EE5103 Computer Control Systems: Homework #2 Solution

Semester 1 Y2025/2026

Q1 Solution

a)

Since the state vector is $x(t) = \left[x_1(t) \quad x_2(t)\right]^T = \left[\dot{y}(t) \quad y(t)\right]^T$, it can be derived that

$$y(t) = x_2(t) \tag{1}$$

$$\dot{x}_2(t) = x_1(t) \tag{2}$$

From the DC motor model, the relationship between voltage and velocity is

$$X_1(s) = \frac{1}{Ts+1}U(s) \tag{3}$$

Then we have

$$TsX_1(s) + X_1(s) = U(s) \tag{4}$$

Inverse Laplace transform gives

$$T\dot{x}_1(t) + x_1(t) = u(t) \tag{5}$$

And further we have

$$\dot{x}_1(t) = -\frac{1}{T}x_1(t) + \frac{1}{T}x_1(t) \tag{6}$$

Thus the state-space model is

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -\frac{1}{T} & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{T} \\ 0 \end{bmatrix} u = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned} \tag{7}$$

Assume the discrete-time state-space model is

$$\begin{aligned} x(k+1) &= \Phi x(\mathrm{k}) + \Gamma u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{aligned} \tag{8}$$

Then we have (*please check HW1 for details on how to solve matrix exponential.*)

$$\Phi = e^{Ah} = e^{\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} h} = \begin{bmatrix} e^{-2h} & 0 \\ 0.5(1-e^{-2h}) & 1 \end{bmatrix} \tag{9}$$

$$\Gamma = \int_0^h e^{At} dtB = \int_0^h \begin{bmatrix} e^{-2t} & 0 \\ 0.5(1-e^{-2t}) & 1 \end{bmatrix} dtB = \begin{bmatrix} -e^{-2h}+1 \\ h+0.5(e^{-2h}-1) \end{bmatrix} \tag{10}$$

Thus the discrete state-space model is

$$\begin{aligned} x(k+1) &= \begin{bmatrix} e^{-2h} & 0 \\ 0.5(1-e^{-2h}) & 1 \end{bmatrix} x(k) + \begin{bmatrix} -e^{-2h}+1 \\ h+0.5(e^{-2h}-1) \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{aligned} \tag{11}$$

b)

For deadbeat controller, the desired poles are all zero. Thus, the closed-loop characteristic equation is $P(\mathrm{z}) = z^2$. The controllability matrix for system (11) is,

$$W_c = [\Gamma, \Phi\Gamma] = \begin{bmatrix} -e^{-2h}+1 & -e^{-4h}+e^{-2h} \\ 0.5e^{-2h}+h-0.5 & 0.5(e^{-4h}-e^{-2h})+h \end{bmatrix} \tag{12}$$

It is apparent that W_c is nonsingular if $h > 0$. Thus this sampled system is controllable. Now we can place the closed-loop poles arbitrarily. Assume the state-feedback gain is L, then the closed-loop characteristic polynomial is

$$\det(\mathrm{zI} - (\Phi - \Gamma L)) = z^2 \tag{13}$$

Let $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}$, then according Ackermann's formula, we have

$$L = \begin{bmatrix} 0 & 1 \end{bmatrix} W_c^{-1} P(\Phi) = \begin{bmatrix} 0 & 1 \end{bmatrix} W_c^{-1} \Phi^2 \tag{14}$$

Substituting (9) and (12), we can get

$$L = \begin{bmatrix} -\dfrac{2h+e^{2h}-e^{4h}}{2h(e^{2h}-1)^2} & \dfrac{e^{2h}}{h(e^{2h}-1)} \end{bmatrix} \tag{15}$$

Thus the deadbeat controller is

$$u(k) = -Lx(\mathrm{k}) \tag{16}$$

c)

