Pricing and profit testing of life insurance

Thesis submitted at the University of Leicester
in partial fulfilment of the requirements for
the MSc degree in Financial Mathematics and Computation

by

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September 2012

Contents

Declaration

All sentences or passages quoted in this project dissertation from other people's work have been specifically acknowledged by clear cross referencing to author, work and page(s). I understand that failure to do this amounts to plagiarism and will be considered grounds for failure in this module and the degree examination as a whole.

Name:

Lemin Wu

Signed:

Date:

Abstract

A summary of the thesis in about 200 words.

Introduction

In this thesis we consider the work of Gauss [1, ch 2] and Hilbert [**?**] on the subject of the title.

1. why doing the project
2. what have I done
3. what have I achieve
4. what work can be done in extension

Convensions

The following variables are used though out the project

$\ddot{a}$

$\ddot{A}$

$\ddot{A}^{(m)}_{[x]}$

$\ddot{a}^{(m)}_{[x]}$

${}_tp_x$

Notation

assurance?annuity

unit-linked

Background

?Do I need background? mathematical background

Chapter 1

Annuity and Assurance

An annuity offers a series of payment to the policy holders, if it is conditioned on the survival of the policy holder[1], then it is called a life annuity. If the annuity is payed for a fixed period, it is called term annuity, but if it continous till the death

In this chapter we are going to explore how will the insurance company[2] gather all the available information

[1]Or sometimes called annuitant.
[2]Insurance company is also called insurer.

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1.1 Background

1.2 Methodology

1.2.1 Equivalence priciple

1.2.2 Annuity and assurance factors

1.2.3 Some assumptions

1.3 Mortality rate

1.3.1 Gomperzt's law

1.3.2 Life table

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1.4 Pricing

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Chapter 2

Unit-linked insurance

In this chapter we are going to introduce the unit-linked insurance contract. Starting from some assumptions to construct a deterministic pricing model, and demonstrate that the deterministic pricing test is not accurate enough for this contract. This is because the uncertainty, or in other words risk, in the investment is not diversifiable.

To solve this problem, a stochastic pricing test will be considered, with the future return on the investment modelled as a random variable, then sensitivity test will be used again to find the best assumption for this contract as well as determine the reserve.

2.1 Background

In some countries unit-linked contract is also called equity-linked contract, it is because the single or regular premium paid by the policy holder will be invested on units of Equity or bonds on their own behalf. Once the insurer receive the premium, it will be payed into policy holder's fund after deduction of expenses and management costs in that year, and the deduction goes to insurer's account. In UK, there is an un-allocated percentage, which is another agreed regular deduction from policy holder's fund to insurer's account.

If the policy holder survival to the maturity date, he/she will receive the greater value between accumulated premiums payed and the fund in policy holder's account. This is called guranteed minimum maturity benefit (GMMB) [2]. But the if the death happens first, there is a guaranteed minimum death benefit, which allows policy holder's estate to receive the policy holder's fund with an extra amount, eg. 10% extra of the policy holder's fund.

2.1.1 Some assumptions

Before we establish the model, we need some assumptions, the following are some ideas from [3] book *Actuarial Mathematics for Life Contingent Risks,* and some online sources [4].

A company is going to issue a new 10 years unit-linked contract to people aged 55 to 60. In the contract, it stated that the annual premium is £5200, with an un-allocated rate of 5% in the first year and 1% in the subsequent years, this will be deducted immediately after receive the premium. Expenses occurs the same time as the premium payed in, which is 13% (including commision) and 0.7% repectively. At the end of each year, there is a further management charge of 0.8% discounted from policy holder's fund. The deductions from policy holder's account will be transferred to insurer's account.

When the contract mature, the policy holder can received the greater value between policy holder's fund and accumulation of premiums payed. But if the death happens before the mature date, policy holder's estate will receive 110% of the policy holder's fund where the extra 10% is from insurer's account.

The insurance company is also prepared to have 10% of policy holders surrender the contract in the first year and 5% in the second year, but no more in the subsequent years. The policy holders who surrender the contract will received their premium(s) after deducted management cost at the end of the year.

In the Chapter 1, we had introduce different methods to calculate the mortality rate, but here we are going to use a constant force of mortality, $\mu_x = 2.2218$, for people aged 55 to 60, which means the mortality rate of policy holders is $q_x = 0.006$ [1] in each year. The reason we set the mortality rate to a constant will be explained later in this chapter.

