Laplace Equation

The Laplace's equation is a second-order partial differential equation named after Pierre-Simon Laplace, who first studied its properties. The homogenous and its inhomogeneous counterpart, the Poisson equation, are governing the mechanical and thermal equilibria of bodies, as well as uid-mechanical and electromagnetic potentials. The Laplace equation is a homogeneous linear dierential equation[1].

As we did with the diffusion equation, we will use the method of separation of variables to solve the homogeneous Laplace equation in a particular boundary-value problem. In the cartesian coordinates, the Laplace equation is

$$\nabla^2 V = \frac{\partial^2}{\partial x^2}(V) + \frac{\partial^2}{\partial y^2}(V) + \frac{\partial^2}{\partial z^2}(V) = 0$$

The theory of potential allows us to solve for V from given boundary conditions. After we obtain the potential V, we can nd the electric eld $\vec{E}$ by calculating the gradient of the potential,

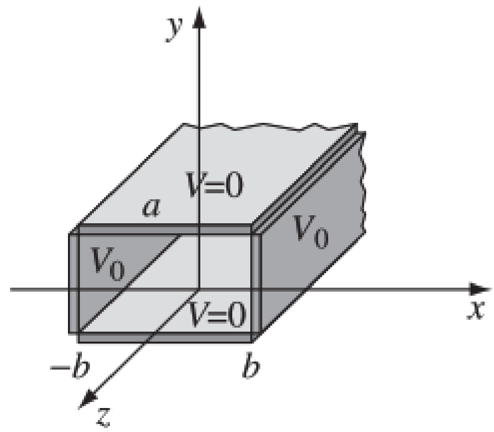
$$\vec{E} = -\nabla V = -\left(\frac{\partial}{\partial x}(V)\hat{x} + \frac{\partial}{\partial y}(V)\hat{y} + \frac{\partial}{\partial z}(V)\hat{z} \right)$$

Here is an illustrative example with a two-dimensional situation[2].

1. Olver, P. J., *Introduction to Partial Equations*, Undergraduate Texts in Mathematics, Springer, 2014, page 9-13.

2. Griffiths, D. J., *Introduction to Electrodynamics*, Fourth Edition, Cambridge University Press, 2017, exemple 3.4 page 136

Four long metallic plates form a shaft along the z direction; two of them are grounded at $y = 0$ and $y = a$, and the other two are maintained at a constant potential V_0 at $x \pm b$. See the following figure.



So the boudary conditions are:

(i) $V = 0$ when $y = 0$,
(ii) $V = 0$ when $y = a$,
(iii) $V = V_0$ when $x = b$,
(iv) $V = V_0$ when $x = -b$.

To solve this problem, we assume that V is independant of z.

Maple has a package specific for solving differential equation like the Laplace equation.

$with(PDEtools):$
$declare(V(x, y))$

$$V(x, y) \text{ will now be displayed as } V \quad \textbf{(1)}$$

$pde := diff(V(x, y), x, x) + diff(V(x, y), y, y) = 0$

$$pde := V_{x,x} + V_{y,y} = 0 \quad \textbf{(2)}$$

Before trying to solve it, we can test if this partial differential equation can be solve by separation of variables. To do this, we use the command *separability* and if the result is *0* , that mean that we can use the form that we have check, here is by the multiplication. That will help us later on.

$separability(pde, V(x, y), \text{`*`})$

$$0 \quad \textbf{(3)}$$

$ics := V(x, 0) = 0, V(x, a) = 0$

$$ics := V(x, 0) = 0, V(x, a) = 0 \quad \textbf{(4)}$$

$sys := [pde, ics]$

$$sys := \left[V_{x,x} + V_{y,y} = 0,\ V(x,0) = 0,\ V(x,a) = 0\right] \tag{5}$$

$pdsolve(sys)$

$$V = \sum_{n=1}^{\infty} \sin\left(\frac{n\,\pi\,y}{a}\right) \mathrm{csgn}\left(\frac{1}{a}\right) \left(\mathrm{e}^{\frac{n\,\pi\,\mathrm{csgn}\left(\frac{1}{a}\right) x}{a}} \_F1(n) + \mathrm{e}^{-\frac{n\,\pi\,\mathrm{csgn}\left(\frac{1}{a}\right) x}{a}} \_F2(n)\right) \tag{6}$$

Of course, Maple need more information on a, that what the csgn mean - cosign. And we have the condition on x that we need to take into account. Moreover, $\_F1(n)$ and $\_F2(n)$ are function to be define that depand on the index n.

