SYS130 - Hand-in 01

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Question 1)

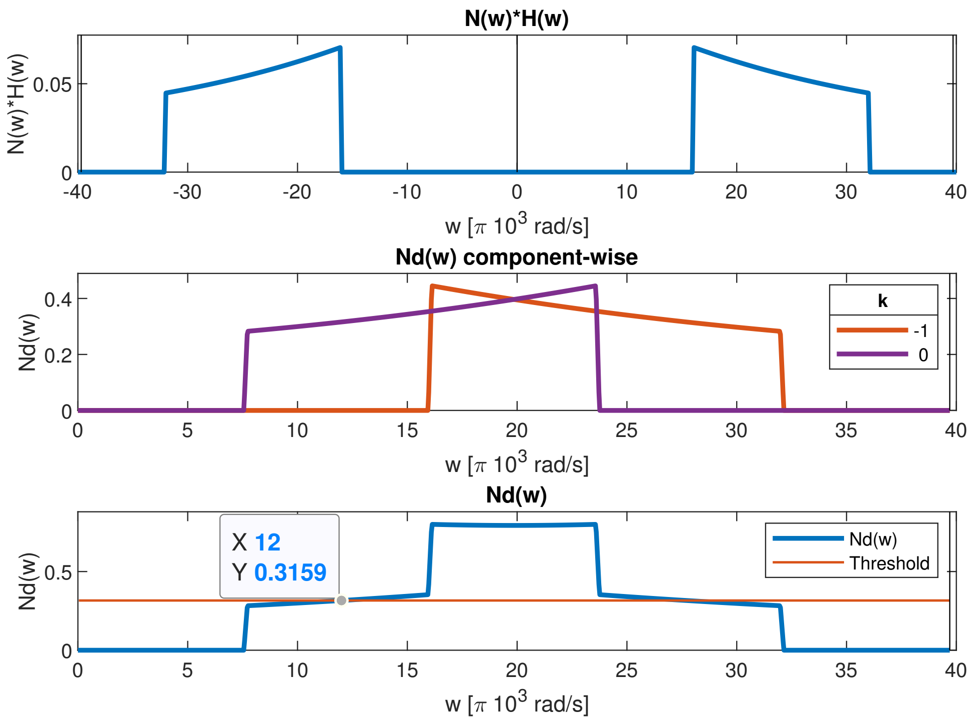

Figure 1: *Case 1:* $w_s > 32\,\pi 10^3$. *Data generated with* $w_s = 39.7\,\pi 10^3$

