EE5103 Computer Control Systems: Homework #3 Solution

Semester 1 Y2025/2026

Q1

The block diagram for this control system is

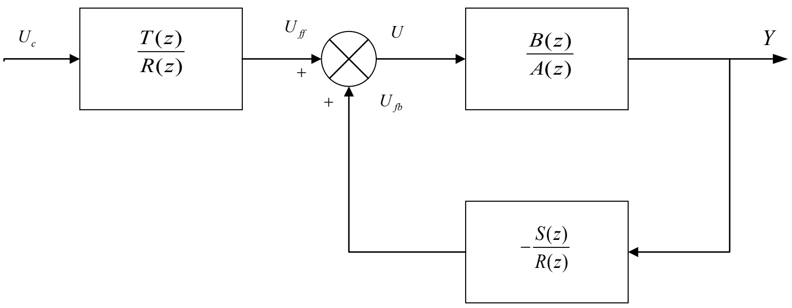

Figure 1 Block diagram of the closed-loop system

And we have

$$\frac{B(z)}{A(z)} = \frac{z+0.9}{z^2 - 1.8z + 0.81} \tag{0.1}$$

a) The polynomial degree of $A(z)$ and $A_m(z)$ are respectively $\deg(A(\mathrm{z})) = 2$ and $\deg(A_m(z)) = 2$. Besides, the closed-loop transfer function from U_{ff} to Y is

$$\frac{Y(z)}{U_{ff}(z)} = \frac{B(z)R(z)}{A(z)R(z) + B(z)S(z)} = \frac{B(z)R(z)}{A_{cl}(z)} \tag{0.2}$$

If the process zero is cancelled, it can be inferred that the closed-loop denominator before cancellation is

$$A_{cl}(z) = (\mathrm{z}+0.9)A_m(z) = (z+0.9)(z^2 - 1.5z + 0.7) \tag{0.3}$$

Obviously, we have $\deg(R(z)) = \deg(A_{cl}(z)) - \deg(A(z)) = 1$. Additionally, since it needs to cancel the zero $z+0.9$, the $R(z)$ must be in this form $R(z) = (z+0.9)$. Further, suppose $S(z) = s_o z + s_1$ due to $\deg(\mathrm{S}(z)) \le \deg(R(z))$, then we have

$$\begin{aligned} A_{cl}(z) &= A(z)R(z) + B(z)S(z) \\ &= (z^2 - 1.8z + 0.81)(z+0.9) + (z+0.9)(s_o z + s_1) \\ &= (z+0.9)[z^2 + (s_0 - 1.8)z + 0.81 + s_1] \end{aligned} \tag{0.4}$$

From (0.3) and (0.4), it can be inferred that

$$rz^2 + (s_0 - 1.8)z + 0.81 + s_1 = A_m(z) = z^2 - 1.5z + 0.7 \tag{0.5}$$

And it can be solved as $s_0 = 0.3$, $s_1 = -0.11$. Hence

$$\frac{S(z)}{R(z)} = \frac{0.3z - 0.11}{z + 0.9} \tag{0.6}$$

Now suppose $T(z) = t_o$, the closed-loop transfer function from u_c to y is

$$G(z) = \frac{Y(z)}{U_c(z)} = \frac{B(z)T(z)}{A(z)R(z) + B(z)S(z)} = \frac{(z+0.9)t_o}{(z+0.9)(z^2 - 1.5z + 0.7)} = \frac{t_o}{z^2 - 1.5z + 0.7} \tag{0.7}$$

And the steady-state gain is one, which requires

$$G(1) = 5t_0 = 1 \tag{0.8}$$

We have

$$T(z) = 0.2 \tag{0.9}$$

The whole controller is designed as follows:

$$\begin{aligned} R(z) &= z + 0.9 \\ S(z) &= 0.3z - 0.11 \\ T(z) &= 0.2 \\ U(z) &= \frac{T(z)}{R(z)}U_c(z) - \frac{S(z)}{R(z)}Y(z) \end{aligned} \tag{0.10}$$

In this case, from (0.7) the closed-loop transfer function is

$$G(z) = \frac{B_m(z)}{A_m(z)} = \frac{Y(z)}{U_c(z)} = \frac{0.2}{z^2 - 1.5z + 0.7} \tag{0.11}$$

b)

If the process zero is not cancelled, based on the above analysis (0.2), the order of the closed-loop characteristic polynomial is $\deg(A_{cl}(z)) = \deg(A(z)R(z) + B(z)S(z)) > \deg(A(z)) = 2$. Then, because the desired one is just $\deg(A_m(z)) = 2$, we have to include another $A_o(z)$ such that the closed-loop characteristic polynomial can be matched as

$$A_{cl}(z) = A_o(z)A_m(z) = zA_m(z) \tag{0.12}$$

Here we choose $A_o(z) = z$ because we want the lowest order and all poles at the origin.

