Numerical Linear Algebra
A.A. 2022/2023

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First Name: .... Last Name: ....

This exam has 2 questions, with subparts. You have **2 hours** to complete the exam. There are a total of 30 points available (sufficiency at 18 points). You cannot consult any notes, books or aids of any kind except for the codes implemented during the lab sessions. Write answers legibly in the additional sheets provided, and show all of your work. Please **write your name** on the exam itself and on the extra sheets. Concerning the implementations using Eigen, upload only the `.cpp` main files. For the exercises requiring LIS, report the bash commands used to perform the computations in a unique `.txt` file. **Upload the files** following the received instructions.

Exercise 1

1. Consider the following problem: find $\boldsymbol{x} \in \mathbb{R}^n$, $A\boldsymbol{x} = \boldsymbol{b}$, where $A \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$ are given. State under which conditions the mathematical problem is well posed.

2. Describe the Cholesky factorization and how it is used to approximately solve the above linear system.

3. State under what conditions Cholesky factorization can be use.

4. Report the main theoretical result and comment on the computational costs.

5. Comment on the main differences with respect to the LU factorization.

6. Download the sparse matrix `Aex1.mtx` from the Exam folder in webeep and save it on the `/shared-folder/iter_sol`++ folder. Load the matrix in a new file `exer1.cpp`. Report on the sheet the matrix size and the Euclidean norm of `Aex1.mtx`. Is the matrix symmetric?

7. Define an `Eigen` vector $\boldsymbol{x}^* = (1, 1, \dots, 1)^{\mathrm{T}}$ with size equal to the number of columns of `Aex1.mtx`. Compute $\boldsymbol{b} = A\boldsymbol{x}^*$ and report on the sheet $\|\boldsymbol{b}\|$.

8. Solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$ using the Cholesky decomposition method available in the `Eigen` library. Report on the sheet the norm of the absolute error.

9. Solve the previous linear system using the Jacobi iterative method implemented in the `jacobi.hpp` template. Set a tolerance sufficient to achieve an error with magnitude similar to the one obtained with the Cholesky method. Report on the sheet the iteration counts and the norm of the absolute error at the last iteration.

10. Repeat the previous point using the Conjugate Gradient method implemented in the `cg.hpp` template combined with the ILU preconditioner provided by the `IncompleteLUT` Eigen function. Report the iteration counts (required to achieve a precision comparable to the previous approximate solutions) and the norm of the absolute error at the last iteration.

Exercise 2

1. Consider the following eigenvalue problem: $A\boldsymbol{x} = \lambda \boldsymbol{x}$, where $A \in \mathbb{R}^{n\times n}$ is given. Describe the power method for the numerical approximation of the largest in modulus eigenvalue of A. Introduce the notation, the algorithm, and the applicability conditions.

2. State the main theoretical results.

3. Comment of the computational costs.

4. Let A be a 81×81 tridiagonal matrix defined such that

$$A = \begin{pmatrix} -3 & 2 & 0 & 0 & \dots & 0 \\ 1 & -3 & 2 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & 1 & -3 & 2 \\ 0 & 0 & \dots & 0 & 1 & -3 \end{pmatrix}.$$

   In a new file called `exer2.cpp`, define the matrix A in the sparse format. Report on the sheet $\boldsymbol{v}^{\mathrm{T}} A \boldsymbol{v}$, where $\boldsymbol{v}$ is such that $v_i = 1$ for all $0 \le i < 81$.

5. Solve the eigenvalue problem $A\boldsymbol{x} = \lambda \boldsymbol{x}$ using the proper solver provided by `Eigen`. Report on the sheet the smallest and largest $\lambda_{min} < \lambda_{max}$ computed eigenvalues of A.

6. Using the `unsupported/Eigen/SparseExtra` module, export matrix A in the matrix market format (save as `Aex2.mtx`) and move it to the folder `lis-2.0.34/test`. Using the proper iterative solver available in the LIS library compute the largest eigenvalue λ_{max} of A up to a tolerance of 10^{-8}. Report the computed eigenvalue and the number of iterations required to achieve the prescribed tolerance.

7. Using the proper iterative solver available in the LIS library compute the smallest eigenvalue λ_{min} of A up to a tolerance of 10^{-8}. Report the computed eigenvalue and the number of iterations required to achieve the prescribed tolerance.

8. Using the Jacobi solver available in LIS, find the approximate solution of the linear system $A\boldsymbol{x} = \boldsymbol{e}_{min}$, where $\boldsymbol{e}_{min}$ is the eigenvector corresponding to the eigenvalue λ_{min} computed in the previous point. Report on the sheet the iteration counts and the relative residual at the last iteration. What is the relation between $\boldsymbol{x}$ and $\boldsymbol{e}_{min}$?