Acadia University

Department of Mathematics and Statistics

INTRODUCTORY CALCULUS 1
(MATH 1013)

ASSIGNMENT 1 Solutions

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1 Functions and Models

1.5 Exponential Functions

14. (a) We say that two functions f and g are reflected about a line $y = a$ if $f(x) - a$ and $g(x) - a$ are equal but have opposite signs for all x. Therefore, in the present case, we need to find a function $f(x)$ such that $f(x) - 4 = 4 - e^x$ and that's $f(x) = 8 - e^x$ (see Fig. 1).

    (b) Two functions f and g are reflected about the line $x = a$ if $f(a+x) = g(a-x)$ for all x. The function we are looking for is $f(x+2) = e^{2-x}$ because $g(x) = e^x$. We finally substitute $z = x+2$ (or $x = z-2$) to obtain $f(z) = e^{4-z}$ (see Fig. 2).

16. Looking at the graph, we can observe that $f(0) = 2$ and $f(2) = \frac{2}{9}$. Since $f(x) = Ca^x$, we have that

$$Ca^0 = 2 \tag{1}$$

and

$$Ca^2 = \frac{2}{9}. \tag{2}$$

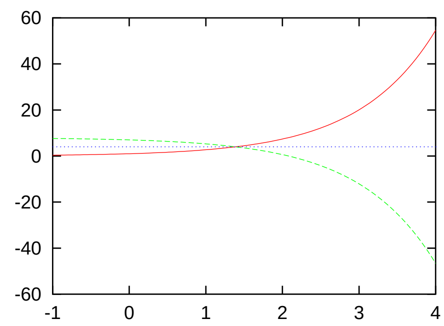

Figure 1: Reflection about the line $y = 4$

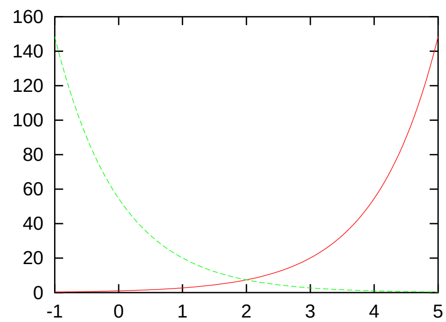

Figure 2: Reflection about the line $x = 2$

From equation 1, we see right away that $C = 2$ and therefore, equation 2 becomes

$$a^2 = \frac{1}{9}$$

which implies that $a = \pm\frac{1}{3}$. However, we must reject the solution $a = -\frac{1}{3}$ because the function $f(x) = 2 \times \left(\frac{-1}{3}\right)^x$ implies $f(1) = \frac{-2}{3}$ which doesn't agree with the graph. We have one remaining solution and it is

$$f(x) = 2 \times \left(\frac{1}{3}\right)^x.$$

18. We will assume that the highest paying job at the end of the month must be chosen. The first job pays $1,000,000$ over a month. How much pays the second one? Setting up the equation is not very difficult, we readily see that it pays[1]

$$pay = \sum_{n=1}^{30} 2^{n-1}. \tag{3}$$

How much is it? The formula 3 is called a *geometric series*. We can solve it rather easily,

[1] We chose a month with 30 days.

we simply evaluate $pay - 2 \times pay$ by using formula 3 which gives

$$\begin{aligned} pay - 2 \times pay = & \sum_{n=1}^{30} 2^{n-1} - 2 \times \sum_{n=1}^{30} 2^{n-1} \\ = & \sum_{n=1}^{30} 2^{n-1} - \sum_{n=2}^{31} 2^{n-1} \\ = & 2^0 - 2^{30} \end{aligned}$$

We then solve for pay in $pay - 2 \times pay = 1 - 2^{30}$ which is very easy since $pay - 2 \times pay = -pay$, and we finally have $pay = 2^{30} - 1$. It should be noted here that 2^{30} is a very big number (about one billion or $1,073,741,824$). What is the lesson here? Exponential functions can increase **very fast**[2].

24. (a) Given that the half-life is 15 hours, after 15 hours, we have $2/2 = 1$ g left (by definition). After another 15 hours (for a total of 30 hours), we have $1/2$ g left, and after yet another 15 hours (for a total of 30 hours), we have $1/4$ g left. Finally, after 60 hours, we'll have $1/8$ g left[3].

    (b) Since the decay is exponential, we have that

$$m(x) = Ca^x. \tag{4}$$

First of all, $m(0) = 2$ and therefore $C = 2$ (see equation 1 on page 1). We have that after 15 hours, the mass must be half, so

$$m(15) = 1. \tag{5}$$

Combining equations 4 and 5, we get that

$$2a^{15} = 1 \tag{6}$$

$2a^{15} = 1$ and we must now solve for a. One way to do it (assuming we don't know about logarithms) is to take the equation 6to the power $\frac{1}{15}$ to get

$$a = \frac{1}{2^{1/15}} \cong \frac{1}{1.047} \cong 0.9548.$$

This means that the mass of our sample goes down according to the equation[4]

$$m(x) = 2\left(\frac{1}{2}\right)^{\frac{x}{15}} = \left(\frac{1}{2}\right)^{\frac{x}{15}-1}. \tag{7}$$

    (c) Amount after 4 days? What you must not do here is substitute 4 in equation 7. The correct reasoning is to first convert 4 days in hours. We have 24 hours for every day, so 4 days is $4 \times 24 = 96$ hours. Therefore, the remaining mass will be

$$m(96) = \left(\frac{1}{2}\right)^{\frac{96}{15}-1} \cong 0.02.$$

[2] Most modern computer architectures are 32 bits (Windows, Linux and even the new gaming consoles). Therefore, the highest integer value most programmers will ever expect is $2^{32} - 1$. Of course, newer computer architectures will probably be 64 bits and integers will go up to $2^{64} - 1$. Will that be a big improvement? How much bigger is 2^{64} with respect to 2^{32}?

