Diffusion Equation

Today, we will analytically solve one of the most important partial differential equations out there, the diffusion equation. It is a fundamental equation that arises in many areas of physics, chemistry, biology, and engineering, and has an enormous range of applications. For instance, in physics, the diffusion equation is used to describe the movement of heat through a solid or fluid, as well as the transport of particles in gases and liquids. Also, the Schrödinger equation is (in principle) a kind of diffusion equation. In biology, the diffusion equation is used to study the movement of nutrients and other substances through living tissues, as well as the spread of diseases. In engineering, the diffusion equation is used to design and optimize processes such as heat exchangers, catalytic converters, and solar cells. It is even applied in economics, where the diffusion equation can be used to model the spread of ideas or innovations through a population. Due to its wide range of applications, it is not surprising that it was found independently by several scientists of the 19th century working on completely different topics. The famous George Gabriel Stokes derived the equation in 1845 to describe the motion of fluids, while the even more famous James Clerk Maxwell derived the equation in 1860 to describe the motion of heat.

The equation is given by:

$$\alpha^2 \cdot \frac{\partial}{\partial t}(u) = \nabla^2 u$$

where t is time, α a constant, and ∇^2 is the Laplacian operator. For concreteness, we will solve the following example application. We have a 10 cm long bar with insulated sides initially at 100° everywhere. Starting at $t=0$, the ends are held at temperature 0°. Our task is to find the temperature distribution at some later time *t*.

The method we will use is the separation of variables, i.e. we use the ansatz

$$u(x, t) = T(t) \cdot X(x)$$

where T and $X$are functions of a single variable t and x, respectively.

We will solve it first by using the *pdsolve* command to get an insight of the general solution. And then, we will solve it anatically by the ansatz that we define.

$with(PDEtools)$

$[CanonicalCoordinates, ChangeSymmetry, CharacteristicQ, CharacteristicQInvariants, ConservedCurrentTest, ConservedCurrents, ConsistencyTest, D\_Dx, DeterminingPDE, Eta\_k, Euler, FirstIntegralSolver, FromJet, FunctionFieldSolutions, InfinitesimalGenerator, Infinitesimals, IntegratingFactorTest, IntegratingFactors, InvariantEquation, InvariantSolutions, InvariantTransformation, Invariants, Laplace, Library, PDEplot, PolynomialSolutions, ReducedForm, SimilaritySolutions, SimilarityTransformation, Solve, SymmetryCommutator, SymmetryGauge, SymmetrySolutions, SymmetryTest, SymmetryTransformation, TWSolutions, ToJet, ToMissingDependentVariable, build, casesplit, charstrip, dchange, dcoeffs, declare, diff\_table, difforder, dpolyform, dsubs, mapde, separability, splitstrip, splitsys, undeclare]$ **(1)**

$declare(u(x,t))$

$$u(x,t) \text{ will now be displayed as } u \quad \textbf{(2)}$$

$pde := \alpha^2 \cdot diff(u(x,t),t) = diff(u(x,t),x,x)$

$$pde := \alpha^2 (u_t) = u_{x,x} \quad \textbf{(3)}$$

$separability(pde, u(x,t), `*`)$

$$0 \quad \textbf{(4)}$$

$ics := u(0,t) = 0, u(L,t) = 0$

$$ics := u(0,t) = 0, u(L,t) = 0 \quad \textbf{(5)}$$

$sys := [pde, ics]$

$$sys := \left[\alpha^2 (u_t) = u_{x,x}, u(0,t) = 0, u(L,t) = 0\right] \quad \textbf{(6)}$$

$pdsolve(sys)$

$$u = \sum_{n=1}^{\infty} \operatorname{csgn}\left(\frac{1}{L}\right) \sin\left(\frac{n \pi x}{L}\right) \mathrm{e}^{-\frac{\pi^2 n^2 t}{L^2 \alpha^2}} \_F1(n) \quad \textbf{(7)}$$

Step-by-step

$U := X(x) * T(t)$

$$U := X T \quad \textbf{(1.1)}$$

with the inital conditions:

$eval(U, x=0) = 0$

$$X(0)\ T = 0 \quad \textbf{(1.2)}$$

$eval(U, x=L) = 0$

$$X(L)\ T = 0 \quad \textbf{(1.3)}$$

$pde1 := eval(pde, u(x,t) = U)$

$$pde1 := \alpha^2 X (T_t) = (X_{x,x}) T \tag{1.4}$$

Rewrite by separating the x and t dependence to opposite sides of the **(1.4)** equation:

