A Family of 4-point Dyadic High Resolution Subdivision Schemes

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Abstract. We present a new family of multistep iterative interpolation schemes generalizing subdivision schemes so that a 4-point interpolation algorithm can reproduce quartic polynomials. Interpolation requires two steps: a coarse scale interpolation followed by a fine scale interpolation. The interpolants are C^1, have good local properties and no additional memory requirement.

§1. Introduction

Subdivision schemes interpolate a discrete set of data points in a local manner, that is, the value of the interpolation function at a given point depends on a small number of nearby data points. The classical dyadic subdivision scheme [5,2] finds the midpoint values by fitting a Lagrange polynomial through the $2N$ closest data points. By repeating this algorithm iteratively, we have a dense set of data points and determine uniquely a smooth interpolation function. Because subdivision schemes relate data points from one scale to another, it is not surprising that they are a key ingredient in the construction of compactly supported wavelets [1,3].

Often, interpolation schemes can be made more local by using more memory. Merrien [14,15,6] introduced Hermite subdivision schemes which have twice the approximation order and better regularity for a given support and vector subdivision schemes in general have received a lot of attention ever since [10,16,12]. In spline theory, adding extra nodes can make spline interpolation local [4]. In this paper, instead of using more memory, we want to make better use of the memory we already have. The strategy we propose is to use at least one step earlier the upcoming nodes in a

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subdivision scheme to record coarse scale guesses (see Fig. 1). Because the new placeholders are used as predictors, the new schemes will be as local as usual subdivision schemes. These schemes are said to be "high resolution" because we no longer consider only the next finer scale, but actually the next two finer scales; alternatively, we could describe these algorithms as "two-step subdivision schemes".

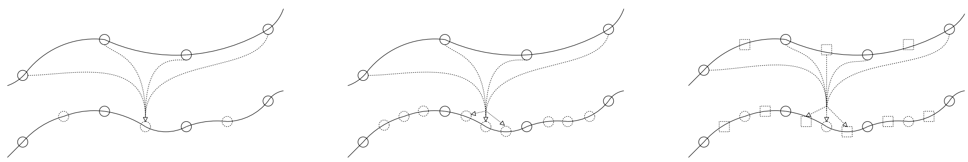

Fig. 1. Diagrams of 4-point subdivision schemes in the dyadic (left) and tetradic (center) cases, and presented HRS schemes (right). Arrows symbolize the interpolation process. The circles are data samples and, in the HRS diagram, the squares represent placeholders recording "guesses" .

The main result of this paper is that by summing up the tetradic (coarse) interpolation and dyadic (fine) interpolation, we get a range of smooth (C^1) high resolution subdivision (HRS) schemes reproducing at least cubic but also quartic polynomials (see Tab. 1). While there exists 5-point quartic subdivision schemes, they are not as local as the presented 4-point HRS scheme.

scheme	regularity	support	polynomials
Dubuc	C^1	$[-3,3]$	cubic
Dyn-Gregory-Levin	up to C^1	$[-3,3]$	up to cubic
Hassan et al.	C^2	$[-5/2,5/2]$	quadratic
presented HRS	C^1	$[-3,3]$ or $[-3,4]$	cubic or quartic

Tab. 1. Comparison between some 4-point iterative interpolation schemes. The support of the fundamental functions is given. The quartic HRS scheme is slightly less local because it requires initialization by a one-step 5-point scheme.

The paper is organized as follows. We begin by a brief review of subdivision schemes and discuss both the dyadic and tetradic 4-point Deslauriers-Dubuc schemes. Combining these subdivision schemes, we present a family of HRS schemes reproducing cubic and prove that some of these schemes are smooth (C^1).

§2. Subdivision Schemes

Let $b > 1$ be an integer, a b-adic number is of the form $x_{j,k} = k/b^j$ for some integers k, j. For a fixed J, given some data $\{y_{J,k}\}_{k\in\mathbb{Z}}$, we want a smooth function f such that $f(x_{J,k}) = y_{J,k}\,\forall k \in \mathbb{Z}$. Starting with $(y_{J,k})$ and using the formula

$$y_{j+1,l} = \sum_{k\in\mathbb{Z}} \gamma_{bk-l} y_{j,k} \tag{1}$$

for some array γ, we get values $y_{j,k}$ for any $j > J$ and since b-adic numbers form a dense set in $\mathbb{R}$, there is at most one continuous function such that $f(x_{j,k}) = y_{j,k}$ for all $k \in \mathbb{Z}, j > J$.

