Derivatives of the Lagrange iterative interpolation and b-adic Cohen-Daubechies-Feauveau wavelets

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Technical Report EPM/RT-97/28, École Polytechnique de Montréal, Montréal,

April 1997

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B-adic biorthogonal wavelets

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Abstract

We build the equivalent of the Cohen-Daubechies-Feauveau biorthogonal wavelets in the b-adic context. Using the existing b-adic Lagrange iterative interpolation schemes, we proceed by applying multiple derivatives on the B-splines and using integration by parts. This approach is flexible because it doesn't require the Fourier transform : it can lead to wavelets on the interval or on irregular subdivisions.

1. Lagrange iterative interpolation

We briefly recall the b-adic interpolation scheme discussed in [3] ($b > 0$ is an integer). We call real numbers of the form k/b^j where k and j are intergers, *b-adic numbers*. If j is the smallest integer such that $x = k/b^j$ for an integer k, then we say that x is a *b-adic number of depth j*. Given a function y defined on the integers, we want to extend that function to all b-adic numbers. For any integers r and n with $0 < r < b$, $y\,(n + r/b)$ is defined as $p\,(n + r/b)$ where p is the Lagrange polynomial of maximal degree $2N - 1$ such that $p\,(k) = y\,(k)$ $\forall k \in [n - N + 1, n + N] \cap \mathbf{Z}$. This can be iterated to all b-adic numbers.

If we fix b and N, starting with $y(k) = \delta_{k,0}$, then we will get, as a final result, the fundamental function $F_{b,2N-1}$. We will also write $F_{2,2N-1} = F_{2N-1}$. $F_{b,2N-1}$ can always be extended to the reals because it is uniformly continuous on b-adic numbers (a dense set) and, given a positive integer k, there exists N_0 large enough so that for $N \geq N_0 \Rightarrow F_{b,2N-1} \in C^k$. In other words, we can increase the regularity of $F_{b,2N-1}$ merely by increasing N. It can also be seen that this interpolation scheme allows perfect reconstruction of polynomials up to degree $2N-1$.

Remark 1. *We use the classical* Lagrange *interpolation scheme for clarity, however all we really need is differentiability and perfect reconstruction of polynomials, up to a certain degree (to get compact supports). For example, numerous $M-$band filters could be used instead [6] but also much more general schemes.*

It can be shown that we have the functional equation

$$F_{b,2k+1}(t) = \sum_n F_{b,2k+1}(n/b)\, F_{b,2k+1}(bt-n)$$

Applying the Fourier transform, we get

$$\widehat{F_{b,2k+1}}(\xi) = p_{b,2k+1}(\xi)\, \widehat{F_{b,2k+1}}(\xi/b)$$

where

$$p_{b,2k+1}(\xi) = \sum_n F_{b,2k+1}(n/b)\, e^{-in\xi}.$$

2. Correlation function

Let $F_{\widetilde{\phi},\phi}$ be the correlation function of the couple $\left(\widetilde{\phi},\phi\right)$

$$F_{\widetilde{\phi},\phi}(y) = \int_{-\infty}^{\infty} \widetilde{\phi}(x)\,\phi(x-y)\,dx$$

We say that $\widetilde{\phi}$ and ϕ are dual functions if

$$\int_{-\infty}^{\infty} \widetilde{\phi}(x)\,\phi(x-k)\,dx = \delta_{0,k}$$

It can be checked easily that whenever $\widetilde{\phi}$ are ϕ dual functions, then $F_{\widetilde{\phi},\phi}(k) = \delta_{k,\mathbf{0}}$ for any integer k. This is why we say that the correlation function of two dual functions is also the fundamental function of an interpolation scheme.

We note the Fourier transform by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx$$

Lemma 2.1. *If $\widetilde{\phi}$, $\phi \in L^2$, then $\widehat{F}_{\widetilde{\phi},\phi}(\xi) = \sqrt{2\pi}\,\overline{\widehat{\phi}(\xi)}\,\widehat{\widetilde{\phi}}(\xi)$*

