طراحی الگوریتم ها

ملکی مجد

مباحث

حل مساله بیشترین جریان

Maximum Flow

Max-flow min-cut theorem

Ford-Fulkerson method

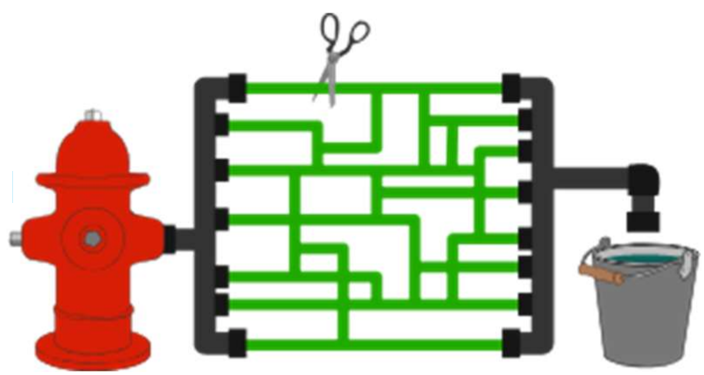
Edmonds-Karp algorithm

Maximum bipartite matching

مبحث Maximum Flow از فصل ۲۶ کتاب CLRS تدریس می شود.

Maximum Flow

+a graph-theoretic definition of flow networks



A ***flow network*** $G = (V, E)$:

is a directed graph in which each edge $(u, v) \in E$ has a nonnegative ***capacity*** $c(u, v) \geq 0$.

If $(u, v) \notin E$, we assume that $c(u, v) = 0$.

In Flow networks

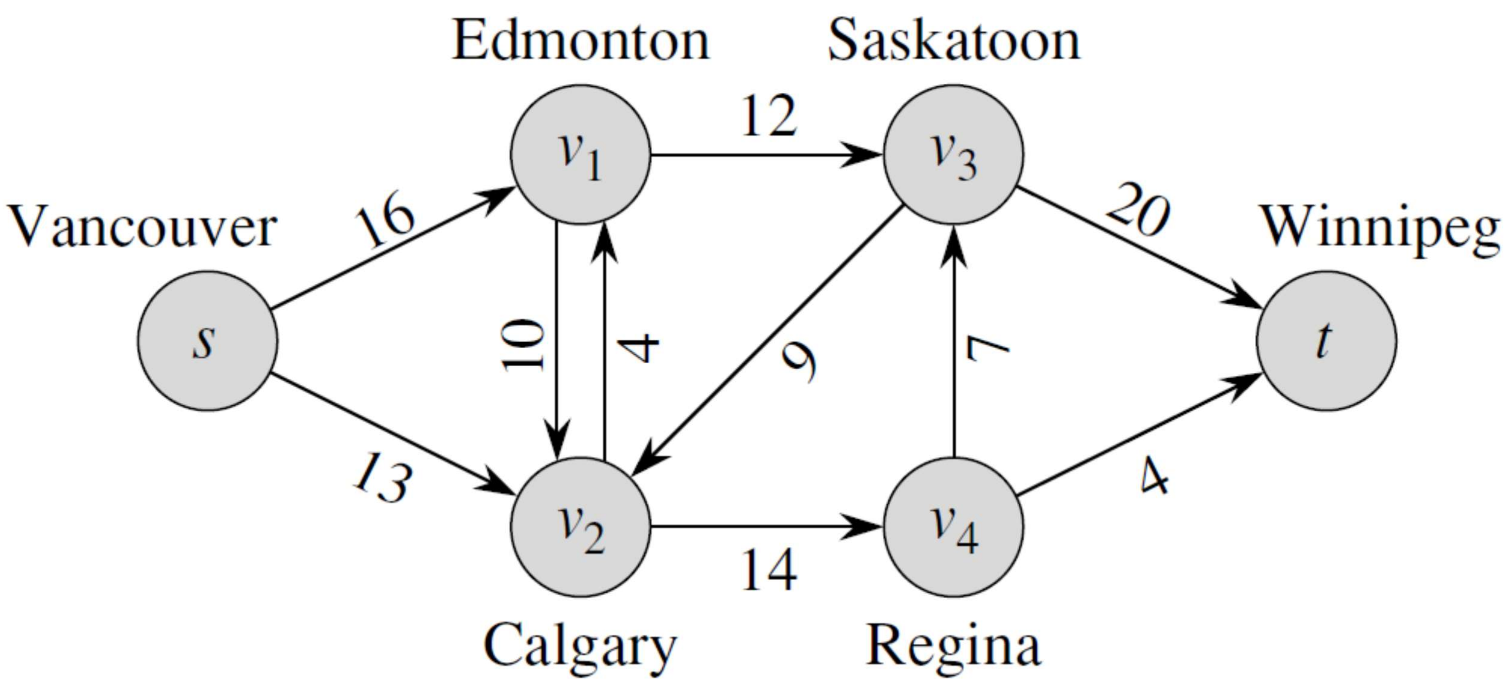
We distinguish : a ***source*** *s* and a ***sink*** *t*

we assume :

every vertex lies on some path from the source to the sink

The graph is therefore connected, and $|E| \geq |V| - 1$.

Example of network



flow

A flow in G is a **real-valued function f** : $V \times V \rightarrow R$ that satisfies the following three properties:

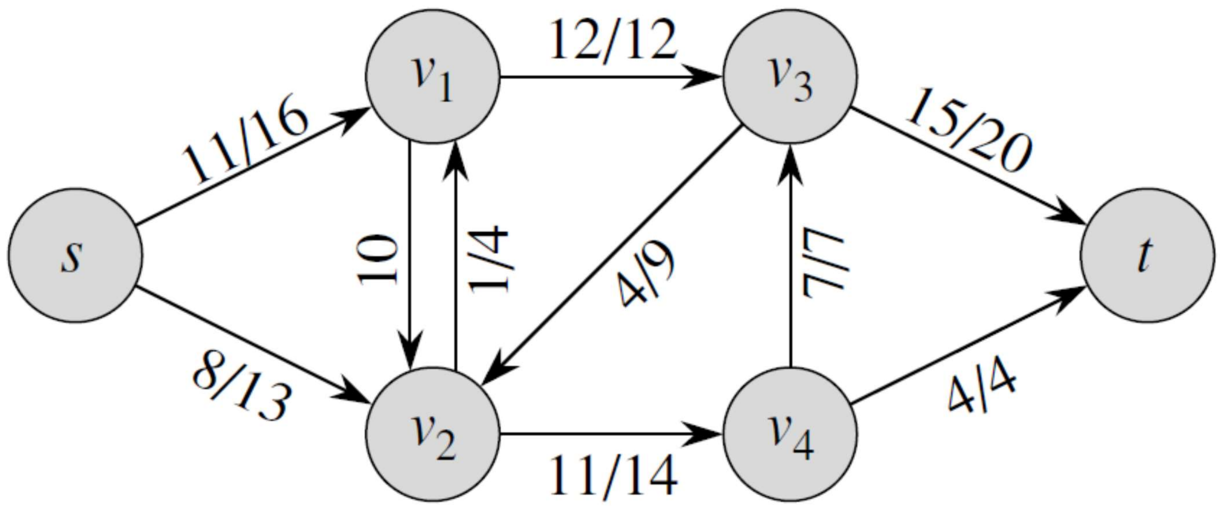
- **Capacity constraint:** For all $u, v \in V$, we require $f(u, v) \leq c(u, v)$.
- **Skew symmetry:** For all $u, v \in V$, we require $f(u, v) = -f(v, u)$.
- **Flow conservation:** For all $u \in V - \{s, t\}$, we require $\sum_{v \in V} f(u, v) = 0$

flow in equals **flow out** for vertex other than source and sink

The ***value*** of a flow f ***:***

total flow out of the source ($|f| = \sum_{v \in V} f(s, v)$)

A sample flow

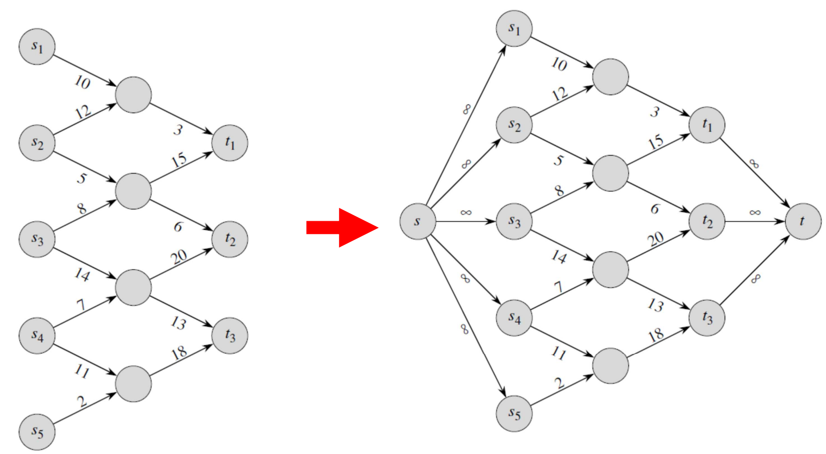


maximum-flow problem

In the ***maximum-flow problem***, we are given a flow network G with source s and sink t, and we wish to **find a flow of maximum value**.

