MTH 5315 NUMERICAL METHODS FOR PDE

MIDTERM 1

By: Max Le

Overview

- Problem Statement

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- Donor Cell Upwinding scheme

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- Corner Transport Upwinding scheme

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- Strang splitting and Min Mod limiter

Problem Statement

- 2D advection equation

$$q_t + u(x,y)q_x + v(x,y)q_y = 0$$

Problem Statement

- 2D advection equation

$$q_t + u(x,y)q_x + v(x,y)q_y = 0$$

$$u(x,y) = 2y \text{ and } v(x,y) = -2x \text{ with } x \in [-1,1], y \in [-1,1]$$

Problem Statement (cont)

- Initial conditions:

$$q(x,y,0) = \begin{cases} 1 & 0.1 < x < 0.6 \text{ and } -0.25 < y < 0.25 \\ 1 - \sqrt{(x+0.45)^2 + y^2}/0.35 & \sqrt{(x+0.45)^2 + y^2} < 0.35 \\ 0 & \text{otherwise} \end{cases}$$

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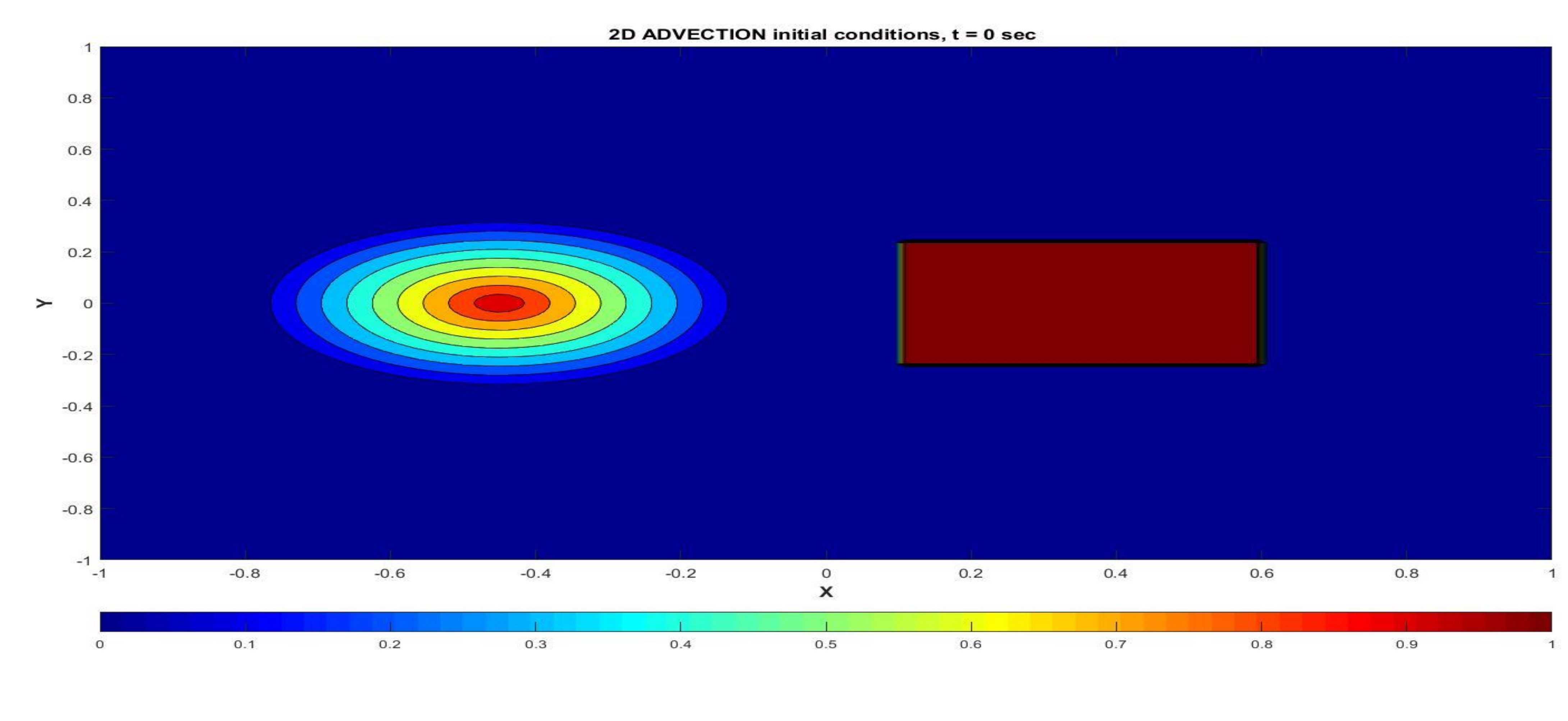
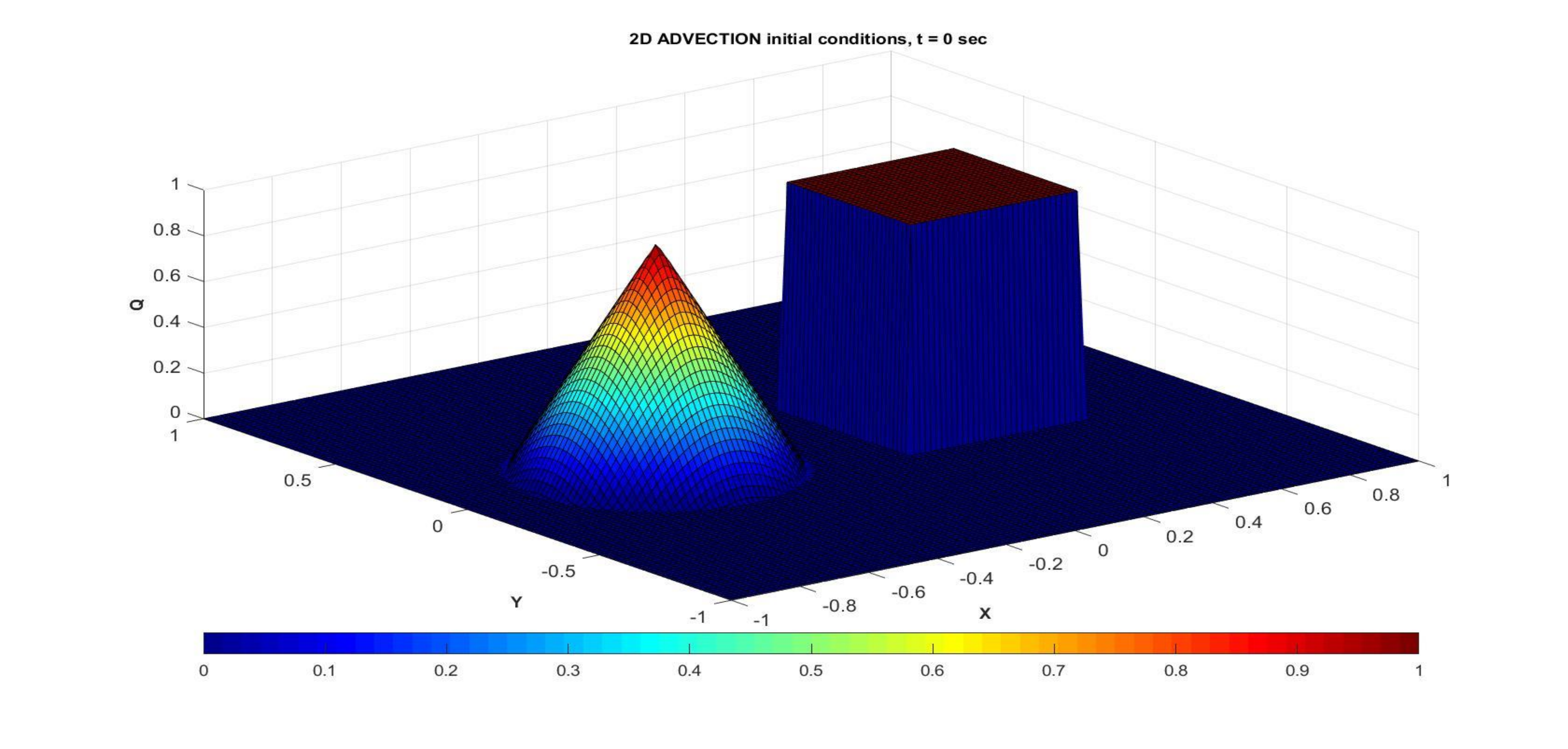
$$\Delta x = \Delta y = 1/64 \text{ and } \Delta t = 0.4\Delta x \text{ and } t_f = \pi$$

