EE5103 Computer Control Systems: Homework #1 Solution

Semester 1 Y2025/2026

Q1 Solution

a) Applying Laplace transform we can get:

$$L(sI(s)-i(0))+RI(s)+\frac{1}{Cs}I(s)=E_i(s) \tag{1.1}$$

$$\frac{1}{C}\frac{1}{s}I(s)=E_o(s) \tag{1.2}$$

Assuming the initial condition is zero, then we have

$$\frac{1}{Cs}\frac{E_i(s)}{Ls+R+\dfrac{1}{Cs}}=E_o(s) \tag{1.3}$$

Then the transfer function is as follows:

$$G(s)=\frac{Y(s)}{U(s)}=\frac{E_o(s)}{E_i(s)}=\frac{1}{LCs^2+R(s)+1}=\frac{1}{s^2+s+1} \tag{1.4}$$

b) Since $u=e_i, y=e_{o,}x_2=\dot{e}_o, x_1=e_o$

$$\frac{1}{C}\int idt=e_o \Rightarrow x_2=\dot{e}_o=\frac{1}{C}i \Rightarrow \frac{di}{dt}=C\dot{x}_2 \tag{1.5}$$

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int idt=e_i \Rightarrow LC\dot{x}_2+RCx_2+x_1=u \tag{1.6}$$

Thus the above equations yield

$$\dot{x}_1=\dot{e}_o=x_2 \tag{1.7}$$

$$\dot{x}_2=-\frac{1}{LC}x_1-\frac{R}{L}x_2+\frac{1}{LC}u \tag{1.8}$$

$$y=e_o=x_1 \tag{1.9}$$

The state space model is

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}&=\begin{bmatrix}0 & 1\\ -\dfrac{1}{LC} & -\dfrac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \dfrac{1}{LC}\end{bmatrix}u=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u\\ y&=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned} \tag{1.10}$$

c) Denote the above state space model as

$$\begin{aligned}x(k+1)&=\Phi x(k)+\Gamma u(k)\\ y(k)&=Cx(k)\end{aligned} \tag{1.11}$$

Then we have

$$\Phi = e^{Ah} = e^{A} \tag{1.12}$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ is the state transition matrix in (1.10). To find e^{A}, we can use the Laplace transform. Let $f(t) = e^{At}$, and its Laplace transform is

$$F(s) = \mathcal{L}[f(t)] = (sI - A)^{-1} = \frac{1}{s^2+s+1}\begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix} \tag{1.13}$$

Then inverse Laplace transform gives

$$f(t) = \mathcal{L}^{-1}[F(s)] = e^{-\frac{t}{2}}\begin{bmatrix} cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}sin\frac{\sqrt{3}}{2}t & \frac{2}{\sqrt{3}}sin\frac{\sqrt{3}}{2}t \\ -\frac{2}{\sqrt{3}}sin\frac{\sqrt{3}}{2}t & cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}sin\frac{\sqrt{3}}{2}t \end{bmatrix} \tag{1.14}$$

Let $t = 1$, we have

$$\Phi = e^{A} = f(1) = \begin{bmatrix} 0.6596 & 0.5335 \\ -0.5335 & 0.1262 \end{bmatrix} \tag{1.15}$$

For the input matrix Γ, since A is nonsingular, we have

$$\begin{aligned}\Gamma &= \int_0^h e^{A\tau} d\tau B = A^{-1} e^{A\tau} \big|_{\tau=0}^{1} B \\ &= A^{-1}(e^{A} - I)B \\ &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^{-1}\left(\begin{bmatrix} 0.6596 & 0.5335 \\ -0.5335 & 0.1262 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.3403 \\ 0.5335 \end{bmatrix}\end{aligned} \tag{1.16}$$

Of course, the above answer can also be obtained by integrating all the elements in f(t) expressed in (1.14).