Networks with multiple sources and sinks

- This problem is no harder than ordinary maximum flow
  - We can **reduce the problem** of determining a maximum flow in a network with multiple sources and multiple sinks to an ordinary maximum-flow problem



- **Add following:**

a ***supersource*** *s*

add a directed edge (s, s_i) with capacity $c(s, s_i) = \infty$ for each source s_i

a ***supersink*** *t*

and add a directed edge (t_i, t) with capacity $c(t_i, t) = \infty$ for each sink t_i

lemma

$$f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$$

Let $G = (V, E)$ be a flow network, and let f be a flow in G. Then the following equalities hold:

1. For all $X \subseteq V$, we have $f(X, X) = 0$.
2. For all $X, Y \subseteq V$, we have $f(X, Y) = -f(Y, X)$.
3. For all $X, Y, Z \subseteq V$ with $X \cap Y = \emptyset$, we have the sums $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ and $f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$.

The value of flow

$$
\begin{aligned}
|f| &= f(s, V) \\
&= f(V, V) - f(V - s, V) \\
&= -f(V - s, V) \\
&= f(V, V - s) \\
&= f(V, t) + f(V, V - s - t) \\
&= f(V, t)
\end{aligned}
$$

The Ford-Fulkerson method

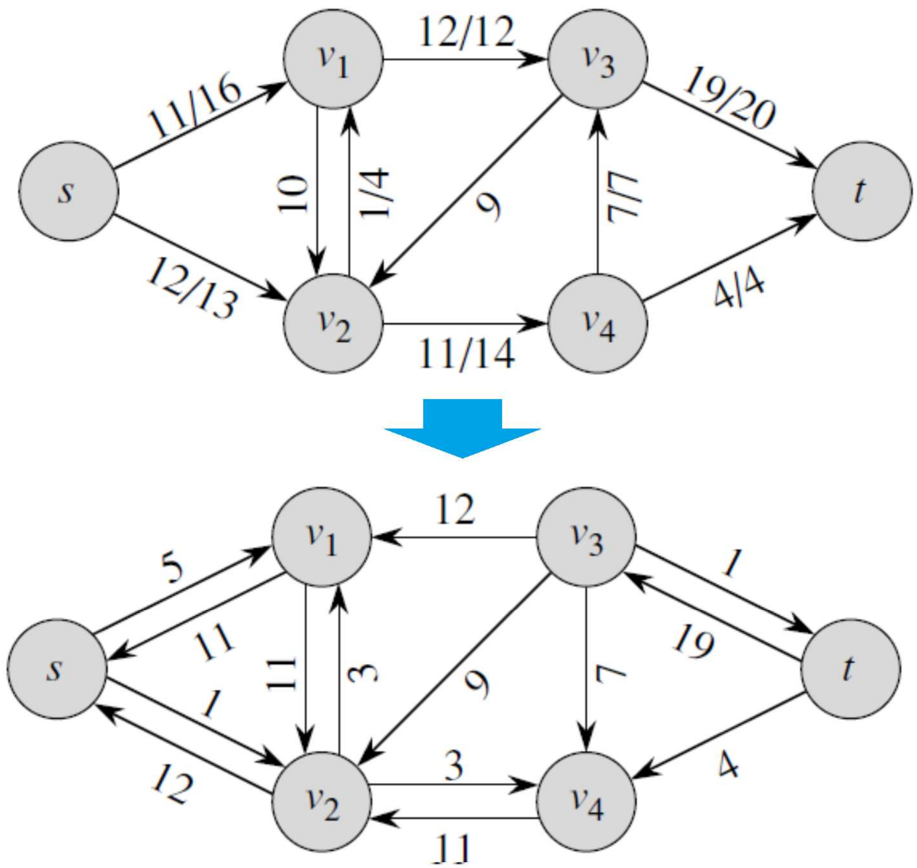
+for solving the maximum-flow problem

Residual networks

- The amount of *additional* flow we can push from *u* to *v* before exceeding the capacity *c(u, v)* is the ***residual capacity*** of *(u, v)*, given by

$$c_f(u, v) = c(u, v) - f(u, v).$$

given a flow network and a flow, the residual network consists of edges that can admit more flow



v1
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5
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11

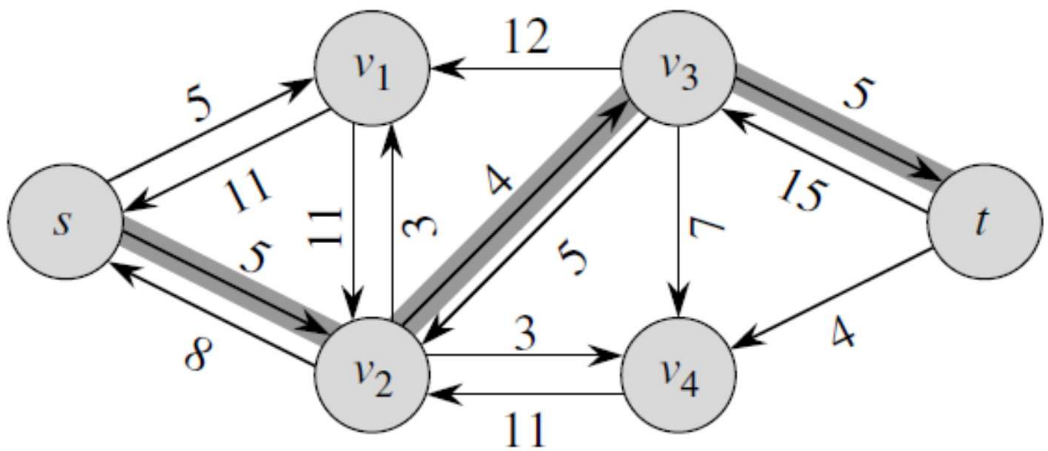
how a flow in a residual network relates to a flow in the original flow network

Let $G = (V, E)$ be a flow network with source s and sink t, and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then the flow sum $f + f'$ defined by equation (26.4) is a flow in G with value $|f + f'| = |f| + |f'|$.

Augmenting paths

Augmenting path is

a path from the source *s* to the sink *t* along which we can send more flow, and then augmenting the flow along this path



❖A flow is maximum if and only if its residual network contains no augmenting path.

Residual capacity of an augmenting path

- The maximum amount by which we can increase the flow on each edge in an augmenting path *p*

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\}$$

residual capacity of an augmenting path can add to the value of flow

Let $G = (V, E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Define a function $f_p : V \times V \to \mathbf{R}$ by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p\ , \\ -c_f(p) & \text{if } (v, u) \text{ is on } p\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{26.6}$$

Then, f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

Corollary

Let $G = (V, E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Let f_p be defined as in equation (26.6). Define a function $f' : V \times V \to \mathbf{R}$ by $f' = f + f_p$. Then f' is a flow in G with value $|f'| = |f| + |f_p| > |f|$.

