Multiple Support Recovery Using Very Few Measurements Per Sample: Supplementary

Lekshmi Ramesh, Chandra R. Murthy, and Himanshu Tyagi

Assumption 1. *For a vector $X \sim \mathrm{P}^{(i)}$, $i \in [\ell]$, and an absolute constant c, we assume that $\mathbb{E}_{\mathrm{P}^{(i)}}\left[XX^T\right] = \mathtt{diag}\left(\lambda_i\right)$ with $\mathtt{supp}\left(\lambda_i\right) = \mathcal{S}_i$, and $X_t \sim subG(c\lambda_{it})$, $t \in [d]$. Furthermore, for each $i \in [\ell]$, $t \in \mathcal{S}_i$, we assume that $\lambda_{it} = \lambda_0 > 0$, and $\mathbb{E}\left[X_t^4\right] = \rho$.*

Assumption 2. *The $m \times d$ measurement matrices $\Phi_1, \ldots, \Phi_n$ are independent, with entries that are independent and zero-mean. Furthermore, $\Phi_i(u,v) \sim subG(c'/m)$, and the moment conditions $\mathbb{E}\left[\Phi_i(u,v)^2\right] = 1/m$ and $\mathbb{E}\left[\Phi_i(u,v)^{2q}\right] = c_q/m^q$ hold for $q \in \{2, \ldots, 4\}$, where c_q and c' are absolute constants.*

I. Proofs from Section IV B

Lemma 1 (Lemma 4 in the main manuscript)**.** *Under Assumptions 1 and 2, $\mathbb{E}\left[T\right]$ exhibits a block structure (under an appropriate permutation of rows and columns) given by*

$$\mathbb{E}\left[T_{uv}\right] = \begin{cases} \mu_0, & \text{if } u = v, \\ \mu_s, & \text{if } u \neq v, (u,v) \in \mathcal{S}_i \times \mathcal{S}_i \text{ for any } i \in [\ell], \\ \mu_d, & \text{otherwise}, \end{cases}$$

where the parameters μ_0, μ_s and μ_d depend on k, m and ℓ and can be explicitly calculated.

Proof. Our goal is to compute the expected value of the clustering matrix, denoted $\mathbb{E}\left[T\right]$, and we will do so by first conditioning on the measurement ensemble Φ_1^n and noting that each entry of T is then of the form $(X^\top AX)^2$, where X is subgaussian and A is a fixed matrix (given Φ_1^n). This conditional expectation can be calculated using Lemma 3. The next step is to average over the distribution of Φ_1^n, and our analysis will require the moment assumptions on the entries of Φ_1^n described in Assumption 2. Although each entry of $\mathbb{E}\left[T\right]$ can be explicitly characterized in terms of the system parameters, we will sometimes only mention the leading terms. In fact, the analysis of our algorithm in Theorem 1 only requires an upper bound on the diagonal entries and tight upper and lower bounds on the off diagonal entries of $\mathbb{E}\left[T\right]$.

Specifically, by the definition of T from (1), we note that

$$\mathbb{E}\left[T_{uv}\right] = \frac{1}{n}\sum_{j=1}^{n}(\Phi_{ju}^\top \Phi_j X_j)^2 \cdot (\Phi_{jv}^\top \Phi_j X_j)^2, \tag{1}$$

for $(u,v) \in \mathcal{S}_{\text{un}} \times \mathcal{S}_{\text{un}}$. The expectation in the expression above is over the joint distribution of X_1^n, Φ_1^n and the labels G_1^n (generating samples from the mixture $\mathrm{P}_{\mathcal{S}} = \frac{1}{\ell}\sum_{i=1}^{\ell}\mathrm{P}^{(i)}$ described in Section II in the main file can be thought of as drawing the label G uniformly from $[\ell]$, and conditioned on $G = g$, drawing a sample from $\mathrm{P}^{(g)}$). We will first condition on the labels (or, equivalently, on the random subsets $\{I_1, \ldots, I_\ell\}$ defined as $I_i \overset{\text{def}}{=} \{j \in [n] : \mathtt{supp}(X_j) = \mathcal{S}_i\}$ and on the measurement matrices. We focus on a single summand in (1), and drop the dependence on the sample index j. With a slight abuse of notation, we let $\mathcal{S} = \mathrm{supp}(X)$ denote the support of the sample we focus on and note that

$$\mathbb{E}_X\left[(\Phi_u^\top \Phi X)^2 \cdot (\Phi_v^\top \Phi X)^2 | \Phi, G\right] = \mathbb{E}_X\left[(X_{\mathcal{S}}^\top \alpha_u \alpha_v^\top X_{\mathcal{S}})^2 | \Phi, G\right], \tag{2}$$

where, $\alpha_u \overset{\text{def}}{=} \Phi_{\mathcal{S}}^\top \Phi_u$, $u \in \mathcal{S}_{\text{un}}$. We can now use Lemma 3 to get

$$\mathbb{E}_X\left[(X_{\mathcal{S}}^\top \alpha_u \alpha_v^\top X_{\mathcal{S}})^2 | \Phi, G\right] = \rho\sum_{i\in\mathcal{S}}\alpha_{ui}^2\alpha_{vi}^2 + \lambda_0^2\sum_{i\neq j}\alpha_{ui}^2\alpha_{vj}^2 + \lambda_0^2\sum_{i\neq j}\alpha_{ui}\alpha_{vi}\alpha_{uj}\alpha_{vj}, \tag{3}$$

where recall $\lambda_0 = \mathbb{E}\left[X_i^2\right]$ and $\rho = \mathbb{E}\left[X_i^4\right]$. We will first handle the $u = v$ case, which will be used to compute the diagonal entries of the mean matrix. We have, for every $u \in \mathcal{S}_{\text{un}}$,

$$\mathbb{E}_{X,\Phi}\left[(X_{\mathcal{S}}^\top \alpha_u \alpha_u^\top X_{\mathcal{S}})^2 | G\right] = \rho\mathbb{E}_\Phi\left[\sum_{i\in\mathcal{S}}\alpha_{ui}^4 | G\right] + 2\lambda_0^2\mathbb{E}_\Phi\left[\sum_{i\neq j}\alpha_{ui}^2\alpha_{uj}^2 | G\right] \tag{4}$$

