + j\,0.1$ has its real part x varing in the range $-\infty$ to $+\infty$. Which one of the following is the locus (shown in thick lines) of $\frac{1}{z}$ in the complex plane? **(IN-2008-2M)**

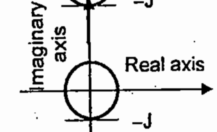
(a)



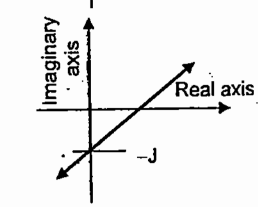
(b)



(c)



(d)



16. The integral $\oint f(z)\,dz$ evaluated around the unit circle on the complex plane for $f(z) = \frac{\cos z}{z}$ is **(ME-2008-2M)**

(a) $2\pi i$ (b) $4\pi i$ (c) $-2\pi i$ (d) 0

17. The value of the expression $\frac{-5+10i}{3+4i}$ is **(PI-2008-1M)**

(a) $1 - 2i$ (b) $1 + 2i$

(c) $2 - i$ (d) $2 + i$

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18. The analytic function has singularities at, where $f(z) = \frac{z-1}{z^2+1}$ **(CE-2009-1M)**

(a) 1 & –1 (b) 1 and i

(c) 1 and –i (d) i and –i

19. The value of the integral $\int_c \frac{\cos(2\pi z)}{(2z-1)(z-3)} dz$ where c is a closed curve given by $|z| = 1$ is **(CE-2009-2M)**

(a) $-\pi i$ (b) $\frac{\pi i}{5}$ (c) $\frac{2\pi i}{5}$ (d) πi

20. If $f(z) = c_0 + c_1 z^{-1}$ then $\oint \frac{1+f(z)}{z} dz$ is given by **(EC-2009-1M)**

(a) $2\pi c_1$ (b) $2\pi(1 + c_0)$

(c) $2\pi j c_1$ (d) $2\pi j(1 + c_0)$

21. If $z = x + jy$, where x and y real then the value of $|e^{jz}|$ is **(IN-2009-1M)**

(a) 1 (b) $e^{\sqrt{x^2+y^2}}$

(c) e^y (d) e^{-y}

22. One of the roots of equation $x^3 = j$, where j is the +ve square root of –1 is **(IN-2009-2M)**

(a) j (b) $\frac{\sqrt{3}}{2} + j\frac{1}{2}$

(c) $\frac{\sqrt{3}}{2} - j\frac{1}{2}$ (d) $\frac{-\sqrt{3}}{2} - j\frac{1}{2}$

23. The value of $\oint \frac{\sin z}{z} dz$, where the contour of integration is a simple closed curve around the origin, is **(IN-2009-1M)**

(a) 0 (b) $2\pi j$

(c) ∞ (d) $\frac{1}{2\pi j}$

24. An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x,y) + i\,v(x,y)$ where $i = \sqrt{-1}$. If $u = xy$ then the expression for v should be **(ME-2009-2M)**

(a) $\frac{(x+y)^2}{2} + k$ (b) $\frac{x^2-y^2}{2} + k$

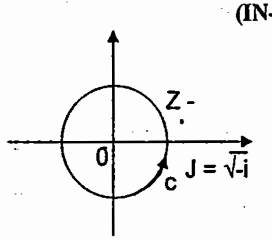
(c) $\frac{y^2-x^2}{2} + k$ (d) $\frac{(x-y)^2}{2} + k$

25. The product of complex numbers (3 – 2i) and (3 + i4) results in **(PI-2009-1M)**

(a) 1 + 6i (b) 9 – 8i

(c) 9 + i8 (d) 17 + i6

26. The contour c in the adjoining figure is described by $x^2 + y^2 = 16$. Then the value of $\oint \frac{z^2+8}{0.5z-1.5j} dz$ is **(IN-2010-1M)**



(a) $-2\pi j$ (b) $2\pi j$

(c) $4\pi j$ (d) $-4\pi j$

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27. The residues of a complex function $X(z) = \frac{1-2z}{z(z-1)(z-2)}$ at it poles are **(EC-2010-2M)**

(a) 1/2, -1/2, 1 (b) 1/2, 1/2, -1

(c) 1/2, 1, -3/2 (d) 1/2, -1, 3/2

28. If $f(x + iy) = x^3 - 3xy^2 + i\,\phi(x,y)$, where $i = \sqrt{-1}$ and $f(x + iy)$ is an analytic function then $\phi(x, y)$ is **(PI-2010-1M)**

