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Revised 2001 D. Lemire

MATHEMATICS 2023 - INFINITE SERIES I
ACADIA UNIVERSITY

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1. TAYLOR POLYNOMIALS

We recall the idea of a Taylor polynomial used as a local approximation to a function. By local, we mean that the approximation is perfect at some point $x = a$. We then say the polynomial is found "at" or "about" this point.

Given a function $f(x)$ that is continuous on some interval I containing the point $x = a$, and has "enough" continuous derivatives on I, the Taylor polynomial of degree n for $f(x)$, expanded about $x = a$, is given by

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(Recall that $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n$ is called “n factorial” and is the product of the first n counting numbers. The symbol $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$.)

Example 1.1. Consider the function $f(x) = e^x$ about the point $x = 0$. Because all of the derivatives of $e^x$are the same, and have a value of 1 at $x = 0$, we have

$$e^x \approx P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$$

Notice that $P_1(x) = 1 + x$ is just the usual linear approximation to e^x at $x = 0$, and $P_2(x) = 1 + x + x^2/2!$ is the quadratic approximation. Taylor polynomials are just the natural extension of these approximations to higher degrees.

Example 1.2. Consider the function $f(x) = \ln(x)$. Since the function is undefined at $x = 0$, we must choose some other point for expansion, say $x = 1$. To find the polynomial of degree n, we calculate the coefficients and look for a pattern.

$$\begin{array}{ll}
f(x) = \ln(x), & f(1) = 0, \\
f'(x) = x^{-1}, & f'(1) = 1, \\
f''(x) = -x^{-2}, & f''(x) = -1, \\
f^{(3)}(x) = 2x^{-3}, & f^{(3)}(1) = 2, \\
f^{(4)}(x) = -3 \cdot 2 \cdot x^{-4}, & f^{(4)}(1) = -3! \\
f^{(5)}(x) = 4!\,x^{-5}, & f^{(5)}(1) = 4!, \\
\vdots & \vdots \\
f^{(n)}(x) = \pm(n-1)!x^{-n}, & f^{(n)}(1) = \pm(n-1)!
\end{array}$$

This gives

$$P_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{2(x-1)^3}{3!} - \frac{3!(x-1)^4}{4!} + \cdots + \frac{(-1)^{n-1}(n-1)!(x-1)^n}{n!}.$$

Using the fact that $\frac{(n-1)!}{n!} = \frac{1}{n}$ (why ??), we have

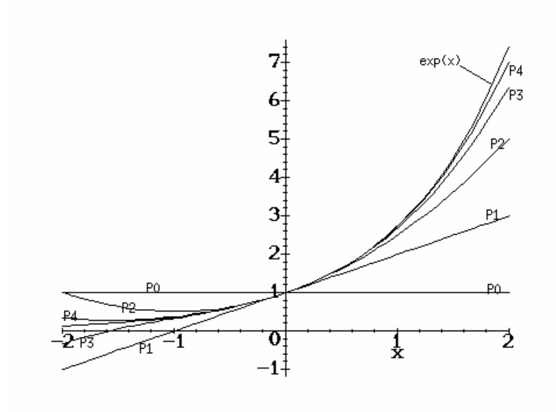
$$P_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n}.$$

Since we are considering the Taylor polynomials as approximations of $f(x)$ near the point $x = a$, it is interesting to see how closely the polynomials we have derived resemble the functions. The fast way to do this is to use Maple. The two important commands here are "taylor" to generate the series for a given function; and "convert" which changes the series (which is thought of as an infinitely long polynomial) to the finite Taylor polynomial we want.

Example 1.3. Here is a Maple session in which we use the “taylor” command to generate the first 5 approximations $P_0(x), P_1(x), \ldots, P_4(x)$, and then plot them all on the same plot. I have used a "do" statement to permit me to do several operations

```
> f:=exp(x); a:=0;
f := exp(x)
a := 0
> for n from 0 to 4 do p[n]:=convert(taylor(f,x=a,n+1),polynom); od;
p[0] := 1
p[1] := 1 + x
p[2] := 1 + x + 1/2 x^2
p[3] := 1 + x + 1/2 x^2 + 1/6 x^3
p[4] := 1 + x + 1/2 x^2 + 1/6 x^3 + 1/24 x^4
```

```
> plot([f,p[k]$k=0..4],x=-2..2,colour=black);
```



Note that for $x > 0$ the approximations all lie below e^x, but seem to get closer to e^x as the degree of the polynomial increases. In fact, on the interval [-1, 1] it is hard to distinguish $P_4(x)$ from the function e^x.

Example 1.4. Now let's return to Example 1.2, where $f(x) = \ln(x)$ and we expanded about $a = 1$. It is just a matter of replacing f with this new function in the Maple commands we used above:

```
> f:=ln(x); a:=1;
> for n from 0 to 4 do: p[n]:=convert(taylor(f,x=a,n+1),polynom); od;
```

f := ln(x)