Proof. Because the Fourier transform is unitary on L^2,

$$\begin{aligned}
\widehat{F}_{\widetilde{\phi},\phi}(\xi) &= \frac{1}{\sqrt{2\pi}}\int e^{-i\xi y}F_{\widetilde{\phi},\phi}(y)\,dy \\
&= \frac{1}{\sqrt{2\pi}}\int \widetilde{\phi}(x)\left(\int e^{-i\xi y}\phi(x-y)\,dy\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int \widetilde{\phi}(x)\left(\int e^{-i\xi(-u+x)}\phi(u)\,du\right)dx \\
&= \overline{\widehat{\phi}(\xi)}\int \widetilde{\phi}(x)\,e^{-i\xi x}dx \\
&= \sqrt{2\pi}\,\overline{\widehat{\phi}(\xi)}\,\widehat{\widetilde{\phi}}(\xi) \quad \blacksquare
\end{aligned}$$

In what follows, we will assume that $\widetilde{\phi}$ and ϕ are dual functions in $L^2(\mathbf{R})$ and that they satisfy

$$\widehat{\phi}(\xi) = m_0(\xi/b)\,\widehat{\phi}(\xi/b) \tag{2.1}$$

$$\widehat{\widetilde{\phi}}(\xi) = \widetilde{m}_0(\xi/b)\,\widehat{\widetilde{\phi}}(\xi/b) \tag{2.2}$$

Lemma 2.2. *If $\widetilde{\phi}$ and ϕ satisfy the equations 2.1 and 2.2, then $F_{\widetilde{\phi},\phi}$ satisfy a similar equation,*

$$\widehat{F}_{\widetilde{\phi},\phi}(\xi) = p(\xi/b)\,\widehat{F}_{\widetilde{\phi},\phi}(\xi/b)$$

where

$$p(\xi) = \overline{m_0}(\xi)\,\widetilde{m}_0(\xi) \tag{2.3}$$

Proof.

$$\begin{aligned}
\widehat{F}_{\widetilde{\phi},\phi}(\xi) &= \sqrt{2\pi}\overline{\widehat{\phi}(\xi)}\widehat{\widetilde{\phi}}(\xi) \\
&= \overline{m_0}(\xi/b)\,\widetilde{m}_0(\xi/b)\,\sqrt{2\pi}\overline{\widehat{\phi}}(\xi/b)\,\widehat{\widetilde{\phi}}(\xi/b) \\
&= \overline{m_0}(\xi/b)\,\widetilde{m}_0(\xi/b)\,\widehat{F}_{\widetilde{\phi},\phi}(\xi/b) \;\blacksquare
\end{aligned}$$

Example 2.3. *Suppose that $\phi = \widetilde{\phi} =_N\phi$ is a Daubechies scaling function. (There is a $b-$adic version of Daubechies wavelets but we only consider the original one [6].) It is known that the corresponding correlation function $F_{N\phi,N\phi}$ is going to be the fundamental function of the Lagrange iterative interpolation ([2, section 6.5], [9], [10]), in fact, $F_{N\phi,N\phi} = F_{2N-1}$. We know the filter of the scaling function $_N\phi$: we have*

$$m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N \mathcal{L}(\xi)$$

where $|\mathcal{L}(\xi)|^2 = L(\xi)$ satisfy

$$L(\xi) = P\left(\sin^2 \xi/2\right)$$

with

$$P(y) = P_N(y)$$

where

$$P_N(y) = \sum_{k=0}^{N-1} \begin{pmatrix} N-1+k \\ k \end{pmatrix} y^k$$

Our formula 2.3 gives

$$p_{2N-1}(\xi) = \cos^{2N}\xi/2 \sum_{k=0}^{N-1} \begin{pmatrix} N-1+k \\ k \end{pmatrix} \sin^{2k}\xi/2 \tag{2.4}$$

Example 2.4. *Now suppose that ϕ and $\widetilde{\phi}$ are Cohen-Daubechies-Feauveau scaling functions [1, section 6] (i.e. $\widetilde{\phi} = {}_{N,\widetilde{N}}\widetilde{\phi}$ and $\phi = {}_N\phi$). Since $N+\widetilde{N}$ is always even, define $n = \left(N+\widetilde{N}\right)/2$. The filters are given by*

$$_N m_0(\xi) = e^{-i\kappa\xi/2}\cos^N\xi/2 \tag{2.5}$$

$$_{N,\widetilde{N}}\widetilde{m}_0(\xi) = e^{-i\kappa\xi/2}\cos^{\widetilde{N}}\xi/2 \sum_{k=0}^{n-1} \begin{pmatrix} n-1+k \\ k \end{pmatrix} \sin^{2k}\xi/2 \tag{2.6}$$

with $\kappa = 1$ if N is odd and $\kappa = 0$ if N is even. By multiplying those formulas, we get

$$\overline{_N m_0}(\xi)_{N,\widetilde{N}}\,\widetilde{m}_0(\xi) = (\cos\xi/2)^{N+\widetilde{N}} \sum_{k=0}^{n-1} \begin{pmatrix} n-1+k \\ k \end{pmatrix} \sin^{2k}\xi/2.$$

