A Family of 4-point Dyadic High Resolution Subdivision Schemes

Daniel Lemire

Research Officer

National Research Council of Canada (NRC)

email: lemire@ondelette.com

Subdivision: why care?

✔ multiscale approach

Subdivision: why care?

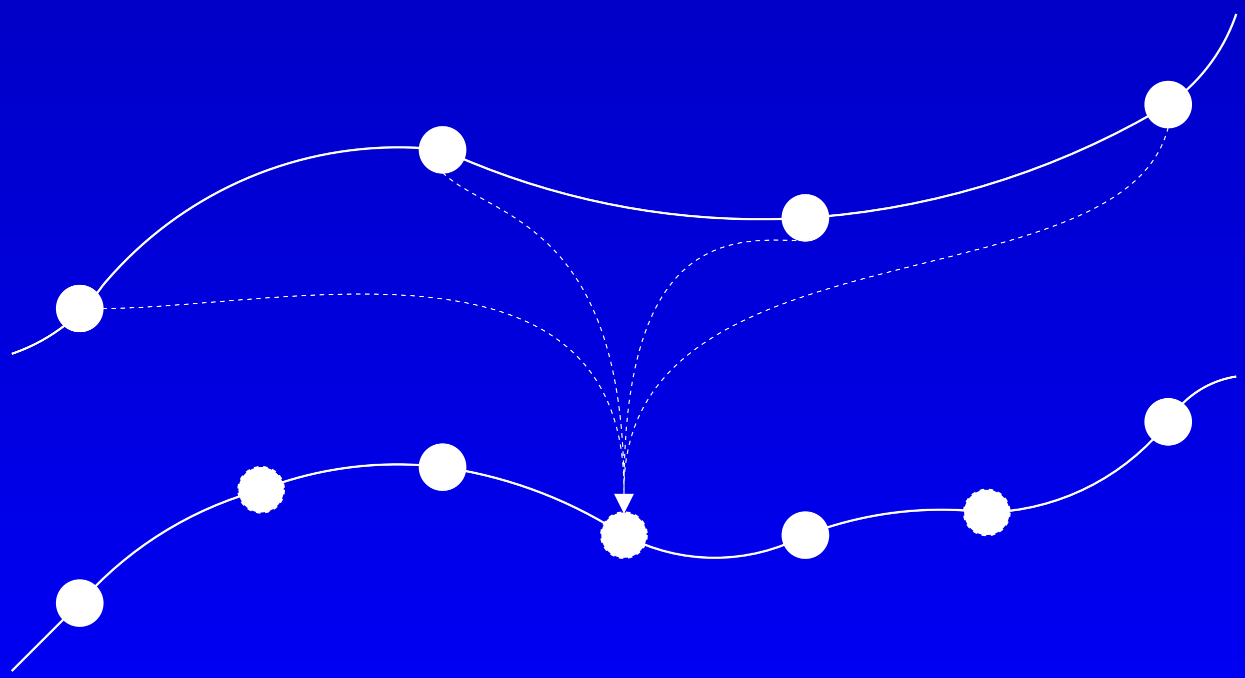
- ✔ multiscale approach
- ✔ local interpolation

Subdivision: why care?

- ✔ multiscale approach
- ✔ local interpolation
- ✔ compactly supported wavelets

How Dubuc did it!

