Assignment 2
MATH 3423 - Numerical Methods 2
Acadia University

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1 Requirements

If you met the requirements (presentation), you were granted 5/5.

2 Linear Splines

2.1 Approximation Order in 1D

1. Given f is a smooth function, let Sf be the linear spline (a function piecewise linear and continuous) interpolating the values $y_k = f(k\Delta x)$ over the nodes $x_k = k\Delta x$.

   (a) If $f(x) = x$ and $\Delta x = 1$, find $\|Sf - f\|_{L^\infty([0,10])} = \max\{|Sf(x) - f(x)| : x \in [0,10]\}$.
   Solution: since f is linear, the error must be zero.

   (b) If $f(x) = \sqrt{x}$ and $\Delta x = 1$, find $\|Sf - f\|_{L^\infty([1,2])} = \max\{|Sf(x) - f(x)| : x \in [1,2]\}$.
   Solution: $Sf = (2-x) + \sqrt{2}(x-1)$ and $f = \sqrt{x}$, hence $Sf - f = (2-x) + \sqrt{2}(x-1) - \sqrt{x}$. We have to find the extrema of $Sf - f$ over $[1,2]$. First, we solve for $f'(z) = 0$ to get $z = \frac{-1}{8\sqrt{2}-12}$. We get that the minimum of $Sf - f$ is about -0.01776695. Hence $\|Sf - f\|_{L^\infty([1,2])}$ is about 0.01776695.

   (c) If $f(x) = \sqrt{x}$ and $\Delta x = 1/2$, find $\|Sf - f\|_{L^\infty([1/2,1])} = \max\{|Sf(x) - f(x)| : x \in [1/2,1]\}$.
   Solution: Same idea again. $Sf - f = \sqrt{2}(1-x) + 2(x-1/2) - \sqrt{x}$, we solve for the extrema and find a minimum of -0.01256313292354 at $x \cong$0.72855339059327 and hence $\|Sf - f\|_{L^\infty([1/2,1])} \cong 0.01256$.

   (d) If $f(x) = \sqrt{x}$ and $\Delta x = 1/4$, find $\|Sf - f\|_{L^\infty([1/4,1/2])} = \max\{|Sf(x) - f(x)| : x \in [1/4,1/2]\}$.
   Solution: Same again... $Sf - f = 2(1/2 - x) + \frac{4}{\sqrt{2}}(x - 1/4) - \sqrt{x}$. This time, we find a minimum of -0.00888347648318 and hence $\|Sf - f\|_{L^\infty([1/4,1/2])} \cong 0.00888$.

   (e) If $f(x) = x^2$ and $\Delta x = 1$, find $\|Sf - f\|_{L^\infty([0,1])} = \max\{|Sf(x) - f(x)| : x \in [0,1]\}$.
   Solution: $Sf = x$ and thus $Sf - f = x - x^2$ and the extrema is at $x = 1/2$. Therefore $\|Sf - f\|_{L^\infty([0,1])} = \frac{1}{4}$.

   (f) If $f(x) = x^2$ and $\Delta x = 1$, find $\|Sf - f\|_{L^\infty([0,10])} = \max\{|Sf(x) - f(x)| : x \in [0,10]\}$.
   Solution: The trick he is that we have 10 elements (11 nodes). On $[k, k+1]$, the spline is given by $Sf(x) = k^2 + (x-k)((k+1)^2 - k^2)$, hence the difference is given by $Sf(x) - f(x) = -x^2 + (2k+1)x - k^2 - k$. Solving for $Sf'(x) - f'(x) = 0$, we get $x_{max} = \frac{2k+1}{2}$ and $Sf(x_{max}) - f(x_{max}) = 1/4$ and hence $\|Sf - f\|_{L^\infty([0,10])} = 1/4$.

2. Given f is a smooth function, let Sf be the linear spline (a function piecewise linear and continuous) interpolating the values $y_k = f(k\Delta x)$ over the nodes $x_k = k\Delta x$.

(a) The function

$$\Phi(t) = f(t) - Sf(t) - \frac{(t-x_k)(t-x_{k+1})}{(x-x_k)(x-x_{k+1})}(f(x) - Sf(x))$$

for any x in (x_k, x_{k+1}) has at least 3 distinct roots in $[x_k, x_{k+1}]$, hence its second derivative has at least one zero by Rolle's theorem. Use this fact to estimate $\|Sf - f\|_{L^\infty([x_k,x_{k+1}])} = \max\{|Sf(x) - f(x)| : x \in [x_k, x_{k+1}]\}$ in terms of the second derivative of f.

