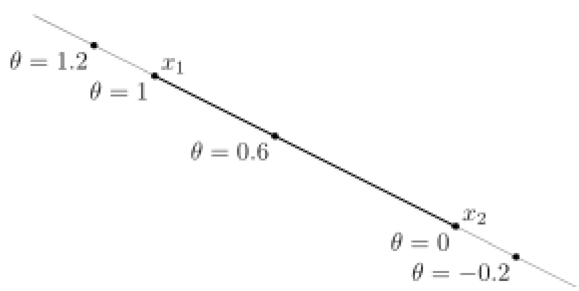
Notes for "Convex Optimization"

Chapter 2: Convex Set

Affine set

Defination:
if $C \in R^n$ is affine set

$$\theta x_1 + (1-\theta)x_2 \in C$$
$$x_1, x_2 \in C, \quad \theta \in R$$



Affine hull

Defination:
The set of all affine combinations of points in some set $C \in R^n$.

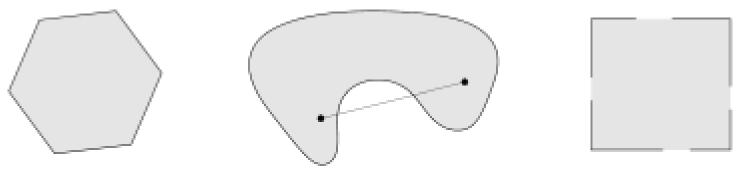
$$\textbf{aff} \quad C = \{\theta_1 x_1 + ... + \theta_k x_k | x_1, ... x_k \in C, \theta_1 + ... \theta_k = 1\}$$

Convex set

Defination:
if $C \in R^n$ is convex set

$$\theta x_1 + (1-\theta)x_2 \in C$$
$$x_1, x_2 \in C, \quad 0 \le \theta \le 1$$

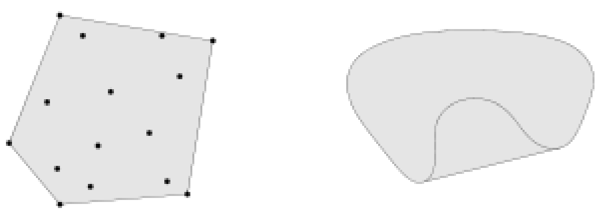
The idea of a convex combination can be generalized to include infinite sums.



Convex hull

Defination:
The convex hull of a set C is the set of all convex combinations pf points in C.

$$\textbf{conv} \quad C = \{\theta_1 x_1 + ... + \theta_k x_k | x_i \in C, \theta_i \geq 0, i = 1, ...k, \theta_1 + ...\theta_k = 1\}$$



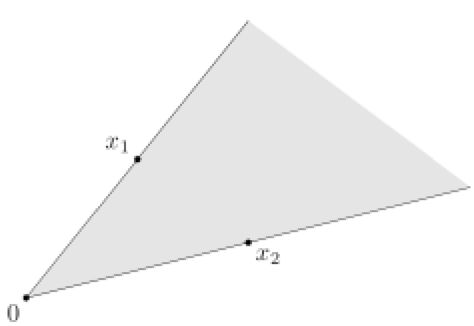
Cones

Defination:
For every $x \in C$ and $\theta \geq 0$, we have $\theta x \in C$.

Convex Cones

Defination:
For any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$, we have

$$\theta_1 x_1 + \theta_2 x_2 \in C$$

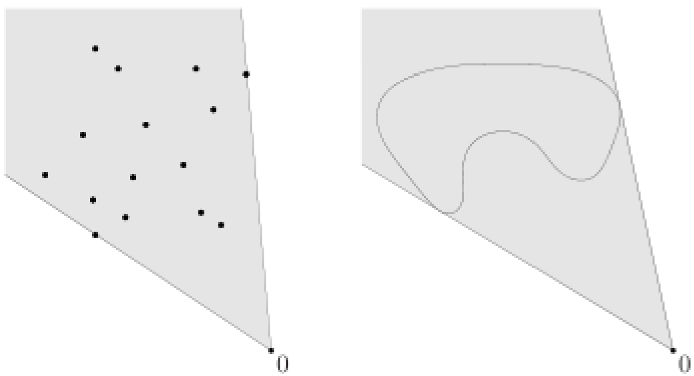


Conic hull

Defination:

The conic hull of a set C is the set of all conic combinations of points in C.

$$\{\theta_1 x_1 + ... + \theta_k x_k | x_i \in C, \theta_i \geq 0, i = 1, ...k,\}$$



