091M4041H-Assignment 4

Algorithm Design and Analysis

December 11, 2019

Notice:

1. Please submit your answers in hard copy **AND** submit a digital version to UCAS website https://sep.ucas.ac.cn.
2. Hard copy should be submitted before 9 a.m. Dec 27 and digital version should be submitted before 11 p.m. Dec 27.
3. You should finish the first five problems. For the last one, you can have a try on your own.
4. Integer Linear Programming is different from the classic Linear Programming that some extra constraints such as

   $$x_i \text{ is an integer, } \forall i = 1, 2, \cdots, n$$

   or

   $$x_i \in \{0, 1\}, \ \forall i = 1, 2, \cdots, n$$

   are added.
5. When you give the formulation of an LP or ILP, you should explain all mathematical symbols you are using if not appearing in the problem, and interpret the constraints if necessary.

1 Linear-inequality Feasibility

Given a set of m linear inequalities on n variables $x_1, x_2, \cdots, x_n$, the **linear-inequality feasibility problem** asks if there is a setting of the variables that simultaneously satisfies each of the inequalities.

Show that if we have an algorithm for linear programming, we can use it to solve the linear-inequality feasibility problem. The number of variables and constraints that you use in the linear-programming problem should be polynomial of n and m.

2 Reformulation Problems with Absolute Values

Consider the problem

$$\begin{aligned} \text{minimize} \quad & 2|x_1| + x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 4 \end{aligned}$$

Reformulate this problem as a LP.

3 Gas Station Placement

Let's consider a long, quiet country road with towns scattered very sparsely along it. Sinopec, largest oil refiner in China, wants to place gas stations along the road. Each gas station is assigned to a nearby town, and the distance between any two gas stations being as small as possible. Suppose there are n towns with distances from one endpoint of the road being $d_1, d_2, \cdots, d_n$. n gas stations are to be placed along the road, one station for one town. Besides, each station is at most r far away from its correspond town. $d_1, \cdots, d_n$ and r have been given and satisfied $d_1 < d_2 < \cdots < d_n$, $0 < r < d_1$ and $d_i + r < d_{i+1} - r$ for all i. The objective is to find the optimal placement such that the maximal distance between two successive gas stations is minimized.

Please formulate this problem as a LP, construct an instance, and try to solve both primal and dual problem using GLPK or Gurobi or other similar tools.

4 Volunteer Recruitment

Suppose you will recruit a group of volunteers for a coming event. It is estimated that this event will take N days to complete, and the $i(th)$ day needs at least A_i volunteers. The number of kinds of volunteers is M. The volunteers of $i(th)$ kind can volunteer from the S_i day to the F_i day and the recruit fee is C_i. In order to do his job well, you hope to recruit enough volunteers with least money. Please formulate this problem as an ILP.

5 Stable Matching Problem

n men $(m_1, m_2, \cdots, m_n)$ and n women $(w_1, w_2, \cdots, w_n)$, where each person has ranked all members of the opposite gender, have to make pairs. You need to give a stable matching of the men and women such that there is no unstable pair. (A matching is *unstable* if: there is an element A of the first matched set which prefers some given element B of the second matched set over the element to which a is already matched, and B also prefers A over the element to which B is already matched.) Please choose one of the two following known conditions, formulate the problem as an ILP.

1. You have known that for every two possible pairs (man m_i and woman w_j, man m_k and woman w_l), whether they are stable or not. If they are stable, then $S_{i,j,k,l} = 1$; if not, $S_{i,j,k,l} = 0$. $(i, j, k, l \in \{1, 2, \cdots, n\})$
2. You have known that for every man m_i, whether m_i likes woman w_j more than w_k. If he does, then $p_{i,j,k} = 1$; if not, $p_{i,j,k} = 0$. Similarly, if woman w_i likes man m_j more than m_k, then $q_{i,j,k} = 1$, else $q_{i,j,k} = 0$. $(i, j, k \in \{1, 2, \cdots, n\})$

6 Simplex Method

Please fulfill the simplex method to solve linear programming problem in standard form.

$$\max \quad c^T x$$

$$\text{s.t.} \quad Ax \leq b$$

$$x_j \geq 0, \quad \forall j = 1, ..., n$$