But since our goal is to solve the Laplace equation by a specific method, we will stop here and concentrate on the solution by separation of variable step-by-step.

Step by Step with $V(x, y) = X(x) \cdot Y(y)$

To avoid confusion with the V, I will use instead the name *U(x,y)*.

$U := X(x) * Y(y)$

$$U := X(x)\ Y(y) \tag{1.1}$$

$eval(U, y = 0) = 0$

$$X(x)\ Y(0) = 0 \tag{1.2}$$

$eval(U, y = a) = 0$

$$X(x)\ Y(a) = 0 \tag{1.3}$$

$pde1 := eval(pde, V(x, y) = U)$

$$pde1 := \left(X_{x,x}\right) Y(y) + X(x) \left(Y_{y,y}\right) = 0 \tag{1.4}$$

$pde2 := expand\left(\frac{pde1}{U}\right)$

$$pde2 := \frac{X_{x,x}}{X(x)} + \frac{Y_{y,y}}{Y(y)} = 0 \tag{1.5}$$

$pde3 := pde2 - \frac{diff(diff(Y(y), y), y)}{Y(y)}$

$$pde3 := \frac{X_{x,x}}{X(x)} = -\frac{Y_{y,y}}{Y(y)} \tag{1.6}$$

$odex := lhs(pde3) = k^2$

$$odex := \frac{X_{x,x}}{X(x)} = k^2 \tag{1.7}$$

$odey := rhs(pde3) = k^2$

$$odey := -\frac{Y_{y,y}}{Y(y)} = k^2 \tag{1.8}$$

$dsolve(odey, Y(y))$

$$Y(y) = \_C1 \sin(k\, y) + \_C2 \cos(k\, y) \tag{1.9}$$

$soly := rhs(\mathbf{(1.9)})$

$$soly := \_C1 \sin(k\, y) + \_C2 \cos(k\, y) \tag{1.10}$$

$eq1 := subs(y = 0, soly) = 0$

$$eq1 := \_C1 \sin(0) + \_C2 \cos(0) = 0 \tag{1.11}$$

$\xrightarrow{\text{simplifier symboliquement}}$

$$\_C2 = 0 \tag{1.12}$$

$\xrightarrow{\text{assigner}}$

$soly$

$$\_C1 \sin(k\, y) \tag{1.13}$$

The other boundary condition gives a constraint on k:

$eq2 := subs(y = a, \mathbf{(1.13)}) = 0$

$$eq2 := \_C1 \sin(k\, a) = 0 \tag{1.14}$$

Of course, the solution $\_C1 = 0$ is trivial. So we need instead that $\sin(k \cdot a) = 0$, which is possible for all value of $n \in \mathbb{Z}^+$ such as:

$k \cdot a = n \cdot \pi$

$$k\, a = n\, \pi \tag{1.15}$$

$\xrightarrow{\text{isoler pour k}}$

$$k = \frac{n\, \pi}{a} \tag{1.16}$$

$\xrightarrow{\text{assigner}}$

So the solution is:

$Y(y) = soly$

$$Y(y) = \_C1 \sin\left(\frac{n\, \pi\, y}{a}\right) \tag{1.17}$$

Now let's solve for X.

$dsolve(odex, X(x))$

$$X(x) = \_C1\, \mathrm{e}^{-\frac{n\, \pi\, x}{a}} + \_C3\, \mathrm{e}^{\frac{n\, \pi\, x}{a}} \tag{1.18}$$

Hmmm… Maple uses the same name $\_C1$ as a constant of integration, so we get a name clash. Let’s rename the constants:

$solx := subs(\{\_C1 = A, \_C3 = B\}, rhs((\mathbf{1.18})))$

$$solx := A\,e^{-\frac{n\pi x}{a}} + B\,e^{\frac{n\pi x}{a}} \tag{1.19}$$

Because the boundary conditions on x is restricted on the range $x \in [-b, b]$, the exponentials are perfectly acceptable.