a := 1

p[0] := 0

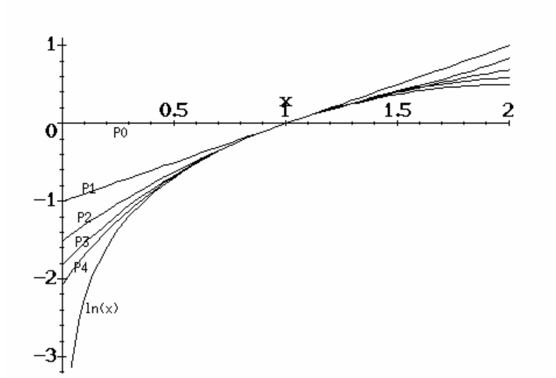
p[1] := x - 1

p[2] := x - 1 - 1/2 (x - 1)^2

p[3] := x - 1 - 1/2 (x - 1)^2+ 1/3 (x - 1)^3

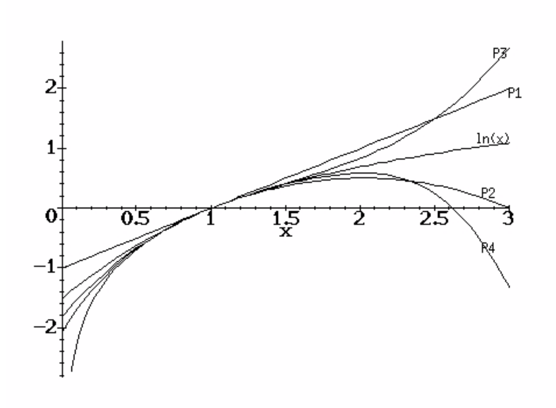
p[4] := x - 1 - 1/2 (x - 1)^2 + 1/3 (x - 1)^3 - 1/4 (x - 1)^4

```
> plot([f,p[k]$k=0..4],x=0..2,colour=black);
```



Once again it appears that the on the interval shown, the polynomials get closer to the function $f(x) = \ln(x)$ as the degree of the polynomials increase. However. if we look a little further away from $x = 1$ by extending the interval from $x = 0$ to $x = 3$, we get quite a different picture:

```
> plot([f,p[k]$k=0..4],x=0..3,colour=black);
```



Here we note that on the interval [2, 3] things do not seem to be going very well for our Taylor polynomials, viewed as approximations to $\ln(x)$. At $x = 3$ it looks as if the polynomials no longer approach $\ln(x)$, but become further away as the degree of the polynomial increases. In fact, we shall show later that these particular approximations for $\ln(x)$ are valid only over the interval (0, 2].

We now pause to explain the Maple commands we have used in the above examples. taylor(f,x=a,n); produces an infinite Taylor series for f expanded about $x = a$, and displays the first n terms of that series. Here f must be what Maple calls an expression (in x), rather than a function defined using the arrow (e.g. $f := \sin(x)$ is all right, but $f := x- > \sin(x)$ won't work). The result is a Taylor polynomial of degree $n-1$ followed by the symbol + O((x-a)^n). This symbol stands for all the remaining terms of the series, and indicates they all contain a factor (x-a)^n. (We read the symbol O((x-a)^n) as " terms of order (x-a)^n".) Note that the first n terms displayed give a polynomial of degree $n-1$, since the first term is degree 0, the second degree 1, etc.

Of course, in our example we wanted Taylor polynomials, not Taylor series. The convert command will convert a Taylor series to a Taylor polynomial by deleting the O((x-a)^n) term at the end. Thus if we wanted the Taylor polynomial of degree 5 for the function $f(x) = \sin(x)$ expanded about $x = a = 0$, we would use the Maple command

```
> t5:=taylor(sin(x),x=0,6);
t5 := x - 1/6 x^3 + 1/120 x^5 + O(x^6)
> p5:=convert(t5,polynom);
p5 := x - 1/6 x^3 + 1/120 x^5
```

If these polynomial approximations are to be of any practical value, we must be able to determine how accurately they approximate the given function. We now consider this problem.

2. How accurate are Taylor polynomials as approximations?

Since $P_n(x) \approx f(x)$ for x near the point $x = a$, we now investigate ways to estimate the size of the error $|f(x) - P_n(x)|$ at any point x.

Theorem 2.1. *(Taylor) Let $f(x)$ have $n+1$ continuous derivatives on the interval I containing the point $x = a$. Let $P_n(x)$ be the Taylor polynomial of degree n for $f(x)$ expanded about the point $x = a$. Then for any point x on the interval I, we have*

$$f(x) - P_n(x) = \frac{f^{(n+1)}(z)(x-a)^{n+1}}{(n+1)!}$$

where z is some (unknown) point between $x$and a. (Note that a can be to the left or right of x.)

Although this result looks to be of limited practical value because of the term $f^{(n+1)}(z)$, which contains an unknown value z, it can be very useful in providing a bound on the error. That is, it will not tell us what the error is, but can tell us the error will be no bigger than a certain value.

Example 2.2. Let $f(x) = e^x$, $a = 0$, $P_8(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^8}{8!}$, and suppose x is on the interval [-1, 1]. The above result then states that

$$e^x - P_8(x) = \frac{e^z x^9}{9!},$$

where z is between 0 and x. Since the maximum value x can have is 1, and thus the maximum value e^z can have is e, we have

$$|e^x - P_8(x)| \leq \frac{e}{9!} \approx 0.00000749....$$

Hence on the interval [0, 1], $P_8(x)$ will provide an approximate value of e^x correct to at least 5 digits.

Because the term $f^{(n+1)}(z)$ can make the above result a bit confusing, sometimes it is presented in the following manner:

Corollary 2.3. *If $P_n(x)$ is used to approximate $f(x)$ over the interval I, then the absolute error $|f(x) - P_n(x)|$ is no greater than $\frac{M|x-a|^{n+1}}{(n+1)!}$, where M is the maximum value of $|f^{(n+1)}(x)|$ on the interval I.*

Example 2.4. Let $f(x) = \ln(x)$, $a = 1$, and (from Example 1.2 on page 2)

$$P_6(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6}$$

Suppose we want to know how large the absolute error $|\ln(x) - P_6(x)|$ could be if x is restricted to the interval [1, 3/2]. (Sometimes we say: find an *upper bound* for the error.) First, to find the value of M in the above result we need the 7^{th} derivative of $\ln(x)$.

$$f(x) = \ln(x),\ f'(x) = x^{-1},\ f''(x) = -x^{-2},\ f^{(3)}(x) = 2!x^{-3}, \ldots, f^{(7)}(x) = 6!x^{-7}$$

and so $M = \max\left\{\frac{6!}{x^7}, 1 \leq x \leq \frac{3}{2}\right\} = \frac{6!}{1^7} = 6!$ Thus

$$M\frac{|x-a|^{n+1}}{(n+1)!} = 6!\frac{|x-1|^7}{7!} \leq 6!\frac{(1/2)^7}{7!} = \frac{1}{2^7 7} = \frac{1}{128 \cdot 7} \cong 0.00116...$$

and the absolute error is bounded by $1.116\text{x}10^{-3}$.