[1] $q_x = \exp\left\{-\int_0^1 \mu_{x+s} ds\right\}$

2.2 Deterministic pricing and reserving

Last section we introduce the unit-linked contract, and have some assumptions to construct the model. But here is something has not been counted in, the uncertainty in the investment return in policy holder's account and insurer's account.

For deterministic pricing, we used conservative interest rates for both accounts, 8% for policy holder's account and 5% for insurer's account, and assume no reserve holds in this case. Since the mortality rate is a constant, as well as the conservative interest rates, the cash flow for different policy holders will be exactly the same. Here is one example.

2.2.1 Cash flow

From all the assumptions, cash flow tables for both policy holder and insurer's account for the 10 years term can be easily construct. Let us explore the policy holder's account first.

		Cash flow for policy holder's account					
Year t	Annual premium	Allocated premium	Fund brought forward	Interest	Fund at time t^-	Manage-ment cost	Fund bring forward
1	5,200	4,940	0	395.2	5,335.2	42.68	5,292.52
2	5,200	5,148	5,292.52	835.24	11,275.76	90.21	11,185.55
3	5,200	5,148	11,185.55	1,306.68	17,640.24	141.12	17,499.12
4	5,200	5,148	17,499.12	1,811.77	24,458.89	195.67	24,263.21
5	5,20,0	5,148	24,263.21	2,352.89	31,764.11	254.11	31,509.99
6	5,200	5,148	31,509.99	2,932.64	39,590.64	316.73	39,273.91
7	5,200	5,148	39,273.91	3,553.75	47,975.67	383.81	47,591.86
8	5,200	5,148	47,591.86	4,219.19	56,959.05	455.67	56,503.38
9	5,200	5,148	56,503.38	4,932.11	66,583.49	532.67	66,050.82
10	5,200	5,148	66,050.82	5,695.91	76,894.73	615.16	76,279.57

Figure 2.1: Deterministic cash flow in one policy holder's account

Notes: Here is how the cash flow table been calculated

Allocated premium: For the first year, the allocated premium in the policy holder's fund will be (1-5%) of the total premium payed, and the other 5% is the agreed un-allocated premium which will be payed into insurer's account. For the subsequent years, the unallocated rate decrease to 1%, hence there will be 99% premium payed into policy holder's fund.

Simon!! CAN YOU PLEASE LINE UP THE TWO EQUATIONS BELOW? THX!!

First year allocated premium: 5,200 $\times$ 0.95 = 4,940

Subsequent years allocated premium: 5,200 $\times$ 0.99 = 5,148

Interest: It is the 8% interest policy holder obtain from the the accumulation of premium payed in at the beginning of the year and fund brought forward from last year.

eg. year t=5, interest = (5,148 + 24263.21) $\times$ 0.08 = 2,352.89

Fund at time t^-: It is the fund in policy holder's account after interest payed in.

Management cost: In our assumption, the management cost is 0.08% of the policy holder's fund, which is the fund at time t^- in our case.

Fund bring forward: It is the fund left in policy holder's account after deduction of management cost.

Table 2.2 is showing the insurer's account cash flow for **one** policy holder[2].

		Cash flow for insurer's account						
Year t	Annual premium	Un-allocated premium	Expenses	Interest	Management cost	Expected death benefit	Profit	Π_t
0	0	0	676	0	0	0	-676	-676
1	5,200	260	36.4	11.18	42.686	3.18	274.29	274.29
2	5,200	52	36.4	0.78	90.21	6.71	99.87	89.35
3	5,200	52	36.4	0.78	141.12	10.50	147.00	124.14
4	5,200	52	36.4	0.78	195.67	14.56	197.49	165.78
5	5,200	52	36.4	0.78	254.11	18.91	251.59	209.92
6	5,200	52	36.4	0.78	316.73	23.56	309.54	256.73
7	5,200	52	36.4	0.78	383.81	28.56	371.63	306.38
8	5,200	52	36.4	0.78	455.67	33.90	438.15	359.05
9	5,200	52	36.4	0.78	532.67	39.63	509.42	414.95
10	5,200	52	36.4	0.78	615.16	45.77	585.77	474.28

Figure 2.2: Deterministic cash flow in insurer's account

[2]If there are N numbers of policy holders exist, then the numbers in the table times N will be the cash flow for insurer in 10 years

Notes:

In insurer's account, un-allocated premium, expense, interest and management cost are calculated in the same way as in policy holder's account. You may notice there is an extra row in the table, year 0 in insurer's cash flow, it is the acquisition expense which will always occur before insurer receive premium, eg. commision, renting, employees' salary etc..