Now, since the situation is symmetric with respect to x, we get that $X(-b) = X(b)$, it follows that $A = B$. To get around this, we are going to be using

$$e^{k\cdot x} + e^{-k\cdot x} = 2\cdot\cosh(k\cdot x)$$

and will be absorbing the 2 and $\_C1$ with the A. Now we can assemble a solution from our base solutions

$V := \sum_{n=1}^{\infty}\left(C_n\cdot\cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\right)$

$$V := \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \tag{1.20}$$

Now, we have to take account to the initial condition on x.

$eq3 := subs(x = b, V) = Vo$

$$eq3 := \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right) = Vo \tag{1.21}$$

This expression is simply a Fourier series. We can solve for C_n by using the fact that the sines are orthogonal.

So multiply both sides by a suitable sine and integrate:

$\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\cdot Int\left(\sin\left(\frac{n\pi y}{a}\right)\cdot\sin\left(\frac{m\pi y}{a}\right), y = 0..a\right) = Vo\cdot Int\left(\sin\left(\frac{m\pi y}{a}\right), y = 0..a\right)$

$$\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right)\left(\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\mathrm{d}y\right) = Vo\left(\int_0^a \sin\left(\frac{m\pi y}{a}\right)\mathrm{d}y\right) \tag{1.22}$$

Now we must tell Maple about our assumptions on m and n.

$assume(n :: \mathbb{Z}, m :: \mathbb{Z}, L :: \mathbb{R})$

Let's verify this:

$$\sin(n\pi)$$

$$0 \tag{1.23}$$

Now we are ready to evaluate the left side of (1.22)

$$\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right) \cdot int\left(\sin\left(\frac{n\pi y}{a}\right)\cdot\sin\left(\frac{m\pi y}{a}\right), y=0..a\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{2\pi(m^2-n^2)}\left(C_n \cosh\left(\frac{n\pi b}{a}\right)(\sin(\pi(m-n))\,m+\sin(\pi(m-n))\,n-\sin(\pi(m+n))\,m+\sin(\pi(m+n))\,n)\,a\right) \tag{1.24}$$

So we have to assume that $m=n$

$$assume(m=n)$$

$$\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi b}{a}\right) \cdot int\left(\sin\left(\frac{n\pi y}{a}\right)\cdot\sin\left(\frac{m\pi y}{a}\right), y=0..a\right)$$

$$\sum_{n\sim=1}^{\infty} \frac{C_{n\sim}\cosh\left(\frac{n\sim\pi b}{a}\right)a\,(2\,n\sim\pi-\sin(2\,n\sim\pi))}{4\,\pi\,n\sim} \tag{1.25}$$

$$\xrightarrow{\text{simplifier symboliquement}}$$

$$\sum_{n\sim=1}^{\infty} \frac{C_{n\sim}\cosh\left(\frac{n\sim\pi b}{a}\right)a}{2} \tag{1.26}$$

Now let's do the right side of (1.22)

$$Vo\cdot int\left(\sin\left(\frac{n\pi y}{a}\right), y=0..a\right) \text{ assuming } n :: odd$$

$$\frac{2\,Vo\,a}{\pi\,n\sim} \tag{1.27}$$

So we have finally

$$C_n \cdot\cosh\left(\frac{n\pi b}{a}\right)=\frac{4\cdot Vo}{n\cdot\pi}$$

$$C_{n\sim}\cosh\left(\frac{n\sim\pi b}{a}\right)=\frac{4\,Vo}{n\sim\pi} \tag{1.28}$$

$$\xrightarrow{\text{isoler pour C[n]}}$$

$$C_{n\sim}=\frac{4\,Vo}{n\sim\pi\cosh\left(\frac{n\sim\pi b}{a}\right)} \tag{1.29}$$

$\xrightarrow{\text{assigner}}$

So the final solution is

$$Vsol := \frac{4 \cdot Vo}{\pi} \cdot \sum_{n=1}^{\infty} \left(\frac{1}{n} \cdot \frac{\cosh\left(\frac{n \pi x}{a}\right)}{\cosh\left(\frac{n \cdot \pi \cdot b}{a}\right)} \cdot \sin\left(\frac{n \pi y}{a}\right) \right)$$

$$Vsol := \frac{4\, Vo \left(\sum_{n\sim=1}^{\infty} \frac{\cosh\left(\frac{n\sim \pi x}{a}\right) \sin\left(\frac{n\sim \pi y}{a}\right)}{n\sim \cosh\left(\frac{n\sim \pi b}{a}\right)} \right)}{\pi} \tag{1.30}$$