The authors are with the Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore 560012, India. Email: {lekshmi, cmurthy, htyagi}@iisc.ac.in.

Note: All the Lemmas, Theorems, and equations that are cross-referenced in this document correspond to results within this document, unless stated otherwise.

$$= \rho \mathbb{E}_{\Phi}\left[\sum_{i\in\mathcal{S}} (\Phi_u^\top \Phi_i)^4 | G\right] + 2\lambda_0^2 \mathbb{E}_{\Phi}\left[\sum_{i\neq j} (\Phi_u^\top \Phi_i)^2 (\Phi_u^\top \Phi_j)^2 | G\right]. \tag{5}$$

When $u \in \mathcal{S}$,

$$\mu_0^s \stackrel{\text{def}}{=} \mathbb{E}_{X,\Phi}\left[(X_{\mathcal{S}}^\top \alpha_u \alpha_u^\top X_{\mathcal{S}})^2 | G\right] \tag{6}$$

$$= \rho \mathbb{E}_{\Phi}\left[\|\Phi_u\|_2^8 + \sum_{i\in\mathcal{S}\backslash\{u\}} (\Phi_u^\top \Phi_i)^4 | G\right] + 2\lambda_0^2 \mathbb{E}_{\Phi}\left[2\|\Phi_u\|_2^4 \sum_{i\in\mathcal{S}\backslash\{u\}} (\Phi_u^\top \Phi_i)^2 + \sum_{i\neq j} (\Phi_u^\top \Phi_i)^2 (\Phi_u^\top \Phi_j)^2 | G\right] \tag{7}$$

$$\le c\rho\left(1 + \frac{k-1}{m^2}\right) + c'\lambda_0^2\left(\frac{k-1}{m} + \frac{(k-1)(k-2)}{m^2}\right), \tag{8}$$

where we used Lemma 3 in the second step and Lemma 4 in the third step, and retained the leading terms.

When $u \in \mathcal{S}_{\text{un}}\backslash\mathcal{S}$, using Lemmas 3 and 4 once again, we have

$$\mu_0^d \stackrel{\text{def}}{=} \mathbb{E}_{X,\Phi}\left[(X_{\mathcal{S}}^\top \alpha_u \alpha_u^\top X_{\mathcal{S}})^2 | G\right] \tag{9}$$

$$= \rho \mathbb{E}_{\Phi}\left[\sum_{i\in\mathcal{S}} (\Phi_u^\top \Phi_i)^4 | G\right] + 2\lambda_0^2 \mathbb{E}_{\Phi}\left[\sum_{i\neq j} (\Phi_u^\top \Phi_i)^2 (\Phi_u^\top \Phi_j)^2 | G\right] \tag{10}$$

$$\le c\rho\left(\frac{k}{m^2}\right) + c'\lambda_0^2 \frac{k(k-1)}{m^2}. \tag{11}$$

We now use these results to bound the diagonal entries of the mean matrix $\mathbb{E}[T]$. Using (1), (8) and (11), we see that for $u \in \mathcal{S}_1$,

$$\mu_0 \stackrel{\text{def}}{=} \mathbb{E}[T_{uu}] = \mathbb{E}_G\left[\mathbb{E}_{X,\Phi}\left[\frac{1}{n}\left(\sum_{j\in I_1} (\Phi_{ju}^\top \Phi_j X_j)^4 + \cdots + \sum_{j\in I_\ell} (\Phi_{ju}^\top \Phi_j X_j)^4\right)\middle| G\right]\right] \tag{12}$$

$$= \mathbb{E}_G\left[\frac{1}{n}\left(|I_1|\mu_0^s + \sum_{i=2}^{\ell} |I_i|\mu_0^d\right)\right] \tag{13}$$

$$= \frac{1}{\ell}\mu_0^s + \frac{\ell-1}{\ell}\mu_0^d \tag{14}$$

$$\le \frac{c}{\ell}\left\{\rho\left(1 + \frac{k-1}{m^2}\right) + \lambda_0^2\left(\frac{k-1}{m} + \frac{(k-1)(k-2)}{m^2}\right)\right\} + \frac{c(\ell-1)}{\ell}\left\{\rho\left(\frac{k}{m^2}\right) + \lambda_0^2\frac{k(k-1)}{m^2}\right\}, \tag{15}$$

where we used $\mathbb{E}_G[|I_i|] = n/\ell$ for all $i \in [\ell]$, under the uniform mixture assumption. The same result holds for $u \in \mathcal{S}_i$ for any $i \in [\ell]$.