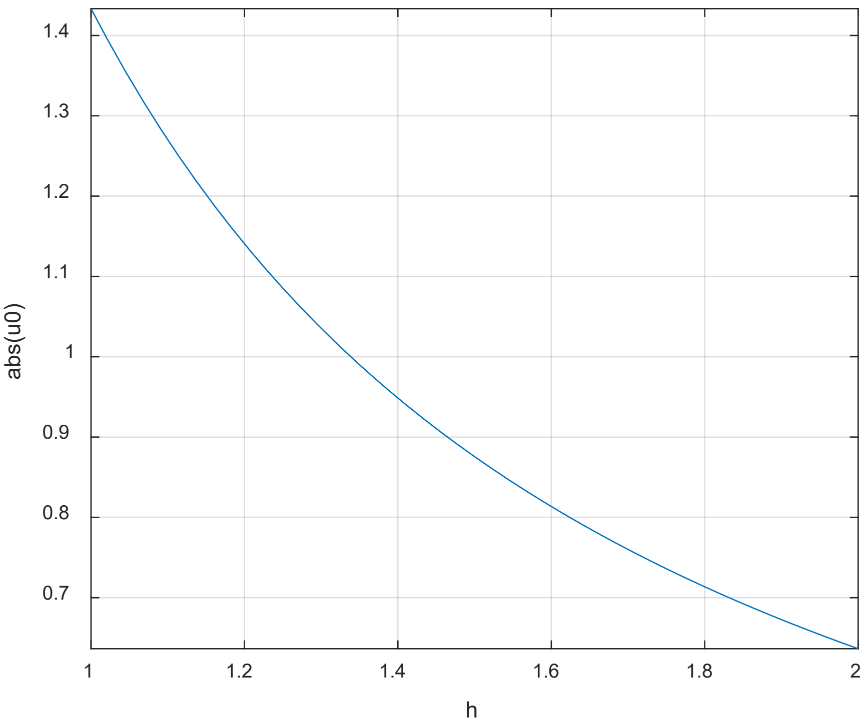
The maximum control signal is

$$u(0) = -Lx(0) = -L\begin{bmatrix}0.5 & 1\end{bmatrix}^T \tag{17}$$

To satisfy $|u(0)| < 1$, substitute equation (15) into (17) and we have

$$\left| \frac{2h + 5e^{2h} - 5e^{4h}}{4h(e^{2h} - 1)^2} \right| < 1 \tag{18}$$

If you plot the left part of (18), it is obviously monotonically decreasing. Focus on the range where $h \in [1, \quad 2]$ and the following figure is acquired:



Therefore, to make $|u(0)| < 1$, it is required that

$$h > 1.34 \tag{19}$$

Q2 Solution

a) The state and v can be measured.

Assume the state feedback gain for x and v is respectively L and L_v, that is

$$u(k) = -\begin{bmatrix} L & L_v \end{bmatrix} \begin{bmatrix} x(k) \\ v(k) \end{bmatrix} = -Lx(k) - L_v v(k) \tag{20}$$

then closed-loop state-space model is

$$\begin{aligned} x(k+1) &= (\Phi - \Gamma L)x(k) + (\Phi_{xv} - \Gamma L_v)v(k) \\ y &= Cx(k) \end{aligned} \tag{21}$$

where $\Phi = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.7 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \Phi_{xv} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

After Z-transform on equation (21), we can get

$$\begin{aligned} zX(z) &= (\Phi - \Gamma L)X(z) + (\Phi_{xv} - \Gamma L_v)V(z) \\ Y(z) &= CX(z) \end{aligned} \tag{22}$$

Then we can get the transfer function between the disturbance and output as

$$H_v(z) = \frac{Y(z)}{V(z)} = C(zI - (\Phi - \Gamma L))^{-1}(\Phi_{xv} - \Gamma L_v) \tag{23}$$

Since the disturbance is constant, to eliminate its influence on the steady-state output, it is required that the DC gain of $H_v(z)$ is zero, i.e.,

$$H_v(1) = 0 \tag{24}$$

The eigenvalues of Φ are $\{1.314, -0.1141\}$, which indicates the original system is unstable. Therefore, there are actually two tasks involved in this question: (1) stabilize the closed-loop system with state feedback; (2) reject the disturbance on steady state.