A subdivision scheme is interpolatory and satisfies $f(x_{J,k}) = y_{J,k}$ if $\gamma_{bk} = 0\,\forall k \in \mathbb{Z}$ except for $\gamma_0 = 1$. We say that a subdivision scheme is stationary if the array γ is constant (does not depend on j). Because γ does not depend explicitly on l but rather on $bk - l$ the scheme is translation invariant or homogeneous. A subdivision scheme is a $2N-$point scheme if $\gamma_l = 0$ for $|l| \geq Nb$. The fundamental function of an interpolatory $2N-$point b-adic scheme has initial data $y_{0,0} = 1$ and $y_{0,k} = 0$ for all $k \neq 0$; it has a compact support of $[-(Nb-1)/(b-1), (Nb-1)/(b-1)]$ (or $[1-2N, 2N-1]$ when $b = 2$).

For $N = 1, 2, 3, ...$ there are corresponding interpolatory $2N-$point Deslauriers-Dubuc subdivision schemes built from the midpoint evaluation of Lagrange polynomial of degree $2N-1$. For $b = 2$ (dyadic case), the 4-point Deslauriers-Dubuc scheme can be defined with the array γ^{DD2} given by $\gamma_0^{DD2} = 1, \gamma_{\pm1}^{DD2} = -9/16, \gamma_{\pm3}^{DD2} = -1/16$ with $\gamma_k^{DD2} = 0$ otherwise; for $b = 4$ (tetradic case), the scheme is defined with the array γ^{DD4} given by $\gamma_{2k}^{DD4} = \gamma_k^{DD2}\,\forall k \in \mathbb{Z}$, $\gamma_{\pm1}^{DD4} = 105/128, \gamma_{\pm3}^{DD4} = 35/128, \gamma_{\pm5}^{DD4} = -7/128, \gamma_{\pm7}^{DD4} = -5/128$, with $\gamma_k^{DD2} = 0$ otherwise.

Because 4-point Deslauriers-Dubuc schemes are derived from cubic Lagrange polynomials, they reproduce cubic polynomials, that is, if the initial data $y_{j,k}$ satisfies $y_{j,k} = p(x_{j,k})\ \forall k \in \mathbb{Z}$ for some cubic polynomial p then the interpolation function f is this same cubic polynomial $f = p$. The two cases presented above (γ^{DD2} and γ^{DD4}) converge to differentiable (C^1) interpolation functions [5,2].

§3. High Resolution Subdivision Schemes

We define stationary HRS schemes by the equation

$$y_{j+1,l} = \sum_{m=1}^{M} \sum_{k\in\mathbb{Z}} \gamma^{(m)}_{Mbk+m-1-l} y_{j,Mk+m-1} \tag{2}$$

where $\gamma^{(1)}, ..., \gamma^{(M)}$ are constant arrays (independent from j). They are b-adic scheme because the number of nodes increases by a factor of b with each iteration. However, because we have $M > 1$ arrays γ, the scheme is said to be a HRS scheme. It is an interpolatory scheme if $y_{j+1,Mbk} = y_{j,Mk}$ and it is a $2N-$point scheme if $\gamma_l^{(m)} = 0$ for $|l| \geq MNb$ and $m = 1, ..., M$.

For $b = M = 2$ the general equation (2) becomes

$$y_{j+1,l} = \sum_{k \in \mathbb{Z}} \gamma^{(1)}_{4k-l} y_{j,2k} + \gamma^{(2)}_{4k+1-l} y_{j,2k+1}. \tag{3}$$

A value $y_{j,k}$ is a stable value if $y_{j,k} = y_{j+1,2k}$; other nodes are said to be temporary or are referred to as placeholders. A HRS scheme on a dyadic grid is an interpolatory scheme if all $y_{j,2k}$ values on even nodes ($x_{j,2k}$) are stable so that $y_{j,2k} = y_{j+1,4k}\, \forall k \in \mathbb{Z}$.