The next step is to bound the off diagonal entries of $\mathbb{E}[T]$. Continuing from (3), we will handle each of the three terms separately. For each of these terms, we will consider the case when both u and v belong to the same support, and when they belong to different supports. Overall, these calculations highlight the block structure of $\mathbb{E}[T]$, with the diagonal entries all being equal, and the off diagonal entries taking two different values based on whether the indices belong to the same support or not.

For the first term in (3), when $(u, v) \in \mathcal{S} \times \mathcal{S}$, $u \neq v$, we have

$$\mathbb{E}_{\Phi}\left[\sum_{i\in\mathcal{S}} \alpha_{ui}^2 \alpha_{vi}^2 | G\right] = \mathbb{E}_{\Phi}\left[\|\Phi_u\|_2^4 (\Phi_u^\top \Phi_v)^2 | G\right] + \mathbb{E}_{\Phi}\left[\|\Phi_v\|_2^4 (\Phi_u^\top \Phi_v)^2 | G\right] + \mathbb{E}_{\Phi}\left[\sum_{i\in\mathcal{S}\backslash\{u\}\cup\{v\}} (\Phi_i^\top \Phi_u)^2 (\Phi_i^\top \Phi_v)^2 \middle| G\right] \tag{16}$$

$$= \frac{2}{m}\left(1 + \frac{3}{m}(c_2 - 1) + \frac{1}{m^2}(c_3 - 3c_2 + 2)\right) + \frac{k-2}{m^2}\left(1 + \frac{1}{m}(c_2 - 1)\right) \stackrel{\text{def}}{=} \gamma_1^s, \tag{17}$$

using Lemma 4. On the other hand, when $(u, v) \in \mathcal{S} \times \mathcal{S}_{\text{un}}\backslash\mathcal{S}$, we have

$$\mathbb{E}_{\Phi}\left[\sum_{i\in\mathcal{S}} \alpha_{ui}^2 \alpha_{vi}^2 | G\right] = \mathbb{E}_{\Phi}\left[\|\Phi_u\|_2^4 (\Phi_u^\top \Phi_v)^2 | G\right] + \mathbb{E}_{\Phi}\left[\sum_{i\in\mathcal{S}\backslash\{u\}} (\Phi_i^\top \Phi_u)^2 (\Phi_i^\top \Phi_v)^2 \middle| G\right] \tag{18}$$

$$= \frac{1}{m}\left(1 + \frac{3}{m}(c_2 - 1) + \frac{1}{m^2}(c_3 - 3c_2 + 2)\right) + \frac{k-1}{m^2}\left(1 + \frac{1}{m}(c_2 - 1)\right) \stackrel{\text{def}}{=} \gamma_1^{sd}. \tag{19}$$

The same result holds when $(u,v) \in \mathcal{S}_{\text{un}}\backslash\mathcal{S} \times \mathcal{S}$. Finally, when $(u,v) \in \mathcal{S}_{\text{un}}\backslash\mathcal{S} \times \mathcal{S}_{\text{un}}\backslash\mathcal{S}$,

$$\mathbb{E}_\Phi\left[\sum_{i\in\mathcal{S}} \alpha_{ui}^2\alpha_{vi}^2 | G\right] = \mathbb{E}_\Phi\left[\sum_{i\in\mathcal{S}} (\Phi_i^\top\Phi_u)^2(\Phi_i^\top\Phi_v)^2 | G\right] \tag{20}$$

$$= \frac{k}{m^2}\left(1+\frac{1}{m}(c_2-1)\right) \stackrel{\text{def}}{=} \gamma_1^d. \tag{21}$$

For the second term in (3), when $(u,v) \in \mathcal{S}\times\mathcal{S}$,

$$\mathbb{E}_\Phi\left[\sum_{i\neq j} \alpha_{ui}^2\alpha_{vj}^2 | G\right] \tag{22}$$

$$= \mathbb{E}_\Phi\left[\|\Phi_u\|_2^4\|\Phi_v\|_2^4 + (\Phi_u^\top\Phi_v)^4 + \|\Phi_u\|_2^4 \sum_{i\in\mathcal{S}\backslash\{u\}\cup\{v\}} (\Phi_v^\top\Phi_i)^2 \middle| G\right]$$

$$+ \mathbb{E}_\Phi\left[\|\Phi_v\|_2^4 \sum_{i\in\mathcal{S}\backslash\{u\}\cup\{v\}} (\Phi_u^\top\Phi_i)^2 + (\Phi_u^\top\Phi_v)^2 \sum_{i\in\mathcal{S}\backslash\{u\}\cup\{v\}} (\Phi_v^\top\Phi_i)^2 \middle| G\right]$$

$$+ \mathbb{E}_\Phi\left[(\Phi_u^\top\Phi_v)^2 \sum_{i\in\mathcal{S}\backslash\{u\}\cup\{v\}} (\Phi_u^\top\Phi_i)^2 + \sum_{\substack{i,j\in\mathcal{S}\backslash\{u\}\cup\{v\}\\ i\neq j}} (\Phi_u^\top\Phi_i)^2\cdot(\Phi_v^\top\Phi_j)^2 \middle| G\right] \tag{23}$$

$$= \left(1+\frac{1}{m}(c_2-1)\right)^2 + \left(\frac{2}{m^2}+\frac{1}{m^3}(c_2^2-2)\right) + 2\left(1+\frac{1}{m}(c_2-1)\right)\frac{k-2}{m}$$
$$+ 2\frac{(k-2)}{m^2}\left(1+\frac{1}{m}(c_2-1)\right) + \frac{(k-2)(k-3)}{m^2} \stackrel{\text{def}}{=} \gamma_2^s, \tag{24}$$

