# A FAMILY OF 4-POINT DYADIC HIGH RESOLUTION SUBDIVISION SCHEMES

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ABSTRACT. By using temporary placeholders on a dense grid, we generalize the 4-point dyadic cubic Deslauriers-Dubuc scheme. Interpolated values require 2 steps to stabilize as they are first interpolated on a coarse scale through a tetradic filter and then on a finer scale using a dyadic filter. The interpolants are $C^1$ and can be chosen to reproduce polynomials of degree 4. These generalized interpolatory subdivision schemes have minimal support and no additional memory requirement. This work has applications in CAGD and wavelet theory.

## 1. INTRODUCTION

Interpolatory subdivision schemes interpolate a discrete set of data points in a local manner, that is, the value of the interpolation function at a given point depends on a small number of nearby data points. The classical dyadic algorithm introduced by Deslauriers and Dubuc [4, 2] finds the midpoint values by fitting a Lagrange polynomial through the $2N$ closest data points. By repeating this algorithm again and again, each time doubling the number of data points or nodes by midpoint interpolation, we eventually have a dense set of data points and we can determine uniquely a smooth interpolation function. Because interpolatory subdivision schemes relate data points from one scale to the data points at another scale, it is not surprising that they are a key ingredient in the construction of compactly supported wavelets [1, 3].

More recently, Merrien [9, 10, 5] introduced Hermite subdivision schemes. Since Merrien subdivision schemes use Hermite nodes, they have have twice the approximation order and better regularity for a given support. For example $2-$point Hermite schemes are differentiable and can reproduce quadratic or cubic polynomials whereas the corresponding Deslauriers-Dubuc scheme (the linear spline) isn't differentiable and can only reproduce linear polynomials.

Since Deslauriers-Dubuc schemes have important applications, it is tempting to add extra nodes to Deslauriers and Dubuc schemes as an attempt to improve them to get "high resolution" schemes. Doubling the number of nodes is costly effectively doubling the memory requirements, however, since a dyadic subdivision scheme doubles its memory usage at each step, we can choose to use right away this upcoming extra storage space without any cost. In effect, we can simply make use of the memory that will be allocated later in any case. Therefore, we can freely increase the number of nodes in intermediate steps. These new placeholders can then be used to record a coarse scale guess (using a tetradic filter) which we can later combine with a finer scale interpolation (using a dyadic filter). As a special case, we may choose to ignore the coarse scale estimate, in which case our approach amounts to a Deslauriers-Dubuc scheme; we can also use this approach to reproduce polynomials of degree 4 by a Richardson extrapolation approach. The main result of this paper is that by summing up the tetradic (coarse) interpolation recorded in placeholders and dyadic (fine) interpolations, we get a range of smooth ($C^1$) high resolutions schemes reproducing cubic polynomials.

## 2. SUBDIVISION SCHEMES

Interpolatory subdivision schemes where first introduced by Deslauriers and Dubuc (quote). Let $b>1$ be an integer, given two integers $k, j$, the number $x_{j,k}=k/b^j$ is said to be $b-$adic (of depth $j$). For a fixed $j$, the $b-$adic numbers form a regularly spaced set of nodes. Given some data $\left\{y_{J,k}\right\}_{k\in\mathbb{N}}$ on the $b-$adic numbers of depth $J$, we want to build a smooth fonction $f$ such that $f\left(x_{J,k}\right)=y_{J,k}\,\forall k\in\mathbb{N}$. Starting with this initial data ($y_{J,k}$) and using the linear formula

$$y_{j+1,l}=\sum_{k\in\mathbb{N}}\gamma_{bk-l}y_{j,k} \tag{2.1}$$

for some constant array $\gamma$, we get values $y_{J,k}$ for any $j>J$ and since $b-$adic numbers form a dense set of the real numbers, there is at most one continuous function such that $f\left(x_{j,k}\right)=y_{j,k}$ for all $k, j>J$.

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A subdivision scheme is interpolatory and will satisfy $f(x_{J,k}) = y_{J,k}$ if $\gamma_{bk} = 0$ except when $k = 0$. We say that a subdivision scheme is stationary if the array $\gamma$ is constant (doesn't depend on $j$). Because $\gamma$ doesn't depend explicitely on $l$ the scheme is translation invariant or homogeneous. An interpolatory subdivision scheme is said to be $2N-$point if $\gamma_l = 0$ for $|l| > Nb$. The interpolation function $f$ computed from a $2N-$point $b-$adic scheme with initial data $y_{0,0} = 1$ and $y_{0,k} = 0$ for all $k \neq 0$ is said to be the fundamental function and has a compact support of $[-(Nb-1)/(b-1),(Nb-1)/(b-1)]$ or $[1-2N,2N-1]$ when $b = 2$. Hence as $N$ increases the support of the fundamental function increases.

