Acadia University

Department of Mathematics and Statistics

Topics from Advanced Calculus
(MATH 2023)

Assignment 1

Due February 1st 2002

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Note: Problems are taken from Multivariable Calculus by McCallum, Hughes-Hallet, Gleason et al. Each student must hand in his own paper, but team work is allowed. You have to give complete solutions!

WARNING: DO NOT WAIT UNTIL THE LAST MINUTE TO DO THIS ASSIGNMENT!!! START EARLY AND COME ASK QUESTIONS!

13 Differentiating Functions of many Variables

13.4 Gradients and Directional Derivatives in the Plane

34. **[5 marks]** The temperature at any point in the plane is given by $T(x,y) = \frac{100}{x^2+y^2+1}$.

    (a) What shape are the level curves of T?

    (b) Where on the plane is it hottest? What is the temperature at that point?

    (c) Find the direction of the greatest increase in temperature at the point $(3,2)$. What is the magnitude of that greatest increase?

    (d) Find the direction of the greatest decrease in temperature at the point $(3,2)$.

    (e) Find a direction at the point $(3,2)$ in which the temperature does not increase or decrease.

35. **[2 marks]** A differentiable function $f(x,y)$ has the property that $f_x(4,1) = 2$ and $f_y(4,1) = -1$. Find the equation of the tangent line to the level curve of f through the point $(4,1)$.

13.5 Gradients and Directional Derivatives in Space

11. **[4 marks]** Consider S to be the surface represented by the equation $F=0$, where $F(x,y,z)=x^2-\left(\frac{y}{z^2}\right)$.

    (a) Find all points on S where a normal vector is parallel to the $xy-$plane.

    (b) Find the tangent plane to S at the points $(0,0,1)$ and $(1,1,1)$.

    (c) Find the unit vectors $\vec{u}_1$ and $\vec{u}_2$ pointing in the direction of maximum increase of F at the points $(0,0,1)$and $(1,1,1)$ respectively.

14. **[4 marks]** A differentiable function $f(x,y)$ has the property that $f(1,3)=7$ and $\nabla f(1,3)=2\vec{i}-5\vec{j}$.

    (a) Find the equation of the tangent line to the level curve of f through the point $(1,3)$.

    (b) Find the equation of the tangent plane to the surface $z=f(x,y)$ at the point $(1,3,7)$.

16. **[3 marks]** Two surfaces are said to be orthogonal to each other at a point P if the normals to their tangent planes are perpendicular at P. Show that the surfaces $z=\frac{1}{2}\left(x^2+y^2-1\right)$ and $z=\frac{1}{2}\left(1-x^2-y^2\right)$ are orthogonal.

18 Line Integrals

18.3 Gradient Fields and Path-Independent Fields

12. **[6 marks]** The line integral of $\vec{F}=(x+y)\vec{i}+x\vec{j}$ along each of the following paths is $3/2$: (i) the path $\left(t,t^2\right)$ with $0\leq t\leq 1$ (ii) the path $\left(t^2,t\right)$ with $0\leq t\leq 1$ (iii) the path (t,t^n) with $n>0$ and $0\leq t\leq 1$. Verify this:

    (a) Using the given parameterization to compute the line integral.

    (b) Using the Fundamental Theorem of Calculus for Line Integrals.

20. **[3 marks]** A particle moves with position vector $\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$. Let $\vec{v}(t)$ and $\vec{a}(t)$ be its velocity and acceleration vectors. Show that $\frac{1}{2}\frac{d}{dt}\|\vec{v}(t)\|^2=\vec{a}(t)\cdot\vec{v}(t)$.

18.4 Path-Dependent Vector Fields and Green's Theorem

12. **[2 marks]** Suppose $\vec{F}=x\vec{j}$. Show that the line integral of $\vec{F}$ around a closed curve in the $xy-$plane, oriented as in Green's theorem, measures the area of the region enclosed by the curve.

15. **[4 marks]** Using the result you just shown, calculate the area of the region with the folium of Descartes, $x^3+y^3=3xy$, parametrized by $x=\frac{3t}{1+t^3}$, $y=\frac{3t^2}{1+t^3}$, for $0\leq t<\infty$.

19 Flux Integrals

19.1 The Idea of a Flux Integral

16. **[3 marks]** Let S be the tetrahedron with vertices at the origin and at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

    (a) Calculate the total flux of the constant vector field $\vec{v} = -\vec{i} + 2\vec{j} + \vec{k}$ out of S by computing the flux through each face separately.

    (b) Calculate the flux out of S in part (a) for any constant vector field $\vec{v}$.

    (c) Do your answers in parts (a) and (b) make sense? Explain.

