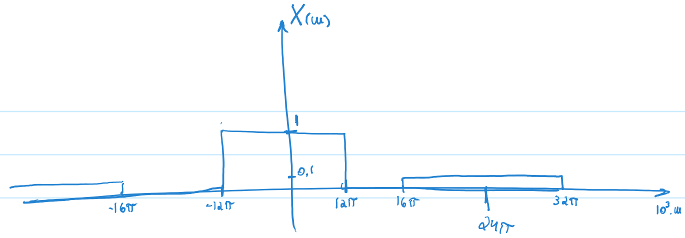
Hand-in 01

Freitag, 23. November 2018 12:06

1

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$



$$H(\omega) = \frac{1}{1 + j\omega/\omega_0} \quad , \quad \omega_0 = 16\pi \cdot 10^3 \text{ rad/s}$$

$$x(t) = x_s(t) + n(t) \qquad \begin{cases} X_s(\omega) = \begin{cases} 1, & |\omega| < 12\pi \cdot 10^3 \text{ rad/s} \\ 0, & \text{otherwise} \end{cases} \\ N(\omega) = \begin{cases} 0{,}1, & 16\pi \cdot 10^3 \leq |\omega| \leq 32\pi \cdot 10^3 \\ 0, & \text{otherwise} \end{cases} \end{cases}$$

a) Signal $x_s(t)$ can be perfectly reconstructed from the sampled signal $x_d(n)$

iff $\quad \omega_{max} \leq \frac{\omega_s}{2} \Rightarrow \omega_s \geq 2 \cdot 12\pi \cdot 10^3 \Rightarrow \boxed{\omega_s \geq 24\pi \cdot 10^3 \text{ rad/s}}$

b) Filtered analog signal : $X_F(\omega) = \begin{cases} \frac{1}{1 + j\omega/\omega_0} & \text{if } |\omega| < 12\pi \cdot 10^3 \text{ rad/s} \\ 0 & \text{otherwise} \end{cases}$

Sampling $X_F(\omega)$: $\quad X_d(\omega) = \frac{1}{\Delta t} \cdot \sum_{k=-\infty}^{\infty} X_F(\omega + \omega_s k)$

at $\omega = 0 \quad X_d(\omega=0) = \frac{1}{\Delta t} \cdot \sum_{k=-\infty}^{\infty} X_F(\omega_s k) = \frac{1}{\Delta t} \Rightarrow \boxed{X_d(\omega=0) = \frac{1}{\Delta t}}$

Filtered Noise signal : $N_F(\omega) = \begin{cases} \frac{0{,}1}{1 + j\frac{\omega}{\omega_0}} & \text{if } 16\pi \cdot 10^3 \leq |\omega| \leq 32\pi \cdot 10^3 \\ 0 & \text{otherwise} \end{cases}$

Sampling $N_F(\omega) \longrightarrow N_d(\omega) = \frac{1}{\Delta t} \cdot \sum_{k=-\infty}^{\infty} N_F(\omega + \omega_s k)$

Case 1: $\omega_s > 32\pi \cdot 10^3 \frac{rad}{s} \longrightarrow$ NO OVERLAP in $|\omega| < 12\pi \cdot 10^3$ and $k = \{-1, 0\}$

$$N_d(\omega) = \frac{1}{\Delta t}\left(\underbrace{N_F(\omega)}_{0\,*} + N_F(\omega - \omega_s)\right) \quad , * \ \forall\ |\omega| \leq 12\pi:\ N_F(\omega) = 0$$

$$\Rightarrow \boxed{N_d(\omega) = \frac{1}{\Delta t} \cdot N_F(\omega - \omega_s) \quad \text{iff } |\omega| \leq 12\pi \cdot 10^3}$$

$$\max_{|\omega| \leq 12\pi}\left(\|N_d(\omega)\|\right) \leq \left\|\frac{X_d(\omega = 0)}{20}\right\| \Rightarrow \max_{|\omega| < 12\pi}\left(\frac{1}{\Delta t} \cdot \frac{0{,}1}{\left\|1 + j \cdot \frac{\omega - \omega_s}{\omega_0}\right\|}\right) \leq \frac{1}{\Delta t} \cdot \frac{1}{20}$$