where we used Lemma 4 in the second step. When $(u,v) \in \mathcal{S}\times\mathcal{S}_{\text{un}}\backslash\mathcal{S}$,

$$\mathbb{E}_\Phi\left[\sum_{i\neq j} \alpha_{ui}^2\alpha_{vj}^2 | G\right] = \mathbb{E}_\Phi\left[\|\Phi_u\|_2^4 \sum_{i\in\mathcal{S}\backslash\{u\}} (\Phi_v^\top\Phi_i)^2 | G\right] + \mathbb{E}_\Phi\left[(\Phi_u^\top\Phi_v)^2 \sum_{i\in\mathcal{S}\backslash\{u\}} (\Phi_u^\top\Phi_i)^2 | G\right]$$

$$+ \mathbb{E}_\Phi\left[\sum_{\substack{i,j\in\mathcal{S}\backslash\{u\}\\ j\neq i}} (\Phi_u^\top\Phi_i)^2\cdot(\Phi_v^\top\Phi_j)^2 | G\right] \tag{25}$$

$$= \left(1+\frac{1}{m}(c_2-1)\right)\frac{k-1}{m} + \frac{(k-1)}{m^2}\left(1+\frac{1}{m}(c_2-1)\right) + \frac{(k-1)(k-2)}{m^2} \stackrel{\text{def}}{=} \gamma_2^{sd}, \tag{26}$$

and the same expression holds when $(u,v) \in \mathcal{S}_{\text{un}}\backslash\mathcal{S}\times\mathcal{S}$. When $(u,v) \in \mathcal{S}_{\text{un}}\backslash\mathcal{S} \times \mathcal{S}_{\text{un}}\backslash\mathcal{S}$,

$$\mathbb{E}_\Phi\left[\sum_{i\neq j} \alpha_{ui}^2\alpha_{vj}^2 | G\right] = \mathbb{E}_\Phi\left[\sum_{\substack{i,j\in\mathcal{S}\\ j\neq i}} (\Phi_u^\top\Phi_i)^2\cdot(\Phi_v^\top\Phi_j)^2 | G\right] = \frac{k(k-1)}{m^2} \stackrel{\text{def}}{=} \gamma_2^d, \tag{27}$$

Finally, for the third term in (3), when $(u,v) \in \mathcal{S}\times\mathcal{S}$,

$$\mathbb{E}_\Phi\left[\sum_{i\neq j} \alpha_{ui}\alpha_{vi}\alpha_{uj}\alpha_{vj} | G\right] = \mathbb{E}_\Phi\left[\|\Phi_u\|_2^2\Phi_u^\top\Phi_v\cdot\|\Phi_v\|_2^2\Phi_u^\top\Phi_v | G\right]$$

$$+ \mathbb{E}_\Phi\left[\|\Phi_u\|_2^2\Phi_u^\top\Phi_v \sum_{j\in\mathcal{S}\backslash\{u\}\cup\{v\}} (\Phi_u^\top\Phi_j)\cdot(\Phi_v^\top\Phi_j) | G\right]$$

$$+ \mathbb{E}_\Phi\left[\|\Phi_v\|_2^2\Phi_u^\top\Phi_v \sum_{j\in\mathcal{S}\backslash\{u\}\cup\{v\}} (\Phi_u^\top\Phi_j)\cdot(\Phi_v^\top\Phi_j) | G\right]$$

$$+ \mathbb{E}_\Phi \left[\sum_{\substack{i,j \in \mathcal{S}\setminus\{u\}\cup\{v\} \\ j \neq i}} (\Phi_u^\top \Phi_i)(\Phi_v^\top \Phi_i)(\Phi_u^\top \Phi_j)(\Phi_v^\top \Phi_j) | G \right] \tag{28}$$

$$= \frac{1}{m}\left(1 + \frac{c_2 - 1}{m}\right)^2 + \frac{2(k-2)}{m^2}\left(1 + \frac{c_2 - 1}{m}\right) + \frac{(k-2)(k-3)}{m^3} \stackrel{\text{def}}{=} \gamma_3^s. \tag{29}$$

When $(u,v) \in \mathcal{S} \times \mathcal{S}_{\text{un}} \setminus \mathcal{S}$,

$$\mathbb{E}_\Phi \left[\sum_{i \neq j} \alpha_{ui}\alpha_{vi}\alpha_{uj}\alpha_{vj} | G \right] = \mathbb{E}_\Phi \left[\|\Phi_u\|_2^2 \Phi_u^\top \Phi_v \sum_{j \in \mathcal{S}\setminus\{u\}} (\Phi_u^\top \Phi_j) \cdot (\Phi_v^\top \Phi_j) | G \right]$$

$$+ \mathbb{E}_\Phi \left[\sum_{\substack{i,j \in \mathcal{S}\setminus\{u\} \\ j \neq i}} (\Phi_u^\top \Phi_i)(\Phi_u^\top \Phi_j)(\Phi_v^\top \Phi_i)(\Phi_v^\top \Phi_j) \middle| G \right] \tag{30}$$

$$= \frac{(k-1)}{m^2}\left(1 + \frac{c_2 - 1}{m}\right) + \frac{(k-1)(k-2)}{m^3} \stackrel{\text{def}}{=} \gamma_3^{sd}, \tag{31}$$

