A StockOpter® *Insight* White Paper
From: Net Worth Strategies. Inc.

Lognormal Random Walk Model for Stock Prices

(Part I)

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StockOpter *Insight* calculates option values using the Black-Scholes option-pricing model. One of the assumptions underlying this model is that the price of a stock follows a lognormal random walk, also known as geometric Brownian motion, with drift. The lognormal random walk model for the behavior of the price of a stock is an industry-standard model that has been found to work well in practice. This document discusses how this process works and why it is a good model for the behavior of the price of a stock.

StockOpter *Insight*'s value-at-risk (VaR) calculations also assume that this process applies, as do its calculations of probabilities associated with various stock prices. For more discussion of the Black-Scholes option-pricing model and VaR calculations, please visit www.NetWorthStrategies.com.

The next three sections of this paper are organized around different pieces of the name for the process.

Random Walk

A *random walk* process is one in which the change in value over any time interval is independent of any changes that have occurred in preceding time intervals, and the size and direction of the changes in value are in some sense random. For stock prices, the applicability of a random walk is based on the assumption that the stock market is efficient, i.e., that the stock price at a given moment reflects all the information available at that moment. Stock prices change for reasons, but changes that are about to happen will be due to new information, which by definition cannot be predicted ahead of time.

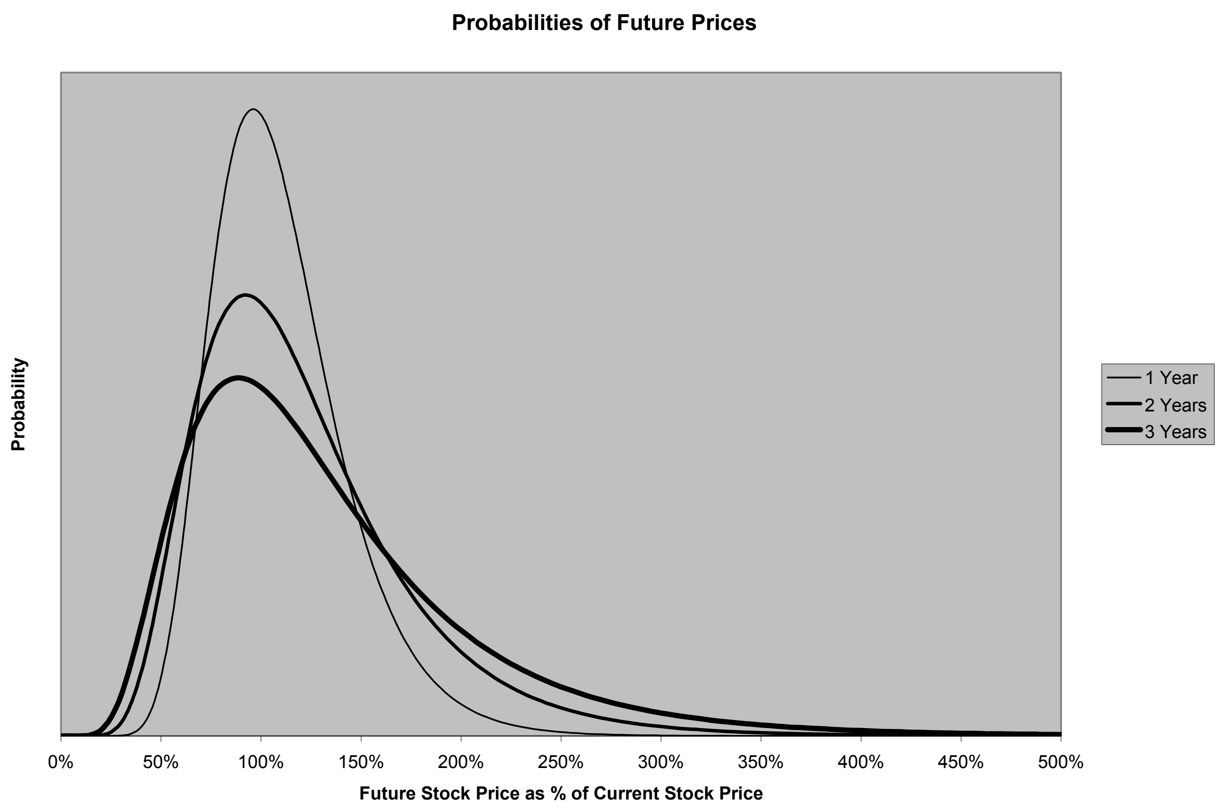
With Drift

With drift refers to the fact that changes in stock prices are not entirely random. There is a systematic component due to the fact that stock prices tend to increase over time. The random walk part of the process represents deviations up and down from that trend.