We see that

$$p_{N+\widetilde{N}-1}\left(\xi\right)=\overline{{}_{N}m_{0}}\left(\xi\right)_{N,\widetilde{N}}\,\widetilde{m}_{0}\left(\xi\right)$$

(see equation 2.4) which implies that

$$F_{N+\widetilde{N}-1}\left(y\right)=\int{}_{N,\widetilde{N}}\widetilde{\phi}\left(x\right)\,{}_{N}\phi\left(x-y\right)dx \tag{2.7}$$

3. Donoho's dyadic average-interpolation

David Donoho [5] build a family of dual multiresolution to the Haar function. It is easy to verify that his wavelets are the first derivatives of the fundamental functions of the Lagrange iterative interpolation (F_n). Indeed, he shows that

$$\frac{d}{dx}F_{n+1}\left(x\right)=\varphi_{n}\left(x+1\right)-\varphi_{n}\left(x\right)$$

where φ_n is the scaling function satisfying

$$\int_{k}^{k+1}\varphi_{n}\left(x\right)dx=\delta_{k,0}$$

and such that it allows perfect reconstruction of polynomials up to degree n. He notices also [5, section 3.2.2] that φ_2,φ_4 and φ_6 have, numerically, the same filters as the Cohen-Daubechies-Feauveau scaling functions[1, section 6] noted by ${}_{1,3}\widetilde{\phi}$, ${}_{1,5}\widetilde{\phi}$ and ${}_{1,7}\widetilde{\phi}$. In fact, we know now that there exists more general results. For example, after rescaling, we have that

$$ {}_{N,\widetilde{N}}\widetilde{\psi} \sim \frac{d^N}{dx^N} F_{\widetilde{N}+N-1} $$

where ${}_{N,\widetilde{N}}\widetilde{\psi}$ is a Cohen-Daubechies-Feauveau wavelets. We will show this result (see theorem 5.1).

4. B-adic multiresolution

Recall that in a b-adic multiresolution [2] is given first by a family of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ satisfying

$$ ...V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset ... $$

and

$$ \overline{\cup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}), $$

$$ \cap_{j\in\mathbf{Z}} V_j = \{0\}. $$

The b-adic component comes in when we require that $f \in V_j \Leftrightarrow f(b^j \cdot) \in V_0$. Suppose now that we have two such multiresolutions $\{V_j\}_{j\in\mathbf{Z}}$ and $\left\{\widetilde{V}_j\right\}_{j\in\mathbf{Z}}$. Moreover, we need two dual *scaling functions* ϕ and $\widetilde{\phi}$ together with $2(b-1)$ wavelets $\{\psi_1, ..., \psi_{b-1}\}$ and $\left\{\widetilde{\psi}_1, ..., \widetilde{\psi}_{b-1}\right\}$. We call ϕ and $\widetilde{\phi}$ *scaling functions* because $\{\phi(\cdot - n)\}_{n\in\mathbf{Z}}$ and $\left\{\widetilde{\phi}(\cdot - n)\right\}_{n\in\mathbf{Z}}$ must be basis for V_0 and $\widetilde{V}_0$ respectively. The wavelets form dual sets $\left\langle \psi_k, \widetilde{\psi}_l \right\rangle = \delta_{k,l}$ and they must be perpendicular to the scaling functions $\left\langle \psi_k, \widetilde{\phi} \right\rangle = \left\langle \phi, \widetilde{\psi}_k \right\rangle = 0 \ \forall k = 1, ..., b-1$. All those functions must have filters, that is

$$
\begin{aligned}
\widehat{\phi}(\xi) &= m(\xi/b)\,\widehat{\phi}(\xi/b), \\
\widehat{\widetilde{\phi}}(\xi) &= \tilde{m}(\xi/b)\,\widehat{\widetilde{\phi}}(\xi/b)
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\psi_k}(\xi) &= g_k(\xi/b)\,\widehat{\phi}(\xi/b), \\
\widehat{\widetilde{\psi}_k}(\xi) &= \tilde{g}_k(\xi/b)\,\widehat{\widetilde{\phi}}(\xi/b)
\end{aligned}
$$

