Acadia University
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Topics from Advanced Calculus
(MATH 2023)

Example for Stokes Theorem

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1 Let's play with Stoke's theorem!

Recall that Stoke's theorem can be written as

$$\int\int_S \nabla \times \underline{F} \cdot d\underline{S} = \int_{\partial S} \underline{F} \cdot d\underline{s}$$

where ∂S is a closed curve defined as the boundary of S.

1.1 An example

Let the surface S be the paraboloid $z = 9 - x^2 - y^2$ defined over the disk in the $xy-$plane of radius 3. Since this surface is of the form $z = f(x,y)$, it has an obvious parametrization using x,y as parameter with a parameter space given by $D = \left\{(x,y)|x^2+y^2 \leq 3\right\}$. The ∂S is the circle $\left\{(x,y,z)|x^2+y^2 = 9, z = 0\right\}$. We will use the vector field $\underline{F} = (2z-y)\underline{i} + (x+z)\underline{j} + (3x-2z)\underline{k}$ to test Stoke's theorem.

We calculate the curl of $\underline{F}$ as

$$\begin{aligned} \nabla \times \underline{F} &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z-y & x+z & 3x-2y \end{vmatrix} \\ &= (-3,-1,2). \end{aligned}$$

Taking the gradient of $z + x^2 + y^2 = 9$, we find that a normal vector to the surface $z = 9 - x^2 - y^2$

is given by $\underline{N} = (2x, 2y, 1)$. From this, we can compute the RHS of Stoke's theorem

$$\begin{aligned}
\int\int_S \nabla \times \underline{F} \cdot d\underline{S} &= \int\int_D (-3,-1,2)\cdot(2x,2y,1)\, dxdy \\
&= \int\int_D (-6x-2y+2)dxdy.
\end{aligned}$$

Consider the first two terms under the integral $-6x-2y$. We see that by symmetry, they must integrate out to zero!!! Therefore

$$\int\int_S \nabla \times \underline{F} \cdot d\underline{S} = 2\int\int_D dxdy = 2 \times area(D) = 2 \times 9\pi.$$

What about the RHS of the Stoke's formula? Well, we need to integrate around the circle of radius 3 centered at the origin. There is only one obvious parametrization for this curve

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = 0 \end{cases}$$

with t ranging from 0 to 2π. Or else, we can write $\underline{x}(t) = (\cos t, \sin t, 0)$, therefore $\underline{x}'(t) = (-\sin t, \cos t, 0)$. We can integrate

$$\begin{aligned}
\int_{\partial S} \underline{F} \cdot d\underline{s} &= \int_0^{2\pi} \underline{F}(\underline{x}(t)) \cdot \underline{x}'(t) dt \\
&= \int_0^{2\pi} (-3\sin t, 3\cos t, 9\cos t - 6\sin t,) \cdot (-3\sin t, 3\cos t, 0)\, dt \\
&= \int_0^{2\pi} 9dt = 18\pi.
\end{aligned}$$