Fourier transform of Hermite interpolatory subdivision schemes

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Abstract

Two subdivision schemes with Hermite data on $\mathbb{Z}$ are studied. These schemes use 2 or 7 parameters respectively depending on whether Hermite data involve only first derivatives or include second derivatives. For a large region in the parameters space, the schemes are C^1 or C^2 convergent or at least are convergent on the space of Schwartz distributions. The Fourier transform of any interpolating function can be computed through products of matrices of order 2 or 3. The Fourier transform is related to a specific system of functional equations whose analytic solution is unique except for a multiplicative constant. The main arguments for these results come from Paley-Wiener-Schwartz theorem on the characterization of the Fourier transforms of distributions with compact support and a theorem of Artzrouni about convergent products of matrices.

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1 Introduction

Hermite interpolatory subdivision schemes have been introduced by Merrien [9, 10]. He and Dyn-Levin [5, 6] studied the convergence of these schemes. What we would like to do here is to compute the Fourier transform of these interpolants in order to give new properties.

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This harmonic analysis provides an additional tool to study these schemes and allows extension to functions which are not necessarily of class C^1 or C^2. A first analysis had been written by Hervé [7] using wavelets in two or more dimensions. But he did not studied the convergence conditions. We do it here.

In Section 2, we study a first Hermite interpolation on $\mathbb{R}$ of a function and its first derivative with data on $\mathbb{Z}$. This scheme called $HS21$ depends on two parameters. We first recall the conditions which insure C^1-convergence on $\mathbb{R}$. Then we evaluate the Fourier transform of the interpolating functions and we give the first properties. The Paley-Wiener-Schwartz theorem on the characterization of the Fourier transform of a distribution with compact support and a convergence theorem of infinite products of matrices proposed by Artzrouni allow us to conclude to the convergence of the scheme in the space of distributions $\mathcal{D}(\mathbb{R})'$ for a large region in the parameters space. Moreover, the Fourier transforms of two basic distributions in the Hermite subdivision scheme is the unique (except for a multiplicative constant) analytic solution of a system of two functional equations.

Section 3 is devoted to the study of a second scheme where we interpolate not only a function and its first derivative but also its second derivative. This scheme is called $HS22$. We prove analogous properties to the previous scheme depending on the parameters introduced in the algorithm. Most proofs are similar to the previous ones. However, a few matrix tools must be improved.

2 The Hermite subdivision scheme $HS21$

We recall Merrien's construction [9]. We suppose that the function f and its first derivative p are known on $\mathbb{Z}$. Precisely, we have two sequences $\{y_k, y'_k\}_{k\in\mathbb{Z}}$ and we suppose that $f(k) = y_k,\ p(k) = y'_k$. We build f and p on $\mathbb{Z}/2^n$ by induction. At step n, set $h = 1/2^n$ and $D_n = \{x = jh, j \in \mathbb{Z}\}$. If $a = jh$ and $b = (j+1)h$ are two consecutive points of D_n, we compute f and p at $x = (a+b)/2$ which is the middle point of $[a, b]$ by the formulae:

$$\left.\begin{array}{rcl} f(x) &=& \dfrac{f(a)+f(b)}{2} + \alpha h[p(b)-p(a)] \\ p(x) &=& (1-\beta)\dfrac{f(b)-f(a)}{h} + \beta\dfrac{p(a)+p(b)}{2} \end{array}\right\} \qquad (1)$$

Hence f and p are defined on D_{n+1}. The construction is depending on two parameters α and β. When we reiterate the process, we define f and p on the set of dyadic numbers $D_\infty = \bigcup D_n$ which is dense in $\mathbb{R}$. For some values of the parameters the functions f and p may be uniformly continuous on D_∞ so that they may be extended to $\mathbb{R}$; sometimes we have in addition: $p = f'$. In these cases the algorithm is said to be C^1- convergent.

This C^1-convergence is given by the following necessary and sufficient condition: the generalized spectral radius of the set $\Sigma = \{\Lambda_1, \Lambda_{-1}\}$ satisfies $\rho(\Sigma) < 1$, where

$\Lambda_\varepsilon = \begin{pmatrix} \frac{1}{2} & \varepsilon(1-\beta) \\ \varepsilon\frac{8\alpha+1}{4} & \frac{1+\beta}{2} \end{pmatrix}, \varepsilon = \pm 1$. An equivalent condition is that there exists a matrix norm $\|\cdot\|$ such that: $\|\Lambda_\varepsilon\| < 1, \varepsilon = \pm 1$. This result can be proved by techniques which are described in [10].

Let us recall that if Σ is a set of matrices in $\mathbb{R}^{n\times n}$ and if we write $\rho(M)$ for the spectral radius of a matrix M, then the generalized spectral radius of $\Sigma, \rho(\Sigma)$, is defined by:

$$\rho(\Sigma) = \limsup_{k\to+\infty}(\rho_k(\Sigma))^{\frac{1}{k}}, \rho_k(\Sigma) = \sup\{\rho(\prod_{i=1}^{k} M_i) : M_i \in \Sigma, 1 \le i \le k\}.$$

More details on generalized spectral radius can be found in Daubechies-Lagarias [2].

Remark 1: If $\alpha = -1/8$, $\beta = -1/2$, then f is the Hermite interpolating cubic polynomial between two consecutive integers. When $\alpha = -1/8$, $\beta = -1$, then f is the quadratic interpolating spline with a node on each middle point of two consecutive integers.

2.1 First properties of $HS21$

We describe some elementary properties of Hermite subdivision schemes. We will use those properties later. The first one is the linearity of the scheme.

Lemma 1 *Let $\{y_k, y'_k, \tilde{y}_k, \tilde{y}'_k\}_{k\in\mathbb{Z}}$ be 4 sequences. Assume that both couples (f,p) and $(\tilde{f},\tilde{p})$ are built by the subdivision scheme from: $f(k) = y_k,\ p(k) = y'_k,\ \tilde{f}(k) = \tilde{y}_k$, $\tilde{p}(k) = \tilde{y}'_k$, then the couple of functions $(f+\tilde{f}, p+\tilde{p})$ are obtained by the subdivision scheme from the sequences: $\{y_k + \tilde{y}_k, y'_k + \tilde{y}'_k\}_{k\in\mathbb{Z}}$.*

Similarly, if $c \in \mathbb{R}$, the couple of functions (cf, cp) is obtained by the scheme from the sequences: $\{cy_k, cy'_k\}_{k\in\mathbb{Z}}$.

Then we have a second lemma about translation and scale change.

Lemma 2 *Let $\{y_k, y'_k\}_{k\in\mathbb{Z}}$ be two sequences from which we build the couple of functions (f,p) by the subdivision scheme with $f(k) = y_k,\ p(k) = y'_k$. If $c \in \mathbb{Z}$, then the couple of functions $(f(x+c), p(x+c))$ is obtained by the scheme from the sequences: $\{y_{k+c}, y'_{k+c}\}_{k\in\mathbb{Z}}$.*

The couple of functions $(f(x/2), p(x/2)/2)$ is built from the sequences: $\{f(k/2), p(k/2)/2\}_{k\in\mathbb{Z}}$.