Thus the state space model for sampled discrete-time system is

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.6596 & 0.5335 \\ -0.5335 & 0.1262 \end{bmatrix} x(k) + \begin{bmatrix} 0.3403 \\ 0.5335 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned} \tag{1.17}$$

d) Applying Z transform on discrete state space model (1.11) yields

$$\begin{aligned} zX(z) - zx(0) &= \Phi X(z) - \Gamma U(z) \\ Y(z) &= CX(z) \end{aligned} \tag{1.18}$$

Then we can get

$$Y(z) = CX(z) = C(zI - \Phi)^{-1}\Gamma U(z) + zC(zI - \Phi)^{-1}x(0) \qquad (1.19)$$

Assuming zero initial conditions we can get the z transfer function as

$$\mathrm{H(z)} = \frac{Y(z)}{U(z)} = C(zI - \Phi)^{-1}\Gamma = \frac{0.3403z + 0.2417}{z^2 - 0.7858z + 0.3679} \qquad (1.20)$$

Therefore, we can get

$$\begin{aligned} &z^2Y(z) - 0.7858zY(z) + 0.3679Y(z) = 0.3403zU(z) + 0.2417U(z) \\ &\Rightarrow y(k+2) - 0.7858y(k+1) + 0.3679y(k) = 0.3403u(k+1) + 0.2417u(k) \end{aligned} \qquad (1.21)$$

The input-output model is

$$y(k+1) = 0.7858y(k) - 0.3679y(k-1) + 0.3403u(k) + 0.2417u(k-1) \qquad (1.22)$$

e) We know that

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (1.23)$$

$$U(z) = Z[u(k)] = \frac{z}{z-1} \qquad (1.24)$$

Substituting $x(0)$, U(z) into equation (1.19) yields

$$\begin{aligned} Y(z) &= C(zI - \Phi)^{-1}\left(\frac{z}{z-1}\begin{bmatrix} 0.3403 \\ 0.5335 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} z - 0.7358 & -0.3679 \\ 0.3679 & z \end{bmatrix}^{-1}\begin{bmatrix} \dfrac{z^2 - 0.7358z}{z-1} \\ \dfrac{0.3679z}{z-1} \end{bmatrix} \\ &= \frac{z}{z-1} \end{aligned} \qquad (1.25)$$

Thus, $y(k) = Z^{-1}[Y(z)] = 1,\ k = 0, 1, 2, \cdots$.

Q2 Solution

a) Poles: $s(s-1) = 0 \Rightarrow s_1 = 0, s_2 = 1$.

Since there is a pole in the right half plane, the system is unstable.

There is no zero. Thus the inverse system is stable.

b) Mapping the above poles onto z-plane, we have poles for the discrete sampled system,

$$z_1 = e^{s_1 h} = e^0 \Rightarrow |z_1| = 1$$
$$z_2 = e^{s_2 h} = e^h \Rightarrow |z_2| = e^h > e^0 = 1 \tag{2.1}$$

Thus, the sampled system is still unstable.

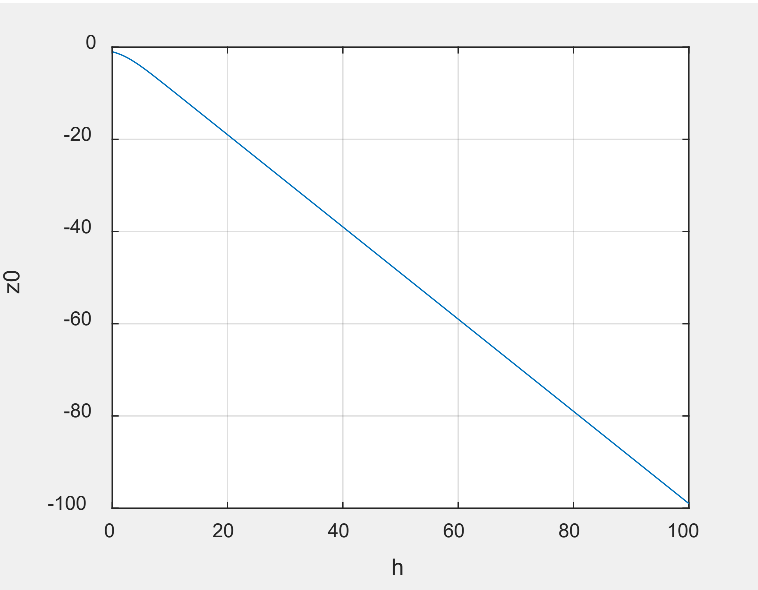
c) Since there are no direct mapping for zeros between s-domain and z-domain, we have to derive the z-transfer function first.