Now it can be inferred that $\deg(R(z)) = \deg(A_{cl}(z)) - \deg(A(z)) = 1$. Similarly, we shall assume $R(z) = z + r_1$, $S(z) = s_o z + s_1$. Then, note that the additional $A_o(z)$ in (0.12) must be cancelled in the final form because the closed-loop should have the characteristic polynomial $A_m(z)$. This is the responsibility of $T(z)$, which is designed to be $T(z) = t_0 z$

Finally, the closed-loop transfer function is

$$\begin{aligned} G(z) = \frac{Y(z)}{U_c(z)} = \frac{B(z)T(z)}{A(z)R(z)+B(z)S(z)} &= \frac{t_0 z(z+0.9)}{(z^2-1.8z+0.81)(z+r_1)+(z+0.9)(s_o z+s_1)} \\ &= \frac{t_0 z(z+0.9)}{z^3+(r_1-1.8+s_0)z^2+(0.81-1.8r_1+s_1+0.9s_0)z+0.81r_1+0.9s_1} \end{aligned} \tag{0.13}$$

According to (0.12) and (0.13), we have

$$z^3+(r_1-1.8+s_0)z^2+(0.81-1.8r_1+s_1+0.9s_0)z+0.81r_1+0.9s_1 = z(z^2-1.5z+0.7) \tag{0.14}$$

Thus, the following equations are derived

$$\begin{aligned} &r_1-1.8+s_0 = -1.5 \\ &0.81-1.8r_1+s_1+0.9s_0 = 0.7 \\ &0.81r_1+0.9s_1 = 0 \end{aligned} \tag{0.15}$$

The solutions is

$$\begin{cases} r_1 = 0.1056 \\ s_0 = 0.1944 \\ s_1 = -0.095 \end{cases} \tag{0.16}$$

And then from (0.13) and (0.14) we get

$$G(z) = \frac{t_0(z+0.9)}{z^2-1.5z+0.7} \tag{0.17}$$

Besides, it is required that $G(1) = 1$ for a unit steady-state gain:

$$G(1) = \frac{1.9t_0}{0.2} = 1 \Rightarrow t_0 = \frac{2}{19} \tag{0.18}$$

After all, the controller is designed as follows

$$\begin{aligned} R(z) &= z + 0.1056 \\ S(z) &= 0.1944z - 0.095 \\ T(z) &= \frac{2}{19}z \\ U(z) &= \frac{T(z)}{R(z)}U_c(z) - \frac{S(z)}{R(z)}Y(z) \end{aligned} \tag{0.19}$$

And the closed-loop transfer function in this case is

$$G(z) = \frac{Y(z)}{U_c(z)} = \frac{2(z+0.9)}{19(z^2 - 1.5z + 0.7)} \tag{0.20}$$

c)

After we finished the design, a Simulink model can be easily built according to Figure 1, which is exhibited in Figure 2. Run the simulation with a sampling period of 1ms.

