20 years of Wavelets : Tales from Industry and Academia

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http ://www.ondelette.com/acadia/

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– Think : FBI Fingerprint, JPEG2000, DSP
– Nice mix of industry-academia work to achieve current result

Goal of this talk

– Justify wavelets from earlier approaches

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- Fully derive Daubechies wavelets using elementary math !

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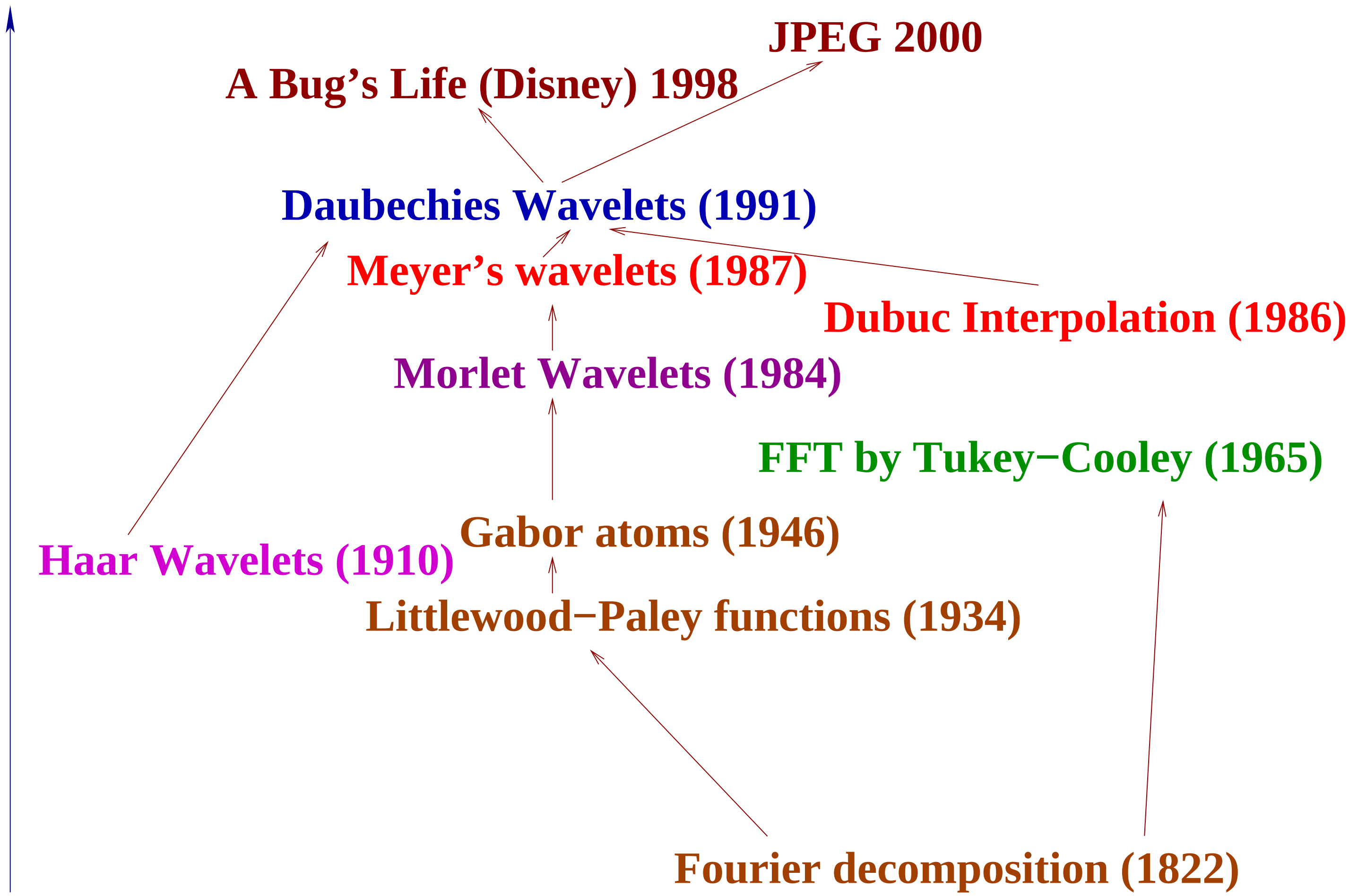
– Fully derive Daubechies wavelets using elementary math !
– Wavelets **don't** have to be hard !