(a) $y^3 - 3x^2y$ (b) $3x^2y - y^3$

(c) $x^4 - 4x^3y$ (d) $xy - y^2$

29. If a complex number w satisfies the equation $w^3 = 1$ then the value of $1 + w + \frac{1}{w}$ is **(PI-2010-1M)**

(a) 0 (b) 1 (c) 2 (d) 4

30. The modulus of the complex number $\left(\frac{3+4i}{1-2i}\right)$ is **(CE-2010-1M)**

(a) 5 (b) $\sqrt{5}$ (c) $\frac{1}{\sqrt{5}}$ (d) $\frac{1}{5}$

31. The value of the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is **(CE-2010-1M)**

(a) $-\pi$ (b) $-\pi/2$ (c) $\pi/2$ (d) π

KEY :

01. c	02. d	03. a
04.	05. a	06. d
07. b	08. a	09. b
10. b	11. a	12. a
13. d	14. a	15.
16. a	17. a	18. d
19. c	20. d	21. d
22. b	23. b	24. c
25. d	26. d	27. c
28. b	29. a	30. b
31. d		

BASIC ENGINEERING MATHEMATICS

TOPIC – 9 **NUMERICAL METHODS**

ROOT FINDING : Consider the equation $f(x) = 0$ .... (1)
If f (x) is continuous in [a, b] and f (a) and f (b) are of opposite signs i.e. $[f(a) . f(b)<0]$ then at least one root of the equation (1) lies between a and b.

BISECTION METHOD :
Iterative formula : If x_i and x_{i-1} enclose the root then

$$x_{i+1} = \frac{(x_i + x_{i-1})}{2}$$

* The Bisection method converges slowly. It is however, the simplest iterative method and is guaranteed to converge. Convergence is slow and steady. This method can not be used for finding complex roots.

REGULA FALSI METHOD : (METHOD OF FALSE POSITION)
Consider, $f(x) = 0$
If $f(a) . f(b) < 0$ then the iterative formula is

$$x_i = \frac{a\ f(b) - b\ f(a)}{f(b) - f(a)}$$

* The false position method is guaranteed to converge.
* It is however, slow as it is first order convergent. i.e. order of convergence = 1.
* It is not necessarily a monotonic convergence to the root but most often it will be superior to Bisection method.

NEWTON - RAPHSON METHOD :
The iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)} \quad (n = 0,1,2,..)$$

Where x_0 is approximate root of the equation $f(x) = 0$
* The Newton's method is useful in cases of large values of $f^1(x)$ i.e, when the graph of f(x) while crossing the x – axis is nearly vertical.
* Sensitive to starting value. Convergence fast if starting point near the root.
 i.e.. The Newton's formula converges provided the initial approximation x_0 is chosen sufficiently close to the root.
* Newton's method is generally used to improve the result obtained by other methods
* Newton's method has a quadratic convergence. i.e. order of convergence = 2.
 i.e. The subsequent error at each step is proportional to the square of the error at previous step
* The number of functions to be evaluated per iteration is 2

Secant method (Modified version of regula falsi or Interpolation method) :

Newton - Raphson method is very powerful, but the evaluation of derivative involved may some times be difficult or expensive.

This suggests the idea of replacing $f^1(x_n)$ by the difference quotient.

$$f^1(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

➢ Instead of choosing two values of x such that the function has opposite signs at these values we choose two values nearest the root

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➢ This method is applicable only if one is sure that there is a root in the vicinity of x_1 the starting value.

* On the average, Secant method is more efficient compared to Newton - Raphson method
* In this method, we approximate the graph of the function $y = f(x)$ in the neighborhood of the root by a straight line (secant) passing through the points (x_{i-1}, f_{i-1}) and (x_i, f_i).