For expected death benefit, profit and profit signature are calculated as below:

Expected death benefit: If the death occurs before the contract mature, the policy holder's estate will receive 110% of the policy holder's fund where the extra 10% is from insurer account. The death benefit we calculate here is the 10% from insurer's account.

At time 0, there policy holder just enter the contract, so the probability for the contract in force is 1.

In year 1, policy holders pay the first premium right after sign the contract, so the probability for the contract in force is still 1.

During the first year, by our assumption, there will be 10% policy holders surender the contract, and the mortality rate for the other 90% is 0.006. Therefore, at the beginning of the second calandar year, insurer will be expecting p_2 = (1-0.006) $\times$(1-0.1) =0.8946 of policy holders to renew their contracts.

Beginning of third year, the insurer will be expecting p_3= (1-0.006-0.5) $\times p_2$= 0.8445024 of policy holders come to renew the contract, since by assumption there are 5% policy holder will surender during the second year.

From the third year, the probability for the contract in force will be p_t = (1-0.006) $\times$ p_{t-1}. (Showing in table 2.3.)

probability in force	
t=	${}_tp_x$
0	1
1	1
2	0.8946
3	0.8445024
4	0.839435386
5	0.834398773
6	0.829392381
7	0.824416026
8	0.81946953
9	0.814552713
10	0.809665397

Figure 2.3: Probability for the contract in force

Profit: For each year, profit is calculated by the fomula

Profit = Un-allocated premium - Expenses + Interest + Management cost - Death benefit

Profit signature: This is profit the insurer can make with probality that the contracts are still in force, which is defined as $\Pi_t = p_x \times$ Profit$_t$

2.2.2 Profitability

From the table 2.2.1, it is obviously a ascending trend on policy holder's fund, and on the contract mature date, it reaches £76,279, which is approximately 1.47 times of the accumulation of all premiums. This means, if the policy holder survive to the mature date, he/she will withdraw the policy holder's fund since it will always be the greater one compares to the accumulation of all the premiums. [3] This means the insurer will have the probability of 0.81 [4] to make profit of £585.77 on each policy holder. And if we discount the profit signature with the risk discount rate of interest, 15%, the net present value of the profit will be

$$NPV = \sum_{i=1}^{10} \Pi_t \times (1+0.15)^{-t} = \pounds 489.59$$

And the profit margin will be

[3]GMMB policy: on the mature date policy holders can withdraw the greater value between policy holder's fund and the accumulation of all the premiums payed.

[4]the probability of this contract in force till the mature date, table 2.3

$$Profit\ Margin = \frac{NPV}{P\ddot{a}_x}^{5} = \frac{489.59}{5200 \times 7.619285226} = 1.24\%$$

Since in the very beginning, we used conservative assumption, eg. expenses, mortality rate, interest rate for insurer's fund, so if in reality the expense and mortality rate are lower than modelled and interest rate is higher, then more profit that the insurer will obtain. Sensitivity test can be used to here to how will the profit change with changing in different factors.

2.2.3 Sensitivity test

Now we change our previous assumptions one by one and see what will happen.

Scenario 1: Lower expenses

The initial expense deduction changed from 13% to 10%, and the renewal changed from 0.7% to 0.5%.

		Cash flow for insurer's account						
Year t	Annual premium	Un-allocated premium	Expenses	Interest	Management cost	Expected death benefit	Profit	Π_t
0	0	0	520	0	0	0	-520	-520
1	5200	260	26	11.7	42.68	3.18	285.21	285.21
2	5200	52	26	1.3	90.21	6.71	110.80	99.12
3	5200	52	26	1.3	141.12	10.50	157.92	133.37
4	5200	52	26	1.3	195.67	14.56	208.41	174.95
5	5200	52	26	1.3	254.11	18.91	262.51	219.035
6	5200	52	26	1.3	316.73	23.56	320.46	265.79
7	5200	52	26	1.3	383.81	28.56	382.55	315.38
8	5200	52	26	1.3	455.67	33.90	449.07	367.99
9	5200	52	26	1.3	532.67	39.63	520.34	423.84
10	5200	52	26	1.3	615.16	45.77	**596.69**	**483.12**