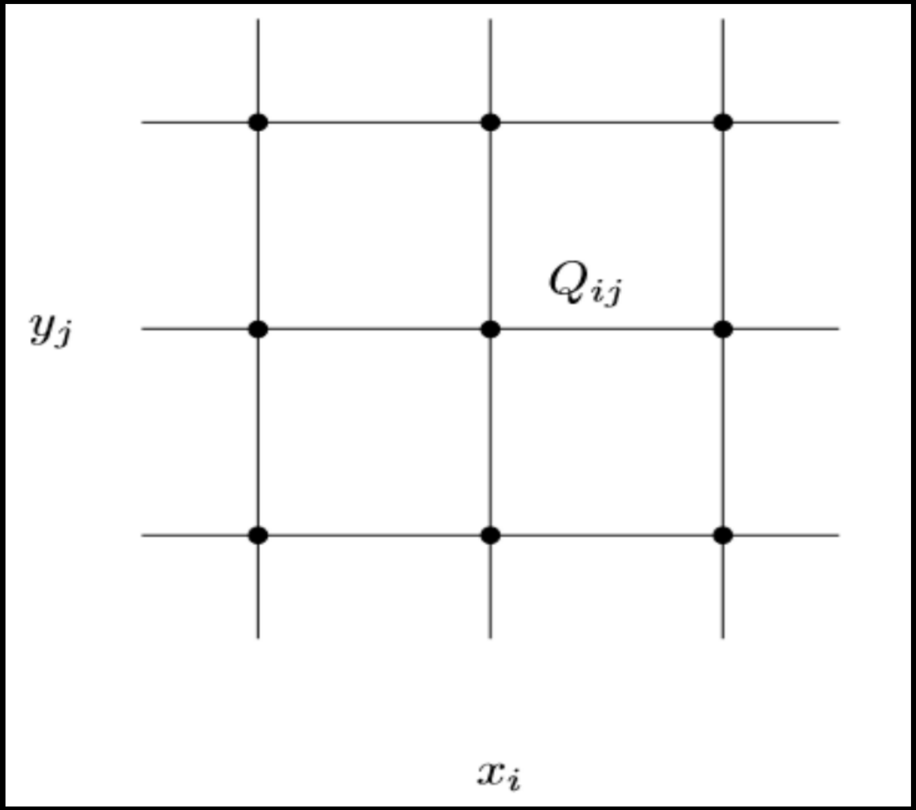
Problem Statement (cont)



Problem Statement (cont)

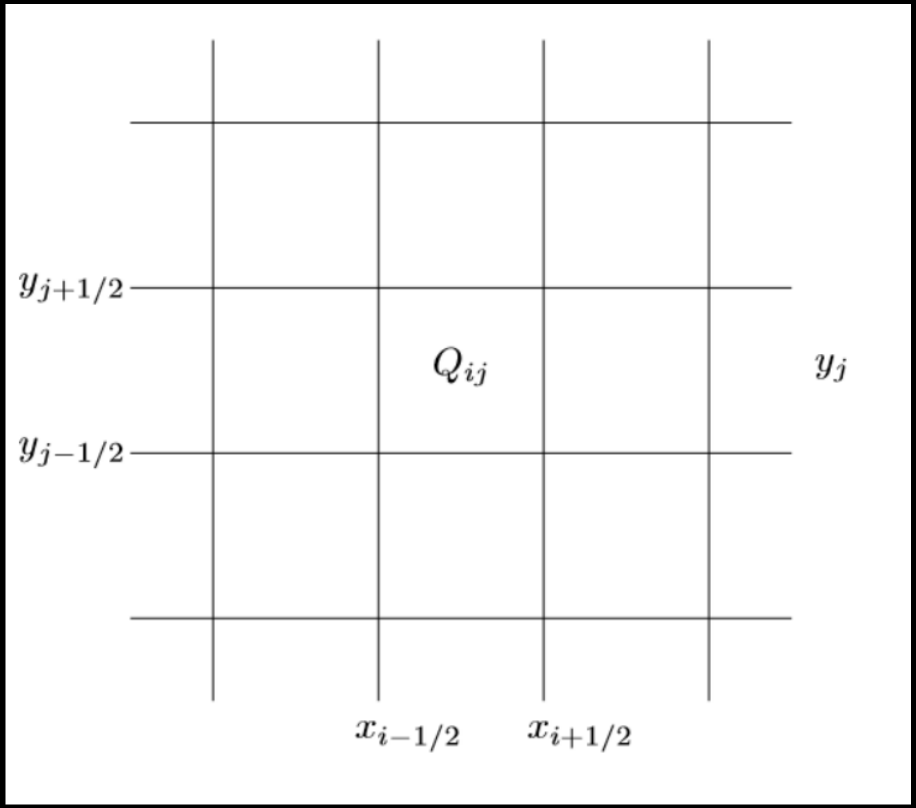


Computational Grid



Flow variables are stored at the same location => Finite Difference Grid

Computational Grid



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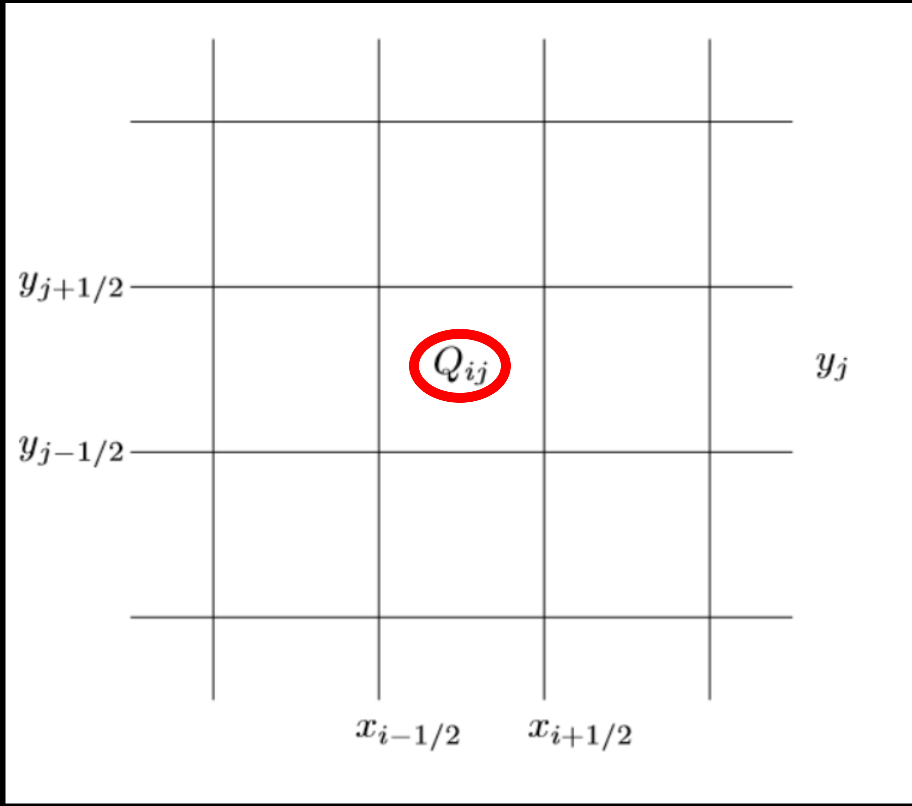
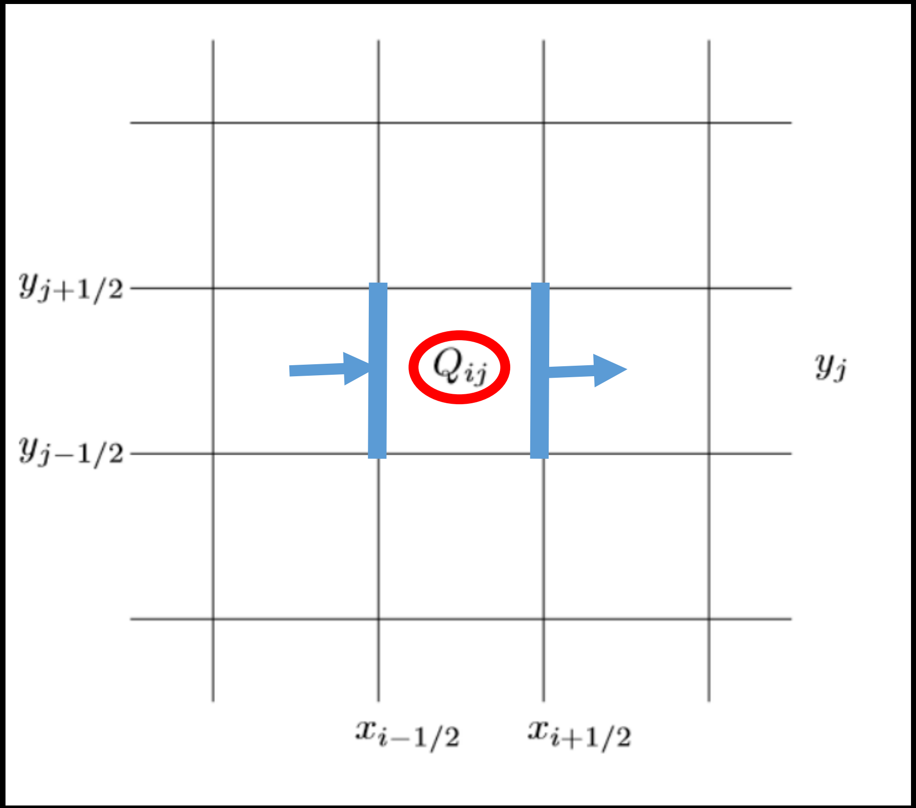