Approximative and Incomplete Timeline



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Orthogonal Transforms and Why They are Good

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- Orthonormality $\Rightarrow$ *preserves* the energy !

Orthogonality by Example

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- Inverse is :$\{0.775, 3.825, 4.575, 1.525\}$

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- $T(x)$, $T(y)$ uncorrelated
- IOW : Orthogonality means things are sane

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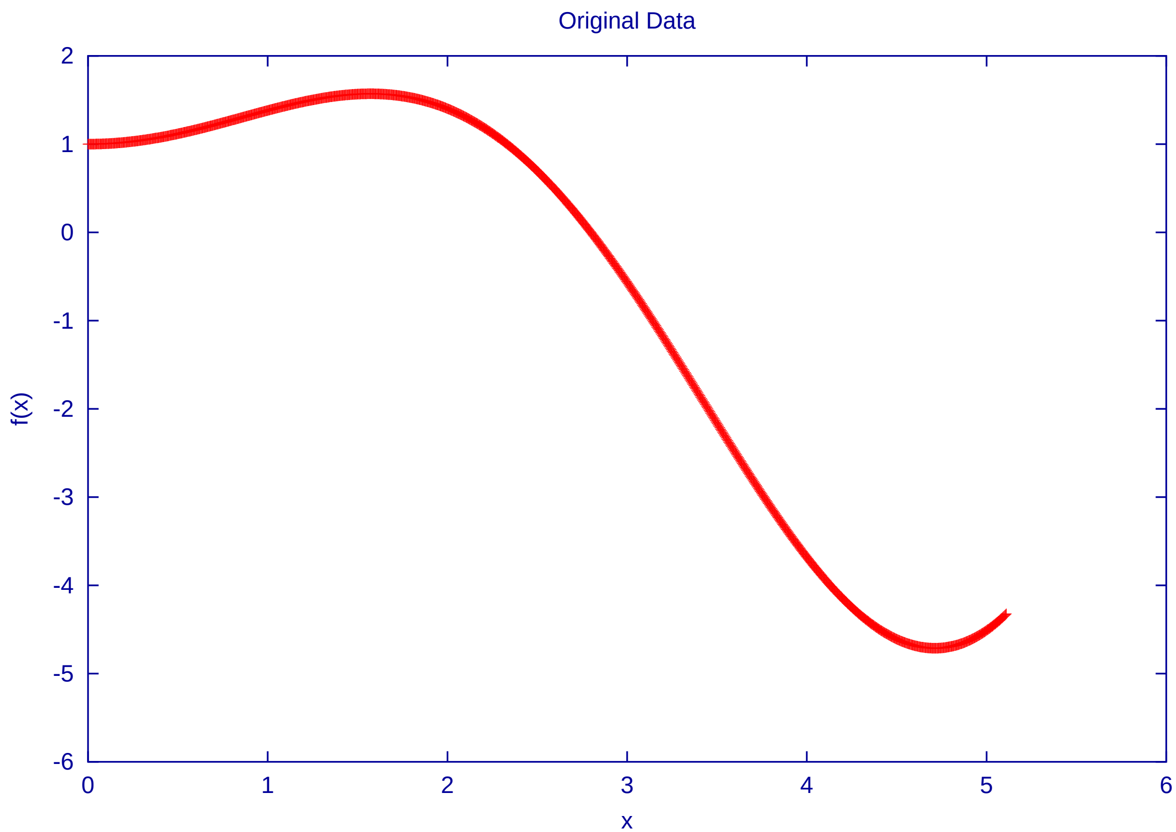
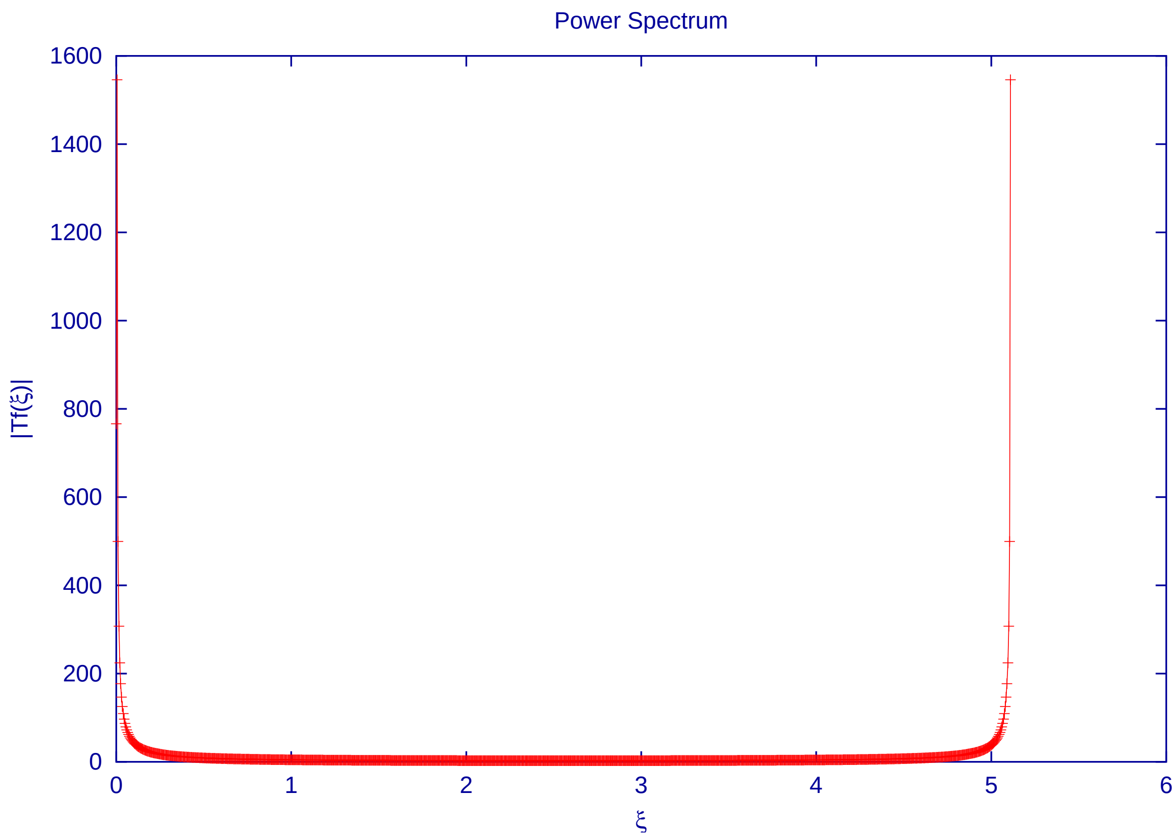
– *Simple* often means **lots of zeros !**
– Result : FFT (Power Spectrum) of smooth signal has lots of zeros !