Solution: The zeroes of $\Phi(t)$ are $t = x_k, x, x_{k+1}$. Hence, there exists ξ such that $\Phi''(\xi) = 0$ and therefore $f(x) - Sf(x) = \frac{(x-x_k)(x-x_{k+1})}{2} f''(\xi)$. If $|f''| \le M$ then $|f(x) - Sf(x)| \le \frac{M(x-x_k)(x-x_{k+1})}{2} \le \frac{M(\Delta x)^2}{8}$.

(b) Let $f(x) = \sqrt{x}$ and $\Delta x = 1/2^{n+1}$, using part (a) find a good upper bound for $\|Sf - f\|_{L^\infty([1/2^{n+1},1/2^n])} = \max\{|Sf(x) - f(x)| : x \in [1/2^{n+1}, 1/2^n]\}$. What do you expect will be $\lim_{n\to\infty} \|Sf - f\|_{L^\infty([1/2^{n+1},1/2^n])}$?

Solution: $f''(x) = \frac{-1}{4x^{3/2}}$ and f'' is bounded by $\frac{1}{4(1/2^{n+1})^{3/2}} = 2^{3n/2+1/2}$ over $[1/2^{n+1}, 1/2^n]$, hence $|f(x) - Sf(x)| \le \frac{M(\Delta x)^2}{8} = \frac{2^{3n/2+1/2}}{8\times 2^{2n+2}} = \frac{1}{32\times 2^{\frac{n+1}{2}}}$. As $n \to \infty$, the error should go down to zero.

(c) Generalize your result from part (a) for $\|Sf - f\|_{L^\infty((-\infty,\infty))} = \max\{|Sf(x) - f(x)| : x \in (-\infty,\infty)\}$ with any Δx.

Solution: The only difference here is that we have several nodes instead of only two. Other than that, the estimate can be chosen to be the same $|f(x) - Sf(x)| \le \frac{M(\Delta x)^2}{8}$.

(d) If $f(x) = x^2$ and $\Delta x = 1$, can you easily give an upper bound for $\|Sf - f\|_{L^\infty((-\infty,\infty))} = \max\{|Sf(x) - f(x)| : x \in (-\infty,\infty)\}$? Explain.

Solution: Given that $f''(x) = 2$, then an easy upper bound is $\frac{M(\Delta x)^2}{8} = \frac{2(1)^2}{8} = \frac{1}{4}$.

2.2 Splines in the plane

1. Given three points in the plane $\underline{x}_1, \underline{x}_2, \underline{x}_3$ with corresponding z values z_1, z_2, z_3 find a linear interpolation of these values $z(x,y) = a + bx + cy$. Hint: you need to generalize Lagrange or Newton interpolation. Use Maple/Matlab/... to check your answer!

   Solution: Let $\underline{x}_1 = (x1, y1)$, $\underline{x}_2 = (x2, y2)$, $\underline{x}_3 = (x3, y3)$, then

$$\begin{bmatrix} 1 & x1 & y1 \\ 1 & x2 & y2 \\ 1 & x3 & y3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} z1 \\ z2 \\ z3 \end{bmatrix}$$

   and

$$\begin{aligned} z(x,y,z) &= \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} 1 & x1 & y1 \\ 1 & x2 & y2 \\ 1 & x3 & y3 \end{bmatrix}^{-1} \begin{bmatrix} z1 \\ z2 \\ z3 \end{bmatrix} \\ &= \frac{\begin{array}{c} ((x1-x)y2 + (x-x2)y1 + (x2-x1)y)z_3 + \\ ((x-x1)y3 + (x3-x)y1 + (x1-x3)y)z_2 + \\ ((x2-x)y3 + (x-x3)y2 + (x3-x2)y)z_1 \end{array}}{(x2-x1)y3 + (x1-x3)y2 + (x3-x2)y1}. \end{aligned}$$

   We observe that it is very close to Lagrange formula, but a lot more complicated at the same time. The lesson here is that life in 2D is a lot more difficult than in 1D.

2. Assume that $f(x,y) = 1 + 2x + 2y + xy + x^2 + y^2$. Let $\underline{x}_1 = (0,0), \underline{x}_2 = (1,0), \underline{x}_3 = (0,1)$. Find

$$\varepsilon = \max\{|z(\underline{x}) - f(\underline{x})| : \underline{x} \in \triangle \underline{x}_1, \underline{x}_2, \underline{x}_3\}.$$

   Solution: $z_1 = f(\underline{x}_1) = 1$, $z_2 = f(\underline{x}_2) = 4$, $z_3 = f(\underline{x}_3) = 4$ and hence we have $z(\underline{x}) = 3y + 3x + 1$, then $z(\underline{x}) - f(\underline{x}) = x + y - xy - x^2 - y^2$. Setting the gradient to zero, we get $(1 - y - 2x, 1 - x - 2y) = 0$ and $x = y = 1/3$. It can be seen to be the maximum and we get $\varepsilon = 1/3$.