General method
How increase the value of flow

FORD-FULKERSON-METHOD*(G, s, t)*

1 initialize flow *f* to 0

2 **while** there exists an augmenting path *p*

3 **do** augment flow *f* along *p*

4 **return** *f*

Definition of cut

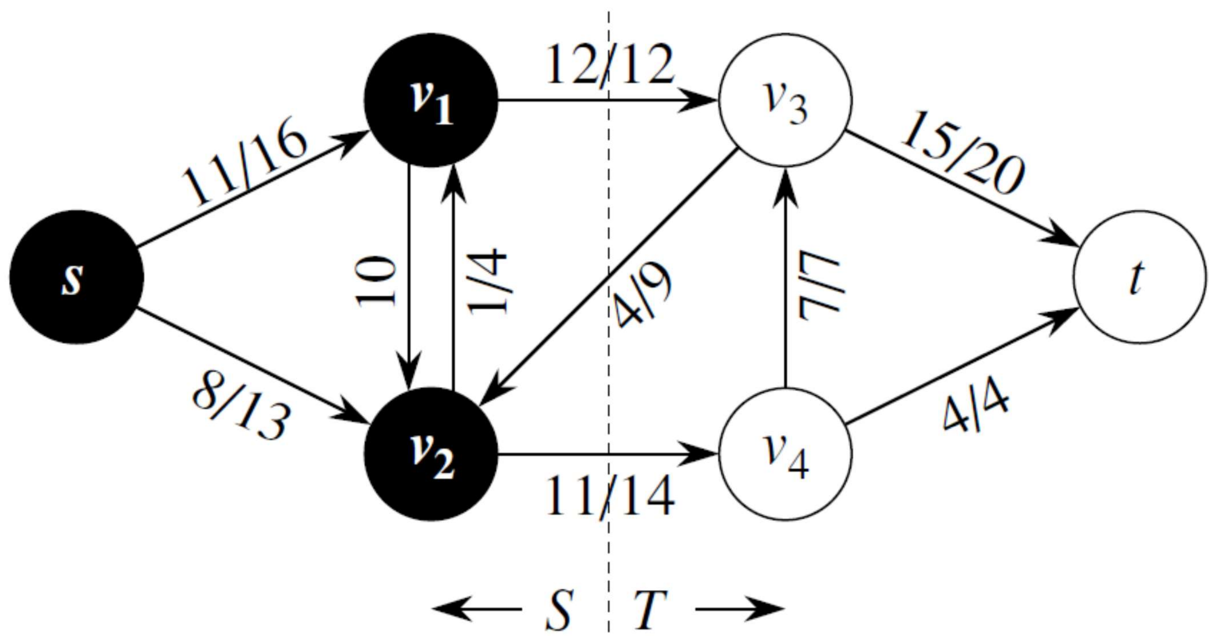
- A ***cut*** *(S, T)* of flow network $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$.

- If f is a flow, then the ***net flow*** across the cut *(S, T)* is defined to be

$$f(S, T).$$

- The ***capacity*** of the cut *(S, T)* is $c(S, T)$.
- A ***minimum cut*** of a network is a cut whose capacity is minimum over all cuts of the network.

A cut with
the net flow $f(S, T) = 19$, and
the capacity $c(S, T) = 26$.



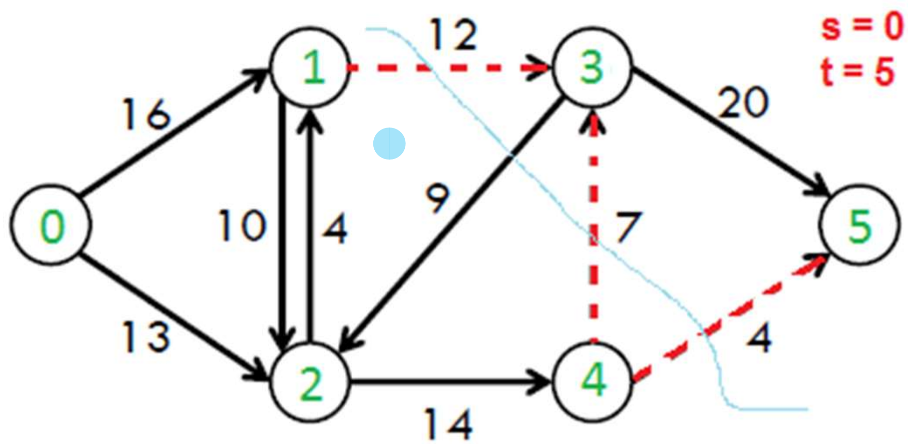
- The value of any flow *f* in a flow network *G* is bounded from above by the capacity of any cut of *G*.

(can proved by definition of flow and cut)

Max-flow min-cut theorem

If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.



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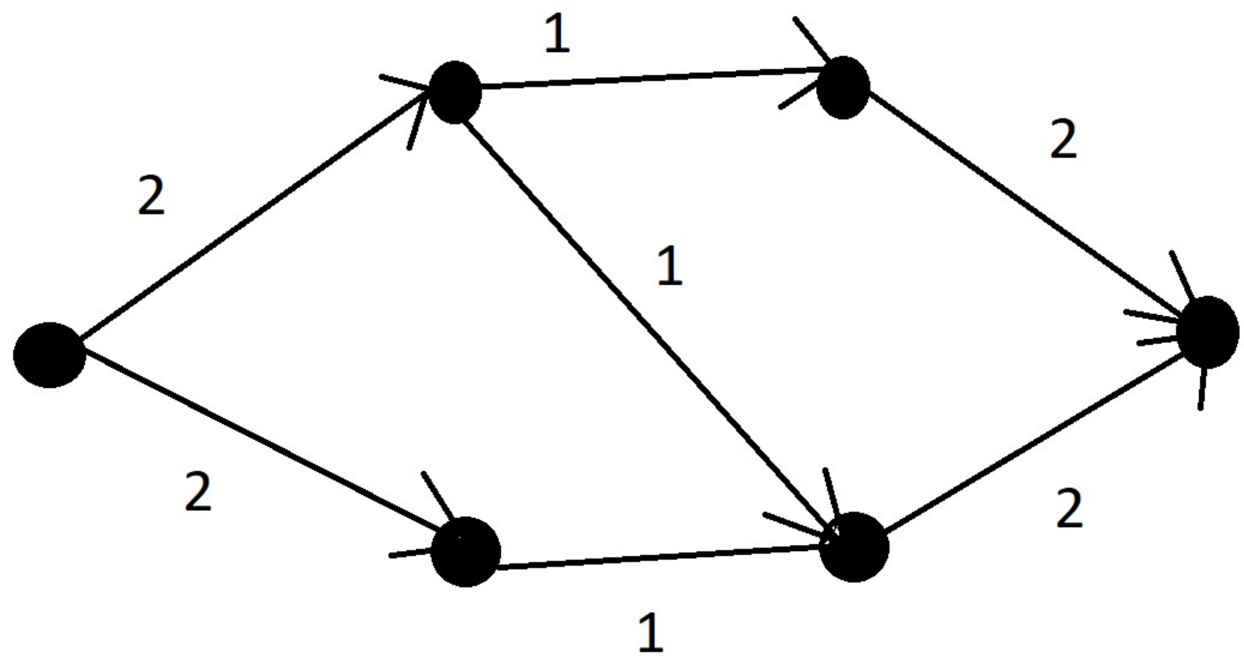
Corollary
The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.

basic Ford-Fulkerson algorithm

expands on the FORD-FULKERSON METHOD

FORD-FULKERSON(G, s, t)