Why Transform ? (an example)



Why Transform ? (result from example)



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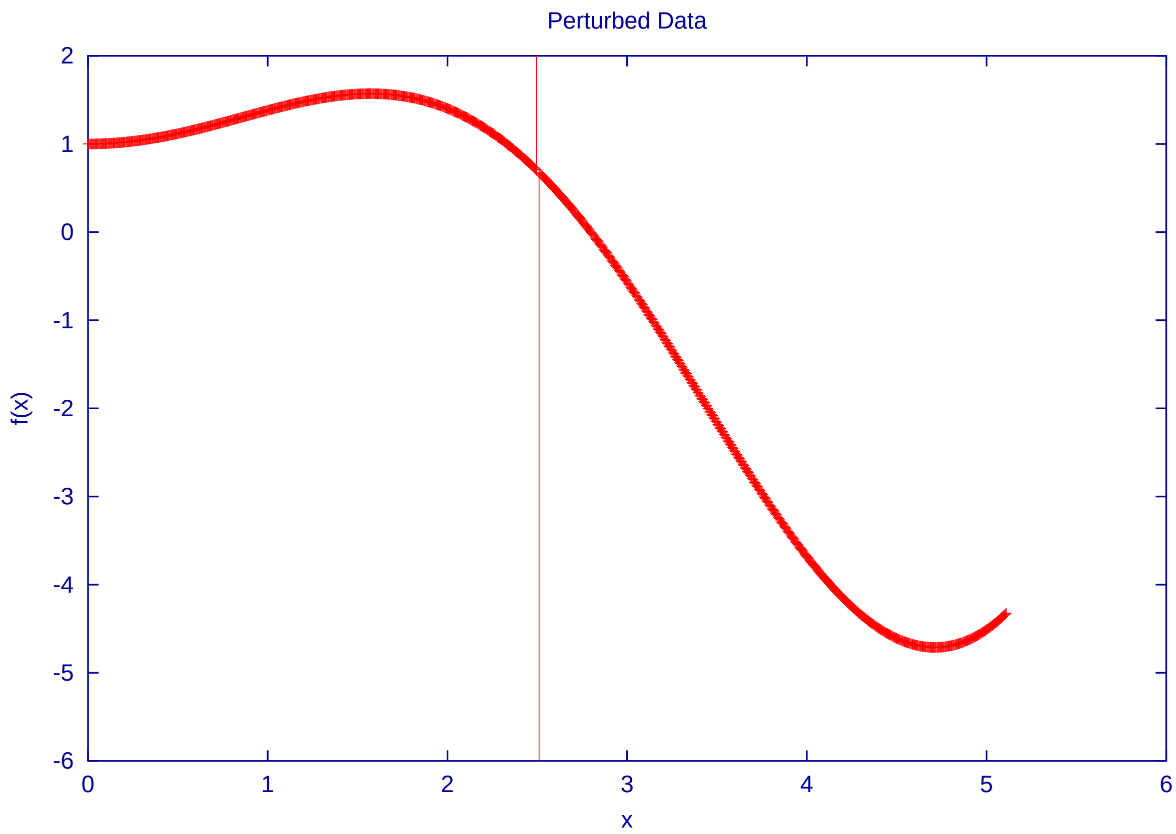
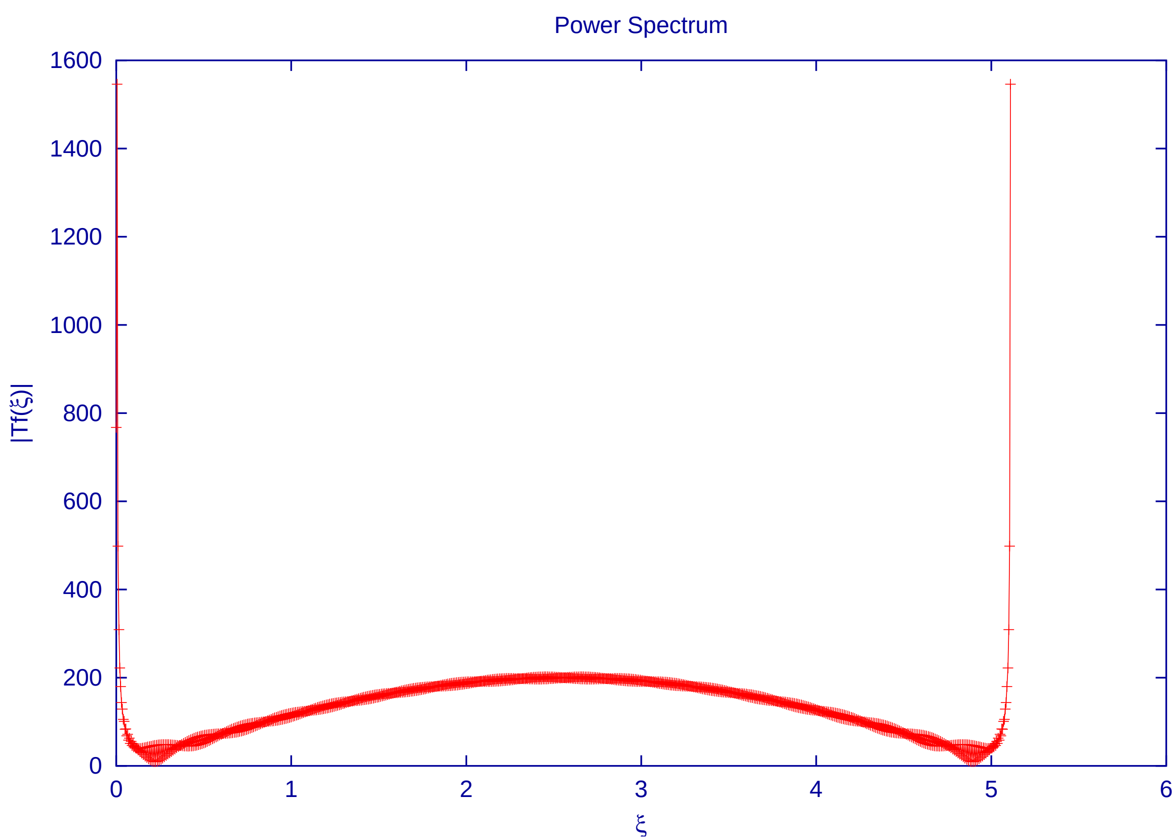
- Lack of locality is a **big**, fundamental flaw !