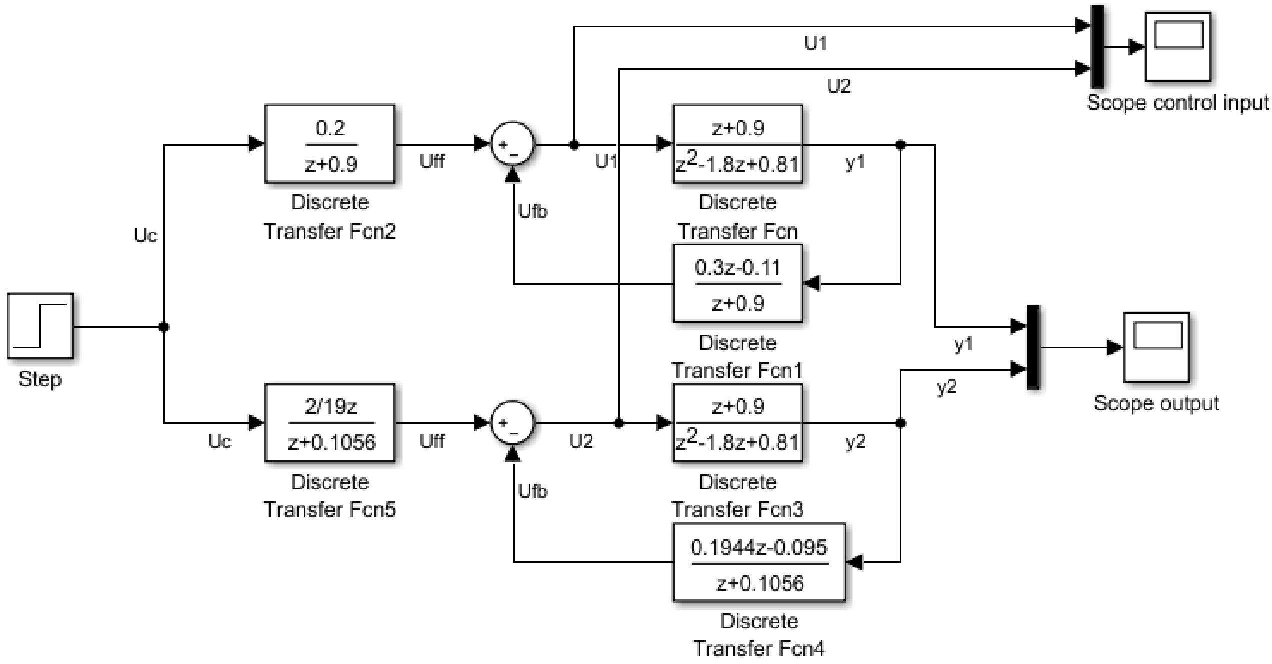

Figure 2 Simulink model

The closed-loop output and control input profiles are shown in Figure 3 and Figure 4 respectively.