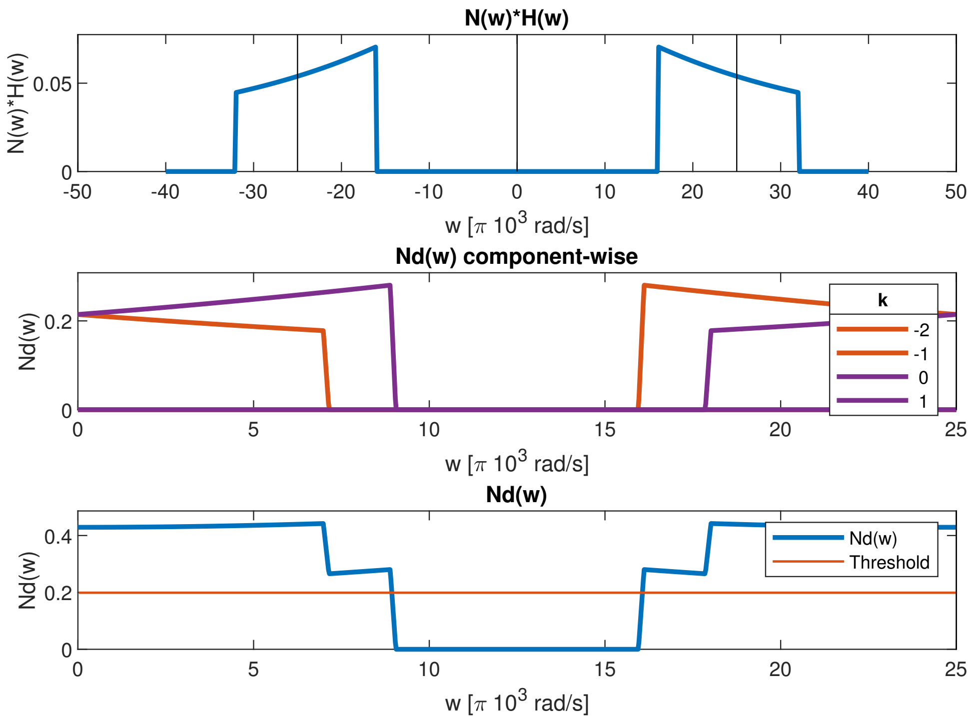

Figure 2: *Case 2:* $24\,\pi 10^3 < w_s < 32\,\pi 10^3$. *Data generated with* $w_s = 25\,\pi 10^3$

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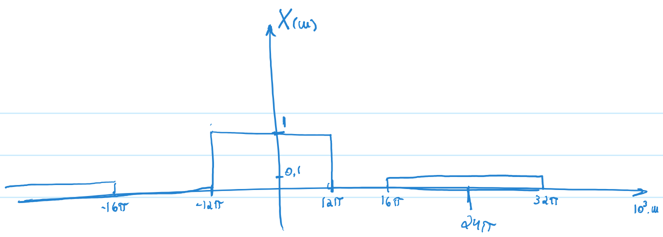
① $X(\omega) \longrightarrow \boxed{h(\omega)} \longrightarrow Y(\omega)$



$$H(\omega) = \frac{1}{1 + j\omega/\omega_0}, \quad \omega_0 = 16\pi \cdot 10^3 \text{ rad/s}$$

$$x(t) = x_s(t) + n(t) \quad \begin{cases} X_s(\omega) = \begin{cases} 1, & |\omega| < 12\pi \cdot 10^3 \text{ rad/s} \\ 0, & \text{otherwise} \end{cases} \\ N(\omega) = \begin{cases} 0{,}1, & 16\pi \cdot 10^3 \leq |\omega| \leq 32\pi \cdot 10^3 \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

ⓐ Signal $x_s(t)$ can be perfectly reconstructed from the sampled signal $x_d(n)$

iff $\omega_{max} \leq \dfrac{\omega_s}{2} \Rightarrow \omega_s \geq 2 \cdot 12\pi \cdot 10^3 \Rightarrow \boxed{\omega_s \geq 24\pi \cdot 10^3 \text{ rad/s}}$

ⓑ Filtered analog signal: $X_F(\omega) = \begin{cases} \dfrac{1}{1 + j\omega/\omega_0} & \text{if } |\omega| < 12\pi \cdot 10^3 \text{ rad/s} \\ 0 & \text{otherwise} \end{cases}$

Sampling $X_F(\omega)$: $X_d(\omega) = \dfrac{1}{\Delta t} \cdot \displaystyle\sum_{k=-\infty}^{\infty} X_F(\omega + \omega_s k)$

at $\omega = 0$: $X_d(\omega=0) = \dfrac{1}{\Delta t} \cdot \displaystyle\sum_{k=-\infty}^{\infty} X_F(\omega_s k) = \dfrac{1}{\Delta t} \Rightarrow \boxed{X_d(\omega=0) = \dfrac{1}{\Delta t}}$

Filtered Noise signal: $N_F(\omega) = \begin{cases} \dfrac{0{,}1}{1 + j\frac{\omega}{\omega_0}} & \text{if } 16\pi \cdot 10^3 \leq |\omega| \leq 32\pi \cdot 10^3 \\ 0 & \text{otherwise} \end{cases}$

Sampling $N_F(\omega) \longrightarrow N_d(\omega) = \dfrac{1}{\Delta t} \cdot \displaystyle\sum_{k=-\infty}^{\infty} N_F(\omega + \omega_s k)$