For $N = 1,2,3,\ldots$ there are corresponding $2N-$point interpolatory Deslauriers-Dubuc subdivision schemes and they are built from the midpoint evaluation of Lagrange polynomial of degree $2N-1$. For $b = 2$ (dyadic case), the $4-$point Deslauriers-Dubuc scheme can be defined by the array $\gamma^{DD2}$ given by $\gamma_0^{DD2} = 1, \gamma_1^{DD2} = \gamma_{-1}^{DD2} = -9/16, \gamma_3^{DD2} = \gamma_{-3}^{DD2} = -1/16$ with $\gamma_k^{DD2} = 0$ otherwise; for $b = 4$ (tetradic case), the scheme is defined by $\gamma^{DD4}$ given by $\gamma_{2k}^{DD4} = \gamma_k^{DD2}\,\forall k \in \mathbb{N}$, $\gamma_{-1}^{DD4} = \gamma_1^{DD4} = 105/128, \gamma_{-3}^{DD4} = \gamma_{-3}^{DD4} = 35/128, \gamma_{-5}^{DD4} = \gamma_5^{DD4} = -7/128, \gamma_{-7}^{DD4} = \gamma_7^{DD4} = -5/128$, with $\gamma_k^{DD2} = 0$ otherwise. Because $4-$point Deslauriers-Dubuc schemes are derived from cubic Lagrange polynomials, they reproduce cubic polynomials, that is, if the initial data $y_{j,k}$ satisfies $y_{j,k} = p(x_{j,k})\ \forall k \in \mathbb{N}$ for some cubic polynomial $p$ then the interpolation function $f$ is this same cubic polynomial $f = p$. The two cases presented above ($\gamma^{DD2}$ and $\gamma^{DD4}$) reproduce cubic polynomials and it can also be shown that they both give differentiable ($C^1$) interpolation functions. Because we later borrow from these two subdivision schemes, we give explicit algorithms for both schemes.

**Algorithm 2.1.** *($4-$point Deslauriers-Dubuc Dyadic Scheme) The following iteration steps depend on $\alpha$, a constant parameter.*

1. *recopy data for $x_{j,k} = x_{j+1,2k}$: $y_{j+1,2k} = y_{j,k}\,\forall k \in \mathbb{N}$ ;*
2. *interpolate midpoint value by the corresponding cubic Lagrange polynomial:*

$$y_{j+1,2k+1} = \frac{-y_{j,k-1} + 9y_{j,k} + 9y_{j,k+1} - y_{j,k+2}}{128}\ \forall k \in \mathbb{N};$$

3. *Repeat with $j \to j+1$ and using $y_{j+1}$ as initial data.*

**Algorithm 2.2.** *($4-$point Deslauriers-Dubuc Tetradic Subdivision Scheme) The following iteration steps depend on $\alpha$, a constant parameter.*

1. *recopy data for $x_{j,k} = x_{j+1,2k}$: $y_{j+1,4k} = y_{j,k}\,\forall k \in \mathbb{N}$ ;*
2. *interpolate quartertile point values by the corresponding cubic Lagrange polynomial:*

$$y_{j+1,4k+1} = \frac{-7y_{j,k-1} + 105y_{j,k} + 35y_{j,k+1} - 5y_{j,k+2}}{128};$$

$$y_{j+1,2k+1} = \frac{-y_{j,k-1} + 9y_{j,k} + 9y_{j,k+1} - y_{j,k+2}}{128};$$

$$y_{j+1,4k+3} = \frac{-5y_{j,k-1} + 35y_{j,k} + 105y_{j,k+1} - 7y_{j,k+2}}{128}\ \forall k \in \mathbb{N};$$

3. *Repeat with $j \to j+1$ and using $y_{j+1}$ as initial data.*

## 3. High resolution subdivsion schemes

3.1. **Definitions.** In this paper, we want to show how the subdivision scheme framework can be extended to build hybrid schemes: mixing tetradic and dyadic subdivision schemes for example. Given some data $\{y_{j-1,k}\}_{k\in\mathbb{N}}$ to interpolate on the $x_{j-1,k}$ grid, we first apply a dyadic subdivision scheme such as the $4-$point Deslauriers-Dubuc scheme to get finer scale data $y_{j,k}$. For any $j'$, the odd nodes $y_{j',2k+1}$will be referred to as "placeholders" because their assigned value will change in general whereas the even nodes are referred to as stable that is, we require $y_{j+1,4k} = y_{j,2k}\,\forall k \in \mathbb{N}$ but not $y_{j+1,4k+2} = y_{j,2k+1}$. Hence the purpose of first applying a dyadic subdivision scheme is to fill the placeholders ($y_{j,2k+1}$). With this setting, the following algorithm is interpolatory.

**Algorithm 3.1.** *($4-$point Dyadic High Resolution Subdivision Scheme) The following iteration steps depend on $\alpha$, a constant parameter.*

1. *recopy stable data: $y_{j+1,4k} = y_{j,2k}\,\forall k \in \mathbb{N}$ ;*

2. *Apply the 4−point Deslauriers-Dubuc tetradic scheme on even (stable) nodes:*

$$y_{j+1,4k+1}=\frac{-7y_{j,2k-2}+105y_{j,2k}+35y_{j,2k+2}-5y_{j,2k+4}}{128};$$

$$y^{temporary}_{j+1,4k+2}=\frac{-y_{j,2k-2}+9y_{j,2k}+9y_{j,2k+2}-y_{j,2k+4}}{128};$$

$$y_{j+1,4k+3}=\frac{-5y_{j,2k-2}+35y_{j,2k}+105y_{j,2k+2}-7y_{j,2k+4}}{128}\,\forall k\in\mathbb{N};$$