The iteration formula is

$$x_{i+1} = \frac{x_{i-1}\ f_i - x_i\ f_{i-1}}{f_i - f_{i-1}}$$

Where x_i and x_{i-1} need not enclose the root.
* The order of convergence is 1.62 converges faster than false position method.
* No guarantee of convergence if not near root. The method fails if $f(x_i) = f(x_{i-1})$
* It may be considered the most economical method giving reasonably rapid convergence at a low cost.
* The amount of computational effort is one function evaluation

SUCCESSIVE APPROXIMATION METHOD :
To find the root of the equation $f(x) = 0$ .... (1) by successive approximation, we rewrite (1) in the form $x = g(x)$
Let $x = x_0$ be the initial approximation of the derived root. The iterative formula is

$$x_i = g(x_{i-1})$$

* Order of convergence is 1
* Easy to program, no guarantee of convergence.
* If (i) α be a root of $f(x) = 0$ which is equivalent to $x = g(x)$
 (ii) 'I' be any interval containing the point $x = \alpha$
 (iii) $|g^1(x)| < 1$ for all x in 'I'

Then the sequence of approximations $x_0, x_1, x_2, \ldots. x_n$ will converge to the root α, provided the initial approximation x_0 is chosen in 'I'
* The smaller the value of g(x), the more rapid will be the convergence.
* This method of iteration is particularly useful for finding the real roots of an equation given in the form of infinite series.

1. Using Newton Raphson iteration formula, find the first approximation to the root of the equation $x^4 - x - 10 = 0$ which is nearer to $x = 2$
 a) 1.671 b) 1.871 c) 2.071 d) 2.271

2. Using, Newton's method, find the first approximation to the root of the equation $3x = \cos x + 1$ (Take $x_0 = 0$ as initial approximation)
 a) 1/3 b) 1/2 c) 2/3 d) 3/4

3. The Newton's iteration formula for finding $\sqrt{N}$ where N is a positive real number is
 a) $x_{n+1} = 1/2\,(x_n + N/x_n)$ b) $x_{n+1} = 1/2\,(x_n - N/x_n)$
 c) $x_{n+1} = x_n\,(2 - N.x_n)$ d) $x_{n+1} = x_n\,(2 + N.x_n)$

4. For N = 18 and $x_0 = 4$, The first approximation to $\sqrt{18}$ by Newton's iteration formula is
 a) 4.20 b) 4.25 c) 4.24 d) 4.2426

5. The Newton's iteration formula for finding 1/N where N is a positive real number is
 a) $x_{n+1} = x_n\,(2 + N/x_n)$ b) $x_{n+1} = x_n\,(2 - N.x_n)$
 c) $x_{n+1} = x_n\,\{2 + (N/x_n)\}$ d) $x_{n+1} = x_n\,\{2 - N/x_n\}$

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6. The Newton – Raphson iteration formula for finding $N^{1/3}$ where N is a positive real number is
   a) $x_{n+1} = 1/3\,(2x_n + N/x_n^2)$ b) $x_{n+1} = 1/3\,(2x_n - N/x_n^2)$
   c) $x_{n+1} = 1/3\,(2x_n^3 + N/x_n^2)$ b) $x_{n+1} = 1/3\,(2x_n^3 - N/x_n^2)$
7. If the initial approximation to a root of the equation $x = e^{-x}$ is $x_0 = 1$, then the first approximation to the root of the equation is
   a) 0.567 b) 0.667 c) 0.767 d) 0.867
8. Using bisection method, find a second approximation to the root of equation $x^3 - 4x - 9 = 0$ between 2 and 3
   a) 2.25 b) 2.5 c) 2.75 d) 2.625
9. Find first approximation to a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position between 2 and 3
   a) 2.0588 b) 2.0466 c) 2.0831 d) 2.0614
10. Using secant method, find first approximation to the root of the equation $x\,e^x = \cos x$ between 0 and 1
   a) 0.31467 b) 0.44673 c) 0.51776 d) 0.6451
11. Find the first approximation to a real root of the equation $\cos x = 3x - 1$ near $x_0 = 0$, using successive approximation method
   a) 1/3 b) 1/2 c) 2/3 d) 3/4
12. Find the first approximation to a real root of the equation $2x - \log_{10} x = 7$ using successive approximation method (Take $x_0 = 3.61$)
   a) 3.77815 b) 3.78863 c) 3.78924 d) 3.78927
13. Which of the following methods can not be applied for locating complex roots of an equation
   a) Bisection method b) Regula falsi
   c) Secant method d) Newton - Raphson method
14. After Newton - Raphson method, which of the following has fastest rate of convergence
   a) Bisection method b) Secant method
   b) Method of false position d) Successive approximation