Lognormal

Lognormal refers to the facts that changes in the natural, or base *e*, logarithm of the stock price are assumed to be normally distributed, and that the resulting distribution of possible stock prices at a particular time is a lognormal distribution. A lognormal distribution is a distribution that becomes a normal distribution if one converts the values of the variable to the natural logarithms, or ln's, of the values of the variable.

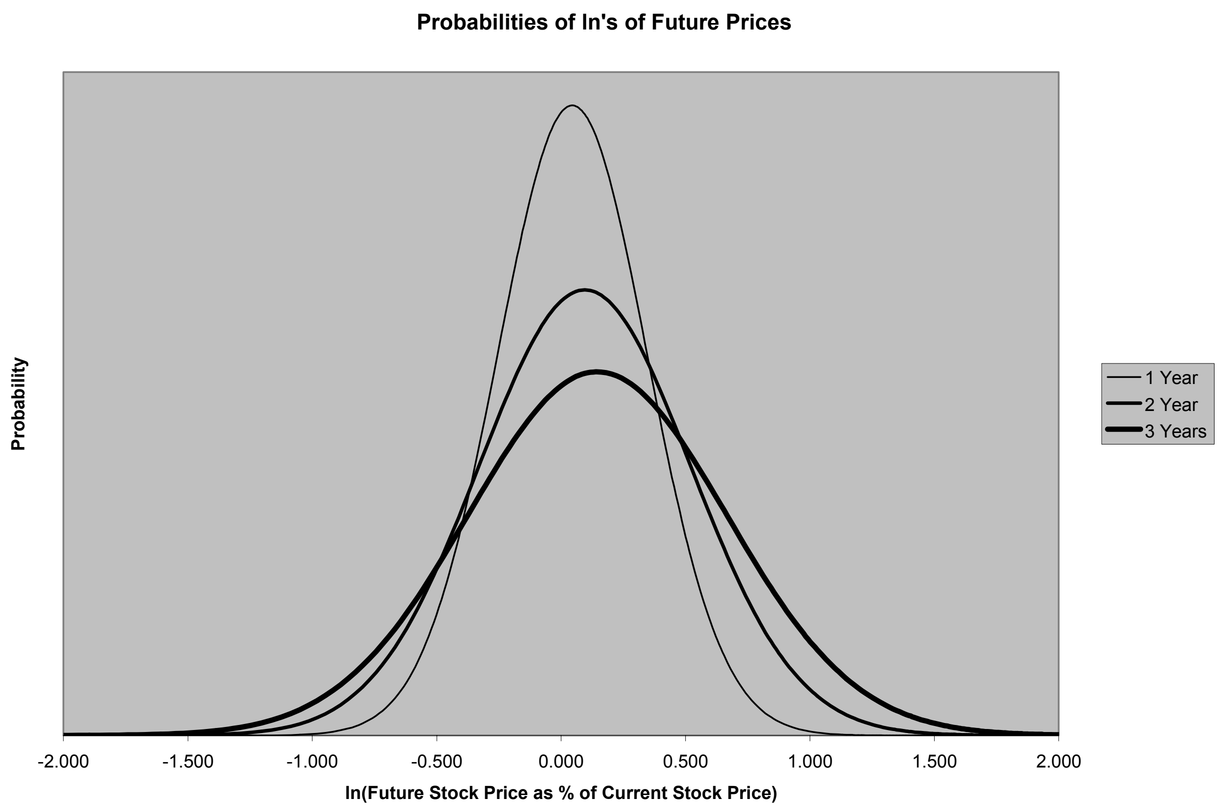
For example, consider a stock for which the expected increase in value per year is 10% and the volatility of the stock price is 30%. The volatility of a stock price is a measure of its variability; for a discussion of volatility please visit www.NetWorthStrategies.com. The following graph shows the probability distribution of possible prices 1, 2, and 3 years from now if the price of the stock follows a lognormal random walk with drift:



As is usually the case, the single most likely price (the peak of the curve) goes *down* as we look further into the future. However, this is more than offset by the increased likelihood of the price being quite high as we look further into the future. The probability-weighted average price for these distributions are 110% of the current price in 1 year, 121% in 2 years, and

133% in 3 years, which is consistent with the expected increase in value being 10% per year.

The following graph shows the probability distribution of ln's of possible prices corresponding to the above graph:



Readers who have taken a statistics course will recognize that the above probability distributions resemble normal distributions, and in fact they are indeed normal distributions. Thus, the probability distributions in the preceding graph are lognormal distributions.

Why the Lognormal Assumption Makes Sense

Consider first what would happen if change in stock price, and thus the end-result stock price, were normally distributed. There are two main problems with this. One is that there is nothing to prevent a random walk of the stock price from producing a negative price. The other is that there is no scaling effect. One would expect that if the current price is $25 the potential changes in price should be half the size that they would be if the current price is $50, all else being equal. If not, one's investment in a particular stock would become much more risky if there were a stock split.