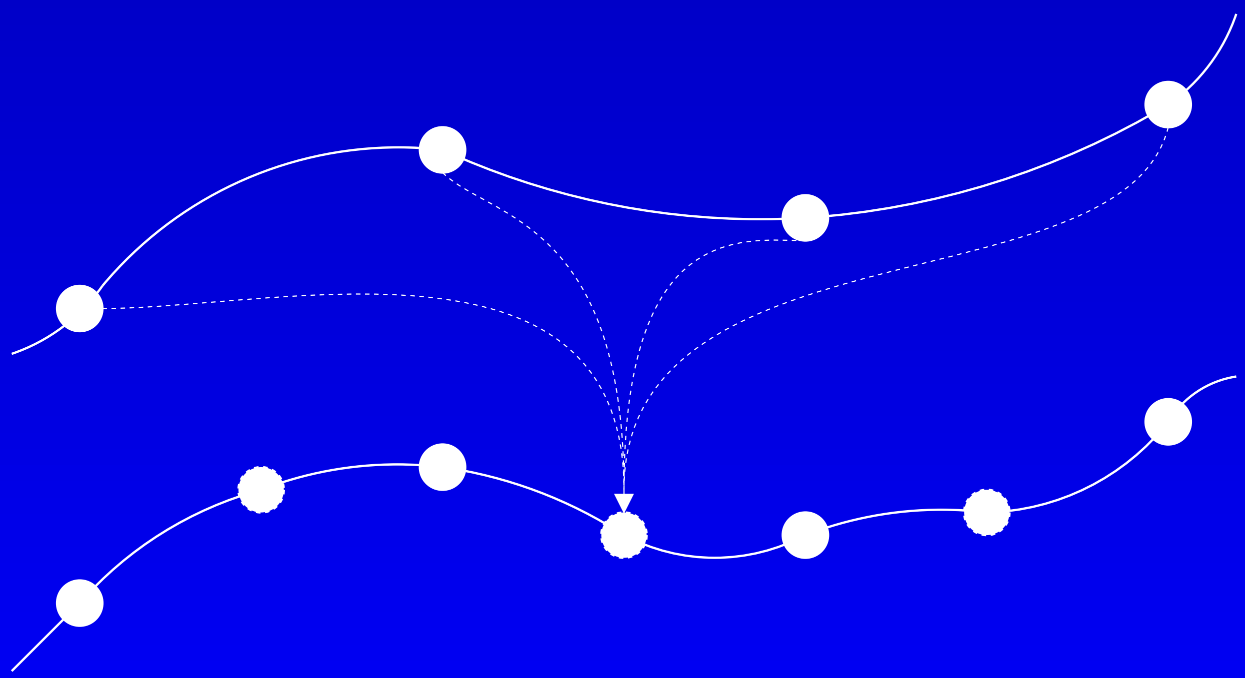
✔ 4 points $\longrightarrow$ cubic polynomial



How Dubuc did it!

