Notice:

1. Please write your name along with student ID.
2. There are 8 sections in the sheet, and for sections 1-6, you can arbitrarily *choose one problem*. If you answer two problems, say 1.1 and 1.2, the higher mark will be *taken in account*.
3. When you are asked to give an algorithm, you should describe your algorithm in *natural language* or pseudo-codes, prove the correctness, and analyze time complexity.
4. You can write answers in either Chinese or English.

1 Divide and Conquer (12 marks)

1.1 Find Minimum in Rotated Sorted Array

Suppose an array sorted in ascending order is rotated at some pivot unknown to you beforehand. (*i.e.*, $[0, 1, 2, 4, 5, 6, 7]$ is an ascending array, then it might be rotated and become $[4, 5, 6, 7, 0, 1, 2]$.) *How to* find the minimum of a rotated sorted array?
(*Hint:* All elements in the array are distinct.)

For example, the minimum of the rotated sorted array $[4, 5, 6, 7, 0, 1, 2]$ is 0.

Please give an algorithm with $O(\log n)$ complexity, prove the correctness and analyze the complexity.

1.2 Local Minimum in Array

A local minimum of an array is defined as the element which is less than its neighbors. The problem is to find one local minimum and report its index. The array may contain several local minimums, in which case you can report any one of them.
(*Hint:* All elements in the array num are distinct, and suppose that $num[-1] = num[n] = +\infty$.)

For example, in array $[2, 3, 0, 1]$, 0 is a local minimal element and your objective is to report the index number 2.

Please give an algorithm with $O(\log n)$ complexity, prove the correctness and analyze the complexity.

2 Dynamic Programming (12 marks)

2.1 Climbing Stairs

You are climbing a stair case. It takes n steps to reach the top. Each time you can either climb 1 or 2 steps. However, since you are too weak, you can not climb 2 steps just after you have climbed 2 steps, you can only climb 1 step to have a rest. Then, in how many distinct ways can you climb to the top?

Please give an algorithm including DP recurrence relations, prove the correctness and analyze the complexity.

2.2 Partition

A palindrome is defined as a string that reads the same backward and forward. For example, the word "refer", is a palindrome. A palindrome partition is that given a string s, cut s into several substrings, such that every substring of the partition is a palindrome. The Palindrome Partition problem is to report a palindrome partition with minimum number of cuts.

For example, given $s =$ "aabaacaa", return 2 since the palindrome partitioning ["aabaa", "c", "aa"] could be produced using 2 cuts.

Please give an algorithm including DP recurrence relations, prove the correctness and analyze the complexity.

3 Greedy Algorithm (12 marks)

3.1 Doing homework

Mike has many jobs to be completed and every job has a deadline. If Mike finishes the job after the deadline, the teacher will give him a penalty. Suppose that every job always takes one day, and how should Mike arrange the order of jobs to minimize the penalty.

For example, there are 4 jobs (denoted as J) in the first line, the deadlines (denoted as D) and the penalty (denoted as P) are in the second and third lines.

J: 1 2 3 4
D: 2 3 1 2
P: 100 80 50 70

So, Mike can finish the first, the second and the fourth job, and finally his penalty is 50.

Please give an algorithm, prove the correctness and analyze the complexity.

3.2 Skiers and Skis

A team wants to engage in a skiing competition. You were asked to help them to match skis to skiers. In particular, skier goes faster with skis whose lengths are close to his height.

The team consists of n members, with heights $h_1, h_2, \ldots, h_n$, and the team gets a delivery of n pairs of skis, with lengths $l_1, l_2, \ldots, l_n$. Your goal is to give an algorithm to assign one pair of skis to each skier to minimize the summation of the absolute differences between the height of each skier h_i and its length of the corresponding ski l_j.

Please give an algorithm, prove the correctness and analyze the complexity.

4 Linear Programming Formulation (12 marks)

Notice: Absolute value ($|\cdot|$) shall never appear in a Linear Programming (LP) formulation, since it is not a linear operation. Neither shall other non-linear operations, such as $\max\{\cdot\}$, *etc. If you want to use your own notation, please explain it first.*

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4.1 Minimum Cost Flow

There are N cities, and M directed roads connecting them. Now you want to transport K units of goods from city 1 to city N. There are many robbers on the road, so you must be very careful. The more goods you carry, the more dangerous it is. To be more specific, for each road i, there is a coefficient $a_i (a_i > 0)$. If you want to carry x units of goods along this road, you should pay $a_i x$ dollars to hire guards to protect your goods. For each road i, there is an upper bound C_i, which means that you cannot transport more than C_i units of goods along this road.