There are two basic solutions of our recursive system (1): the first one is the couple (f_0, p_0) which is solution of (1) with data $f_0(k) = \delta_{k,0},\ p_0(k) = 0, k \in \mathbb{Z}$ and the second one is the couple (f_1, p_1) solution of (1) with data $f_1(k) = 0, p_1(k) = \delta_{k,0}, k \in \mathbb{Z}$. These two couples are important because with linear combinations of their tranlates we can get all the solutions (f,p) of (1). For any dyadic number x:

$$\left.\begin{array}{rcl} f(x) & = & \sum_{k=-\infty}^{\infty}[f(k)f_0(x-k)+p(k)f_1(x-k)] \\ p(x) & = & \sum_{k=-\infty}^{\infty}[f(k)p_0(x-k)+p(k)p_1(x-k)] \end{array}\right\} \tag{2}$$

Notice that all these sum are finite as the supports of $f_i, p_i, i = 0, 1$ lie in the set $[-1, 1]$.

Now, using relation (1) which is applied to the couple of functions $(f(x) = f_0(x/2), p(x) = p_0(x/2)/2)$ and then to the couple $(f_1(x/2), p_1(x/2)/2)$, after evaluations of the functions f_0, p_0, f_1, p_1 at the half-integers, we obtain a system of functional equations for f_0, p_0, f_1, p_1.

$$\left.\begin{array}{rcl} f_0(x/2) & = & \frac{1}{2}f_0(x-1)+f_0(x)+\frac{1}{2}f_0(x+1)-\frac{1-\beta}{2}f_1(x-1)+\frac{1-\beta}{2}f_1(x+1), \\ p_0(x/2)/2 & = & \frac{1}{2}p_0(x-1)+p_0(x)+\frac{1}{2}p_0(x+1)-\frac{1-\beta}{2}p_1(x-1)+\frac{1-\beta}{2}p_1(x+1), \\ f_1(x/2) & = & -\alpha f_0(x-1)+\alpha f_0(x+1)+\frac{\beta}{4}f_1(x-1)+\frac{1}{2}f_1(x)+\frac{\beta}{4}f_1(x+1), \\ p_1(x/2)/2 & = & -\alpha p_0(x-1)+\alpha p_0(x+1)+\frac{\beta}{4}p_1(x-1)+\frac{1}{2}p_1(x)+\frac{\beta}{4}p_1(x+1). \end{array}\right\} \tag{3}$$

2.2 Fourier transform of $HS21$

Let us start with a computation without trying proper justification. We will suppose that the system (3) of functional equations is valid not only whenever x is a dyadic number, but also whenever x is an arbitrary real number. We must suppose that f_0, p_0, f_1, p_1 have been extended by continuity on $\mathbb{R}$. Now, we compute the Fourier transform $\hat{f}$ of a function f by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx.$$

Using this Fourier operator on each equation of the system (3), we get:

$$\begin{pmatrix} \hat{f}_0(\xi) \\ \hat{f}_1(\xi) \end{pmatrix} = A(\xi/2)\begin{pmatrix} \hat{f}_0(\xi/2) \\ \hat{f}_1(\xi/2) \end{pmatrix}, \begin{pmatrix} \hat{p}_0(\xi) \\ \hat{p}_1(\xi) \end{pmatrix} = 2A(\xi/2)\begin{pmatrix} \hat{p}_0(\xi/2) \\ \hat{p}_1(\xi/2) \end{pmatrix}, \tag{4}$$

where

$$A(\xi) = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}\cos\xi & i\frac{1-\beta}{2}\sin\xi \\ i\alpha\sin\xi & \frac{1}{4}+\frac{\beta}{4}\cos\xi \end{pmatrix}$$

We have two vector equations in (4). To study them, we now look at the product of matrices:

$$P_n(\xi) = A(\xi/2)A(\xi/4)\ldots A(\xi/2^n). \tag{5}$$

Precisely, we look for conditions on the parameters α, β to get convergence of the sequence of matrices $P_n(\xi)$. This convergence should happen for every real or complex value of ξ. The study of this sequence for complex values of ξ is motivated by a generalization of Paley-Wiener theorem as proposed by Schwartz [11].

Theorem 3 (Schwartz) *Let F be a continuous function on the real axis. F is the Fourier transform of a distribution with bounded support $[-C, C]$ if and only if F may be extended on the complex plane to an analytic entire function of exponential type $\leq C$.*

We recall that a entire function $F(z)$ is of exponential type $\leq C$ if

$$\limsup_{|z|\to\infty} \frac{\log |F(z)|}{|z|} \leq C.$$

To study the convergence of the matrix products $P_n(\xi)$, we will need a lemma to bound the moduli of the elements of $P_n(\xi)$. We will also need a convergence criterion which has been found by Artzrouni [1].

Lemma 4 *Let $||\cdot||$ be a matrix norm on the space of complex matrices of order d, $\mathbb{C}^{d\times d}$, and let A_n be a sequence of this space such that $||A_n|| \leq 1+\varepsilon_n$, with $\varepsilon_n \geq 0$. Set $P_0 = I, P_n = P_{n-1}A_n, n = 1, 2, \ldots$. Then $||P_n|| \leq e^{\sum_{k=1}^{n}\varepsilon_k}$ and the modulus of each component of P_n is bounded by $Ce^{\sum_{k=1}^{n}\varepsilon_k}$ where C is a constant which depends only on the matrix norm $||\cdot||$.*

Proof: Since we notice that $1+\varepsilon_n \leq e^{\varepsilon_n}$, we get the first result by induction.

Then, for $A = (a_{ij}) \in \mathbb{C}^{d\times d}$, we choose a new vector norm $N(A) = \max\{|a_{ij}|\}$. We know that there exists a constant C_N such that for all $A \in \mathbb{C}^{d\times d}$, $N(A) \leq C_N\|A\|$. Then we get the second upper bound. $\square$

Theorem 5 (Artzrouni) *Let $||\cdot||$ be a matrix norm on $\mathbb{C}^{d\times d}$. Let M_n be a sequence in $\mathbb{C}^{d\times d}$ such that for all $n \in \mathbb{N}, ||M_n|| = 1$ and for all $m \in \mathbb{N}$ the sequence of matrices $M_m M_{m+1} \ldots M_n$, $n \geq m$ converges. If N_n is a sequence in $\mathbb{C}^{d\times d}$ such that $\sum_{n=1}^{\infty} ||N_n - M_n|| < \infty$, then the sequence of matrices $N_1 N_2 \ldots N_n$ converges.*

We are now ready for the main result.

Theorem 6 *If $-5 < \beta \leq 3$, then for all complex number ξ the sequence of matrices $P_n(\xi)$ defined in (5) converges and the convergence is uniform whenever ξ lies in the disk $|\xi| \leq R$. As functions of ξ, the four components of the limit matrix $P(\xi)$ are entire functions of exponential type ≤ 1.*

Proof: Let us start by proving that the moduli of all components of $P_n(z)$ are uniformly bounded whenever z is in any given disk $|z| \leq R$.

If $A = (a_{ij})$ is a matrix in $\mathbb{C}^{2\times 2}$, we recall that $||A||_\infty = \max_i\{|a_{i1}| + |a_{i2}|\}$. Then we set:

$$\varepsilon(R) = \max[0, \sup_{|z|\leq R}(||A(z)||_\infty - 1)].$$

Then, we use Lemma 4 with the matrix $A_n = A(z/2^n)$ and the number $\varepsilon_n = \varepsilon(R/2^n)$. If the components of the matrix $A(z)$ are $a_{ij}(z)$, these functions are analytic at $z = 0$, therefore $\varepsilon_n = O(1/2^n)$ (remembering the hypothesis $-5 < \beta \leq 3$). Lemma 4 shows that the moduli of all components of the matrices $P_n(z)$ are uniformly bounded whenever z lies in the disk $|z| \leq R$.