We will find an even easier way to bound the absolute error for polynomials whose terms alternate in sign as does this one, but that comes later when we consider "alternating" series. For now, we move on to consider what happens to a Taylor polynomial when we take an infinite number of terms. Before we tackle this, we need some ideas about limits and sequences, however.

3. About Sequences and Limits

In dealing with series precisely we need to have a fairly clear idea of what is meant by a limit. You have used these extensively in calculus already. We need to look at limits of *sequences*, which are a little different from limits of functions.

A *sequence* is an ordered list of real numbers. For example, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ and $2, 4, 6, 8, 10, \ldots$ are both infinite sequences. In each or these example, we can write the *general term* of the sequence: the first has general term $a_n = \frac{1}{n}$ and the second has $a_n = 2n$. We will often describe a sequence by its general term.

Not every sequence has a general term. For example a random sequence is still a sequence. It can also be hard to find the general term, for example as with Conway's sequence 1,11,21,1211,111221, 312211, 13112221, 1113213211, ...

In our first example, the sequence has a (finite) limit, namely 0, while in the second example there is no finite limit (the sequence tends to infinity). Some sequences have no limit without becoming infinite, simply because they are not heading anywhere definite. An example of this kind is the sequence with general term $a_n = (-1)^n$.

Informally we say $\lim_{n\to\infty} a_n = L$ if we can make the terms of the sequence as close as we wish to L by going far enough along in the sequence (by taking n large enough). Thus we see that we can make $1/n$ within, for example, 0.002 of its limit 0 by going to the 501st term of the sequence, since $\left|\frac{1}{n} - 0\right| < 0.002 = \frac{1}{500}$when $n \geq 501$. In this expression, the distance between the sequence value and the limit is written inside the absolute value bars.

More formally we say that $\lim_{n\to\infty} a_n = L$ if and only if for any possible positive *error* ε between the sequence term and the limit we can find a term number N in the sequence so that the error remains within ε for the rest of the sequence. Symbolically:

Definition 3.1. We say that $\lim_{n\to\infty} a_n = L$ if and only if for any $\varepsilon > 0$ we can find an N such that $|a_n - L| < \varepsilon$ whenever $n \geq N$.

This is generally known as the hardest definition in undergraduate mathematics. Compare it carefully to the example we just wrote down.

Proposition 3.2. $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Proof. For simplicity, we begin by picking a specific value for the error: 1/100. Let's find a value for N such that when n, the subscript or term number, is further along in the sequence than N, the error (or absolute value of the difference between the function and the limit) is less than the error (1/100). To find this, we need to solve the following inequality for n: $\left|\frac{n}{n+1} - 1\right| < \frac{1}{100}$.Dropping the absolute values makes this $1 - \frac{n}{n+1} < \frac{1}{100}$, or$\frac{1}{n+1} < \frac{1}{100}$, which is true if and only if 100 < n+1, or when n is greater than or equal to 100. Thus if we pick any value for N which is greater than or equal to 100, the condition that the error be less than 1/100 is satisfied. We didn't need to do the example just shown, but doing it makes it clear how to proceed for any value of the error. Given any error $\varepsilon > 0$, we seek N so that $\left|\frac{n}{n+1} - 1\right| < \varepsilon$ whenever $n \geq N$. We solve the inequality for n: $\left|\frac{n}{n+1} - 1\right| < \varepsilon$if and only if $\frac{1}{n+1} < \varepsilon$, which is true if and only if $n > \frac{1}{\varepsilon} - 1$. Thus if we take N to be the next positive integer greater than $\frac{1}{\varepsilon} - 1$ the inequality required in the definition will hold. $\square$

Sometimes solving the inequality can be rather hard, and we will not dwell on such cases. However, for our purposes in studying series, we need to prove one other important limit.

Lemma 3.3. *(Power Sequence) If 0<a <1, then* $\lim_{n\to\infty} a^n = 0$.

Proof. Here you can convince yourself easily with a calculator, or just common sense. Our proof will use natural logarithms. Given any $\varepsilon > 0$, we need to find N such that $a^n < \varepsilon$ for n greater than or equal to N. (We could drop the absolute value sign because $a$is positive.) But we can preserve this inequality by taking the logarithm of both sides, since ln(x) is strictly increasing.

Thus $n\ln(a) < \ln\varepsilon$. Note that since a <1, ln(a)<0, so solving for n gives

$$n > \frac{\ln\varepsilon}{\ln a}.$$

Therefore if we let N be the next integer greater than $\frac{\ln\varepsilon}{\ln a}$, $n \geq N$ leads to the required inequality, namely that for big enough values of n, $a^n$is closer to 0 than ε (where ε is a positive number that is a small as we like). □

More generally, we could prove that for $|a| < 1$ a power sequence will converge and it will diverge for $|a| \geq 1$.