3. *Update midpoint (which then becomes stable):*

$$y_{j+1,4k+2}=(1-\alpha)y^{temporary}_{j+1,4k+2}+\alpha y_{j,2k+1};$$

4. *Repeat with $j\to j+1$ and using $y_{j+1}$ as initial data.*

This new algorithm is not a subdivision scheme and thus we need to propose a more general definition: subdivision schemes (equation 2.1) can be generalized by the linear equation

$$y_{j+1,l}=\sum_{m=1}^{M}\sum_{k\in\mathbb{N}}\gamma^{(m)}_{Nbk+m-1-l}y_{j,Nk+m-1} \tag{3.1}$$

where $\gamma^{(1)},\ldots,\gamma^{(M)}$ are constant arrays (independent from $j$). It can be said to be $b-$adic because the number of nodes is increasing by a factor of $b$ with each iteration but because we have $M$ arrays $\gamma$, the scheme is said to be a high resolution subdivision scheme if $M>1$. For $b=M=2$ the general equation 3.1 becomes

$$y_{j+1,l}=\sum_{k\in\mathbb{N}}\gamma^{(1)}_{4k-l}y_{j,2k}+\gamma^{(2)}_{4k+1-l}y_{j,2k+1}. \tag{3.2}$$

Algorithm 3.1 amounts to choosing $\gamma^{(1)}$ and $\gamma^{(2)}$ to be:

$$\gamma^{(1)}_{2k}=\gamma^{DD4}_{2k}+\alpha\left(\delta_{k,0}-\gamma^{DD2}_{k}\right),\quad \gamma^{(1)}_{2k+1}=\gamma^{DD4}_{2k+1}\,\forall k\in\mathbb{N} \tag{3.3}$$

$$\gamma^{(2)}_{0}=\alpha,\qquad \gamma^{(2)}_{k}=0\,otherwise \tag{3.4}$$

for some parameter $\alpha\in\mathbb{R}$.

In the simplest case, $\alpha=0$, equation 3.2becomes

$$y_{j+1,l}=\sum_{k\in\mathbb{N}}\gamma^{DD4}_{4k-l}y_{j,2k}. \tag{3.5}$$

Because $\gamma^{(2)}=0$ in this case, we see that the placeholders (odd nodes) are effectively ignored. Indeed, we observe that this last equation discards odd nodes at each step: $y_{j+1,l}$ depends only on even nodes ($y_{j,2k}$ ) and not at all on the odd nodes ( $y_{j,2k+1}$). Hence, we can replace equation 3.5 by

$$y_{j+1,2l}=\sum_{k\in\mathbb{N}}\gamma^{DD4}_{4k-2l}y_{j,2k}$$

but because $\gamma^{DD4}_{2k}=\gamma^{DD2}_{k}$ , this last equation becomes $y_{j+1,2l}=\gamma^{DD2}_{2k-l}y_{j,2k}$ and if we define $\widetilde{y}_{j,k}=y_{j,2k}$ then

$$\widetilde{y}_{j+1,2l}=\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-l}\widetilde{y}_{j,2k} \tag{3.6}$$

which we recognize as the cubic Deslauriers-Dubuc scheme. We could also show the same result by looking at algorithm 3.1.

**Proposition 3.2.** *For $\alpha=0$, the high resolution scheme given by algorithm 3.1 (or equations 3.2, 3.3, and 3.4) is equivalent to the 4−point dyadic Deslauriers-Dubuc subdivision scheme (discarding the odd nodes or placeholders in the first iteration).*

In general, since $\gamma^{DD4}_{2k}=\gamma^{DD2}_{k}$, we can rewrite equation 3.2 for even and odd terms. Firstly, setting $l=2s$ ($l$ even), we have

$$\begin{aligned} y_{j+1,2s} &= \textstyle\sum_{k\in\mathbb{N}} \quad \gamma^{(1)}_{4k-2s}y_{j,2k}+\gamma^{(2)}_{4k+1-2s}y_{j,2k+1}\\ &= \textstyle\sum_{k\in\mathbb{N}} \quad \left(\gamma^{DD4}_{4k-2s}-\alpha\gamma^{DD2}_{2k-s}+\alpha\delta_{4k,2s}\right)y_{j,2k}+\alpha\delta_{4k+1,s}y_{j,2k+1}\\ &= \textstyle\sum_{k\in\mathbb{N}} \quad \left((1-\alpha)\gamma^{DD2}_{2k-s}+\alpha\delta_{2k,s}\right)y_{j,2k}+\delta_{2k+1,s+1}\alpha y_{j,2k+1} \end{aligned}$$

so that when $s$ is even ($l=2s=4r$), we have the interpolatory condition

$$y_{j+1,4r}=y_{j,2r} \tag{3.7}$$

otherwise, when $s$ is odd $(l=2s=4r+2)$

$$y_{j+1,4r+2}=\alpha y_{j,2r+1}+(1-\alpha)\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-2r-1}y_{j,2k}. \tag{3.8}$$

Secondly, if $l$ is odd ($l=2s+1$), we have

$$\begin{aligned} y_{j+1,2s+1} &= \sum_{k\in\mathbb{N}}\gamma^{(1)}_{4k-2s-1}y_{j,2k}+\gamma^{(2)}_{4k-2s-1}y_{j,2k+1}\\ &= \sum_{k\in\mathbb{N}}\gamma^{DD4}_{4k-2s-1}y_{j,2k}. \end{aligned} \tag{3.9}$$

Equations 3.7, 3.8, and 3.9 can be used to describe the chosen high resolution schemes: while equation 3.7 is the interpolatory condition, equation 3.9 fills the placeholders with tetradic (coarse scale) interpolated values whereas equation 3.8 combines the value stored in the placeholder with the newly available interpolated value (fine scale) given by the summation term which we recognize from the dyadic Deslauriers-Dubuc interpolation.