and the same expression holds when $(u,v) \in \mathcal{S}_{\text{un}} \setminus \mathcal{S} \times \mathcal{S}$. When $(u,v) \in \mathcal{S}_{\text{un}} \setminus \mathcal{S} \times \mathcal{S}_{\text{un}} \setminus \mathcal{S}$,

$$\mathbb{E}_\Phi \left[\sum_{i \neq j} \alpha_{ui}\alpha_{vi}\alpha_{uj}\alpha_{vj} | G \right] = \mathbb{E}_\Phi \left[\sum_{\substack{i,j \in \mathcal{S} \\ j \neq i}} (\Phi_u^\top \Phi_i)(\Phi_u^\top \Phi_j)(\Phi_v^\top \Phi_i)(\Phi_v^\top \Phi_j) \middle| G \right] \tag{32}$$

$$= \frac{k(k-1)}{m^3} \stackrel{\text{def}}{=} \gamma_3^d, \tag{33}$$

We have thus computed the expected values of each of the three terms in (3).

Thus, combining (17), (24) and (29) and using (3) and (1), we have for $(u,v) \in \mathcal{S}_1 \times \mathcal{S}_1$, $u \neq v$,

$$\mathbb{E}[T_{uv}] = \mathbb{E}_G \left[\frac{1}{n} \left(\sum_{j \in I_1} \rho\gamma_1^s + \lambda_0^2(\gamma_2^s + \gamma_3^s) + \sum_{j \in I_2} \rho\gamma_1^d + \lambda_0^2(\gamma_2^d + \gamma_3^d) \right.\right. \tag{34}$$

$$\left.\left. + \cdots + \sum_{j \in I_\ell} \rho\gamma_1^d + \lambda_0^2(\gamma_2^d + \gamma_3^d) \right) \right]$$

$$= \frac{1}{\ell}\left(\rho\gamma_1^s + \lambda_0^2(\gamma_2^s + \gamma_3^s)\right) + \frac{\ell - 1}{\ell}\left(\rho\gamma_1^d + \lambda_0^2(\gamma_2^d + \gamma_3^d)\right) \stackrel{\text{def}}{=} \mu_s, \tag{35}$$

where again we used $\mathbb{E}_G[|I_i|] = n/\ell$ for all $i \in [\ell]$. This holds for $(u,v) \in \mathcal{S}_i \times \mathcal{S}_i$, for every $i \in [\ell]$.

For the case when $(u,v) \in \mathcal{S}_1 \times \mathcal{S}_2$ or when $(u,v) \in \mathcal{S}_2 \times \mathcal{S}_1$,

$$\mathbb{E}[T_{uv}] = \mathbb{E}_G \left[\frac{1}{n} \left(\sum_{j \in I_1} \rho\gamma_1^{sd} + \lambda_0^2(\gamma_2^{sd} + \gamma_3^{sd}) + \sum_{j \in I_2} \rho\gamma_1^{sd} + \lambda_0^2(\gamma_2^{sd} + \gamma_3^{sd}) \right.\right. \tag{36}$$

$$\left.\left. + \sum_{j \in I_3} \rho\gamma_1^d + \lambda_0^2(\gamma_2^d + \gamma_3^d) + \cdots + \sum_{j \in I_\ell} \rho\gamma_1^d + \lambda_0^2(\gamma_2^d + \gamma_3^d) \right) \right] \tag{37}$$

$$= \frac{2}{\ell}\left(\rho\gamma_1^{sd} + \lambda_0^2(\gamma_2^{sd} + \gamma_3^{sd})\right) + \frac{\ell - 2}{\ell}\left(\rho\gamma_1^d + \lambda_0^2(\gamma_2^d + \gamma_3^d)\right) \stackrel{\text{def}}{=} \mu_d. \tag{38}$$

Again, the same expression holds for $\mathbb{E}[T_{uv}]$ whenever $(u,v) \in \mathcal{S}_i \times \mathcal{S}_j$, $i,j \in [\ell]$, $i \neq j$. The mean matrix $\mathbb{E}[T]$ thus has a block structure with μ_0 on the diagonal, μ_s on the remaining entries in the diagonal blocks and μ_d on the off diagonal blocks as depicted in Figure 1. □

Lemma 2 (Lemma 7 in the main manuscript)**.** *Under Assumptions 1 and 2, we have*

$$\|\mathbb{E}[T]\|_{op} \leq \rho\frac{k^2\ell}{m^2} + \lambda_0^2\frac{k^3\ell}{m^2}, \textit{ and } \Delta_\ell \geq \frac{\lambda_0^2 k}{\ell}.$$

Proof. Using the structure of $\mathbb{E}[T]$ derived in Lemma 1, we have,

$$\|\mathbb{E}[T]\|_{op} = \mu_0 + (k-1)\mu_s + k(\ell-1)\mu_d \tag{39}$$

$$
\mathbb{E}\left[T\right] = \left[\begin{array}{cccc} \mu_0 & \mu^s & \mu^d & \mu^d \\ \mu^s & \mu_0 & \mu^d & \mu^d \\ \mu^d & \mu^d & \mu_0 & \mu^s \\ \mu^d & \mu^d & \mu^s & \mu_0 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\}\mathcal{S}_1 \\ \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\}\mathcal{S}_2 \end{array}
$$

Fig. 1: Block structure of the expected clustering matrix when $\ell = 2$ and the supports are disjoint, under appropriate permutation of rows and columns.

$$
\leq \rho \frac{k^2 \ell}{m^2} + \lambda_0^2 \frac{k^3 \ell}{m^2}, \tag{40}
$$

where we have used the definitions in (15), (35) and (38), and simplified.