KEY

1. b 2. c 3. a 4. b 5. b 6. a 7. a 8. c 9. a 10. a 11. c 12. a 13. a 14. b

PROBLEMS

1) Using secant method find a root of $f(x) = x^3 + x^2 - 2x - 4 = 0$ between 1 and 2, after two iterations.
   a) 1.5 b) 1.6279 c) 1.6623 d) 1.6589
2) The Newton - Raphson iteration formula for finding inverse square root of N is
   a) $x_{n+1} = ½(3x_n - Nx_n^3)$ b) $x_{n+1} = ½(3x_n + Nx_n^2)$
   c) $x_{n+1} = ½(3x_n + Nx_n^3)$ d) $x_{n+1} = ½(3x_n - Nx_n^2)$

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3) Newton - Raphson method is used to find the root of the equation $x^2 - 2 = 0$. If the iterations are started from -1, the iterations will (GATE-97')
   a) converge to –1 b) converge to $\sqrt{2}$ c) converge to $-\sqrt{2}$ d) not converge
4) Which of the following statements applies to bisection method used for finding roots of function. (GATE-98')
   a) converges within few iterations
   b) guaranteed to work for all continuous functions
   c) is faster than Newton - Raphson method
   d) requires that there will be no error in determining the sign of functions

ADDITIONAL PROBLEMS ON NUMERICAL METHODS

5) Starting from $x_0 = 1$, one step of Newton - Raphson method in solving equation $x^3 + 3x - 7 = 0$ gives the next value x_1 as *(GATE'05[ME])*
   a) $x_1 = 0.5$ b) $x_1 = 1.406$ c) $x_1 = 1.5$ d) $x_1 = 2$
6) Consider the following iterative root finding methods and convergence properties

Method	Property
Q : False Position	I) Order of convergence is 1.62
R : Newton - Raphson	II) Order of convergence is 2
S : Secant	III) Order of convergence is 1 with guarantee of convergence
T : Successive Approximation	IV) Order of convergence = 1 with no guarantee of convergence

   The correct matching of methods and properties is *(GATE'04[IT])*
   a) Q – II, R – IV, S – III, T – I b) Q – III, R – II, S – I, T – IV
   c) Q – II, R – I, S – IV, T – III d) Q – I, R – IV, S – II, T – III
7) The Newton - Raphson iteration $x_{n+1} = (x_n/2) + (3/2)x_n$ can be used to solve the equation
   a) $x^2 = 3$ b) $x^3 = 3$ c) $x^2 = 2$ d) $x^3 = 2$ *(GATE'02)*
8) The real root of the equation $x\,.\,e^x = 2$ is evaluated using Newton – Raphson method. If the first approximation of the value is 0.8679, the second approximation of the value of x correct to three decimal places is *(GATE'05[PI])*
   a) 0.865 b) 0.853 c) 0.849 d) 0.838
9) Given a>0, we wish to calculate its reciprocal value (1/a) by using Newton - Raphson method for f(x) = 0. The Newton - Raphson iteration formula for the function will be
   a) $x_{k+1} = ½(x_k + a/x_k)$ b) $x_{k+1} = x_k + a/2\ x_k^2$
   c) $x_{k+1} = 2x_k - a\,x_k^2$ d) $x_{k+1} = x_k - a/2\,x_k^2$ [GATE-05-CE]
10) Using the iteration formula in the last example, for a = 7 and starting with $x_0 = 0.2$, the first two iterations will be
   a) 0.11, 0.1299 b) 0.12, 0.1392 c) 0.12, 0.1416 d) 0.13, 0.1426

KEY

1. b 2. a 3. b 4. d 5 c 6 b 7. a 8. b 9. c 10. b

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NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Single & Multi – Step Methods:

- A one step method is a method, that in each step uses only values obtained in a single step, viz. in the preceding step.
- Some of the one – step methods are: Euler's method, Heun's method, Runge's method, Runge – Kutta method and Taylor series method.
- A multi – step method is a method, that in each step uses values from more than one of the preceding steps. The reason for using the additional information might increase the accuracy.
- Some of the multi – step methods are: Milne's method, Simpson's method, Adams – Bash forth – Moulton methods.

