A Family of 4-point Dyadic High Resolution Subdivision Schemes

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Subdivision: why care?

- ✔ multiscale approach

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- ✔ local interpolation

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- ✔ multiscale approach
- ✔ local interpolation
- ✔ lead to compactly supported wavelets

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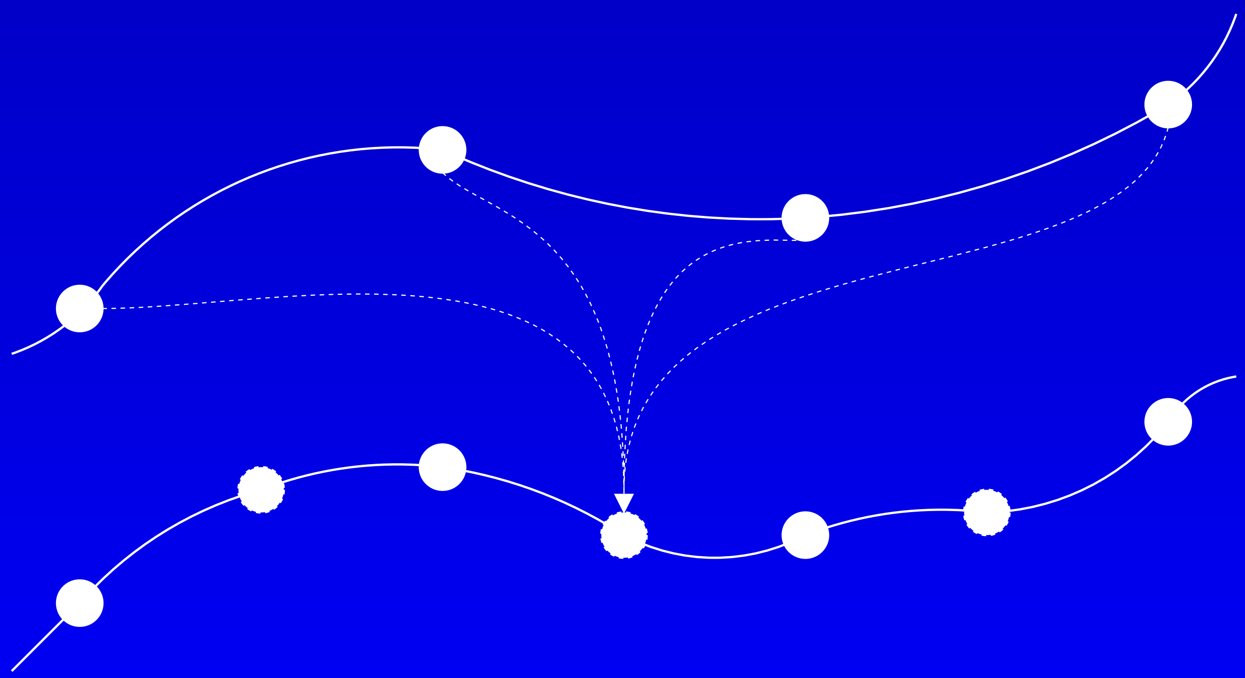

One follow-up...

Same story but...