Flow variables are stored at the same location => **FINITE DIFFERENCE GRID**

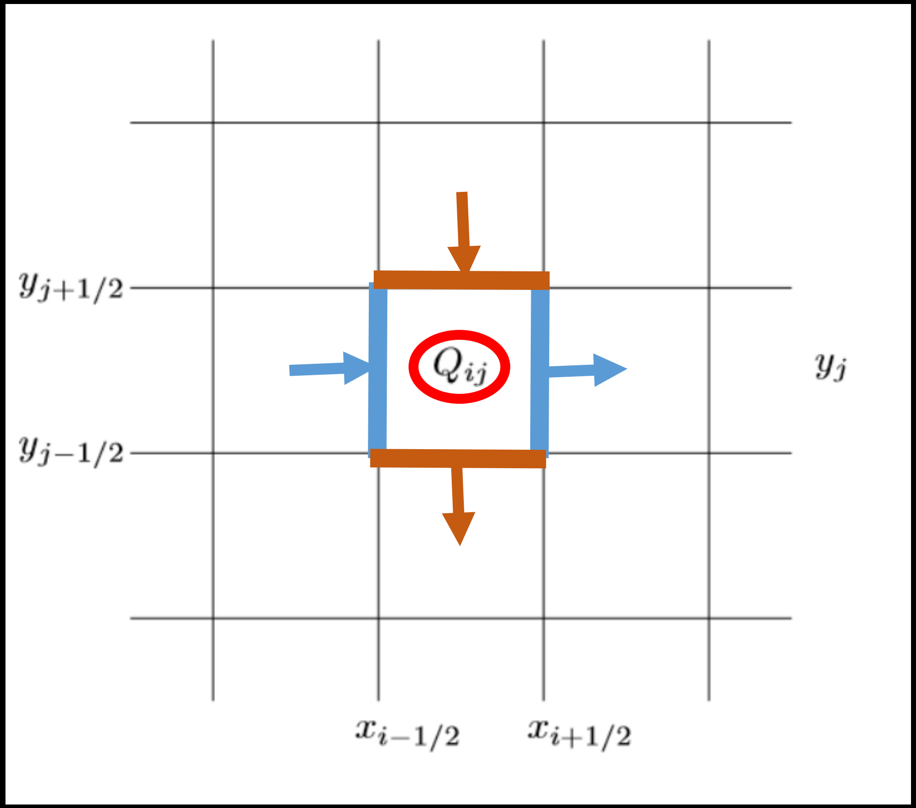
Computational Grid



Substance Q is defined at center

Velocities are defined on the edge

Computational Grid



Substance Q is defined at center

Velocities are defined on the edge

=> **STAGGERED GRID**

(IR)ROTATIONAL FLOW

$$\nabla \times \vec{V} = 0 \text{ is irrotational}$$

$$\nabla \times \vec{V} = \left(\frac{\partial}{\partial x} V_y - \frac{\partial}{\partial y} V_x \right) \hat{k}$$

$$\nabla \times \vec{V} = \left(\frac{\partial}{\partial x} (-2x) - \frac{\partial}{\partial y} (2y) \right) \hat{k}$$

$$\nabla \times \vec{V} = (-2 - 2)\hat{k} = -4\hat{k} \neq 0.$$

GENERAL SCHEME FOR SPATIAL FIELDS

$$
\begin{aligned}
Q_{i,j}^{n+1} &= Q_{i,j} - \frac{\Delta t}{\Delta x}(A^+\Delta Q_{i-1/2,j} + A^-\Delta Q_{i+1/2,j}) - \frac{\Delta t}{\Delta y}(B^+\Delta Q_{i,j-1/2} + B^-\Delta Q_{i,j+1/2}) \\
&\quad - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j}) - \frac{\Delta t}{\Delta y}(\tilde{G}_{i,j+1/2} - \tilde{F}_{i,j-1/2})
\end{aligned}
$$

DONOR CELL TRANSPORT UPWINDING

$$
\begin{aligned}
Q_{i,j}^{n+1} = Q_{i,j} - \frac{\Delta t}{\Delta x}(A^{+}\Delta Q_{i-1/2,j} + A^{-}\Delta Q_{i+1/2,j}) - \frac{\Delta t}{\Delta y}(B^{+}\Delta Q_{i,j-1/2} + B^{-}\Delta Q_{i,j+1/2}) \\
\cancel{- \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j})} \cancel{- \frac{\Delta t}{\Delta y}(\tilde{G}_{i,j+1/2} - \tilde{F}_{i,j-1/2})}
\end{aligned}
$$

DONOR CELL TRANSPORT UPWINDING

$$Q_{i,j}^{n+1} = Q_{i,j} - \frac{\Delta t}{\Delta x}(A^+ \Delta Q_{i-1/2,j} + A^- \Delta Q_{i+1/2,j}) - \frac{\Delta t}{\Delta y}(B^+ \Delta Q_{i,j-1/2} + B^- \Delta Q_{i,j+1/2})$$

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$$Q_{i,j}^{n+1} = Q_{i,j} - \frac{\Delta t}{\Delta x}(A^+\Delta Q_{i-1/2,j} + A^-\Delta Q_{i+1/2,j}) - \frac{\Delta t}{\Delta y}(B^+\Delta Q_{i,j-1/2} + B^-\Delta Q_{i,j+1/2})$$

$$A^{+/-}\Delta Q_{i-1/2,j} = u^{+/-}(Q_{i,j} - Q_{i-1,j})$$

$$B^{+/-}\Delta Q_{i,j-1/2} = v^{+/-}(Q_{i,j} - Q_{i,j-1})$$

DONOR CELL TRANSPORT UPWINDING

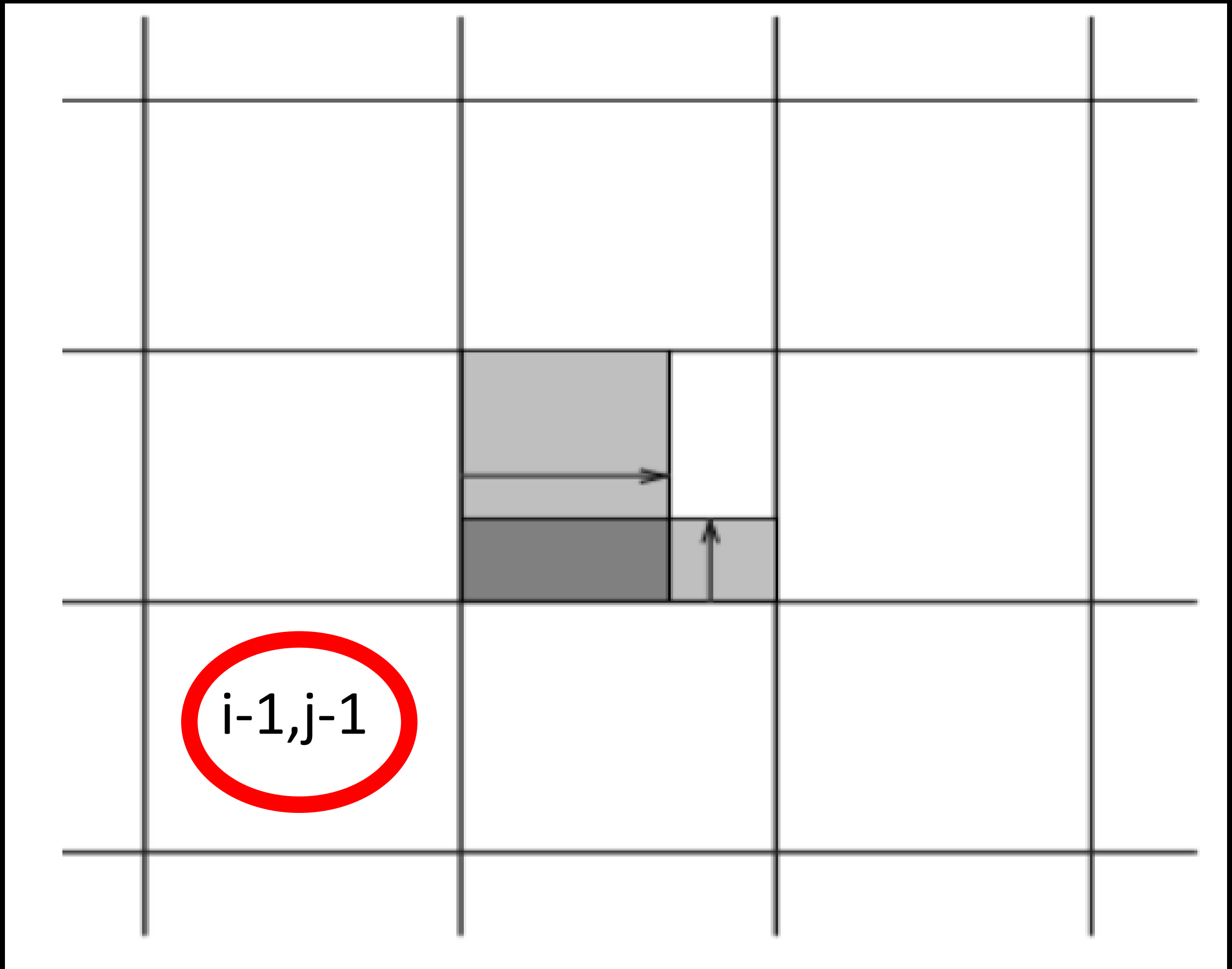
$$Q_{i,j}^{n+1} = Q_{i,j} - \frac{\Delta t}{\Delta x}(A^+ \Delta Q_{i-1/2,j} + A^- \Delta Q_{i+1/2,j}) - \frac{\Delta t}{\Delta y}(B^+ \Delta Q_{i,j-1/2} + B^- \Delta Q_{i,j+1/2})$$