for each edge $(u, v) \in E[G]$
    do $f[u, v] \leftarrow 0$
        $f[v, u] \leftarrow 0$
while there exists a path p from s to t in the residual network G_f
    do $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
        for each edge (u, v) in p
            do $f[u, v] \leftarrow f[u, v] + c_f(p)$
                $f[v, u] \leftarrow -f[u, v]$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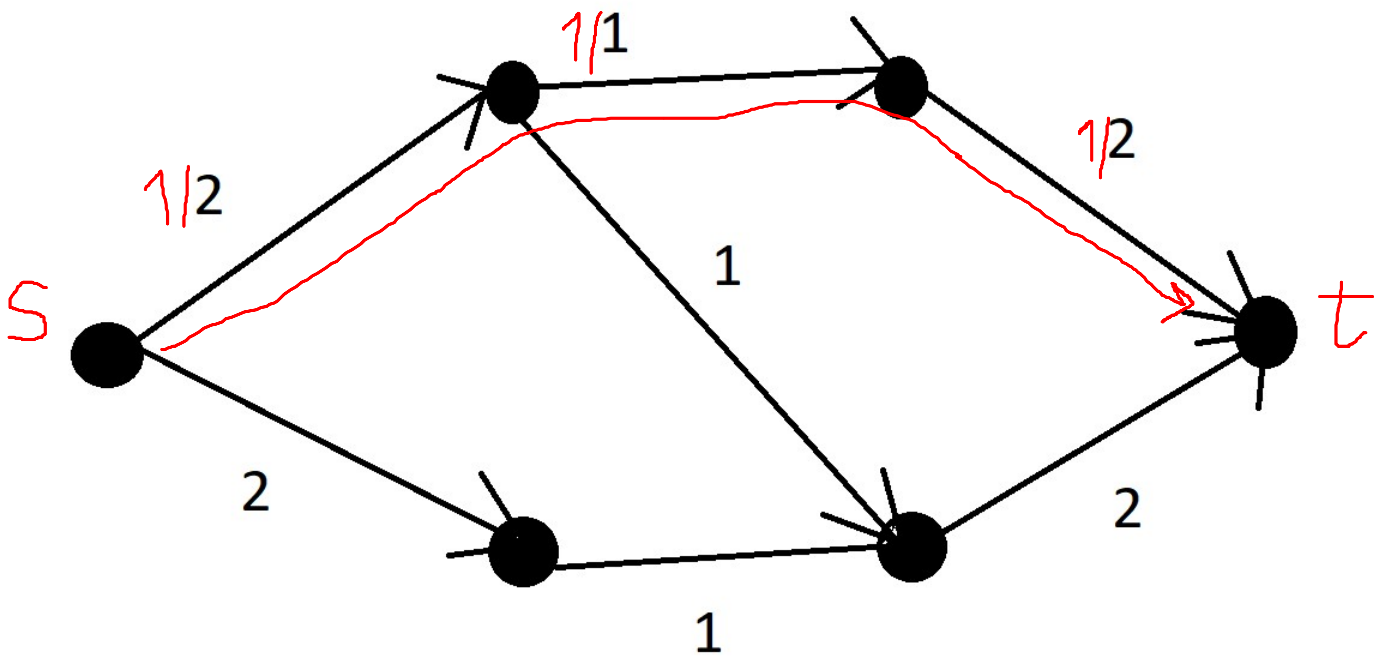
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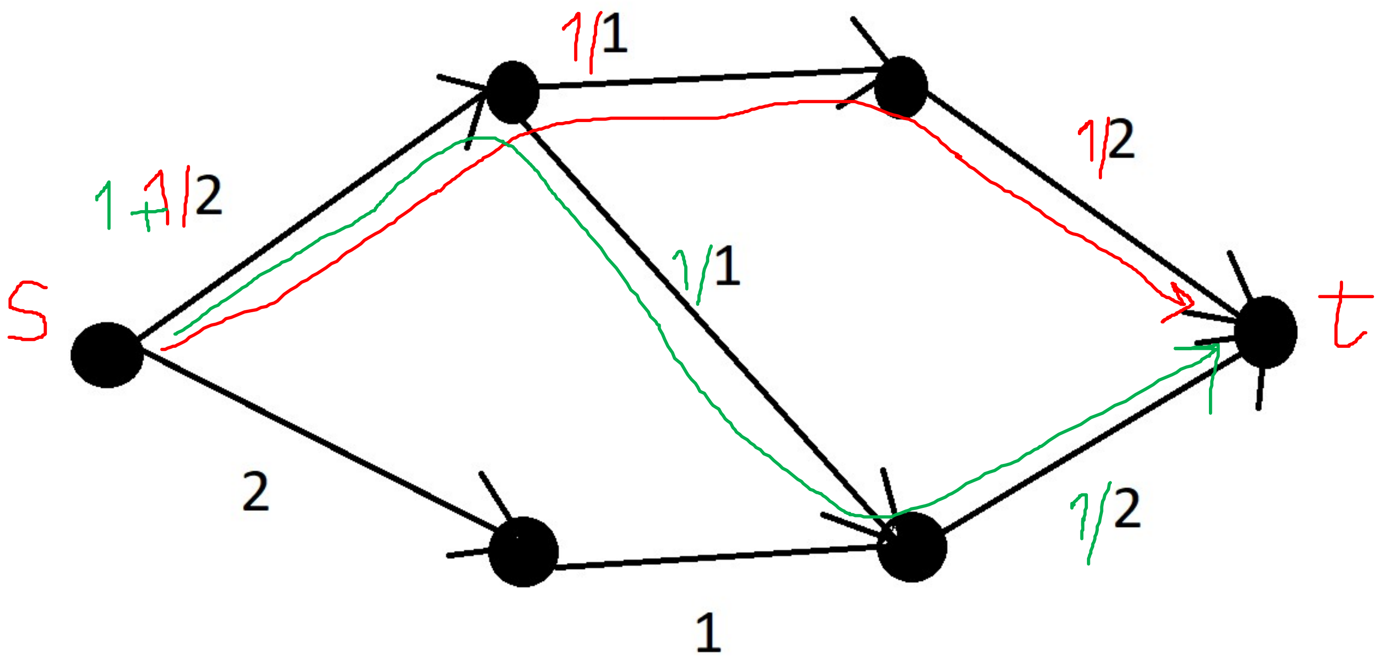