Why Locality ? (a spike in the data)



Why Locality ? (a very different spectrum)



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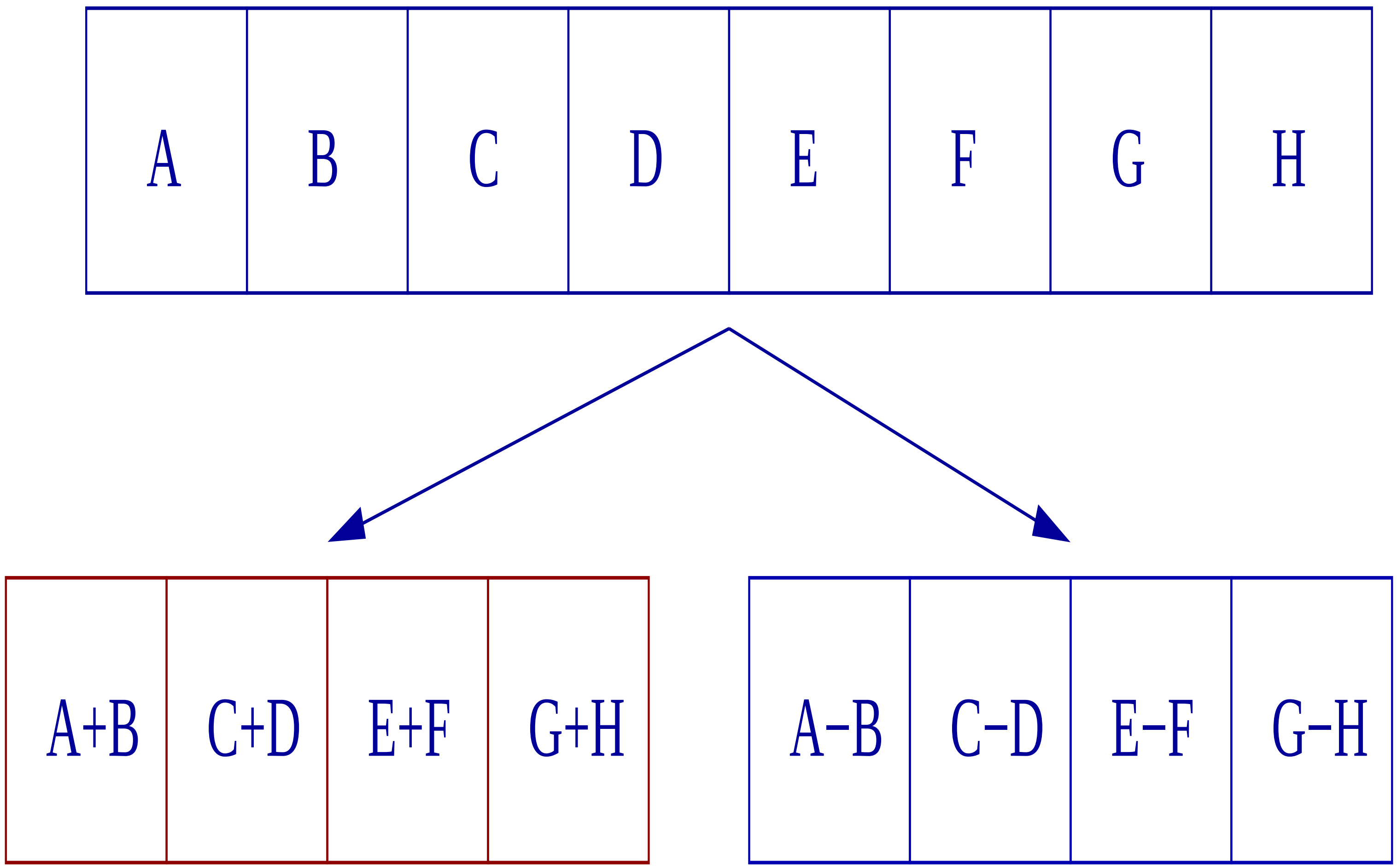
- Lots of events : use short window (low freq resolution)
- Few events : use large window (better freq resolution)
- Suppose you want to track both (think Doppler signal) ?

A simple alternative : Haar (1910)



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- A smooth signal will have a lot of **zeros** : $x_i - x_{i+1}$ is small when sampling is high and signal is smooth
- Example : $1, 1.2, 1.3, 1.2$ transforms to $-0.2, -0.1$ and $2.2, 2.5$.

