HERMITE WAVELETS

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1. Introduction

Our goal is to build wavelets to process Hermite data. Specifically, we want to derive from some classes of Hermite interpolation schemes [DuLeMe] a wavelet basis. This approach is similar to the way Daubechies derived its wavelets from the Deslauriers-Dubuc interpolation scheme [Dau, DeDu]. However, the theoritical foundation of the wavelet transform need to be extended to encompass Hw

2. Multiple Scaling Functions And Hermite Interpolation

2.1. **Multiple scaling functions.** Our goal is to generalize the classical scaling function theory to the case where we have more than one scaling function and then extend to the case where we take into account derivatives of some of the scaling functions when defining duality. Let $m(\xi)$ and $\widetilde{m}(\xi)$ be $n \times n$ matrices of trigonometric polynomials (2π-periodic in ξ). Noting the Fourier transform of f by $\hat{f}$, we want to find n-dimensional vectors $\hat{\phi}$ and $\hat{\gamma}$,

$$\hat{\phi} = \begin{bmatrix} \hat{\phi}_1 \\ \vdots \\ \hat{\phi}_n \end{bmatrix}, \hat{\gamma} = \begin{bmatrix} \hat{\gamma}_1 \\ \vdots \\ \hat{\gamma}_n \end{bmatrix}$$

such that $\hat{\phi}(\xi) = m(\xi/2)\hat{\phi}(\xi/2)$ and $\hat{\gamma}(\xi) = \widetilde{m}(\xi/2)\hat{\gamma}(\xi/2)$. To see what these equations imply, we write $m(\xi) = \sum_k c_k e^{ik\xi}$, and then observe that $\hat{\phi}(\xi) = m(\xi/2)\hat{\phi}(\xi/2) \Rightarrow \phi(x) = \sum_k c_k \phi(2x-k)$ and similarly for γ, so that if $V_j = span_k\{\phi(2^j \cdot -k), \gamma(2^j \cdot -k)\}$ then $V_{j-1} \subset V_j$ and $f \in V_j \Rightarrow f(2\cdot) \in V_{j+1}$.

Lemma 2.1. *Using the fact that $m(\xi)$ and $\widetilde{m}(\xi)$ are 2π-periodic in ξ, we have that* $\int \phi_i(x-l)\gamma_j(x-k)dx = \delta_{k,l}\delta_{i,j} \Leftrightarrow m(\xi/2+\pi)\overline{\widetilde{m}(\xi/2+\pi)^T} + m(\xi/2)\overline{\widetilde{m}(\xi/2)^T} = I$ *a.e.*

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Proof. **Something is wrong with my proof since $\hat{\gamma}_1(\xi)$ should come out and not$\overline{\hat{\gamma}_1(\xi)}$.**

$$I = \begin{pmatrix} \delta_{k,0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_{k,0} \end{pmatrix} = \begin{pmatrix} \int \phi_1(x)\gamma_1(x-k)dx & \cdots & \int \phi_1(x)\gamma_n(x-k)dx \\ \vdots & \ddots & \vdots \\ \int \phi_n(x)\gamma_1(x-k)dx & \cdots & \int \phi_n(x)\gamma_n(x-k)dx \end{pmatrix}$$
$$= \begin{pmatrix} \int \hat{\phi}_1(\xi)\overline{\hat{\gamma}_1(\xi)}e^{ik\xi}d\xi & \cdots & \int \hat{\phi}_1(\xi)\overline{\hat{\gamma}_n(\xi)}e^{ik\xi}d\xi \\ \vdots & \ddots & \vdots \\ \int \hat{\phi}_n(\xi)\overline{\hat{\gamma}_1(\xi)}e^{ik\xi}d\xi & \cdots & \int \hat{\phi}_n(\xi)\overline{\hat{\gamma}_n(\xi)}e^{ik\xi}d\xi \end{pmatrix}$$
$$= \int_0^{2\pi} \sum_l \hat{\phi}(\xi+2\pi l)\overline{\hat{\gamma}^T(\xi+2\pi l)}e^{ik\xi}d\xi$$