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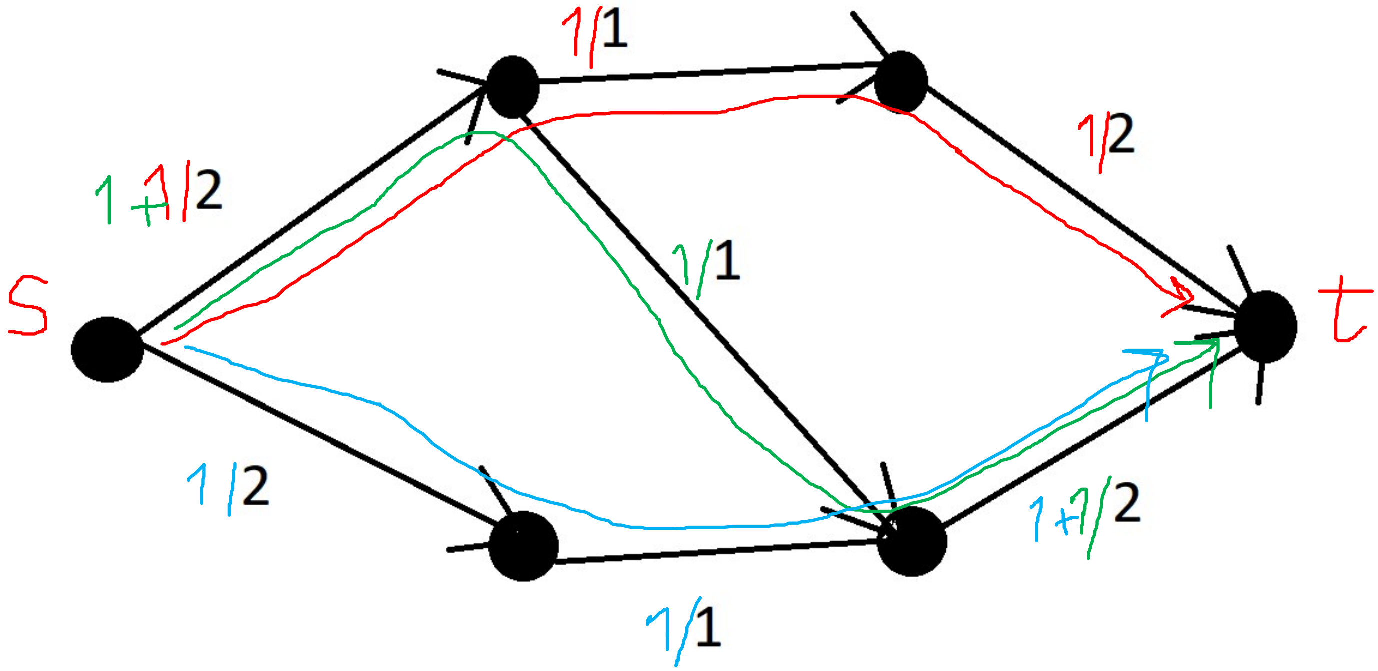
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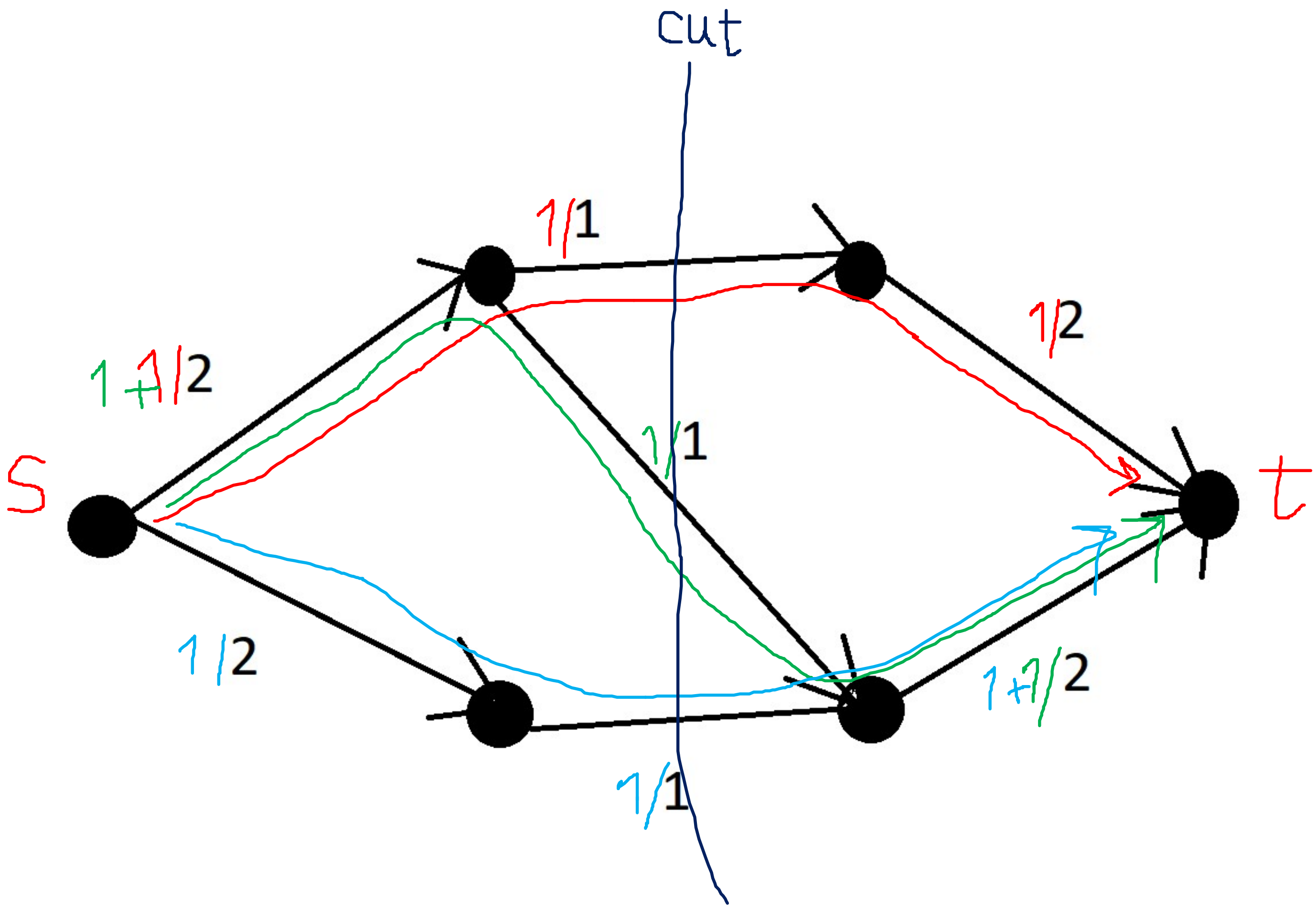
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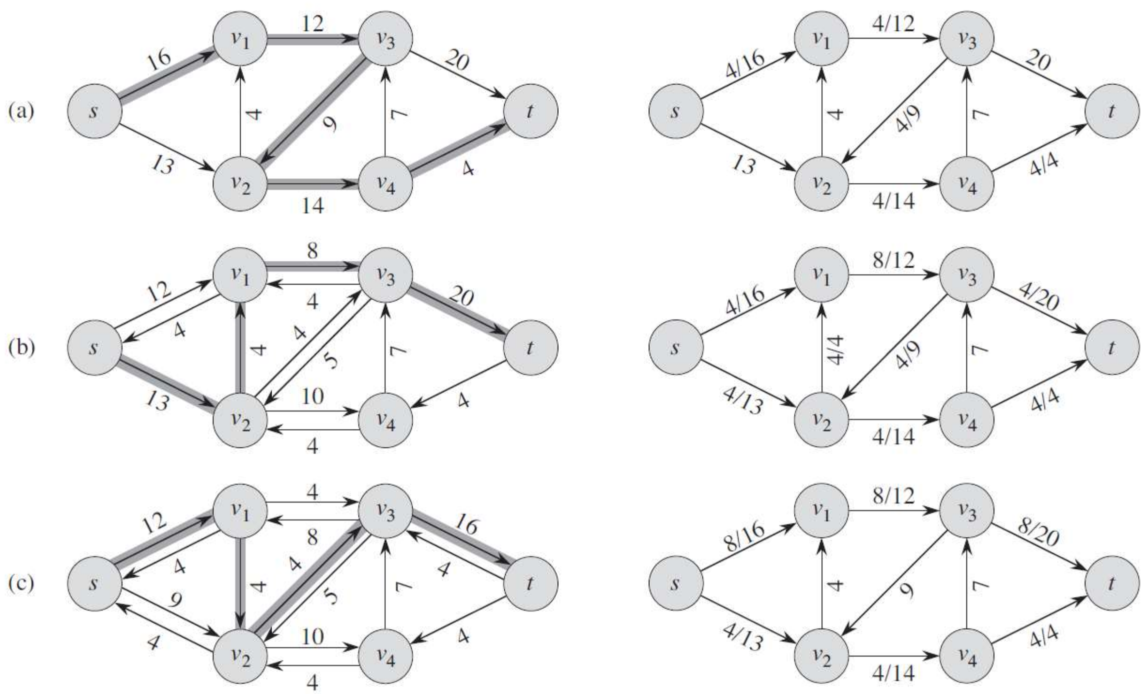
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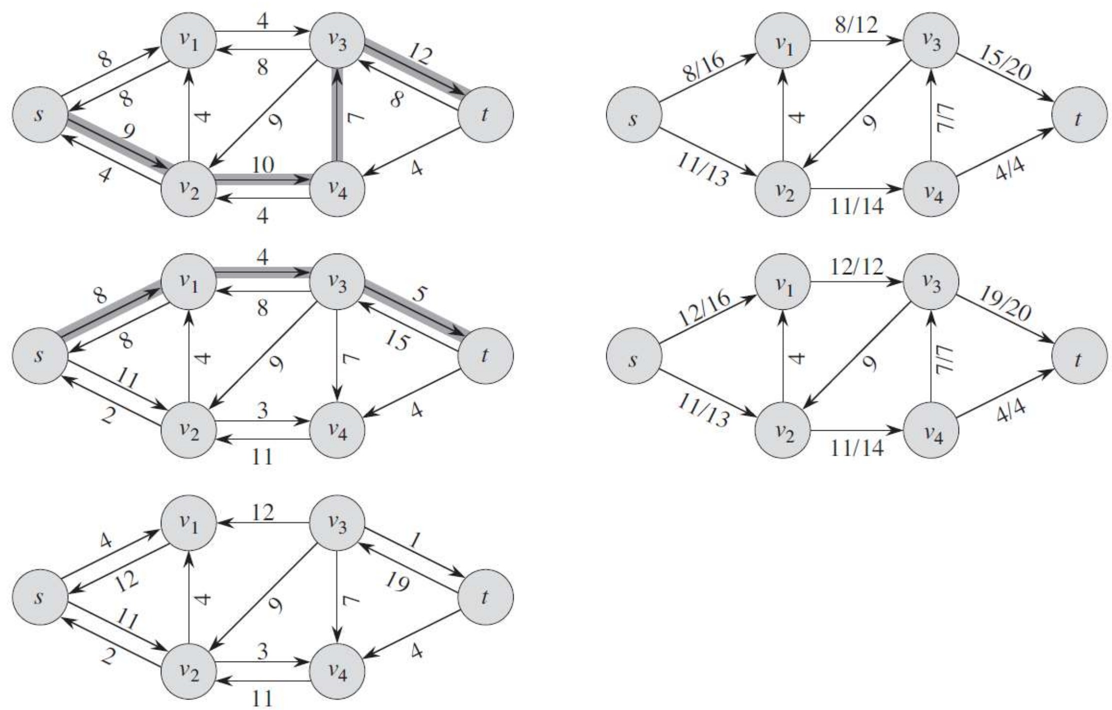
(d)

(e)

(f)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Time complexity

- The running time of FORD-FULKERSON depends on how the augmenting path *p* in line 4 is determined.

- where is the maximum flow found by the algorithm:

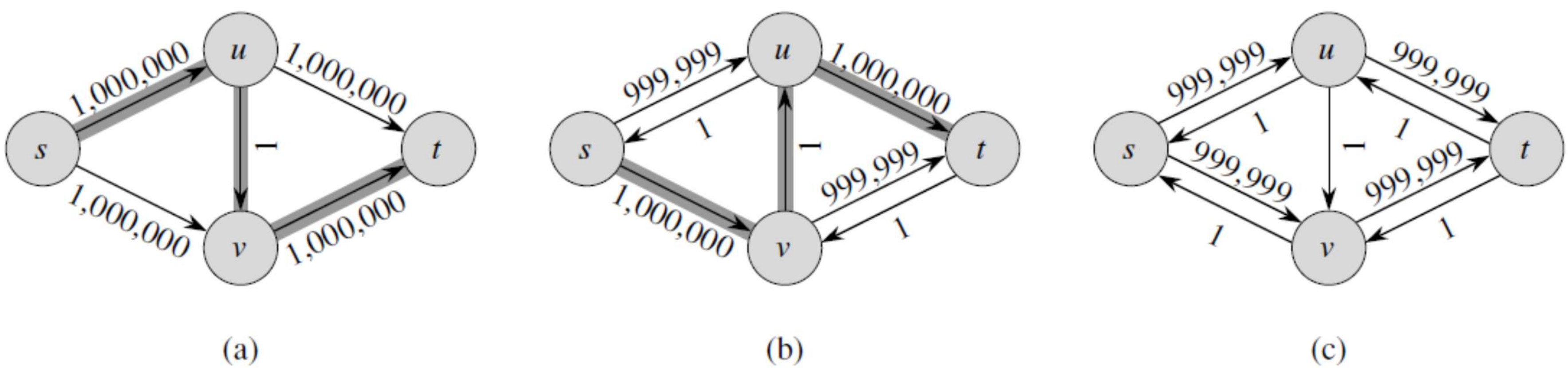
  a straightforward implementation runs in time $O(E\ |f^*|)$

- Prove Hint:

  the flow value increases by at least one unit in each iteration

$O(E\ |f^*|)$ can be bad!