Figure 2.4: Sensitivity test for lower expense rate

The table shows the profit increased about £10 the mature date, and the new NPV of profit becomes

[5] $\ddot{a}_x = \sum_{i=1}^{10} v^t {}_tp_x = 7.619285226$, where ${}_tp_x$ read from table ?!?!?!? and $v = \frac{1}{1+0.05}$

$$NPV = \sum_{i=1}^{10} \Pi_t \times (1+0.15)^{-t} = £693.24$$

and the profit margin is:

$$Profit\ Margin = \frac{NPV}{P\ddot{a}_x}^{6} = \frac{693.24}{5200 \times 7.619285226} = 1.7\%$$

This shows if we can lower the cost in the company, the more profit we can make, but in reality, this is really hard to achieve, since some certain cost is unavoidable, eg. renting, salary etc..

Scenario 2: Lower mortality rate

Constant mortality rate change from 0.006 to 0.005.

		Cash flow for insurer's account						
Year t	Annual premium	Un-allocated premium	Expenses	Interest	Management cost	Expected death benefit	Profit	Π_t
1	5,200	260	36.4	11.18	42.68	2.65	274.82	274.81
2	5,200	52	36.4	0.78	90.21	5.59	100.99	90.44
3	5,200	52	36.4	0.78	141.12	8.75	148.75	125.88
4	5,200	52	36.4	0.78	195.67	12.13	199.92	168.34
5	5,200	52	36.4	0.78	254.11	15.75	254.74	213.42
6	5,200	52	36.4	0.78	316.73	19.64	313.47	261.31
7	5,200	52	36.4	0.78	383.81	23.80	376.39	312.20
8	5,200	52	36.4	0.78	455.67	28.25	443.80	366.27
9	5,200	52	36.4	0.78	532.67	33.03	516.02	423.74
10	5,200	52	36.4	0.78	615.16	38.14	**593.40**	**484.85**

Figure 2.5: Sensitivity test for mortality rate change

Here are the new net present value of profit and profit margin obtianed from the new cash flow:

$$NPV = \sum_{i=1}^{10} \Pi_t \times (1+0.15)^{-t} = £491.43$$

$$Profit\ Margin = \frac{NPV}{P\ddot{a}_x}^{7} = \frac{491.43}{5200 \times 7.619285226} = 1.24\%$$

[6] $\ddot{a}_x = \sum_{i=1}^{10} v^t {}_tp_x = 7.619285226$, where ${}_tp_x$ read from table ?!?!?!? and $v = \frac{1}{1+0.05}$

[7] $\ddot{a}_x = \sum_{i=1}^{10} v^t {}_tp_x = 7.619285226$, where ${}_tp_x$ read from table ?!?!?!? and $v = \frac{1}{1+0.05}$

The lower mortality rate will only affect the insurer's account. The table shows that the profit will increase from £585.77 to £593.40, the raise in net present value of the profit is less than £2, and the profit margin will not change. This means the change in mortality rate is not sensitive, and should not be count as a important factor in our model, so a constant assumption for the mortality rate is acceptable.

Scenario 3: Higher interest rate

Interest rate change from 8% to 9%.

		Cash flow for policy holder and insurer's account				
Year t	Interest	Fund at time t^-	Management cost	Fund bring for-ward	Profit	Π_t
0					-676	-676
1	444.6	5,384.6	43.08	5,341.5232	274.65	274.65
2	944.06	11,433.58	91.47	11,342.11	101.04	90.39
3	1,484.11	17,974.22	143.79	17,830.43	149.48	126.23
4	2,068.06	25,046.49	200.37	24,846.11	201.84	169.44
5	2,699.47	32,693.58	261.55	32,432.04	258.47	215.67
6	3,382.20	40,962.24	327.7	40,634.54	319.7	265.15
7	4,120.43	49,902.97	399.22	49,503.75	385.9	318.14
8	4,918.66	59,570.4	476.56	59,093.84	457.49	374.9
9	5,781.77	70,023.6	560.19	69,463.42	534.89	435.67
10	6,715.03	81,326.44	650.61	**80,675.83**	618.59	500.85