$$A^{+/-} \Delta Q_{i-1/2,j} = u^{+/-}(Q_{i,j} - Q_{i-1,j})$$

$$B^{+/-} \Delta Q_{i,j-1/2} = v^{+/-}(Q_{i,j} - Q_{i,j-1})$$

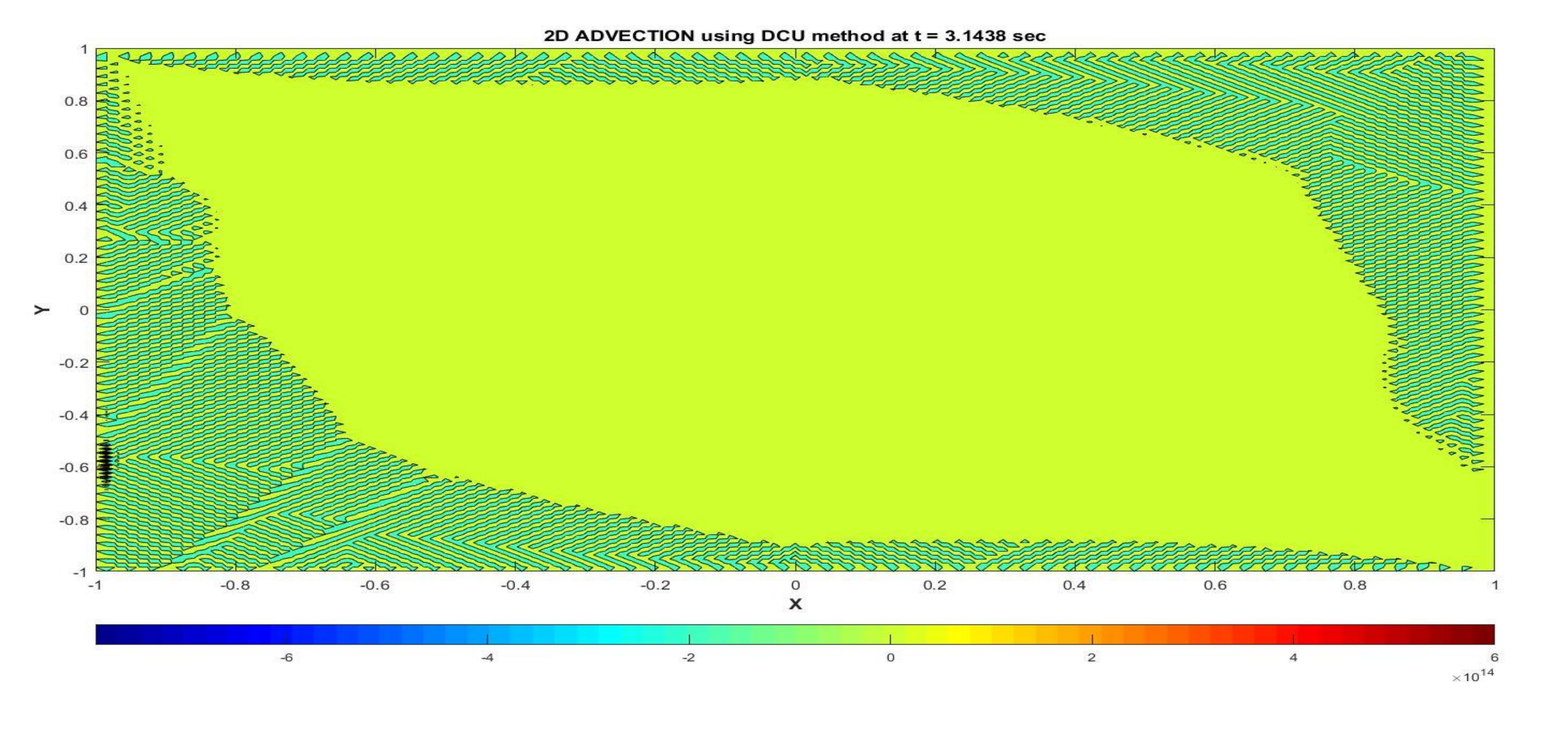
$$\begin{aligned} u^+ &= max(0, u) \\ u^- &= min(0, u) \\ v^+ &= max(0, v) \\ v^- &= min(0, v) \end{aligned}$$