Finally, we can plot it. But of course, we have to cut off the series at some point. We will approximate the solution by the first 15 terms. For graphical purpose, we will use $Vo = 1$, $b = 1$ and $a = 1$ so our function to plot become (keeping in mind that n is odd)

$n :=$'n':

$Vo, a, b := 1, 1, 1$

$$Vo, a, b := 1, 1, 1 \tag{1.31}$$

$N := seq(n, n = 1..15, 2)$

$$N := 1, 3, 5, 7, 9, 11, 13, 15 \tag{1.32}$$

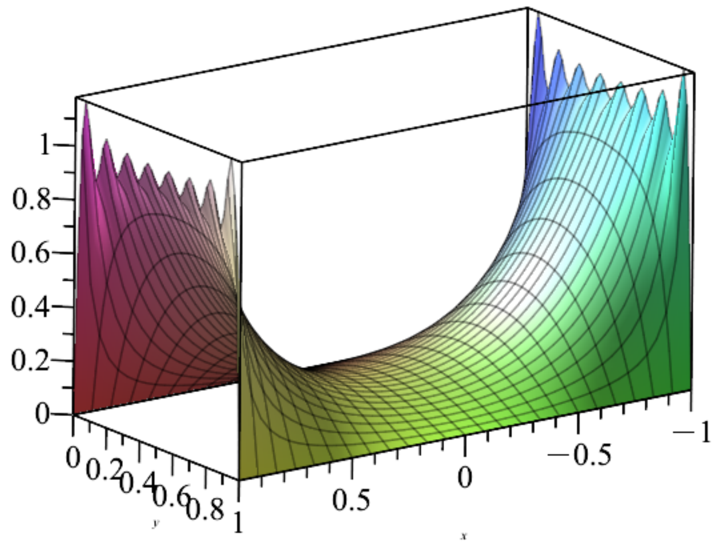
$$Vsol\_coeff := \frac{4}{\pi} \cdot \frac{1}{n} \cdot \frac{\cosh\left(\frac{n \cdot \pi \cdot x}{a}\right)}{\cosh\left(\frac{n \pi b}{a}\right)} \cdot \sin\left(\frac{n \pi y}{a}\right)$$

$$Vsol\_coeff := \frac{4 \cosh(n \pi x) \sin(n \pi y)}{\pi n \cosh(n \pi)} \tag{1.33}$$

$Vsol\_approx := add(Vsol\_coeff, n = N)$

$$Vsol\_approx := \frac{4 \cosh(\pi x) \sin(\pi y)}{\pi \cosh(\pi)} + \frac{4 \cosh(3 \pi x) \sin(3 \pi y)}{3 \pi \cosh(3 \pi)} + \frac{4 \cosh(5 \pi x) \sin(5 \pi y)}{5 \pi \cosh(5 \pi)} + \frac{4 \cosh(7 \pi x) \sin(7 \pi y)}{7 \pi \cosh(7 \pi)} + \frac{4 \cosh(9 \pi x) \sin(9 \pi y)}{9 \pi \cosh(9 \pi)} + \frac{4 \cosh(11 \pi x) \sin(11 \pi y)}{11 \pi \cosh(11 \pi)} + \frac{4 \cosh(13 \pi x) \sin(13 \pi y)}{13 \pi \cosh(13 \pi)} + \frac{4 \cosh(15 \pi x) \sin(15 \pi y)}{15 \pi \cosh(15 \pi)} \tag{1.34}$$

$plot3d(Vsol\_approx, x = -1..1, y = 0..1, scaling = constrained)$



Now, we are ready to derive the electric field **E** with the relation $\vec{E} = -\vec{\nabla} V$.