Figure 2.6: Sensitivity test for interest rate change

In the table, it shows that once interest change 1%, the fund in the policy holder's account will increase about 4.6% every year, the profit for insurer will increase 5.3% on the date of mature. And the new net present value of profit and will be:

$$NPV = \sum_{i=1}^{10} \Pi_t \times (1 + 0.15)^{-t} = £522.72$$

The profit margin becomes:

$$Profit\ Margin = \frac{NPV}{P\ddot{a}_x} = \frac{522.72}{5200 \times 7.619285226} = 1.3\%$$

There is a big raise in both NPV of profit and profit margin. NPV of profit increases from £489.59 to £522.72, and the profit margin shoots up from 1.24% to 1.3%. This is a very sensitive case, little change in interest will have huge impact in profit. Therefore a good interest model will be very essential in this case.

2.2.4 Little summary for deterministic test

So far everything looks fine, but is this a good model for unit-linked contract? The answer will be no, from our sensitivity test, we can see that the interest rate change will lead to a huge fluctuate in profit, so the asssumption of constant interest rate is not good enough in this case. This is because the constant interest rate does not contain enough information when modelling, since the uncertainty of investment on equity and bond is not diversifiable. By law, we do not have an arbitrage market, no one can make profit without risk, and the interest on return will be affected by a few different factors, [5] eg. inflation, quality of information, governmental policy, change in bond market or equity market etc.. And once the factors affecting the market at the same time, the risk will be non-diversifiable. So now we should consider to model the interest rate non-linearly, eg. as a random variable.

2.3 Stochastic pricing

From the sensitivity test in last section, we conclude that the change in interest rate is non-diversifiable and extremely sensitive, which a constant assumption is not suitable. So now we are going to introduce stochastic test which can model the interest rate as a random variable, so that it can reflect the reality realistically. (simon! I DON'T LIKE THE LAST SENTENSE, CAN YOU FIGURE OUT SOMETHING BETTER???)

2.3.1 Central Limit Theorem and Monte Carlo simulation

In our basic assumptions [8], we assume that the policy holders will be able to choose different units of equities or bonds or portfolios [9] to invest their premium. Which means the the

[8]assumption in 2.1.1

[9]Portfolio: mixed equities and bonds

more equities and bonds available, the less similarity in the interest return each year for different policy holders.

By [6]Central Limit Theorem, which obtained by Mood, Graybill, and Boes in 1974, if we have a number of independent identically distributed random variables [10], we can observe a new series of random variables following normal distribution by summing or multiplying the old ones. The theorem can be used in finance, compound return or equity return follows a log-normal distribution[11], which means the logarithm of interest return I_t follows a normal distribution with mean μ and variance σ^2. This theorem can be applied in the new model, then we will have

$$logI_t = \xi \sim N(\mu, \sigma^2)$$

$$where\ \mu = 0.74928, \sigma = 0.15$$

Now we can use Monte Carlo simulation[7], generate some random numbers [12] ξ_t' from the Standard Normal distribution with mean 0 and variance 1, and transform them to the specified Normal distribution $N(0.74928, 0.15^2)$ via

$$\xi_t = 0.74928 + 0.15 \times \xi_t'$$

then, our annual accumulation[13] factor will be simulated by

$$I_t = exp\{\xi_t\}$$

2.3.2 Modelling

Now we assume there are 1,000 policy holders [14] purchased unit-linked contract with different portfolios. Using the generated random variable from the Standard Normal distribution, we can construct our new cash flow tables. I used C^{++} to help me with constructing the

[10]The random variables can follow any distribution
[11]log-normal distribution is also called Galton's distribution, after Francis Galton
[12]I used C^{++} boost random number generator,
[13]It is the interest rate $r_t + 1$
[14]The larger number the better statistic result we will get, here we just set 1,000 as an example.

tables [15]. Since there are 1,000 policy holdders and interest rate will be different from year to year for all of them. Therefore, 10,000 ξ_t' need to be generated from the Standard Normal distribution.

