量子信息与量子密码 第三次作业

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11. 对等式，分别证明如下： 对双量子比特系统，共有4种不同情况，$|00\rangle,|01\rangle,|10\rangle,|11\rangle$

① $CX_1C = X_1X_2$.

(1) $\begin{cases} CX_1C|00\rangle = CX_1|00\rangle = C|10\rangle = |11\rangle \\ X_1X_2|00\rangle = X_1|01\rangle = |11\rangle. \end{cases}$

(2) $\begin{cases} CX_1C|01\rangle = CX_1|01\rangle = C|11\rangle = |10\rangle \\ X_1X_2|01\rangle = X_1|00\rangle = |10\rangle. \end{cases}$

(3) $\begin{cases} CX_1C|10\rangle = CX_1|11\rangle = C|01\rangle = |01\rangle \\ X_1X_2|10\rangle = X_1|11\rangle = |01\rangle. \end{cases}$

(4) $\begin{cases} CX_1C|11\rangle = CX_1|10\rangle = C|00\rangle = |00\rangle. \\ X_1X_2|11\rangle = X_1|10\rangle = |00\rangle \end{cases}$

∴ 得证 $CX_1C = X_1X_2$.

②. $CY_1C = Y_1X_2$

(1) $\begin{cases} CY_1C|00\rangle = CY_1|00\rangle = C\,i|10\rangle = i|11\rangle. \\ Y_1X_2|00\rangle = Y_1|01\rangle = i|11\rangle. \end{cases}$

(2) $\begin{cases} CY_1C|01\rangle = CY_1|01\rangle = C\,i|11\rangle = i|10\rangle. \\ Y_1X_2|01\rangle = Y_1|00\rangle = i|10\rangle. \end{cases}$

(3) $\begin{cases} CY_1C|10\rangle = CY_1|11\rangle = C(-i)|01\rangle = -i|01\rangle \\ Y_1X_2|10\rangle = Y_1|11\rangle = -i|01\rangle. \end{cases}$

(4) $\begin{cases} CY_1C|11\rangle = CY_1|10\rangle = C(-i)|00\rangle = -i|00\rangle \\ Y_1X_2|11\rangle = Y_1|10\rangle = -i|00\rangle. \end{cases}$

∴ 得证 $CY_1C = Y_1X_2$.

③ $CZ_1C = Z_1$

(1) $\begin{cases} CZ_1C|00\rangle = CZ_1|00\rangle = C|00\rangle = |00\rangle \\ Z_1|00\rangle = |00\rangle \end{cases}$

(2) $\begin{cases} CZ_1C|01\rangle = CZ_1|01\rangle = C|01\rangle = |01\rangle. \\ Z_1|01\rangle = |01\rangle. \end{cases}$

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(3) $\begin{cases} CZ_1C|10\rangle = CZ_1|11\rangle = C-|11\rangle = -|10\rangle. \\ Z_1|10\rangle = -|10\rangle \end{cases}$

(4) $\begin{cases} CZ_1C|11\rangle = CZ_1|10\rangle = C-|10\rangle = -|11\rangle \\ Z_1|11\rangle = -|11\rangle \end{cases}$

$\therefore CZ_1C = Z_1$

④ $CX_2C = X_2$.

$CX_2C|00\rangle = CX_2|00\rangle = C|01\rangle = |01\rangle = X_2|00\rangle$

$CX_2C|01\rangle = CX_2|01\rangle = C|00\rangle = |00\rangle = X_2|01\rangle$

$CX_2C|10\rangle = CX_2|11\rangle = C|10\rangle = |11\rangle = X_2|10\rangle$

$CX_2C|11\rangle = CX_2|10\rangle = C|11\rangle = |10\rangle = X_2|11\rangle$

$\therefore$ 得证 $CX_2C = X_2$.

⑤ $\begin{cases} CY_2C|00\rangle = CY_2|00\rangle = C\,i|01\rangle = i|01\rangle \\ Z_1Y_2|00\rangle = Z_1\,i|01\rangle = i|01\rangle \end{cases}$

$\begin{cases} CY_2C|01\rangle = CY_2|01\rangle = C-i|00\rangle = -i|00\rangle \\ Z_1Y_2|01\rangle = Z_1-i|00\rangle = -i|00\rangle \end{cases}$

$\begin{cases} CY_2C|10\rangle = CY_2|11\rangle = C-i|10\rangle = -i|11\rangle. \\ Z_1Y_2|10\rangle = C_1\,i|11\rangle = -i|11\rangle \end{cases}$

$\begin{cases} CY_2C|11\rangle = CY_2|10\rangle = C\,i|11\rangle = i|10\rangle \\ Z_1Y_2|11\rangle = Z_1-i|10\rangle = i|10\rangle. \end{cases}$

故 $CY_2C = Z_1Y_2$.

⑥ $CZ_2C|00\rangle = CZ_2|00\rangle = C|00\rangle = |00\rangle = Z_1Z_2|00\rangle$.

$CZ_2C|01\rangle = CZ_2|01\rangle = C-|01\rangle = -|01\rangle = Z_2|01\rangle = Z_1Z_2|01\rangle$.