✔ 4 points ⟶ cubic polynomial

⟶midpoint value *(dyadic interpolation)*

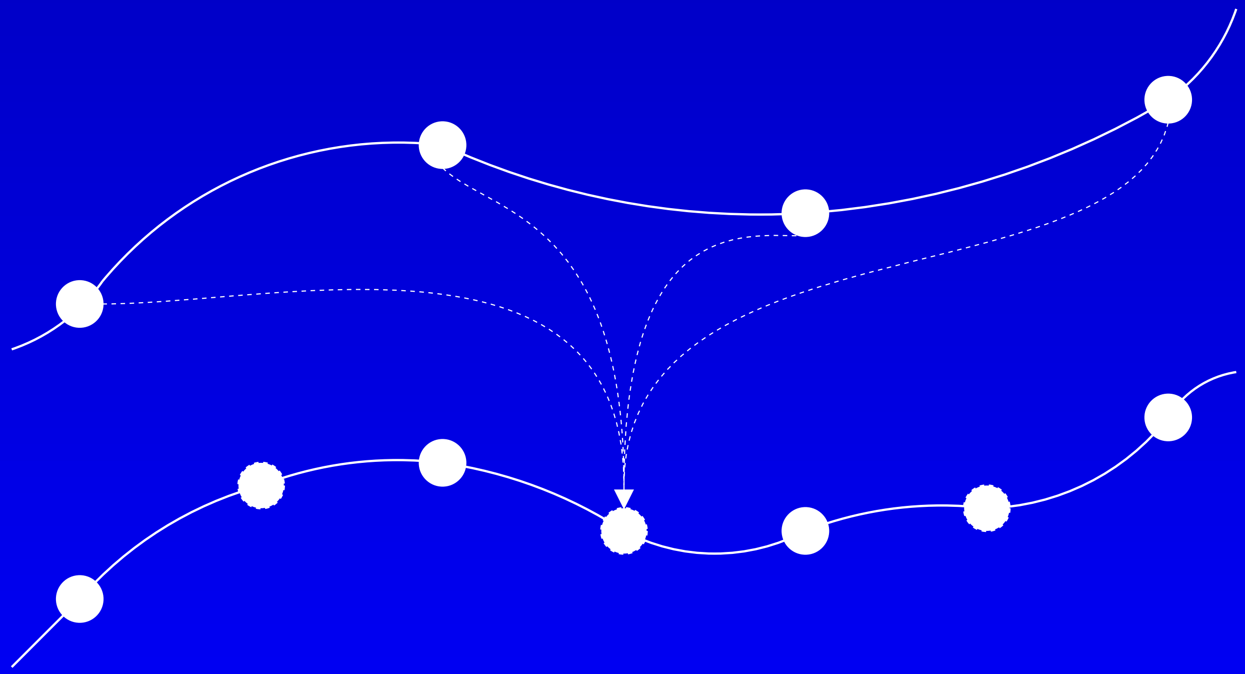


How Dubuc did it!

✔ 4 points $\longrightarrow$ cubic polynomial

$\longrightarrow$midpoint value *(dyadic interpolation)*

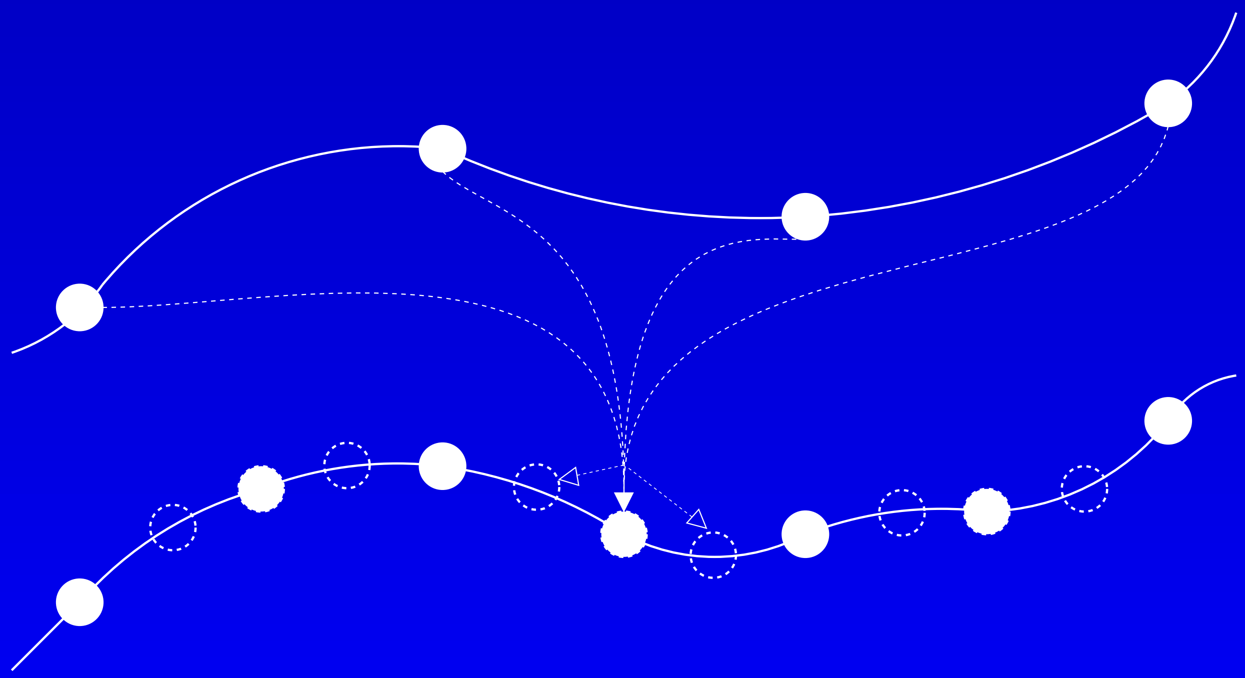
✔ repeat at finer scales



One follow-up...

Same story but...