For the rest of the paper, we will consider the schemes HRS_α where $\gamma^{(1)}$ and $\gamma^{(2)}$ are chosen to be $\gamma^{(1)}_{2k} = \gamma^{DD4}_{2k} + \alpha\left(\delta_{k,0} - \gamma^{DD2}_k\right)$, $\gamma^{(1)}_{2k+1} = \gamma^{DD4}_{2k+1}\, \forall k \in \mathbb{Z}$, $\gamma^{(2)}_0 = \alpha$, and $\gamma^{(2)}_k = 0$ otherwise for some parameter $\alpha \in \mathbb{R}$. Since $\gamma^{DD4}_{2k} = \gamma^{DD2}_k\, \forall k \in \mathbb{Z}$, we can rewrite equation (3) for even and odd terms. Firstly, setting $l = 2s$ (l even), we have

$$y_{j+1,2s} = \sum_{k \in \mathbb{Z}} \left((1-\alpha)\gamma^{DD2}_{2k-s} + \alpha\delta_{2k,s}\right) y_{j,2k} + \delta_{2k+1,s+1} \alpha y_{j,2k+1}$$

so that when s is even ($l = 2s = 4r$), we have the interpolatory condition

$$y_{j+1,4r} = y_{j,2r}, \tag{4}$$

otherwise, when s is odd ($l = 2s = 4r + 2$)

$$y_{j+1,4r+2} = \alpha y_{j,2r+1} + (1-\alpha) \sum_{k \in \mathbb{Z}} \gamma^{DD2}_{2k-2r-1} y_{j,2k}. \tag{5}$$

Secondly, if l is odd ($l = 2s + 1$), we have

$$y_{j+1,2s+1} = \sum_{k \in \mathbb{Z}} \gamma^{DD4}_{4k-2s-1} y_{j,2k}. \tag{6}$$

Equations (4), (5), and (6) can be used to describe HRS_α: while equation (4) is the interpolatory condition, equation (6) fills the placeholders with tetradic (coarse scale) interpolated values whereas equation (5) combines the value stored in the placeholder with the newly available interpolated value (fine scale) given by the summation term which we recognize from the dyadic Deslauriers-Dubuc interpolation.

In the simplest case, $\alpha = 0 \Rightarrow \gamma^{(2)} = 0$ and equation (3) becomes $y_{j+1,l} = \sum_{k \in \mathbb{Z}} \gamma^{DD4}_{4k-l} y_{j,2k}$. This last equation discards odd nodes at each step: $y_{j+1,l}$ depends only on even nodes ($y_{j,2k}$) and not at all on the odd nodes ($y_{j,2k+1}$). Hence, we have $y_{j+1,2l} = \sum_{k \in \mathbb{Z}} \gamma^{DD4}_{4k-2l} y_{j,2k}$ but because $\gamma^{DD4}_{2k} = \gamma^{DD2}_k$, this last equation becomes $y_{j+1,2l} = \sum_{k \in \mathbb{Z}} \gamma^{DD2}_{2k-l} y_{j,2k}$ and if we define $\widetilde{y}_{j,k} = y_{j,2k}$ we have that HRS_0 is equivalent to the 4-point dyadic Deslauriers-Dubuc subdivision scheme.

§4. Reproduced polynomials

Assume that for some j, $y_{j,k} = p_3(x_{j,k})\ \forall k \in \mathbb{Z}$ where p_3 is a cubic polynomial. Because 4-point Deslauriers-Dubuc schemes reproduce cubic polynomials, we have

$$\sum_{k\in\mathbb{Z}} \gamma^{DD2}_{2k-2r-1} y_{j,2k} = y_{j,2r+1} = p_3(x_{j,2r+1})$$

and thus equation (5) becomes $y_{j+1,4r+2} = p_3(x_{j,2r+1})$ for any $\alpha \in \mathbb{R}$. Similarly, equation (6) implies $y_{j+1,2s+1} = p_3(x_{j+1,2s+1})$. We conclude that $y_{j+1,k} = p_3(x_{j+1,k})\ \forall k \in \mathbb{Z}$ if $y_{j,k} = p_3(x_{j,k})\ \forall k \in \mathbb{Z}$ and thus HRS_α schemes reproduce cubic polynomials. For practical implementations of a HRS scheme, it is necessary to first apply a one-step subdivision scheme. Let $\{y_{j,k}\}_k$ be some initial data. As a first step, we apply Deslauriers-Dubuc's equation

$$y_{j+1,l} = \sum_{k\in\mathbb{Z}} \gamma^{DD2}_{2k-l} y_{j,2k} \tag{7}$$

followed by HRS_α with $j+1$. This algorithm is as local as the corresponding Deslauriers-Dubuc subdivision scheme in the sense that the fundamental function has support $[-3,3]$. By induction on j, we get the following lemma.