For the eigengap computation, we first note from the definitions in (35) and (38) that

$$
\begin{aligned}
\mu_s - \mu_d =& \frac{\rho}{\ell}(\gamma_1^s + \gamma_1^d - 2\gamma_1^{sd}) + \frac{\lambda_0^2}{\ell}(\gamma_2^s + \gamma_2^d - 2\gamma_2^{sd} + \gamma_3^s + \gamma_3^d - 2\gamma_3^{sd}) && (41)\\
=& \frac{\rho}{\ell} \cdot 0 + \frac{\lambda_0^2}{\ell}\left\{\left(1 + \frac{c_2 - 1}{m}\right)^2 + \frac{1}{m^2}\left(2 + \frac{c_2^2 - 2}{m}\right)\right. && (42)\\
&\left. + \frac{1}{m}\left(1 + \frac{c_2 - 1}{m}\right)^2 - \frac{2}{m}\left(1 + \frac{c_2 - 1}{m}\right)\left(1 + \frac{2}{m}\right) + \frac{4}{m^2}\right\} && (43)\\
\geq& \frac{\lambda_0^2}{\ell}. && (44)
\end{aligned}
$$

We therefore have

$$
\Delta_\ell = \nu_\ell - \nu_{\ell+1} = k(\mu_s - \mu_d) \geq \frac{\lambda_0^2 k}{\ell}. \tag{45}
$$

□

II. Moment and Concentration Bounds For Subgaussian Random Variables

Lemma 3. *Let $X \in \mathbb{R}^d$ be a mean zero random vector with independent entries such that $\mathbb{E}\left[X_i^2\right] = \lambda_0$ and $\mathbb{E}\left[X_i^4\right] = \rho$ for all $i \in [d]$. Then, for every $a, b \in \mathbb{R}^d$,*

$$
\mathbb{E}\left[(X^\top a b^\top X)^2\right] = \rho \sum_{i=1}^d a_i^2 b_i^2 + \lambda_0^2 \sum_{i \neq j} (a_i^2 b_j^2 + a_i b_i a_j b_j). \tag{46}
$$

In particular,

$$
\mathbb{E}\left[(X^\top a a^\top X)^2\right] = \rho \sum_{i=1}^d a_i^4 + 2\lambda_0^2 \sum_{i \neq j} a_i^2 a_j^2. \tag{47}
$$

Remark 1. If the second and fourth moments are related as $\rho = 2\lambda_0^2 = 2c$ for some absolute constant c, then the result simplifies to $\mathbb{E}\left[(X^\top a b^\top X)^2\right] = c((a^\top b)^2 + \|a\|_2^2 \|b\|_2^2)$.

Proof. To start with, we note that the quadratic form $X^\top a b^\top X$ is a subexponential random variable since X is subgaussian. Although this fact can be used to derive upper bounds on the moments of $X^\top a b^\top X$, we would like to explicitly compute the second moment. We have,

$$
\begin{aligned}
\mathbb{E}\left[(X^\top a b^\top X)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^d a_i b_i X_i^2 + \sum_{i \neq j} a_i b_j X_i X_j\right)^2\right] && (48)\\
&= \mathbb{E}\left[\left(\sum_{i=1}^d a_i b_i X_i^2\right)^2 + \left(\sum_{i \neq j} a_i b_j X_i X_j\right)^2 + 2\sum_{i=1}^d a_i b_i X_i^2 \sum_{i \neq j} a_i b_j X_i X_j\right] && (49)\\
&= \mathbb{E}\left[\sum_{i=1}^d a_i^2 b_i^2 X_i^4 + \sum_{i \neq j} a_i b_i a_j b_j X_i^2 X_j^2 + \sum_{i \neq j} a_i^2 b_j^2 X_i^2 X_j^2\right]. && (50)
\end{aligned}
$$

Using $\mathbb{E}\left[X_i^2\right]=\lambda_0$ and $\mathbb{E}\left[X_i^4\right]=\rho$, we get

$$\mathbb{E}\left[(X^\top ab^\top X)^2\right]=\rho\sum_{i=1}^{d}a_i^2b_i^2+\lambda_0^2\sum_{i\neq j}(a_i^2b_j^2+a_ib_ia_jb_j). \tag{51}$$

□

Lemma 4. *Let X, Y, Z and W be independent random vectors taking values in $\mathbb{R}^m$, with independent entries that are zero mean with variance $1/m$. Additionally, for every $i\in[m]$, let $\mathbb{E}\left[Z_i^{2q}\right]=c_q/m^q$, for q=2, 3, 4 and a constant c_q that depends only on q. Then, the following results hold:*