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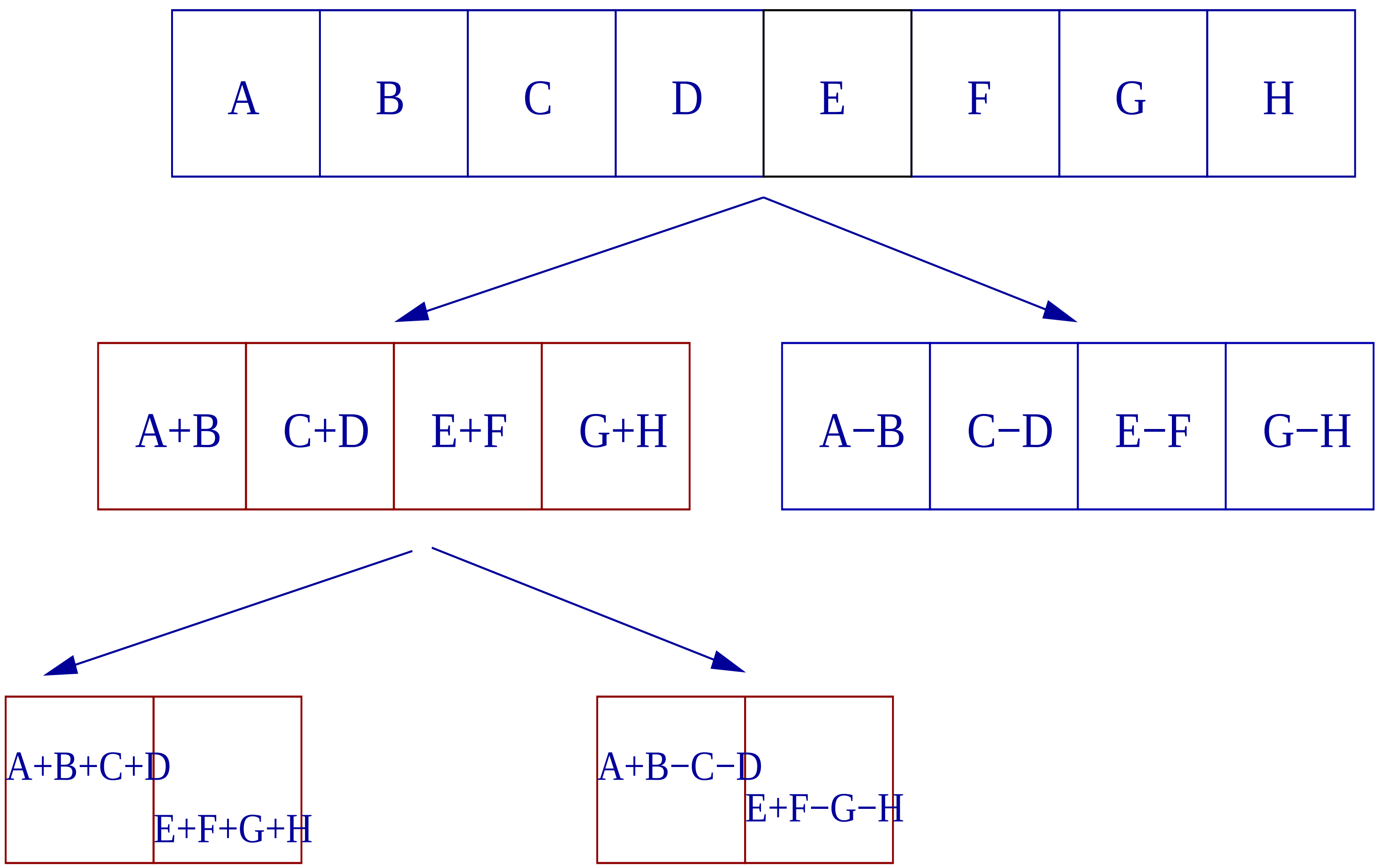
- Yes! Just repeat!
- $1, 1.2, 1.3, 1.2 \longrightarrow -0.2, -0.1$ and $2.2, 2.5$.
- $2.2, 2.5 \longrightarrow -0.3$ and 4.7
- This is magical : everything goes to near zero!!!

Two-scale Haar



Going multiscale

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- And then we could apply the transform to the output of the output of the ouput...
- Kind of like a **mathematical zooming out**!
- This leads to self-similarity (fractals) and multiscale analysis...

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- A French engineer (Morlet) *invented* wavelets, and the math was done by Meyer
- However, wavelets really took off thanks to **Ingrid Daubechies**
- Her work can be seen as a generalization of Haar whereas Morlet's wavelets were not

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- Needs really high sampling frequency
- Not very realistic

Adding two terms

– So, Haar is of the form $ax_i + bx_{i+1}$ and $cx_i + dx_{i+1}$ where $a = 1, b = 1, c = 1, d = -1$ (plus normalization...)