$$G(s) = \frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s} = -\frac{-1}{s-1} - \frac{1}{s} \tag{2.2}$$

From the z transform table, we can find the sampled system is

$$G(z) = -\frac{1-e^h}{z-e^h} - \frac{h}{z-1} = -\frac{(1-e^h+h)z-(1-e^h+he^h)}{(z-1)(z-e^h)} \tag{2.3}$$

The only zero is $z_0 = \frac{1-e^h+he^h}{1-e^h+h}$. Plot the cure of z_0 with respect to h in MATLAB and we get the following figure. It shows that no matter how small h is, the only zero will always be smaller than -1, which means the inverse system is unstable.



As can be seen, sampling will not change the stability of the system due to the pole mapping relation between continuous-time and discrete-time domain. However, there is no simple relation between the zeros. Further, even the original system has no zeros, after sampling, it may have zeros.

Q3 Solution

a) The state transition matrix, input matrix and output matrix are

$$\Phi = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, \Gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.1}$$

The characteristic equation and the poles are

$$\det(\lambda I - \Phi) = \lambda^2 - 0.3\lambda = 0 \Rightarrow \lambda_1 = 0.3, \lambda_2 = 0 \tag{3.2}$$

Since all the poles are in the unit circle of the z-plane, the system is stable.

Controllability matrix is

$$W_c = \begin{bmatrix}\Gamma & \Phi\Gamma\end{bmatrix} = \begin{bmatrix}1 & 0.1\\ 0 & 0.1\end{bmatrix} \tag{3.3}$$

Since W_c is nonsingular, the system is controllable.

The observability matrix is

$$W_o = \begin{bmatrix}C\\ C\Phi\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0.1 & 0.2\end{bmatrix} \tag{3.4}$$

Since W_o is nonsingular, the system is observable.

b) The z transfer function is

$$\begin{aligned}H(z) &= C(zI - \Phi)^{-1}\Gamma\\ &= \frac{z - 0.2}{z^2 - 0.3\,\mathrm{z}}\end{aligned} \tag{3.5}$$

Since $H(z) = \dfrac{Y(z)}{U(z)}$, we can get

$$z^2Y(z) - 0.3zY(z) = zU(z) - 0.2U(z) \tag{3.6}$$

Applying inverse z transform yields

$$y(k+2) - 0.3y(k+1) = u(k+1) - 0.2u(\mathrm{k}),\ \mathrm{k} = 0,1,2,\cdots \tag{3.7}$$

That is,

$$y(k+1) = 0.3y(k) + u(k) - 0.2u(k-1) \tag{3.8}$$

c) Applying z transform to the controller signal gives

$$U(z) = Z[\mathrm{u(k)}] = \mathrm{K}(\mathrm{U}_c(z) - Y(z)) \tag{3.9}$$

Since we have known $Y(z) = U(z)H(z)$, then it is known that

$$Y(\mathrm{z}) = \mathrm{K}(\mathrm{U}_c(z) - Y(z))H(z) \tag{3.10}$$

Thus, it can be derived that

$$\frac{Y(z)}{U_c(z)} = \frac{KH(z)}{1+KH(z)} = \frac{Kz-0.2K}{z^2+(K-0.3)z-0.2K} \tag{3.11}$$

d) The characteristic polynomial is $z^2+(K-0.3)z-0.2K$. For simplicity, let $M=0.2K-0.02$. Then Jury's test can be listed in a table as follows.