Case 1: $\omega_s > 32\pi \cdot 10^3 \frac{rad}{s}$ $\longrightarrow$ NO OVERLAP in $|\omega| < 12\pi \cdot 10^3$ and $k = \{-1, 0\}$

$$N_d(\omega) = \frac{1}{\Delta t}\left(\underbrace{N_F(\omega)}_{0\,*} + N_F(\omega - \omega_s)\right) \quad , * \ \forall\ |\omega| < 12\pi: \ \underline{N_F(\omega) = 0}$$

$$\Rightarrow \boxed{N_d(\omega) = \frac{1}{\Delta t} \cdot N_F(\omega - \omega_s) \quad \text{iff } |\omega| < 12\pi \cdot 10^3}$$

$$\max_{|\omega| < 12\pi}\left(\left\| N_d(\omega) \right\|\right) \leqslant \left\| \frac{X_d(\omega = 0)}{20} \right\| \Rightarrow \max_{|\omega| < 12\pi}\left(\frac{1}{\Delta t} \cdot \frac{0{,}1}{\left\| 1 + j \cdot \frac{\omega - \omega_s}{\omega_0} \right\|}\right) \leqslant \frac{1}{\Delta t} \cdot \frac{1}{20}$$

Clearly $\frac{0{,}1}{\left\| 1 + j\frac{\omega - \omega_s}{\omega_0} \right\|}$ has maximum value for $|\omega| < 12\pi \cdot 10^3$ when $\omega = 12 \cdot \pi \cdot 10^3$

$$\Rightarrow \cancel{\frac{1}{\Delta t}} \cdot \frac{0{,}1}{\left\| 1 + j \cdot \left(\frac{12 - \omega_s}{16}\right) \right\|} \leqslant \cancel{\frac{1}{\Delta t}} \cdot \frac{1}{20} \Rightarrow 1 + \left(\frac{12 - \omega_s}{16}\right)^2 \geqslant 4$$

$$\Rightarrow \left(\frac{12 - \omega_s}{16}\right)^2 \geqslant 3$$

Solutions: $\begin{cases} \omega_s \geqslant 12 + 16\sqrt{3} = 39{,}7\,\pi \cdot 10^3\,rad/s > 32\pi \cdot 10^3 \\ \cancel{\omega_s \leqslant 12 - 16\sqrt{3} = -15{,}7\,\pi \cdot 10^3\,rad/s} > -32\pi \cdot 10^3 \end{cases}$

Ok !!

✗ DOES NOT SATISFY Case 1: $|\omega| > 32\pi \cdot 10^3$

Case 2: $24\pi.10^3 \frac{rad}{s} < \omega_s < 32\pi.10^3 \frac{rad}{s} \longrightarrow$ OVERLAP In $|\omega| < 12\pi.10^3$ and $k=\{-2,-1,0,1\}$

$$N_d(\omega) = \frac{1}{\Delta t}\left(\underbrace{N_F(\omega-2\omega_s)}_{0\,*} + N_F(\omega-\omega_s) + \underbrace{N_F(\omega)}_{0\,*} + N_F(\omega+\omega_s)\right) \quad , * \ \forall \ |\omega| < 12\pi.10^3 \ , \ N_F(\omega) = N_F(\omega-2\omega_s) = 0$$

$$\Rightarrow \boxed{N_d(\omega) = \frac{1}{\Delta t}\left(N_F(\omega-\omega_s) + N_F(\omega+\omega_s)\right) \quad \text{iff} \ |\omega| < 32\pi.10^3}$$

$\text{Max}(\|N_d(\omega)\|)$

By visually inspection:

$0 < \omega \leq 32\pi.10^3 - \omega_s \qquad k=\{-1,1\}$

$$\max_\omega \|N_d(\omega)\| = \max_\omega \left\| \frac{N_F(\omega-\omega_s) + N_F(\omega+\omega_s)}{\Delta t} \right\|$$

$$= \max_\omega \left(\frac{1}{\Delta t}\cdot\left(\frac{1}{\left\|1+j\frac{\omega-\omega_s}{\omega_0}\right\|} + \frac{1}{\left\|1+j\frac{\omega+\omega_s}{\omega_0}\right\|}\right)\right) \leq \left\|\frac{X_d(\omega=0)}{20}\right\| = \frac{1}{\Delta t.20}$$