✔ 4 points $\longrightarrow$ cubic polynomial

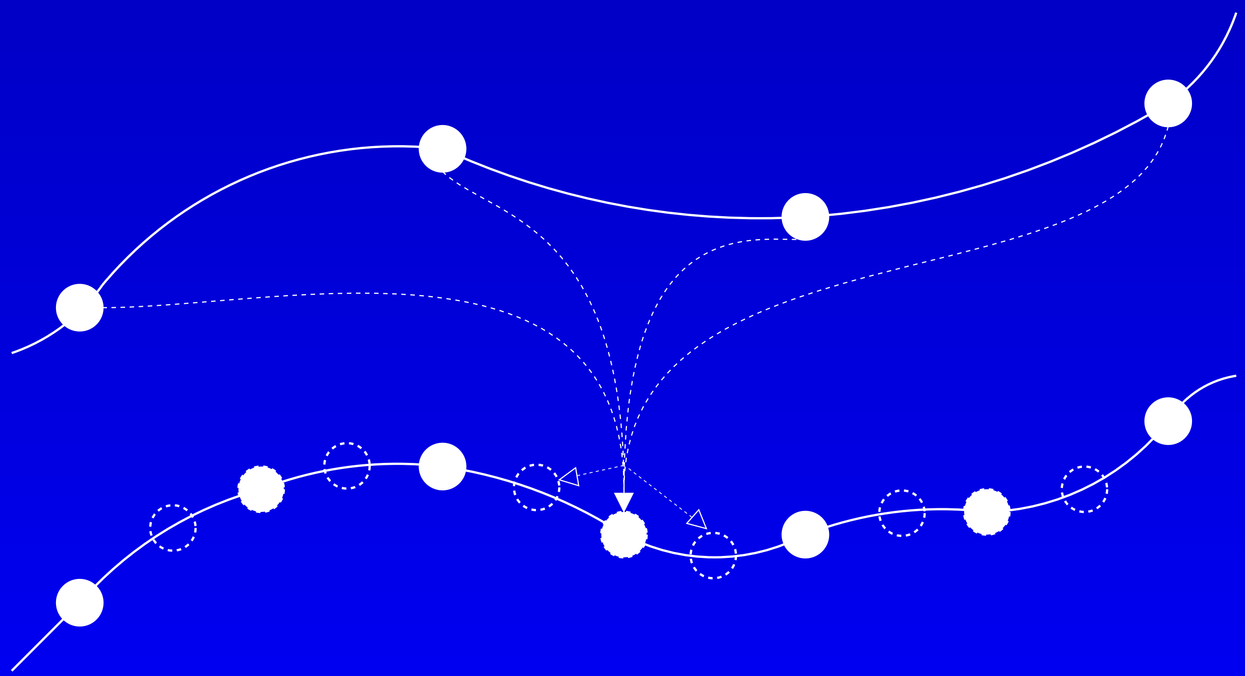


One follow-up...

Same story but...

✔ 4 points ⟶ cubic polynomial

⟶midpoint **and quartile** values *(tetradic interpolation)*



Question

Given a $4-$point subdivision scheme, can we reproduce more than just cubic polynomials?

One more simple trick?

Recall Richardson's extrapolation:

1. *an error or order p*

$$a_{\Delta x} = a_{true} + \underbrace{k\left(\Delta x\right)^{p} + O\left(\Delta x^{p+1}\right)}_{\text{some error}}$$

One more simple trick?

Recall Richardson's extrapolation:

1. *an error or order p*

$$a_{\Delta x} = a_{true} + \underbrace{k\left(\Delta x\right)^p + O\left(\Delta x^{p+1}\right)}_{\text{some error}}$$