Clearly $\frac{0{,}1}{\left\|1 + j\frac{\omega - \omega_s}{\omega_0}\right\|}$ has maximum value for $|\omega| < 12\pi \cdot 10^3$ when $\omega = 12 \cdot \pi \cdot 10^3$

$$\Rightarrow \frac{1}{\cancel{\Delta t}} \cdot \frac{0{,}1}{\left\|1 + j \cdot \left(\frac{12 - \omega_s}{16}\right)\right\|} \leq \frac{1}{\cancel{\Delta t}} \cdot \frac{1}{20} \Rightarrow 1 + \left(\frac{12 - \omega_s}{16}\right)^2 \geq 4$$

$$\Rightarrow \left(\frac{12 - \omega_s}{16}\right)^2 \geq 3$$

Solutions: $\begin{cases} \omega_s \geq 12 + 16\sqrt{3} = 39{,}7\,\pi \cdot 10^3\,rad/s > 32\pi \cdot 10^3 & \text{Ok !!} \\ \cancel{\omega_s \leq 12 - 16\sqrt{3} = -15{,}7\,\pi \cdot 10^3\,rad/s} > -32\pi \cdot 10^3 & \text{DOES NOT SATISFY Case 1: } |\omega| > 32\pi \cdot 10^3 \end{cases}$

Case 2: $24\pi \cdot 10^3 \frac{rad}{s} < \omega_s < 32\pi \cdot 10^3 \frac{rad}{s} \longrightarrow$ OVERLAP in $|\omega| < 12\pi \cdot 10^3$ and $k = \{-2, -1, 0, 1\}$

$$N_d(\omega) = \frac{1}{\Delta t}\left(\underbrace{N_F(\omega - 2\omega_s)}_{0\,*} + N_F(\omega - \omega_s) + \underbrace{N_F(\omega)}_{0\,*} + N_F(\omega + \omega_s)\right), \quad * \ \forall\ |\omega| < 12\pi \cdot 10^3, \ N_F(\omega) = N_F(\omega - 2k) = 0$$

$$\Rightarrow \boxed{N_d(\omega) = \frac{1}{\Delta t}\left(N_F(\omega - \omega_s) + N_F(\omega + \omega_s)\right) \quad \text{iff } |\omega| < 32\pi \cdot 10^3}$$

$\mathrm{Max}(\|N_d(\omega)\|)$

By visually inspection:

$0 < \omega \leq 32\pi \cdot 10^3 - \omega_s \qquad k = \{-1, 1\}$

$$\max_{\omega} \|N_d(\omega)\| = \max_{\omega} \left\| \frac{N_F(\omega - \omega_s) + N_F(\omega + \omega_s)}{\Delta t} \right\|$$

$$= \max_{\omega} \left(\frac{1}{\Delta t} \cdot \left(\frac{1}{\left\|1 + j\frac{\omega - \omega_s}{\omega_0}\right\|} + \frac{1}{\left\|1 + j\frac{\omega + \omega_s}{\omega_0}\right\|} \right) \right) \leq \left\| \frac{X_d(\omega = 0)}{20} \right\| = \frac{1}{\Delta t \cdot 20}$$

Solutions: $\begin{cases} \omega_s > 640\pi \cdot 10^3 \\ \omega_s < -640\pi \cdot 10^3 \end{cases}$ X DO NOT SATISFY Case 2 $24\pi \cdot 10^3 < \omega_s < 32\pi \cdot 10^3$

$32\pi \cdot 10^3 - \omega_s \leq \omega \leq 16\pi \cdot 10^3 \qquad k = \{-1\}$

$$\max_{\omega}(\|N_d(\omega)\|) = \max_{\omega}\left(\left\| \frac{N_F(\omega - \omega_s)}{\Delta t} \right\|\right) = \max_{\omega}\left(\left\| \frac{1}{\Delta t} \frac{1}{\left(1 + j\frac{\omega - \omega_s}{\omega_0}\right)} \right\|\right)$$

max at $\omega = 16\pi \cdot 10^3$

$$\Rightarrow \frac{1}{\cancel{\Delta t}} \cdot \frac{1}{\left\|1 + j\frac{16 - \omega_s}{\omega_0}\right\|} \leq \left\| \frac{X_d(\omega = 0)}{20} \right\| = \frac{1}{\cancel{\Delta t} \cdot 20}$$

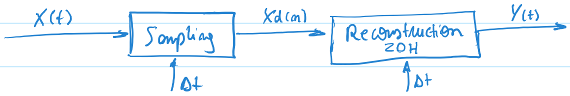
Solutions: $\begin{cases} \omega_s > 335\pi \cdot 10^3 \\ \omega_s < -303\pi \cdot 10^3 \end{cases}$ X DO NOT SATISFY Case 2 $24\pi \cdot 10^3 < \omega_s < 32\pi \cdot 10^3$