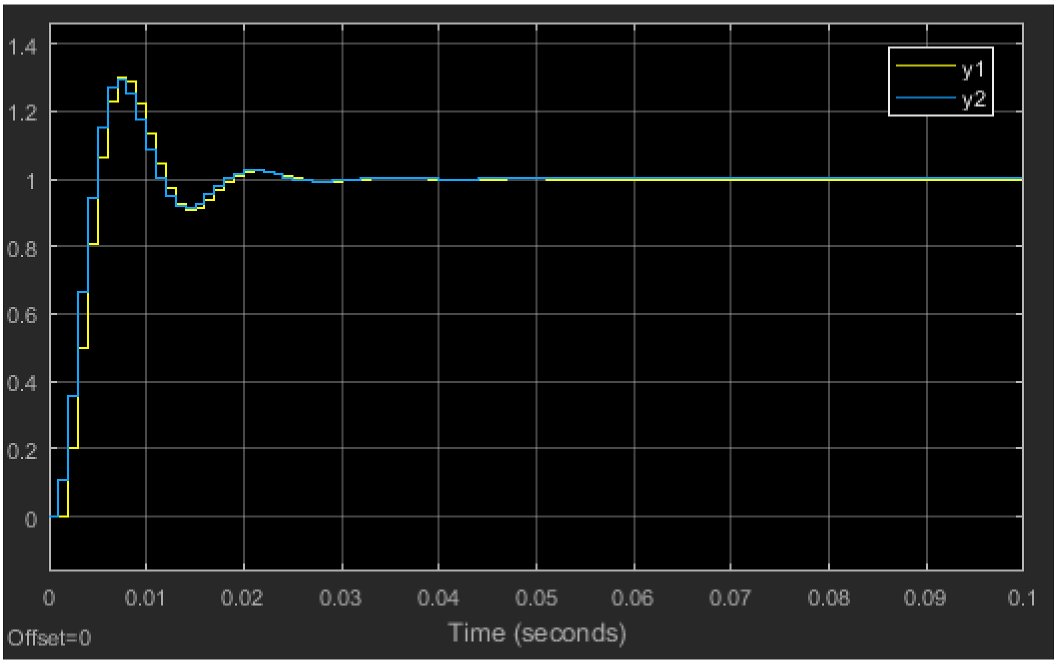

Figure 3 System output for the two cases

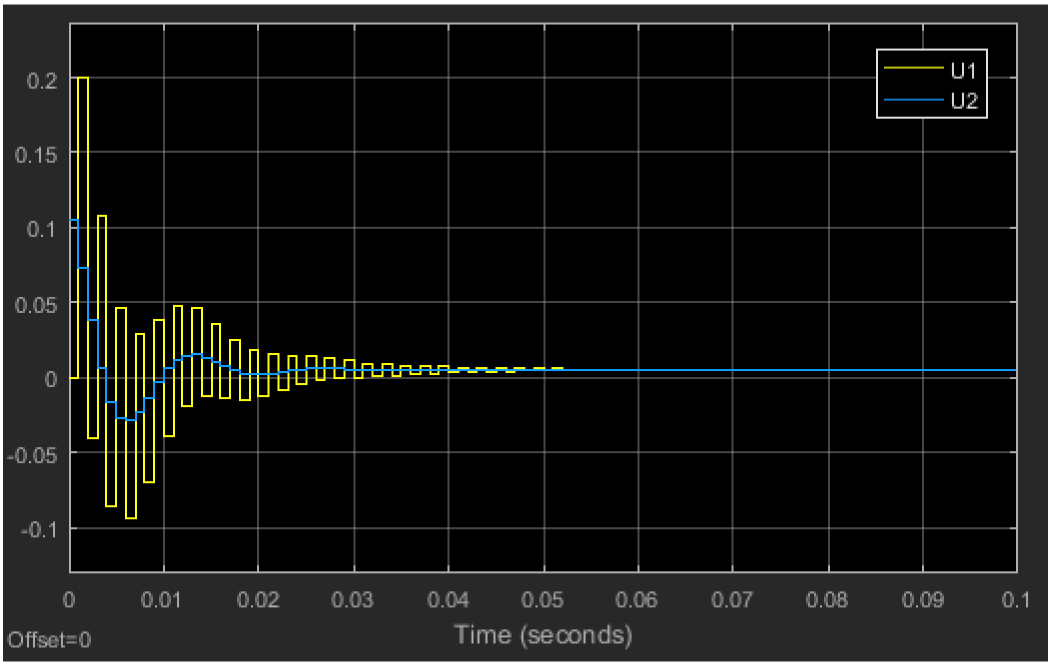

Figure 4 Control input signals for the two cases

As can be seen, the outputs in the two cases are quite similar. However, in Figure 4 it is noted that the control input signal for case 1 (with zero cancellation) is more seriously oscillating before arriving at the steady state. In practice, we would favor a smoother control input. Hence with the comparison, the second controller (without zero cancellation) is preferred.

Let us compare the transfer function from the command signal to the control signal.

$$H_c(z) = \frac{T(z)A(z)}{A_{cl}(z)}$$

$$H_{c1}(z) = \frac{U(z)}{U_c(z)} = \frac{0.2(z^2 - 1.8z + 0.81)}{(z^2 - 1.5z + 0.7)(z + 0.9)} \quad for \quad design \quad a)$$

$$H_{c2}(z) = \frac{U(z)}{U_c(z)} = \frac{\frac{2}{19}(z^2 - 1.8z + 0.81)}{(z^2 - 1.5z + 0.7)} \quad for \quad design \quad \text{b)}$$

The poles of the above two transfer function are located on different position. The design in a) has a pole on $z = -0.9$. A negative pole would cause some oscillation. Although the stable zero (z+0.9) is cancelled out in the transfer function from the command to the output, it is still there in the transfer function from the command to the control signal. When the stable zero is negative, or complex, the zero cancellation will lead to more oscillations in the control signal, which is not desired from the implementation point of view.

Q2

a)

Applying Z transform to the state-space representation with zero initial conditions yields

$$\begin{aligned} zX(z) &= \Phi X(z) + \Gamma U(z) + \Phi_{xv} V(z) \\ Y(z) &= CX(z) \end{aligned} \tag{1.1}$$

where $\Phi = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.7 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$, $\Phi_{xv} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Then we can get the transfer function

$$Y(z) = C(zI - \Phi)^{-1}\Gamma U(z) + C(zI - \Phi)^{-1}\Phi_{xv} V(z) \tag{1.2}$$

Based on the superposition principle, we can assume the disturbance to be zero, then the transfer function from input to output is

$$G_u(z) = \frac{Y(z)}{U(z)} = C(zI - \Phi)^{-1}\Gamma = \frac{0.2z - 0.04}{z^2 - 1.2z - 0.15} \tag{1.3}$$

In the same way, assume the input is zero, and the transfer function from the disturbance to the output is

$$G_v(\mathrm{z}) = \frac{Y(z)}{V(z)} = C(zI - \Phi)^{-1}\Phi_{xv} = \frac{z - 0.7}{z^2 - 1.2z - 0.15} \tag{1.4}$$

Now we let $B(z) = 0.2z - 0.04$, $B_v(z) = z - 0.7$ and $A(z) = z^2 - 1.2z - 0.15$ such that $G_v(z) = \dfrac{B_v(z)}{A(z)}$. Then the disturbance rejection controller can be designed as Figure 5 with output feedback.