From this, we get that $\sum_l \hat{\phi}(\xi+2\pi l)\overline{\hat{\gamma}^T(\xi+2\pi l)} = I/2\pi\, a.e.$ and since $\hat{\phi}(\xi)\hat{\gamma}^T(\xi) = m(\xi/2)\hat{\phi}(\xi/2)\overline{\hat{\gamma}(\xi/2)^T\widetilde{m}(\xi/2)^T}$, we have $\sum_l m(\xi/2+\pi l)\hat{\phi}(\xi/2+\pi l)\overline{\hat{\gamma}(\xi/2+\pi l)^T\widetilde{m}(\xi/2+\pi l)^T} = I/2\pi\, a.e.$. Now, splitting positive and negative l-values, we have

$$\sum_l m(\xi/2+\pi)\hat{\phi}(\xi/2+\pi l)\overline{\hat{\gamma}(\xi/2+\pi l)^T\widetilde{m}(\xi/2+\pi)^T} + \sum_l m(\xi/2)\hat{\phi}(\xi/2+\pi l)\overline{\hat{\gamma}(\xi/2+\pi l)^T\widetilde{m}(\xi/2)^T} = I/2\pi\, a.e.$$

or $m(\xi/2+\pi)\overline{\widetilde{m}(\xi/2+\pi)^T} + m(\xi/2)\overline{\widetilde{m}(\xi/2)^T} = I\, a.e.$. □

Using the fact that

$$\delta_{k,0} = \int \phi(x)\gamma'(x-k)dx = \int \hat{\phi}(\xi)\widetilde{\hat{\gamma}(\xi)}i\xi e^{ik\xi}dx = \int_0^{2\pi}\sum_l i(\xi+2\pi l)\hat{\phi}(\xi)\widetilde{\hat{\gamma}(\xi)}e^{ik\xi}dx,$$

we have the corresponding result that

$$I = \begin{pmatrix} \delta_{k,0} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_{k,0} \end{pmatrix} = \begin{pmatrix} \int \phi_1(x)\gamma_1(x-k)dx & \cdots & \int \phi_1(x)\gamma_n^{(n)}(x-k)dx \\ \vdots & \ddots & \vdots \\ \int \phi_n(x)\gamma_1(x-k)dx & \cdots & \int \phi_n(x)\gamma_n^{(n)}(x-k)dx \end{pmatrix}$$

(where $f^{(n)} = d^n f/dx^n$) implies

$$m(\xi/2+\pi)\overline{\widetilde{m}(\xi/2+\pi)^T} + m(\xi/2)\overline{\widetilde{m}(\xi/2)^T} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/n \end{bmatrix} a.e.$$

Lemma 2.2. *If we write* $I^{(n)} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/n \end{bmatrix}$, *we have that* $\int \phi_i(x-l)\gamma_j^{(j)}(x-k)dx = \delta_{k,l}\delta_{i,j} \Leftrightarrow m(\xi/2+\pi)\overline{\widetilde{m}(\xi/2+\pi)^T} + m(\xi/2)\overline{\widetilde{m}(\xi/2)^T} = I^{(n)}\ a.e.$

2.2. **Hermite interpolation.** Let f and p be defined over the integers. We want to extend f and p so that $f' = p$. One iterative way of doing it is to apply the equations

$$f(x+1/2^n) = \frac{f(x)+f(x+1/2^{n-1})}{2} + \alpha\frac{(p(x+1/2^{n-1})-p(x))}{2^n}$$
$$p(x+1/2^n) = (1-\beta)2^n(f(x+1/2^{n-1})-f(x)) + \beta\frac{p(x+1/2^n)+p(x)}{2}$$

and of course, we could also work within irregular grids instead of dyadic numbers. This scheme depends on two parameters α and β which must be specified (convenient values include $\alpha = -1$ and $\beta = -1$ or $\beta = -1/2$). We define the first fundamental distribution F_0 to be the interpolant we get from the Hermite data $f(k) = \delta_{k,0},\ p(k) = 0\,\forall k \in \mathbf{N}$ and the second fundamental distribution F_1 to come from $f(k) = 0,\ p(k) = \delta_{k,0}\ \forall k \in \mathbf{N}$.

Using the Fourier transform, we can write

$$\begin{aligned} \begin{pmatrix} \hat{F}_0(\xi) \\ \hat{F}_1(\xi) \end{pmatrix} &= \begin{pmatrix} \frac{1+\cos\xi/2}{2} & i\frac{1-\beta}{2}\sin\xi/2 \\ i\alpha\sin\xi/2 & \frac{1+\beta\cos\xi/2}{4} \end{pmatrix} \begin{pmatrix} \hat{F}_0(\xi/2) \\ \hat{F}_1(\xi/2) \end{pmatrix} \\ &= A(\xi/2) \begin{pmatrix} \hat{F}_0(\xi/2) \\ \hat{F}_1(\xi/2) \end{pmatrix} \end{aligned}$$

which is much more convenient for what follows.