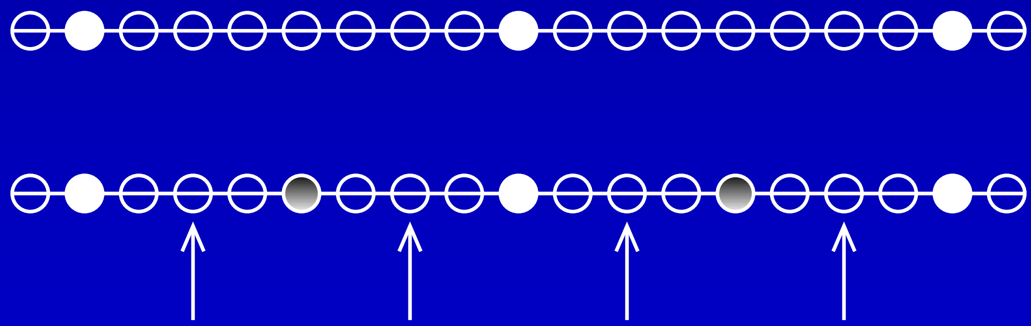
2. $\Delta x \rightarrow \Delta x/2$

$$a_{\frac{\Delta x}{2}} = a_{true} + \underbrace{k\left(\frac{\Delta x}{2}\right)^p + O\left(\Delta x^{p+1}\right)}_{\text{somewhat smaller error}}$$

3. Combine $a_{\Delta x}$ and $a_{\frac{\Delta x}{2}}$,

$$\frac{2^p \times a_{\frac{\Delta x}{2}} - a_{\Delta x}}{2^p - 1} = a_{true} + \underbrace{O\left(\Delta x^{p+1}\right)}_{\text{we gain an order!}}$$

Guessing early or coarsing it up



✔ Ample storage: why not use it early?

HRS Algorithm

1. recopy stable data: $y_{j+1,4k} = y_{j,2k}$;

HRS Algorithm

1. recopy stable data: $y_{j+1,4k} = y_{j,2k}$;

2. Tetradic scheme:

✔ cubic p interpolates $y_{j,2k-2}, y_{j,2k}, y_{j,2k+2}, y_{j,2k+4}$

HRS Algorithm

1. recopy stable data: $y_{j+1,4k} = y_{j,2k}$;

2. Tetradic scheme:

✔ cubic p interpolates $y_{j,2k-2}, y_{j,2k}, y_{j,2k+2}, y_{j,2k+4}$

✔ $y^{\text{coarse}}_{j+1,4k+1} = p\left(x_{j+1,4k+1}\right), y^{\text{fine}}_{j+1,4k+2} = p\left(x_{j+1,4k+2}\right), y^{\text{coarse}}_{j+1,4k+3} = p\left(x_{j+1,4k+3}\right)$

HRS Algorithm

1. recopy stable data: $y_{j+1,4k} = y_{j,2k}$;

2. Tetradic scheme:

✔ cubic p interpolates $y_{j,2k-2}, y_{j,2k}, y_{j,2k+2}, y_{j,2k+4}$

✔ $y^{\text{coarse}}_{j+1,4k+1} = p\left(x_{j+1,4k+1}\right), y^{\text{fine}}_{j+1,4k+2} = p\left(x_{j+1,4k+2}\right), y^{\text{coarse}}_{j+1,4k+3} = p\left(x_{j+1,4k+3}\right)$

1. Update midpoint :

$$y_{j+1,4k+2} = (1-\alpha)y^{\text{fine}}_{j+1,4k+2} + \alpha y^{\text{coarse}}_{j,2k+1}.$$

Smoothness

- $\triangleright$ $\alpha = 0 \Rightarrow$ Dubuc

Smoothness

- $\alpha = 0 \Rightarrow$Dubuc
- Dubuc$\Rightarrow C^1$.
- For $-25/56 < \alpha < 15/32$, the HRS schemes are C^1.

Reproduced polynomials

- HRS always reproduce cubic polynomials.

Reproduced polynomials

- ▷ HRS always reproduce cubic polynomials.
- ▷ Reproduce quartic polynomials when $\alpha = -3/32$.

Reproduced polynomials

▷ HRS always reproduce cubic polynomials.

▷ Reproduce quartic polynomials when $\alpha = -3/32$.

$y_{j-1,k} = p_4\left(x_{j-1,k}\right)$ where $p_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, then

✔ $y^{\text{coarse}}_{j,2k+1} = p_4\left(x_{j,2r+1}\right) - \frac{105a_4}{2^{4j}}$

✔ $y^{\text{fine}}_{j,2k+1} = p_4\left(x_{j,2r+1}\right) - \frac{9a_4}{2^{4j}}$

Initialisation

Need subdivision scheme such that $y_{j-1,k} = p_4\left(x_{j-1,k}\right) \Longrightarrow$

✔ $y_{j,2k+1} = p_4\left(x_{j,2k+1}\right) - \frac{105a_4}{16\times 2^{4j}}$

✔ $y_{j,2k} = p_4\left(x_{j,2k}\right)$

(Possible with a $5-$point s.s.)