4. Monotone and Cauchy sequences

One practical problem with limit of sequences is illustrated by proposition 3.2 and lemma 3.3 on the preceding page: it seems like we need to know the value of the limit of a sequence to prove that it converges. However,it is quite hard to determine the actual limit of a sequence in general.

Looking back on proposition 3.2 and lemma 3.3 on the facing page, we notice that they have some common properties. We say that a sequence x_k is bounded from above if the exist M such that $x_k < M$ for all k, and similarly, we say that a sequence is bounded from below if there exist N such that $x_k > N$. For example, if 0<a <1, a^n is bounded from above and from below (respectively by 1 and 0). Of course, $n/(n+1)$ is also bounded from above and below (by 1 and 0 respectively). We say that that a sequence is monotone increasing if $a_k \geq a_l$ whenever $k > l$ and we say that the sequence is monotone decreasing if $a_k \leq a_l$ whenever $k < l$. Both a^n and $n/(n+1)$ are monotone sequences as you can easily verify by subtracting successive terms. It turns out that monotone sequences which are also bounded converges.

Theorem 4.1. *If x_k is a monotone increasing sequence that is bounded above, then the sequence must converge. Similarly, if x_k is a monotone decreasing sequence that is bounded below, then the sequence must converge.*

However, we don't need monotonicity for a sequence to converge. More generally, it is enough for the values to get closer and closer together.

Definition 4.2. We say that the sequence a_k is Cauchy if for each $\varepsilon > 0$ there is an integer $N > 0$ such that if $i, j > N$ then $\left|a_j - a_i\right| < \varepsilon$.

The first fact we can prove is that Cauchy sequences are bounded sequences which means that they are bounded from below and from above.

Theorem 4.3. *(Completeness Theorem) Let a_k be a Cauchy sequence of real numbers. Then the sequence is bounded.*

Proof. See exercise 4. □

Intuitively, it makes sense that a Cauchy sequence would converge and it does. More importantly, maybe, is the fact that this is a necessary and sufficient condition for a sequence to converge. Hence, saying that a sequence is Cauchy is the same thing as saying it converges.

Theorem 4.4. *Let a_k be a sequence of real numbers. The sequence is Cauchy if and only if it converges to some limit a.*

Proof. (⇐)Assume that the sequence converges to some limit a. Take any $\varepsilon > 0$ then there exists $N > 0$ such that if $j > N$ then $\left|a_j - a\right| < \varepsilon/2$. For $k > j$, we have

$$\left|a_j - a_k\right| \leq \left|a_j - a\right| + |a_k - a| < \varepsilon.$$

and therefore the sequence is Cauchy.

($\Rightarrow$) This direction is much harder and left out. $\square$

5. Infinite Series

Now we can tackle questions involving series. We have found that if $f(x) = e^x$, then the Taylor polynomial of degree n for e^x, expanded about x = 0, is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$$

and the absolute error $|e^x - P_n(x)|$ is given by

$$|e^x - P_n(x)| = \frac{e^z\,|x|^{n+1}}{(n+1)!},$$

with z between 0 and x.

(This example is particularly easy to deal with since all of the derivatives of e^x are the same.)

Since the only constraint on this process is that the function have lots of derivatives, and since e^x can be differentiated as many times as we like, it is easy to imagine continuing the process of making a Taylor polynomial indefinitely, to get an infinite number of terms. The result in this case is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The idea here is that by going to infinitely many terms, we get something that is (or might be) actually equal to the function on some interval, diminishing the error to zero. As we shall see, sometimes this works perfectly for every value of x, sometimes not.

There are two related issues:

1. does the infinite expression (the infinite *series*) add up to a finite number for a given value of x?
2. is the value of the series equal to the value of the function?

To investigate these questions, we will begin by looking at what happens to the error when n gets large.

Suppose we fix the value of x at any arbitrary number and ask what happens to the absolute error as we increase n. Since x is fixed, and z is somewhere between 0 and x, the e^z factor in the error term is bounded and we really only need to know what happens to the term $\frac{|x|^{n+1}}{(n+1)!}$ as n gets bigger and bigger. You might expect that the factorial term $(n+1)!$ would eventually grow much faster than $|x|^{n+1}$, and the term would approach 0. Indeed, this is just what happens. To see why, remember that x is a fixed number for the moment. Let k be the first positive integer greater than $|x|$, (so k is fixed too) and suppose $n > k$. Then

$$\begin{aligned}
\frac{|x|^n}{n!} &= \frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \frac{|x|}{3} \cdots \frac{|x|}{k} \cdot \frac{|x|}{k+1} \cdot \frac{|x|}{k+2} \cdots \frac{|x|}{n} \\
&\leq \frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \frac{|x|}{3} \cdots \frac{|x|}{k} \cdot \frac{|x|}{k} \cdot \frac{|x|}{k} \cdots \frac{|x|}{k} \\
&\leq \frac{|x|^k}{k!} \left[\frac{|x|}{k}\right]^{n-k}.
\end{aligned}$$

and since $\frac{|x|}{k} < 1$ (remember $k > |x|$), we see that $\lim_{n\to\infty}\left[\frac{|x|}{k}\right]^{n-k} = 0$. Hence we have shown that $\lim_{n\to\infty}\frac{|x|^n}{n!} = 0$ for any real number x, which in turn means that the error goes to 0 as the number of terms becomes infinite. This means that

$$e^x = \lim_{n\to\infty}(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!})$$

for all values of x. We usually write this as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots,$$

or $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$. We refer to the expression on the right as an "**infinite series**", because it is the sum of an infinite number of terms. Often we omit the word "**infinite**" and just call an infinite sum a "**series**". When the series involves powers of x (or more generally, powers of $(x-a)$) , we call it a "**power series**". Hence a general power series is any expression of the form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

If a power series comes from a Taylor polynomial of a function $f(x)$ expanded about the point $x = a$, then we usually refer to it as a **Taylor series**. Hence in the above example we have the Taylor series for $f(x) = e^x$ about the point $x = 0$. In fact, we have shown that the series "converges" to the value e^x for any x. By this we mean that for any value of x we choose, if we sum enough terms of the Taylor series, we will get a value arbitrarily close to e^x.

