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WRITTEN TEST 04/09/2023

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First Name: ..... Last Name: .....

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This exam has 2 questions, with subparts. You have **2 hours** to complete the exam. There are a total of 30 points available (sufficiency at 18 points). You cannot consult any notes, books or aids of any kind except for the codes implemented during the lab sessions. Write answers legibly in the additional sheets provided, and show all of your work. Please **write your name** on the exam itself and on the extra sheets. Concerning the implementations using Eigen, upload only the `.cpp` main files. For the exercises requiring LIS, report the bash commands used to perform the computations in a unique `.txt` file. **Upload the files** following the received instructions.

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### Exercise 1

1. Consider the following problem: find  $\mathbf{x} \in \mathbb{R}^n$ ,  $A\mathbf{x} = \mathbf{b}$ , where  $A \in \mathbb{R}^{n \times n}$  and  $\mathbf{b} \in \mathbb{R}^n$  are given. State under which conditions the mathematical problem is well posed.
2. Describe the general form of a linear iterative method for the approximate solution of  $A\mathbf{x} = \mathbf{b}$  and describe the stopping criteria.
3. State the necessary and sufficient condition for convergence.
4. Describe the Generalized Minimal Residual Method (GMRES). Recall the interpretation of the scheme as a Krylov subspace method and the main theoretical results.
5. Download the matrix `Aex1.mtx` from the Exam folder in webeep and save it on the `/shared-folder/iter_sol++` directory. Load the matrix in a new file called `exer1.cpp` using the `unsupported/Eigen/SparseExtra` module and check if the matrix is symmetric. Report on the sheet  $\|A\|$  where  $\|\cdot\|$  denotes the Euclidean norm.
6. Define the Eigen vector  $\mathbf{b} = (1, 1, \dots, 1)^T$  and find the approximate solution  $\mathbf{x}_j$  of  $A\mathbf{x} = \mathbf{b}$  using the Jacobi method (implemented in the `jacobi.hpp` template). Fix a maximum number of iterations which is sufficient to reduce the (relative) residual below than  $10^{-5}$  and take  $\mathbf{x}_0 = \mathbf{b}$  as initial guess. Report on the sheet the iteration counts and  $\|\mathbf{x}_j\|$ .
7. Compute the approximate solution  $\mathbf{x}_g$  of  $A\mathbf{x} = \mathbf{b}$  obtained using the GMRES method without *restart*. Fix a maximum number of iterations which is sufficient to reduce the residual below than  $10^{-10}$  considering  $\mathbf{x}_j$  (computed in the previous point) as initial guess. Use the Eigen diagonal preconditioner. Report the iteration counts and  $\|\mathbf{x}_j - \mathbf{x}_g\|$ .
8. Repeat the previous points using the iterative solvers available in the LIS library. First, compute the approximate solution obtained with the Jacobi method prescribing a tolerance of  $10^{-5}$ . Then, using the GMRES solver compute the approximate solution up to a tolerance of  $10^{-10}$ . Find a preconditioning strategy that yield a decrease in the number of required iterations with respect to the GMRES method without preconditioning. Report on the sheet the iteration counts and the relative residual at the last iteration.

## Exercise 2

1. Consider the rectangular linear system  $A\mathbf{x} = \mathbf{b}$ , where  $A$  is an  $m \times n$  matrix,  $m \geq n$ . Provide the definition of the solution in the least-square sense and state under which condition the problem is well posed.
2. Describe the QR factorization of the rectangular matrix  $A$ .
3. Discuss how the QR factorization can be employed to solve the above linear system in the least-square sense.
4. Download the matrix `Aex2.mtx` from the Exam folder in webeep and save it on the `/shared-folder/iter_sol++` directory. Load the matrix in a new file called `exer2.cpp` using the `unsupported/Eigen/SparseExtra` module. Report on the sheet  $\|A^T A\|$ .
5. Define an `Eigen` vector  $\mathbf{b} = A\mathbf{x}^*$ , where  $\mathbf{x}^* = (1, 1, \dots, 1)^T$ . Report on the sheet  $\|\mathbf{b}\|$ .
6. Use the `SparseQR` solver available in the `Eigen` library to compute the approximate solution of the least-square problem associated to  $A\mathbf{x} = \mathbf{b}$ . Report on the sheet the Euclidean norm of the error  $\|\mathbf{x}_{SQR} - \mathbf{x}^*\|$ , where  $\mathbf{x}_{SQR}$  is the obtained approximate solution.
7. Use the `LeastSquaresConjugateGradient` solver available in the `Eigen` library to compute the approximate solution of the previous least-square problem up to a tolerance of  $10^{-10}$ . Report on the sheet the iteration counts and the error norm  $\|\mathbf{x}_{LSCQ} - \mathbf{x}^*\|$ , where  $\mathbf{x}_{LSCQ}$  is the obtained approximate solution.
8. Export the matrix  $A^T A$  in the matrix market format (save it as `AtA.mtx`) and move it to the `lis-2.0.34/test` folder. Using the Conjugate Gradient iterative solver available in LIS compute the approximate solution of the linear system  $A^T A\mathbf{x} = A^T \mathbf{b}$  up to a tolerance of  $10^{-10}$ . Report the iteration counts and the relative residual at the last iteration.