20 Calculus of Vector Fields

20.1 The Divergence of a Vector Field

12. **[4 marks]** Show that if $g(x,y,z)$ is a scalar valued function and $\vec{F}(x,y,z)$ is a vector field, then $\nabla \cdot \left(g\vec{F}\right) = \nabla g \cdot \vec{F} + g\nabla \cdot \vec{F}$.

18. **[3 marks]** Let $\vec{F}(x,y,z) = z\vec{k}$.

    (a) Calculate $\nabla \cdot \vec{F}$.

    (b) Sketch $\vec{F}$. Does it appear to be diverging? Does it agree with your answer to part (a)?

19. **[4 marks]** Let $\vec{F}(\vec{r}) = \vec{r}/\left\|\vec{r}\right\|^3$ (in 3-space), $\vec{r} \neq \vec{0}$.

    (a) Calculate $\nabla \cdot \vec{F}$.

    (b) Sketch $\vec{F}$. Does it appear to be diverging? Does it agree with your answer to part (a)?

27. **[2 marks]** If $f(x,y,z)$ and $g(x,y,z)$ are functions with continuous second partial derivatives, show that $\nabla \cdot (\nabla f \times \nabla g) = 0$.

20.2 The Divergence Theorem

6. **[4 marks]** Use the Divergence Theorem to evaluate the flux integral $\int_S \left(x^2\vec{i} + (y-2xy)\vec{j} + 10z\vec{k}\right) \cdot d\vec{A}$ where S is the sphere of radius 5 centered at the origin, oriented outward.

12. **[4 marks]** The gravitational field, $\vec{F}$, of a planet of mass m at the origin is given by $\vec{F} = -Gm\frac{\vec{r}}{\|\vec{r}\|^3}$. Use the Divergence Theorem to show that the flux of the gravitational field through the sphere of radius a is independent of a. (HINT: Consider the region bounded by two concentric spheres.)

For the next questions, recall that a function f is said to be harmonic if $\nabla \cdot \nabla f = \nabla^2 f = 0$.

18. **[4 marks]** What is the condition on the constant coefficients a, b, c, d, e, f such that $ax^2 + by^2 + cz^2 + dxy + exz + fyz$ is harmonic?

21. **[4 marks]** Show that a nonconstant harmonic function cannot have a local mininmum and that it can achieve a minimum value in a closed region only on the boundary.

22. **[4 marks]** Show that if ϕ is a harmonic function, then $\nabla \cdot (\phi \nabla \phi) = \|\nabla \phi\|^2$.

20.3 The Curl of a Vector Field

14. **[3 marks]** For any constant field $\vec{c}$, and any vector field, $\vec{F}$, show that $\nabla \cdot \left(\vec{F} \times \vec{c}\right) = \vec{c} \cdot \nabla \times \vec{F}$.

16. **[3 marks]** If $\vec{F}$ is any vector field whose components have continuous second partial derivatives, show that $\nabla \cdot \left(\nabla \times \vec{F}\right) = 0$.

17. **[4 marks]** Show that $\nabla \times \left(\phi \vec{F}\right) = \phi \nabla \times \vec{F} + \nabla \phi \times \vec{F}$ for a scalar function ϕ and a vector field $\vec{F}$.

22. **[4 marks]** Show that if ϕ is a harmonic function, the $\nabla \phi$ is both curl free and divergence free.

24. **[4 marks]** Express $(3x + 2y)\vec{i} + (4x + 9y)\vec{j}$ as the sum of a curl free vector field and a divergence free vector field.

27. **[4 marks]** Let $\vec{F}$ be a smooth vector field and let $\vec{u}$ and $\vec{v}$ be constant vectors. using the definitions of $\nabla \times \vec{F}$ in Cartesian coordinates, show that $\nabla\left(\vec{F} \cdot \vec{v}\right) \cdot \vec{u} - \nabla\left(\vec{F} \cdot \vec{u}\right) \cdot \vec{v} = \left(\nabla \times \vec{F}\right) \cdot \vec{u} \times \vec{v}$.

20.4 Stoke's Theorem

6. **[4 marks]** Compute the line integral $\int_C \left(\left(yz^2 - y\right)\vec{i} + \left(xz^2 - x\right)\vec{j} + 2xyz\vec{k}\right) \cdot d\vec{r}$ where C is the circle of radius 3 in the $xy-$plane, centered at the origin, oriented counterclockwise as viewed from the positive $z-$axis. Do it two ways: (a) directly (b) using Stoke's Theorem.

17. **[4 marks]** Let $\vec{F} = -y\vec{i} + x\vec{j} + \cos(xy)z\vec{k}$ and let S be the surface of the lower unit hemisphere $x^2 + y^2 + z^2 = 1, z \leq 0$, oriented with outward pointing normal. Find $\int_S \nabla \times \vec{F} \cdot d\vec{A}$.