DONOR CELL TRANSPORT UPWINDING

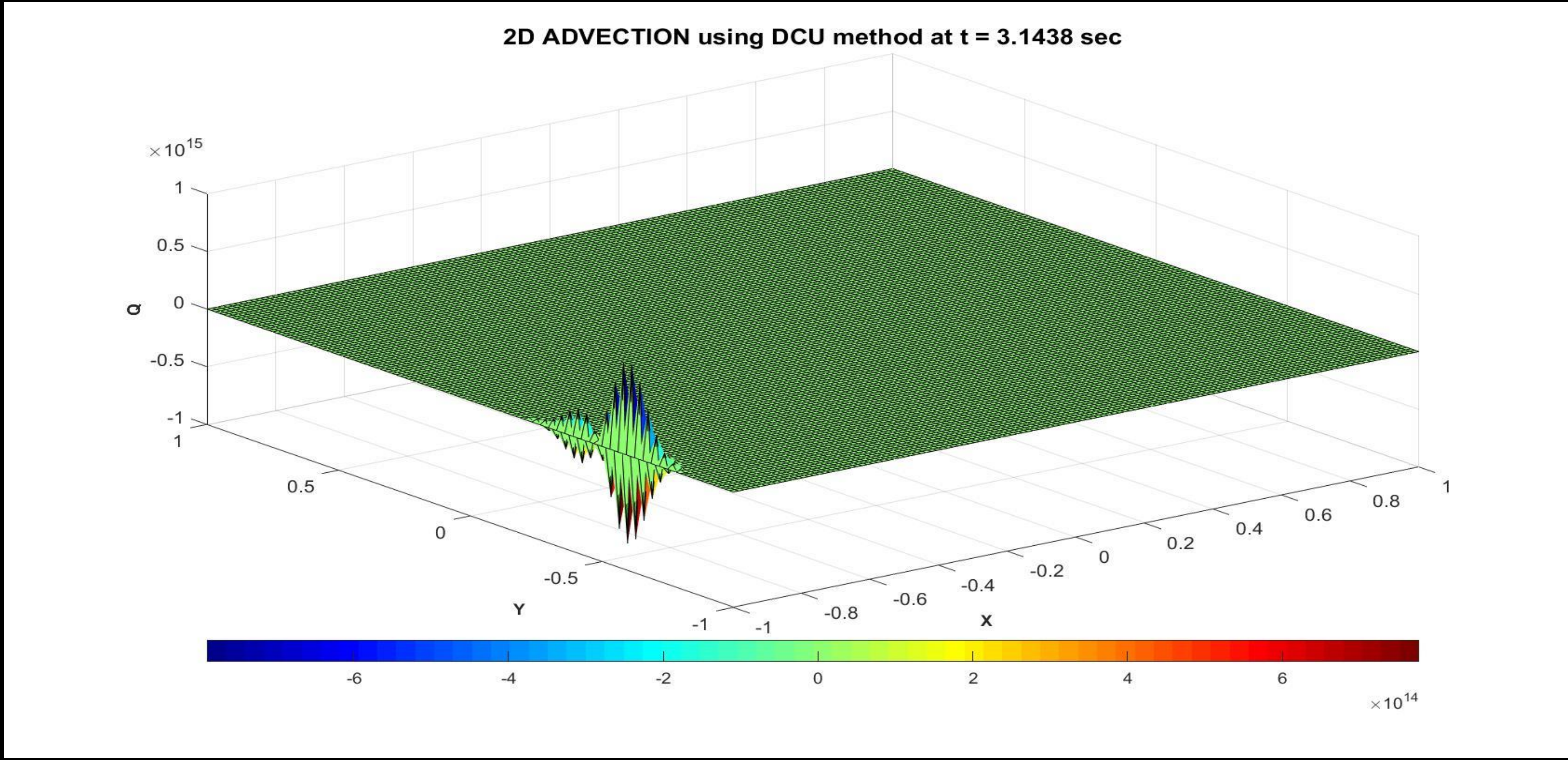


DCU- show video

DONOR CELL TRANSPORT UPWINDING



DONOR CELL TRANSPORT UPWINDING



DONOR CELL TRANSPORT UPWINDING

$$\left|\frac{u\Delta t}{\Delta x}\right| + \left|\frac{v\Delta t}{\Delta y}\right| \leq 1$$

CFL condition

Base on our problem, $\Delta t = 0.4\Delta x$ and $\Delta x = \Delta y$, this implies:

$$\left|\frac{0.4u\Delta x}{\Delta x}\right| + \left|\frac{0.4v\Delta x}{\Delta x}\right| \leq 1$$

DONOR CELL TRANSPORT UPWINDING

$$\left|\frac{u\Delta t}{\Delta x}\right| + \left|\frac{v\Delta t}{\Delta y}\right| \leq 1$$

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$$\left|\frac{0.4u\Delta x}{\Delta x}\right| + \left|\frac{0.4v\Delta x}{\Delta x}\right| \leq 1$$

$$|0.4u| + |0.4v| \leq 1$$

DONOR CELL TRANSPORT UPWINDING

$$\left|\frac{u\Delta t}{\Delta x}\right| + \left|\frac{v\Delta t}{\Delta y}\right| \leq 1$$

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$$\left|\frac{0.4u\Delta x}{\Delta x}\right| + \left|\frac{0.4v\Delta x}{\Delta x}\right| \leq 1$$

$$|0.4u| + |0.4v| \cancel{\leq} 1$$

DONOR CELL TRANSPORT UPWINDING

$$
\begin{aligned}
2\left(\frac{\Delta t}{\Delta x}\right) + 2\left(\frac{\Delta t}{\Delta x}\right) &\leq 1 \\
4\left(\frac{\Delta t}{\Delta x}\right) &\leq 1 \\
\frac{\Delta t}{\Delta x} &\leq \frac{1}{4} \\
\frac{\Delta t}{1/64} &\leq \frac{1}{4}
\end{aligned}
$$

DONOR CELL TRANSPORT UPWINDING

$$2(\frac{\Delta t}{\Delta x}) + 2(\frac{\Delta t}{\Delta x}) \leq 1$$

$$4(\frac{\Delta t}{\Delta x}) \leq 1$$

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{4}$$

$$\frac{\Delta t}{1/64} \leq \frac{1}{4}$$

Therefore, $\Delta t \leq \frac{1}{256} \leq 0.00390625$ second, which is very small.

DONOR CELL TRANSPORT UPWINDING

Dissipation

$$|\rho|^2 = \sqrt{1^2 + (R_x(1 - e^{-i\zeta\Delta x}))^2 + (R_y(1 - e^{-i\eta\Delta y}))^2}$$

High frequency modes

$$|\rho|^2 = \sqrt{1 + (\frac{2 * 0.00390625}{1/64}(1 - -1))^2 + (\frac{2 * 0.00390625}{1/64}(1 - -1))^2}$$
$$= \sqrt{3}$$

DONOR CELL TRANSPORT UPWINDING

Dissipation

$$|\rho|^2 = \sqrt{1^2 + (R_x(1 - e^{-i\zeta\Delta x}))^2 + (R_y(1 - e^{-i\eta\Delta y}))^2}$$

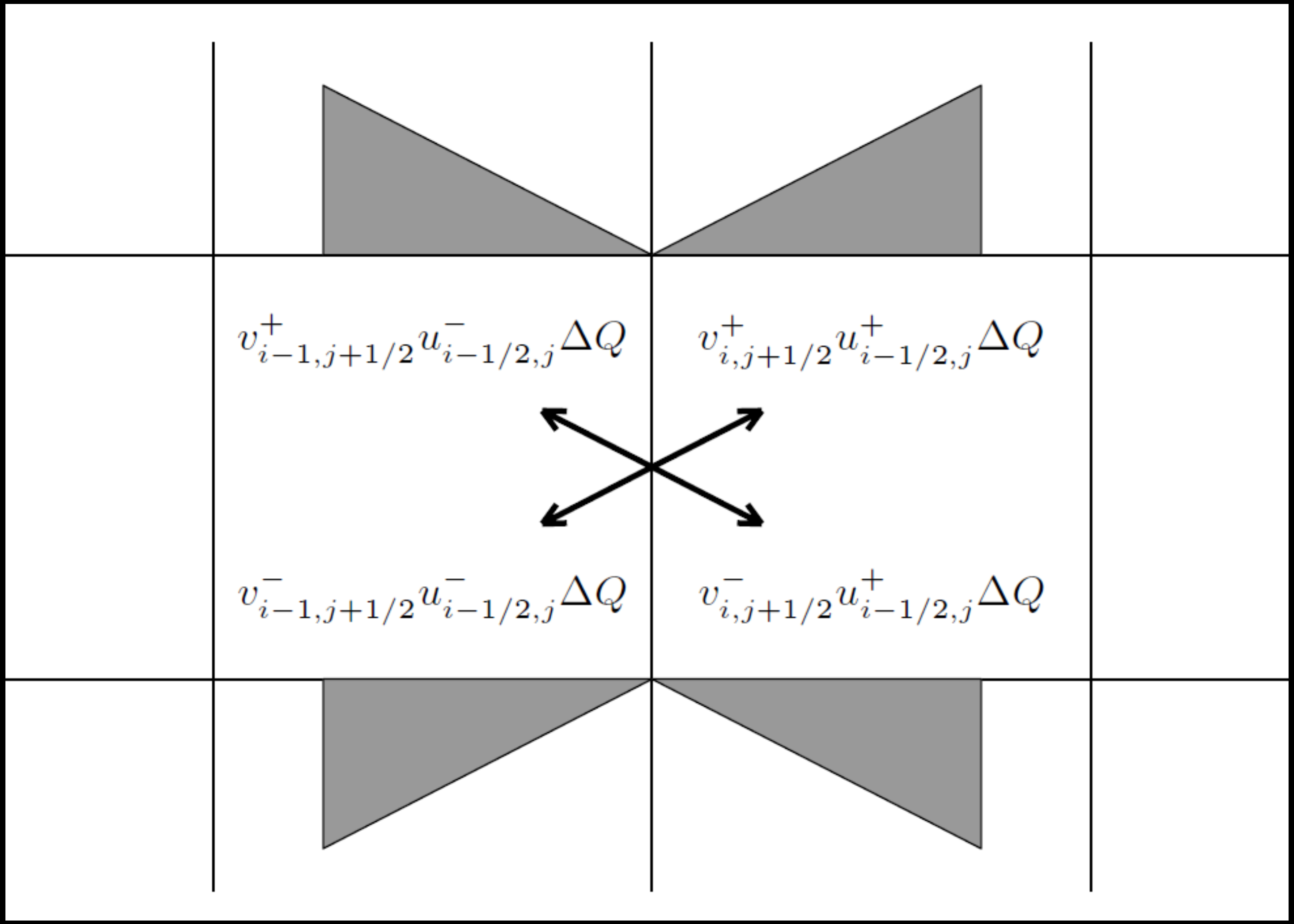
Low frequency modes

$$|\rho|^2 = \sqrt{1 + (\frac{2 * 0.00390625}{1/64}(1-1))^2 + (\frac{2 * 0.00390625}{1/64}(1-1))^2}$$
$$= 1$$