3.2. **Reproduced polynomials.** Assume that for some $j$, $y_{j,k}=p_3\left(x_{j,k}\right)$ for some cubic polynomial $p_3$, because $4-$point Deslauriers-Dubuc schemes reproduce cubic polynomials, we have

$$\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-2r-1}y_{j,2k}=y_{j,2r+1}=p_3\left(x_{j,2r+1}\right)$$

and thus equation 3.8 becomes $y_{j+1,4r+2}=p_3\left(x_{j,2r+1}\right)$ . Similarly, equation 3.9 implies $y_{j+1,2s+1}=p_3\left(x_{j+1,2s+1}\right)$. We can conclude that $y_{j+1,k}=p_3\left(x_{j+1,k}\right)$ if $y_{j,k}=p_3\left(x_{j,k}\right)$ and thus the high resolution scheme reproduce cubic polynomials. For practical implementations of a high subdivision scheme, it is necessary to first apply a one-step subdivision scheme since in general, we don't have placeholder values precomputed. As we've seen this can be solved by a one-step dyadic Deslauriers-Dubuc interpolation. Let $\left\{y_{j,k}\right\}_k$ be some initial data. As a first step, we apply equation

$$y_{j+1,l}=\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-l}y_{j,2k} \tag{3.10}$$

followed by equation 3.2 with $j=j+1$, and so on (we follow algorithm 3.1). By induction, we get the following lemma.

**Lemma 3.3.** *High resolution schemes given by algorithm 3.1 (or equations 3.2, 3.3, and 3.4) using a one step 4-point dyadic Deslauriers-Dubuc interpolation (equation 3.10) as an initialization step reproduce cubic polynomials.*

We can also get a stronger result by choosing a specific $\alpha$. Let $p_4(x)=a_4x^4+p_3(x)$ where $p_3$ is some cubic polynomial. Suppose that for some $j$, $y_{j,2k}=p_4\left(x_{j,2k}\right), y_{j-1,k}=p_4\left(x_{j-1,k}\right)\ \forall k\in\mathbb{N}$. We can write $y_{j+1,4r+2}$ in terms of this initial data ($y_j$ and $y_{j-1}$) by substituting equation 3.9 into 3.8 to get

$$\begin{aligned} y_{j+1,4r+2} &= \alpha y_{j,2r+1}+(1-\alpha)\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-2r-1}y_{j,2k}\\ &= \alpha\sum_{k\in\mathbb{N}}\gamma^{DD4}_{4k-2r-1}y_{j-1,2k}+(1-\alpha)\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-2r-1}y_{j,2k}. \end{aligned} \tag{3.11}$$

We want to show that $y_{j+1,4r+2}=p_4\left(x_{j,2r+1}\right)$ for some $\alpha$ and so we substitute $y_{j,2k}=p_4\left(x_{j,2k}\right), y_{j-1,k}=p_4\left(x_{j-1,k}\right)\ \forall k\in\mathbb{N}$ into the two sums of this last equation. Because of the easily verified results

$$\begin{aligned} \frac{-9}{2^{4j}} &= \frac{-\left(x_{j,2r-2}\right)^4+9\left(x_{j,2r}\right)^4+9\left(x_{j,2r+2}\right)^4-\left(x_{j,2r+4}\right)^4}{128}-\left(x_{j,2r+1}\right)^4\\ \frac{-105}{2^{4j}} &= \frac{-7\left(x_{j,2r-4}\right)^4+105\left(x_{j,2r}\right)^4+35\left(x_{j,2r+4}\right)^4-5\left(x_{j,2r+8}\right)^4}{128}-\left(x_{j,2r+1}\right)^4\\ &= \frac{-5\left(x_{j,2r-4}\right)^4+35\left(x_{j,2r}\right)^4+105\left(x_{j,2r+4}\right)^4-7\left(x_{j,2r+8}\right)^4}{128}-\left(x_{j,2r+1}\right)^4, \end{aligned}$$

we can compute each of the sums explicitely

$$\begin{aligned}\sum_{k\in\mathbb{N}}\gamma^{DD4}_{4k-2r-1}y_{j-1,2k} &= p_3\left(x_{j,2r+1}\right)+a_4\sum_{k\in\mathbb{N}}\gamma^{DD4}_{4k-2r-1}\times\left(x_{j-1,2k}=x_{j,4k}\right)^4\\ &= p_4\left(x_{j,2r+1}\right)-\frac{105a_4}{2^{4j}}\end{aligned}$$

and

$$\begin{aligned}\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-2r-1}y_{j,2k} &= p_3\left(x_{j,2r+1}\right)+a_4\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-2r-1}\times\left(x_{j,2k}\right)^4\\ &= p_4\left(x_{j,2r+1}\right)-\frac{9a_4}{2^{4j}}\end{aligned}$$

Hence, setting $\alpha=-3/32$ in equation 3.11, we get

$$y_{j+1,4r+2}=p_4\left(x_{j,2r+1}\right)-\frac{105\alpha+9(1-\alpha)}{2^{4j}}a_4=p_4\left(x_{j+1,4r+2}\right)$$

since for $\alpha=-3/32$, $105\alpha+9(1-\alpha)=0$. Therefore, the scheme reproduces polynomials of degree 4.