Here are Taylor series for some well known elementary functions:

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$for all x
- $\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}\cdots$for $0 < x < 2$
- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\cdots$for all x
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$for all x
- $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$for $-1 < x < 1$

Of course, convergence may well depend upon the value of x. For example, recall the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This equation only makes sense if -1 < x < 1. If you choose an x outside of this interval, $\lim_{n\to\infty}(1 + x + x^2 + x^3 + \cdots + x^n)$ does not exist. We say that for $|x| \geq 1$ the **series diverges**. This may mean that the sum gets arbitrarily large (try $x = 2$) or it may just "not settle down", as in the case of $x = -1$.

We are able to prove these results about the convergence of the geometric series because we can express the sum of the first n terms of the series in a simple expression. To see this, let S_n be the sum of the first n terms of the series, i.e.

$$S_n = 1 + x + x^2 + x^3 + \cdots + x^{n-1}$$

Multiply by x, $xS_n = x + x^2 + x^3 + \cdots + x^{n-1} + x^n$ and subtract to get $S_n - xS_n = 1 - x^n$. Therefore we have

$$S_n = \frac{1-x^n}{1-x}$$

Hence, if $|x| < 1$, $x^n \to 0$ as $n \to \infty$ and $S_n \to \frac{1}{1-x}$. In fact, S_n behaves like a power sequence. This proof is a good illustration of what we mean by ***convergence***. We call S_n the "n^{th} partial sum" of the geometric series since it is the sum of the first n terms. In this example, we were able to show that

$$\lim_{n\to\infty} S_n = \frac{1}{1-x}.$$

We base our **definition of convergence** on just this idea. Let $a_1 + a_2 + a_3 + \cdots$ be an infinite series and $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ the n^{th} partial sum. If

$$\lim_{n\to\infty} S_n = L$$

then we say that the series **converges** to L. Note this is a sequence limit. We summarize our findings in the next proposition.

Proposition 5.1. *The geometric series* $\sum_{k=0}^{\infty} x^k$ *converges to* $\frac{1}{1-x}$ *for* $|x| < 1$ *and diverges otherwise.*

If we are dealing with Taylor series, the remainder term is often sufficient to determine the values for which the series converges. Taylor series are special series in the sense that we start with a known function and from that we generate the series. In such a case it is not surprising that when the series converges, it usually converges to the function we started with (but not always!) Often, however, we arrive at a power series by quite a different route, and do not know if we have a Taylor series for a common function. In such a situation it is often important to have some test that will quickly give us the limits on the values of x for which we can expect convergence. The easiest thing is to try to somehow test the *terms* of the series rather than the partial sums (which can be very hard to find).

6. The Ratio Test. Absolute and Conditional Convergence.

One of the simplest series is the geometric series given by $\sum_{k=0}^{\infty} a^k$. You may notice that geometric series converge if and only if the ratio of successive terms (which is simply a) is smaller than 1 in absolute value. It turns out that this result is true in general.

Theorem 6.1. *(Ratio Test) Given the infinite series* $a_0 + a_1 + a_2 + a_3 + \cdots$ *and* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = R$,

- *if* $R < 1$, *the series converges,*
- *if* $R > 1$ *the series diverges,*
- *if* $R = 1$ *the test is inconclusive and fails.*

This is a particularly easy test to apply, as long as you can see what the general term a_n looks like. Here are a few examples:

Example 6.2. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ Here we see that $a_n = \frac{1}{2^n}$, $n = 0, 1, 2, 3, \ldots$ and so

$$\lim_{n\to\infty} \frac{(1/2)^{n+1}}{(1/2)^n} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}$$

and since $R = \frac{1}{2} < 1$, the series converges. Of course, this series is a geometric series with ratio $r = \frac{1}{2}$, and since we know a geometric series with $|r| < 1$ always converges, we really did not need to use the Ratio Test.

Next is a series that is not a geometric series.

Example 6.3. $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$. (Note this is a power series) Here we see that $a_n=\frac{x^n}{n!}$, n = 0, 1, 2, ... and so

$$\lim_{n\to\infty}\frac{|x|^{n+1}/(n+1)!}{|x|^n/n!}=\lim_{n\to\infty}\frac{|x|^{n+1}}{|x|^n}\frac{n!}{(n+1)!}=|x|\lim_{n\to\infty}\frac{1}{n+1}=0$$

which means that the series converges, regardless of the value of x.

Example 6.4. $(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+(-1)^{n-1}\frac{(x-1)^n}{n}+\cdots$ This is another power series, with the n^{th} term given for us. We apply the ratio test, noting that the term $(-1)^{n-1}$ just makes the signs alternate, and disappears when we take absolute values:

$$\lim_{n\to\infty}\frac{|x-1|^{n+1}/(n+1)}{|x-1|^n/n}=|x-1|\lim_{n\to\infty}\frac{n}{n+1}=|x-1|=R$$

We know that the series converges if $R<1$ and diverges if $R>1$, so we see that if

$$|x-1|<1$$

we will have convergence. This means $-1<x-1<1$ or $0<x<2$. Hence we conclude that if x is in the open interval (0, 2) the series converges, and if $x>2$ or $x<0$ then the series diverges. Since the ratio test is inconclusive when $R=1$, we can conclude nothing about the two points $x=0$ and $x=2$. We call the interval (0, 2) the **interval of convergence** for this power series.