$\forall k = 1, ..., b-1$. $\{\psi_k(\cdot - n)\}_{n\in\mathbf{Z}}$ and $\left\{\widetilde{\psi}_k(\cdot - n)\right\}_{n\in\mathbf{Z}}$ are basis for spaces W_0^k and $\widetilde{W}_0^k$ that satisfy

$$V_{-1} = V_0 \oplus W_0^1 \oplus ... \oplus W_0^{b-1},$$

$$\widetilde{V}_{-1} = \widetilde{V}_0 \oplus \widetilde{W}_0^1 \oplus ... \oplus \widetilde{W}_0^{b-1}.$$

We can verify that B-splines of degree N can be used as scaling functions where the corresponding V_j's are the spline spaces with knots at the b-adic numbers of depth j. Moreover, as we will show, a natural dual multiresolution to the spline multiresolution is generated by

$$\widetilde{V}_0 = \overline{span\left\{F_b^{(N)}(\cdot - n) : n \in \mathbf{Z}\right\}}$$

where $F_b^{(N)}$ is the N^{th} derivatives of a b-adic fundamental function. One can see how natural this is, merely by using integration by parts and realizing that the N^{th} derivative of a B-spline of degree N is a combinaison of Dirac delta functions on the integers. As it turns out, in the dyadic case, this leads to Cohen-Daubechies-Feauveau wavelets.

5. B-adic CDF Wavelets

Remark 2. *Note that in the dyadic case, some of what follows can be done using Lemarié's "formule de commutation" [7, Proposition 2].*

If ${}_N\phi$ is the *b-spline* of degree N, we have

$$\frac{d^N}{dx^N}\,{}_N\phi\,(x) = \sum_{l=0}^{N} \begin{pmatrix} N \\ l \end{pmatrix} (-1)^l\,\delta\,(x - l + [N/2]) \tag{5.1}$$

Suppose that $\widetilde{N}$ is chosen sufficiently large so that we have $F_{N+\widetilde{N}-1} \in C^N$. We apply N derivatives to the equation 2.7, we then have

$$\begin{aligned} \frac{d^N}{dy^N} F_{N+\widetilde{N}-1}\,(y) &= \int {}_{N,\widetilde{N}}\widetilde{\phi}\,(x)\,\frac{d^N}{dy^N}\,{}_N\phi\,(x-y)\,dx \\ &= \sum_{l=0}^{N} \begin{pmatrix} N \\ l \end{pmatrix} (-1)^{N+l}\;{}_{N,\widetilde{N}}\widetilde{\phi}\,(y + l - [N/2]) \end{aligned} \tag{5.2}$$

So we can take

$${}_{N,\widetilde{N}}\widetilde{\psi}_k\,(z) = aF^{(N)}_{N+\widetilde{N}-1}\,(bz-k) \;\text{ where } k = 1, ..., b-1$$

for $a \in \mathbf{R}/\{0\}$ a normalisation coefficient. Indeed, using integration by parts, it can be seen that

$$\int {}_N\phi(x)\, F^{(N)}_{N+\widetilde{N}-1}(bx-k)\, dx = 0$$

since $F_{N+\tilde{N}-1}(bn-k) = 0$ for $n \in \mathbf{Z}$ and $k = 1, ..., b-1$. Finally, it is convenient to choose

$$a = 1/\left\| F^{(N)}_{N+\widetilde{N}-1}(b\cdot) \right\|_{L^2}.$$

Corresponding filters can be easily computed (see tables I and II where we chose $a = 1$ for simplicity).