Here is an example we get from the program. [16]

Year t	ξ_t	Cash flow for policy holder and insurer's fund						Profit	Π_t
		R_t	Premium	Allocated Premium	Fund at t^-	Management	Fund at t		
0	0	0	0	0	0	0	0	-676	-676
1	0.213436	1.11287	5,200	4,940	5,497.58	43.9807	5,453.6	313.709	313.71
2	-0.49558	1.00059	5,200	5,148	10,607.9	84.863	10,523	94.9292	84.92
3	1.57538	1.36511	5,200	5,148	21,392.6	171.141	21,221.5	174.788	147.61
4	-1.0592	**0.919475**	5,200	5,148	**24,246.1**	193.969	**24,052.1**	195.917	164.46
5	1.83927	1.42023	5,200	5,148	41,470.8	331.767	41,139.1	323.463	269.90
6	1.88577	1.43017	5,200	5,148	66,198.4	529.587	65,668.8	506.566	420.14
7	0.604675	1.18014	5,200	5,148	83,573.4	668.588	82,904.9	635.225	523.69
8	-0.365983	1.02023	5,200	5,148	89,834.4	718.675	89,115.7	681.586	558.54
9	-0.578264	0.988258	5,200	5,148	93,156.9	745.255	92,411.6	706.188	575.22
10	-0.634376	0.979975	5,200	5,148	95,606	764.848	94,841.1	724.323	586.46

Figure 2.7: Stochastic cash flow 1

Notes:

ξ_t**:** This column is the generated random variables from Standard Normal distribution.

R_t**:** This is simulated annual accumulation factor, **the annual interest rate** is $r_t = R_t - 1$.

Other columns in the table: The table was calculated using the same method in deterministic cash flow, the only difference is the interest rate had changed.

From the table 2.8 we can see that, the annual accumulation factor is sometimes less than one, which indecates that annual interest $r_t = R_t - 1$ is negative, then there is a loss in the policy holder's fund in that year. But for the insurer in this case, the net present value of profit and profit margin[17] are ideal:

[15] codes will be included in the appendix

[16] To curious readers, this is the 1st cash flow table we get from the program.

[17] Profit margin is calculated using risk discount factor 15%.

$$NPV = £1,011.86 \quad \& \quad Profit\ margin = 2.6\%$$

Now showing another example from the 1,000 which is not that ideal for the insurer.

Year t	ξ_t	Cash flow for policy holder and insurer's fund						Profit	Π_t
		R_t	Premium	Allocated Premium	Fund at t^-	Management	Fund at t		
0	0	0	0	0	0	0	0	-676	-676
1	-1.33578	0.88211	5,200	4,940	4357.62	34.86	4,322.76	305.27	305.27
2	-0.412251	1.01318	5,200	5,148	9,595.55	76.76	9518.79	87.43	78.22
3	-1.09414	0.91467	5,200	5,148	13,415.3	107.32	13,307.9	115.72	97.72
4	0.847198	1.22386	5,200	5,148	22,587.4	180.7	22,406.7	183.64	154.15
5	-0.581987	0.987706	5,200	5,148	27,216	217.73	26,998.3	217.91	181.82
6	0.43257	1.15006	5,200	5148	36,970.1	295.76	36,674.3	290.14	240.64
7	0.0178033	1.08069	5,200	5148	45,196.9	361.58	44,835.4	351.05	289.412
8	-2.10093	0.786462	5,200	5148	39,310	314.48	38,995.5	307.46	251.96
9	-1.4693	0.864618	5,200	5148	38,167.3	305.34	37,862	299.00	243.55
10*	-0.207703	1.04474	5,200	5148	44,934.4	359.48	**44,575**	**349.11**	-4,609.47
10	-0.207703	1.04474	5,200	5148	44,934.4	359.48	**44,575**	**-7,031.38**	-5,693.06

Figure 2.8: Stochastic cash flow2

Note:

Row 10* & 10: It is showing that the fund in policy holder's account at the mature date, £4,4575, is less than the accumalation of 10 premiums, £52,000, therefore the difference will be deducted to pay off the insurer's liability with probability 0.994[18].

$$(52,000 - 44,575) \times 0.994 = £7,380.45$$

The profit showing in row 10* is the profit in year 10 before the deduction, the actual profit the insurer obtaind at the mature date is £349.11 - £7,380.45 = -£7031.3 showing in row 10. And the NPV of the profit in this case is -£995.422, and the profit margin is -2.5%.

Now there are two examples for both situation, making profit and lossing, it will be really important for the company to know the statistic numbers, eg. how many contracts have negative NPV of profit, how much profit can the company make, what is the probability for the company to lose etc..

[18] The probability that the contract is still in force from year 9, table 2.3.