Now, we prove that for all $z \in \mathbb{C}$, the sequence of matrices $P_n(z)$ converges. Let M_n be the constant sequence of matrices $M = M_n = A(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+\beta}{4} \end{pmatrix}$ and define the sequence $N_n = A(z/2^n)$. We know that $||N_n - M_n||_\infty = O(1/2^n)$. Moreover $||M||_\infty = 1$ and with the hypothesis on β, the sequence $M^k, k \geq 0$ converges. All the hypothesis of Theorem 5 are satisfied, therefore the sequence $P_n(z)$ converges.

As the sequence $P_n(z)$ converges to a matrix $P(z)$ and as the moduli of the components of the matrices $P_n(z)$ are uniformly bounded whenever $|z| \leq R$, the Lebesgue dominated convergence theorem and Cauchy formula give us the proof that all the components of the matrix $P(z)$ are analytic in z and that the convergence of the sequence $P_n(z)$ to $P(z)$ is uniform on the disk $|z| \leq R$.

Finally, let us verify that each element of the matrix $P(z)$ is an entire function of exponential type. Firstly there exists a real positive number C, which depends on the parameters α, β such that for all $z \in \mathbb{C}$, $||A(z)||_\infty \leq Ce^{|z|}$. Secondly, we know that there exists a real positive number M (depending on the parameters again) such that for all $z \in \mathbb{C}, |z| \leq 1, ||P(z)||_\infty \leq M$.

Let z be a complex number such that $2^n \leq |z| \leq 2^{n+1}$.
As $P(z) = A(z/2)A(z/4)\ldots A(z/2^n)P(z/2^{n+1})$, we obtain the bound

$$||P(z)||_\infty \leq ||P(z/2^{n+1}||_\infty \prod_{k=1}^{n} ||A(z/2^k)||_\infty \leq M \prod_{k=1}^{n}[Ce^{z/2^k}] \leq MC^n e^{|z|}.$$

So that $\limsup_{|z|\to\infty} \dfrac{\log ||P(z)||_\infty}{|z|} \leq 1$. Then the functions composing the matrix $P(z)$ are entire function of exponential type ≤ 1. □

The Schwartz version of Paley-Wiener theorem allows us to have a corollary of Theorem 6.

Corollary 7 *Let us assume that* $-5 < \beta \leq 3$. *Then each function composing the limit matrix* $P(z) = \lim P_n(z)$ *is the Fourier transform of a distribution whose support lies in the interval* $[-1, 1]$.

2.3 Schwartz distributions associated to the scheme

We will link the computation of Fourier transforms of the previous subsection with the limit matrix $P(\xi)$. This link will come from four sequences of Schwartz distributions. We set

$$
\begin{aligned}
T_i^{(n)} &= \frac{1}{2^n}\sum_m f_i(m/2^n)\delta_{m/2^n}, i=0,1,\\
U_i^{(n)} &= \frac{1}{2^n}\sum_m p_i(m/2^n)\delta_{m/2^n}, i=0,1,
\end{aligned}
$$

where δ_h is the Dirac distribution at point h defined by $\delta_h(\phi)=\phi(h)$.

Notice that these sums are finite and that the distributions are compactly supported, the supports of $f_i, p_i, i=0,1$ being in $[-1,1]$.

If we evaluate the Fourier transform of these four distributions:

$$\hat{T}_i^{(n)}(\xi) = T_i^{(n)}(e^{-i\xi x}), \hat{U}_i^{(n)}(\xi) = U_i^{(n)}(e^{-i\xi x}), i=0,1.$$

Hence using the equalities (3), we verify that two simple inductions link both Fourier transforms through the matrix $A(\xi)$. Indeed,

$$\hat{T}_0^{(n+1)}(\xi) = \frac{1}{2^{n+1}}\sum_m f_0(m/2^{n+1})e^{-i\xi m/2^{n+1}}.$$

In this last equation, we substitute to $f_0(m/2^{n+1})$ the right member of the first equation of system (3) with $x=m/2^n$ to obtain a first recursion:

$$\hat{T}_0^{(n+1)}(\xi) = [\frac{1}{2}+\frac{1}{2}\cos(\frac{\xi}{2})]\hat{T}_0^n(\xi/2) + i\frac{1-\beta}{2}\sin(\frac{\xi}{2})\hat{T}_1^n(\xi/2)$$

Similarly, we can evaluate $\hat{T}_1^{(n+1)}(\xi)$ using the third equation of system (3) again at $x=m/2^n$ and we get a second recursion:

$$\hat{T}_1^{(n+1)}(\xi) = i\alpha\sin(\frac{\xi}{2})\hat{T}_0^n(\xi/2) + [\frac{1}{4}+\frac{\beta}{4}\cos(\frac{\xi}{2})]\hat{T}_1^n(\xi/2)$$

Both recursions may be linked in a single vector recursion through the matrix $A(\xi/2)$:

$$\begin{pmatrix} \hat{T}_0^{(n+1)}(\xi) \\ \hat{T}_1^{(n+1)}(\xi) \end{pmatrix} = A(\xi/2)\begin{pmatrix} \hat{T}_0^{(n)}(\xi/2) \\ \hat{T}_1^{(n)}(\xi/2) \end{pmatrix} \tag{6}$$

Similarly, with the second and fourth equations of system (3), we obtain a second vector recursion:

$$\begin{pmatrix} \hat{U}_0^{(n+1)}(\xi) \\ \hat{U}_1^{(n+1)}(\xi) \end{pmatrix} = 2A(\xi/2)\begin{pmatrix} \hat{U}_0^{(n)}(\xi/2) \\ \hat{U}_1^{(n)}(\xi/2) \end{pmatrix} \tag{7}$$

Since $\begin{pmatrix} \hat{T}_0^{(0)}(\xi) \\ \hat{T}_1^{(0)}(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \hat{U}_0^{(0)}(\xi) \\ \hat{U}_1^{(0)}(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get

$$\begin{pmatrix} \hat{T}_0^{(n)}(\xi) \\ \hat{T}_1^{(n)}(\xi) \end{pmatrix} = P_n(\xi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8}$$

and

$$\begin{pmatrix} \hat{U}_0^{(n)}(\xi) \\ \hat{U}_1^{(n)}(\xi) \end{pmatrix} = 2^n P_n(\xi) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9}$$

The vector sequence $\begin{pmatrix} \hat{T}_0^{(n)}(\xi) \\ \hat{T}_1^{(n)}(\xi) \end{pmatrix}$ converges to the first column of the matrix $P(\xi)$. Now, Schwartz theory tells us that the Fourier transform in the space of tempered distributions is a linear continuous operator and its inverse is its conjugate ([11] p. 107 of Vol. 2). Therefore both sequences $T_0^{(n)}, T_1^{(n)}$ converge respectively to the distributions T_0, T_1, respective components of the inverse Fourier transform applied to the first column of the matrix $P(\xi)$.

Theorem 8 *If $-5 < \beta \leq 3$, then both sequences of distributions $T_0^{(n)}, T_1^{(n)}$ converge respectively to the distributions T_0, T_1, the respective components of the inverse Fourier transform applied to the first column of the matrix $P(\xi)$.*

We are ready to prove that the subdivision scheme is always convergent in the space of distributions $\mathcal{D}(\mathbb{R})'$ whenever $-5 < \beta \leq 3$. In the following, we use Schwartz notation for the translation operator τ_h where h is a real number. If ϕ is a function in C_0^∞ and if T is a distribution, then $\tau_h\phi(x) = \phi(x-h)$ and $\tau_h T(\phi) = T(\tau_h\phi)$.