Solutions: $\begin{cases} \omega_s > 640\pi.10^3 \\ \omega_s < -640\pi.10^3 \end{cases}$ X DO NOT SATISFY Case 2 $24\pi.10^3 < \omega_s < 32\pi.10^3$

$32\pi.10^3 - \omega_s \leq \omega \leq 16.\pi.10^3 \qquad k=\{-1\}$

$$\max_\omega(\|N_d(\omega)\|) = \max_\omega\left(\left\|\frac{N_F(\omega-\omega_s)}{\Delta t}\right\|\right) = \max_\omega\left(\left\|\frac{1}{\Delta t}\frac{1}{\left(1+j\frac{\omega-\omega_s}{\omega_0}\right)}\right\|\right)$$

max at $\underline{\underline{\omega = 16\pi.10^3}}$

$$\Rightarrow \frac{1}{\cancel{\Delta t}}\cdot\frac{1}{\left\|1+j\frac{16-\omega_s}{\omega_0}\right\|} \leq \left\|\frac{X_d(\omega=0)}{20}\right\| = \frac{1}{\cancel{\Delta t}.20}$$

Solutions: $\begin{cases} \omega_s > 335\pi.10^3 \\ \omega_s < -303\pi.10^3 \end{cases}$ X DO NOT SATISFY Case 2 $24\pi.10^3 < \omega_s < 32\pi.10^3$

Only one solution among all cases satisfies requirement:

$$\boxed{\omega_s \geqslant (12+16\sqrt{3}).\pi.10^3 = 39,7.\pi.10^3 \, rad/s}$$

Question 2)