- continue, choosing the augmenting path $s \rightarrow u \rightarrow v \rightarrow t$ in the odd-numbered iterations and the augmenting path $s \rightarrow v \rightarrow u \rightarrow t$ in the even-numbered iterations.



The Edmonds-Karp algorithm

- The bound on FORD-FULKERSON can be improved if we implement the computation of the augmenting path *p* in line 4 **with a breadth-first search** (each edge has unit distance (weight))

the augmenting path is a *shortest* path from *s* to *t* in the residual network

the running time of the Edmonds-Karp algorithm is $\boldsymbol{O(V\ E^2)}$

the running time of the Edmonds-Karp algorithm is $O(VE^2)$

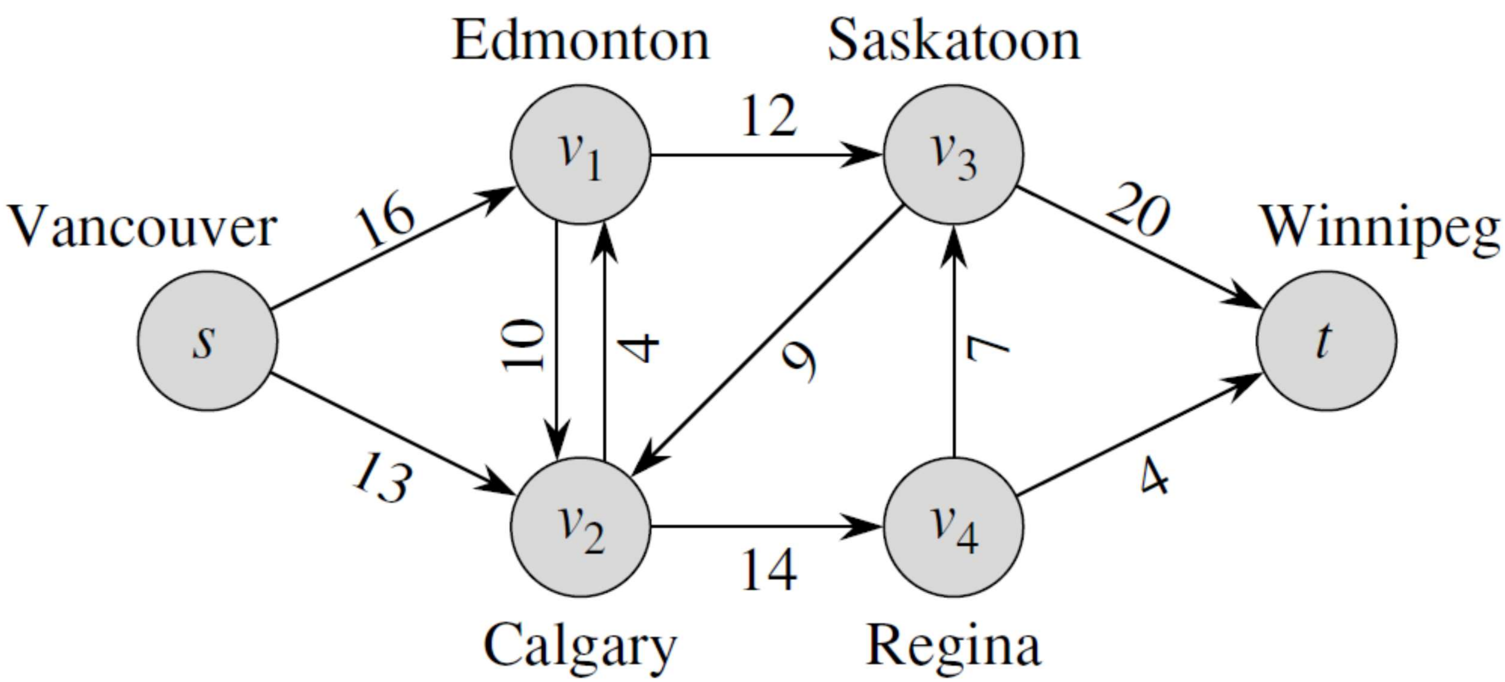
Lemma 26.8

If the Edmonds-Karp algorithm is run on a flow network $G = (V, E)$ with source s and sink t, then for all vertices $v \in V - \{s, t\}$, the shortest-path distance $\delta_f(s, v)$ in the residual network G_f increases monotonically with each flow augmentation.

Theorem 26.9

If the Edmonds-Karp algorithm is run on a flow network $G = (V, E)$ with source s and sink t, then the total number of flow augmentations performed by the algorithm is $O(VE)$.

Show the execution of the Edmonds-Karp algorithm on the flow network



Sample Problem

Suppose you are given a flow network G with ***integer*** edge capacities and an ***integer*** maximum flow f^* in G. Describe algorithms for the following operations:

(a) INCREMENT(e): Increase the capacity of edge e by 1 and update the maximum flow.

(b) DECREMENT(e): Decrease the capacity of edge e by 1 and update the maximum flow.

Both algorithms should modify f^* so that it is still a maximum flow, more quickly than recomputing a maximum flow from scratch.

Maximum bipartite matching

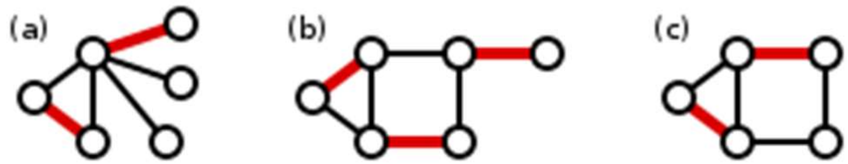
+solve using flow network

The maximum matching problem

Given an undirected graph $G = (V, E)$,

Matching is a subset of edges $M \subseteq E$ such that for all vertices $v \in V$, at most one edge of M is incident on v.

Maximum matching is a matching of maximum cardinality



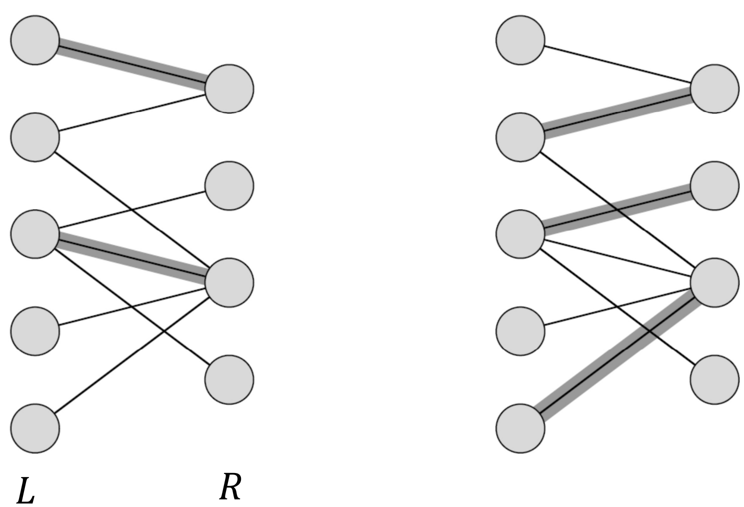
a vertex $v \in V$ is ***matched*** by matching M if some edge in M is incident on v;

otherwise, v is ***unmatched***.

The maximum-bipartite-matching problem

- **Bipartite graphs**: the vertex set can be partitioned into $V = L \cup R$, where L and R are disjoint and all edges in E go between L and R.

(further assume that every vertex in V has at least one incident edge.)



Finding a maximum bipartite matching

- We can **use the Ford-Fulkerson method** to find a maximum matching in an **undirected bipartite graph** $G = (V, E)$ in time **polynomial in $|V|$ and $|E|$**.