Of course, this last result assumes that we initialize the data so that $y_{j,2k+1}=p_4\left(x_{j,2k+1}\right)-\frac{105a_4}{16\times2^{4j}}$ and $y_{j,2k}=p_4\left(x_{j,2k}\right)$ for all $k\in\mathbb{N}$. We can get this result naturally by having $y_{j-1,k}=p_4\left(x_{j,k}\right)$ and applying first the high resolution scheme (equation 3.2) with $\alpha=1$ , since equations 3.7 and 3.8 will guarantee $y_{j,2k}=p_4\left(x_{j,k}\right)$ whereas equation 3.9 will initialize the placeholders properly. Indeed, the case $\alpha=1$ essentially relies only on the tetradic interpolation and discard the finer scale guesses (dyadic). We have show the following result.

**Proposition 3.4.** *For any given $j$, if $y_{j-1,k}=p_4\left(x_{j-1,k}\right)$ where $p_4$ is a polynomial of degree 4 then applying the high resolution scheme given by algorithm 3.1 (or equations 3.2, 3.3, and 3.4) first with $\alpha=1$ as an initialization step and then with $\alpha=-3/32$ will guarantee for the following iterations that $y_{j',2k}=p_4\left(x_{j',2k}\right)$ for $\forall k\in\mathbb{N}$ and all $j'\geq j-1$.*

3.3. **Sufficient conditions for regularity.** To study the regularity of high resolution schemes, it is convenient rewrite formula 3.2 in terms of (trigonometric) polynomials. Given some data $y_{j,k}$, define $P^j(z)=\sum_{k\in\mathbb{N}}y_{j,k}z^k$. If $P_2(z)=\sum_{k\in\mathbb{N}}\gamma^{DD2}_k z^k$, then the equation of the 4−point dyadic Deslauriers-Dubuc scheme (equation 3.10), can be rewritten $P^{j+1}(z)=P_2(z)P^j(z^2)$. Similarly, if $P_4(z)=\sum_{k\in\mathbb{N}}\gamma^{DD4}_k z^k$, then the tetradic subdivision scheme is given by $P^{j+1}(z)=P_4(z)P^j(z^2)$. It can be shown that we can rewrite the general equation for high resolution subdivision schemes as

$$P^{j+1}(z)=\sum_{i=1}^{M}\Phi_i(z)P^j\left(e^{2\pi i/b}z^b\right).$$

where $\Gamma_i$ must be Laurent polynomials and similarly for dyadic schemes ($b=2$),

$$P^{j+1}(z)=\Phi_1(z)P^j\left(z^2\right)+\Phi_2(z)P^j\left(-z^2\right).$$

The equation of symbols for the 4−point cubic high resolution sheme is (see equation 3.2 and algorithm 3.1)

$$\begin{aligned}P^{j+1}(z) &= \left\{P_4(z)-\alpha P_2\left(z^2\right)+\alpha\right\}\left(\frac{P^j\left(z^2\right)+P^j\left(-z^2\right)}{2}\right)+\alpha\left(\frac{P^j\left(z^2\right)-P^j\left(-z^2\right)}{2}\right)\\ &= \left\{\frac{P_4(z)-\alpha P_2\left(z^2\right)}{2}+\alpha\right\}P^j\left(z^2\right)+\frac{P_4(z)-\alpha P_2\left(z^2\right)}{2}P^j\left(-z^2\right)\\ &= \Gamma_1(z)P^j\left(z^2\right)+\Gamma_2(z)P^j\left(-z^2\right). \qquad (3.12)\end{aligned}$$

Because $\gamma^{DD4}_{2k}=\gamma^{DD2}_k\,\forall k\in\mathbb{N}$, we observe that $P_2$ is uneeded and everything can be written in terms of $P_4$, indeed

$$P_4(z)-\alpha P_2\left(z^2\right)=\frac{P_4(z)-P_4(-z)}{2}+(1-\alpha)\frac{P_4(z)+P_4(-z)}{2}$$

and thus, the symbols $\Gamma_1$and $\Gamma_2$ can be written

$$\begin{aligned}\Gamma_1(z) &= \Gamma_2(z)+\alpha\\ \Gamma_2(z) &= \frac{P_4(z)-P_4(-z)}{4}+(1-\alpha)\frac{P_4(z)+P_4(-z)}{4}.\end{aligned}$$

When $\alpha = 0$ (Deslauriers-Dubuc case), $\Gamma_1(z) = \Gamma_2(z) = \frac{P_4(z)}{2}$ and equation 3.2 becomes

$$P^{j+1}(z) = P_4(z)\left(\frac{P^j\left(z^2\right) + P^j\left(-z^2\right)}{2}\right)$$

and it can be shown to be equivalent to the Deslauriers-Dubuc dyadic scheme by using the last equation for averaging $P^{j+1}(z)$ and $P^{j+1}(-z)$,

$$\begin{aligned}\frac{P^{j+1}(z) + P^{j+1}(-z)}{2} &= \left(\frac{P_4(z) - P_4(-z)}{2}\right)\left(\frac{P^j\left(z^2\right) + P^j\left(-z^2\right)}{2}\right) \\ &= P_2\left(z^2\right)\left(\frac{P^j\left(z^2\right) + P^j\left(-z^2\right)}{2}\right).\end{aligned}$$

which can be used to prove that when $\alpha = 0$ the high resolution subdivision scheme becomes the dyadic Deslauriers-Dubuc scheme (proposition 3.2).