2.3.3 Statistic analysis

In the C^{++} program, the NPV can be sorted in order, and counted how many negative values are there, the table below showing the statistic results.

```
NPV median = 677.104
NPV fifth percentile = -927.485
NPV ninety fifth percentile = 1035.04
NPV mean = 534.232
NPV sd = 687.702
NPV count negative values = 85
95% CI for NPV = (491.607,576.856)
```

Notes:

NPV median: This is the sample median of all NPV of profit.

NPV 5th percentile: The 50^{th} in the ascending order 1,000 NPV of profit.

NPV 95th percentile: The 950^{th} in the ascending order 1,000 NPV of profit.

NPV mean: This is the sample mean, which calculated by $\frac{\sum_{i=1}^{1000} NPV_i}{1000}$

NPV sd: This is the sample standard deviation.

NPV count negative values: This is the number of negative NPV of profit out of 1,000.

95% CI for NPV: The 95% of all NPV of profit will drop in this interval.

There are 85 negative NPV of profit out of 1,000, which means the probability for this company to lose on this contract is 8.5%, which is considerablely high. And the 5^{th} percentile, $-£927.485$, is a very large lose as well. In conclusion, this contract has a high probability of losing a large amount of money, which should not be issued in reality. This is caused by the gauranteed benefit increased the risk for insurer. In the next section we are going to disscuss how to lower the risk and make a profit.

2.3.4 Stochastic pricing

Recall in Chapter 1, the equivalence principle were used to find the premium, but it can not be used in this unit-linked contract since there is a non-diversifiable risk. But instead, we can use stochastic simulation with quantile premium principle[8].

For quantile premium principle, the idea is to obtain a profit with a certain probability. If the 5^{th} percentile is positive and probability of loss is no greater than 5% then it is a good model and can be taken to practice. Based on the previous model, changing in some assumptions may let us achieve this target.

Scenario 1: Increase the premium

Increase the premium from £5,200 to £5,400

```
NPV median = 703.147
NPV fifth percentile = -963.16
NPV ninety fifth percentile = 1074.85
NPV mean = 554.779
NPV sd = 714.152
NPV count negative values = 85
95% CI for NPV = (510.515,599.042)
```

There is an increase on the NPV sample mean and median but the number of negative NPV of profit did not change, and there is a decrease on 5^{th} percentile. This is because when we increase the premium, we increase the guaranteed minimum maturity benefit (GMMB) as well, which means we increase the liability on the mature date of the contract.

Scenario 2: Increase the expenses deduction

Initial expenses increase from 13% to 15%, renewal expenses increase from 0.7% to 0.9%

```
NPV median = 638.943
NPV fifth percentile = -965.646
NPV ninety fifth percentile = 996.881
NPV mean = 496.07
NPV sd = 687.702
NPV count negative values = 86
95% CI for NPV = (453.446,538.695)
```

Scenario 3: Lower the mortality rate

Mortality rate decreases from 0.006 to 0.005

```
NPV median = 689.194
NPV fifth percentile = -919.938
NPV ninety fifth percentile = 1051.93
NPV mean = 546.463
NPV sd = 689.761
NPV count negative values = 85
95% CI for NPV = (503.711,589.214)
```

In the above two sensitivity tests, we can see a growth in the sample mean and median too, but the number of negative NPV of profit did not change or even raise a little. The 5^{th} percentiles are still negative.

Scenario 4: Lower the GMMB

The guaranteed minimum maturity benefit decrease from 100% to 91% premiums paid.

```
NPV median = 677.337
NPV fifth percentile = 3.53572
NPV ninety fifth percentile = 1035.04
NPV mean = 617.827
NPV sd = 480.784
NPV count negative values = 50
95% CI for NPV = (588.028,647.627)
```

In this scenario, we lowered the guaranteed minimum maturity benefit in order to decrease the liability to meet on the date of mature. The new assumption will enable the insurer to have 5% probability to lose, and the 5^{th} percentile becomes positive.

This is an ideal contract for the insuer, but may not be ideal for the policy holders because of the decrement on maturity benefit. In order to be more competitive, the insurer can hedge partial risk in the financial market at the same time lower the GMMB.