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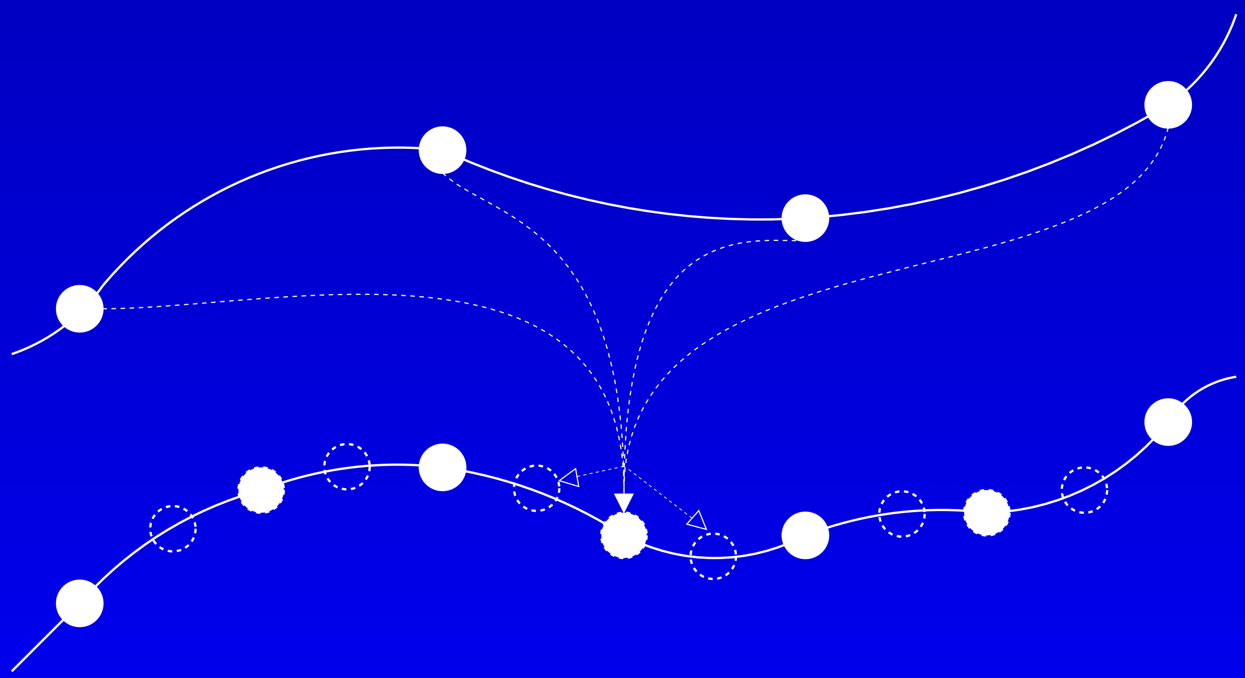

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cubic (coarse) + cubic (fine) = cubic + 1 = quartic

Different kind of multiscale

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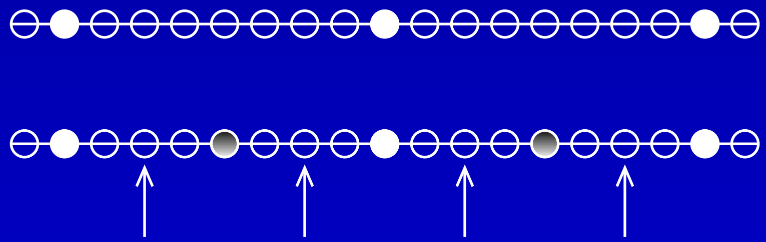
✔ “High Resolution Subdivision” (HRS) Scheme

Guessing early or coarsing it up



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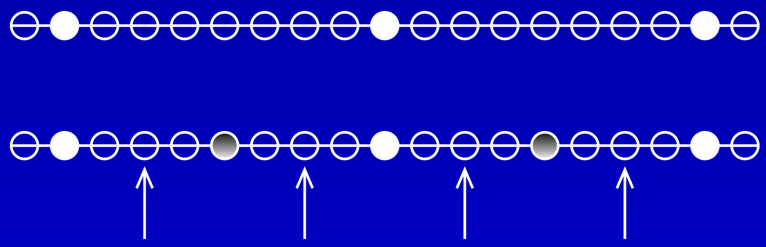
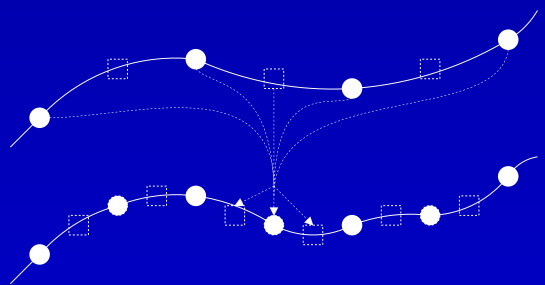


- ✔ Ample storage: why not use it early?
- ✔ Good guesses for the quartiles

Crunch the tetradic tree into a dyadic one



How does it compare to Vector Subdivision Schemes?

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$$y_{j+1,4k+3} = \frac{-5y_{j,2k-2} + 35y_{j,2k} + 105y_{j,2k+2} - 7y_{j,2k+4}}{128};$$

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✔ $\alpha = 1 \longrightarrow$ only $y_{j,2k+1}$

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$$P^{j+1}(z) = \Gamma_1(z)P^j\left(z^2\right) + \Gamma_2(z)P^j\left(-z^2\right)$$

Smoothness

- $\alpha = 0 \Rightarrow$ Dubuc

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- Dubuc$\Rightarrow C^1$.
- For $-25/56 < \alpha < 15/32$, the HRS schemes are C^1.