For the first task, we can place the closed-loop poles both at 0 (a deadbeat controller). Then the closed-loop characteristic equation is

$$A_m(z) = z^2 \tag{25}$$

The controllability matrix is

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.17 \end{bmatrix} \tag{26}$$

It is nonsingular. We can place the closed-loop poles arbitrarily with state feedback. From Ackermann's formula, we have

$$L = \begin{bmatrix} 0 & 1 \end{bmatrix} W_c^{-1} A_m(\Phi) = \begin{bmatrix} 3.2143 & 5.5714 \end{bmatrix} \tag{27}$$

Then substitute L into $H_v(z)$ in (23) and equation (24) yields

$$\begin{bmatrix} 0.8571 & -0.1143 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - L_v \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} \right) = 0 \tag{28}$$

That is,

$$\frac{6}{7} - \frac{4L_v}{25} = 0 \tag{29}$$

Thus, we can solve

$$L_v = \frac{75}{14} \approx 5.3571 \tag{30}$$

Thus, the state feedback controller is

$$u(k) = -Lx(k) - L_v v(k) = -\begin{bmatrix} 3.2143 & 5.5714 \end{bmatrix} x(k) - 5.3571 v(k) \tag{31}$$

b) The state can be measured

Since the disturbance cannot be measured, we need to design some kind of observer to estimate the disturbance for state feedback. Note that the state can be measured and the disturbance is constant. From the state transition equation, it holds that

$$\Phi_{xv} v(k) = x(k+1) - \Phi x(k) - \Gamma u(k) \tag{32}$$

Since $v(k)$ is constant, i.e., $v(k) = v, \forall k \ge 0$. Inserting $k = 0$ into (32), we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} v = x(1) - \Phi x(0) - \Gamma u(0) = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \tag{33}$$

Since the state $x(1)$ and $x(0)$ can be measured and $u(0)$ is known, the right side of equation (33) can be acquired. Suppose it is $\begin{bmatrix} \gamma & 0 \end{bmatrix}^T$. Then the constant scalar disturbance v can be solved.

$$v(k) = v = \gamma, \quad \forall k \ge 0 \tag{34}$$

After the estimation of the disturbance is available, the controller is designed as (31), i.e.,

$$u(k) = -\begin{bmatrix} 3.2143 & 5.5714 \end{bmatrix} x(k) - 5.3571 v \tag{35}$$

c) Only the output can be measured.

Since only the output can be measured, we should implement an observer for both the state variables and the disturbance. Define the augmented state variable as

$$z(k)=\begin{bmatrix} x(k) \\ v(k) \end{bmatrix} \tag{36}$$

And the state-space model now turns into

$$\begin{aligned} z(k+1) &= \begin{bmatrix} \Phi & \Phi_{xv} \\ 0 & 1 \end{bmatrix} z(k) + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} C & 0 \end{bmatrix} z(k) \end{aligned} \tag{37}$$

For convenience, define some symbols as follows.

$$\Phi_z = \begin{bmatrix} \Phi & \Phi_{xv} \\ 0 & 1 \end{bmatrix}_{3\times 3}, \Gamma_z = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}_{3\times 1}, C_z = \begin{bmatrix} C & 0 \end{bmatrix}_{1\times 3} \tag{38}$$

The augmented state observer is described as

$$\begin{aligned} \hat{z}(k+1) &= \Phi_z \hat{z}(k) + \Gamma_z u(k) + K(y-\hat{y}) \\ \hat{y} &= C_z \hat{z}(k) \end{aligned} \tag{39}$$

The state estimation error is defined as $e(k)=z(k)-\hat{z}(k)$ and it follows that

$$e(k+1) = (\Phi_z - KC_z)e(k) \tag{40}$$

The objective is to make the error dynamics (40) stable and it should be faster than the controller dynamics. The observation matrix of system (Φ_z, C_z) is

$$W_o = \begin{bmatrix} C_z \\ C_z\Phi_z \\ C_z\Phi_z^2 \end{bmatrix} = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0.5000 & 1.0000 & 1.0000 \\ 0.7500 & 1.2000 & 1.5000 \end{bmatrix} \tag{41}$$

which is nonsingular. Therefore, we can place the poles for (40) as we like.