Euler's Method:

Consider $dy/dx = f(x, y)$ with initial condition $y(x_0) = y_0$. We want to find out the value of y at $x = l$.
Divide (x_0, l) into 'n' equal parts of width 'h'. Let $x_1, x_2, \ldots\ldots x_{n-1}, (x_n = l)$ be the intermediate points. Now in (x_0, x_1) we have

$$y_1 = y_0 + h\,f(x_0, y_0)$$
$$y_2 = y_1 + h\,f(x_1, y_1)$$
$$\vdots$$
$$y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1})$$

- The method is stable if $|1 + h(\partial f/\partial y)| < 1$ then the errors will be damp down with successive iterations. Otherwise, the errors increase in successive iterations and the procedure will be unstable.
- It is based on the linear term in the Taylor's expansion of f(x, y).
- In practice, the error build up in using the method is substantial and the method is rarely used.
- This method is also called Runge – Kutta first order method.

PROBLEMS

1. Given $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, with initial condition $y(0) = 1$. Find $y(0.1)$ by using Euler's method and taking $h = 0.05$.
Ans: 1.0954

2. Given that $(dy/dx) + xy = 0$, $y(0) = 1$. Find $y(0.1)$ with $h = 0.05$ using Euler's method.
Ans: 0.9975

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Heun's Method:

Consider $dy/dx = f(x, y)$ with the initial condition $y(x_0) = y_0$. Initially y_1 is computed by Euler's formula.
$y_1^p = y_0 + h\,f(x_0, y_0)$ [where p indicates predictor]
then modified value of y_1 is given by
$y_1^c = y_0 + (h/2)[f(x_0, y_0) + f(x_1, y_1^p)]$ [where c indicates corrector]
Similarly, we can find $y_2, y_3, \ldots\ldots y_n$

- This method is also called modified Euler's method / second order Runge – Kutta method.

PROBLEMS

1. Given $dy/dx = \log(x + y)$ with $y = 1$ when $x = 0$. Find $y(0.2)$ using Heun's method with $h = 0.2$.
Ans: 1.008

2. Given $dy/dx = x + y$, $y(0) = 1$. Find $y(0.02)$ and $y(0.04)$ by Heun's method (taking $h = 0.02$).
Ans: 1.0204 and 1.0416

Runge's Method (R – K method of order – 3):

To solve $dy/dx = x + y$ with $y(x_0) = y_0$. Calculate successively

$$k_1 = h\,f(x_0, y_0)$$
$$k_2 = h\,f(x_0 + h/2, y_0 + k_1/2)$$
$$k^1 = h\,f(x_0 + h, y_0 + k_1)$$
$$k_3 = h\,f(x_0 + h, y_0 + k^1)$$

Finally, compute $k = 1/6(k_1 + 4k_2 + k_3)$ and the solution is $y_1 = y_0 + k$.

PROBLEM

1. Apply Runge's method to find an appropriate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.
Ans: 1.2426

Runge – Kutta Method(R – K method of order – 4):

To solve $dy/dx = f(x, y)$ with the condition $y(x_0) = y_0$. Let 'h' denotes the interval between equidistant values of x.
If the initial values are (x_0, y_0) then the first increment in y is computed from the formula given by

$$k_1 = h\,f(x_0, y_0)$$
$$k_2 = h\,f(x_0 + h/2, y_0 + k_1/2)$$
$$k_3 = h\,f(x_0 + h/2, y_0 + k_2/2)$$
$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$
$$\Delta y = 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_1 = y_0 + \Delta y$$

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PROBLEM

1. If $dy/dx = x + y$, $y(0) = 1$, find $y(0.1)$ by taking $h = 0.1$ and using R – K method of order – 4.

Ans: 1.11034

Properties:

1. R – K methods do not require prior computations of the higher derivatives of $y(x)$ as the Taylor method does.
2. The R – K formulae involve the computation of $f(x, y)$ at various positions and this function occurs in the given equation.
3. To evaluate y_{n+1}, we need information only at y_n. Information at y_{n-1}, y_{n-2} etc not directly required. Thus R – K methods are one step methods.
4. These methods agree with Taylor's series solution upto the terms of h^r, where 'r' differs from method to method and is known as the order of R – K method.

