Silent Disco Glamping Network

A Concave Fractional Integer Optimisation Approach via Outer Approximation

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1 Introduction

The design of spatially diverse facility networks is a fundamental problem in optimisation, with applications spanning tourism, logistics, telecommunications, and emergency services. Recent advances in diversity optimisation, particularly in the Euclidean max-sum diversity problem and the max-mean dispersion problem, have significantly enhanced optimisation methodologies in these application domains. Building on these developments, this project investigates a fictional Silent Disco Glamping Network problem, where k locations are selected from the remote airstrips across Australia to host glamping (glamorous camping) sites. The objective is to maximise geographical dispersion among selected locations while accounting for infrastructure costs, capacity requirements, and minimum separation constraints.

Mathematically, this problem leads to a fractional quadratic integer program, in which a quadratic dispersion is normalised by a linear cost term. Problems of this type are challenging due to the simultaneous presence of integrability, nonlinearity, and a fractional objective. Classical linearisation techniques or decomposition approaches are generally insufficient, motivating the need for more specialised methods.

This project develops a rigorous mathematical formulation, proves the concavity of a transformed objective via a perspective mapping, and proposes an Outer Approximation (OA) algorithm to compute globally optimal solutions, and determines the maximum minimum-separation distance using a binary search. The method scales to realistic problem sizes and is implemented using Gurobi.

2 Motivation for the Chosen Solution Approach

At first glance, the problem resembles a classical diversity optimisation problems such as the Max-Sum Diversity Problem (MSDP). However, the fractional objective introduces additional complexity that renders standard linearisation and decomposition techniques ineffective.

In particular:

- *Benders decomposition* is not applicable, as the problem is purely integer with no natural continuous subproblem.
- Standard linearisation approaches require introducing auxiliary variables for each bilinear term $x_i x_j$. This leads to a model with $O(m^2)$ additional variables and constraints, resulting in weak linear relaxations and poor computational scalability.

- Exact approaches for related problems in Garrafa, Della Croce, and Salassa (2017) based on semidefinite programming, have been proposed for the max-mean dispersion problem. While theoretically powerful, these methods rely on large matrix variables and branch-and-bound schemes that scale poorly beyond small problem instances, limiting their practical applicability.

Outer Approximation is well suited to this setting because the objective can be reformulated as a *concave function* over the feasible region and concave functions admit global linear upper bounds via supporting hyperplanes.

3 Problem formulation

Given a set of potential locations $\mathcal{L}$ (set of airstrips in Australia), $\mathcal{L} = 1, 2, ..., m$.

Each element $i \in \mathcal{L}$ corresponds to one location point in $\mathbb{R}^2$, where each point is indexed by $l_i = (x_i, y_i) \in \mathbb{R}^2$.

3.1 Parameters

- d_{ij} Haversine distance distance between location i and j ($d_{ii} = 0$)
- $w_i > 0$: weighted cost factor of location i
- a_i: maximum number of tents at location i
- k: number of sites to select
- δ: minimum distance between any two selected sites
- N: total number of tents across all sites

3.2 Decision variable

* $x_i \in \{0, 1\}$:

$$x_i = \begin{cases} 1, & \text{if location} i \text{is selected,} \\ 0, & \text{otherwise} \end{cases}$$

3.3 Mathematical formulation

$$\begin{aligned} \max_{x} \quad & \frac{\sum_{i,j \in \mathcal{L}} d_{ij} x_i x_j}{\sum_{i \in \mathcal{L}} w_i x_i} \\ \text{s.t.} \quad & \sum_{i \in \mathcal{L}} x_i = k \\ & x_i + x_j \leq 1 \quad \forall (i, j) : d_{ij} < \delta \\ & \sum_{i \in \mathcal{L}} a_i x_i \geq N \\ & x_i \in \{0, 1\} \end{aligned}$$

The numerator represents "total isolation", the bigger this number, the more the startup can increase the experience of customers. On the other hand, the denominator penalises expensive cost of construction in remote areas. This results in a max-mean type objective like that introduced in Garraffa et al. 2016.

4 Perspective Reformulation and Concavity Prove

Define the total cost variable

$$t = \sum_{i=1}^{m} w_i x_i, \quad t > 0,$$

and the quadratic dispersion function

$$f(x) = \sum_{i,j \in \mathcal{L}}^{m} d_{ij} x_i x_j$$

The objective becomes

$$\max_x \frac{f(x)}{t}$$

Since $f(x)$ is quadratic, $f(\frac{x}{t}) = \frac{1}{t^2} f(x)$.

Therefore, maximising $\frac{f(x)}{t}$ is equivalent to maximising $g(x,t) = tf(\frac{x}{t})$.