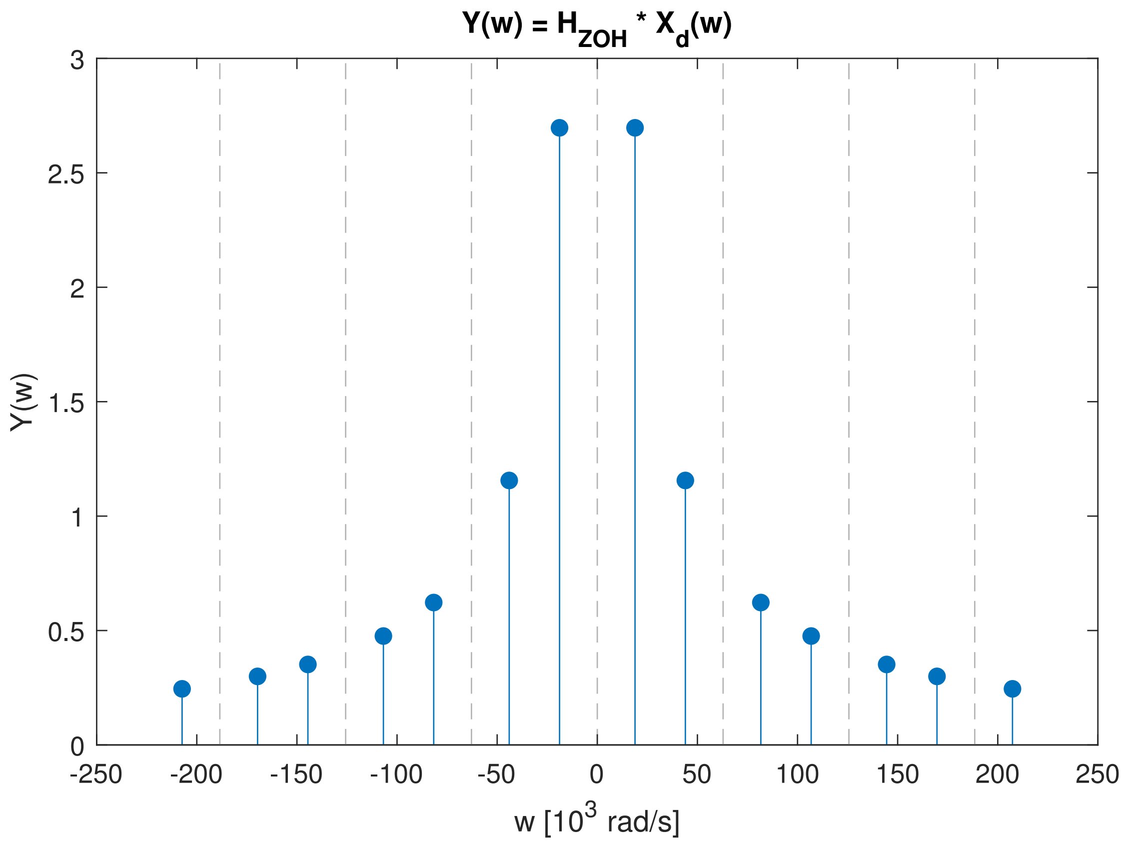


Figure 3: *Frequency response of the sampled and reconstructed $Y(w)$*

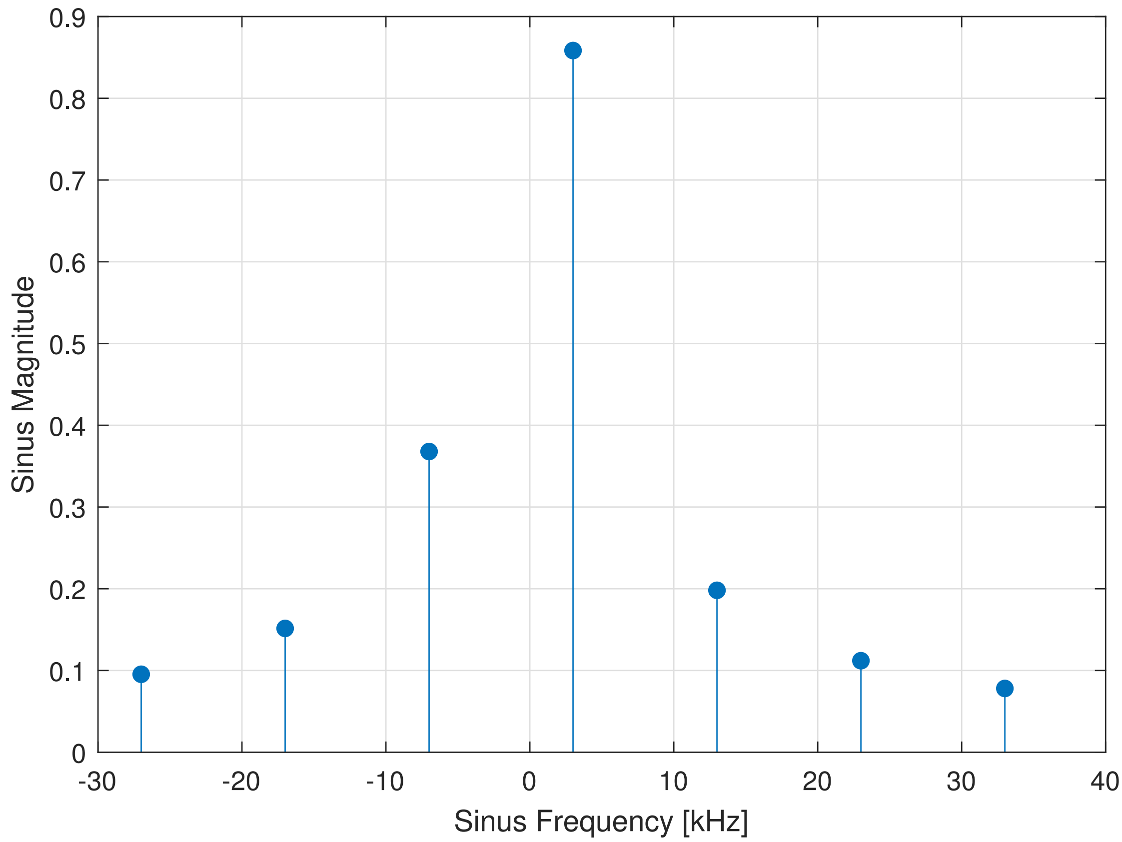


Figure 4: *Magnitude and frequency for the 7 first harmonics of $y(t)$*

② $x_d(n) = \sin(2\pi n\, f_0/f_s) = \sin(\omega_0 n.\Delta t) = \frac{e^{i\omega_0 n\Delta t} - e^{-i\omega_0 n\Delta t}}{2i}$

$$x(t) \longrightarrow \boxed{\text{Sampling}} \xrightarrow{x_d(n)} \boxed{\text{Reconstruction ZOH}} \xrightarrow{y(t)}$$