(1)	Get coefficients	1	K-0.3	-0.2K
(2)	Reverse	-0.2K	K-0.3	1
(3)	(1)+(2)*(0.2K)	1-0.04K^2	(0.2K+1)(K-0.3)	
(4)	Reverse	(0.2K+1)(K-0.3)	1-0.04K^2	
(5)	(3) - (4)* $\frac{(0.2K+1)(K-0.3)}{1-0.04K^2}$	$1-0.04K^2-\frac{((0.2K+1)(K-0.3))^2}{1-0.04K^2}$		

According to Jury's test, if the system is stable, the first element of all the **odd** rows should be positive (in blue), i.e.,

$$\begin{cases} 1-0.04K^2>0 \\ 1-0.04K^2-\dfrac{((0.2K+1)(K-0.3))^2}{1-0.04K^2}>0 \end{cases} \tag{3.12}$$

The second inequality in (3.12) can be simplified to be

$$1-\frac{(0.2K+1)^2(K-0.3)^2}{(0.2K+1)^2(1-0.2K)^2} = 1-\frac{(K-0.3)^2}{(1-0.2K)^2}>0 \tag{3.13}$$

That is

$$0.96K^2-0.2K-0.91<0 \tag{3.14}$$

Finally, the answer for (3.12) is

$$\begin{cases} -5<K<5 \\ -\dfrac{7}{8}<K<\dfrac{13}{12} \end{cases} \Rightarrow -\frac{7}{8}<K<\frac{13}{12} \tag{3.15}$$

e) The z transform for $u_c(k)$, a unit step input, is

$$U_c(z)=\frac{z}{z-1} \tag{3.16}$$

And the output $Y(z)$ is got according to equation (3.11)

$$Y(z)=\frac{Kz-0.2K}{z^2+(K-0.3)z-0.2K}\frac{z}{z-1} \tag{3.17}$$

Assuming the system is stable, according to the final value theorem, the stable state value of $y(k)$ is

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}\frac{Kz-0.2K}{z^2+(K-0.3)z-0.2K} = \frac{0.8K}{0.8K+0.7} \tag{3.18}$$

Thus, the steady-state error is

$$\lim_{k\to\infty}(\mathrm{u}_c(k)-y(k)) = 1-\frac{0.8K}{0.8K+0.7} = \frac{0.7}{0.8K+0.7} \tag{3.19}$$

Q4 Solution

a) Applying z transform yields

$$zY(z)-zy(0) = -Y(z)+z^{-1}Y(z)+z^{-2}Y(z)+U(z)+2z^{-1}U(z)+z^{-2}U(z) \tag{4.1}$$

Assuming zero initial conditions, the transfer function is

$$H(z)=\frac{Y(z)}{U(z)}=\frac{1+2z^{-1}+z^{-2}}{z+1-z^{-1}-z^{-2}}=\frac{z^2+2z+1}{z^3+z^2-z-1}=\frac{(z+1)^2}{(z+1)^2(z-1)} \tag{4.2}$$

(Note: here you should NOT cancel the unstable poles with zeros.)

Poles: $(z+1)^2(z-1)=0 \Rightarrow z_1=1, z_2=z_3=-1$

Since there are two same poles on the unit circle, the system is unstable.

Zeros: $(z+1)^2=0 \Rightarrow z_1=z_2=-1$.

Two identical zeros of magnitude 1 exist, thus the inverse system is also unstable.

b)

(First note that since there are common poles/zeros in the transfer function (4.2), the system will be either uncontrollable or unobservable or both uncontrollable and unobservable.)