Note that the power series in Example 6.4 was in fact the Taylor series for $f(x)=\ln(x)$, expanded about the point $a=1$, and a is the center of the interval of convergence. This means that if the series is to converge, x must remain within a distance 1 of a. In this case we say the **radius of convergence** is 1. The radius of convergence for a power series is always one-half the width of the interval of convergence.

Definition 6.5. *Radius of convergence* $=\frac{1}{2}$*interval of convergence.*

By using the ratio test on the general power series

$$c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\cdots,$$

we find

$$\lim_{n\to\infty}\frac{|c_{n+1}(x-a)^{n+1}|}{|c_n(x-a)^n|}=|x-a|\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}=|x-a|L$$

where $\lim_{n\to\infty}\frac{|c_{n+1}|}{|c_n|}=L$. Thus if $|x-a|L<1$ the series will converge, with a **radius of convergence** $R=1/L$. Therefore exactly one of the following is always true:

1. $L=0$ and the power series converges for all values of x, with a radius of convergence$R=\infty$
2. $L>0$ and the power series converges for $|x-a|<R$, with a radius of convergence $R=\frac{1}{L}$.
3. $L=\infty$ and the power series converges only if $x=a$, with a radius of convergence $R=0$.

We note that the ratio test gives us nearly all of the information we need regarding the convergence of a power series. At most, we are missing information at the two end-points of the interval of convergence.

We now continue on Example 6.4 to illustrate that each end-point must be considered separately. We found that

$$(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\cdots+(-1)^{n-1}\frac{(x-1)^n}{n}+\cdots$$

$n = 1$, $S_1 = 1.$	$n = 2$, $S_2 = .5000000000$
$n = 3$, $S_3 = .8333333333$	$n = 4$, $S_4 = .5833333333$
$n = 5$, $S_5 = .7833333333$	$n = 6$, $S_6 = .6166666667$
$n = 7$, $S_7 = .7595238095$	$n = 8$, $S_8 = .6345238095$
$n = 9$, $S_9 = .7456349206$	$n = 10$, $S_{10} = .6456349206$
$n = 11$, $S_{11} = .7365440115$	$n = 12$, $S_{12} = .6532106782$
$n = 13$, $S_{13} = .7301337551$	$n = 14$, $S_{14} = .6587051837$
$n = 15$, $S_{15} = .7253718504$	$n = 16$, $S_{16} = .6628718504$
$n = 17$, $S_{17} = .7216953798$	$n = 18$, $S_{18} = .6661398242$
$n = 19$, $S_{19} = .7187714032$	$n = 20$, $S_{20} = .6687714032$

converged on the interval (0, 2). What happens at the "end points" $x = 0$ and $x = 2$? At $x = 0$ the series becomes

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \cdots \; = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\right)$$

The series 1 + 1/2 + 1/3 + 1/4 + . . . + 1/n + . . . is known as the **harmonic series**. This series **diverges**. There are several ways of showing this. Perhaps the most elementary is to group the terms as follows:

$$1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] +$$
$$\left[\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right] +$$
$$\left[\frac{1}{17} + \cdots + \frac{1}{32}\right] + \cdots$$

Note that

$$\left[\frac{1}{3} + \frac{1}{4}\right] > \left[\frac{1}{4} + \frac{1}{4}\right] = \frac{2}{4} = \frac{1}{2},$$
$$\left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] > \left[\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right] = \frac{4}{8} = \frac{1}{2},$$
$$\left[\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right] > \left[\frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16}\right] = \frac{8}{16} = \frac{1}{2},$$

and so on, so each group adds up to a value greater than $\frac{1}{2}$. As a result, a partial sum can be made arbitrarily large by adding a sufficient number of groups. This not an unexpected result when you consider that our series is the Taylor series for the function $f(x) = \ln(x)$, and as $x \to 0^+$, $\ln(x) \to -\infty$. Hence at $x = 0$ it seems appropriate that the series diverges to $-\infty$.

Let's look at the other end-point, $x = 2$. Now the series becomes

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots,$$

which is known as the **"alternating" harmonic series**, since the terms alternate in sign. This series actually converges. Here are the first 20 partial sums:

Note that the table of partial sums are arranged so that it is immediately apparent that the odd partial sums (the first column) form decreasing sequence and the even partial sums form an increasing sequence, both (apparently) converging toward a common limit. As you might guess.

this limit is ln(2) = .693147... In the next section we consider series that have signs that alternate, and find an extremely easy test for the convergence of such series.

Let us conclude this section with a few more examples using the ratio test.

Example 6.6. Find the radius of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2\,3^n} = \frac{(x-2)}{3} + \frac{(x-2)^2}{2^2 3^2} + \frac{(x-2)^3}{3^2 3^3} + \frac{(x-2)^4}{4^2 3^4} + \cdots$$

To find the radius of convergence, we use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2 3^{n+1}} \frac{n^2 3^n}{(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)}{3} \frac{n^2}{(n+1)^2} \right| = \frac{|x-2|}{3} \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2 = \frac{|x-2|}{3}$$

For convergence we know that $\frac{|x-2|}{3} < 1$, which means $|x-2| < 3$. Thus $-3 < x-2 < 3$ or $-1 < x < 5$. The radius of convergence is 3 and the series converges on the open interval (t1, 5).