Deriving the Electric Field

To do so, we will now set the Cartesian system of Coodinates and use the package VectorCalculus

restart
with(*VectorCalculus*) : *with*(*plots*) : *with*(*LinearAlgebra*) :

Then copy-paste the solution that we have found in the preceding section

$$Vsol\_approx := \frac{4 \cosh(\pi x) \sin(\pi y)}{\pi \cosh(\pi)} + \frac{4 \cosh(3 \pi x) \sin(3 \pi y)}{3 \pi \cosh(3 \pi)} + \frac{4 \cosh(5 \pi x) \sin(5 \pi y)}{5 \pi \cosh(5 \pi)}$$

$$+\frac{4\cosh(7\pi x)\sin(7\pi y)}{7\pi\cosh(7\pi)}+\frac{4\cosh(9\pi x)\sin(9\pi y)}{9\pi\cosh(9\pi)}+\frac{4\cosh(11\pi x)\sin(11\pi y)}{11\pi\cosh(11\pi)}$$
$$+\frac{4\cosh(13\pi x)\sin(13\pi y)}{13\pi\cosh(13\pi)}+\frac{4\cosh(15\pi x)\sin(15\pi y)}{15\pi\cosh(15\pi)}$$

$$Vsol\_approx := \frac{4\cosh(\pi x)\sin(\pi y)}{\pi\cosh(\pi)}+\frac{4\cosh(3\pi x)\sin(3\pi y)}{3\pi\cosh(3\pi)}+\frac{4\cosh(5\pi x)\sin(5\pi y)}{5\pi\cosh(5\pi)}+\frac{4\cosh(7\pi x)\sin(7\pi y)}{7\pi\cosh(7\pi)}+\frac{4\cosh(9\pi x)\sin(9\pi y)}{9\pi\cosh(9\pi)}+\frac{4\cosh(11\pi x)\sin(11\pi y)}{11\pi\cosh(11\pi)}+\frac{4\cosh(13\pi x)\sin(13\pi y)}{13\pi\cosh(13\pi)}+\frac{4\cosh(15\pi x)\sin(15\pi y)}{15\pi\cosh(15\pi)} \quad \textbf{(2.1)}$$

And now we are ready to calculate the Gradient

$Efield := Gradient(-Vsol\_approx, [x, y])$

$$Efield := \left(-\frac{4\sinh(\pi x)\sin(\pi y)}{\cosh(\pi)}-\frac{4\sinh(3\pi x)\sin(3\pi y)}{\cosh(3\pi)}-\frac{4\sinh(5\pi x)\sin(5\pi y)}{\cosh(5\pi)}-\frac{4\sinh(7\pi x)\sin(7\pi y)}{\cosh(7\pi)}-\frac{4\sinh(9\pi x)\sin(9\pi y)}{\cosh(9\pi)}-\frac{4\sinh(11\pi x)\sin(11\pi y)}{\cosh(11\pi)}-\frac{4\sinh(13\pi x)\sin(13\pi y)}{\cosh(13\pi)}-\frac{4\sinh(15\pi x)\sin(15\pi y)}{\cosh(15\pi)}\right)\bar{e}_x+\left(-\frac{4\cosh(\pi x)\cos(\pi y)}{\cosh(\pi)}-\frac{4\cosh(3\pi x)\cos(3\pi y)}{\cosh(3\pi)}-\frac{4\cosh(5\pi x)\cos(5\pi y)}{\cosh(5\pi)}-\frac{4\cosh(7\pi x)\cos(7\pi y)}{\cosh(7\pi)}-\frac{4\cosh(9\pi x)\cos(9\pi y)}{\cosh(9\pi)}-\frac{4\cosh(11\pi x)\cos(11\pi y)}{\cosh(11\pi)}-\frac{4\cosh(13\pi x)\cos(13\pi y)}{\cosh(13\pi)}-\frac{4\cosh(15\pi x)\cos(15\pi y)}{\cosh(15\pi)}\right)\bar{e}_y \quad \textbf{(2.2)}$$

$NormEfield := Normalize(Efield, 2)$:

Of course, the answer is a vector normalisezed. We first create a fieldplot of our normalized field. We terminate with a : because we don't want to plot it immediately.

$p1 := fieldplot([NormEfield[1], NormEfield[2]], x=-1..1, y=0..1)$:

We will draw the equipotential lines on the xy plane with the direction of the electric field. For that we use an impliciplot function with 10 equipotential line. We have to plot y form 0 to 1.1 to be able to see the line $x=-1..1, y=1$.

$p2 := implicitplot\left(\left\{seq\left(Vsol\_approx = \frac{k}{10}, k = 0..10\right)\right\}, x = -1..1, y = 0..1.1, numpoints = 400\right):$

and now we can display both plots

$display([p1, p2], scaling = constrained)$