The concavity result for the Euclidean max-sum diversity function $f(x) = \sum_{i,j \in \mathcal{L}}^{m} d_{ij} x_i x_j$ is establised in Spiers, Bui, and Loxton (2023) over the feasible set

$$K = \{x \in \{0,1\}^m : \sum_{i=1}^{m} = k\}$$

In the Silent Disco Glamping Network problem, the feasible region is given by

$$\mathcal{F} = \{x \in S : x_i + x_j \leq 1 \quad \forall (i,j) : d_{ij} < \delta, \sum_{i=1}^{m} a_i x_i \geq N\}$$

Clearly, $\mathcal{F} \subseteq S$.

The concavity of f on K implies that for any $y \in \mathcal{F}$, the upper bound of the f is given by

$$f(x) \leq f(y) + \langle \nabla f(y), x - y \rangle, \quad \forall x \in K$$

Since the feasible region $\mathcal{F}$ is a subset of K, all supporting hyperplanes constructed on K remain valid on $\mathcal{F}$. Consequently, the restriction of f to $\mathcal{F}$ preserves the concavity structure required for Outer Approximation.

Moreover, the perspective of a concave fuction remains concave, hence, $g(x,t)$ is concave over the feasible region. This validates the Outer Approximation algorithm.

5 Methodology

5.1 Computation of Optimal Solution via Outer Approximation Algorithm

To apply Outer Approximation, an auxiliary variable θ is introduced to represent the objective value.

5.1.1 Supporting hyperplane

Recall:

$$f(x) = \frac{1}{2}x^T Qx \Rightarrow \nabla f(x) = Qx$$

where $Q = [q_{ij}]$ is the distance matrix and $Q_{ii} = 0 \quad \forall i$

Now, $g(x,t) = tf(\frac{x}{t})$, we have

$$\nabla_x g(x,t) = \nabla f(\frac{x}{t}) = Q\frac{x}{t}$$

Hence,

$$\begin{aligned}
\frac{\partial g}{\partial t} &= f\Big(\frac{x}{t}\Big) + t\left\langle \nabla f\Big(\frac{x}{t}\Big), \frac{\partial}{\partial t}\Big(\frac{x}{t}\Big)\right\rangle \\
&= f\Big(\frac{x}{t}\Big) + t\left\langle \nabla f\Big(\frac{x}{t}\Big), \Big(-\frac{x}{t^2}\Big)\right\rangle \\
&= f\Big(\frac{x}{t}\Big) - \left\langle \nabla f\Big(\frac{x}{t}\Big), \frac{x}{t}\right\rangle
\end{aligned}$$

$$\frac{\partial g}{\partial t}(x,t) = f\Big(\frac{x}{t}\Big) - \left\langle Q\frac{x}{t}, \frac{x}{t}\right\rangle$$

Let (y,s) be a feasible point with $s > 0$.

The supporting hyperplane of g at (y,s) is

$$h_{(y,s)}(x,t) = g(y,s) + \langle \nabla_x g(y,s),\, x-y\rangle + \frac{\partial g}{\partial t}(y,s)\,(t-s).$$

$$\begin{aligned}
h_{(y,s)}(x,t) = g(y,s) &+ \left\langle Q\frac{y}{s},\, x-y\right\rangle \\
&+ \left[f\Big(\frac{y}{s}\Big) - \left\langle Q\frac{y}{s},\, \frac{y}{s}\right\rangle\right](t-s).
\end{aligned}$$

Since g is concave, $g(x,t) \le h_{(y,s)}(x,t) \quad \forall (x,t)$.

5.1.2 Master Problem

$$
\begin{aligned}
\max_{\theta,x,t} \quad & \theta \\
\text{s.t.} \quad & \sum_{i=1}^{m} x_i = k, \\
& t = \sum_{i=1}^{m} w_i x_i, \\
& x_i + x_j \le 1, \quad \forall (i,j) : d_{ij} < \delta, \\
& \sum_{i=1}^{m} a_i x_i \ge N, \\
& \theta \le h_{y,s}(x,t), \quad \forall (y,s) \in \mathcal{C}, \\
& x_i \in \{0,1\}, \quad i = 1,\dots,m.
\end{aligned}
$$

Here, $h_(y,s)$ denotes a supporting hyperplane of g at a previously evaluated point (y,s)

5.1.3 Algorithm

Outer Approximation for the Silent Disco Glamping Network

Input: Distance matrix Q, cost vector w, capacity vector a, cardinality k, minimum separation δ, capacity target N, tolerance $\varepsilon > 0$.

