Supplemental Material for 2-designs and Redundant Syndrome Extraction for Quantum Error Correction

Vickram N. Premakumar,[1] Hele Sha,[1] Daniel Crow,[1] Eric Bach,[2] and Robert Joynt[1,3]

[1] *Physics Department, University of Wisconsin-Madison, 1150 Univ. Ave., Madison, WI, USA*

[2] *Computer Science Department, University of Wisconsin-Madison, Madison, WI, USA*

[3] *Kavli Institute for Theoretical Sciences, University of Chinese Academy of Sciences, Beijing 100190, China*

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I. CONSTRAINTS ON BIBD PARAMETERS FOR QEC

A. Proof of constraint 1

We must show that all the measured stabilizers S commute with all the logical operators L if and only if w is even. An example of such a commutator is

$$[S,L] = [Z_{i_1} Z_{i_2} ... Z_{i_w}, X_1 X_2 ... X_n],$$

where $\{i_1, i_2, ...i_w\} \subset \{1, 2, ..., n\}$. We then define the set of indices $\{k_1, k_2 ... k_{(n-w)}\}$ such that $\{i_1, i_2, ...i_w\} \cup \{j_1, j_2 ... j_{(n-w)}\} = \{1, 2 ... n\}$ This can be simplified as follows:

$$\begin{aligned}
[S,L] &= [Z_{i_1} Z_{i_2} ... Z_{i_w}, X_1 X_2 ... X_n] \\
&= X_{k_1} X_{k_2} ... X_{k_{(n-w)}} [Z_{i_1} Z_{i_2} ... Z_{i_w}, X_{i_1} X_{i_2} ... X_{i_w}] \\
&= X_{k_1} X_{k_2} ... X_{k_{(n-w)}} \times \\
&\quad (Z_{i_1} Z_{i_2} ... Z_{i_w} X_{i_1} X_{i_2} ... X_{i_w} - X_{i_1} X_{i_2} ... X_{i_w} Z_{i_1} Z_{i_2} ... Z_{i_w}) \\
&= X_{k_1} X_{k_2} ... X_{k_{(n-w)}} \times \\
&\quad (Z_{i_1} X_{i_1} Z_{i_2} X_{i_2} ... Z_{i_w} X_{i_w} - X_{i_1} Z_{i_1} X_{i_2} Z_{i_2} ... X_{i_w} Z_{i_w}) \\
&= X_{k_1} X_{k_2} ... X_{k_{(n-w)}} \times \\
&\quad ((-1)^w X_{i_1} Z_{i_1} X_{i_2} Z_{i_2} ... X_{i_w} Z_{i_w} - X_{i_1} Z_{i_1} X_{i_2} Z_{i_2} ... X_{i_w} Z_{i_w}) \\
&= X_{k_1} X_{k_2} ... X_{k_{(n-w)}} [(-1)^w - 1] X_{i_1} Z_{i_1} X_{i_2} Z_{i_2} ... X_{i_w} Z_{i_w},
\end{aligned}$$

which is zero if and only if w is even. This proof clearly holds for all stabilizer/logical operator pairs.

B. Proof of constraint 2

For the CSS-DBR codes, the same design is used twice, once for the X-stabilizers and once for the Z-stabilizers. The full group must be abelian, so one must check that the X-stabilizers commute with the Z-stabilizers. The relevant commutators have the form

$$C = [X_{i_1} X_{i_2} ... X_{i_w}, Z_{j_1} Z_{j_2} ... Z_{j_w}]$$

Here $\{i_1, i_2, ...i_w\} \subset \{1, 2, ..., n\}$. and $\{j_1, j_2, ...j_w\} \subset \{1, 2, ..., n\}$. Inspection of the proof for constraint 1 then indicates that if we define

$$x = |\{i_1, i_2, ...i_w\} \cap \{j_1, j_2, ...j_w\}|,$$

then $C = 0$ if and only if x is even. This is true for all choices of the index sets if λ is even.

II. FAILURE RATES

In the main text we showed the failure rates of traditional QEC compared with two proposed extensions by plotting the difference as a function of both physical and measurement error rates. To emphasize the region of interest for near-threshold quantum computers, we restricted p_q, p_m to a small range close to zero. For completeness, we include the full comparison over the entire feasible parameter range. When the curve where $F_{QEC} - F_{MR/DBR} = 0$ does not

