# Numerical Methods 2 (MATH 3423) - MidTerm

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1. [25 marks] Given the following $(x,y)$ couples, find an $O\left((\Delta x)^2\right)$ estimation for the derivative at 0 using Taylor series: $(-1,2),(0,1),(3,0)$. Show your work.
Solution: This problem was part of assignment 1, the solution was given as a handout and it was also done in class. I even went as far as to mention it was a good test question! We "Taylor expand twice about $x=0$". $2=f(-1)=1+f'(0)(-1)+f''(0)(-1)^2/2+O\left((\Delta x)^3\right)$ (backward) and $0=f(3)=1+f'(0)(3)+f''(0)(3)^2/2+O\left((\Delta x)^3\right)$ (forward). Solving for $f'(0)$, we get $9(2)=9-1+(-9-3)f'(0)+O\left((\Delta x)^3\right)$ hence $f'(0)=\frac{18-9+1}{-12}+O\left((\Delta x)^2\right)=\frac{-5}{6}+O\left((\Delta x)^2\right)$.

2. [25 marks] Using the Gaussian Quadrature, how many nodes $(n=?)$ do I need to integrate $\int_a^b x^2+x^5+x^7dx$ exactly (except for unavoidable numerical errors)? How many nodes do I need if I use the composite Simpson's method? Which method is best?
Solution: Gaussian Quadrature has accuracy $2n-1$ and we have a polynomial of degree 7, we solve $2n-1=7$ to get $n=4$. Simpson's rule (composed or not) has accuracy order 3 (cubic) and cannot be used to compute exactly polynomials of degree 7. In fact, the only practical way of doing so is to use Gaussian Quadrature since Newton-Cotes method are not very good at integrating very high degrees polynomials.

3. [25 marks] You want to integrate
$$f(x)=\begin{cases} x^4 & x>0 \\ x^2+1 & x\leq 0\end{cases}$$
from $-1$ to 1. Which type of method is best, composite Newton-Cotes or Gaussian Quadrature? Explain.
Solution: Composite Newton-Cotes method is better because $f$ is discontinuous. While Gaussian Quadrature assumes we have a smooth function, composite Newton-Cotes allows for some discontinuities. Just think of integrating a step function using the midpoint formula.

4. [25 marks] Show that if $f$ satisfies $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(x+y)}{2}$ for any $x,y$ and $\max_x\{|f(x)-f(x+1)|\}=M<\infty$, then $f$ must be continuous (a sketch of the proof is sufficient).
Solution: $f(x+1/2)=\frac{f(x)+f(x+1)}{2}$ hence $f(x+1/2)-f(x)=\frac{-f(x)+f(x+1)}{2}$ or $\left|f\left(\frac{x+1}{2}\right)-f(x)\right|\leq\frac{M}{2}$ repeating this, you get $|f(x+1/2^n)-f(x)|\leq M/2^n$ which leads you to a proof that $f$ is continuous.