Since the ratio test takes the absolute value of the ratio of successive terms, it is clear that the test does not take into consideration the signs of the terms. Suppose we are using the ratio test to test a series that has both positive and negative terms, and we find that the series converges. Then changing the signs of any of the terms will not alter the convergence (although it will alter the value to which the series converges). We give this type of convergence a special name – we say the series is **absolutely convergent,** meaning that if we were to take the absolute value of all the terms, the series would still converge. It is not difficult to show the following is true:

If a series is absolutely convergent, then it is convergent.

The converse of this result is **not** true, as we saw above, i.e. the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

converges, but is not absolutely convergent, since taking absolute values of the terms gives the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \cdots ,$$

which diverges.

Sometimes we use the name **conditionally convergent** for a series that converges, but is not absolutely convergent. Hence the alternating harmonic series is conditionally convergent.

You might wonder why we would make the distinction between absolute and conditional convergence. The following result gives one good reason:

- If a series is absolutely convergent, you can rearrange its terms in any way you wish and it still converges to the same value.
- If a series is only conditionally convergent, a rearrangement of terms may produce a series that converges to a different value.

For example, we have

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \cdots = \ln(2),$$

whereas if we regroup by taking two positive terms followed by one negative one, we have

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}\ln(x)$$

(To see that this series does converge to $\frac{3}{2}\ln(2)$, add the following:

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \cdots$$
$$\frac{1}{2}\ln(2) = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots)$$

In fact, we can make a "rearranged" alternating harmonic series converge to any number we want!

7. Alternating Series

We now present the second test we shall consider , the **alternating series test**. We consider a series of the form

$$S = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots$$

where $a_k > 0$ for all k. The test for convergence is surprisingly simple:

Theorem 7.1. *(Alternating Series Test) For $S = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots$ with$a_k > 0$ for all k, if*

(1) the terms decrease in size (i.e. $a_{n+1} < a_n$)

and

(2) $\lim_{n\to\infty} a_n = 0$,

*then **the series converges.** In addition, if the partial sum*

$$S_n = a_0 - a_1 + a_2 + \cdots \pm a_n$$

*is used as an approximation to S, then $|S_n - S| < a_{n+1}$. (i.e. the error in using $S_n$in place of S is bounded by the magnitude of the first term **not** used in forming S_n.)*

Example 7.2. The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

satisfies the two conditions:

- the terms decrease in size since $\frac{1}{n+1} < \frac{1}{n}$, and
- $\lim_{n\to\infty} \frac{1}{n} = 0$,

and so by the **AST** (alternating series test) the series converges. In fact it converges to $\ln(2) = 0.693147\ldots$, and so if we were to use the first 10 terms of the series as an approximation to $\ln(2)$, the absolute error in this approximation would be no bigger than 1/11 = 0.090909... We could calculate the actual error, assuming that the series does converge to $\ln(2)$, i.e.

$$S_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} = \frac{1627}{2520} = 0.6456349206...$$
$$\ln(2) = 0.6931471806...$$

This gives $|\ln(2) - S_{10}| = 0.04751226....$which is indeed less than our bound 0.090909...

The proof of the **AST** is geometrical, we start at the origin, move a_0 units to the right to get S_0, then back up a_1 units to get to S_1, then forward a_2 units to get to S_2, then back up a_3 units to get to S_3, etc... This shunting back and forth continues, and because $a_{n+1} < a_n$, we from an increasing sequence $S_1, S_3, S_5, \ldots$ from the right and a decreasing sequence $S_0, S_2, S_4, \ldots$ from the left. Finally we note that since $|S_n - S_{n-1}| = a_n$, and $a_n \to 0$, we have both sequences converging to a common limit S.

Example 7.3. Consider the Taylor series for $\sin(x)$:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

The ratio test tells us the series converges for all x. Suppose we were to use the first 4 terms of the series to approximate $\sin(x)$ over the interval $[0, \frac{\pi}{2}]$,i.e.

$$\sin(x) \approx P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

The two conditions of the alternating series test are satisfied and hence we have an error bounded by

$$|\sin(x) - P_7(x)| < \left|\frac{x^9}{9!}\right| \leq \frac{(\pi/2)^9}{9!} = 0.000160441...$$

We see that it is easier to use the alternating series result to bound the error in an approximation than the Taylor remainder. However, we must ensure our series actually is an alternating series before we apply this result. A tricky example is given by

Example 7.4. Suppose we approximate e^{-x} on the interval [-1, 1] with a Taylor polynomial of degree 10, i.e.

$$e^{-x} \cong P_{10}(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + \frac{x^{10}}{10!} \quad , \quad 1 \leq x \leq 1.$$

At first glance it may appear that we have an alternating series and hence can apply the **AST**. However, x may range over the interval [-1, 1], and we see that when x is negative, all of the terms of our series become positive and we no longer have an alternating series. Hence we cannot use the AST to bound the error in our approximation, and must resort to Taylor's result.

$$M = \max\left\{\, |\, f^{(11)}(x)\,|, -1 \leq x \leq 1 \right\} = \max\left\{\, e^{-x}, -1 \leq x \leq 1 \,\right\} = e$$

and so

$$\left|\, e^{-x} - P_{10}(x) \,\right| \leq e\frac{|x|^{11}}{11!} \leq \frac{e}{11!} = 68098... \times 10^{-8}.$$

8. Operations with Power Series

Power series can be viewed as an alternate way of expressing a function. Power series can usually be manipulated just as if they were functions.

Example 8.1. The Taylor series for $\sin(x)$, expanded about $x = 0$, is given by

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Because the series has an infinite radius of convergence, and converges to $\sin(x)$,we can write

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots .$$

Differentiating the series term by term, we have

$$1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots ,$$

which simplifies to give

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

We see that this is just the Taylor series for $\cos(x)$, as expected.