- First:

define the ***corresponding flow network*** $G' = (V', E')$ for the bipartite graph G

corresponding flow network

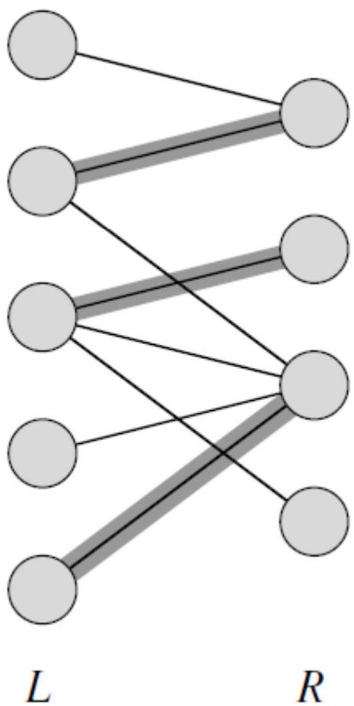
- Add source *s* and sink *t* as new vertices
  - let $V' = V \cup \{s, t\}$.

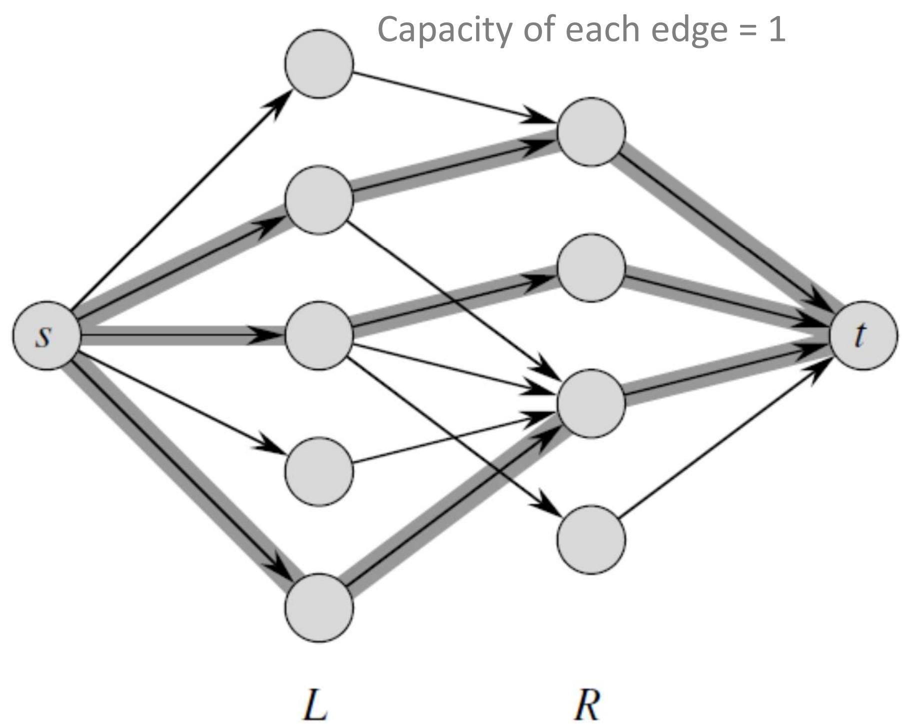
- The directed edges of G' are :
  - the edges of *E*, directed from L to R ($V = L \cup R$), along with *V* new edges:

$$
\begin{aligned}
E' \quad = \quad & \{(s, u) : u \in L\} \\
& \cup \{(u, v) : u \in L,\ v \in R,\ \text{and } (u, v) \in E\} \\
& \cup \{(v, t) : v \in R\}\ .
\end{aligned}
$$

  - assign unit capacity to each edge in E'.

The flow network corresponding to a bipartite graph





The order of E' ?

$$|E| \leq |E'|$$

$$|E'| = |E| + |V| \leq 3\,|E|, \; (|E| \geq |V|/2)$$

$$\boldsymbol{so}\ |\boldsymbol{E'}| = \boldsymbol{\Theta(E)}$$

a **matching** in *G* corresponds directly to
a **flow** in *G*'s corresponding flow network

Lemma 26.10

Let $G = (V, E)$ be a bipartite graph with vertex partition $V = L \cup R$, and let $G' = (V', E')$ be its corresponding flow network. If M is a matching in G, then there is an integer-valued flow f in G' with value $|f| = |M|$. Conversely, if f is an integer-valued flow in G', then there is a matching M in G with cardinality $|M| = |f|$.

maximum matching in a bipartite graph *and* the value of a maximum flow

Theorem 26.11 (Integrality theorem)
If the capacity function c takes on only integral values, then the maximum flow f produced by the Ford-Fulkerson method has the property that $|f|$ is integer-valued. Moreover, for all vertices u and v, the value of $f(u, v)$ is an integer.

Corollary 26.12
The cardinality of a maximum matching M in a bipartite graph G is the value of a maximum flow f in its corresponding flow network G'.

Whole picture

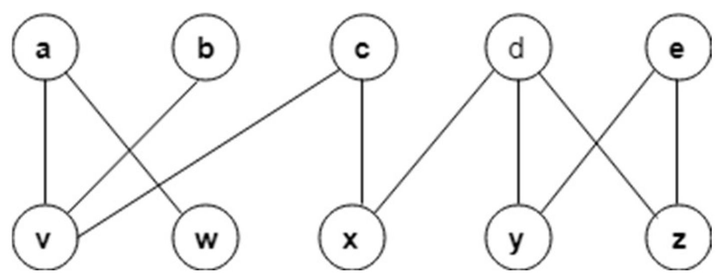
- Given a bipartite undirected graph G,
- Create the flow network G'
- Run the Ford-Fulkerson method
- Obtaining a maximum matching *M* from the integer-valued maximum flow

Ford-Fulkrson time complexity $= \boldsymbol{O}(\boldsymbol{E}\ |\boldsymbol{f}^*|)$

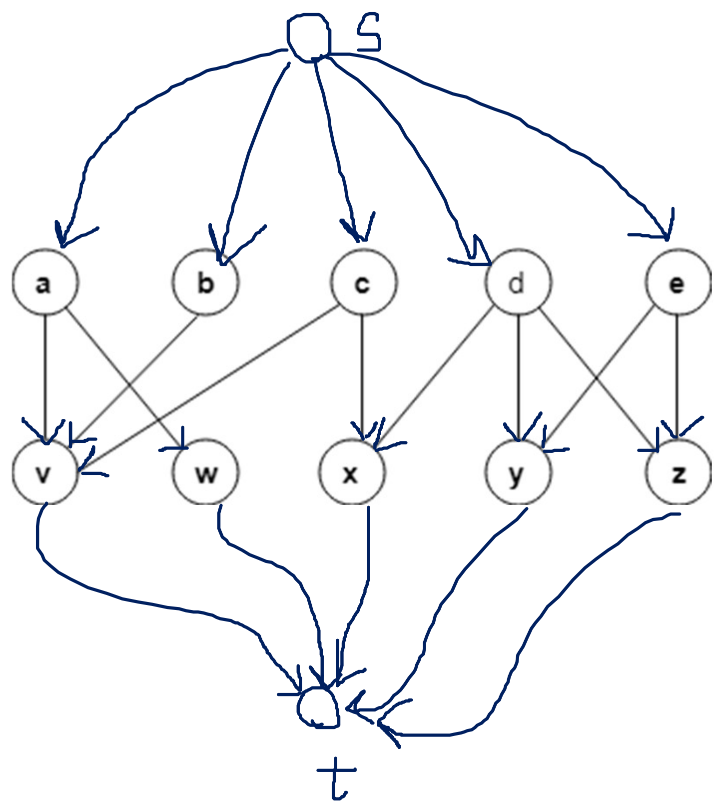
- The value of the maximum flow in G' is $O(V)$.
  - Since any matching in a bipartite graph has cardinality at most $\min(L, R) = O(V)$,

Time complexity of finding a maximum matching in a bipartite graph ($|E'| = \Theta(E)$) is $\boldsymbol{O}(\boldsymbol{V}\ \boldsymbol{E})$.