Here, for convenience, we can design a deadbeat state observer, that is, let all the eigenvalues of (40) be zero. Then the characteristic polynomial is $f(z)=z^3$. The feedback gain K is

$$K = f(\Phi_z)W_o^{-1}\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T = \begin{bmatrix}2.2000 \\ -0.6433 \\ 3.3333\end{bmatrix} \tag{42}$$

From the above observer (39), we can estimate both the state and the disturbance to be $\hat{x}(k)$ and $\hat{v}(k)$. Then the controller is designed as (31) using the estimated variables, that is

$$u(k) = -L\hat{x}(k) - L_v\hat{v}(k) = -\begin{bmatrix}3.2143 & 5.5714\end{bmatrix}\hat{x}(k) - 5.3571\hat{v}(k) \tag{43}$$

Q3 Solution

a) Assume the continuous-time reference model is

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{44}$$

where ζ is the damping ration and ω_n is the natural frequency.

According the performance specifications, overshoot satisfies

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.05 \tag{45}$$

Then we have $0.69 < \zeta < 1$, let $\xi = 0.7$. (*You can choose any value inside the range.)*

The 1% settling time is required that

$$t_s \cong \frac{4.6}{\zeta\omega_n} < 8 \tag{46}$$

Thus, we have

$$\omega_n > \frac{4.6}{8\xi} \approx 0.82 \tag{47}$$

We choose $\omega_n = 0.82$. The reference model in the continuous-time is

$$G(s) = \frac{0.67}{s^2 + 1.15s + 0.67} \tag{48}$$

Sampling period is $h = 0.1$. From the table, we can get the discrete-time reference model as

$$H_m(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2} = \frac{0.003223z + 0.003102}{z^2 - 1.885z + 0.8914} \tag{49}$$

b)

First we find the continuous-time state-space model. Since $x(t)=[y(t),\dot{y}(t)]^T$, we have

$$\dot{x}_1(t)=x_2(t) \tag{50}$$

And the dynamics is $m\dot{x}_2(t)+bx_2(t)=u(t)$, it can be derived that

$$\dot{x}_2(t)=-\frac{b}{m}x_2(t)+\frac{1}{m}u(t) \tag{51}$$

Thus, the continuous-time state-space model is

$$\begin{aligned}\dot{x}(t)&=\begin{bmatrix}0 & 1\\ 0 & -\dfrac{b}{m}\end{bmatrix}x(t)+\begin{bmatrix}0\\ \dfrac{1}{m}\end{bmatrix}u(t)=\begin{bmatrix}0 & 1\\ 0 & -0.25\end{bmatrix}x(t)+\begin{bmatrix}0\\ 0.00125\end{bmatrix}u(t)\\ y(t)&=\begin{bmatrix}1 & 0\end{bmatrix}x(t)\end{aligned} \tag{52}$$

For the sampled system, the state matrix and input matrix are respectively

$$\Phi=e^{Ah}=\begin{bmatrix}1 & 0.0988\\ 0 & 0.9753\end{bmatrix} \tag{53}$$

$$\Gamma=\int_0^{0.1}e^{At}dtB=\begin{bmatrix}6.1982\times10^{-6}\\ 1.2345\times10^{-4}\end{bmatrix} \tag{54}$$

Thus, the state-space model for the sampled system is

$$\begin{aligned}x(k+1)&=\Phi x(k)+\Gamma u(k)\\ y(k)&=\begin{bmatrix}1 & 0\end{bmatrix}x(k)\end{aligned} \tag{55}$$

c)

From reference model (49) we can get the closed-loop characteristic polynomial is

$$P(\mathrm{z})=z^2-1.851z+0.8607 \tag{56}$$

Now considering the state-space model (55), the controllability matrix is

$$W_c=[\Gamma,\Phi\Gamma]=\begin{bmatrix}0.0062 & 0.0184\\ 0.1235 & 0.1204\end{bmatrix}\times10^{-3} \tag{57}$$