5.1. Dyadic case

Now we take the Fourier transform of 5.2 to obtain

$$\widehat{F^{(N)}_{N+\widetilde{N}-1}}(\xi) = \left(\sum_{l=0}^{N} \begin{pmatrix} N \\ l \end{pmatrix} (-1)^{N+l} e^{i\xi(l-[N/2])} \right) {}_{N,\widetilde{N}}\widehat{\widetilde{\phi}}(\xi) \tag{5.3}$$

Let $q_N(\xi)$ be the function defined as

$$\widehat{F^{(N)}_{N+\widetilde{N}-1}}(\xi) = q_N(\xi)\, {}_{N,\widetilde{N}}\widehat{\widetilde{\phi}}(\xi) \tag{5.4}$$

The equation 5.3 gives a formula for $q_N(\xi)$

$$
\begin{aligned}
q_N(\xi) &= \sum_{l=0}^{N} \begin{pmatrix} N \\ l \end{pmatrix} (-1)^{N+l} e^{-i\xi(-l+[N/2])} \\
&= z^{[N/2]} \sum_{l=0}^{N} \begin{pmatrix} N \\ l \end{pmatrix} (-1)^{N+l} z^{-l}
\end{aligned}
$$

with $z = e^{-i\xi}$. We can then compute that

$$
\begin{aligned}
q_N(\xi) &= (-1)^N z^{[N/2]} \left(1 - z^{-1}\right)^N \\
&= (-1)^N z^{-\kappa/2} \left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)^N \\
&= (-2i)^N e^{i\kappa\xi/2} \sin^N \xi/2
\end{aligned}
$$

with $\kappa = 1$ if N is odd and $\kappa = 0$ if N is even.

Recall that Cohen, Daubechies and Feauveau chose the following filters for their wavelets :

$$
\begin{aligned}
{}_{N,\widetilde{N}}m_1(\xi) &= \overline{{}_{N,\widetilde{N}}\widetilde{m}_0(\xi+\pi)} e^{-i\xi} \\
{}_{N}\widetilde{m}_1(\xi) &= \overline{{}_{N}m_0(\xi+\pi)} e^{-i\xi}
\end{aligned}
$$

We are especially interested by the filter of the dual wavelet,

$$\begin{aligned} {}_N\widetilde{m}_1(\xi) &= i^\kappa e^{i\kappa\xi/2} e^{-i\xi} \cos^N\left(\frac{\xi+\pi}{2}\right) \\ &= (-1)^N e^{-i\xi} i^\kappa e^{i\kappa\xi/2} \sin^N \xi/2 \\ &= (-1)^{[N/2]} 2^{-N} e^{-i\xi} q_N(\xi) \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \widehat{{}_{N,\widetilde{N}}\widetilde{\psi}}(\xi) &= (-1)^{[N/2]} 2^{-N} e^{-i\xi/2} q_N(\xi/2) \; {}_{N,\widetilde{N}}\widehat{\widetilde{\phi}}(\xi/2) \\ &= (-1)^{[N/2]} 2^{-N} e^{-i\xi/2} \; \widehat{F^{(N)}_{N+\widetilde{N}-1}}(\xi/2) \end{aligned}$$

We can then take the inverse transform to get an explicit formula.

Theorem 5.1. *Assume that $\widetilde{N}$ is sufficiently large to have $F_{N+\widetilde{N}-1} \in C^N$, if ${}_{N,\widetilde{N}}\widetilde{\psi}$ is a Cohen-Daubechies-Feauveau wavelet and if $F^{(N)}_{N+\widetilde{N}-1}$ is the N^{th} derivative of the fundamental function of the Lagrange iterative interpolation of degree $N+\widetilde{N}-1$ then*

$${}_{N,\widetilde{N}}\widetilde{\psi}(z) = (-1)^{[N/2]} 2^{1-N} \; F^{(N)}_{N+\widetilde{N}-1}(2z-1)$$

We can use this result to understand how the regularity and geometry of the wavelets ${}_{N,\widetilde{N}}\widetilde{\psi}$ will vary according to N and $\tilde{N}$ (see figures 5.1, 5.2, 5.3 and 5.4).

For example, we can now see clearly why on increasing $\tilde{N}$ for fixed N the shape of ${}_{N,\widetilde{N}}\widetilde{\psi}$ remains the same for $\widetilde{N}$ large (see [2] p.271).