Only one solution among all cases satisfies requirement:

$$\boxed{\omega_s \geq (12 + 16\sqrt{3}) \cdot \pi \cdot 10^3 = 39{,}7 \cdot \pi \cdot 10^3 \ rad/s}$$

2

$$x_d(n) = \sin(2\pi n\, f_0/f_s) = \sin(\omega_0 n\cdot\Delta t) = \frac{e^{i\omega_0 n\Delta t} - e^{-i\omega_0 n\Delta t}}{2i}$$



Applying DTFT in xd we get:

$$X_d(\omega) = DTFT\left\{\frac{e^{i\omega_0 n\Delta t} - e^{-i\omega_0 n\Delta t}}{2i}\right\} = \omega_s\cdot\left(\frac{\tilde{\delta}(\omega-\omega_0) - \tilde{\delta}(\omega+\omega_0)}{2i}\right)$$

As known: $H_{ZOH}(\omega) = \Delta t\cdot e^{-j\pi\frac{\omega}{\omega_s}}\cdot\frac{\sin(\pi\cdot\omega/\omega_s)}{\pi\cdot\omega/\omega_s}$

So the reconstruction of xd(n) in the frequency domain will be:

$$Y(\omega) = X_d(\omega)\cdot H_{ZOH}(\omega) = \omega_s\cdot\left(\frac{\tilde{\delta}(\omega-\omega_0) - \tilde{\delta}(\omega+\omega_0)}{2i}\right)\cdot\Delta t\cdot e^{-j\pi\frac{\omega}{\omega_s}}\cdot\frac{\sin(\pi\cdot\omega/\omega_s)}{\pi\cdot\omega/\omega_s}$$

$$\Rightarrow Y(\omega) = 2\pi\cdot\sum_{k=-\infty}^{\infty}\left(\frac{\delta(\omega-\omega_0+k\omega_s) - \delta(\omega+\omega_0+k\omega_s)}{2i}\right)\cdot e^{-j\pi\frac{\omega}{\omega_s}}\cdot sinc(\omega/\omega_s)$$

Applying now the inverse Fourier IFT:

Changed this signal without any problem

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\cdot e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\left(\frac{\delta(\omega-\omega_0+k\omega_s) - \delta(\omega+\omega_0-k\omega_s)}{2i}\right)e^{j\omega(t-\Delta t/2)}\cdot sinc(\omega/\omega_s)\,d\omega$$

$$= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\delta(\omega-\omega_0+k\omega_s) - \delta(\omega+\omega_0-k\omega_s)}{2i}\right)\cdot e^{j\omega(t-\Delta t/2)}\cdot sinc(\omega/\omega_s)\,d\omega$$

$$= \sum_{k=-\infty}^{\infty}\frac{1}{2i}\cdot\left[e^{j(\omega_0-k\omega_s)(t-\Delta t/2)}\cdot sinc\left(\frac{\omega_0-k\omega_s}{\omega_s}\right) - e^{-j(\omega_0-k\omega_s)(t-\Delta t/2)}\cdot \underbrace{sinc\left(\frac{-\omega_0+k\omega_s}{\omega_s}\right)}_{\text{Symetric} \Rightarrow sinc\left(\frac{\omega_0-k\omega_s}{\omega_s}\right)}\right]$$

$$\boxed{y(t) = \sum_{k=-\infty}^{\infty} sinc\left(\frac{\omega_0-k\omega_s}{\omega_s}\right)\cdot\sin\left((\omega_0-k\omega_s)(t-\Delta t/2)\right)}$$

y(t) is a sum of sinus with decreasing magnitude as ω increases !!!

	Magnitude $= \left\|\mathrm{sinc}\left(\frac{\omega_0 - k\omega_s}{\omega_s}\right)\right\|$ $= \left\|\mathrm{sinc}\left(\frac{3 - k.10}{10}\right)\right\|$	Frequency $= (\omega_0 - k\omega_s)$ $= (3 - k.10)$ [kHz]	
$k=-3$	0,078	33	
$k=-2$	0,112	23	
$k=-1$	0,1981	13	2nd harmonic
$k=0$	0,858	3 $= f_0$	Fundamental
$k=1$	0,367	-7	1st harmonic
$k=2$	0,151	-17	3rd harmonic
$k=3$	0,095	-27	