Theorem 9 *Let us assume that $-5 < \beta \leq 3$. If we build the couple (f, p) by the subdivision scheme (1) from the data $\{y_k, y'_k\}_{k\in\mathbb{Z}}$, then the sequence of distributions $F_n = \dfrac{1}{2^n}\displaystyle\sum_{m=-\infty}^{\infty} f(m/2^n)\delta_{m/2^n}$ converges to the distribution*

$$F = \sum_{k=-\infty}^{\infty} [y_k \tau_{-k} T_0 + y'_k \tau_{-k} T_1].$$

Proof: Let ϕ be a function in C^∞ which support is in $[-N, N]$, then

$F_n(\phi) = \frac{1}{2^n} \sum_{m=-N2^n}^{N2^n} f(m/2^n)\phi(m/2^n)$. We use relation (2) to get:

$$\begin{aligned}
2^n F_n(\phi) &= \sum_{m=-N2^n}^{N2^n} \sum_{k=-N-1}^{N+1} [y_k f_0(m/2^n - k) + y'_k f_1(m/2^n - k)]\phi(m/2^n) \\
&= \sum_{k=-N-1}^{N+1} \sum_{m=-N2^n}^{N2^n} [y_k f_0(m/2^n) + y'_k f_1(m/2^n)]\phi(m/2^n + k) \\
&= 2^n \sum_{k=-N-1}^{N+1} [y_k T_0^{(n)} + y'_k T_1^{(n)}](\tau_{-k}\phi).
\end{aligned}$$

As n tends to infinity, the limit of the sequence $F_n(\phi)$ is:

$$\sum_{k=-\infty}^{\infty} [y_k \tau_{-k} T_0 + y'_k \tau_{-k} T_1](\phi).$$

□

Theorem 10 *If $-3 < \beta < 1$, then both sequences of distributions $U_0^{(n)}, U_1^{(n)}$ converge respectively to the distributions T'_0, T'_1 which are the derivatives of the distributions T_0, T_1.*

Proof: Using both relations (5) and (7), we have:

$$\begin{pmatrix} \hat{U}_0^{(n)}(\xi) \\ \hat{U}_1^{(n)}(\xi) \end{pmatrix} = 2^n P_{n-1}(\xi) A(\xi/2^n) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \hat{U}_0^{(n)}(\xi) \\ \hat{U}_1^{(n)}(\xi) \end{pmatrix} = 2^n P_{n-1}(\xi) \begin{pmatrix} i\frac{1-\beta}{2} \sin(\xi/2^n) \\ \frac{1}{4} + \frac{\beta}{4} \cos(\xi/2^n) \end{pmatrix}$$

$$\begin{pmatrix} \hat{U}_0^{(n)}(\xi) \\ \hat{U}_1^{(n)}(\xi) \end{pmatrix} = i\frac{1-\beta}{2}\xi \begin{pmatrix} \hat{T}_0^{(n-1)}(\xi) \\ \hat{T}_1^{(n-1)}(\xi) \end{pmatrix} + \frac{1+\beta}{2} \begin{pmatrix} \hat{U}_0^{(n-1)}(\xi) \\ \hat{U}_1^{(n-1)}(\xi) \end{pmatrix} + O(1/2^n)$$

$$\begin{pmatrix} \hat{U}_0^{(n)}(\xi) \\ \hat{U}_1^{(n)}(\xi) \end{pmatrix} = i\frac{1-\beta}{2}\xi \sum_{k=0}^{n-1} (\frac{1+\beta}{2})^k \begin{pmatrix} \hat{T}_0^{(n-1-k)}(\xi) \\ \hat{T}_1^{(n-1-k)}(\xi) \end{pmatrix} + O(n[\max(\frac{1}{2}, \frac{|1+\beta|}{2})]^n)$$

If $|1+\beta| < 2$ which is the hypothesis, then the right member of the last vector equation tends to the column vector whose components are $i\xi\hat{T}_0(\xi), i\xi\hat{T}_1(\xi)$. They are the two respective limits of the sequences $\hat{U}_0^{(n)}(\xi), \hat{U}_0^{(n)}(\xi)$. Using the inverse Fourier transform on each sequence, it is clear that $U_0^{(n)}, U_1^{(n)}$ converge respectively to the distributions T'_0, T'_1. □

Theorem 11 *Let us assume that* $-3 < \beta < 1$. *If we build the couple* (f, p) *by the subdivision scheme (1) from the data* $\{y_k, y'_k\}_{k\in\mathbb{Z}}$, *then the sequence of distributions* $G_n = \frac{1}{2^n} \sum_{m=-\infty}^{\infty} p(m/2^n)\delta_{m/2^n}$ *converges to the distribution*

$$G = \sum_{k=-\infty}^{\infty} [y_k \tau_{-k} T'_0 + y'_k \tau_{-k} T'_1].$$

Proof: The proof is similar to that of Theorem 9. □

2.4 A characterization of Fourier transforms $\hat{T}_0, \hat{T}_1$

In this subsection, we characterize the couple of functions $\phi_0(\xi) = \hat{T}_0(\xi)$, $\phi_1(\xi) = \hat{T}_1(\xi)$ without computing all the subdivision scheme for two systems of initial data. We have seen in equation (4), that this couple of functions satisfies the system of two functional equations:

$$\begin{pmatrix} \phi_0(2\xi) \\ \phi_1(2\xi) \end{pmatrix} = A(\xi) \begin{pmatrix} \phi_0(\xi) \\ \phi_1(\xi) \end{pmatrix}. \tag{10}$$

The hypothesis on the parameters α, β is $-5 < \beta \leq 3$. With it, we know that the vector sequence

$$\begin{pmatrix} \hat{T}_0^{(n)}(\xi) \\ \hat{T}_1^{(n)}(\xi) \end{pmatrix} = A(\xi/2)A(\xi/4)\ldots A(\xi/2^n) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{11}$$

converges to $(\hat{T}_0(\xi), \hat{T}_1(\xi))^T$. For $\xi = 0$, we have for all $n \in \mathbb{N}$, $(\hat{T}_0^{(n)}(0), \hat{T}_1^{(n)}(0)) = (1, 0)$, so that $\phi_0(0) = 1, \phi_1(0) = 0$.

Theorem 12 *If* $-5 < \beta < 3$, *then there exists one and only one couple of analytic functions* $(\phi_0(\xi), \phi_1(\xi))$ *solution of the functional system (10) and satisfying* $\phi_0(0) = 1$.

Proof: Let (ϕ_0, ϕ_1) be a couple of analytic functions solution of (10) and such that $\phi_0(0) = 1$. By setting $\xi = 0$ in equation (10), we have that $\phi_1(0) = 0$ (because $\beta \neq 3$). We expand the functions $\phi_0(\xi), \phi_1(\xi)$ and the components $a_{ij}(\xi)$ of the matrix $A(\xi)$ in power series of ξ:
$\phi_0(\xi) = \sum_{n=0}^{\infty} \phi_0^{(n)} \xi^n$, $\phi_1(\xi) = \sum_{n=0}^{\infty} \phi_1^{(n)} \xi^n$, $a_{ij}(\xi) = \sum_{n=0}^{\infty} a_{ij}^{(n)} \xi^n$.
Then we substitute these expansions in system (10). We develop the products and we reorganize the result in terms of powers of ξ. We use the hypothesis $\phi_0(0) = 1, \phi_1(0) = 0$ to compute the Maclaurin series of the functions ϕ_0, ϕ_1 recursively, for $n = 1, 2, 3, \ldots$:

$$\phi_{i-1}^{(n)} = (\sum_{j=1}^{2} \sum_{k=1}^{n} a_{ij}^{(k)} \phi_{j-1}^{(n-k)})/(2^n - a_{ii}(0)), i = 1, 2 \tag{12}$$

Therefore ϕ_0 and ϕ_1 are completely known. $\square$
Remark 2: In this last theorem, the analyticity hypothesis on the functions ϕ_0, ϕ_1 is critical. If one just use the continuity hypothesis combined with the values of $\phi_0(0), \phi_1(0)$ one does not have a unique solution of system (10). Kuczma proved in a general way in his book ([8], p. 245, Theorem 12.1), that the set of solutions of a system of type (10) depends on an arbitrary function.