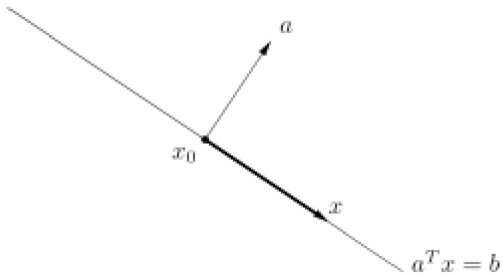
Hyperplanes

Defination:

$$\{x | a^T x = b, a \in R^n, a \neq 0\}$$

This formulation can be expressed as:

$$\{x | a^T (x - x_0) = 0, a \in R^n, a \neq 0\}$$



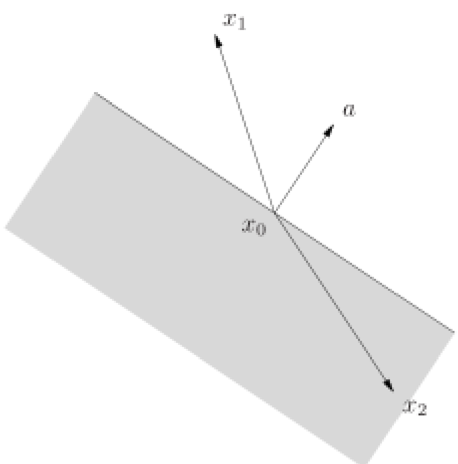
Halfspace

Defination:

$$\{x|a^T x \leq b, a \in R^n, a \neq 0\}$$

which can also be expressed as:

$$\{x|a^T(x-x_0) \leq 0, a \in R^n, a \neq 0\}$$



Euclidean balls and ellipsoids

Defination:

A ball in R^n has the form

$$B(x_c, r) = \{x| \parallel x - x_c \parallel_2 \leq r\} = \{x|(x-x_c)^T(x-x_c)\ leqr^2\}$$

which can be expressed as:

$$B(x_c, r) = \{x_c + ru| \parallel u \parallel_2 \leq 1\}$$

A Euclidean ball is a convex set.

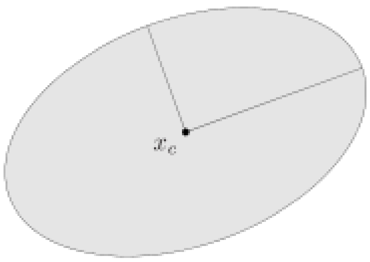
Ellipsoids is a convex set, which have the form

$$\varepsilon = \{x|(x-x_c)^T P(x-x_c) \leq 1\}$$

where $P = P^T \succ 0$. It can be also expressed as

$$\varepsilon = \{x_c + Au| \parallel u \parallel_2 \leq 1\}$$

A is symmetric and positive definite. $A = P^{1/2}$



Polyhedra

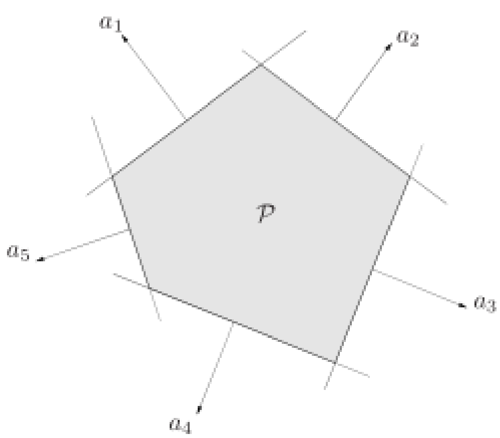
Defination:
A polyhedra is the solution set of a finite number of linear equalities and ineuqalities.

$$\mathbf{P} = \{x | a_j^T x \leq b_j, j = 1, ...m, c_j^T x = d_j, j = 1, ...p\}$$

Also can be expressed as

$$\mathbf{P} = \{x | Ax \preceq b, Cx = d\}$$

where A and C are matrix.
Polyhedra is a convex set.



Simplexes:
Suppose the points $v_0, ...v_k \in R^n$ are affinely independent ($v1_v0, ...v_k - v_0$ are linear independent)

$$C = \mathbf{conv}\{v_0, ..., v_k\} = \{\theta_1 x_1 + ... + \theta_k x_k | \theta \succeq 0, 1^T \theta = 0\}$$

Convex hull description of polyhedra:

$$\mathbf{conv}\{v_0, ..., v_k\} = \{\theta_1 x_1 + ... + \theta_k x_k | \theta \succeq 0, 1^T\theta = 0\}$$

The difference between ployhedra and simplexes is that ployhedra doesn't require the affinely independent on the points.
The difference between ployhedra and convex hull is that convex hull requires the first m coefficients to sum to one.

$$\mathbf{conv}\{v_0, ..., v_k\} = \{\theta_1 x_1 + ... + \theta_k x_k | \theta \succeq 0, \theta_1 + ... + \theta_m = 1\}$$

where $m \leq k$.