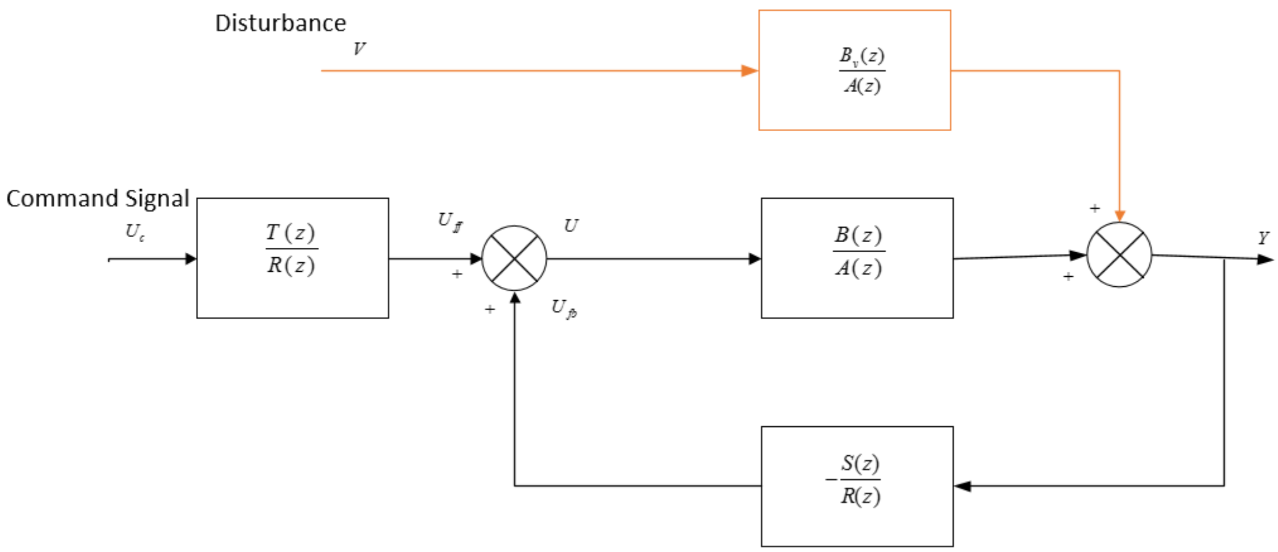

Figure 5 Block diagram for disturbance rejection

Assuming $u_c(t) = 0$, we can get the closed-loop transfer function between disturbance and the output

$$H_v(\mathrm{z}) = \frac{Y(z)}{V(z)} = \frac{B_v(z)R(\mathrm{z})}{A(z)R(z) + B(z)S(z)} \tag{1.5}$$

To eliminate the step disturbance influence on the steady state output, we require the D.C. gain be zero, that is $H_v(1) = 0$. This can be achieved by letting $R(\mathrm{z})$ contain a factor $(z-1)$. However, which order of $R(\mathrm{z})$ should we choose?

We know the order of $A(z)$ is 2. If we choose a first order $R(z)$, the we have $R(z) = (z-1)$ and $S(z) = s_0 z + s_1$. In this case, in (1.5) the total order of the denominator $A_{cl}(z) = A(z)R(z) + B(z)S(z)$ would be 3. To match $A_{cl}(z)$ with an arbitrarily specified characteristic polynomial $A_m(z)$, it will generate 3 equations. However, we have only 2 unknowns, i.e., s_0 and s_1. Generally, there will be no solutions.

Following the same idea as above, we can find that the lowest order of R(z) should be 2. Thus, they are defined to be

$$R(z) = (z + r_1)(z - 1) \quad S(z) = s_0 z^2 + s_1 z + s_2 \tag{1.6}$$

Because the desired closed-loop characteristic polynomial (CP) is not given, we can randomly select one with the requirements that the closed-loop poles for both the command signal $U_c(z)$ and disturbance signal $V(z)$ should be stable. The simplest one should be $A_m(z) = z^4$.

The closed-loop polynomial is

$$\begin{aligned} A_{cl}(z) &= A(z)R(z) + B(z)S(z) \\ &= z^4 + \left(r_1 - \frac{11}{5} + \frac{s_0}{5}\right)z^3 + \left(\frac{21}{20} - \frac{11}{5}r_1 - \frac{s_0}{25} + \frac{s_1}{5}\right)z^2 \\ &+ \left(\frac{3}{20} + \frac{21}{20}r_1 - \frac{s_1}{25} + \frac{s_2}{5}\right)z + \frac{3}{20}r_1 - \frac{s_2}{25} \end{aligned} \tag{1.7}$$