3 The Hermite subdivision scheme $HS22$

We will see how we can generalize the previous results to the scheme $HS22$ which is based on bissections that use not only the values of a function and of its first derivative but also the values of its second derivative. We recall Merrien's construction [10]. If the values of a function f, and of its first and second derivative p, s are known on $\mathbb{Z}$, we build the functions f, p and s by a recursion on n as in the previous section. If $a = j/2^n$ and $b = (j+1)/2^n$ are two successive points of D_n, we compute f, p et s at the middle point $x = (a+b)/2$ by the formulae:

$$\left.\begin{array}{rcl} f(x) & = & 1/2[f(b)+f(a)] + \alpha_2 h[p(b)-p(a)] + \alpha_3 h^2[s(b)+s(a)] \\ p(x) & = & \beta_1 \dfrac{f(b)-f(a)}{h} + \beta_2[p(a)+p(b)] + \beta_3 h[s(b)-s(a)] \\ s(x) & = & \gamma_2 \dfrac{p(b)-p(a)}{h} + \gamma_3[s(b)+s(a)] \end{array}\right\} \qquad (13)$$

The algorithm is said to be C^2-convergent if, for any data $\{y_k, y'_k, y''_k\}_{k\in\mathbb{Z}}$, the three functions f, p, s can be extended from D_∞ to $\mathbb{R}$ with $f \in C^2(\mathbb{R})$, $p = f'$ and $s = f''$. A necessary condition to get C^2-convergence is:

$$8\alpha_2 + 16\alpha_3 = -1, \beta_1 + 2\beta_2 = 1, \gamma_2 + 2\gamma_3 = 1.$$

We will find again the last two conditions later in the convergence theorems in the space of distributions. If these conditions are satisfied, then the algorithm depends only on 4 parameters: $\alpha_2, \beta_1, \beta_3$ and γ_2.

As for $HS21$, with this necessary hypothesis, a necessary and sufficient condition to get C^2-convergence is $\hat{\rho}(\Sigma) < 1$ where $\Sigma = \{\Lambda_1, \Lambda_{-1}\}$ and

$$\Lambda_\varepsilon = \begin{pmatrix} \frac{1}{2} & \varepsilon\gamma_2 & 0 \\ \varepsilon(\frac{1}{4} + 2\beta_3 - \frac{\beta_1}{6}) & 1 - \frac{\gamma_2}{2} & \varepsilon 2\beta_1 \\ -\frac{1}{8} + \beta_3 - \frac{\beta_1}{12} & \varepsilon(\frac{1}{2} + 4\alpha_2 + \frac{\gamma_2}{12}) & 2 - \beta_1 \end{pmatrix}, \varepsilon \pm 1.$$

An equivalent condition is that there exists a matrix norm $\|\cdot\|$ such that: $\|\Lambda_\varepsilon\| < 1, \varepsilon = \pm 1$. The proof of this result is in [10].
Remark 3: For some values of the parameters, the function f on each interval $[k, k+1], k \in \mathbb{Z}$ is a usual polynomial or a piecewise polynomial. For example:

1. $\alpha_2 = -5/32, \alpha_3 = 1/64, \beta_1 = 15/8, \beta_2 = -7/16, \beta_3 = 1/32, \gamma_2 = 3/2$, $\gamma_3 = -1/4$, then f is the Hermite interpolating quintic polynomial on each interval $[k, k+1], k \in \mathbb{Z}$,

2. $\alpha_2 = -23/144, \alpha_3 = 5/288, \beta_1 = 9/4, \beta_2 = -5/8, \beta_3 = 1/16, \gamma_2 = 3/2$, $\gamma_3 = -1/4$, then f is the cubic spline with two knots at 1/3 and 2/3 of the interval $[k, k+1]$,

3. $\alpha_2 = -5/32, \alpha_3 = 1/64, \beta_1 = 2, \beta_2 = -1/2, \beta_3 = 1/24, \gamma_2 = 3/2, \gamma_3 = -1/4$, then f is the quartic spline with one knot at the middle point of the interval $[k, k+1]$.

3.1 Elementary properties of $HS22$

Of course, we have lemmas equivalent to Lemma 1 and 2 on linearity, translation and scale change for the scheme $HS22$; they are not written again.

As for $HS21$, we introduce three basic solutions for the recursive system (13): the triplets $f_i, p_i, s_i, i = 0, 1, 2$ which are solutions of (13) with data,

$$\begin{array}{rl} & f_0(k) = \delta_{k,0}, p_0(k) = 0, s_0(k) = 0, \\ \forall k \in \mathbb{Z}, & f_1(k) = 0, p_1(k) = \delta_{k,0}, s_1(k) = 0, \\ & f_2(k) = 0, p_2(k) = 0, s_2(k) = \delta_{k,0}. \end{array}$$

Then, for any construction produced by (13), we have for any dyadic number x:

$$\left.\begin{array}{rcl} f(x) & = & \sum_{k=-\infty}^{\infty} [f(k)f_0(x-k) + p(k)f_1(x-k) + s(k)f_2(x-k)] \\ p(x) & = & \sum_{k=-\infty}^{\infty} [f(k)p_0(x-k) + p(k)p_1(x-k) + s(k)p_2(x-k)] \\ s(x) & = & \sum_{k=-\infty}^{\infty} [f(k)s_0(x-k) + p(k)s_1(x-k) + s(k)s_2(x-k)] \end{array}\right\} \tag{14}$$

From these formulae we can deduce a system of functional equations:

$$\left.\begin{array}{rcl} f_0(x/2) & = & f_0(x-1)/2 + f_0(x) + f_0(x+1)/2 + \frac{\beta_1}{2}[-f_1(x-1) + f_1(x+1)] \\ f_1(x/2) & = & \alpha_2[-f_0(x-1) + f_0(x+1)] + \frac{\beta_2}{2}f_1(x-1) + \frac{1}{2}f_1(x) + \frac{\beta_2}{2}f_1(x+1) \\ & & +\frac{\gamma_2}{4}[-f_2(x-1) + f_2(x+1)] \\ f_2(x/2) & = & \alpha_3[f_0(x-1) + f_0(x+1)] + \frac{\beta_3}{2}[-f_1(x-1) + f_1(x+1)] \\ & & +\frac{\gamma_3}{4}f_2(x-1) + \frac{1}{4}f_2(x) + \frac{\gamma_3}{4}f_2(x+1)] \end{array}\right\} \tag{15}$$

and similar equations in terms of p_i et s_i (cf (3)).