Reproduced polynomials

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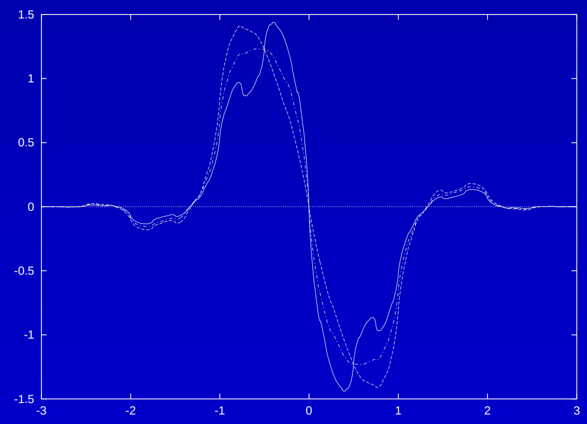
Gotcha! Need proper initialization!

Conclusion (a comparative table)

scheme	regularity	reproduced polynomials
Dubuc[5]	C^1	cubic
Deslauriers-Dubuc[2]	C^1	cubic
Dyn-Gregory-Levin[8]	up to C^1	up to cubic
Hassan et al.[10]	C^2	quadratic
presented HRS	up to C^1	cubic to quartic

Bonus material 1 - nice pictures



Derivatives of the fundamental functions for $\alpha = -0.2$ (continuous line), $\alpha = 0$ (dash-dot line), and $\alpha = 0.15$ (dashed line).

Bonus material 2 - intermediate result

- (Dyn) Given trigonometric polynomials $\Gamma_1(z)$ and $\Gamma_2(z)$, the HRS scheme defined by

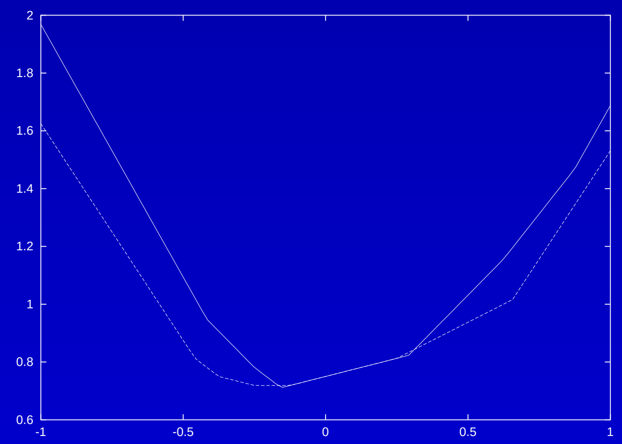
$$P^{j+1}(z) = \Gamma_1(z)P^j\left(z^2\right) + \Gamma_2(z)P^j\left(-z^2\right)$$

is C^n if the symbol corresponding to finite differences of order $n+1$

$$dH_n^j(z) = \frac{2^{jn}(1-z)^{n+1}}{z^{n+1}}P^j(z)$$

is the symbol of a HRS scheme converging uniformly to zero for all bounded initial data.

Bonus material 3 - more on proof



For a given α, an HRS scheme is differentiable if

$$\lambda_{HR}(\alpha) = \max\{\lambda_1(\alpha), \lambda_2(\alpha)\} < 1.$$

Bonus material 4 - proper initialization

1. recopy data at $x_{j+1,2k} = x_{j,k}$: $y_{j+1,2k} = y_{j,k}$;

2. extrapolate $y_{j,k+4}$ using $y_{j,k-2}, y_{j,k-1}, y_{j,k}, y_{j,k+1}, y_{j,k+2}$ by the formula

$$\gamma_{j,k} = 5y_{j,k-2} - 24y_{j,k-1} + 45y_{j,k} - 40y_{j,k+1} + 15y_{j,k+1}; \qquad (1)$$

3. interpolate midpoint :

$$y_{j+1,2k+1} = \frac{-7y_{j,k-2} + 105y_{j,k} + 35y_{j,k+2} - 5\gamma_{j,k}}{128}.$$

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