Now match (1.7) with $A_m(z) = z^4$ and we get

$$\begin{cases} r_1 - \dfrac{11}{5} + \dfrac{s_0}{5} = 0 \\ \dfrac{21}{20} - \dfrac{11}{5}r_1 - \dfrac{s_0}{25} + \dfrac{s_1}{5} = 0 \\ \dfrac{3}{20} + \dfrac{21}{20}r_1 - \dfrac{s_1}{25} + \dfrac{s_2}{5} = 0 \\ \dfrac{3}{20}r_1 - \dfrac{s_2}{25} = 0 \end{cases} \Rightarrow \begin{cases} r_1 = -0.1943 \\ s_0 = 11.9714 \\ s_1 = -4.9929 \\ s_2 = -0.7286 \end{cases} \tag{1.8}$$

Substituting (1.8) into (1.6) will give $R(z)$ and $S(z)$. Since there is no other requirement, you can choose any appropriate $T(z)$ such that steady state gain is unity.

For example let $T(z) = t_o$, compute the steady state gain as

$$H_{cl}(z) = \frac{Y(z)}{U_c(z)} = \frac{B(z)R(z)}{A(z)R(z) + B(z)S(z)}$$

$$H_{cl}(1) = \frac{B(1)t_o}{A(1)R(1) + B(1)S(1)} = \frac{t_o}{S(1)} = \frac{t_o}{6.25} = 1$$

$$t_o = 6.25$$

The final controller is

$$R(z) = (z-1)(z-0.1943), \quad S(z) = 11.9714z^2 - 4.9929z - 0.7286, \quad T(z) = 6.25 \quad (1.9)$$

b)

Since the Problem 2 of Homework 2 requires that the state be estimated for state feedback, we have to design an observer besides the controller. In this problem, only the output feedback is needed, and there is no need for a state observer. Thus, this transfer function based approach is much simpler.

Q3

a)

The controller design diagram is the same with Figure 6, where $A(z) = z^2 - 2z + 0.6$ and $B(z) = z + 0.5$.

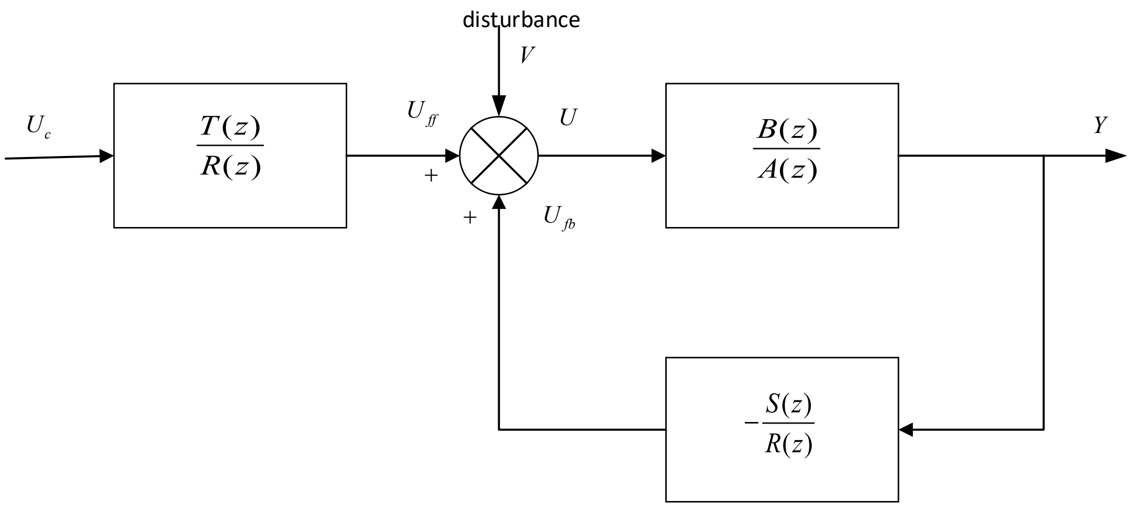

Figure 6 Block diagram for controller design

The closed-loop transfer function from the reference to the output is

$$G(z) = \frac{Y(z)}{U_c(z)} = \frac{B(z)T(z)}{A(z)R(z) + B(z)S(z)} \quad (2.1)$$

From the comparison of $H_m(z)$ and $G(z)$, we can find the zero of $G(z)$ derived from $B(z) = z + 0.5 = 0$ has been cancelled. Thus, we need $R(z) = \bar{R}(z+0.5)$ for zero cancellation.

The closed-loop transfer function from the disturbance to the output is

$$G_v(\mathrm{z}) = \frac{Y(z)}{V(z)} = \frac{B(z)R(z)}{A(z)R(z) + B(z)S(z)} \tag{2.2}$$

To reject constant disturbance, it is required that $G_v(1) = 0 \Rightarrow R(1) = 0$. Thus, the lowest order controller should be $R(z) = (z-1)(z+0.5)$.

