Notes for Paper 3

Title:Convex relaxation of optimal power flow—Part I: Formulations and equivalence

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0. Summary

1. Power flow models

1.1 Bus injection model

$$I_{jk} = y_{jk}(V_j - V_k)$$
$$s_j = \sum_{k:j\sim k} y_{jk}^H V_j (V_j^H - V_k^H), j \in N^+$$

where superscript H refers the conjugate transpose, s_j is the power flow on node j, $j \sim k$ refers that node j is connected with node k.
Bus 0 is the slack bus, which voltage is fixed and we assume that $V_0 = 1\angle 0^\circ$; s_j is the net complex power injection at bus $j \in N^+$.
So, the solution for the power flow model $V \in C^{n+1}$, where C is the complex numbers.

Bus type

1. slack bus. V_0 is given, s_0 is variable.
2. generator bus. $Re(s_j) = p_j$ and $|V_j|$ are known, $Im(s_j) = q_j$ and $\angle V_j$ are unknown.
3. load bus. s_j is specified and V_j is variable.

Each bus is characterized by two complex variables V_j and s_j (or four real variables). As described above, two variables will be given at each bus (slack, generator, load), then we can solve the $n+1$ complex equations, or $2(n+1)$ real number equations, to get the remaining $2(n+1)$ variables.

1.2 Branch Flow Model

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j}(S_{ij} - z_{ij}|I_{ij}|^2) + s_j, j \in N^+$$
$$I_{ij} = y_{jk}(V_j - V_k), j \to k \in \widetilde{E}$$
$$S_{jk} = V_j I_{jk}^H, j \to k \in \widetilde{E}$$

The solution $\widetilde{x} := (S, I, V) \in C^{2m+n+1}$, where m is the number of directed edges, s_j is the net complex power injection at bus j. The total equation number is $(n+1)+m+m = 2m+n+1$, so the equation group is closed.

2. convert OPF into QCQP

Let I_j be the net injection current from bus j to the rest of the network:

$$I_j = \sum_{k:j\sim k} y_{ik}(V_k - V_j)$$

Then we can construct a sysmmetric matrice to let $I = YV$:

$$Y_{ij} = \begin{cases} \sum_{k:k\sim i} y_{ik}, & if \quad i = j \\ -y_{ij} & if \quad i \neq j \quad and \quad i \sim j \\ 0, & otherwise. \end{cases}$$

so BIM is equivalent to:

$$s_j = V_j I_j^H = (e_j^H V)(I^H e_j)$$

where e_j is the $(n+1)$ dimensional vector with 1 in the jth entry and 0 elsewhere. Because sj is scalar variable, we have

$$s_j = tr(s_j) = tr(e_j^H V V^H Y^H e_j)$$

Because the shape of $e_j^H V V^H$ is the same with that of $(Y^H e_j)^T$, we have

$$s_j = tr(e_j^H V V^H Y^H e_j) = tr(Y^H e_j e_j^H V V^H)$$

Then we have (**why?**)

$$s_j = tr(Y^H e_j e_j^H V V^H) = tr(Y^H e_j e_j^H) V V^H = V^H Y_j^H V$$

where $Y_j := e_j e_j^H Y$

Then

$$Re(s_j) = 1/2 V^H (Y_j^H + Y_j) V$$
$$Im(s_j) = 1/(2i) V^H (Y_j^H - Y_j) V$$

$$\text{Re } s_j = V^H \Phi_j V \quad \text{and} \quad \text{Im } s_j = V^H \Psi_j V.$$

Let their upper and lower bounds be denoted by

$$\underline{p}_j := \text{Re } \underline{s}_j \quad \text{and} \quad \overline{p}_j := \text{Re } \overline{s}_j$$
$$\underline{q}_j := \text{Re } \underline{s}_j \quad \text{and} \quad \overline{q}_j := \text{Re } \overline{s}_j.$$

Let $J_j := e_j e_j^H$ denote the Hermitian matrix with a single 1 in the (j, j)th entry and 0 everywhere else, then OPF (7) can be written as a standard form QCQP

$$\min_{V \in \mathbb{C}^{n+1}} \; V^H C V \tag{10a}$$
$$\text{s.t.} \quad V^H \Phi_j V \le \overline{p}_j, \quad V^H(-\Phi_j)V \le -\underline{p}_j \tag{10b}$$
$$V^H \Psi_j V \le \overline{q}_j, \quad V^H(-\Psi_j)V \le -\underline{q}_j \tag{10c}$$
$$V^H J_j V \le \overline{v}_j, \quad V^H(-J_j)V \le -\underline{v}_j \tag{10d}$$

where $j \in N^+$ in (10).