$\uparrow \Delta t \qquad \uparrow \Delta t$

Applying DTFT in x_d we get:

$$X_d(\omega) = DTFT\left\{\frac{e^{i\omega_0 n\Delta t} - e^{-i\omega_0 n\Delta t}}{2i}\right\} = \omega_s.\left(\frac{\tilde{\delta}(\omega-\omega_0) - \tilde{\delta}(\omega+\omega_0)}{2i}\right)$$

As known: $H_{ZOH}(\omega) = \Delta t.\, e^{-j\pi\frac{\omega}{\omega_s}} . \frac{\sin(\pi.\omega/\omega_s)}{\pi.\omega/\omega_s}$

So the reconstruction of $x_d(n)$ in the frequency domain will be:

$$Y(\omega) = X_d(\omega).H_{ZOH}(\omega) = \omega_s.\left(\frac{\tilde{\delta}(\omega-\omega_0) - \tilde{\delta}(\omega+\omega_0)}{2i}\right).\ \Delta t.\, e^{-j\pi\frac{\omega}{\omega_s}} . \frac{\sin(\pi.\omega/\omega_s)}{\pi.\omega/\omega_s}$$

$$\Rightarrow Y(\omega) = 2\pi.\sum_{k=-\infty}^{\infty}\left(\frac{\delta(\omega-\omega_0+k\omega_s) - \delta(\omega+\omega_0+k\omega_s)}{2i}\right).\, e^{-j\pi\frac{\omega}{\omega_s}} . \operatorname{sinc}(\omega/\omega_s)$$

Applying now the inverse Fourier IFT:

Changed this signal without any problem

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega).\, e^{j\omega t}\, d\omega = \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left(\frac{\delta(\omega-\omega_0+k\omega_s) - \delta(\omega+\omega_0-k\omega_s)}{2i}\right) e^{j\omega(t-\Delta t/2)} . \operatorname{sinc}(\omega/\omega_s).d\omega$$

$$= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\delta(\omega-\omega_0+k\omega_s) - \delta(\omega+\omega_0-k\omega_s)}{2i}\right).\, e^{j\omega(t-\Delta t/2)} . \operatorname{sinc}(\omega/\omega_s).d\omega$$

$$= \sum_{k=-\infty}^{\infty}\frac{1}{2i}\left[e^{j(\omega_0-k\omega_s)(t-\Delta t/2)} . \operatorname{sinc}\left(\frac{\omega_0-k\omega_s}{\omega_s}\right) - e^{-j(\omega_0-k\omega_s)(t-\Delta t/2)} . \underbrace{\operatorname{sinc}\left(\frac{-\omega_0+k\omega_s}{\omega_s}\right)}_{\text{Symetric} \Rightarrow \operatorname{sinc}\left(\frac{\omega_0-k\omega_s}{\omega_s}\right)}\right]$$

$$\boxed{y(t) = \sum_{k=-\infty}^{\infty}\operatorname{sinc}\left(\frac{\omega_0-k\omega_s}{\omega_s}\right) . \sin\left((\omega_0-k\omega_s)(t-\Delta t/2)\right)}$$

$y(t)$ is a sum of sinus with decreasing magnitude as ω increases !!!

	Magnitude $= \left\|\mathrm{sinc}\left(\frac{\omega_0 - k\omega_s}{\omega_s}\right)\right\|$ $= \left\|\mathrm{sinc}\left(\frac{3 - k.10}{10}\right)\right\|$	Frequency $= (\omega_0 - k\omega_s)$ $= (3 - k.10)$ [kHz]	
$k=-3$	0,078	33	
$k=-2$	0,112	23	
$k=-1$	0,1981	13	→ 2nd harmonic
$k=0$	0,858	$3 \Rightarrow f_0$	→ Fundamental
$k=1$	0,367	-7	→ 1st harmonic
$k=2$	0,151	-17	→ 3rd harmonic
$k=3$	0,095	-27	