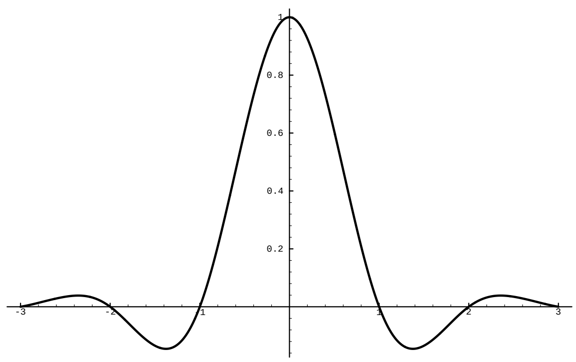

Figure 5.1: $F_9\,(x)$

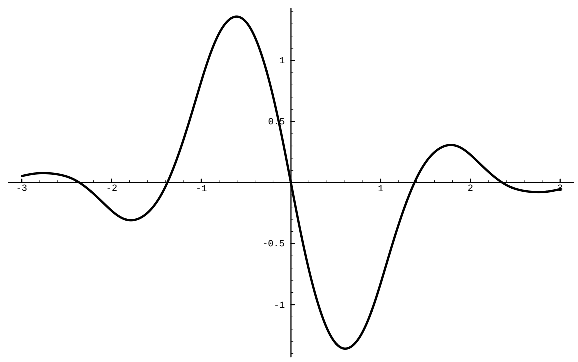

Figure 5.2: $\frac{d}{dx}F_9\left(x\right) = {}_{1,9}\widetilde{\psi}\left(\frac{x+1}{2}\right)$

6. ***Interpolative lifting*** of the ***B-splines***

In general, we can choose

$$
{}_{N,\widetilde{N}}\widetilde{\phi}\left(x\right) = \sum_{k=0}^{\infty} \left(\begin{array}{c} N-1+k \\ \\ N-1 \end{array} \right) F^{(N)}_{N+\widetilde{N}-1}\left(x-k-\kappa-[N/2]\right)
$$

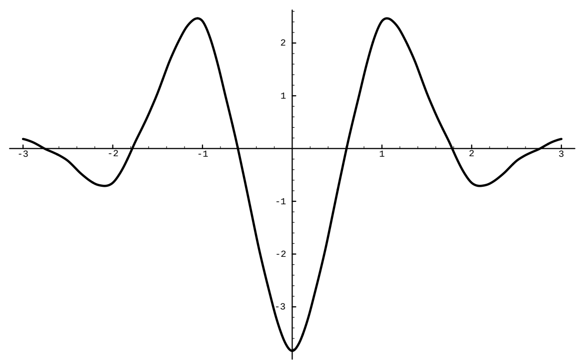

Figure 5.3: $\frac{d^2}{dx^2}F_9(x) = -2\ {}_{2,8}\widetilde{\psi}\left(\frac{x+1}{2}\right)$

since integration by parts allows to check easily that $\int {}_N\phi(x)_{N,\widetilde{N}}\widetilde{\phi}(x-n)\,dx = \delta_{0,n}$ (use equation 5.1). If we write $D^{-1}f(x) = \int_{-\infty}^{x} f(s)\,ds$ and ${}_N\psi^{\mathbf{0}}(x) = \frac{(-1)^N}{a}D^{-N}\delta(x)$, then we can obtain the primary wavelets by the general formula

$$_{N,\widetilde{N}}\psi_k(x) = {}_N\psi^{\mathbf{0}}(bx-k) - \sum_n {}_N\phi(x-n)\int {}_N\psi^{\mathbf{0}}(by-k)_{N,\widetilde{N}}\widetilde{\phi}(y-n)\,dy$$

where $k = 1, ..., b-1$ since ${}_N\psi^{\mathbf{0}}(bx-l)$ is dual with ${}_{N,\widetilde{N}}\widetilde{\psi}_k$.

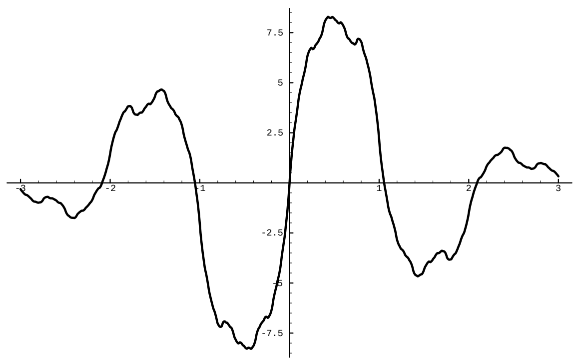

Figure 5.4: $\frac{d^3}{dx^3}F_9(x) = -4\ {}_{3,7}\widetilde{\psi}\left(\frac{x+1}{2}\right)$

Remark 3. *We can show that* ${}_{N,\widetilde{N}}\widetilde{\phi}$ *and therefore* ${}_{N,\widetilde{N}}\psi_k$ *are compactly supported. See the proof of proposition 6.1.*

6.1. Dyadic case

We are left to show how we can write the ${}_{N,\widetilde{N}}\widetilde{\phi}$ (the Cohen-Daubechies-Feauveau dual scaling function) as the N^{th} derivative of the Lagrange iterative interpolation. However, since we take N derivatives, it is understood that such a formula is not unique.