In order to study the regularity and stability of the chosen high resolution schemes, we need to find corresponding schemes for the (forward) finite differences. Let $dx_j = 1/2^j$ and write

$$D_{j,k} = \frac{dy_{j,k}}{dx_j} = 2^j\left(y_{j,k+1} - y_{j,k}\right),$$

and define higher order finite differences recursively

$$D^n_{j,k} = d^{(n)}y_{j,k}/\left(dx_j\right)^n = d\left(d^{(n-1)}y_{j,k}\right)/\left(dx_j\right)^n = 2^{jn} \times d^{(n)}y_{j,k}.$$

Let $H_1^j$ be the symbol for $dy_{j,k}/dx_j$, then

$$\begin{aligned}H_1^j(z) &= \sum_{k\in\mathrm{N}} 2^j\left(y_{j,k+1} - y_{j,k}\right)z^k \\ &= \sum_{k\in\mathrm{N}} 2^j y_{j,k} z^{k-1} - \sum_{k\in\mathrm{N}} 2^j y_{j,k} z^k \\ &= 2^j(1/z - 1)P^j(z) = 2^j(1-z)P^j(z)/z,\end{aligned}$$

and thus $P^j\left(z^2\right) = z^2 2^j H_1^j\left(z^2\right)/\left(1-z^2\right)$, $P^j\left(-z^2\right) = -z^2 2^j H_1^j\left(-z^2\right)/\left(1+z^2\right)$, and $P^{j+1}(z) = z2^{j+1}H_1^{j+1}(z)/(1-z)$. Substituting these three equations into $P^{j+1}(z) = \Gamma_1(z)P^j\left(z^2\right) + \Gamma_2(z)P^j\left(-z^2\right)$ (equation 3.12) gives

$$H_1^{j+1}(z) = \frac{2z(1-z)}{\left(1-z^2\right)}\Gamma_1(z)H_1^j\left(z^2\right) - \frac{2z(1-z)}{\left(1+z^2\right)}\Gamma_2(z)H_1^j\left(-z^2\right).$$

Similarly, the higher order finite differences are given by

$$H_n^j(z) = \frac{2(1-z)}{z}H_{n-1}^j(z) = \left(\frac{2(1-z)}{z}\right)^n P^j(z)$$

where $H_0(z) = P(z)$ and it can be seen that they can be computed by

$$H_n^{j+1}(z) = \left(\frac{2z}{1+z}\right)^n \Gamma_1(z)H_n^j\left(z^2\right) + \left(\frac{-2z(1-z)}{1+z^2}\right)^n \Gamma_2(z)H_n^j\left(-z^2\right). \tag{3.13}$$

$H_n$ is said to be the symbol of a high resolution subdivision scheme if $\Gamma_1(z)/(1+z)$ and $\Gamma_2(z)/(1+z^2)$ are Laurent polynomials. We have

$$P_4(z) = \frac{-(1+z)^4\left(1+z^2\right)^4\left(5z^2 - 12z + 5\right)}{128z^7}$$

so that $\Gamma_1(z)/(1+z)^n$ and $\Gamma_2(z)/(1+z^2)^n$ are Laurent polynomial for n $= 1,2,3,4$. Therefore, $H_n$ is the symbol of a high resolution subdivision scheme if $n = 1,2,3,4$.

**Lemma 3.5.** *For high resolution subdivision schemes given by algorithm 3.1 (or equations 3.2, 3.3, and 3.4), the finite differences $d^{(n)}y_{j,k}$ can be computed by a corresponding high resolution subdivision scheme for $n = 1,2,3,4$.*

We can define $dH_n^j$ as the symbol of

$$dD_{j,k}^{n-1} = d\left(\frac{d^{(n-1)}y_{j,k}}{(dx_j)^{n-1}}\right) = \frac{d^n y_{j,k}}{(dx_j)^{n-1}} = \frac{D_{j,k}^n}{2^j}$$

or $dH_n^j(z) = H_{n+1}^j(z)/2^j$ and thus

$$dH_{n-1}^j(z) = \frac{(1-z)}{z}H_{n-1}^j(z) = \frac{2^{j(n-1)}(1-z)^n}{z^n}P^j(z). \tag{3.14}$$

Replacing $H_{n-1}$ by $dH_{n-1}$ in equation 3.13, we find

$$dH_{n-1}^{j+1}(z) = \frac{1}{2}\left\{\left(\frac{2z}{1+z}\right)^n \Gamma_1(z)dH_{n-1}^j\left(z^2\right) + \left(\frac{-2z(1-z)}{1+z^2}\right)^n \Gamma_2(z)dH_{n-1}^j\left(-z^2\right)\right\}. \tag{3.15}$$

And because $dH_n^j(z) = H_{n+1}^j(z)/2^j$, $dH_{n-1}$ is the symbol of a high resolution subdivision scheme for $n = 1,2,3,4$.