Positive semidefinite Cone

$\mathbf{S}^n$ is used to denote the set of symmetric $n \times n$ matrices.
The positive semidefinite matrices:

$$\mathbf{S}^n_+ = \{X \in \mathbf{S}^n | X \succ 0\}$$

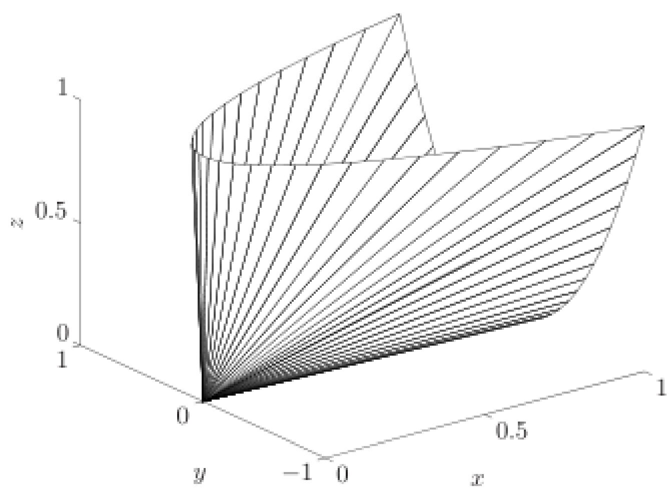


Figure 2.12 Boundary of positive semidefinite cone in $\mathbf{S}^2$.

Operations that preserve convexity

Intersection

If S_1 and S_2 are convex, then $S_1 \cup S_2$ is convex.

Linear-fractional and perspective function

A *linear-fractional function* is formed by composing the perspective function with an affine function. Suppose $g : \mathbf{R}^n \to \mathbf{R}^{m+1}$ is affine, *i.e.*,

$$g(x) = \begin{bmatrix} A \\ c^T \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}, \tag{2.12}$$

where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$, and $d \in \mathbf{R}$. The function $f : \mathbf{R}^n \to \mathbf{R}^m$ given by $f = P \circ g$, *i.e.*,

$$f(x) = (Ax + b)/(c^T x + d), \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}, \tag{2.13}$$

Generalized inequalities

Proper cones

A cone $K \subseteq R^n$ is called a proper cone if:

1. K is convex.
2. K is closed.
3. K has nonempty interior
4. K is point (it contains no line)

Generalized inequality defination

$$x \preceq_K y \Leftrightarrow y - x \in K$$

When the symbol appears between vectors, it means:

$$x \preceq_K y$$
$$x_i < y_i, i = 1, ..., n$$

When the symbol appears between matrices, it means:
$X \preceq_K Y$ means $Y - X$ is positive semidefinite.

Minimum elements

A point $x \in S$ is the minimum element of S if and only if

$$S \subseteq x + K$$

where $x + K$ denotes all the points that are comparable to x and greater than or equal to x ($x \preceq_K y$).

Minimal elements

A point $x \in S$ is the minimal element of S if and only if

$$(x - K) \cup S = \{x\}$$

where $x - K$ denotes all the points that are comparable to x and less than or equal to x ($y \preceq_K x$).

The difference between minimum and minimal elements

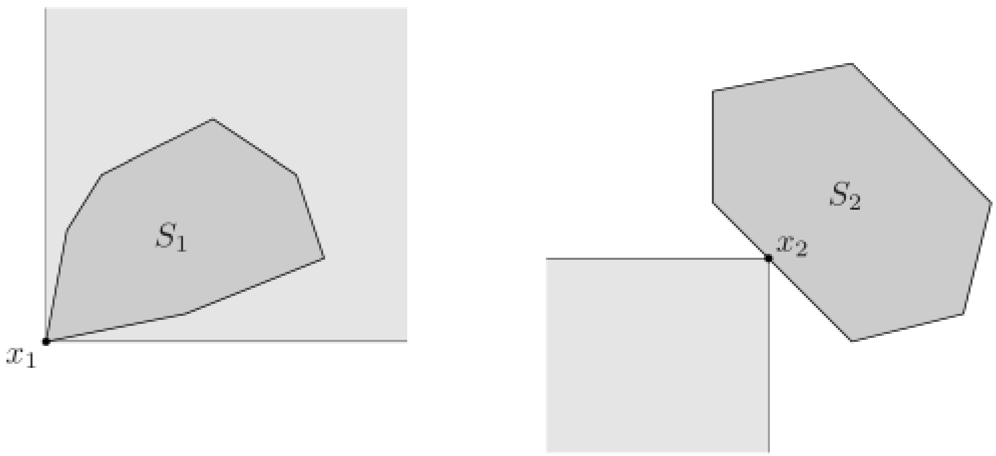
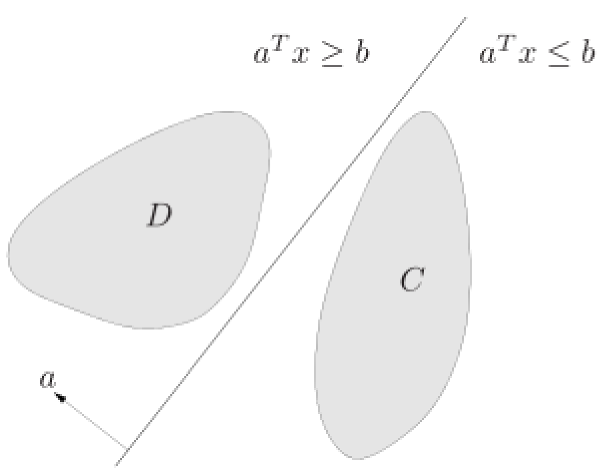