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- Must satisfy $c + d = 0$ (constant goes to zero)
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthonormal implies...
- $ac + bd = 0$, $a^2 + b^2 = 1$, and $c^2 + d^2 = 1$

Setting up the equations

– So let's look at things of the form $a_0 x_i + a_1 x_{i+1} + a_2 x_{i+2} + a_3 x_{i+3}$

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- We want orthonormality which implies, just like with Haar,
- Set $a_0a_2 + a_1a_3 = 0$ (orthogonality condition)

Using Maxima (free software)

```
(C1) solve([a_1+2*a_2+3*a_3=0,a_0+a_1+a_2+a_3=0],[a_
                A2 + 3 a_0          A2 - a_0
(D1) [[A1 = - ---------, A3 = - -------]]
                   2                 2
(C2) ratsimp(expand(subst(D1,a_0*a_2+a_1*a_3)));
          2                 2
        A2  + 6 A0 A2 - 3 A0
(D2)  ---------------------
                4
(C3) solve(%=0,A2);
(D3) [A2=(- 2 SQRT(3) - 3)a_0, A2=(2 SQRT(3)-3)a_0]
```

Solution...

$$[a_0, a_1, a_2, a_3] =$$

$$\left[a_0, \frac{-\left(2\sqrt{3}-3\right)a_0 - 3\,a_0}{2}, \left(2\sqrt{3}-3\right)a_0, \frac{a_0 - \left(2\sqrt{3}-3\right)a_0}{2}\right]$$

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Notice how everything on the right depends linearly on a_0 ?

Finishing...

– Remains to set a_0, we require that $a_0^2 + a_1^2 + a_2^2 + a_3^3 = 1$ (conservation of energy)

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- We've fully derived Daubechies 2 wavelet (can go much higher)
- Can use Cohen's formula to derive lowpass filter : $-a_3, a_2, -a_1, a_0$ (this is orthogonal to a_0, a_1, a_2, a_3)

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- use 4 conditions : constant signal goes to zero, linear slope goes to zero, and orthonormality
- enough to derive wavelet filters up to normalization
- get normalization by requiring conservation of energy
- That is all !

So ?

– We’ve fully derived the Daubechies wavelets !

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– Anyone is crying ?

Daubechies needs multiscale too !

– For the same reason Haar needed to go multiscale...

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– For the same reason Haar needed to go multiscale...
– Daubechies needs to be multiscale
– So we get more zeroes !

Possible Applications

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- unassuming denoising : wavelet shrinkage, much better than lowpass because doesn't kill specific frequencies
- compression : embedded zerotrees, JPEG2000, FBI, better than Fourier because localization buil-tin

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- We have to live with uncertainty principle : can't have both the frequency and the position !

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– As you increase the number of coefficients (taps), you get more zeroes
– In real life, beyond 16 coefficients, there is no more gains...

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- Likely not to be a critical question in practice.

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- Same stuff that's used in JPEG2000 and other wavelet software

Question 1

Can you really compute the highpass filter on their own ?

Yep. You can worry about the lowpass filters only at the very end (Cohen’s formula).

Question 2

You say that orthogonality implies $a_0a_2 + a_1a_3 = 0$*, where does that come from ?*

We require the following vectors to be orthogonal...

$$(a_0, a_1, a_2, a_3, 0, 0)(0, 0, a_0, a_1, a_2, a_3)$$

Question 3

You say that energy conservation implies $a_0^2+a_1^2+a_2^2+a_3^2=1$*, shouldn't that be* $a_0^2+a_1^2=1$ *and* $a_2^2+a_3^2=1$*?*

Nope. We require these vectors to have unit norm...

$$(a_0, a_1, a_2, a_3, 0, 0)(0, 0, a_0, a_1, a_2, a_3)$$

Question 4

What's Cohen's formula ?

Simply put, it is the observation that these vectors are orthogonal...

$$(a_0, a_1, a_2, a_3)(-a_3, a_2, -a_1, a_0)$$

Question 4

How do you generalize to the case where you have 8 coefficients to solve for ?

Do the same kind of computations, but with these vectors (must all be orthonormal)

$$(\quad a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a8, 0, 0, 0, 0, 0, 0 \quad)$$
$$(\quad 0, 0, a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, 0, 0, 0, 0 \quad)$$
$$(\quad 0, 0, 0, 0, a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, 0, 0 \quad)$$
$$(\quad 0, 0, 0, 0, 0, 0, a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7 \quad)$$

Rest of the answer left as an exercise.