3.2 Fourier transform of $HS22$

When the Fourier operator is applied to each equation of the system (15) and to similar equations for p_i, s_i, we obtain

$$\begin{pmatrix} \hat{f}_0(\xi) \\ \hat{f}_1(\xi) \\ \hat{f}_2(\xi) \end{pmatrix} = A(\xi/2) \begin{pmatrix} \hat{f}_0(\xi/2) \\ \hat{f}_1(\xi/2) \\ \hat{f}_2(\xi/2) \end{pmatrix}, \begin{pmatrix} \hat{p}_0(\xi) \\ \hat{p}_1(\xi) \\ \hat{p}_2(\xi) \end{pmatrix} = 2A(\xi/2) \begin{pmatrix} \hat{p}_0(\xi/2) \\ \hat{p}_1(\xi/2) \\ \hat{p}_2(\xi/2) \end{pmatrix},$$

$$\begin{pmatrix} \hat{s}_0(\xi) \\ \hat{s}_1(\xi) \\ \hat{s}_2(\xi) \end{pmatrix} = 4A(\xi/2) \begin{pmatrix} \hat{s}_0(\xi/2) \\ \hat{s}_1(\xi/2) \\ \hat{s}_2(\xi/2) \end{pmatrix}, \tag{16}$$

where

$$A(\xi) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\cos\xi & i\frac{\beta_1}{2}\sin\xi & 0 \\ i\alpha_2 \sin\xi & \frac{1}{4} + \frac{\beta_2}{2}\cos\xi & i\frac{\gamma_2}{4}\sin\xi \\ \alpha_3\cos\xi & i\frac{\beta_3}{2}\sin\xi & \frac{1}{8} + \frac{\gamma_3}{4}\cos\xi \end{pmatrix}$$

Therefore, again, we will have to study the sequence of matrix products:

$$P_n(\xi) = A(\xi/2)A(\xi/4)\dots A(\xi/2^n). \tag{17}$$

Using Lemma 4 and Theorem 5, we prove a similar theorem to Theorem 6 and its corollary. But before that, we must change the matrix norm.

Let $\Theta = (\theta_1, \theta_2, \dots, \theta_d)$ be an element of $\mathbb{R}^d$ with positive θ_i. We define a new norm $\|\cdot\|_\Theta$ on $\mathbb{C}^d$ by $\|X\|_\Theta = \max_{i=1,d}(\theta_i|x_i|), X \in \mathbb{C}^d$. It is easy to prove that the associated matrix norm satisfies $\|A\|_\Theta = \max_i \sum_j \theta_i/\theta_j|a_{ij}|$ whenever $A = (a_{ij})$ lies in $\mathbb{C}^{d\times d}$. Obviously if $\theta_1 = \dots = \theta_d = 1$, we obtain the usual norm $\|\ \|_\infty$.

Theorem 13 *If $-5/2 < \beta_2 \leq 3/2$ and if $-9/2 < \gamma_3 < 7/2$, then for any complex number ξ, the sequence of matrices $P_n(\xi)$ defined in* (17) *converges and the convergence is uniform whenever ξ lies in the disq $|\xi| \leq R$. The nine components of the limit matrix $P(\xi)$ are entire functions of exponential type ≤ 1.*

Proof: We proceed as in Theorem 6 except that on $\mathbb{C}^3$, we use the norm $\|X\|_\Theta$ with $\Theta = (1, 1, \theta_3)$ where θ_3 is choosen small enough to get $\theta_3|\alpha_3| + |1/8 + \gamma_3/4| \leq 1$. Then it is easy to prove that $\|A(z/2^n)\|_\Theta \leq 1 + O(1/2^n)$ in the disk $|z| \leq R$.

Notice that $A(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} + \frac{\beta_2}{2} & 0 \\ \alpha_3 & 0 & \frac{1}{8} + \frac{\gamma_3}{4} \end{pmatrix}$ so that $\|A(0)\|_\Theta = 1$ with the above condition on θ_3.

By induction, we prove that

$$[A(0)]^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\frac{1}{4} + \frac{\beta_2}{2})^n & 0 \\ \frac{\alpha_3}{\frac{1}{8}+\frac{\gamma_3}{4}-1}[(\frac{1}{8} + \frac{\gamma_3}{4})^n - 1] & 0 & (\frac{1}{8} + \frac{\gamma_3}{4})^n \end{pmatrix}, n \geq 1,$$

therefore, using the hypothesis, this sequence converges to the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{\alpha_3}{\frac{1}{8}+\frac{\gamma_3}{4}-1} & 0 & 0 \end{pmatrix}$.

We can complete the proof as in the previous section because the hypothesis of Theorem 5 are satisfied. □

Corollary 14 *If $-5/2 < \beta_2 \leq 3/2$ and if $-9/2 < \gamma_3 < 7/2$, then each component of the limit matrix $P(z) = \lim P_n(z)$ is the Fourier transform of a distribution with support in the interval $[-1, 1]$.*

3.3 Schwartz distributions associated to $HS22$

We introduce 9 sequences of distributions in Schwartz sense:

$$\begin{array}{rcl} T_i^{(n)} & = & \frac{1}{2^n}\sum_m f_i(m/2^n)\delta_{m/2^n}, i = 0,1,2, \\ U_i^{(n)} & = & \frac{1}{2^n}\sum_m p_i(m/2^n)\delta_{m/2^n}, i = 0,1,2, \\ V_i^{(n)} & = & \frac{1}{2^n}\sum_m s_i(m/2^n)\delta_{m/2^n}, i = 0,1,2. \end{array}$$

When we compute the Fourier transform of these distributions, we obtain:

$$\hat{T}_i^{(n)}(\xi) = T_i^{(n)}(e^{-i\xi x}), \hat{U}_i^{(n)}(\xi) = U_i^{(n)}(e^{-i\xi x}), \hat{V}_i^{(n)}(\xi) = V_i^{(n)}(e^{-i\xi x}).$$

Then using the equations (15), we may verify that three recursions link these Fourier transform through the matrix $A(\xi)$:

$$\begin{pmatrix} \hat{T}_0^{(n+1)}(\xi) \\ \hat{T}_1^{(n+1)}(\xi) \\ \hat{T}_2^{(n+1)}(\xi) \end{pmatrix} = A(\xi/2) \begin{pmatrix} \hat{T}_0^{(n)}(\xi/2) \\ \hat{T}_1^{(n)}(\xi/2) \\ \hat{T}_2^{(n)}(\xi/2) \end{pmatrix}, \begin{pmatrix} \hat{U}_0^{(n+1)}(\xi) \\ \hat{U}_1^{(n+1)}(\xi) \\ \hat{U}_2^{(n+1)}(\xi) \end{pmatrix} = 2A(\xi/2) \begin{pmatrix} \hat{U}_0^{(n)}(\xi/2) \\ \hat{U}_1^{(n)}(\xi/2) \\ \hat{U}_2^{(n)}(\xi/2) \end{pmatrix},$$

$$\begin{pmatrix} \hat{V}_0^{(n+1)}(\xi) \\ \hat{V}_1^{(n+1)}(\xi) \\ \hat{V}_2^{(n+1)}(\xi) \end{pmatrix} = 4A(\xi/2) \begin{pmatrix} \hat{V}_0^{(n)}(\xi/2) \\ \hat{V}_1^{(n)}(\xi/2) \\ \hat{V}_2^{(n)}(\xi/2) \end{pmatrix}. \tag{18}$$

As $\begin{pmatrix} \hat{T}_0^{(0)}(\xi) \\ \hat{T}_1^{(0)}(\xi) \\ \hat{T}_2^{(0)}(\xi) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \hat{U}_0^{(0)}(\xi) \\ \hat{U}_1^{(0)}(\xi) \\ \hat{U}_2^{(0)}(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \hat{V}_0^{(0)}(\xi) \\ \hat{V}_1^{(0)}(\xi) \\ \hat{V}_2^{(0)}(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we deduce that

$$\begin{pmatrix} \hat{T}_0^{(n)}(\xi) \\ \hat{T}_1^{(n)}(\xi) \\ \hat{T}_2^{(n)}(\xi) \end{pmatrix} = P_n(\xi) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \hat{U}_0^{(n)}(\xi) \\ \hat{U}_1^{(n)}(\xi) \\ \hat{U}_2^{(n)}(\xi) \end{pmatrix} = 2^n P_n(\xi) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \hat{V}_0^{(n)}(\xi) \\ \hat{V}_1^{(n)}(\xi) \\ \hat{V}_2^{(n)}(\xi) \end{pmatrix} = 4^n P_n(\xi) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{19}$$

Like Theorems 8 and 9 for $HS21$, we have the two following theorems whose proofs are similar.