2.3. **Hermite basis.** We want that $\forall f \in \Lambda \subset C^1$ a closed subspace, $f = \sum_k \langle f, \Phi_k \rangle \phi_k + \langle f', \Gamma_k \rangle \eta_k$. We have $\phi_k, \eta_k \in \Lambda\ \forall k \in \mathbf{N} \Leftrightarrow \phi_l = \sum_k \langle \phi_l, \Phi_k \rangle \phi_k + \langle \phi_l', \Gamma_k \rangle \eta_k$ and $\eta_l = \sum_k \langle \eta_l, \Phi_k \rangle \phi_k + \langle \eta_l', \Gamma_k \rangle \eta_k$. It suggests the following orthonormality condition (a sufficient condition)

$$\begin{bmatrix} \langle \phi_l, \Phi_k \rangle & \langle \phi_l', \Gamma_k \rangle \\ \langle \eta_l, \Phi_k \rangle & \langle \eta_l', \Gamma_k \rangle \end{bmatrix}_{l,k,l',k'} = I.$$

Assuming $\begin{bmatrix} \hat{\Phi}(\xi) \\ \hat{\Gamma}(\xi) \end{bmatrix} = m(\xi/2) \begin{bmatrix} \hat{\Phi}(\xi/2) \\ \hat{\Gamma}(\xi/2) \end{bmatrix}$ and $\begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\eta}(\xi) \end{bmatrix} = \widetilde{m}(\xi/2) \begin{bmatrix} \hat{\phi}(\xi/2) \\ \hat{\eta}(\xi/2) \end{bmatrix}$ and using the corresponding lemme, we see that we need $m(\xi/2+\pi)\overline{\widetilde{m}(\xi/2+\pi)^T} + m(\xi/2)\overline{\widetilde{m}(\xi/2)^T} = I^{(2)}$ *a.e.*.

We will suppose that ϕ and γ are chosen to be the fundamental distribution of the Hermite scheme defined previously, $\phi_l = F_0(\cdot - l)$ and $\gamma_l = F_1(\cdot - l)$ so that $\begin{pmatrix} \hat{\phi}_l(\xi) \\ \hat{\gamma}_l(\xi) \end{pmatrix} = A(\xi/2) \begin{pmatrix} \hat{\phi}_l(\xi/2) \\ \hat{\gamma}_l(\xi/2) \end{pmatrix}$. From our previous results, we know that we need to find m such that $m(\xi)A(\xi) + m(\xi+\pi)A(\xi+\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$ *a.e.* One solution is to choose $\widetilde{m}(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since $A(\xi) + A(\xi+\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$. This solution amounts to choosing Dirac functions for Φ and Γ: $\Phi_k = \delta_k = \Gamma_k$.

It does suggest however that an orthogonal solution is possible if we can find B such that $B(\xi)\overline{B(\xi)^T} = A(\xi)$. This means that A has to be a positive matrix (eigenvalues are positive)? If we set $B(\xi) = \begin{bmatrix} a \times (1+e^{i\xi/2}) & b \times (1+e^{i\xi/2}) \\ c \times (1-e^{i\xi/2}) & d \times (1-e^{i\xi/2}) \end{bmatrix}$ for some constants a, b, c, d, we have that

$$B(\xi)\overline{B(\xi)^T} = 2 \begin{bmatrix} \left(|a|^2 + |b|^2\right)(1+\cos(\xi/2)) & (a\overline{c} + b\overline{d})i\sin(\xi/2) \\ -\overline{(a\overline{c} + b\overline{d})}i\sin(\xi/2) & \left(|c|^2 + |d|^2\right)(1-\cos(\xi/2)) \end{bmatrix}.$$

This solution is useful if $\beta = -1$, we then have to satisfy

$$\begin{bmatrix} |a|^2 + |b|^2 & a\overline{c} + b\overline{d} \\ -\overline{(a\overline{c} + b\overline{d})} & |c|^2 + |d|^2 \end{bmatrix} = \begin{pmatrix} 1/4 & 1/2 \\ 1/2 & 1/8 \end{pmatrix}$$

However, it would appear that this equation has no solution... so that even in a very simple case,

In the usual (Lagrange) case, we have the the polynomials are non-negative... clearly, we need a similar requirement on A... I think we need it to be "positive" as well...

References

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