Using results from Dyn [6], we have the following theorem.

**Theorem 3.6.** *(Dyn) If $dH_n$ as in equations3.14 and 3.15 is the symbol of a high resolution subdivision scheme converging uniformly to zero for all bounded initial data, then the corresponding scheme P as in equation 3.12 is $C^n$, that is, all interpolation functions f are $C^n$*

*Proof.* See theorems 4.2 and 4.4 in [6] as they apply to this high resolution subdivision schemes. □

In general, given $y_{j+1,l} = \sum_{k\in\mathbb{N}} \gamma_{2k-l}y_{j,k}$, a sufficient condition for $y_{j,k} \to 0$ uniformly as $j \to \infty$ is that $\lambda = \max_{l=0,1}\left\{\sum_{k\in\mathbb{N}} |\gamma_{2k-l}|\right\} < 1$, indeed, if $M_j = \sup\left\{\left|y_{j,k}\right| : k \in \mathbb{N}\right\}$ then $M_{j+1} \le \max_{l=0,1}\left\{\sum_{k\in\mathbb{N}} |\gamma_{2k-l}|\right\}M_j$ because $y_{j+1,2l} = \sum_{k\in\mathbb{N}} \gamma_{2k-2l}y_{j,k}$ and $y_{j+1,2l+1} = \sum_{k\in\mathbb{N}} \gamma_{2k-2l-1}y_{j,k}$. For a high resolution subdivision scheme given by $y_{j+1,l} = \sum_{k\in\mathbb{N}} \gamma_{4k-l}^{(1)}y_{j,2k} + \gamma_{4k+1-l}^{(2)}y_{j,2k+1}$, we proceed in the same manner. firstly $y_{j+1,2l} = \sum_{k\in\mathbb{N}} \gamma_{4k-2l}^{(1)}y_{j,2k} + \gamma_{4k+1-2l}^{(2)}y_{j,2k+1}$ and secondly $y_{j+1,2l+1} = \sum_{k\in\mathbb{N}} \gamma_{4k-2l-1}^{(1)}y_{j,2k} + \gamma_{4k-2l}^{(2)}y_{j,2k+1}$. Thus if $\lambda_{HR} = \max_{l=0,1}\left\{\sum_{k\in\mathbb{N}}\left|\gamma_{2k-l}^{(1)}\right| + \left|\gamma_{2k+1-l}^{(2)}\right|\right\}$ then $M_{j+1} \le \lambda M_j$. Given a symbol $Q(z) = \sum_k q_k z^k$, define $\|Q(z)\|_{sup} = \sup_k\{|q_k|\}$ and $\|Q(z)\|_{\Sigma} = \max\left\{\sum_k |q_k|\right\}$. For high resolution subdivision schemes, starting with $P^{j+1}(z) = \Phi_1(z)P^j\left(z^2\right) + \Phi_2(z)P^j\left(-z^2\right)$ ,we see that $\lambda_{HR}$ is given by

(3.16)

$$\lambda_{HR} = \max\{\lambda_1,\lambda_2\} = \max\left\{\left\|\frac{\Phi_1(z)+\Phi_1(-z)+\Phi_2(z)-\Phi_2(-z)}{2}\right\|_{\Sigma}, \left\|\frac{\Phi_1(z)-\Phi_1(-z)+\Phi_2(z)+\Phi_2(-z)}{2}\right\|_{\Sigma}\right\}$$

and $\left\|P^{j+1}(z)\right\|_{sup} \le \lambda_{HR}\left\|P^j(z)\right\|_{sup}$.

**Lemma 3.7.** *A high resolution subdivision scheme given by the symbol equation $P^{j+1}(z) = \Phi_1(z)P^j\left(z^2\right) + \Phi_2(z)P^j\left(-z^2\right)$ converges uniformly to zero for all bounded initial values if $\lambda_{HR} < 1$ where $\lambda_{HR}$ is as in equation .*

We are now ready to prove the next theorem.

**Theorem 3.8.** *For $-25/56 < \alpha < 15/32$, the high resolution subdivision scheme given by equation3.12 are $C^1$.*

*Proof.* The symbol of the high resolution subdivision scheme $dD_{j,k}$, $dH_1$ is given by (see equation 3.15)

$$dH_1^{j+1}(z) = 2\left(\frac{z}{1+z}\right)^2 \Gamma_1(z)dH_1^j\left(z^2\right) + 2\left(\frac{-z(1-z)}{1+z^2}\right)^2 \Gamma_2(z)dH_1^j\left(-z^2\right)$$

By theorem 3.6, it is enough to show that $dD_{j,k}$ converges uniformly to zero for all bounded initial data. However, using lemma 3.7, we know that it is sufficient to prove that $\lambda_{HR} < 1$ with $\Phi_1(z) = 2z^2\Gamma_1(z)/(1+z)^2$ and $\Phi_2(z) = 2z^2(1-z)^2\Gamma_2(z)/\left(1+z^2\right)^2$. We get

$$\begin{aligned} \lambda_1 &= \frac{5+2|4\alpha+1|+2|7-8\alpha|+2|5+12\alpha|+|32\alpha+5|+|5-8\alpha|+|24\alpha-7|}{64} \\ \lambda_2 &= \frac{5+2|4\alpha+1|+2|3+8\alpha|+2|1-4\alpha|+|21-32\alpha|+|1+8\alpha|+|24\alpha+11|}{64}. \end{aligned}$$