CORNER TRANSPORT UPWINDING

$$Q_{i,j}^{n+1} = Q_{i,j} - \frac{\Delta t}{\Delta x}(A^+ \Delta Q_{i-1/2,j} + A^- \Delta Q_{i+1/2,j}) - \frac{\Delta t}{\Delta y}(B^+ \Delta Q_{i,j-1/2} + B^- \Delta Q_{i,j+1/2})$$
$$- \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2,j} - \tilde{F}_{i-1/2,j}) - \frac{\Delta t}{\Delta y}(\tilde{G}_{i,j+1/2} - \tilde{F}_{i,j-1/2})$$

CORNER TRANSPORT UPWINDING



CORNER TRANSPORT UPWINDING

TOP RIGHT

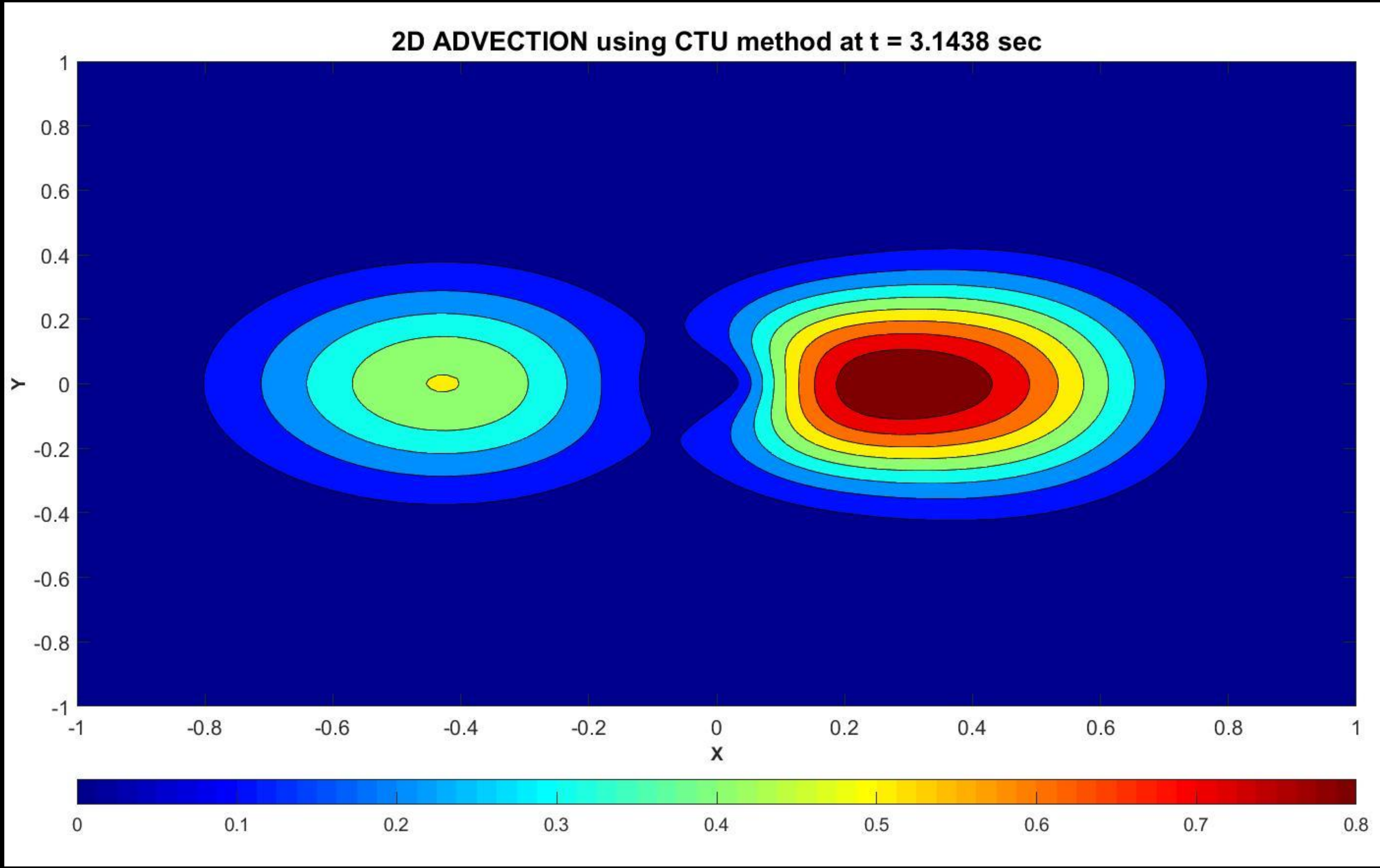
$$\tilde{F}_{i+1/2,j} := -\frac{1}{2}\frac{\Delta t}{\Delta y}u^{+}_{i+1/2,j}v^{+}_{i,j-1/2}(Q_{i,j} - Q_{i,j-1})$$

BOTTOM RIGHT

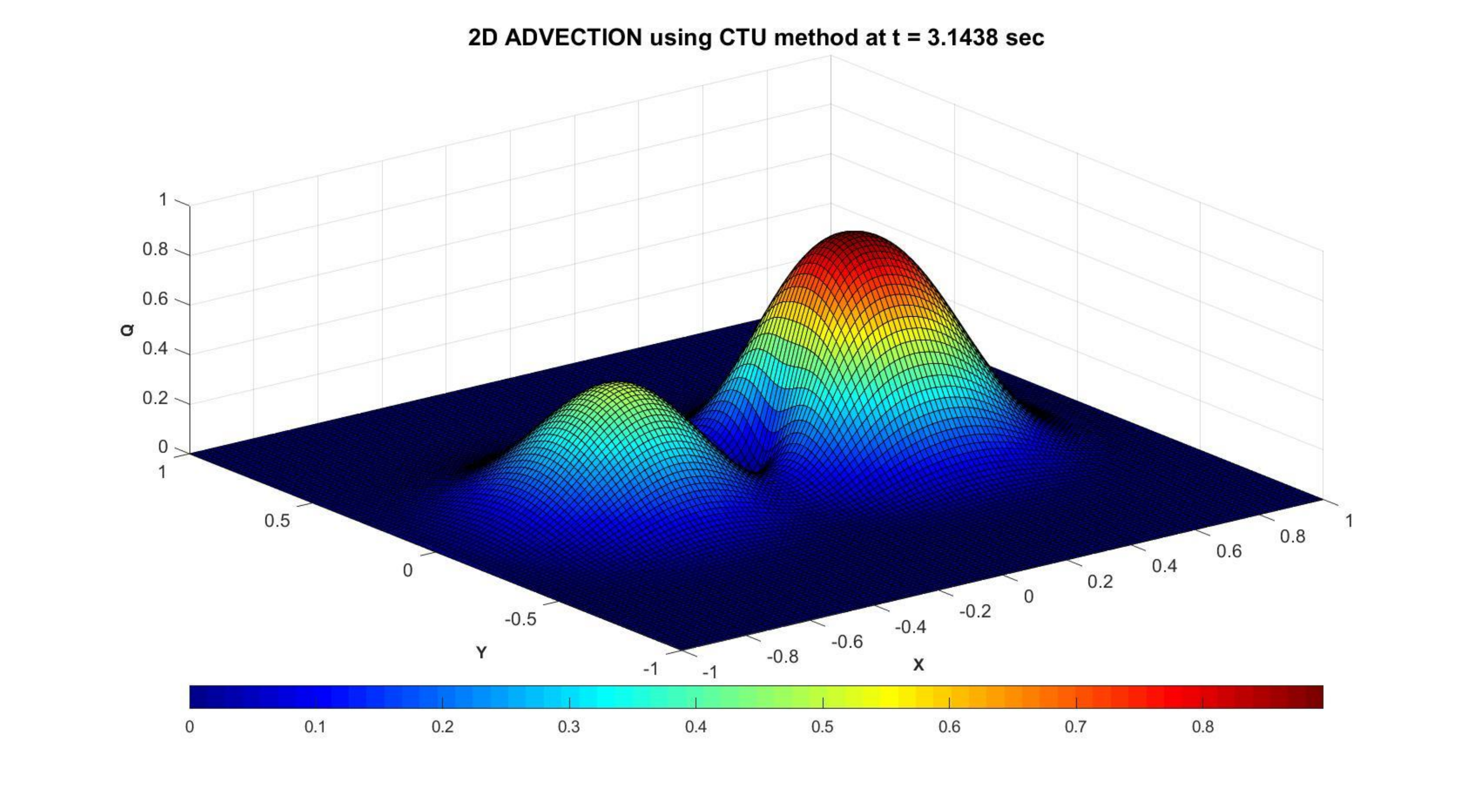
$$\tilde{F}_{i+1/2,j-1} := -\frac{1}{2}\frac{\Delta t}{\Delta y}u^{+}_{i+1/2,j-1}v^{-}_{i,j-1/2}(Q_{i,j} - Q_{i,j-1})$$