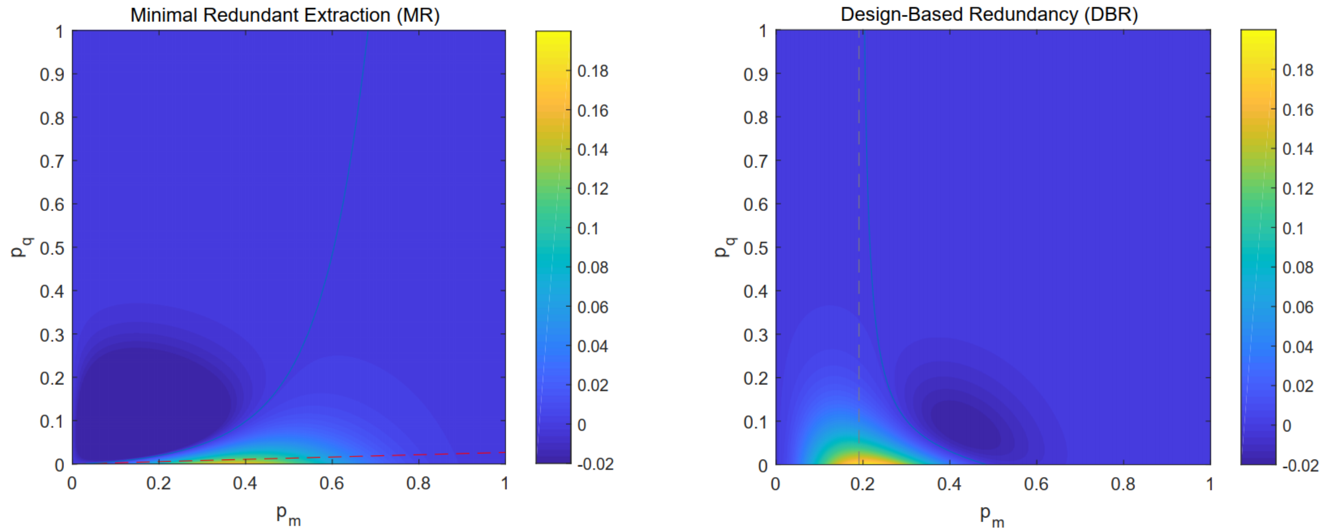

Figure 1. $F_{QEC} - F_{MR}$ and $F_{QEC} - F_{DBR}$ for the [[5,1,3]] Perfect code. This is a zoomed-out version of the plot in the main text showing the results for all possible error rates.

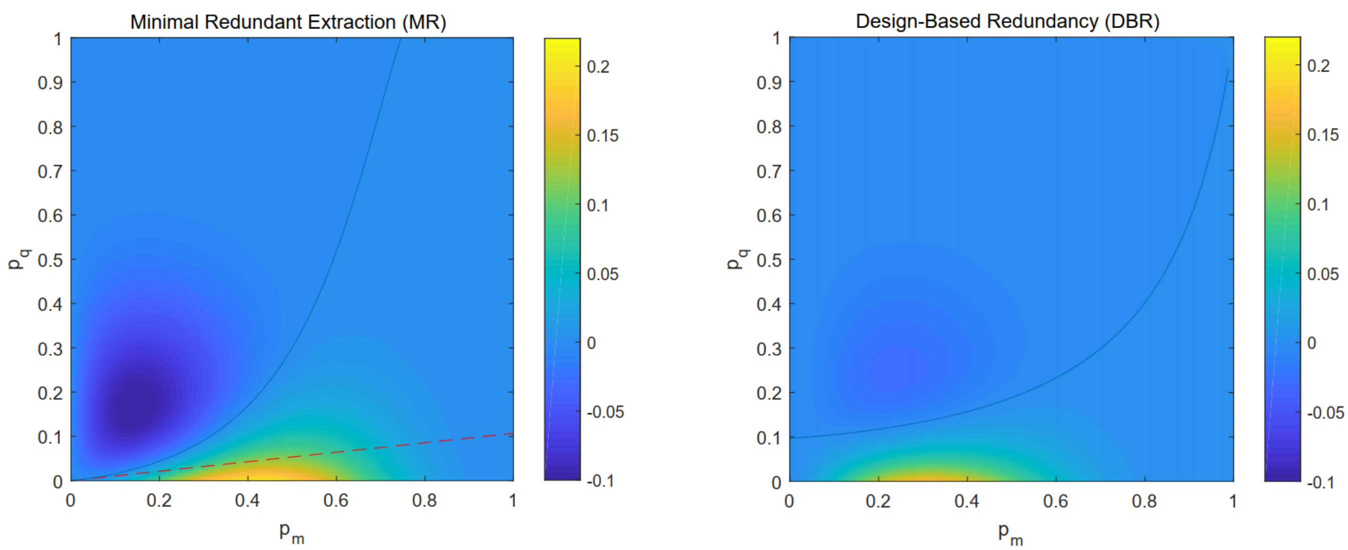

Figure 2. $F_{QEC} - F_{MR}$ and $F_{QEC} - F_{DBR}$ for the [[7,1,3]] Steane code. This is a zoomed-out version of the plot in the main text showing the results for all possible error rates.

pass through the origin, this reflects a difference in the power of the leading order p_q or p_m between the two rates. That is to say, one method can tolerate more errors of a certain type than the other when this is the case.

We also provided truncated analytic forms for the failure rates as a function of p_q, p_m, here we include the full expressions.