One can try to salvage a normal model by introducing relative changes that are expressed as percentages of the current stock price. There are problems with this as well, however. One is that prices can still go negative. If we ignore drift, a price change of -125% will be just as likely as a price change of +125% if we assume that the changes are normally distributed. Yet a deviation of more than 100% in the negative direction is impossible, since it will lead to a negative price, while a deviation of more than 100% in the positive direction is quite plausible; there is no reason that the stock price cannot more than double.

Another problem with modeling percent changes is that one runs into problems with consecutive time intervals. Suppose that we are modeling consecutive one-year changes. If one assumes that percent changes are normally distributed, prices at the end of the first year will be normally distributed, since all of the changes had the same starting price. However, the dollar change in price during the second year for a given percent change in price depends on the price at the end of the first year. It turns out that the distribution of prices at the end of the second year modeled in this way will be much closer to a lognormal distribution than to a normal distribution. This should make us have doubts about how we modeled the first year. Should we have broken that down into shorter intervals as well? If so, what intervals? Unfortunately, the results will depend on the intervals chosen.

Using a lognormal model solves all of these problems. Change in ln of price is similar to percent change in price, but the differences are telling. The following table compares given changes in ln to comparable percent changes:

Change in ln		Corresponding % Change		Change in ln Expressed as %	
Down	Up	Down	Up	Down	Up
-0.010	0.010	-1.0%	1.0%	-1.0%	1.0%
-0.020	0.020	-2.0%	2.0%	-2.0%	2.0%
-0.050	0.050	-4.9%	5.1%	-5.0%	5.0%
-0.100	0.100	-9.5%	10.5%	-10.0%	10.0%
-0.200	0.200	-18.1%	22.1%	-20.0%	20.0%
-0.500	0.500	-39.3%	64.9%	-50.0%	50.0%
-1.000	1.000	-63.2%	171.8%	-100.0%	100.0%
-2.000	2.000	-86.5%	638.9%	-200.0%	200.0%
-5.000	5.000	-99.3%	14741.3%	-500.0%	500.0%

The *Change in ln* and *Change in ln Expressed as %* columns consist of the same two columns of numbers, just expressed differently. For example, 0.010 expressed as a percentage is 1.0%. The *Corresponding % Change* columns show what the resulting percentage change is for a given change in ln. For example, if the ln of a number increases by 0.100, the number increases by 10.5%.

The above table shows that when the change in ln is quite small, the % change in a number corresponding to a given change in ln (middle pair of columns) is almost the same as the change in ln expressed as a percent (last pair of columns). However, as the change in ln gets larger, the percent change corresponding to the change in ln starts to be substantially larger than the change in ln expressed as a percent if the change is an increase, and the percent change corresponding to the change in ln starts to be substantially smaller (in absolute value, i.e., ignoring the fact that the numbers are negative) than the change in ln expressed as a percent if the change is a decrease.

There are two very important desirable features of these results. One is that, as the above table would make one suspect, there is an inherent floor on a decrease in the value of a number, in our case the stock price, of -100%, and it takes a very large decrease in ln to get close to that floor. This is due to the fact that the ln's [negative infinity, 0, positive infinity] correspond to the numbers [0, 1, positive infinity], respectively. All ln's greater than negative infinity correspond to positive numbers; a negative ln just means the number is less than 1. Thus, by modeling the ln's of prices, we guarantee that prices remain positive, though they can become quite small.

The other desirable feature is that, unlike percent changes, a change in ln of +x followed by a change of –x, or the other way around, gets one back to the same ln of the number, and thus the same number, that one started with. We can also show this using the results in the above table. For example, if the ln of the price increases by 0.2 and then decreases by 0.2, the resulting price is (1-0.181)(P(1+0.221)) = 1.000P, or 100% of the starting price. By contrast, if the price increases by 20% and then decreases by 20%, the resulting price is (1-0.2)(P(1+0.2)) = 0.96P, or 96% of the starting price. More generally, a decimal fraction change of +x followed by a change of –x, or the other way around, results in a price that is $(1-x)(1+x) = 1-x^2$ times the starting price.

These two desirable features of changes in ln are related to the fact that for a lognormal random walk model, one can model changes over various time intervals and have the results be internally consistent. One can, for example, model a two month change directly as such and obtain the same probability distribution, only far more easily, as one would if one modeled consecutive one month changes with the same inputs. Demonstrating this mathematically is best left for the <u>Part II</u> material, but it is a very useful result.

These desirable features of a lognormal random walk with drift model help explain why this model works well in practice and why it has been assumed in deriving the Black-Scholes model, among others.