Lemma 1. *HRS_α schemes reproduce cubic polynomials and are interpolatory when using a one step interpolatory 4-point dyadic Deslauriers-Dubuc interpolation as initialization.*

We get a stronger result by choosing a specific α. We can write any quartic polynomial p_4 as $p_4(x) = a_4x^4 + p_3(x)$ where p_3 is some cubic polynomial. Because of the Generalized Rolle's theorem and because $\frac{p_4}{4!} = a_4$, given any 4 points ξ_1,ξ_2,ξ_3, and ξ_4, the corresponding cubic polynomial $p_{Lagrange3}$ approximates p_4 with error

$$p_4(x) - p_{Lagrange3}(x) = a_4(x-\xi_1)(x-\xi_2)(x-\xi_3)(x-\xi_4) \tag{8}$$

for some ξ. In other words, the error depends only on a_4 and the geometry of the sample points ξ_i with respect to x. This makes the task of canceling out the errors convenient as we shall see.

Suppose that for some j, $y_{j,2k} = p_4(x_{j,2k})$ and $y_{j-1,k} = p_4(x_{j-1,k})\ \forall k \in \mathbb{Z}$. We can write $y_{j+1,4r+2}$ for any $r \in \mathbb{Z}$ in terms of this initial data (y_j and y_{j-1}) by substituting equation (6) into (5) to get

$$\begin{aligned} y_{j+1,4r+2} &= \alpha y_{j,2r+1} + (1-\alpha)\sum_{k\in\mathbb{Z}} \gamma^{DD2}_{2k-2r-1} y_{j,2k} \\ &= \alpha \sum_{k\in\mathbb{Z}} \gamma^{DD4}_{4k-2r-1} y_{j-1,2k} + (1-\alpha)\sum_{k\in\mathbb{Z}} \gamma^{DD2}_{2k-2r-1} y_{j,2k}. \end{aligned} \tag{9}$$

We want to show that $y_{j+1,4r+2} = p_4(x_{j,2r+1})$ for some $\alpha \in \mathbb{R}$ and so we substitute $y_{j,2k} = p_4(x_{j,2k})$ and $y_{j-1,k} = p_4(x_{j-1,k})$ into the two sums of this last equation. We compute both sums in equation (9) explicitly using equation (8) (Rolle's):

$$\sum_{k\in\mathbb{Z}} \gamma^{DD4}_{4k-2r-1} y_{j-1,2k} = p_4(x_{j,2r+1}) - \frac{105a_4}{2^{4j}} \tag{10}$$

and similarly $\sum_{k\in\mathbb{Z}} \gamma^{DD2}_{2k-2r-1} y_{j,2k} = p_4(x_{j,2r+1}) - \frac{9a_4}{2^{4j}}$. Hence, setting $\alpha = -3/32$ in equation (9), we get

$$y_{j+1,4r+2} = p_4(x_{j,2r+1}) - \frac{105\alpha + 9(1-\alpha)}{2^{4j}} a_4 = p_4(x_{j+1,4r+2})$$

Therefore, $HRS_{-3/32}$ reproduces quartic polynomials once the data has been properly initialized. While there are no 4-point subdivision scheme capable of interpolating $y_{j-1,k} = p_4(x_{j,k})$ into $y_{j,2k+1} = p_4(x_{j,2k+1}) - \frac{105a_4}{16\times 2^{4j}}$ and $y_{j,2k} = p_4(x_{j,2k})$ for all $k \in \mathbb{Z}$, there exist 5-point subdivision schemes such as the subdivision scheme described by the next algorithm.