Reproducing quartic polynomials

- ▷ subdivision $\Longrightarrow 5-$point scheme

Reproducing quartic polynomials

- $\triangleright$ subdivision $\Longrightarrow$ $5-$point scheme
- $\triangleright$ HRS ($\alpha = -3/32$) $\Longrightarrow$ $5-$point initialization + $4-$point for the rest

How does it compare to Vector Subdivision Schemes?

- ✔ Vector $\Rightarrow$several fixed value per node

How does it compare to Vector Subdivision Schemes?

- ✔ Vector $\Rightarrow$several fixed value per node
- ✔ HRS$\Rightarrow$one value per node but **allowed to changed over time**

Conclusion (a comparative table)

scheme	regularity	reproduced polynomials
Dubuc	C^1	cubic
Deslauriers-Dubuc	C^1	cubic
Dyn-Gregory-Levin	up to C^1	up to cubic
Hassan et al.	C^2	quadratic
presented HRS	up to C^1	cubic to quartic

Key Mathematical Argument

Fourier $\Longrightarrow$

$$P(z) = \sum_k y_{j,k} z^k$$

Key Mathematical Argument

Fourier $\Longrightarrow$

$$P(z) = \sum_k y_{j,k} z^k$$

and

$$\Gamma(z) = 1 + \frac{9\left(z + z^{-1}\right)}{16} - \frac{9\left(z^3 + z^{-3}\right)}{16}$$

then

$$P^{j+1}(z) = \Gamma(z) P^j\left(z^2\right)$$

Key Mathematical Argument

Fourier $\Longrightarrow$

$$P(z) = \sum_k y_{j,k} z^k$$

and

$$\Gamma(z) = 1 + \frac{9\left(z + z^{-1}\right)}{16} - \frac{9\left(z^3 + z^{-3}\right)}{16}$$

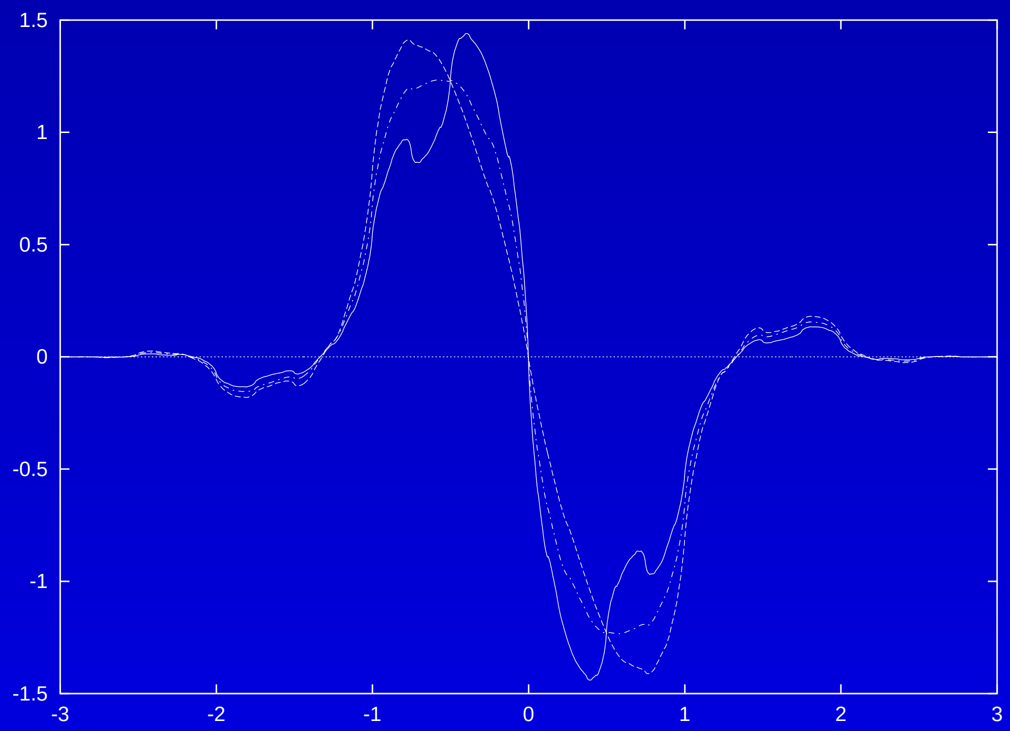
then

$$P^{j+1}(z) = \Gamma(z) P^j\left(z^2\right)$$

HRS requires 2 trig. poly.