(i) $\mathbb{E}\left[\|Z\|_2^4\right]=1+\frac{1}{m}(c_2-1)$
(ii) $\mathbb{E}\left[\|Z\|_2^6\right]=1+\frac{3}{m}(c_2-1)+\frac{1}{m^2}(c_3-3c_2+2)$
(iii) $\mathbb{E}\left[\|Z\|_2^8\right]=1+\frac{6}{m}(c_2-1)+\frac{1}{m^2}(11-18c_2+6c_2^2+4c_3)+\frac{1}{m^3}(c_4-4c_3-6c_2^2+12c_2-6)$
(iv) $\mathbb{E}\left[(X^\top Y)^4\right]=\frac{2}{m^2}+\frac{1}{m^3}(c_2^2-2)$
(v) $\mathbb{E}\left[\|Z\|_2^4(Z^\top W)^2\right]=\frac{1}{m}\left(1+\frac{3}{m}(c_2-1)+\frac{1}{m^2}(c_3-3c_2+2)\right)$
(vi) $\mathbb{E}\left[(X^\top Z)^2(X^\top W)^2\right]=\frac{1}{m^2}\left(1+\frac{1}{m}(c_2-1)\right)$
(vii) $\mathbb{E}\left[\|Z\|_2^2\|W\|_2^2(Z^\top W)^2\right]=\frac{1}{m}\left(1+\frac{1}{m}(c_2-1)\right)^2$
(viii) $\mathbb{E}\left[\|Z\|_2^2(W^\top Z)(X^\top Z)(X^\top W)\right]=\frac{1}{m^2}\left(1+\frac{1}{m}(c_2-1)\right)$
(ix) $\mathbb{E}\left[(Z^\top X)(Z^\top Y)(W^\top X)(W^\top Y)\right]=\frac{1}{m^3}$
(x) $\mathbb{E}\left[(X^\top Y)^2\right]=\frac{1}{m}$.

Proof. (i)

$$\mathbb{E}\left[\|Z\|_2^4\right]=\mathbb{E}\left[\sum_{i=1}^m Z_i^4+\sum_{i\neq j}Z_i^2Z_j^2\right] \tag{52}$$

$$=\frac{c_2}{m}+\frac{m-1}{m}=1+\frac{1}{m}(c_2-1). \tag{53}$$

(ii)

$$\mathbb{E}\left[\|Z\|^6\right]=\mathbb{E}\left[(Z_1^2+\ldots+Z_m^2)^2(Z_1^2+\ldots+Z_m^2)\right] \tag{54}$$

$$=\mathbb{E}\left[\left(\sum_{i=1}^m Z_i^4+\sum_{i\neq j}Z_i^2Z_j^2\right)\left(\sum_{t=1}^m Z_t^2\right)\right] \tag{55}$$

$$=\mathbb{E}\left[\sum_{i=1}^m Z_i^4\sum_{t=1}^m Z_t^2+\sum_{t=1}^m Z_t^2\sum_{i\neq j}Z_i^2Z_j^2\right]. \tag{56}$$

For the first term,

$$\mathbb{E}\left[\sum_{i=1}^m Z_i^4\sum_{t=1}^m Z_t^2\right]=\mathbb{E}\left[\sum_{i=1}^m Z_i^6+\sum_{i\neq t}Z_i^4Z_t^2\right]$$

$$=m\frac{c_3}{m^3}+m(m-1)\frac{c_2}{m^2}\frac{1}{m}=\frac{1}{m^2}(c_3-c_2)+\frac{c_2}{m}, \tag{57}$$

and for the second term,

$$\mathbb{E}\left[\sum_{t=1}^m Z_t^2\sum_{i\neq j}Z_i^2Z_j^2\right]=\mathbb{E}\left[2\sum_{t\neq i}Z_t^4Z_i^2+\sum_{t\neq i\neq j}Z_t^2Z_i^2Z_j^2\right] \tag{58}$$

$$=2m(m-1)\frac{c_2}{m^2}\frac{1}{m}+m(m-1)(m-2)\frac{1}{m^3} \tag{59}$$

$$=1+\frac{1}{m}(2c_2-3)-\frac{2}{m^2}(c_2-1) \tag{60}$$

Thus,

$$\mathbb{E}\left[\|Z\|^6\right]=1+\frac{3}{m}(c_2-1)+\frac{1}{m^2}(c_3-3c_2+2). \tag{61}$$

(iii)

$$\begin{aligned}\mathbb{E}\left[\|Z\|^8\right] =&\mathbb{E}\left[(Z_1^2+\cdots+Z_m^2)^4\right]\\ =&m\mathbb{E}\left[Z_1^8\right]+\binom{m}{2}\frac{4!}{3!}2\mathbb{E}\left[Z_1^6Z_2^2\right]+\binom{m}{2}\frac{4!}{2!2!}2\mathbb{E}\left[Z_1^4Z_2^4\right]+\binom{m}{3}\frac{4!}{2!}3\mathbb{E}\left[Z_1^4Z_2^2Z_3^2\right]+\binom{m}{4}4!\mathbb{E}\left[Z_1^2Z_2^2Z_3^2Z_4^2\right]\\ =&1+\frac{6}{m}(c_2-1)+\frac{1}{m^2}(11-18c_2+6c_2^2+4c_3)+\frac{1}{m^3}(c_4-4c_3-6c_2^2+12c_2-6).\end{aligned}$$

(iv) To compute $\mathbb{E}\left[(X^\top Y)^4\right]$, we first note that

$$\mathbb{E}\left[(X^\top Y)^4|X\right]=\mathbb{E}\left[(Y^\top XX^\top Y)^2|X\right] \tag{62}$$

$$=\mathbb{E}\left[Y_1^4\right]\sum_{i=1}^m X_i^4+2(\mathbb{E}\left[Y_1^2\right])^2\sum_{i\neq j}X_i^2X_j^2 \tag{63}$$

$$=\frac{c_2}{m^2}\sum_{i=1}^m X_i^4+2\left(\frac{1}{m}\right)^2\sum_{i\neq j}X_i^2X_j^2, \tag{64}$$

where we used Lemma 3 in the second step. This gives

$$\mathbb{E}\left[(X^\top Y)^4\right]=\frac{c_2}{m}\mathbb{E}\left[X_1^4\right]+\frac{2(m-1)}{m}(\mathbb{E}\left[X_1^2\right])^2 \tag{65}$$

$$=\frac{c_2^2}{m^3}+\frac{2(m-1)}{m^3}=\frac{2}{m^2}+\frac{1}{m^3}(c_2^2-2). \tag{66}$$