In this case, the characteristic polynomial $A_{cl}(z)$ of (2.1) would be of 3rd order ($\deg(A(z)) + \deg(R(z)) - 1 = 3$ due to one zero cancellation), which is higher than the reference model. Thus, for simplicity, we can suppose $A_o(z) = z$ for $A_{cl}(z) = A_o(z)A_m(z)$. Then, let $T(z) = A_o(z)\mathrm{B}_m(z)$

Now the closed-loop transfer function (2.1) is rewritten as

$$G(z) = \frac{B(z)T(z)}{A(z)R(z) + B(z)S(z)} = \frac{B(z)A_o(z)B_m(z)}{B(z)A_o(z)A_m(z)} = H_m(z) \tag{2.3}$$

where $H_m(z) = \dfrac{B_m(z)}{A_m(z)} = \dfrac{z-0.5}{z^2 - 1.8z + 0.9}$.

Assume $S(z) = s_0 z^2 + s_1 z + s_2$ we have

$$\begin{aligned}
&AR + BS = A_{cl} \\
&(z^2 - 2z + 0.6)(z-1)(z+0.5) + (z+0.5)(s_0 z^2 + s_1 z + s_2) = (z+0.5)z(z^2 - 1.8z + 0.9) \\
&(z^2 - 2z + 0.6)(z-1) + s_0 z^2 + s_1 z + s_2 = z(z^2 - 1.8z + 0.9) \\
&z^3 + (s_0 - 3)z^2 + (2.6 + s_1)z - 0.6 + s_2 = z^3 - 1.8z^2 + 0.9z
\end{aligned} \tag{2.4}$$

Compare the coefficients we have
$$\begin{aligned} s_0 &= 1.2 \\ s_1 &= -1.7 \\ s_2 &= 0.6 \end{aligned}$$

(2.5)

Thus, the controller is designed as

$$\begin{aligned} &R(z) = (z-1)(z+0.5) = z^2 - 0.5z - 0.5 \\ &S(z) = 1.2z^2 - 1.7z + 0.6 \\ &T(z) = z^2 - 0.5z \end{aligned} \tag{2.6}$$

Then, from (2.2) the transfer function from the disturbance to the output is

$$G_v(z) = \frac{(z-1)(z+0.5)}{z(z^2 - 1.8z + 0.9)} \tag{2.7}$$

$G_v(z)$ is stable and $G_v(1) = 0$ which means the controller can reject the constant disturbance.

Hence, the controller is

$$U(z) = \frac{T(z)}{R(z)}U_c(z) - \frac{S(z)}{R(z)}Y(z) = \frac{z^2 - 0.5z}{z^2 - 0.5z - 0.5}U_c(z) - \frac{1.2z^2 - 1.7z + 0.6}{z^2 - 0.5z - 0.5}Y(z) \tag{2.8}$$

(Three key points: (1) disturbance rejection; (2) zero cancellation which is required for this type of controller and (3) the final closed-loop transfer function should match both $A_m(z)$ and $B_m(z)$.)

b)

Now the closed-loop transfer function from the reference to the output is

$$G(z) = \frac{Y(z)}{U_c(z)} = \frac{B(z)R(z)}{A(z)R(z) + B(z)S(z)}H_{ff}(z) \tag{2.9}$$

The closed-loop transfer function from the disturbance to the output is still

$$G_v(z) = \frac{Y(z)}{V(z)} = \frac{B(z)R(z)}{A(z)R(z) + B(z)S(z)} \tag{2.10}$$

To reject constant disturbance, we need $R(z) = (z-1)\overline{R}$. If $R(z)$ is chosen to be 1st order, there will be 3 unknowns and 4 equations, which usually has no solution. Therefore, the lowest order of $R(z)$ should be 2. Assume $R(z) = (z-1)(z-r_0)$ and $S(z) = s_0z^2 + s_1z + s_2$, then we have

$$A(z)R(z) + B(z)S(z) = z^4 + (s_0 - r_0 - 3)z^3 + (s_1 + 0.5s_0 + 3r_0 + 2.6)z^2 + (s_2 + 0.5s_1 - 0.6 - 2.6r_0)z + 0.6r_0 + 0.5s_2 \tag{2.11}$$