Initialisation:

- Choose an initial feasible solution $x^0 \in \mathcal{F}$.
- Set $t^0 \leftarrow \sum_{i=1}^{m} w_i x_i^0$.
- Set $LB_0 \leftarrow g(x^0, t^0)$, $UB_0 \leftarrow +\infty$.
- Set $A^0 \leftarrow \{(x^0, t^0)\}$ and $k \leftarrow 0$.

$UB_k - LB_k > \varepsilon$

1. Solve the master MILP:
$$\max\ \theta \quad \text{s.t.}\ (x,t) \in \mathcal{F},\ \theta \le h_{(y,s)}(x,t)\ \forall (y,s) \in A^k.$$
Obtain solution $(x^{k+1}, t^{k+1}, \theta^{k+1})$.
2. Update upper bound:
$$UB_{k+1} \leftarrow \theta^{k+1}.$$
3. Evaluate the nonlinear objective:
$$LB_{k+1} \leftarrow \max\{LB_k,\ g(x^{k+1}, t^{k+1})\}.$$
4. Add the new supporting point:
$$A^{k+1} \leftarrow A^k \cup \{(x^{k+1}, t^{k+1})\}.$$

5. Increment iteration counter:

$$k \leftarrow k + 1.$$

Output: Optimal solution $(x^\star, t^\star)$ and objective value $g(x^\star, t^\star)$.

5.2 Determination of the Maximum Feasible Separation Distance

In addition to solving the Silent Disco Glamping Network problem for a fixed minimum separation distance δ, an important practical question is to determine the largest value of δ for which the problem remains feasible given a fixed number of sites selection.

5.2.1 Feasibility of the Separation Constraint

For a given value of δ, the minimum separation constraint enforces

$$x_i + x_j \leq 1 \quad \forall (i,j) \quad s.t. \quad d_{ij} < \delta$$

As δ increases, the number of such constraints increases monotonically, thereby shrinking the feasible region. Therefore, if the problem is infeasible for some δ, it remains infeasible for all larger values of δ.

5.2.2 Feasibility Checking Problem

For a candidate value of δ, feasibility is assessed by solving the following problem to determine whether a feasible solution exists without optimising the objective function.

$$\begin{aligned} \text{find} \quad & x \\ \text{s.t.} \quad & \sum_{i=1}^{m} x_i = k, \\ & \sum_{i=1}^{m} a_i x_i \geq N, \\ & x_i + x_j \leq 1, \quad \forall (i,j) : d_{ij} < \delta, \\ & x_i \in \{0,1\}, \quad i = 1, \ldots, m. \end{aligned}$$

5.2.3 Binary Search Procedure

To compute the maximum feasible separation distance $\delta^\star$, a binary search algorithm is employed over the interval

$$[0, \delta_{\max}], \quad \text{where } \delta_{\max} = \max_{i,j} d_{ij}.$$

The procedure is as follows:

1. Initialise the lower and upper bounds as

$$\delta_{\text{low}} = 0, \qquad \delta_{\text{high}} = \delta_{\max}.$$

2. At each iteration, test the midpoint
$$\delta = \frac{\delta_{\text{low}} + \delta_{\text{high}}}{2}.$$
3. If the feasibility problem admits a solution, update
$$\delta_{\text{low}} \leftarrow \delta.$$
4. Otherwise, update
$$\delta_{\text{high}} \leftarrow \delta.$$
5. Terminate when
$$\delta_{\text{high}} - \delta_{\text{low}}$$
is below a prescribed tolerance.

The final value $\delta^\star = \delta_{\text{low}}$ is reported as the maximum feasible separation distance.

6 Computation and Results

6.1 Experimental Setup

The set of candidate locations is constructed from airstrip data in Australia, restricted to small and medium airports to reflect the remoteness and exclusivity required for high-end glamping sites. All geographic distances are computed using the Haversine distance, measured in kilometres.

Choice of parameters:

- Number of sites: $k = 10$ represent a small number of exclusive glamping locations suitable for a premium tourism offering.
- Minimum separation distance: $\delta = 1000$ to enforce strong spatial exclusivity between selected sites
- fixed cost: $w_i = \alpha d_i^h + \beta \text{size\_penalty}_i + \epsilon_i$
  1. d_i^h: Distance to the nearest capital (proxy for construction and maintenance cost)
  2. size_penalty: 1 for smaller airports and 0.5 for medium airports
  3. ϵ: Small random noise to avoid symmetry. 4. α, β: weights of each factor. Here, I chose $\alpha = 5, \beta = 3$
- Capacity a_i: small airport can have 15 tents, medium airport can have 40 tents
- Number of tents required: $N = \gamma \frac{k}{m} \sum_{i=1}^{m} a_i$ for $\gamma = 0.6$ to ensure that the total capacity requirement scales proportionally with the number of selected sites while remaining feasible (Here, $\frac{\sum_{i=1}^{m} a_i}{m}$ gives average number of tents per location).