Algorithm 2. *(5-point "Initialization" Subdivision Scheme) For a given integer j, begin with some initial y values $y_{j,k}$ $k \in \mathbb{Z}$ over dyadic numbers $x_{j,k} = k/2^j$,*

1) *recopy data at $x_{j+1,2k} = x_{j,k}$: $y_{j+1,2k} = y_{j,k}$ $\forall k \in \mathbb{Z}$;*
2) *extrapolate $y_{j,k+4}$ using $y_{j,k-2}, y_{j,k-1}, y_{j,k}, y_{j,k+1}, y_{j,k+2}$ by the formula $\gamma_{j,k} = 5y_{j,k-2} - 24y_{j,k-1} + 45y_{j,k} - 40y_{j,k+1} + 15y_{j,k+1}$, $\forall k \in \mathbb{Z}$;*
3) *interpolate midpoint value using the tetradic Deslauriers-Dubuc formula $y_{j+1,2k+1} = \frac{-7y_{j,k-2}+105y_{j,k}+35y_{j,k+2}-5\gamma_{j,k}}{128}$ $\forall k \in \mathbb{Z}$.*

To see that algorithm 2 properly initializes the placeholders, observe that if we assume $y_{J,k} = p_4(x_{J,k})$, then we only need to check that $y_{J+1,2k+1} = p_4(x_{J+1,2k+1}) - \frac{105a_4}{16\times 2^{4(J+1)}}$. However, if $y_{J,k} = p_4(x_{J,k})$ is satisfied, then $\gamma_{J,k} = p_4(x_{J,k+4})$ since it can be derived by finding the quartic polynomial $p_{J,k}$ satisfying $p_{J,k}(x_{j,l}) = y_{J,l}$ for $l = k-2, ..., k+2$ and setting $\gamma_{J,k} = p_{J,k}(x_{J,k+4})$. Hence, by equation (10), we have the following lemma.

Lemma 3. *Algorithm 2 describes a 5-point dyadic subdivision scheme such that with $y_{j-1,k} = p_4(x_{j-1,k})$ where $p_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ is a quartic polynomial, $y_{j,2k+1} = p_4(x_{j,2k+1}) - \frac{105a_4}{16\times 2^{4j}}$ and $y_{j,2k} = p_4(x_{j,2k})$ for all $k \in \mathbb{Z}$.*

Then, because we have a proper initialization scheme, we can reproduce quartic polynomials as the next proposition states.

Proposition 4. *$HRS_{-3/32}$ with algorithm 2 as an initialization step, reproduce quartic polynomials.*

Only subdivision schemes using at least 5 points can interpolate quartic polynomials and the support of the fundamental function is at least of *size 8* whereas the algorithm described by proposition 4 ($HRS_{-3/32}$) leads to fundamental functions having compact support of *size 7* taking into account the 5-point initialization scheme.

§5. Sufficient conditions for regularity

Given that HRS_0 is equivalent to the Deslauriers-Dubuc subdivision which is C^1, it is reasonable to expect HRS_α to be C^1 for some range of α values. Moreover, motivated by proposition 4, we need to show that this range of values includes $\alpha = -3/32$. At this point, it is convenient to rewrite equation (3) in terms of (trigonometric) Laurent polynomials. Given some data $y_{j,k}$, define $P^j(z) = \sum_{k\in\mathbb{Z}} y_{j,k} z^k$. If $P_2(z) = \sum_{k\in\mathbb{Z}} \gamma_k^{DD2} z^k$, then the equation of the 4-point dyadic Deslauriers-Dubuc scheme (equation (7)), can be rewritten $P^{j+1}(z) = P_2(z)P^j(z^2)$. Similarly, if $P_4(z) = \sum_{k\in\mathbb{Z}} \gamma_k^{DD4} z^k$, then the tetradic subdivision scheme is given by $P^{j+1}(z) = P_4(z)P^j(z^2)$. We can rewrite the general equation for $b-$adic HRS schemes as

$$P^{j+1}(z) = \sum_{i=1}^{M} \Gamma_i(z) P^j\left(e^{2\pi i/b} z^b\right)$$

where the Γ_i must be Laurent polynomials. For dyadic HRS schemes ($b = 2$), this equation becomes

$$P^{j+1}(z) = \Gamma_1(z)P^j\left(z^2\right) + \Gamma_2(z)P^j\left(-z^2\right). \tag{11}$$