$$P^{j+1}(z) = \Gamma_1(z) P^j\left(z^2\right) + \Gamma_2(z) P^j\left(-z^2\right)$$

Bonus material 1 - nice pictures



Derivatives of the fundamental functions for $\alpha = -0.2$ (continuous line), $\alpha = 0$ (dash-dot line), and $\alpha = 0.15$ (dashed line).

Bonus material 2 - intermediate result

▷ (Dyn) Given trigonometric polynomials $\Gamma_1(z)$ and $\Gamma_2(z)$, the HRS scheme defined by

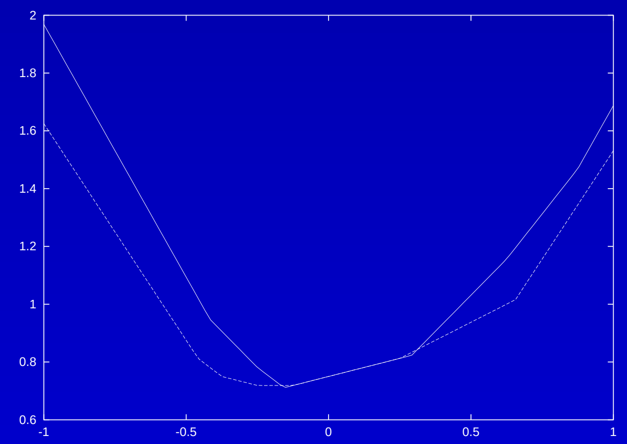
$$P^{j+1}(z) = \Gamma_1(z)P^j\left(z^2\right) + \Gamma_2(z)P^j\left(-z^2\right)$$

is C^n if the symbol corresponding to finite differences of order $n+1$

$$dH_n^j(z) = \frac{2^{jn}(1-z)^{n+1}}{z^{n+1}}P^j(z)$$

is the symbol of a HRS scheme converging uniformly to zero for all bounded initial data.

Bonus material 3 - more on proof



For a given α, an HRS scheme is differentiable if

$$\lambda_{HR}(\alpha) = \max\{\lambda_1(\alpha), \lambda_2(\alpha)\} < 1.$$

Bonus material 4 - proper initialization

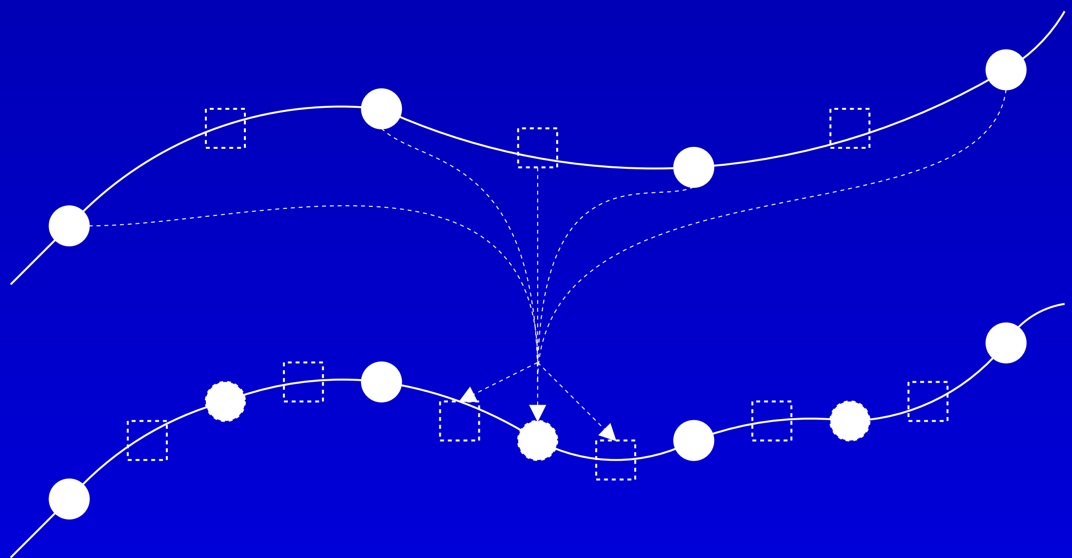
1. recopy data at $x_{j+1,2k} = x_{j,k}$: $y_{j+1,2k} = y_{j,k}$;
2. extrapolate $y_{j,k+4}$ using $y_{j,k-2}, y_{j,k-1}, y_{j,k}, y_{j,k+1}, y_{j,k+2}$ by the formula