Figure 2.17 *Left.* The set S_1 has a minimum element x_1 with respect to componentwise inequality in $\mathbf{R}^2$. The set $x_1 + K$ is shaded lightly; x_1 is the minimum element of S_1 since $S_1 \subseteq x_1 + K$. *Right.* The point x_2 is a minimal point of S_2. The set $x_2 - K$ is shown lightly shaded. The point x_2 is minimal because $x_2 - K$ and S_2 intersect only at x_2.

Separating hyperplane theorem

Suppose two convex sets C and D is disjoint, i.e., $C \cap D = \emptyset$. Then there exist $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$. The hyperplane $\{x | a^T x = b\}$ is called a separating hyperplane for C and D.



Supporting hyperplanes

If $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for $x \in C$, then the hyperplane $\{x|a^T x = a^T x_0\}$ is called a supporting hyperplane to C at point x_0.

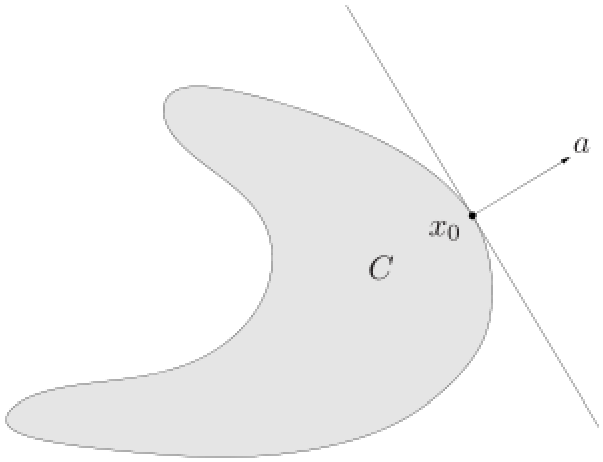


Figure 2.21 The hyperplane $\{x \mid a^T x = a^T x_0\}$ supports C at x_0.

Dual cones

K is a cone. The set

$$K^* = \{y|x^T y \geq 0 for all x \in K\}$$

is called the dual cone of K. K^* **is always convex even when the cone K is not.**

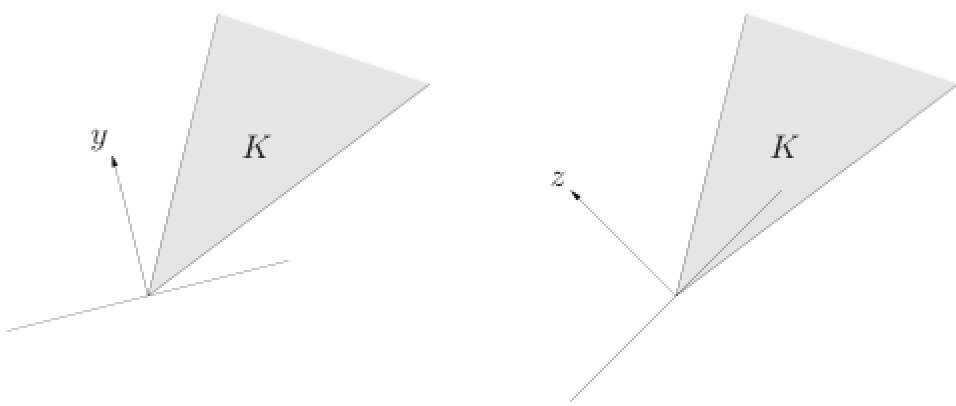


Figure 2.22 *Left.* The halfspace with inward normal y contains the cone K, so $y \in K^*$. *Right.* The halfspace with inward normal z does not contain K, so $z \notin K^*$.

How to construct Geometries?

line:$\{x+tv|x,v\in R^n,t\in R\}$
intersetion between line and set: substitude the line formulation as variable into the set.
affine set: $D=\{Fu+g|F\in R^{n\times m},u\in R^m\}$
Polyhedra: $\{x|Ax\preceq h\}$ or $\mathbf{conv}\{x_1,...x_k|x_i\in R^n\}$