Here is the general result:

Theorem 8.2. *Let*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

be a power series convergent to $f(x)$ on the interval $(a-R, a+R)$ with a positive radius of convergence R. Then

1. *the series $c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$ converges to $f'(x)$ with a radius of convergence R and*
2. *the series $c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + c_3\frac{(x-a)^4}{4} + \cdots$*

converges to $\int_a^x f(t)dt$ with radius of convergence R.

Example 8.3. The series

$$1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \frac{x^{12}}{12!} + \cdots$$

converges with an infinite radius of convergence, since by the ratio test

$$\lim_{n\to\infty}\left|\frac{x^{3n+3}/(3n+3)!}{x^{3n}/(3n)!}\right| = |x|^3 \lim_{n\to\infty}\frac{3n!}{(3n+3)!} = |x|^3 \lim_{n\to\infty}\frac{1}{(3n+1)(3n+2)(3n+3)} = 0.$$

Hence the series defines a function, say $f(x)$, that is differentiable with derivative

$$f'(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots$$

If we wished to evaluate $\int_0^1 f(x)dx$, then we would integrate the series for $f(x)$ term by term to give

$$\begin{aligned}\int_0^1 f(x)dx &= \int_0^1 \left(1 + \tfrac{x^3}{3!} + \tfrac{x^5}{5!} + \tfrac{x^7}{7!} + \tfrac{x^9}{9!} + \cdots\right) dx \\ &= \left[x + \tfrac{x^4}{3!4} + \tfrac{x^7}{6!7} + \tfrac{x^{10}}{9!10} + \cdots\right]_0^1 \\ &= 1 + \tfrac{1}{4!} + \tfrac{1}{7!} + \tfrac{1}{10!} + \cdots\end{aligned}$$

If we were to use Maple to evaluate this, we find

```
> sum(1/(3*n+1)!,n=0..infinity);
hypergeom([], [4/3, 2/3], 1/27)
> evalf(");
1.041865355
```

Note that the exact value of the sum is given as a "hypergeometric" series, which we evaluate with the evalf command. This leads us to believe that the series we started off with was known to Maple as a special case of something called a "hypergeometric series". To see that this is indeed the case, we might ask Maple

```
> sum(x (3*n)/((3*n)!*(3*n+1)),n=0..infinity);
hypergeom([], [4/3, 2/3], 1/27 x^3 )
```

This function cannot be expressed in terms of the "elementary" functions we are used to dealing with. Infinite series form the basis of definitions for many "special" functions that occur in many branches of science, especially physics.

9. Exercises

1. Find the first 5 terms of the Taylor series for $f(x) = \ln(x)$ expanded about the point $x = e$. From these terms predict what the n^{th} term will be and use the ratio test to determine the open interval on which the series converges.
2. Suppose we approximate $e^x \cong 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} = P_5(x)$
   (a) Use the Taylor remainder term to bound the error $|e^x - P_5(x)|$ in this estimate if we restrict $-2 \leq x \leq 2$.
   (b) If we take $x = 1$, then we have $e \cong 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} = \frac{163}{60}$. Use the Taylor remainder term to bound the error in this approximation.
   (c) Use your calculator to calculate the error in $e \cong \frac{163}{60}$ and compare it with the bound found in part (b).
3. To find the Taylor series for $\arctan(x)$ expanded about the point $x = 0$ you could proceed to differentiate $\arctan(x)$many times, but the derivatives quickly become cumbersome. (Try a few and see!) An alternate way to find this series is to note that
$$\int_0^x \frac{1}{1+z^2} dz = \arctan(x).$$
   But we can expand $\frac{1}{1+z^2}$in a geometric series
$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$
   Now integrate this series from 0 to x term by term, to arrive at the series for $\arctan(x)$ given on page 9.
4. Using only the definition, prove that a Cauchy sequence is bounded.
5. Use the ratio test to determine the radius of convergence and open interval on which the following series converge:
   (a) $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \cdots$.
   (b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} \cdots$
   (c) $\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = 1 + \frac{3x}{1!} + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots$.
   (d) $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{2n^3} = \frac{(2x-1)}{2\cdot 1^3} + \frac{(2x-1)^2}{2\cdot 2^3} + \frac{(2x-1)^3}{2\cdot 3^3} + \frac{(2x-1)^4}{2\cdot 4^3} \cdots$
   (e) $\sum_{n=0}^{\infty} \frac{n!x^n}{2^n} = 1 + \frac{1!x}{2} + \frac{2!x^2}{2^2} + \frac{3!x^3}{2^3} + \cdots$
6. Consider the series 4 - 4/3 + 4/5 - 4/7 + 4/9 + . . . = $\sum\limits_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1}$. Show that the conditions of the alternating series test are satisfied and hence the series converges. It is known that this series converges to the value of π. Suppose we wanted to approximate π by using the n^{th} partial sum of the series. If we wanted the error to be less than 0.001, how many terms of the series do we need to use?
7. The function $si(x) = \int\limits_0^x \frac{\sin(t)}{t} dt$ is called the "sine integral" function. Since the anti-derivative of $\frac{\sin(t)}{t}$ is not expressible in terms of elementary functions, we can't use the fundamental theorem of integral calculus to evaluate the definite integral. To find the Taylor series for $si(x)$ proceed as follows:
   (a) Using the series for $\sin(t)$, divide it by t to get the series for $\frac{\sin(t)}{t}$.
   (b) Integrate the series for $\frac{\sin(t)}{t}$ to evaluate the definite integral in the definition of $si(x)$.

(c) What is the radius of convergence of the series for *si(x)*?

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