(v) Similar to the previous calculation, we first compute the conditional expectation to get

$$\mathbb{E}\left[\|Z\|_2^4(Z^\top W)^2|Z\right]=\|Z\|_2^4\left(\sum_{i=1}^m\mathbb{E}\left[Z_i^2W_i^2|Z\right]+\sum_{i\neq j}\mathbb{E}\left[Z_iW_iZ_jW_j|Z\right]\right)=\|Z\|_2^4\frac{\|Z\|_2^2}{m}, \tag{67}$$

which gives

$$\mathbb{E}\left[\|Z\|_2^4(Z^\top W)^2\right]=\frac{1}{m}\mathbb{E}\left[\|Z\|_2^6\right]=\frac{1}{m}\left(1+\frac{3}{m}(c_2-1)+\frac{1}{m^2}(c_3-3c_2+2)\right). \tag{68}$$

(vi) We have

$$\mathbb{E}\left[(X^\top Z)^2(X^\top W)^2|X\right]=\mathbb{E}\left[(X^\top Z)^2|X\right]\mathbb{E}\left[(X^\top W)^2|X\right]=\frac{\|X\|_2^2}{m}\cdot\frac{\|X\|_2^2}{m}. \tag{69}$$

Thus,

$$\mathbb{E}\left[(X^\top Z)^2(X^\top W)^2\right]=\frac{1}{m^2}\left(1+\frac{1}{m}(c_2-1)\right). \tag{70}$$

(vii)

$$\mathbb{E}\left[\|Z\|_2^2\|W\|_2^2(Z^\top W)^2|Z\right]=\|Z\|_2^2\ \mathbb{E}\left[\|W\|_2^2(Z^\top W)^2|Z\right] \tag{71}$$

$$=\|Z\|_2^2\left(\sum_{i=1}^m\mathbb{E}\left[\|W\|_2^2Z_i^2W_i^2|Z\right]+\sum_{i\neq j}\mathbb{E}\left[\|W\|_2^2W_iW_jZ_iZ_j|Z\right]\right) \tag{72}$$

$$=\|Z\|_2^2\sum_{i=1}^m Z_i^2\mathbb{E}\left[W_i^4+\sum_{l\neq i}W_i^2W_l^2\right] \tag{73}$$

$$+\|Z\|_2^2\sum_{i\neq j}Z_iZ_j\mathbb{E}\left[W_i^3W_j+W_j^3W_i+\sum_{l\neq i,\ l\neq j}W_l^2W_iW_j\right] \tag{74}$$

$$=\|Z\|_2^2\sum_{i=1}^m Z_i^2\left(\frac{c_2}{m^2}+\frac{m-1}{m^2}\right)=\|Z\|_2^4\left(\frac{1}{m}+\frac{c_2-1}{m^2}\right). \tag{75}$$

Thus,

$$\mathbb{E}\left[\|Z\|_2^2\|W\|_2^2(Z^\top W)^2\right]=\frac{1}{m}\left(1+\frac{c_2-1}{m}\right)^2. \tag{76}$$

(viii)

$$\mathbb{E}\left[\|Z\|_2^2(W^\top Z)(X^\top Z)(X^\top W)|Z,W\right] = \|Z\|_2^2(W^\top Z)\mathbb{E}\left[X^\top W Z^\top X|W,Z\right] \tag{77}$$

$$= \|Z\|_2^2(W^\top Z)\frac{Z^\top W}{m}. \tag{78}$$

Using similar arguments as in the proof of (v),

$$\mathbb{E}\left[\|Z\|_2^2(W^\top Z)(X^\top Z)(X^\top W)\right] = \frac{1}{m^2}\left(1 + \frac{c_2 - 1}{m}\right). \tag{79}$$

(ix)

$$\mathbb{E}\left[(Z^\top X)(Z^\top Y)(W^\top X)(W^\top Y)|X,Y,W\right] = (W^\top X)(W^\top Y)\mathbb{E}\left[Z^\top X Y^\top Z|X,Y\right] \tag{80}$$

$$= (W^\top X)(W^\top Y)\frac{X^\top Y}{m} \tag{81}$$

Thus,

$$\mathbb{E}\left[(Z^\top X)(Z^\top Y)(W^\top X)(W^\top Y)\right] = \frac{1}{m}\mathbb{E}_{X,Y}\left[\mathbb{E}_W\left[(W^\top X)(W^\top Y)(X^\top Y)|X,Y\right]\right] \tag{82}$$

$$= \frac{1}{m}\mathbb{E}_{X,Y}\left[(X^\top Y)\mathbb{E}_W\left[W^\top X Y^\top W|X,Y\right]\right] \tag{83}$$

$$= \frac{1}{m^2}\mathbb{E}_{X,Y}\left[(X^\top Y)^2\right] = \frac{1}{m^3}. \tag{84}$$

(x)

$$\mathbb{E}\left[(X^\top Y)^2\right] = \sum_{i=1}^{m}\mathbb{E}\left[X_i^2 Y_i^2\right] + \sum_{i \neq j}\mathbb{E}\left[X_i Y_i X_j Y_j\right] = \frac{1}{m}. \tag{85}$$

$\square$