2.3.5 Reserving

From the scenario test 4 in last section, we have a model which enable the insurer have less risk to make profit on the contract. But even the risk is much lower, if the number of contracts issued is large, the insurer may still have a probability to lose. In this case, the insurer may have the following solutions[9]:

1. Hedge the risk in the financial market.

2. Reinsure (purchase insurance from one or more other insurance companies to pass on the risk) [10].

3. Reserving.

In this section, we are going to disscuss the third option, reserving. To calculate the reserve for the contract, the non-diversifiable risk should be taken into account, the methodology which enable us to do this is risk measure[11], one popular type is called Value at

Risk, it is defined in terms of a parameter α which is the probability that the loss L will not exceed ℓ. Therefore, we have

$$P[L \leq \ell] = \alpha$$

Continue on scenario 4, set $\alpha = 0.95$, the reserve ${}_tV$ at time t = 0 will equal to the 95^{th} percentile[19] of the distribution of L_i where

$$L_i = -\sum_{\mathbf{t=1}}^{10} \frac{{}_{t-1}p_x Pr_{t,i}}{(1+j)^t}$$

It is calculated similar to NPV of profit, but with different discount rate $j = 0.05$ and the first acquisition cost will not be included.

$${}_0V = \pounds 1,552.99$$

In practice, the insurer may review the reserve annually, and make adjustment based on changes in that year.

2.4 Deterministic VS Stochastic test

For unit-linked contract, the mortality rate is diversifiable and using simple assumption will save the insurer money in terms of time and cost[20]. But there is an uncertainty factor on the investment performance, then risk on interest rate is non-diversifiable. In this case, the simple deterministic test will not be able to model the profit realistically. Therefore, the stochastic model should be used.

Once the stochastic model has been found, the insurer will be interested to know the statistical numbers which reflect the protential risk, and then try different test to reduce or even eliminate the risk. None of these can not be achieved by deterministic pricing, so in unit-linked contract, the stochastic test is very essential! The better model the insurer can find, the more money they are going to make.

[19] if the 95^{th} percentile is negative, the reserve will be set to zero.

[20] Of course you can use more accurate mortality rate, but it is not that important comparing with the investment risk factor.

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[11] Andrey Mudrov, *Financial Risk, Lecture 7*, page 1-8.

Appendix 1: Optional Extra

A PROGRAM TO COMPUTE EIGENVALUES

```
100 GOTO 200

200 END
```

Appendix 2: More Extra

Table 2.1: Lowercase Greek Letters.

α	`\alpha`	θ	`\theta`	o	`o`	υ	`\upsilon`
β	`\beta`	ϑ	`\vartheta`	π	`\pi`	ϕ	`\phi`
γ	`\gamma`	ι	`\iota`	ϖ	`\varpi`	φ	`\varphi`
δ	`\delta`	κ	`\kappa`	ρ	`\rho`	χ	`\chi`
ϵ	`\epsilon`	λ	`\lambda`	ϱ	`\varrho`	ψ	`\psi`
ε	`\varepsilon`	μ	`\mu`	σ	`\sigma`	ω	`\omega`
ζ	`\zeta`	ν	`\nu`	ς	`\varsigma`		
η	`\eta`	ξ	`\xi`	τ	`\tau`		

Table 2.2: Uppercase Greek Letters.

Γ	`\Gamma`	Λ	`\Lambda`	Σ	`\Sigma`	Ψ	`\Psi`
Δ	`\Delta`	Ξ	`\Xi`	Υ	`\Upsilon`	Ω	`\Omega`
Θ	`\Theta`	Π	`\Pi`	Φ	`\Phi`		

Table 2.3: Math Alphabets.

Example	Command	Required package
$\mathrm{ABCDE abcde 1234}$	`\mathrm{ABCDE abcde 1234}`	
$\mathit{ABCDE abcde 1234}$	`\mathit{ABCDE abcde 1234}`	
$\mathnormal{ABCDE abcde 1234}$	`\mathnormal{ABCDE abcde 1234}`	
$\mathcal{ABCDE}\dashv\lfloor\rfloor\lceil\rceil\infty\in\ni\triangle$	`\mathcal{ABCDE abcde 1234}`	
$\mathfrak{ABCDE abcde 1234}$	`\mathfrak{ABCDE abcde 1234}`	amsfonts or amssymb
$\mathbb{ABCDE}\Game\nVdash\nvDash\nVDash\ntrianglerighteq$	`\mathbb{ABCDE abcde 1234}`	amsfonts or amssymb