$$\gamma_{j,k} = 5y_{j,k-2} - 24y_{j,k-1} + 45y_{j,k} - 40y_{j,k+1} + 15y_{j,k+1}; \qquad (1)$$

3. interpolate midpoint :

$$y_{j+1,2k+1} = \frac{-7y_{j,k-2} + 105y_{j,k} + 35y_{j,k+2} - 5\gamma_{j,k}}{128}.$$

Bonus material 5 - Crunch the tetradic tree into a dyadic one



Some references

References

[1] I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure & Appl. Math. 41, pp. 909–996, 1988.

[2] G. Deslauriers and S. Dubuc, Symmetric iterative interpolation processes, Constr. Approx., 5, pp. 49-68, 1989.

[3] G. Deslauriers, S. Dubuc, and D. Lemire, Une famille d’ondelettes biorthogonales sur l’intervalle obtenue par un schéma d’interpolation itérative, Ann. Sci. Math. Québec 23 no. 1, pp. 37-48, 37-48, 1999.

[4] F. Dubeau and R. Gervais, Procédures locale et non locale d'interpolation à l'aide de fonctions splines quadratiques, Ann. Sci. Math. Québec 23 no. 1, pp. 49-61, 1999.

[5] S. Dubuc, Interpolation through an iterative scheme, J. Math. Anal. Appl., 114, pp. 185-204,1986.

[6] S. Dubuc, D. Lemire, J.-L. Merrien, Fourier analysis of 2-point Hermite interpolatory subdivision schemes, J. of Fourier Anal. Appl., 7 no. 5, 2001.

[7] N. Dyn, Subdivision schemes in computer-aided geometric design, Advances in numerical analysis (W. Light, ed.), vol. 2, Clarendon Press, pp. 36-104, 1992.

[8] N. Dyn, J.A. Gregory, and D. Levin, A 4-point interpolatory subdivision

scheme for curve design. Comput. Aided Geom. Design, 4, pp. 257-268, 1987.

[9] B. Han, Approximation Properties and Construction of Hermite Interpolants and Biorthogonal Multiwavelets, Journal of Approximation Theory, 110 no.1, pp. 18-53, 2001.

[10] M.F. Hassan, I.P. Ivrissimitzis, N.A. Dodgson, and M.A. Sabin, An Interpolating $4-$Point C^2 Ternary Stationary Subdivision Scheme, submitted for publication to CAGD (December 18, 2001).

[11] C. Heil and D. Colella, Matrix refinement equations and subdivision schemes, J. Fourier Anal. Appl., 2, pp. 363-377, 1996.

[12] F. Kuijt and R. van Damme, Stability of subdivision schemes, Mem-

orandum no. 1469, Faculty of Applied Mathematics, University of Twente, the Netherlands.

[13] J.-L. Merrien, A family of Hermite interpolants by bissection algorithms. Numer. Algorithms 2, pp. 187-200, 1992.

[14] J.-L. Merrien, Interpolants d'Hermite C^2 obtenus par subdivision. M2An Math. Model. Numer. Anal. 33, pp. 55-65, 1999.

[15] G. Plonka, Approximation order provided by refinable function vectors, Constr. Approx. 13, pp. 221-244, 1997.