$pde2 := \frac{pde1}{U}$

$$pde2 := \frac{\alpha^2 (T_t)}{T} = \frac{X_{x,x}}{X} \tag{1.5}$$

The key observation here is that the left side depends only on t, whereas the right side depends only on x. So, when you change t, only the left side changes but not the right side. Conversely, if you change x, the right side changes, but not the left side. The only way how this can be true is that both sides are equal to a constant. Let's call that constant $-k^2$:

$odex := rhs(pde2) = -k^2$

$$odex := \frac{X_{x,x}}{X} = -k^2 \tag{1.6}$$

$odet := lhs(pde2) = -k^2$

$$odet := \frac{\alpha^2 (T_t)}{T} = -k^2 \tag{1.7}$$

Clearing the denominator in each leads to:

$eqx := odex \cdot X(x)$

$$eqx := X_{x,x} = -k^2 X \tag{1.8}$$

$eqt := odet \cdot T(t)$

$$eqt := \alpha^2 (T_t) = -T k^2 \tag{1.9}$$

and bringing all terms to the left side puts the ODEs into the form:

$eqx1 := lhs(eqx) - rhs(eqx) = 0$

$$eqx1 := k^2 X + X_{x,x} = 0 \tag{1.10}$$

$eqt1 := lhs(eqt) - rhs(eqt) = 0$

$$eqt1 := \alpha^2 (T_t) + T k^2 = 0 \tag{1.11}$$

We will solve for the function $T(t)$ first:

$solt := dsolve(eqt1, T(t))$

$$solt := T = \_C1\, e^{-\frac{k^2 t}{\alpha^2}} \tag{1.12}$$

and for the function $X(x)$:

$solx := dsolve(eqx1, X(x))$

$$solx := X = \_C1 \sin(k\, x) + \_C2 \cos(k\, x) \tag{1.13}$$

Hmmm… Maple uses the same names of the constants of integration, so we get a name clash. Let's rename the constants:

$sol\_X := subs(\{\_C1 = A, \_C2 = B\}, rhs(solx))$

$$sol\_X := A \sin(k\, x) + B \cos(k\, x) \tag{1.14}$$

Now consider the first boundary conditions at $x = 0$. This gives a value for the constant B:

$eq1 := subs(x = 0, sol\_X) = 0$

$$eq1 := A \sin(0) + B \cos(0) = 0 \tag{1.15}$$

$\xrightarrow{\text{simplifier symboliquement}}$

$$B = 0 \tag{1.16}$$

$\xrightarrow{\text{assigner}}$

sol_X

$$A \sin(k\, x) \tag{1.17}$$

The other boundary condition gives a constraint on k:

$eq2 := subs(x = L, \textbf{(1.17)}) = 0$

$$eq2 := A \sin(k\, L) = 0 \tag{1.18}$$

Of course, the solution $A = 0$ is trivial. So we need instead that $\sin(k\, L) = 0$, which is possible for all value of $n \in \mathbb{Z}^+$ such as:

$k \cdot L = n \cdot \pi$

$$k\, L = n\, \pi \tag{1.19}$$

$\xrightarrow{\text{isoler pour k}}$

$$k = \frac{n\, \pi}{L} \tag{1.20}$$

$\xrightarrow{\text{assigner}}$

So the solution is:

$X(x) = sol\_X$

$$X = A\sin\left(\frac{n\pi x}{L}\right) \tag{1.21}$$

Now we can assemble a solution from our base solutions.

$$u(x, t) = \sum_{n=1}^{\infty}\left(a_n \cdot \mathrm{e}^{-\frac{k^2 t}{\alpha^2}} \cdot \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$u = \sum_{n=1}^{\infty} a_n\, \mathrm{e}^{-\frac{\pi^2 n^2 t}{L^2 \alpha^2}} \sin\left(\frac{n\pi x}{L}\right) \tag{1.22}$$

where we have absorbed the coefficient C and A into the new coefficients a_n.

