1 Expectred Prediction Error (EPE) - Buzz exercise, Week 1

Show that

$$E(y-\hat{f})^2 = \sigma^2 + (E(\hat{f}) - f)^2 + E((\hat{f} - E\hat{f})^2)$$

given

$$y = f + e, \quad E(e) = 0 \text{ and } E(e^2) = \sigma^2$$

Which uses the short notation:

$$\begin{aligned} f &= f(x_0), \\ \hat{f} &= \hat{f}(x_0; D), \\ E() &= E_{y,D}() \end{aligned}$$

we have that

$$\begin{aligned} Ey &= f, \\ Ef &= f, \end{aligned}$$

also remember that

$$\begin{aligned} E(X^2) &= E(X)^2, \\ E(X+Y) &= E(X) + E(Y), \end{aligned}$$

Starting from the back, the three terms: $\sigma^2 + (E\hat{f} - f)^2 + E((\hat{f} - E\hat{f})^2)$ can be written as:

$$\begin{aligned} e^2 =& E(y-f)^2 = Ey^2 + Ef^2 - 2EyEf = Ey^2 - f^2 \\ (E\hat{f} - f)^2 =& (E\hat{f} - f)^2 = (E\hat{f})^2 + f^2 - 2fE\hat{f} \\ E((\hat{f} - E\hat{f})^2) =& E\hat{f}^2 + E\hat{f}^2 - 2E\hat{f}E\hat{f} = 2E(\hat{f}^2) - 2(E\hat{f})^2 \end{aligned}$$

Adding the three terms, we get

$$\begin{aligned} \sigma^2 + (E\hat{f} - f)^2 + E((\hat{f} - E\hat{f})^2) =& Ey^2 - f^2 + (E\hat{f})^2 + f^2 - 2fE\hat{f} + 2E(\hat{f}^2) - 2(E\hat{f})^2 \\ =& Ey^2 + E\hat{f}^2 - 2EfE\hat{f} \\ =& Ey^2 + E\hat{f}^2 - 2yE\hat{f} \\ =& E(y - \hat{f})^2 \end{aligned}$$