Theorem 15 *If $-5/2 < \beta_2 \leq 3/2$ and if $-9/2 < \gamma_3 < 7/2$, then the three sequences of distributions $T_0^{(n)}, T_1^{(n)}, T_2^{(n)}$ converge respectively to the distributions T_0, T_1, T_2. They are the three respective components of the inverse Fourier transform applied to the first column of the matrix $P(\xi)$.*

Theorem 16 *We suppose that $-5/2 < \beta_2 \leq 3/2$ et $-9/2 < \gamma_3 < 7/2$. If we build the triplet (f, p, s) by the subdivision scheme (13) from the data $\{y_k, y'_k, y''_k\}_{k\in\mathbb{Z}}$, then the sequence of distributions $F_n = \dfrac{1}{2^n} \displaystyle\sum_{m=-\infty}^{\infty} f(m/2^n)\delta_{m/2^n}$ converges to the distribution*

$$F = \sum_{k=-\infty}^{\infty} [y_k \tau_{-k} T_0 + y'_k \tau_{-k} T_1 + y''_k \tau_{-k} T_2].$$

To study the convergence of the sequences $\hat{U}_i^{(n)}$ et $\hat{V}_i^{(n)}, i = 0, 1, 2$, we need a new lemma which gives the convergence of an infinite product of matrices.

Lemma 17 *Let $M = \begin{pmatrix} 1 & 0 & 0 \\ a & b & 0 \\ 0 & c & d \end{pmatrix} \in \mathbb{C}^{3\times 3}$ with $|b| < 1, |d| < 1$ then*

$$\lim_{n\to+\infty} M^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{a}{1-b} & 0 & 0 \\ \frac{ac}{(1-b)(1-d)} & 0 & 0 \end{pmatrix}$$

Proof: By induction, we obtain that for all $n \geq 2$ and $b \neq d$

$$M^n = \begin{pmatrix} 1 & 0 & 0 \\ a\frac{1-b^n}{1-b} & b^n & 0 \\ \frac{ac}{b-d}\left(\frac{1-b^n}{1-b} - \frac{1-d^n}{1-d}\right) & c\frac{b^n-d^n}{b-d} & d^n \end{pmatrix}$$

and for $b = d$,

$$M^n = \begin{pmatrix} 1 & 0 & 0 \\ a\frac{1-b^n}{1-b} & b^n & 0 \\ ac\frac{(n-1)b^n - nb^{n-1}+1}{(1-b)^2} & cnb^{n-1} & b^n \end{pmatrix}$$

It is now easy to conclude. □

Theorem 18 *If $-3/2 < \beta_2 < 1/2$ et $-3/2 < \gamma_3 < 1/2$ then both sequences $U_i^{(n)}$ and $V_i^{(n)}, i = 0,1,2$ converge respectively to U_i and V_i.*

If we add $\beta_1 + 2\beta_2 = 1$, then $U_i = T_i', i = 0,1,2$

Moreover, if we add $\gamma_2 + 2\gamma_3 = 1$, then $V_i = T_i'', i = 0,1,2$.

Proof: Let us set $t_n = \begin{pmatrix} \hat{T}_0^{(n)} \\ \hat{T}_1^{(n)} \\ \hat{T}_2^{(n)} \end{pmatrix}, u_n = \begin{pmatrix} \hat{U}_0^{(n)} \\ \hat{U}_1^{(n)} \\ \hat{U}_2^{(n)} \end{pmatrix}, v_n = \begin{pmatrix} \hat{V}_0^{(n)} \\ \hat{V}_1^{(n)} \\ \hat{V}_2^{(n)} \end{pmatrix}$. With (18), we get

$$\begin{aligned} t_n &= P_{n-1}A(\xi/2^n)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = P_{n-1}\begin{pmatrix} 1/2 + \cos(\xi/2^n)/2 \\ i\alpha_2 \sin(\xi/2^n) \\ \alpha_3 \cos(\xi/2^n) \end{pmatrix} \\ &= [1/2 + \cos(\xi/2^n)/2]t_{n-1} + \tfrac{1}{2^{n-1}} i\alpha_2 \sin(\xi/2^n)u_{n-1} + \tfrac{1}{4^{n-1}}\alpha_3 \cos(\xi/2^n)v_{n-1}. \end{aligned}$$

Similarly, we obtain:

$$\begin{aligned} u_n &= i\frac{\beta_1}{2}2^n \sin(\xi/2^n)t_{n-1} + [1/2 + \beta_2 \cos(\xi/2^n)]u_{n-1} + i\frac{\beta_3}{2}2^n \sin(\xi/2^n)\frac{1}{4^{n-1}}v_{n-1}, \\ v_n &= i\frac{\gamma_2}{2}2^n \sin(\xi/2^n)u_{n-1} + [1/2 + \gamma_3 \cos(\xi/2^n)]v_{n-1}. \end{aligned}$$

These equations can be written in a vector form: $\begin{pmatrix} t_n \\ u_n \\ v_n \end{pmatrix} = N_n \begin{pmatrix} t_{n-1} \\ u_{n-1} \\ v_{n-1} \end{pmatrix}$ where

$$N_n = \begin{pmatrix} 1/2 + \cos(\xi/2^n)/2 & \frac{1}{2^{n-1}} i\alpha_2 \sin(\xi/2^n) & \frac{1}{4^{n-1}} i\alpha_3 \cos(\xi/2^n) \\ i\frac{\beta_1}{2}2^n \sin(\xi/2^n) & 1/2 + \beta_2 \cos(\xi/2^n) & i\beta_3 \frac{1}{2^{n-1}} \sin(\xi/2^n) \\ 0 & i\frac{\gamma_2}{2}2^n \sin(\xi/2^n) & 1/2 + \gamma_3 \cos(\xi/2^n) \end{pmatrix}$$

Setting $M = \begin{pmatrix} 1 & 0 & 0 \\ i\frac{\beta_1}{2}\xi & 1/2 + \beta_2 & 0 \\ 0 & i\frac{\gamma_2}{2}\xi & 1/2 + \gamma_3 \end{pmatrix}$ and using the previous lemma and the hypothesis on β_2 and γ_3, we get that the sequence M^n converges.

We choose a matrix norm $\|\cdot\|_\Theta = (1, \theta_2, \theta_3)$ with θ_2 and θ_3 such that: $\theta_2|\frac{\beta_1}{2}\xi| + |1/2 + \beta_2| < 1$ then $\dfrac{\theta_3}{\theta_2}|\dfrac{\gamma_2}{2}\xi| + |1/2 + \gamma_3| < 1$.