And the rank of W_c is 2, thus the system is controllable, which means we can place the poles arbitrarily through state feedback. And from Ackermann's formula, we have

$$L = \begin{bmatrix} 0 & 1 \end{bmatrix} W_c^{-1} P(\Phi) \tag{58}$$

From equations (57) and (56),(53), we can get

$$\mathrm{L} = [517.95, 703.083] \tag{59}$$

And the state feedback controller is

$$u_{fb}(k) = -Lx(k) \tag{60}$$

d)

We first derive the transfer function from $u_{ff}(k)$ to $y(k)$. It is obvious that the closed-loop system is

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma\left(u_{ff}(k) - Lx(k)\right) = \left(\Phi - \Gamma L\right)x(k) + \Gamma u_{ff}(k) \\ y(k) &= Cx(k) \end{aligned} \tag{61}$$

where L is the state feedback law we found in c). The transfer function is

$$\frac{Y(z)}{U_{ff}(z)} = C[zI - (\Phi - \Gamma L)]^{-1}\Gamma = \frac{B(z)}{A(z)} \tag{62}$$

Note that in the above state feedback c), we have designed the feedback gain L to match the poles of the reference model $H_m(z) = \frac{B_m(z)}{A_m(z)}$ in (49). Therefore, we already have $A(z) = A_m(z)$.

Since $U_{ff}(z) = U_c(z)\mathrm{H}_{ff}(z)$, it is further derived from (62) that

$$\frac{Y(z)}{U_c(z)} = C[zI - (\Phi - \Gamma L)]^{-1}\Gamma H_{ff}(z) = \frac{B(z)H_{ff}(z)}{A(z)} = \frac{B(z)H_{ff}(z)}{A_m(z)} \tag{63}$$

Now it can be seen that the feedforward controller $H_{ff}(z)$ is used to match the denominator of the reference model. It follows that

$$B(z)H_{ff}(z) = B_m(z) \quad \Rightarrow \quad H_{ff}(z) = \frac{B_m(z)}{B(z)} \tag{64}$$

From equations (49) and (62), we can get $H_{ff}(z)$.

$$H_{ff}(z) = \frac{3223z+3102}{6.198z+6.148}$$

(Summary: in this case, state feedback is to match poles and feed-forward controller is to match zeros; you need to conduct these two tasks one by one.)

e)

Since we have used state feedback, if only the output is measurable, then we have to try to estimate the state with an observer.

$$\begin{aligned} \hat{x}(k+1) &= \Phi\hat{x}(k) + \Gamma u(k) + K[y(k) - \hat{y}(k)] \\ \hat{y}(k) &= C\hat{x}(k) \end{aligned} \tag{65}$$

And define the estimation error as $e(k) = x(k) - \hat{x}(k)$. Equation (65) and (55) yields

$$e(k+1) = (\Phi - KC)e(k) \tag{66}$$

To make $e(k)$ converge to zero, it is required to place the eigenvalues of $\Phi - KC$ inside the unit circle on the z plane. And this requires the system (Φ, C) to be observable. The observability matrix is

$$W_o = \begin{bmatrix} C \\ C\Phi \end{bmatrix} = \begin{bmatrix} 1.0000 & 0 \\ 1.0000 & 0.0988 \end{bmatrix} \tag{67}$$

W_o is nonsingular, so this system is observable. We can make a deadbeat observer, that is , to place all the eigenvalues at origin. In this case, the expected characteristic polynomial is

$$f(\mathrm{z}) = z^2 \tag{68}$$

And the observer gain K is

$$K = f(\Phi)W_o^{-1}\begin{bmatrix} 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1.9753 & 9.6317 \end{bmatrix}^T \tag{69}$$

Then we can use the above K and the observer (65) to estimate the state vector though it is not directly measurable. Hence it is still possible to use this two-degree-of-freedom controller to meet the performance specification with estimated state feedback by a state observer.

In conclusion, the two-degree-of-freedom control method is designed as

$$u(k) = H_{ff}(q)u_c(k) - L\hat{x}(k) \tag{70}$$

where q is the time shift operator.

Note:

For all these problems, you can choose any poles and parameter values as long as they are satisfactory. This solution just uses some specific values to demonstrate the design process.