Taylor's Series Method:

Consider $dy/dx = f(x, y)$, $y(x_0) = y_0$. If the solution curve $y(x)$ is expanded in a Taylor series around $x = x_0$. We obtain

$$y(x) = y_0 + (x - x_0)y_0^{1} + \frac{(x - x_0)^2}{2!} y_0^{11} + \frac{(x - x_0)^3}{3!} y_0^{111} + \text{---} \quad (1)$$

- In equation (1), if we take only the first two terms then it corresponds to the Euler method of extrapolation. Thus the errors due to the truncation of the series would be of the order of h^2.
- An improvement of the Euler method would thus be to include the h^2 term in the above expansions. Then the truncation error in the above formula would of the order of h^3.
- This method is not applicable in general, because the partial derivatives f_x, f_y are not always easy to obtain and considerable computation effort is involved.
- If $f(x, y)$ is given in a tabular form then this method is not applicable.
- R – K methods are equivalent to Taylor series method but will use only the values of $f(x, y)$ at specified values of x and y and will not require the derivative to be evaluated.
- R – K methods agree with Taylor's series solution upto the term in h^r where r differs from method to method and is called the order of that method.

PROBLEMS

1. Consider $dy/dx = x + y$, $y(1) = 0$. Using Taylor's series expansion upto h^2 terms evaluate $y(1.1)$ and $y(1.2)$. (with $h = 0.1$)

Ans: 0.11 and 0.242

2. Find by Taylor's series method, the values of y at $x = 0.1$ from $dy/dx = x^2 y - 1$, $y(0) = 1$. (Consider the Taylor's series expansion upto h^3 terms). (with $h = 0.1$)

Ans: $y(0.1) = 0.90033$

3. Solve $y^1 = y^2 + x$, $y(0) = 1$. Using Taylor series method, compute $y(0.1)$ and $y(0.2)$. (with $h = 0,1$)

Ans: 1.1164 and 1.2724

4. Given $dy/dx = 1 + xy$, $y(0) = 1$. Compute $y(0.1)$, using Taylor series method. (with $h = 0.1$)

Ans: 1.1053

MULTISTEP METHODS

- Methods that use information from more than one previous points to compute the next point are called multistep methods.
- Sometimes, a pair of multistep methods are used in conjunction with each other ; one for predicting the value of y_{i+1} and one for correcting the predicted value of y_{i+1}, such methods are called predictor – corrector methods.
- One major problem with multistep methods is that, they are not self starting. They need more information than initial value condition. If a method uses four previous points ; say y_0, y_1, y_2 and y_3, then all these values must be obtained before the method is actually used. These values can be obtained using R – K methods.
- The degree of accuracy of the single step method must match that of the multistep method to be used. For example, a fourth order R – K method is normally used for implementing fourth order multistep method.

Milne – Simpson Method:

The formula used in this method is based on the fundamental theorem of calculus.

$$y(x_{i+1}) = y(x_j) + \int_{x_j}^{x_{i+1}} f(x, y)\, dx \qquad \text{where } j = i - 3$$

we have $y^{p}_{i+1} = y_{i-3} + (4h/3)\,[2f_{i-2} - f_{i-1} + 2f_i]$ (Milne's predictor formula)

Similarly, when $j = i - 1$, we have $y^{c}_{i+1} = y_{i-1} + (h/3)\,[f_{i-1} + 4f_i + f_{i+1}]$ (Simpson's corrector formula)

PROBLEM

1. Given the equation, $y^1 = (2y/x)$ with $y(1) = 2$, $y(1.25) = 3.13$, $y(1.5) = 4.5$, $y(1.75) = 6.13$ then estimate $y(2) =$ ....

Ans: 8

Note: In some cases, Milne's method is not stable. The errors do not tend to zero as step size is made smaller in those cases. Because of this problem Milne's method is not widely used.

Adam's – Bash forth – Moulton Method:

This method is another popular fourth order predictor – corrector method (most widely used).

The A – B predictor formula is given by

$$y_{i+1} = y_i + (h/24)\,[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}]$$

The A – M corrector formula is given by

$$y_{i+1} = y_i + (h/24)\,[f_{i-2} - 5f_{i-1} + 19f_i + 9f_{i+1}]$$

PROBLEM

1. Solve the equation by using A – B – M method, $y^1 = (2y/x)$ with $y(1) = 2$, $y(1.25) = 3.13$, $y(1.5) = 4.5$, $y(1.75) = 6.13$ then estimate $y(2) =$ ....
Ans: 8.0079

Note:

1. A – B (predictor) formula involves extrapolation, hence it becomes unstable. Therefore, it is used mostly as a predictor.
2. A – M (corrector) formula is generally much more accurate.
3. A – B – M method is numerically stable.