CTU-show video

CORNER TRANSPORT UPWINDING



CORNER TRANSPORT UPWINDING



CORNER TRANSPORT UPWINDING

$$max\left[\left|\frac{u\Delta t}{\Delta x}\right|,\left|\frac{v\Delta t}{\Delta y}\right|\right]\leq 1$$

CFL condition

Base on our problem, $\Delta t = 0.4\Delta x$ and $\Delta x = \Delta y$, this implies:

$$max\,[0.4(2), 0.4(2)] \leq 1$$

$$2(0.4) \leq 1$$

$$0.8 \leq 1 \Rightarrow \text{which is always true}$$

CORNER TRANSPORT UPWINDING

Dissipation

$$\rho(\zeta,\eta) = \left[1 - R_x(1 - e^{-i\zeta\Delta x})\right]\left[1 - R_y(1 - e^{-i\eta\Delta y})\right]$$

High frequency modes

$$\begin{aligned} |\rho|^2 &= \sqrt{(1 - 0.8(1 - -1))(1 - 0.8(1 - -1))} \\ &= \sqrt{1.6} \\ &= 0.4 \end{aligned}$$

CORNER TRANSPORT UPWINDING

Dissipation

$$\rho(\zeta, \eta) = \left[1 - R_x(1 - e^{-i\zeta\Delta x})\right]\left[1 - R_y(1 - e^{-i\eta\Delta y})\right]$$

Low frequency modes

$$\begin{aligned}
|\rho|^2 &= \sqrt{(1 - 0.4(1-1))(1 - 0.4(1-1))} \\
&= \sqrt{1} \\
&= 1
\end{aligned}$$

STRANG SPLITTING

$$Q_t + uQ_x + vQ_y = 0$$

STRANG SPLITTING

$$Q_t + uQ_x + vQ_y = 0$$

$$\begin{cases} Q_t + uQ_x = 0 \\ Q_t^* + vQ_y^* = 0 \end{cases}$$

This is **Godunov's splitting**

STRANG SPLITTING

$$Q_t + uQ_x + vQ_y = 0$$

$$\begin{cases} Q_t + uQ_x = 0, \text{at dt}/2 \\ Q_t^* + vQ_y^* = 0, \text{at dt} \\ Q_t^{**} + uQ_x^{**} = 0, \text{at dt}/2 \end{cases}$$

This is **Strang's splitting**

LIMITER SCHEME

$$F^n_{i-1/2} = F_L(Q_{i-1}, Q_i) + \phi^n_{i-1/2}[F_H(Q_{i-1}, Q_i) - F_L(Q_{i-1}, Q_i)]$$

$$F_L(Q_{i-1}, Q_i) = u^+(Q_i - Q_{i-1})$$

LIMITER SCHEME

$$F^n_{i-1/2} = F_L(Q_{i-1}, Q_i) + \phi^n_{i-1/2}[F_H(Q_{i-1}, Q_i) - F_L(Q_{i-1}, Q_i)]$$

$$F_H(Q_{i-1}, Q_i) = (A^- Q^n_i + A^+ Q^n_{i-1}) + \frac{1}{2}|A|(I - \frac{\Delta t}{\Delta x}|A|)(Q^n_i - Q^n_{i-1})$$

LIMITER SCHEME

$$F^n_{i-1/2} = F_L(Q_{i-1}, Q_i) + \phi^n_{i-1/2}[F_H(Q_{i-1}, Q_i) - F_L(Q_{i-1}, Q_i)]$$

$$\phi(\theta) = max(0, min(\theta, 1))$$

LIMITER SCHEME

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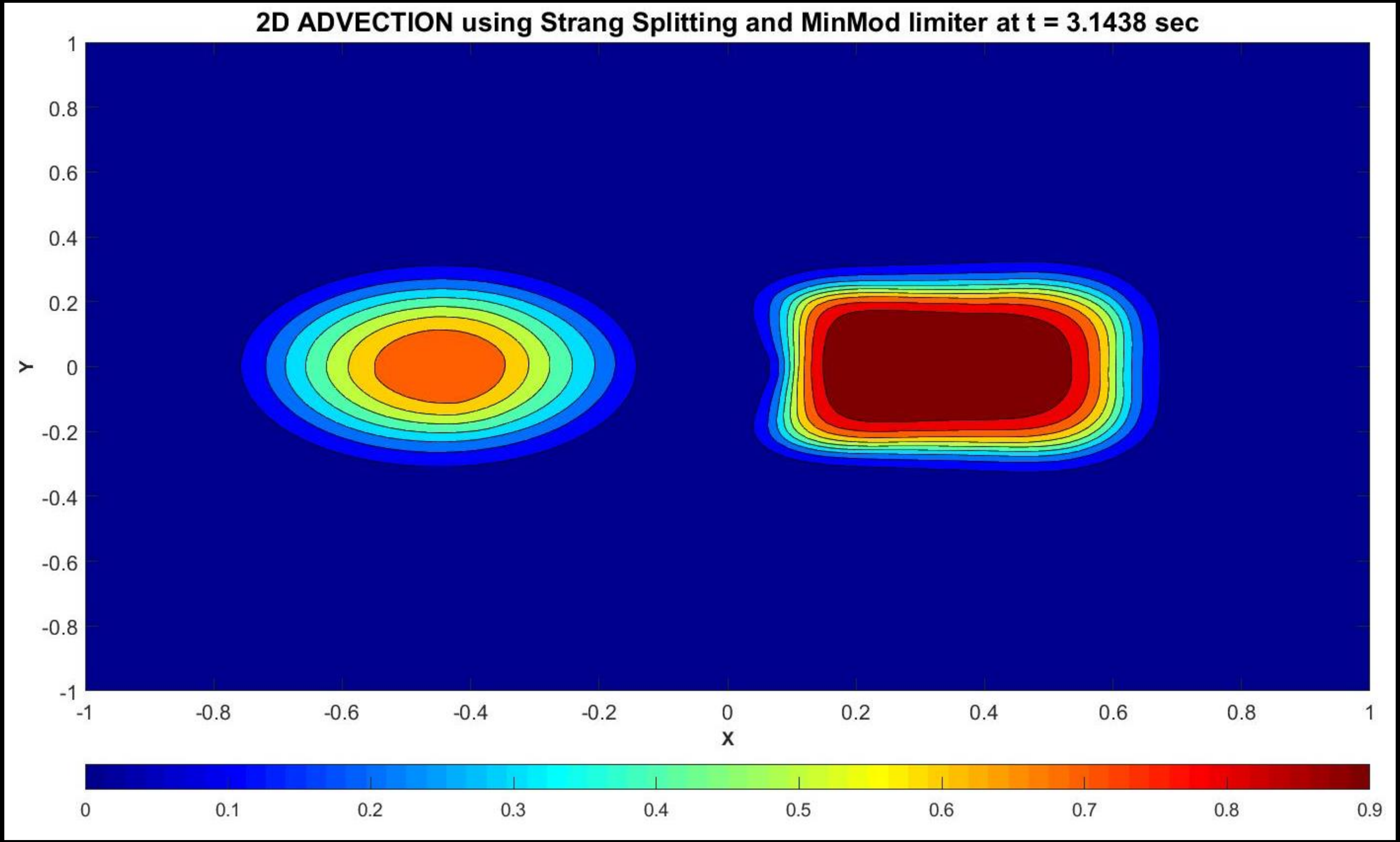
$$\phi(\theta) = max(0, min(\theta, 1))$$