Proposition 6.1. *Assume that* $\widetilde{N}$ *is sufficiently large to have* $F_{N+\widetilde{N}-1} \in C^N$, *we then have*

$$_{N,\widetilde{N}}\widetilde{\phi}\,(x) = f_{N,\widetilde{N}}\,(x)$$

where

$$f_{N,\widetilde{N}}\,(x) = \sum_{k=0}^{\infty} \begin{pmatrix} N-1+k \\ N-1 \end{pmatrix} F_{N+\widetilde{N}-1}^{(N)}\,(x - k - \kappa - [N/2])$$

Proof. Firstly, we have to show that $f_{N,\widetilde{N}}\,(x) \in L^2\,(\mathbf{R})$. Since it is a continuous function, it is sufficient to show it has compact support. To do so, we observe that

$$g_N\,(k) = \begin{pmatrix} N-1+k \\ N-1 \end{pmatrix}$$

is a polynomial of degree $N-1$ over $\mathbf{N}$. The Lagrange iterative interpolation of degree $N+\widetilde{N}-1$ preserves polynomials of the corresponding degree and fundamental functions always have compact support, therefore $f_{N,\widetilde{N}}\,(x)$ must have compact support.

We can now take the Fourier transform of $f_{N,\widetilde{N}}$

$$\widehat{f_{N,\widetilde{N}}}(\xi) = \sum_{k=0}^{\infty} \left(\begin{array}{c} N-1+k \\ \\ N-1 \end{array} \right) z^{k+\kappa+[N/2]} \, \widehat{F^{(N)}_{N+\widetilde{N}-1}}(\xi)$$

where $z = e^{-i\xi}$. If we substitute formula 5.4, we then have

$$\widehat{f_{N,\widetilde{N}}}(\xi) = \alpha(\xi) \; {}_{N,\widetilde{N}}\widehat{\widetilde{\phi}}(\xi)$$

where

$$\alpha(\xi) = q_N(\xi) \sum_{k=0}^{\infty} \left(\begin{array}{c} N-1+k \\ \\ N-1 \end{array} \right) z^{k+\kappa+[N/2]}$$

Recall that

$$\begin{aligned} q_N(\xi) &= (-1)^N z^{[N/2]} \left(1 - z^{-1}\right)^N \\ &= z^{-[N/2]-\kappa} (1-z)^N \end{aligned}$$

and our sum is the Taylor series of $(1-z)^{-N}$ so we have that for $z \neq 1$,

$$(1-z)^N \times \sum_{k=0}^{\infty} \left(\begin{array}{c} N-1+k \\ \\ N-1 \end{array} \right) z^k = 1$$

Therefore $\alpha(\xi) = 1$ for $\xi \neq 0$ which is sufficient to write that in L^2,

$$\widehat{f_{N,\widetilde{N}}}(\xi) =_{N,\widetilde{N}} \widehat{\widetilde{\phi}}(\xi)$$

We are left to take the inverse transform to show the result. ■

7. Conclusion

While it relies on well-understood ideas (see for example [7]), we think that our derivation of the Cohen-Daubechies-Feauveau wavelets makes clearer a few of their properties. In particular, since the regularity of the Lagrange iterative interpolation is known [3], the regularity of the corresponding wavelet is automatically established. We also get a different algorithm to compute those functions. We know [3] that the derivatives of the fundamental functions can be computed exactly over b-adic numbers of a given depth in a finite number of steps by the formula

$$F^{(r)}\left(k/b^j\right)/b^{rj} = \sum_n F\left(n/b^j\right) F^{(r)}(k-n)$$

Note that we can easily compute $F^{(r)}$ over integers by using the properties of Lagrange iterative interpolation.

It would be interesting to study other C^k iterative interpolation schemes and to see what type of wavelets they could give (M-band filters but also cases where the Fourier transform fails because of the lack of symmetry). This type of work is closely related to Swelden's lifting scheme [11]. Indeed, in another paper [4], we showed that those ideas can lead to the construction of wavelets on the interval without the use of a Gram-Schmidt process using some iterative interpolation schemes over the interval (see [8] and [5]). This is possible because the Fourier transform isn't built into the scheme. Using remark 1, all you have to do is to choose an iterative interpolation scheme which "lives" on your subdivision. For example, this can lead to a FWT well-adapted for irregular sampling (in preparation).