For $-25/56 < \alpha < 15/32$, we have $\lambda_1 < 1$, whereas for $-7/12 < \alpha < 5/8$, $\lambda_2 < 1$ Hence, we have that $\lambda_{HR} = \max\{\lambda_1,\lambda_2\} < 1$ for $-25/56 < \alpha < 15/32$ or $-\sim 0.45 < \alpha <\sim 0.47$ (see Fig. 3.2). □

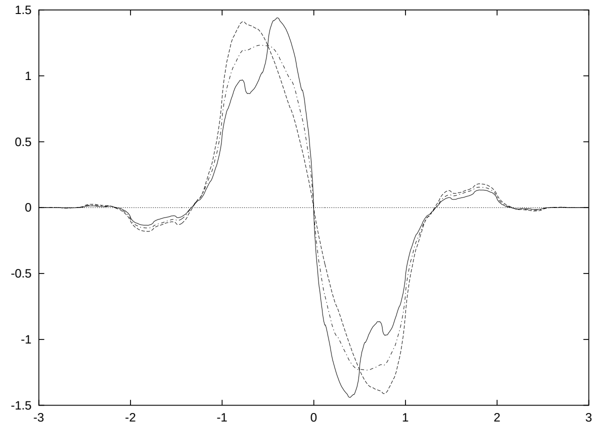


FIGURE 3.1. Derivatives of the fundamental functions for $\alpha = -0.2$ (continuous line), $\alpha = 0$ (dash-dot line), and $\alpha = 0.15$ (dashed line). The fundamental functions are defined as the interpolation of $y_{0,k} = \delta_{k,0}$ by the high resolution subdivision scheme initialized with the $4-$point Deslauriers-Dubuc dyadic scheme. Derivatives were estimated using first-order forward finite differences after 8 iterations of the high resolution scheme (discarding the placeholders at the last iteration). The $\alpha = 0$ case is in fact the derivative of the Deslauriers-Dubuc fundamental function.

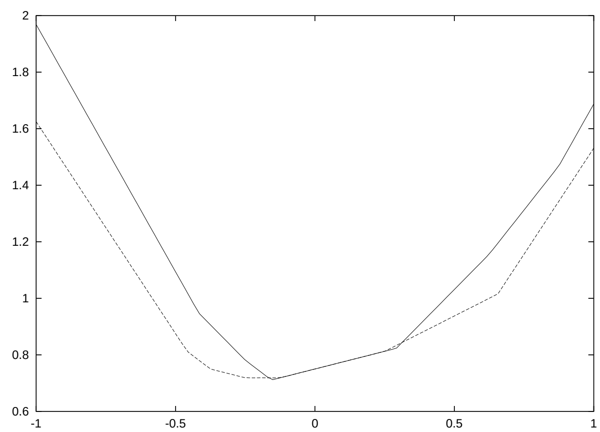


FIGURE 3.2. $\lambda_1(\alpha)$ (continuous line) and $\lambda_2(\alpha)$ (dashed line) as in the proof of theorem 3.8). The high resolution scheme is differentiable if $\lambda_{HR} = \max\{\lambda_1, \lambda_2\} < 1$.

Theorem 3.8 is illustrated by Fig. 3.1 where the derivative of three interpolants are given for $\alpha = -0.2, 0, 0.15$ respectively.

As a corrolary to theorem 3.8, we have that for $-25/56 < \alpha < 15/32$, high resolution subdivision schemes are stable that is, given $\left|y_{j,k} - \widetilde{y}_{j,k}\right| < \delta \forall k$ then $\left|y_{j,k+n} - \widetilde{y}_{j,k+n}\right| < \lambda^n \delta < \delta \forall i$ for all $n > 0$. We are interested in measuring how well a given subdivsion scheme can approximation functions. One such measure is given by the approximation order of the scheme [8, definition 2]. We say that a stationary and homogeneous subdivision scheme has approximation order $p$ if given given any smooth function $g \in C^p([0,1])$, the interpolation function $f$ satisfying $f(x_{j,k}) = g(x_{j,k})\ \forall k \in \mathbb{N}$ for some $j$ is such that $\|f - g\|_{L^\infty([0,1])} \leq Cb^{-jp}$ for a constant $C$ (independent of $j$). For a continuous subdivision scheme reproducing polynomials of degree $p$, it is sufficient for the scheme to converge to a continous function to have approximation order $p+1$ [8]. Specifically, this means that $4-$point Deslauriers-Dubuc schemes have approximation order 4.

**Theorem 3.9.** *Local, stationary, and homogeneous high resolution schemes that reproduce polynomials of order $p$ have approximation order $p+1$.*

*Proof.* See [8, Theorem 10]. Let $g \in C^p([0,1])$ , then the $p$ terms Taylor expansion of $g$ about $x_{j,k}$, $Taylor_p(g)$ is such that

$$\|g - Taylor_p(g)\|_{L^\infty([x_{j,k}, x_{j,k+b}])} \leq \frac{g^{(p)}(\xi) b \times b^{-jp}}{p!}.$$

Using this last theorem, we can conclude with the following corrolary. □

**Corollary 3.10.** *For* $-25/56 < \alpha < 15/32$*, high resolutions schemes have approximation order 4.*

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