MATHEMATICS 2023 - INFINITE SERIES I (EXAMPLES)
ACADIA UNIVERSITY

1. Questions

Exercise 2. Compute the Taylor expansion of $1/x$ about 1. Find the radius of convergence. Using Taylor Theorem, find a bound on $|1/x - S_2(x)|$ where S_2 is the sum of the first two terms over the interval $[1,2)$. Can you bound the error over $[1,2)$ using another theorem? What result do you get?

Exercise 3. Will $\sum_{k=0}^{\infty} \frac{e^k}{k!}$ converge???

Exercise 4. Will $\sum_{k=0}^{\infty} \frac{k!}{(2k)!}$ converge???

Exercise 5. Will $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ converge?

6. SOLUTIONS

Example. $1/x = 1-(x-1)+(x-1)^2-(x-1)^3+\ldots$. The ratio test gives us $R = |x-1|$ and thus the ratio of convergence is 1. Taylor Theorem bounds the error after two terms by $\frac{\frac{2}{\xi}(x-1)^2}{2} = \frac{(x-1)^2}{\xi} < \frac{(1)^2}{1} = 1$. Using AST, we check that this is an alternating series (assuming $1 > (x-1) > 0$ makes this easy), hence the error after two terms is given by $(x-1)^2 < 1$.

Example. We apply the ratio test, $\frac{e^{k+1}}{(k+1)!}\frac{k!}{e^k} = \frac{e}{k+1}$ which goes to zero when k becomes big.

Example. Again, we apply the ratio test, $\frac{(k+1)!}{(2k+2)!}\frac{(2k)!}{k!} = \frac{k+1}{(2k+2)(2k+1)}$ which clearly goes to zero as k becomes big.

Example. Applying the ratio test, we get $\frac{\ln k}{\ln(k+1)}$ which tends to 1 as k becomes big (just use l'Hospital rule!) However, since $\frac{1}{\ln(k+1)} < \frac{1}{\ln k}$ (ln is monotone increasing!) the series is alternating and thus, it converges!