Table I : tryadic ($b = 3$) scaling filters ($z = e^{-i\xi}$)

$\widetilde{N}$	N	primary scaling filters (${}_{\widetilde{N}}\widetilde{m}_0$)	dual scaling filters (${}_{\widetilde{N},N}m_0$)
1	3	$\frac{1}{3} + \frac{z}{3} + \frac{z^2}{3}$	$-\frac{4}{81z^3} - \frac{1}{81z^2} + \frac{5}{81z} + \frac{26}{81} + \frac{29}{81}z$ $+\frac{26}{81}z^2 + \frac{5}{81}z^3 - \frac{1}{81}z^4 - \frac{4}{81}z^5$
1	5	$\frac{1}{3} + \frac{z}{3} + \frac{z^2}{3}$	$\frac{7}{729z^8} + \frac{1}{729z^7} - \frac{8}{729z^6} - \frac{49}{729z^5} - \frac{13}{729z^4}$ $+\frac{62}{729z^3} + \frac{77}{243z^2} + \frac{89}{243z} + \frac{77}{243} + \frac{62}{729}z$ $-\frac{13}{729}z^2 - \frac{49}{729}z^3 - \frac{8}{729}z^4 + \frac{1}{729}z^5 + \frac{7}{729}z^6$
2	2	$\frac{1}{9z^2} + \frac{2}{9z} + \frac{1}{3} + \frac{2}{9}z + \frac{1}{9}z^2$	$-\frac{4}{27z^3} + \frac{1}{9z^2} + \frac{2}{9z} + \frac{17}{27} + \frac{2}{9}z$ $+\frac{1}{9}z^2 - \frac{4}{27}z^3$
2	4	$\frac{1}{9z^2} + \frac{2}{9z} + \frac{1}{3} + \frac{2}{9}z + \frac{1}{9}z^2$	$\frac{7}{243z^6} - \frac{2}{81z^5} - \frac{1}{27z^4} - \frac{34}{243z^3}$ $+\frac{10}{81z^2} + \frac{22}{81z} + \frac{5}{9} + \frac{22}{81}z + \frac{10}{81}z^2$ $-\frac{34}{243}z^3 - \frac{1}{27}z^4 - \frac{2}{81}z^5 + \frac{7}{243}z^6$

Table II : tryadic ($b = 3$) dual wavelet filters

$\widetilde{N}$	filters ${}_N\widetilde{m}_1$
1	$1 - z, z - z^2$
2	$1 - 2z + z^2, z - 2z^2 + z^3$

Table III : tryadic ($b = 3$) primary wavelet filters

$\widetilde{N}$	N	filters ${}_{\tilde{N},N}m_1$
2	2	$\frac{4}{81z^2} + \frac{8}{81z} - \frac{4}{27} - \frac{14}{81}z - \frac{40}{81}z^2 + \frac{5}{27}z^3 + \frac{10}{81}z^4 + \frac{5}{81}z^5$
		$\frac{5}{81z^2} + \frac{10}{81z} + \frac{5}{27} - \frac{40}{81}z - \frac{14}{81}z^2 + \frac{4}{27}z^3 + \frac{8}{81}z^4 + \frac{4}{81}z^5$
2	4	$-\frac{8}{729z^5} - \frac{16}{729z^4} - \frac{8}{243z^3} + \frac{38}{729z^2} + \frac{100}{729z} + \frac{2}{9} - \frac{112}{243}z$
		$-\frac{35}{243}z^2 + \frac{14}{81}z^3 + \frac{77}{729}z^4 + \frac{28}{729}z^5 - \frac{7}{243}z^6 - \frac{14}{729}z^7 - \frac{7}{729}z^8$
		$-\frac{7}{729z^5} - \frac{14}{729z^4} - \frac{7}{243z^3} + \frac{28}{729z^2} + \frac{77}{729z} + \frac{14}{81} - \frac{35}{243}z$
		$-\frac{112}{243}z^2 + \frac{2}{9}z^3 + \frac{100}{729}z^4 + \frac{38}{729}z^5 - \frac{8}{243}z^6 - \frac{16}{729}z^7 - \frac{8}{729}z^8$

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