All these parameters can be tuned to make better decisions.

6.2 Solution Procedure

The problem is solved using the Outer Approximation algorithm developed in earlier sections and implemented in Python using Gurobi as the mixed-integer solver. The algorithm converges in a small number of iterations, demonstrating the effectiveness of the algorithm.

6.3 Results

Figure 1 illustrates the optimal selection of glamping locations obtained by the Outer Approximation algorithm. Selected locations are highlighted in red, while unselected candidates are shown in blue.

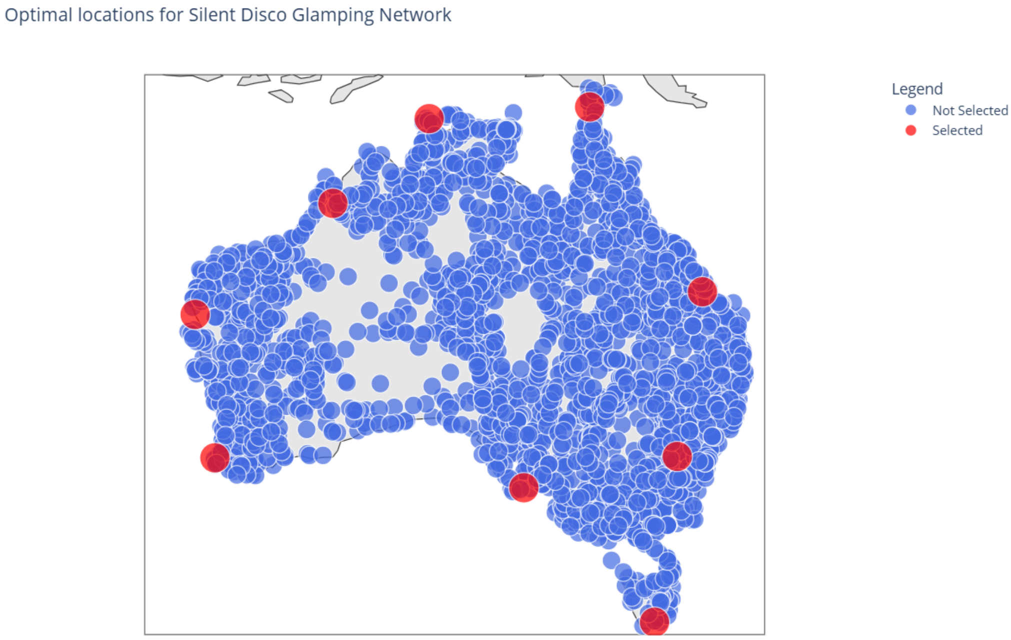

Figure 1: Optimal locations for the Silent Disco Glamping Network

The selected locations are well-dispersed across Australia, confirming that the minimum separation constraint is active and effective. The solution balances geographic diversity with cost and capacity considerations, achieving a high objective value while respecting all operational constraints.

6.4 Maximum Feasible Separation Distance

Using a binary search combined with a mixed-integer feasibility check, the largest value δ^* for which the problem remains feasible is identified as 1321.1 km holding all other constraints constant.

7 Conclusion

The project adopts the concavity results of the max-sum problem and applies them to the max-mean setting via a perspective map. Compared to existing exact approaches based on semidefinite programming, the proposed method offers improved scalability and practical applicability. The framework is flexible and can be extended to other facility location and network design problems involving fractional objectives.

Regarding the project, the selected airstrips are located along the coast of Australia where most capital cities are located since the optimisation algorithm finds location close to the Capital city to minimise logistic cost. However, the target of the startup is to find locations where guests feel completely isolated from the rest of the world, the camping site near those populated area is not ideal. The future work would incorporate an additional constraint where the location cannot be within a certain distance from the capital city, further reinforcing the isolation requirement.

8 References

Garraffa, M., Della Croce, F., & Salassa, F. (2017). An exact semidefinite programming approach for the max–mean dispersion problem. *Journal of Combinatorial Optimization, 34*(1), 71–93. https://doi.org/10.1007/s10878-016-0065-1

Spiers, S., Bui, H. T., & Loxton, R. (2023). An exact cutting plane method for the Euclidean max-sum diversity problem. *European Journal of Operational Research, 311*(2), 444–454. https://doi.org/10.1016/j.ejor.2023.03.010