Now, we have to take account to the initial condition that at $t=0$, the temperature is 100°. This gives us:

$eq4 := subs(t=0, rhs(\mathbf{(1.22)})) = 100$

$$eq4 := \sum_{n=1}^{\infty} a_n\, \mathrm{e}^{0} \sin\left(\frac{n\pi x}{L}\right) = 100 \tag{1.23}$$

We can solve for a_n by using the fact that the sines are orthogonal. So multiply both sides by a suitable sine and integrate:

$$\sum_{n=1}^{\infty} Int\left(a_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{m\pi x}{L}\right), x=0..L\right) = Int\left(100 \cdot \sin\left(\frac{m\pi x}{L}\right), x=0..L\right)$$

$$\sum_{n=1}^{\infty}\left(\int_0^L a_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x\right) = \int_0^L 100 \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x \tag{1.24}$$

Now we must tell Maple about our assumptions on m and n.

$assume(n :: \mathbb{Z}, m :: \mathbb{Z}, L :: \mathbb{R})$

Let's verify this:

$\sin(n\pi)$

$$0 \tag{1.25}$$

Now we are ready to evaluate the equation **(1.24)** .

$$\int_0^L 100\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x$$

$$\int_0^{L\sim} 100\sin\left(\frac{m\sim\pi x}{L\sim}\right)\mathrm{d}x \quad (1.26)$$

$=$

$$-\frac{100\,L\sim\left(\left(-1\right)^{m\sim}-1\right)}{m\sim\pi} \quad (1.27)$$

We first assume that $m \neq n$:

$$\sum_{n=1}^{\infty} Int\left(a_n\cdot\sin\left(\frac{n\pi x}{L}\right)\cdot\sin\left(\frac{m\pi x}{L}\right), x=0..L\right)$$

$$\sum_{n\sim=1}^{\infty}\left(\int_0^{L\sim} a_{n\sim}\sin\left(\frac{n\sim\pi x}{L\sim}\right)^2\mathrm{d}x\right) \quad (1.28)$$

$=$

$$0 \quad (1.29)$$

That is normal since the sinuses are orthogonal. Now redo the integration for $m=n$:

$assume(m=n)$

$$\sum_{n=1}^{\infty} Int\left(a_n\cdot\sin\left(\frac{n\pi x}{L}\right)\cdot\sin\left(\frac{m\pi x}{L}\right), x=0..L\right)$$

$$\sum_{n\sim=1}^{\infty}\left(\int_0^{L\sim} a_{n\sim}\sin\left(\frac{n\sim\pi x}{L\sim}\right)^2\mathrm{d}x\right) \quad (1.30)$$

$=$

$$\sum_{n\sim=1}^{\infty}\frac{L\sim\left(n\sim\pi-\cos(n\sim\pi)\sin(n\sim\pi)\right)a_{n\sim}}{2\,n\sim\pi} \quad (1.31)$$

$\overset{\text{simplifier}}{=}$

$$\sum_{n\sim=1}^{\infty}\frac{L\sim a_{n\sim}}{2} \quad (1.32)$$

$n := m$

$$n := m\sim \quad (1.33)$$

So finally, we have the following equation relation:

$$\textbf{(1.27)} = \frac{L\sim a_{n\sim}}{2}$$

$$-\frac{100\,L\sim\left(\left(-1\right)^{m\sim}-1\right)}{m\sim\,\pi} = \frac{L\sim a_{m\sim}}{2} \qquad \textbf{(1.34)}$$

$\xrightarrow{\text{isoler pour a[m]}}$

$$a_{m\sim} = -\frac{200\left(\left(-1\right)^{m\sim}-1\right)}{\pi\,m\sim} \qquad \textbf{(1.35)}$$

$\xrightarrow{\text{assigner}}$

And now, that is the exact solution to our problem.