The transfer function (4.2) is rewritten as

$$H(z)=\frac{z^2+2z+1}{z^3+z^2-z-1} \tag{4.3}$$

Assume the state space realization is

$$\begin{cases} x(k+1)=\Phi x(k)+\Gamma u(k) \\ y(k)=Cx(k) \end{cases} \tag{4.4}$$

- **We first try the controllable canonical form.**

For the controllable canonical form, we have

$$\Phi = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \Gamma = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \tag{4.5}$$

Check the observability of the state space model (4.5). The observability matrix is

$$W_o = \begin{bmatrix} C \\ C\Phi \\ C\Phi^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \tag{4.6}$$

Since W_o is not full column rank, the system is unobservable.

- **We can also try the observable canonical form.**

$$\Phi = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \Gamma = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{4.7}$$

Its controllability matrix is computed to be

$$W_c = \begin{bmatrix} \Gamma & \Phi\Gamma & \Phi\Gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \tag{4.8}$$

It is obvious that the model (4.8) is not controllable.

As can be seen now, we have provided two realizations: one is controllable but unobservable; the other is observable but uncontrollable. **Which one is right?** To verify which one is valid, we have to reconstruct the transfer function from the state space model and see whether it is the same with the original transfer function.

For model (4.5), it is easy to get

$$\begin{aligned} x_1(k+1) &= -x_1(k) + x_2(k) + x_3(k) + u(k) \\ x_2(k+1) &= x_1(k) \\ x_3(k+1) &= x_2(k) \\ y(k) &= x_1(k) + 2x_2(k) + x_3(k) \end{aligned} \tag{4.9}$$

Now consider the original transfer function. We express $y(k+1)$ by equation (4.9)

$$\begin{aligned} y(k+1) &= x_1(k+1) + 2x_2(k+1) + x_3(k+1) \\ &= -x_1(k) + x_2(k) + x_3(k) + u(k) + 2x_1(k) + x_2(k) \\ &= x_1(k) + 2x_2(k) + x_3(k) + u(k) \\ &= y(k) + u(k) \end{aligned} \tag{4.10}$$

There, the actual difference equation for the first realization (4.5) is different from our original input-output difference equation, which means (4.5) is not a valid realization. In fact, from (4.10), we can see that the transfer function for (4.5) is actually $\frac{1}{z-1}$, which shows (4.5) is actually a nonminimum realization of $\frac{1}{z-1}$.

Similarly, we can check the 2nd realization in (4.7). It follows that

$$\begin{aligned} x_1(k+1) &= -x_1(k)+x_2(k)+u(k) \\ x_2(k+1) &= x_1(k)+x_3(k)+2u(k) \\ x_3(k+1) &= x_1(k)+u(k) \\ y(k) &= x_1(k) \end{aligned} \tag{4.11}$$

Then, it is derived that

$$\begin{aligned} y(k+1) = x_1(k+1) & \\ &= -x_1(k)+x_2(k)+u(k) \\ &= -y(k)+x_1(k-1)+x_3(k-1)+2u(k-1)+u(k) \\ &= -y(k)+y(k-1)+x_1(k-2)+u(k-2)+2u(k-1)+u(k) \\ &= -y(k)+y(k-1)+y(k+2)+u(k-2)+2u(k-1)+u(k) \end{aligned} \tag{4.12}$$

which is identical with the original difference equation.

Now we can conclude that the 2nd realization is right. **The system is observable but uncontrollable.**

This problem also shows that for any given transfer function, you can "realize" it in a canonical controllable form or observable form. That is to say, even the original system is uncontrollable, you can still write a controllable canonical form for it according to the formulas. However, this realization may be wrong, especially when there are common poles and zeros. You need to verify it.

If you define the state variables explicitly as the method given in the lecture notes, you can construct a state space model which is observable and uncontrollable directly. Such answer is also correct (with full marks). Anyway, here we just want to show you the effect of common poles/zeros when you realize a transfer function model into a state space model.