You should find out the minimum cost to transport all the goods safely. Please formulate this problem as an ILP (9 marks) and explain the meaning of every constrain (3 marks).

Note that you can only carry integral unit of goods along each road.

4.2 Single pair shortest path

In the single-pair shortest-path problem, we are given: a weighted directed graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}^+$ mapping edges to positive real-valued weights, a source vertex s, and a destination vertex t. For each edge (u, v), its weight is denoted as $w(u, v)$, and $w(u, v) > 0$. We wish to compute the minimal weight path from s to t.

Please formulate this problem as an ILP using the notations below (9 marks) and explain the meaning of every constrain (3 marks).

5 Network Flow Formulation (12 marks)

5.1 Maximum-Weight Closure

Given a vertex-weighted directed graph(weight can be either positive or negative), find the maximum-weight closure in this graph.
(*Hint:* A closure of a directed graph is a set of vertices with no outgoing edges. That is, the graph should have no edges that start within the closure and end outside the closure)

Please give an algorithm and prove the correctness.

5.2 Choose Numbers

Given a matrix $M = (M_{ij})^{n \times m}$ where $M_{ij} > 0$, for every two neighbor elements, at least one of them should be chosen. You are asked minimize the sum of chosen elements that meets the conditions.
(*Hint:* Neighbor means share common edge. That is, the neighbors of $M_{i,j}$ are $M_{i-1,j}, M_{i,j-1}, M_{i,j+1}, M_{i+1,j}$).

Please give an algorithm and prove the correctness.

6 NP-completeness Reduction (12 marks)

Notice: Only the following problems which have been proved in NP-complete in slides of this course can be used as for proof in this section, in order to avoid circular reasoning:

SAT, 3SAT, Clique, Independent-Set, Vertex-Cover, Set-Cover, Subset-Sum, 3-Coloring, Hamiltonian-Cycle

6.1 Quarter-3SAT

In the Quarter-3SAT problem, given a 3SAT formula ϕ with n variables and m clauses, where m is a multiple of 4. Determine whether there exists an assignment to the variables of ϕ such that exactly a quarter of the clauses evaluate to true and three quarters of the clauses evaluate to false or not.

Prove that Quarter-3SAT problem is in NP-complete.

6.2 Mine-sweeper

This problem is inspired by the single-player game Mine-sweeper, generalized to an arbitrary graph. Let G be an undirected graph, where each node either contains a single, hidden mine or is empty. The player chooses nodes, one by one. If the player chooses a node containing a mine, the player loses. If the player chooses an empty node, the player learns the number of neighboring nodes containing mines. (A neighboring node is one connected to the chosen node by an edge.). The player wins if and when all empty nodes have been so chosen.

In this mine consistency problem, you are given a graph G, along with numbers labeling some of G's nodes. You must determine whether a placement of mines on the remaining nodes is possible, so that any node v that is labeled m has exactly m neighboring nodes containing mines.

Prove that mine consistency problem is in NP-complete.

7 Unique Paths (14 marks)

A robot is located at the top-left corner of a $n \times m$ grid which cells are labeled by 0 and 1. The robot is trying to reach the bottom-right corner of the grid. However, the robot can only move either down or right at any point in time, and he can only pass the cell labeled by 0.

1. You want to know whether the robot can reach the bottom-right corner or not. There is a greedy strategy that the robot always chooses to move right if right cell is 0, otherwise he chooses to move down. Please give a case to prove this strategy is wrong. (4 marks)
2. Give an algorithm to calculate the number of possible unique paths. (6 marks)
3. After arriving at the bottom-right corner, the robot needs to return the starting point(top-left corner) and he can only move either up or left at any point in time when he returns. If the robot starts at top-left corner and comes back when he arrives bottom-right corner and each cell can be passed at most once, how many possible unique paths there are? (4 marks)

8 String (14 marks)

Given two strings s_1 and s_2, please construct a new string T, of which s_1 and s_2 are both subsequence. A subsequence is a sequence that can be derived from another sequence by deleting some elements without changing the order of the remaining elements.

1. If s_1 and s_2 are both in ascending order, please find the shortest length of T. (2 marks)
2. If s_1 and s_2 are regular strings, please find the shortest length of T. (4 marks)
3. If $s_1 =$ "$aaa \ldots aaa$", $s_2 =$ "$bbb \ldots bbb$" and $|s_1| > |s_2|$, it is obvious that the shortest length of T is $|s_1| + |s_2|$. But if in any prefix of T, the number of 'a' should be no less than the number of 'b', how many different T could we construct? (4 marks)
(*Hint:* $|s_1|$ is the length of s_1.)