(1.22)

$$u = \sum_{m\sim=1}^{\infty}\left(-\frac{200\left(\left(-1\right)^{m\sim}-1\right)\mathrm{e}^{-\frac{\pi^2 m\sim^2 t}{L\sim^2\alpha^2}}\sin\left(\frac{m\sim\pi x}{L\sim}\right)}{\pi\,m\sim}\right) \qquad \textbf{(1.36)}$$

Finally, we can plot it. But of course, we have to cut off the series at some point. We will approximate the solution by the first 30 terms and taked a rod of length $L = 10$ and $\alpha = 1$:

$L := 10 : \alpha := 1$

$$\alpha := 1 \qquad \textbf{(1.37)}$$

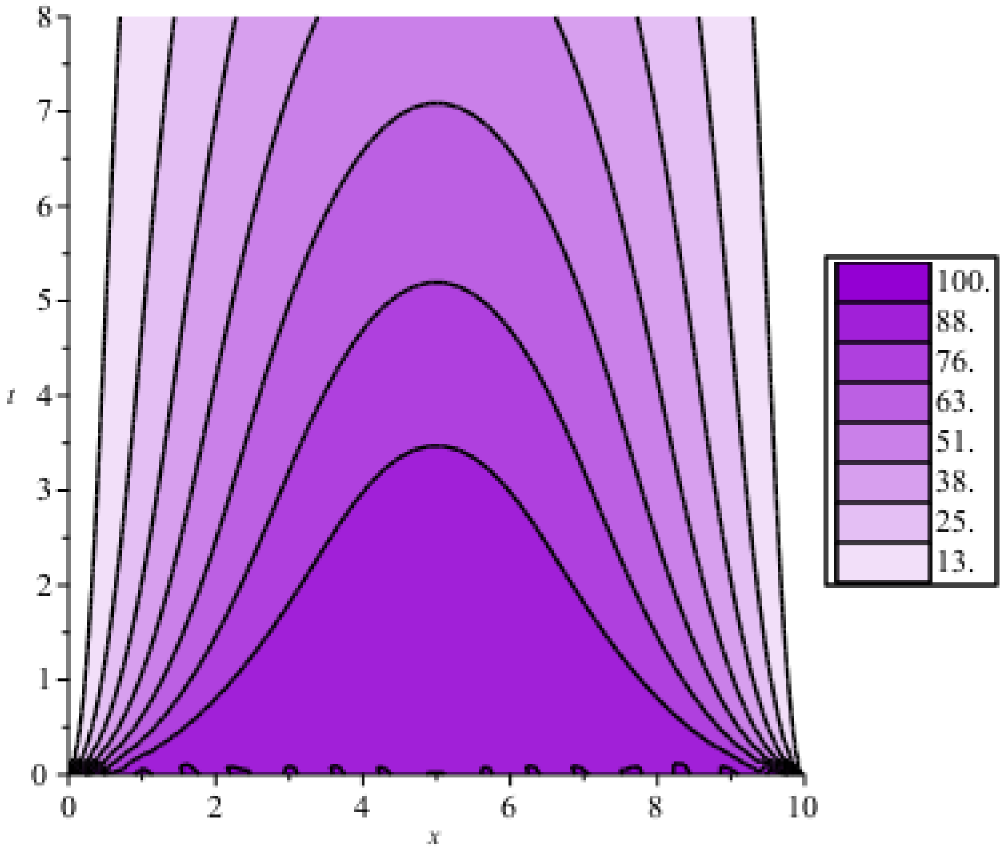
$$sol\_approx := sum\left(-\frac{200\left(\left(-1\right)^{m}-1\right)\mathrm{e}^{-\frac{\pi^2 m^2\cdot t}{L\sim^2\alpha^2}}\sin\left(\frac{m\ \pi x}{L\sim}\right)}{\pi\cdot m}, m = 1..30\right):$$

$with(plots):$

$contourplot\Big(sol\_approx, x = 0..10, t = 0..8, filledregions = true, coloring = [\text{"White"}, \text{"DarkViolet"}],$

$\quad legendstyle = [location = right], legend = [digits = 2], caption = typeset\Big(\text{"A contourplot of "}, u(x, t)$

$$= \sum_{m=1}^{\infty}\left(-\frac{200\left(\left(-1\right)^{m}-1\right)\mathrm{e}^{-\frac{\pi^2 m^2 t}{L^2\alpha^2}}\sin\left(\frac{m\,\pi x}{L}\right)}{\pi\,m}\right), \text{"."}\Big)\Big)$$



A contourplot of $u = \sum_{m=1}^{\infty} \left(-\frac{200\left((-1)^m - 1\right) e^{-\frac{1}{100}\pi^2 m^2 t} \sin\left(\frac{1}{10} m \pi x\right)}{m \pi}\right.$

Thank you very.

This document has been largely inspired by an article in Medium by **Mathcube** .

My next goal is to do it in a Maple Notebook. Stay tune!