$CZ_2C|10\rangle = CZ_2|11\rangle = C-|11\rangle = -|10\rangle = Z_1Z_2|10\rangle$

$CZ_2C|11\rangle = CZ_2|10\rangle = C|10\rangle = |11\rangle = Z_1Z_2|11\rangle$.

$\therefore CZ_2C = Z_1Z_2$

故，全部得证.

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12. 假设两个量子比特为 $|c\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$, $|t\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

对于电路而言，则，电路1为：

$$U_1|c\rangle|t\rangle = U_1(\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle)$$

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle - \beta_1\beta_2|11\rangle$$

对于电路2而言：

$$U_2|c\rangle|t\rangle = U_2(\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle)$$

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle - \beta_1\beta_2|11\rangle$$

$\therefore U_1|c\rangle|t\rangle = U_2|c\rangle|t\rangle$

$\therefore$ 电路1和2相等： $U_{z_1} = U_{z_2}$.

13. 假设电路1表示为 $U_1 = H^{(2)}C_2H^{(2)}$, $U_2 = C_1$.

同样，假设两个量子比特为 $|c\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$, $|t\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$.

$\therefore$ ~~$U_1|c\rangle$~~

$\therefore U_1|c\rangle|t\rangle = H^{(2)}C_2H^{(2)}|c\rangle|t\rangle$

$$= H^{(2)}C_2\frac{(\alpha_1+\beta_1)(\alpha_2+\beta_2)|00\rangle + (\alpha_1+\beta_1)(\alpha_2-\beta_2)|01\rangle + (\alpha_1-\beta_1)(\alpha_2+\beta_2)|10\rangle + (\alpha_1-\beta_1)(\alpha_2-\beta_2)|11\rangle}{2}$$

$$= H^{(2)}\frac{(\alpha_1+\beta_1)(\alpha_2+\beta_2)|00\rangle + (\alpha_1+\beta_1)(\alpha_2-\beta_2)|01\rangle + (\alpha_1-\beta_1)(\alpha_2-\beta_2)|10\rangle + (\alpha_1-\beta_1)(\alpha_2+\beta_2)|11\rangle}{2}$$

$$\cancel{= \frac{1}{4}\left((\alpha_1+\beta_1)(\alpha_2+\beta_2) + (\alpha_1+\beta_1)(\alpha_2-\beta_2) + (\alpha_1-\beta_1)(\alpha_2-\beta_2) + (\alpha_1-\beta_1)(\alpha_2+\beta_2)\right)|00\rangle}$$

$$= \frac{1}{4}(4\alpha_1\alpha_2|00\rangle + 4\beta_1\beta_2|01\rangle + 4\beta_1\alpha_2|10\rangle + 4\alpha_1\beta_2|11\rangle)$$

$$= \alpha_1\alpha_2|00\rangle + \beta_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \alpha_1\beta_2|11\rangle$$

$$U_2|c\rangle|t\rangle = C_1(\alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|11\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|01\rangle)$$

$$= \alpha_1\alpha_2|00\rangle + \beta_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \alpha_1\beta_2|11\rangle$$

$\therefore U_1 = U_2$

得证.

14. (1) 证明如下：

$$e^{iAx} = \sum_{k=0}^{\infty}\frac{(Ax)^{2k}}{(2k)!} + i\sum_{k=0}^{\infty}\frac{(Ax)^{2k+1}}{(2k+1)!}$$

$$= I\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!} + iA\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$

$$= \cos x\cdot I + i\sin x\cdot A.$$

(2) $$e^{-i\theta X/2} = I\cdot\cos\frac{\theta}{2} + -X\cdot i\sin\frac{\theta}{2}.$$

$$= \begin{pmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}. = R_x(\theta)$$

$$R_y(\theta) = I\cdot\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\cdot Y$$

$$= \begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$$

$$R_z(\theta) = I\cdot\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}Z$$

$$= \begin{pmatrix}\cos\frac{\theta}{2} - i\sin\frac{\theta}{2} & 0\\ 0 & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix}e^{-i\frac{\theta}{2}} & 0\\ 0 & e^{i\frac{\theta}{2}}\end{pmatrix}.$$

15. 证明如下，已知 A 为正规矩阵，

① 充分性：若 A 的特征值为实数，则根据正规矩阵相似于对角矩阵 S

设 T 为酉矩阵，$T^{+} = T^{-1}$，则有

$$TAT^{-1} = \begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \\ 0 & & \lambda_n\end{bmatrix} = S \quad 则\ A = TST^{-1}$$

$$A^{+} = (TST^{-1})^{+} = TS^{+}T^{-1}$$

$\because$ S 中所有值为实数，$\therefore S = S^{+}$，则 $A = A^{+}$.

② 必要性，若 $A = A^{+}$，根据之前的分析有，

$S = S^{+}$，即对任意 λ_i；有 $\lambda_i = \lambda_i^{*}$

故 λ_i 为实数，$\therefore$ A 的特征值为实数.