PREVIOUS GATE QUESTIONS - "NUMERICAL METHODS"

1. The accuracy of Simpson's rule quadrature for a step is size h is *(GATE'03)*
a) $O(h^2)$ b) $O(h^3)$ c) $O(h^4)$ d) $O(h^5)$

2. Starting from $x_0 = 1$, one step of Newton-Raphson method in solving the equatioin $x^3 + 3x - 7 = 0$ gives the next value (x_1) as *(GATE'05)*
a) $x_1 = 0.5$ b) $x_1 = 1.406$ c) $x_1 = 1.5$ d) $x_1 = 2$

KEY

1. c 2. c

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PREVIOUS GATE QUESTIONS

01. Using the given data points tabulated below, a straight line passing through the origin is fitted using least squares method. The slope of the line is

x	1	2	3
y	1.5	2.2	2.7

IN-2005-2M

(a) 0.9 (b) 1
(c) 1.1 (d) 1.5

02. Newton – Raphson formula to find the roots of an equation f(x) = 0 is given by PI-2005-1M

(a) $x_{n+1} = x_n - \left[\dfrac{f(x_n)}{f^l(x_n)}\right]$

(b) $x_{n+1} = x_n + \left[\dfrac{f(x_n)}{f^l(x_n)}\right]$

(c) $x_{n+1} = \dfrac{f(x_n)}{\left[x_n f^l(x_n)\right]}$

(d) $x_{n+1} = \dfrac{x_n f(x_n)}{\left[f^l(x_n)\right]}$

03. For solving algebraic and transcendental equation, which one of the following is used? PI-2005-1M
(a) Coulomb's theorem
(b) Newton – Raphson Method
(c) Euler's theorem
(d) Stoke's theorem

04. The polynomial $p(x) = x^5 + x + 2$ has IN-2007-2M
(a) all real roots
(b) 3 real and 2 complex roots
(c) 1 real and 4 complex roots
(d) all complex roots

05. Identify the Newton – Raphson iteration scheme for the finding the square root of 2 IN-2007-2M

(a) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{2}{x_n}\right)$

(b) $x_{n+1} = \dfrac{1}{2}\left(x_n - \dfrac{2}{x_n}\right)$

(c) $x_{n+1} = \dfrac{2}{x_n}$

(d) $x_{n+1} = \sqrt{2 + x_n}$

06. Given that one root of the equation $x^3 - 10x^2 + 31x - 30 = 0$ is 5 the other tow roots are CE-2007-2M
(a) 2 and 3 (b) 2 and 4
(c) 3 and 4 (d) – 2 and – 3

07. The following equation needs to be numerically solved using the Newton – Raphson method $x^3 + 4x - 9 = 0$. The iterative equation for this purpose is (k indicates the iteration level) CE-2007-2M

(a) $x_{k+1} = \dfrac{2x_k^3 + 9}{3x_k^2 + 4}$

(b) $x_{k+1} = \dfrac{3x_k^3 + 4}{2x_k^2 + 9}$

(c) $x_{k+1} = x_k - 3x_k^2 + 4$

(d) $x_{k+1} = \dfrac{4x_k^2 + 3}{9x_k^2 + 2}$

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08. Matching exercise choose the correct one out of the alternatives A, B, C, D

Group – I

P. 2nd order differential equations
Q. Non – linear algebraic equations
R. Linear algebraic equations
S. Numerical integration

Group – II

1. Range – kulta method
2. Newton – Raphson method
3. Gauss elimination
4. Simpson's rule **PI-2007-2M**

(a) P – 3, Q – 2, R – 4, S – 1
(b) P – 2, Q – 4, R – 3, S – 1
(c) P – 1, Q – 2, R – 3, S – 4
(d) P – 1, Q – 3, R – 2, S – 4

09. The differential equation $\frac{dx}{dy}=\frac{1-x}{\tau}$ is discretised using Euler's numerical integrations method with a time step $\Delta T > 0$. What is the maximum permissible value of ΔT to ensue stability of the solution of the corresponding discrete time equation? **EE-2007-2M**

(a) 1 (b) $\tau/2$
(c) τ (d) 2τ

10. The equation $x^3 - x^2 + 4x - 4 = 0$ is to be solved using the Newton – Raphson method. If x = 2 is taken as the initial approximation of the solution then the first approximation using this method will be **EC-2008-2M**