The HRS_α symbols are

$$\Gamma_1(z) = \Gamma_2(z) + \alpha$$
$$\Gamma_2(z) = \frac{P_4(z) - P_4(-z)}{4} + (1-\alpha)\frac{P_4(z) + P_4(-z)}{4}.$$

Following Dyn [7], we want to find corresponding schemes for the (forward) finite differences. Let $dx_j = 1/2^j$ and write

$$D_{j,k} = \frac{dy_{j,k}}{dx_j} = 2^j\left(y_{j,k+1} - y_{j,k}\right),$$

and define higher order finite differences recursively

$$D^n_{j,k} = d^{(n)} y_{j,k} / \left(dx_j\right)^n = 2^{jn} d^{(n)} y_{j,k}.$$

Because $\sum_{k\in\mathbb{Z}} 2^j\left(y_{j,k+1}-y_{j,k}\right)z^k = \sum_{k\in\mathbb{Z}} 2^j y_{j,k} z^{k-1} - \sum_{k\in\mathbb{Z}} 2^j y_{j,k} z^k$, we have

$$H_n^j(z) = \frac{2(1-z)}{z} H_{n-1}^j(z) = \left(\frac{2(1-z)}{z}\right)^n P^j(z)$$

where $H_0(z) = P(z)$ and they can be computed by

$$\begin{aligned} H_n^{j+1}(z) = & \left(\frac{2z}{1+z}\right)^n \Gamma_1(z) H_n^j\left(z^2\right) \\ & + \left(\frac{-2z(1-z)}{1+z^2}\right)^n \Gamma_2(z) H_n^j\left(-z^2\right). \end{aligned} \tag{12}$$

H_n is the symbol of a HRS scheme if $n = 1, 2, 3, 4$ because $\Gamma_1(z)/(1+z)^n$ and $\Gamma_2(z)/(1+z^2)^n$ are Laurent polynomials

We define $dH_n^j(z) = H_{n+1}^j(z)/2^j$ as symbols of $dD_{j,k}^{n-1} = d\frac{dy_{j,k}}{dx_j}$ and since $dH_n^j(z) = H_{n+1}^j(z)/2^j$, dH_{n-1} is the symbol of a HRS scheme for $n = 1, 2, 3, 4$. Using results from Dyn [7], we have the following theorem.

Theorem 5. *(Dyn-Levin) Given trigonometric polynomials $\Gamma_1(z)$ and $\Gamma_2(z)$, the HRS scheme defined by $P^{j+1}(z) = \Gamma_1(z) P^j\left(z^2\right) + \Gamma_2(z) P^j\left(-z^2\right)$ is C^n if the symbol corresponding to finite differences of order $n+1$, $dH_n^j(z) = \frac{2^{jn}(1-z)^{n+1}}{z^{n+1}} P^j(z)$ is the symbol of a HRS scheme converging uniformly to zero for all bounded initial data.*

Proof: See the proof of theorem 3.4 in [7] or section 4.2 in [9] as it applies to HRS schemes. The key point being that for an iterative interpolation scheme (even a nonstationary one) to be C^n, it is sufficient for the finite differences $d^{n+1}y_{j,k}/\left(dx_j\right)^n$ to converge uniformly to zero. □

In general, given $y_{j+1,l} = \sum_{k\in\mathbb{Z}} \gamma_{2k-l} y_{j,k}$, a sufficient condition for $y_{j,k} \to 0$ uniformly as $j \to \infty$ is that $\lambda = \max_{l=0,1}\left\{\sum_{k\in\mathbb{Z}} \left|\gamma_{2k-l}\right|\right\} < 1$. For a HRS scheme given by $y_{j+1,l} = \sum_{k\in\mathbb{Z}} \gamma_{4k-l}^{(1)} y_{j,2k} + \gamma_{4k+1-l}^{(2)} y_{j,2k+1}$, $\lambda_{HR} = \max_{l=0,1}\left\{\sum_{k\in\mathbb{Z}} \left|\gamma_{2k-l}^{(1)}\right| + \left|\gamma_{2k+1-l}^{(2)}\right|\right\}$ implies $M_{j+1} \leq \lambda_{HR} M_j$ where $M_j = \sup\left\{\left|y_{j,k}\right| : k \in \mathbb{Z}\right\}$. To write this statement in symbols, define $\|Q(z)\|_{sup} = \sup_k\left\{\left|q_k\right|\right\}$ and $\|Q(z)\|_\Sigma = \max\left\{\sum_k \left|q_k\right|\right\}$ where $Q(z) = \sum_k q_k z^k$. Now, if $P^{j+1}(z) = \tilde{\Gamma}_1(z) P^j\left(z^2\right) + \tilde{\Gamma}_2(z) P^j\left(-z^2\right)$ and

