Acadia University
Department of Mathematics and Statistics

Topics from Advanced Calculus
(MATH 2023)

Assignment 4 - solutions

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April 5, 2002

The marking has to reflect the fact that students were given the answers (in the back of the book). The answer alone is worth nothing. Out of 130 marks.

6 Applications of First-Order Differential Equations

6.61. [5 marks] The differential equation is $2v' = 2g + 50v$ or $v' = g + 25v$. The limiting velocity is reached when $v' = 0$ or $g + 25v = 0$ or when $v = -g/25 = -9.8/25 = -0.392$ m/sec where we set $g = 9.8$ m/sec^2. Alternatively, one could solve the full differential equation and find the maximum velocity...
6.71. (a) [20 marks] The differential equation is $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$ or $\frac{dq}{dt} + \frac{1}{10\times 10^{-2}}q = 0.5$ or $\frac{dq}{dt} + 10q = 0.5$ The solution is $q(t) = 4.95 \times e^{-10t} + 0.05$. The current is given by $I(t) = \frac{dq}{dt} = -49.5e^{-10t}$.
(b) [5 marks] When t approaches ∞, we get 0.

7 Linear Differential Equations

7.36. [5 marks] $W = \begin{vmatrix} 3x & 4x \\ 3 & 4 \end{vmatrix} = 0$. The set of functions is linearly dependent.
7.45. [5 marks] $W = \begin{vmatrix} x & 1 & 2x-7 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$. The set of functions is linearly dependent.
7.48. [10 marks]

$$\begin{aligned} W &= \begin{vmatrix} e^{-x} & e^{x} & e^{2x} \\ -e^{-x} & e^{x} & 2e^{2x} \\ e^{-x} & e^{x} & 4e^{2x} \end{vmatrix} \\ &= e^{-x}\begin{vmatrix} e^{x} & 2e^{2x} \\ e^{x} & 4e^{2x} \end{vmatrix} + e^{x}\begin{vmatrix} 2e^{2x} & -e^{-x} \\ 4e^{2x} & e^{-x} \end{vmatrix} + e^{2x}\begin{vmatrix} -e^{-x} & e^{x} \\ e^{-x} & e^{x} \end{vmatrix} \\ &= 2e^{2x} + 6e^{2x} - 2e^{2x} \\ &= 6e^{2x}. \end{aligned}$$

The set of functions is linearly independent. (For full marks, the computation of the determinant must be explicit: it is impossible to do such a computation in the air...)

8 Second-Order Linear Homogeneous Differential Equations

8.17. [10 marks] We substitute $y(x) = e^{\lambda x}$ to get the characteristic equation $\lambda^2 - 1 = 0$ hence $\lambda = \pm 1$ and the solution is $y = c_1 e^x + c_2 e^{-x}$.
8.24. [20 marks] We substitute $y(x) = e^{\lambda x}$ to get the characteristic equation $\lambda^2 + 2\lambda + 3 = 0$ hence $\lambda = \pm\sqrt{2}i - 1$ and the solution is $y = c_1 e^{-x} \cos\sqrt{2}x + c_2 e^{-x} \sin\sqrt{2}x$.
8.38. [20 marks] We substitute $y(t) = e^{\lambda t}$ to get the characteristic equation $\lambda^2 - 5\lambda + 7 = 0$ hence $\lambda = \pm\frac{\sqrt{3}}{2}i + \frac{5}{2}$ and the solution is $Q = c_1 e^{5t/2} \cos\frac{\sqrt{3}}{2}t + c_2 e^{5t/2} \sin\frac{\sqrt{3}}{2}t$.

9 n^{th} order Linear Homogeneous Equations

9.16. [20 marks] We substitute $y(x) = e^{\lambda x}$ to get the characteristic equation $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$ hence $\lambda = -1, 1, 2$ and the solution is $y = c_1 e^{-x} + c_2 e^x + c_3 e^{2x}$.
9.32. [20 marks] We substitute $y(x) = e^{\lambda x}$ to get the characteristic equation $\lambda^3 + \lambda^2 + 2\lambda + 3 = 0$ hence $\lambda = \pm\sqrt{2}i - 1$ and the solution is $q = c_1 e^x + c_2 e^{-x} + c_3 e^{\sqrt{2}x} + c_4 e^{-\sqrt{2}x}$.

10 The Method of Undetermined Coefficients

10.44. [20 marks] We substitute $y(x) = e^{\lambda x}$ in the corresponding homogeneous differential equation (setting the RHS to zero, $y'' = 2y' + y = 0$). The characteristic equation is $\lambda^2 - 2\lambda + 1$ or $(\lambda - 1)^2$, hence the solution to the homogeneous problem is $y(x) = c_1 e^x + c_2 x e^x$. The inhomogeneous term is a polynomial, hence we guess a polynomial solution of the form $ax^2 + bx + c$. Substituting and collecting the terms we get $a = 1$, $b = 4$, and $c = 5$. Hence the general solution is $y(x) = c_1 e^x + c_2 x e^x + x^2 + 4x + 5$.
10.46. [20 marks] We substitute $y(x) = e^{\lambda x}$ in the corresponding homogeneous differential equation (setting the RHS to zero, $y'' - 2y' + y$). The characteristic equation is $\lambda^2 - 2\lambda + 1$ or $(\lambda - 1)^2$, hence the solution to the homogeneous problem is $y(x) = c_1 e^x + c_2 x e^x$. (This is all the same as in question 10.44.) The inhomogeneous term is a trigonometric function, hence we guess a trigonometric solution of the form $A\cos x + B\sin x$. Substituting and collecting the terms we get $A = 0$ and $B = -2$. Hence the general solution is $y = c_1 e^x + c_2 x e^x - 2\sin x$.
10.49. [20 marks] We substitute $y(x) = e^{\lambda x}$ in the corresponding homogeneous differential equation (setting the RHS to zero, $y' - y$). The characteristic equation is $\lambda - 1$, hence the solution to the homogeneous problem is $y(x) = c_1 e^x$. The inhomogeneous term is an exponential, hence we guess an exponential solution of the form e^x. This first guess will fail since this is also the solution to the homogeneous problem. Therefore, we modifiy our guess by multiplying it by x. We try Axe^x, substituting we get $Ae^x + Axe^x - Axe^x = e^x$ or $Ae^x = e^x$ or $A = 1$. Hence the general solution is $y = c_1 e^x + x e^x$.