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$
(c) 1 (d) $\frac{3}{2}$

11. A differential equation $\frac{dx}{dt}=e^{-2t}u(t)$ has to solved using trapezoidal rule of integration with a step size h = 0.01 s. Function u(t) indicates a unit step function if $x(0^-) = 0$, then value of x at t = 0.01 s will be given by **EE-2008-2M**

(a) 0.00099 (b) 0.00495
(c) 0.0099 (d) 0.0198

12. Equation $e^x - 1 = 0$ is required to be solved using Newton's method with an initial guess $x_0 = -1$. Then after one step of Newton's method, estimate x_1 of the solution will be given by **EE-2008-2M**

(a) 0.71828 (b) 0.36784
(c) 0.20587 (d) 0.00000

13. It is known that two roots of the non – linear equation $x^3 - 6x^2 + 11x - 6 = 0$ are 1 and 3. The third root will be **IN-2008-2M**

(a) j (b) – j
(c) 2 (d) 4

14. The recursion relation to solve $x = e^{-x}$ using Newton – Raphson method is **EC-2008-2M**

(a) $x_{n+1}=e^{-x_n}$

(b) $x_{n+1}=x_n - e^{-x_n}$

(c) $x_{n+1}=(1+x_n)\frac{e^{-x_n}}{1+e^{-x_n}}$

(d) $x_{n+1}=\frac{x_n^2-e^{-x_n}(1+x_n)-1}{x_n-e^{-x_n}}$

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15. The differential equation $\frac{dx}{dt}=\frac{4-x}{\tau}$ with x(0) = 0 and the constant $\tau > 0$ is to be numerically integrated using the forward Euler method with a constant integration time step T. The maximum value of T such that the numerical solution of x converges is **IN-2009-2M**

(a) $\tau/4$ (b) $\tau/2$
(c) τ (d) 2τ

16. Let $x^2 - 117 = 0$. The iterative steps for the solution using Newton's – Raphson's method is given by **EE-2009-2M**

(a) $x_{k+1}=\frac{1}{2}\left(x_k+\frac{117}{x_k}\right)$

(b) $x_{k+1}=x_k-\frac{117}{x_k}$

(c) $x_{k+1}=x_k-\frac{x_k}{117}$

(d) $x_{k+1}=x_k-\frac{1}{2}\left(x_k+\frac{117}{x_k}\right)$

17. A cubic polynomial with real coefficients **EE-2009-2M**
(a) can possibly have no extrema and no zero crossings
(b) may have ~~up~~ to three extrema and upto 2 zero crossings
(c) can not have more than two extrema and more than three zero crossings
(d) will always have an equal number of extrema and zero crossings

18. During the numerical solution of a first under differential equation using the Euler (also known as Euler canchy) method with step size h the local truncation error is of the order of

PI-2009-1M

(a) h^2 (b) h^3
(c) h^4 (d) h^5

19. Consider a differential equation $\frac{dy(x)}{dx}-y(x)=x$ with initial condition y(0) = 0. Using Euler's first order method with a step size of 0.1, the value of y(0.3) is **EC-2010-2M**

(a) 0.01 (b) 0.031
(c) 0.0631 (d) 0.1

20. Newton – Raphson method is used to compute a root of the equation $x^2 - 13 = 0$ with 3.5 as the initial value. The approximation after one iteration is **CS-2010-2M**

(a) 3.575 (b) 3.677
(c) 3.667 (d) 3.607

21. Euler's Method of integration is applied to the initial value problem $\frac{dx}{dy}=2x$, y(0) = 0. If the step size h = 0.2 then the error in computation (in percentage) after 5 steps would be **PI-2010-2M**

(a) 0 (b) 10
(c) 20 (d) 30

KEY:

01. b	02. a	03. b
~~04. c~~	05. a	06. a
07. a	08. c	09.
10. b	11.	12. a
13. c	14. c	15.
16. a	17.	18. a
19. b	20. d	21. c

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01	C VEERAIAH	EE-1010022	5
02	THEJOSWAMY NAIDU B	EE-1490267	7

INSTRUMENTATION ENGINEERING TOPPERS

S.NO	STUDENT NAME	GATE H.T NO	RANK
01	KAMANI DEEPESH HASMUKHLAL	IN-1290044	6
02	KARTHIKEYAN S	IN-1290364	8

CIVIL ENGINEERING TOPPERS

S.NO	STUDENT NAME	GATE H.T NO	RANK
01	ACHARYA PRERANA U	CE-1270825	9

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