Compared with $A_{cl}(z) = A_o(z)A_m(z) = z^2(z^2 - 1.8z + 0.9)$, we can have

$$\begin{cases} s_0 - r_0 - 3 = -1.8 \\ s_1 + 0.5s_0 + 3r_0 + 2.6 = 0.9 \\ s_2 + 0.5s_1 - 0.6 - 2.6r_0 = 0 \\ 0.6r_0 + 0.5s_2 = 0 \end{cases} \Rightarrow \begin{cases} r_0 = -0.3153 \\ s_0 = 0.8847 \\ s_1 = -1.1964 \\ s_2 = 0.3784 \end{cases} \tag{2.12}$$

Thus, the feedback controller is designed as

$$\frac{S(z)}{R(z)} = \frac{0.8847z^2 - 1.1964z + 0.3784}{(z-1)(z+0.3153)} \tag{2.13}$$

Now we have $R(z)$ and $S(z)$, and from (2.9) we know the closed-loop transfer function is

$$\frac{B(z)R(z)}{A(z)R(z) + B(z)S(z)} H_{ff}(z) = \frac{B(z)R(z)}{A_{cl}(z)} H_{ff}(z) = \frac{B_m(z)}{A_m(z)} = H_m(z) \tag{2.14}$$

to match the reference model.

Thus

$$\begin{aligned} H_{ff}(z) &= \frac{B_m(z)A_{cl}(z)}{A_m(z)B(z)R(z)} = \frac{B_m(z)A_o(z)}{B(z)R(z)} \\ &= \frac{(z-0.5)z^2(z^2 - 1.8z + 0.9)}{(z^2 - 1.8z + 0.9)(z+0.5)(z-1)(z+0.3153)} \\ &= \frac{(z-0.5)z^2}{(z+0.5)(z-1)(z+0.3153)} \end{aligned} \tag{2.15}$$

Hence the controller is

$$\begin{aligned} U(z) &= H_{ff}(z)U_c(z) - \frac{S(z)}{R(z)}Y(z) \\ &= \frac{(z-0.5)z^2}{(z+0.5)(z-1)(z+0.3153)}U_c(z) - \frac{0.8847z^2 - 1.1964z + 0.3784}{(z-1)(z+0.3153)}Y(z) \end{aligned} \tag{2.16}$$

Note:

1. For a second thought, you may find that if we keep the $R(z)$ and $S(z)$ in the solution to question a) unchanged and let $H_{ff}(z) = \frac{T(z)}{R(z)}$, we can also meet the requirement. That is to

say, the control configuration in a) is exactly a special case of b). However, the $H_{ff}(z)$ structure in b) will give your more freedom since there is no requirement on its denominator and it is a more general control configuration. For exercise and examination purpose, you should follow the different procedures to finish the design in these two cases, although a) can be regarded as a special case of b).

2. In this case b), $R(z)$ is only responsible for disturbance rejection. All the other work will be done by $H_{ff}(z)$ since it has more freedom than case a).

Q4

a)

The given input-output model is

$$y(k+1)=\sin(y(k))+u(k-1)+cu(k-2) \tag{3.1}$$

From the input-output model we can get

$$u(k)=y(k+2)-\sin(y(k+1))-cu(k-1) \tag{3.2}$$

Substituting (3.1) into (3.2) gives

$$u(k)=y(k+2)-\sin\left(\sin(y(k))+u(k-1)+cu(k-2)\right)-cu(k-1) \tag{3.3}$$

Let $y(k+2)=r(k+2)$, then from (3.3) we get the one-step-ahead controller

$$u(k)=r(k+2)-\sin\left(\sin(y(k))+u(k-1)+cu(k-2)\right)-cu(k-1) \tag{3.4}$$

b)

We need to assure that the plant output $y(k)$ and the controller output $u(k)$ are both bounded for perfect tracking.

Since $y(k)=r(k)$, the plant output is bounded as long as the reference $r(k)$ is bounded.

From (3.2), in the case of perfect tracking, the controller output is

$$u(k)=r(k+2)-\sin\left(r(k+1)\right)-cu(k-1) \tag{3.5}$$

Let $v(k)=r(k+2)-\sin\left(r(k+1)\right)$. It is obvious that $v(k)$ is bounded provided $r(k),k=1,2,3\cdots$ is bounded. Now (3.5) turns out to be

$$u(k)=v(k)-cu(k-1). \tag{3.6}$$

Applying z transform we get

$$U(z) = V(z) - z^{-1} c U(z) \quad \Rightarrow \quad \frac{U(z)}{V(z)} = \frac{z}{z+c} \tag{3.7}$$

Since $v(k)$ is bounded, to make $u(k)$ also bounded we require the transfer function in (3.7) to be a BIBO one. That is,

$$|-c| < 1 \quad \Rightarrow \quad |c| < 1 \tag{3.8}$$

In summary, to achieve perfect tracking the condition of c is $|c| < 1$.