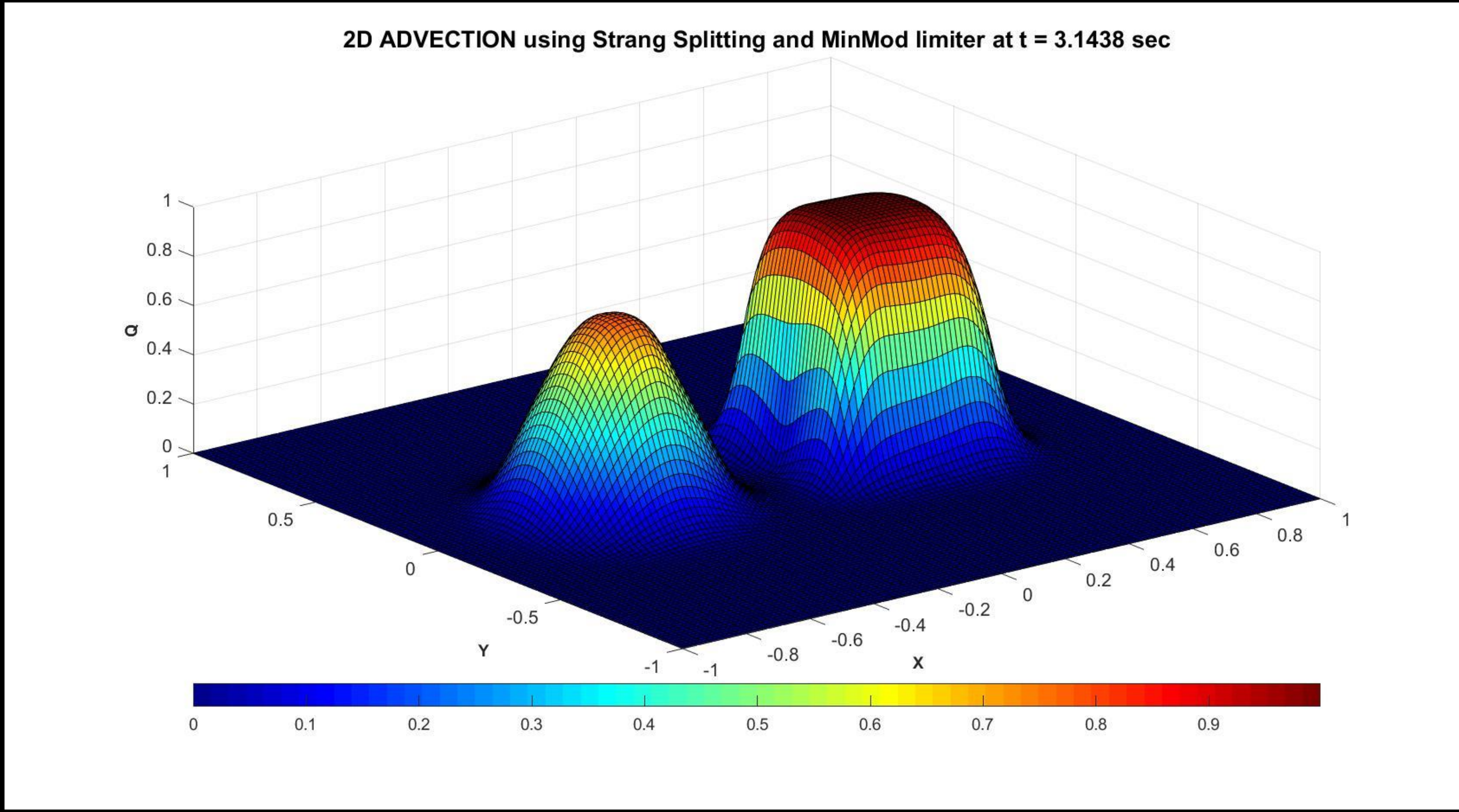
$$\theta^n_{i-1/2} = \frac{\Delta Q_{I-1/2}n}{\Delta Q_{i-1/2}n}$$

Strang,MinMod, show video

STRANG SPLITTING AND MIN MOD



STRANG SPLITTING AND MIN MOD



STRANG SPLITTING AND MIN MOD

$$
\begin{aligned}
E &= \frac{\Delta t^2}{2}(AB - BA) \\
&= \frac{\Delta t^2}{2}\left(\left(u\frac{\partial}{\partial x}v\frac{\partial}{\partial y}\right) - \left(v\frac{\partial}{\partial y}u\frac{\partial}{\partial x}\right)\right)
\end{aligned}
$$

STRANG SPLITTING AND MIN MOD

$$
\begin{aligned}
E &= \frac{\Delta t^2}{2}(AB - BA) \\
&= \frac{\Delta t^2}{2}\left(\left(u\frac{\partial}{\partial x}v\frac{\partial}{\partial y}\right) - \left(v\frac{\partial}{\partial y}u\frac{\partial}{\partial x}\right)\right)
\end{aligned}
$$

Recall

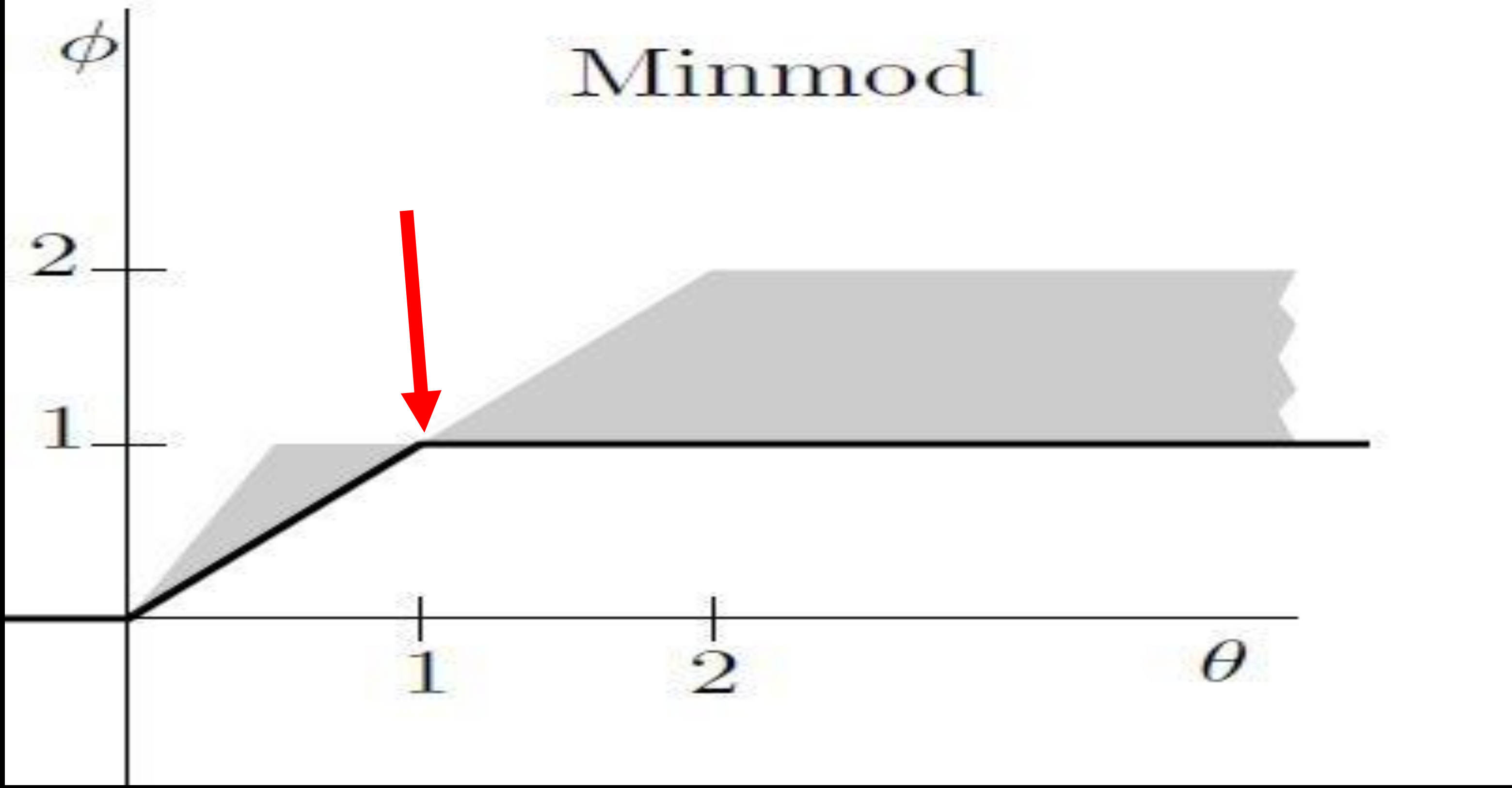
$$u(x, y) = 2y \text{ and } v(x, y) = -2x$$

STRANG SPLITTING AND MIN MOD

$$E = \frac{\Delta t^2}{2}(AB - BA)$$
$$= \frac{\Delta t^2}{2}\left(\left(u\frac{\partial}{\partial x}v\frac{\partial}{\partial y}\right) - \left(v\frac{\partial}{\partial y}u\frac{\partial}{\partial x}\right)\right)$$
$$= \frac{\Delta t^2}{2}\left(\left(u\frac{\partial}{\partial x}v\frac{\partial}{\partial y}\right) - \left(u\frac{\partial}{\partial x}v\frac{\partial}{\partial y}\right)\right)$$
$$= 0$$

STRANG SPLITTING AND MIN MOD

Dissipation



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- R. J. LeVeque. Finite Volume Methods for Hyperbolic Problems. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2002. doi: 10.1017/CBO9780511791253.

QUESTIONS?