$$\lambda_{HR} = \max\left\{\lambda_1, \lambda_2\right\} \tag{13}$$

where $2\lambda_{1/2} = \left\|\tilde{\Gamma}_1(z) \pm \tilde{\Gamma}_1(-z) + \tilde{\Gamma}_2(z) \mp \tilde{\Gamma}_2(-z)\right\|_\Sigma$, then

$$\left\|P^{j+1}(z)\right\|_{sup} \leq \lambda_{HR} \left\|P^j(z)\right\|_{sup}.$$

Lemma 6. *A HRS scheme given by the symbol equation* $P^{j+1}(z) = \tilde{\Gamma}_1(z)P^j\left(z^2\right) + \tilde{\Gamma}_2(z)P^j\left(-z^2\right)$ *converges uniformly to zero for all bounded initial values if* $\lambda_{HR} < 1$ *where* λ_{HR} *is as in equation (13).*

We are now ready to prove the following theorem which shows that HRS_α are smooth for α near 0 (see Fig. 2).

Theorem 7. *For* $-25/56 < \alpha < 15/32$, HRS_α *interpolants are* C^1.

Proof: By theorem 5, it is enough to show that $dD^1_{j,k}$ converges uniformly to zero for all bounded initial data. The symbol of the HRS scheme $dD^1_{j,k}$, dH_1 is given by

$$dH_1^{j+1}(z) = \tilde{\Gamma}_1 dH_1^j\left(z^2\right) + \tilde{\Gamma}_2 dH_1^j\left(-z^2\right)$$

where $\tilde{\Gamma}_1(z) = 2z^2\Gamma_1(z)/\left(1+z\right)^2$ and $\tilde{\Gamma}_2(z) = 2z^2(1-z)^2\Gamma_2(z)/\left(1+z^2\right)^2$ (see equation (12)). By lemma 6, it is sufficient to prove that $\lambda_{HR} < 1$. We have that $\lambda_1 < 1$ for $-25/56 < \alpha < 15/32$ and $\lambda_2 < 1$ for $-7/12 < \alpha < 5/8$, $\lambda_2 < 1$ hence the result. $\square$

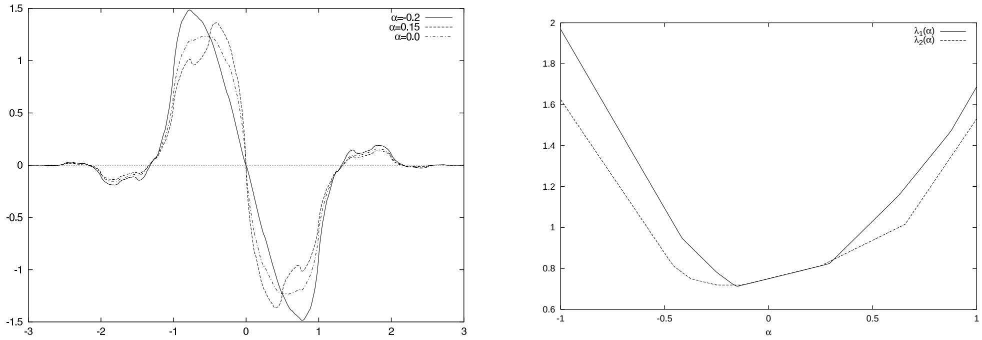

Fig. 2. Derivatives of HRS_α fundamental functions (left). HRS_α is differentiable if $\lambda_{HR}(\alpha) = \max\{\lambda_1(\alpha), \lambda_2(\alpha)\} < 1$ (right).

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