Now $\|M\|_\Theta = 1$ and $\|M - N_n\|_\Theta = O(1/2^n)$. Using again Theorem 5 we conclude that the sequence $N_n N_{n-1} \dots N_1$ converges, so that the sequence $(t_n, u_n, v_n)^T$ converges to $(t, u, v)^T$.

When we reach the limit, then $u = i\frac{\beta_1}{2}\xi t + (1/2 + \beta_2)u$ and $v = i\frac{\gamma_2}{2}\xi u + (1/2 + \gamma_3)v$. This can be written $u = i\frac{\beta_1}{1-2\beta_2}\xi t$ and $v = i\frac{\gamma_2}{1-2\gamma_3}\xi u$. By the inverse Fourier transform, we have the convergence result for $U_i^{(n)}$ and $V_i^{(n)}, i = 0,1,2$.

With the additional hypothesis $\beta_1 + 2\beta_2 = 1$, we have $u = i\xi t$, therefore $U_i = T_i', i = 0, 1, 2$.

Finally, if moreover $\gamma_2 + 2\gamma_3 = 1$, then $v = i\xi u$ so that $V_i = T_i'', i = 0, 1, 2$. □

Now we have a last theorem on the convergence of derivatives distributions as we had for $HS21$.

Theorem 19 *We suppose that $-3/2 < \beta_2 < 1/2, -3/2 < \gamma_3 < 1/2$ and $\beta_1 + 2\beta_2 = 1, \gamma_2 + 2\gamma_3 = 1$. If we build the triplet (f, p, s) by the subdivision scheme (13) from the data $\{y_k, y_k', y_k''\}_{k\in\mathbb{Z}}$, then the sequence of distributions $G_n = \dfrac{1}{2^n}\displaystyle\sum_{m=-\infty}^{\infty} p(m/2^n)\delta_{m/2^n}$ converges to the distribution $G = \displaystyle\sum_{k=-\infty}^{\infty}[y_k\tau_{-k}T_0' + y_k'\tau_{-k}T_1' + y_k''\tau_{-k}T_2']$, and the sequence of distributions $H_n = \dfrac{1}{2^n}\displaystyle\sum_{m=-\infty}^{\infty} s(m/2^n)\delta_{m/2^n}$ converges to the distribution $H = \displaystyle\sum_{k=-\infty}^{\infty}[y_k\tau_{-k}T_0'' + y_k'\tau_{-k}T_1'' + y_k''\tau_{-k}T_2'']$.*

3.4 A characterization of Fourier transforms $\hat{T}_0, \hat{T}_1, \hat{T}_2$

We characterize the triplet of functions $\phi_i(\xi) = \hat{T}_i(\xi), i = 0, 1, 2$ without computing all the subdivision scheme for three triplets of initial data. This triplet of functions satisfies the functional system:

$$\begin{pmatrix} \phi_0(2\xi) \\ \phi_1(2\xi) \\ \phi_2(2\xi) \end{pmatrix} = A(\xi)\begin{pmatrix} \phi_0(\xi) \\ \phi_1(\xi) \\ \phi_2(\xi) \end{pmatrix}, \tag{20}$$

Theorem 20 *If $-5/2 < \beta_2 < 3/2$ and $-9/2 < \gamma_3 < 7/2$, then there exists one and only one triplet of analytic functions $(\phi_0(\xi), \phi_1(\xi)), \phi_2(\xi))$ solution of the functional system (20) and satisfying $\phi_0(0) = 1$.*

Proof: One must first notice that the matrix $A(0)$ is a triangular lower matrix whose diagonal components are successively $1, 1/4 + \beta/2, 1/8 + \gamma_3/4$. Under the hypothesis $-5/2 < \beta_2 < 3/2, -9/2 < \gamma_3 < 7/2$, we know that the vector sequence

$$\begin{pmatrix} \hat{T}_0^{(n)}(\xi) \\ \hat{T}_1^{(n)}(\xi) \\ \hat{T}_2^{(n)}(\xi) \end{pmatrix} = A(\xi/2)A(\xi/4)\dots A(\xi/2^n)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

converges to $(\hat{T}_0(\xi), \hat{T}_1(\xi), \hat{T}_2(\xi))^T$. In the vector equation (21), if we substitute 0 to ξ, we verify that for all $n, \hat{T}_0^{(n)}(0) = 1$. Therefore $\hat{T}_0(0) = 1$. Setting $\phi_i(\xi) = \hat{T}_i(\xi)$, we obtain that the triplet (ϕ_0, ϕ_1, ϕ_2) is an analytic solution of the equations (20) and $\phi_0(0) = 1$.

Now, let (ϕ_0, ϕ_1, ϕ_2) be a triplet of analytic functions solution of (20) and such that $\phi_0(0) = 1$. As in Theorem 12, we expand the functions ϕ_0, ϕ_1, ϕ_2 and the components of $A(\xi)$ in power series of ξ:
$\phi_i(\xi) = \sum_{n=0}^{\infty} \phi_i^{(n)} \xi^n, i = 1, 2, 3$; $A(\xi) = \sum_{n=0}^{\infty} \xi^n A_n$.

When we substitute 0 to ξ in equation (20), we obtain that $(\phi_0(0), \phi_1(0), \phi_2(0))^T$ is a column eigenvector for the eigenvalue 1 of the matrix $A(0)$. Since $A(0)$ is triangular with only one 1 on its main diagonal, this eigenvector is unique when its first component is 1. So $\phi_1(0) = 0, \phi_2(0) = 8\alpha_3/(7 - 2\gamma_3)$. Hence $\phi_i(0) = \hat{T}_i(0), i = 0, 1, 2$. By replacing ϕ_i with their powers series in the system (20), by developing the products and reorganizing the results in terms of powers of ξ, we obtain the series of equations depending on $n = 1, 2, 3, \ldots$:

$$2^n \begin{pmatrix} \phi_0^{(n)} \\ \phi_1^{(n)} \\ \phi_2^{(n)} \end{pmatrix} = \sum_{k=0}^{n} A_k \begin{pmatrix} \phi_0^{(n-k)} \\ \phi_1^{(n-k)} \\ \phi_2^{(n-k)} \end{pmatrix}, \tag{22}$$

Since for all $n > 0$, the matrix $2^n I - A(0)$ has an inverse, all the components in the expansions of $\phi_i^{(n)}, i = 0, 1, 2$ are uniquely definite. $\square$

4 Conclusion

As we saw it, it is possible to define Hermite interpolatory subdivision schemes on the space of Schwartz distributions. The study of the convergence of these schemes on the space $\mathcal{D}(\mathbb{R})'$ can be done through the Fourier transform provided that sufficiently general results about convergence of infinite products of matrices are available. We came to this approach after previous works of Deslauriers-Dubuc [3] and of Deslauriers-Dubois-Dubuc [4]. These authors considered interpolatory subdivision schemes which nevertheless were not of Hermite type. In this situation, the Fourier transform allowed the study of the convergence of some schemes with the help of products of trigonometrical polynomials (and not of matrices).

After defining Hermite subdivision schemes on the space of distributions, we may think to many other questions. One of these is the following. In the scheme $HS11$, can we characterize the largest region of the parameters α, β such that for the corresponding Fourier transforms $\hat{T}_0, \hat{T}_1$,we get $\int_{-\infty}^{\infty} (1 + \xi^2)\hat{T}_i(\xi)|^2 d\xi < \infty, i = 0, 1$? This characterization would be useful to specifying all interpolating functions f of the scheme such that f and f' are in $L^2(\mathbb{R})$.

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