Compute the maximum bipartite matching



Compute the maximum bipartite matching



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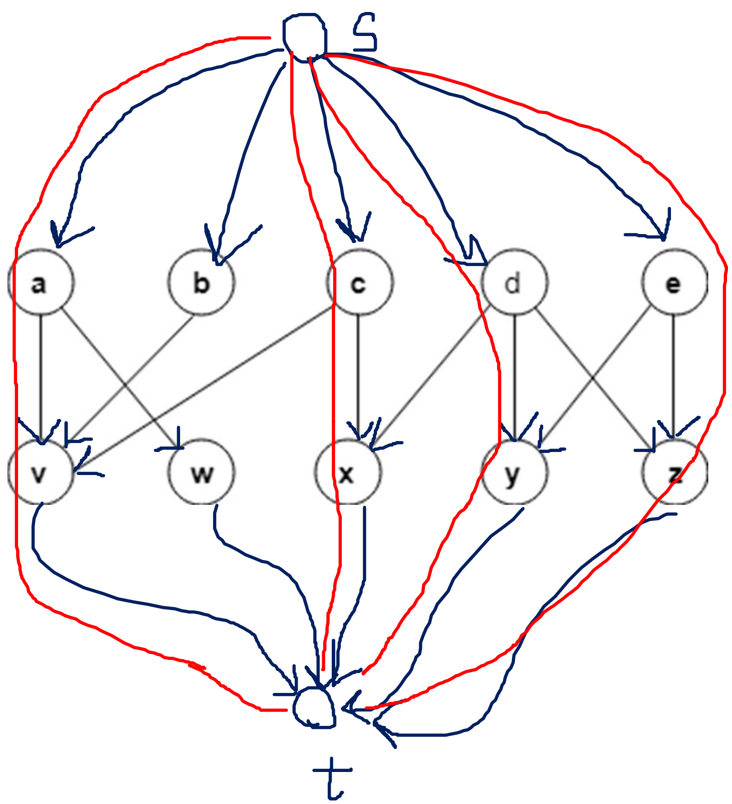

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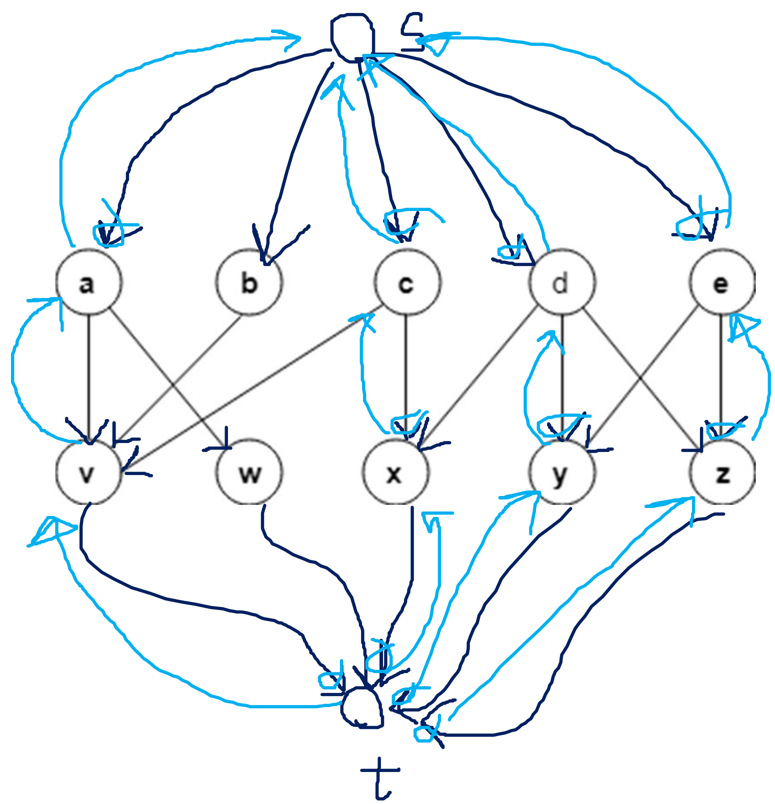

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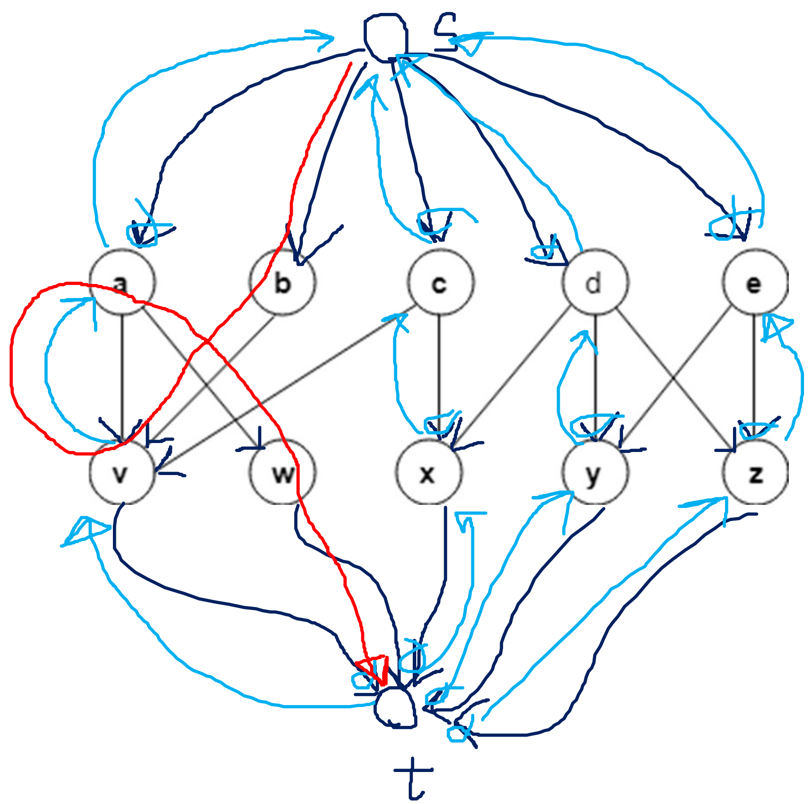
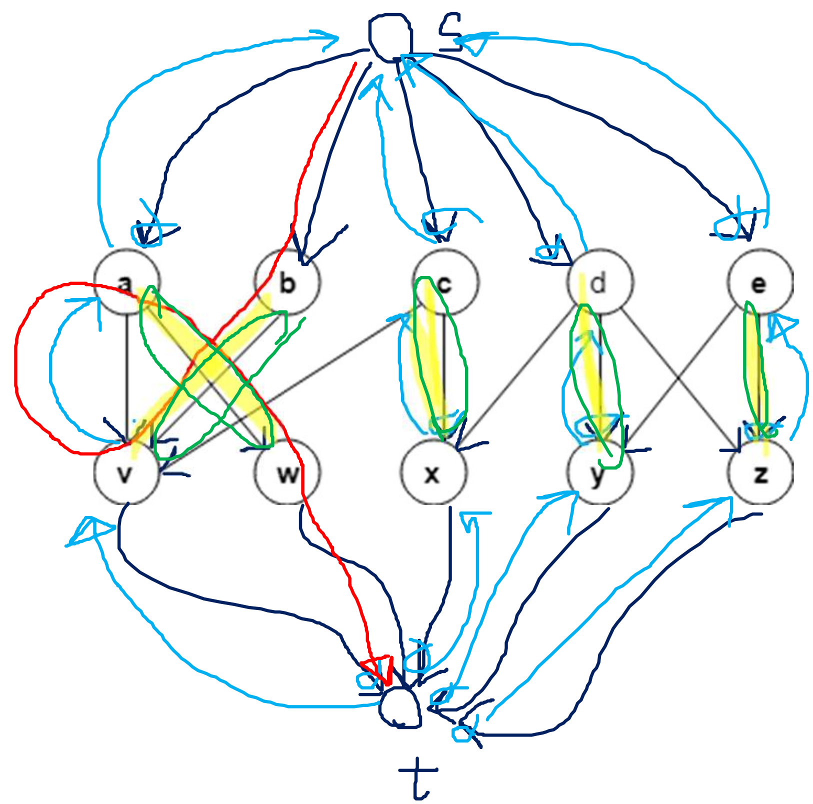


Compute the maximum bipartite matching



Some sample problems

1- If all edges in a graph have distinct capacities, is there a unique maximum flow?

2- Let $G = (V, E)$ be a bipartite graph with vertex partition $V = L \cup R$, and let G' be its corresponding flow network. Give a good upper bound on the length of any augmenting path found in G' during the execution of FORD-FULKERSON.

3- A ***perfect matching*** is a matching in which every vertex is matched.

Let $G = (V, E)$ be an undirected bipartite graph with vertex partition $V = L \cup R$, where $|L| = |R|$. For any $X \subseteq V$, define the ***neighborhood*** of X as $N(X) = \{y \in V : (x, y) \in E \text{ for some } x \in X\}$, that is, the set of vertices adjacent to some member of X.

Prove ***Hall's theorem***: there exists a perfect matching in G if and only if $|A| \leq |N(A)|$ for every subset $A \subseteq L$.