量子信息与量子密码 第六次作业

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32. 构造如下的三个元素构成 POVM:

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle\langle 1|$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2}$$

$$E_3 = I - E_1 - E_2$$

显然, $\sum_m E_m = I$.

考虑两个量子比特的状态

其中, $\langle \phi_1 | E_1 | \phi_1 \rangle = 0$

$\langle \phi_2 | E_2 | \phi_2 \rangle = 0$

故若测量 E_1, 则表明得到的量子比特状态为 $|\phi_2\rangle$

若测量得到 E_2, 则表明得到的量子比特状态为 $|\phi_1\rangle$

而测量得到 E_3, 则不能对状态作出判断.

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36. 证明如下：

即：$(Q\otimes S + R\otimes S + R\otimes T - Q\otimes T)^2$

$= (Q\otimes S)^2 + (R\otimes S)^2 + (R\otimes T)^2 + (Q\otimes T)^2 + (R\otimes S)(Q\otimes S) + (R\otimes S)(R\otimes T)$
$- (R\otimes S)(Q\otimes T) + (Q\otimes S)(R\otimes S) - (Q\otimes S)(Q\otimes T) + (Q\otimes S)(R\otimes T)$
$+ (R\otimes T)(Q\otimes S) + (R\otimes T)(R\otimes S) - (R\otimes T)(Q\otimes T) - (Q\otimes T)(Q\otimes S)$
$- (Q\otimes T)(R\otimes S) - (Q\otimes T)(R\otimes T)$

其中 $(Q\otimes S)^2 = Q^2 \otimes S^2$，$Q^2 = I$，$S^2 = I$，$R^2 = I$，$T^2 = I$

$\therefore (Q\otimes S)^2 = I$.，同理，则 $(R\otimes S)^2 = I$，$(R\otimes T)^2 = I$，$(Q\otimes T)^2 = I$

且有：$(Q\otimes S)[(R\otimes S) - (Q\otimes T)] + (R\otimes S)[(Q\otimes S) + (R\otimes T)]$
$+ (R\otimes T)[(R\otimes S) - (Q\otimes T)] - (Q\otimes T)[(Q\otimes S) + (R\otimes T)]$
$= [(Q\otimes S) + (R\otimes T)][(R\otimes S) - (Q\otimes T)] - [(R\otimes S) - (Q\otimes T)][(Q\otimes S) + (R\otimes T)]$
$= 0$

$\therefore$ 原式 $= 4I + (Q\otimes S)(R\otimes T) - (R\otimes S)(Q\otimes T) - (Q\otimes T)(R\otimes S) + (R\otimes T)(Q\otimes S)$
$= 4I + QR\otimes ST - RQ\otimes ST - QR\otimes TS + RQ\otimes TS$
$= 4I + (QR - RQ)\otimes(ST - TS)$
$= 4I + [Q,R]\otimes[S,T]$

进一步，知，$\langle R\rangle^2 \le \langle R^2\rangle$

$\Rightarrow (\langle Q\otimes S\rangle + \langle R\otimes S\rangle + \langle R\otimes T\rangle - \langle Q\otimes T\rangle)^2$
$\le \langle (Q\otimes S + R\otimes S + R\otimes T - Q\otimes T)^2\rangle = \langle\varphi|4I + [Q,R]\otimes[S,T]|\varphi\rangle$
$= 4 + \langle\varphi|[Q,R]\otimes[S,T]|\varphi\rangle$

知 $|\langle\varphi|[A,B]|\varphi\rangle|^2 \le 4\langle\varphi|A^2|\varphi\rangle\langle\varphi|B^2|\varphi\rangle$

若 $A^2 = I$，$B^2 = I \Rightarrow \langle\varphi|[A,B]|\varphi\rangle \le 2$

$\therefore \langle\varphi|[Q,R]\otimes[S,T]|\varphi\rangle = \langle\varphi_1|[Q,R]|\varphi_1\rangle\langle\varphi_2|[S,T]|\varphi_2\rangle$
$\le 2\cdot 2 = 4$

$\therefore \langle Q\otimes S\rangle + \langle R\otimes S\rangle + \langle R\otimes T\rangle - \langle Q\otimes T\rangle \le \sqrt{8} = 2\sqrt{2}$.

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38. 证明如下，

知 $\rho^{A} = tr_B(\rho^{AB})$

$\rho^{A'} = tr_{B'}(\rho^{A'B'})$

又因有 $\rho^{A'B'} = U_B \rho^{AB}$

不妨假设 $\rho^{AB} = |a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|$

知 $U_B \rho^{AB} = |a_1\rangle\langle a_2| \otimes U_B|b_1\rangle\langle b_2|U_B^{+}$

而 $tr_B(\rho^{AB}) = |a_1\rangle\langle a_2|\, tr_B(|b_1\rangle\langle b_2|)$

$= |a_1\rangle\langle a_2|\, \langle b_2|b_1\rangle$

$tr_{B'}(\rho^{A'B'}) = |a_1\rangle\langle a_2|\, tr_{B'}(U_B|b_1\rangle\langle b_2|U_B^{+})$

$= |a_1\rangle\langle a_2|\, \langle b_2|U_B^{+}U_B|b_1\rangle$

$= |a_1\rangle\langle a_2|\, \langle b_2|b_1\rangle$

∴ 故 $\rho^{A} = \rho^{A'}$

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39. $\rho = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

~~故 $S(\rho)$ =~~ 其特征值均为 $\frac{1}{2}$

故 $S(\rho) = -\sum_i \lambda_i \log \lambda_i$

$= -(\frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\log\frac{1}{2})$

$= \log 2.$

40. 假设 ρ 的 D 个非零本征值为 λ_i，$i=1,\dots,D$

则 $S(\rho) = -\sum_{i=1}^{D} \lambda_i \log \lambda_i$

证明如下：令 $p_i = D\lambda_i \Rightarrow \lambda_i = \frac{p_i}{D}$

故 $S(\rho) = -\sum_{i=1}^{D} \frac{p_i}{D} \lg \frac{p_i}{D}$

$$= -\frac{1}{D} \sum_{i=1}^{D} p_i (\log p_i - \lg D)$$

$$= -\frac{1}{D} \left(\sum_{i=1}^{D} p_i \log p_i - \log D \sum_i p_i \right)$$

知 $\ln x \le x - 1$

$\Rightarrow \ln \frac{1}{x} \le \frac{1}{x} - 1$

$\Rightarrow x \ln \frac{1}{x} \le 1 - x$

$\Rightarrow -x \ln x \le 1 - x$

代入可得

$$S(\rho) \le -\frac{1}{D} \left(\sum_{i=1}^{D} (1 - p_i) - \log D \sum_i p_i \right)$$

$$= -1 + \frac{1}{D} (1 + \log D) \sum_i p_i$$

对一个密度矩阵 $\sum_i \lambda_i = 1$

$\Rightarrow \sum_i p_i = D$

$\therefore S(\rho) \le \log D$

当且仅当 $p_i = 1 \Rightarrow$ 所有 λ_i 相等时

$S(\rho) = \log D$.