[3] It is also possible to deduce this result from equation 7.

[4] Actually, your calculator or mathematical software is likely to evaluate this function using the formula $e^{(1-x/15)\ln 2}$ instead. We will come back to logarithms and their applications later in the course.

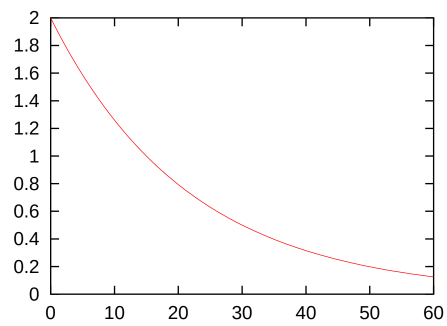

Figure 3: Mass of sodium vs time according to equation 7

1.6 Inverse Functions and Logarithms

18. (a) We have to solve the equation

$$3 = 3 + x^2 + \tan(\pi x/2)$$

which simplifies to

$$\tan(\pi x/2) = -x^2.$$

Now, some of you might think that life is pretty hard (and it sometimes is!). But wait! What is $\tan(0)$? 0 of course! And what is x^2 evaluated at 0? 0 of course! So we found a solution which we will assume to be unique and we conclude that $f^{-1}(3) = 0$ (See Fig. 4).

(b) You must not try to evaluate $f^{-1}(5)$! We write $x_5 = f^{-1}(5)$ and x_5 is defined by the equation $f(x_5) = 5$. And thus

$$f\left(f^{-1}(5)\right) = f(x_5) = 5.$$

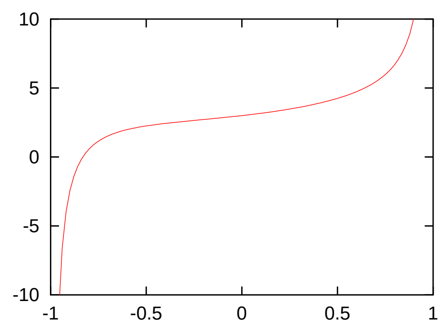

Figure 4: $3 + x^2 + \tan(\pi x/2)$

28. We want to compute the inverse of the following function

$$f(x) = (1 + e^x)/(1 - e^x) \tag{8}$$

It is worth noting that when $x = 0$, we get $f(0) = 2/0 = \infty$. Since we know that $\ln(e^x) = x$, we have

$$\begin{aligned} \frac{1+e^x}{1-e^x} = y \quad &\Longrightarrow \quad (1-e^x)y = 1+e^x \\ &\Longrightarrow \quad (1+y)e^x = y-1 \\ &\Longrightarrow \quad e^x = \frac{y-1}{y+1} \\ &\Longrightarrow \quad x = \ln\left(\frac{y-1}{y+1}\right). \end{aligned}$$

Of course, this function isn't defined for $\frac{y-1}{y+1} \leq 0$ or $-1 \leq y \leq 1$. One could verify that $f(x)$ in equation 8 never takes these values (see also Fig. 5).

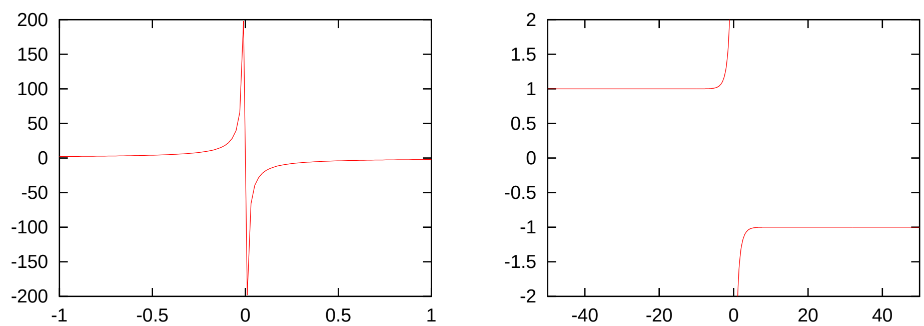

Figure 5: $f(x) = (1+e^x)/(1-e^x)$

36. Recall that $\log a^b = b \log a$ and $\log_a a = 1$.

    (a) $\log_8 2 = \log_8\left(8^{\frac{1}{3}}\right) = \frac{1}{3} \times \log_8 8$ because $\log a^b = b \log a$, and since $\log_a a = 1$,

    $$\frac{1}{3} \times \log_8 8 = \frac{1}{3}$$

    so that $\log_8 2 = \frac{1}{3}$.

    (b) $\ln e^{\sqrt{2}} = \sqrt{2} \ln e$ because $\ln a^b = b \ln a$, and since $\ln e = 1$, we have $\ln e^{\sqrt{2}} = \sqrt{2}$.

38. Recall that $a^{b+c} = a^b a^c$ and $a^{\log_a b} = b$ (see equation 7 in [Stewart] on page 68).

    (a) $2^{\log_2 3 + \log_2 5} = 2^{\log_2 3} \times 2^{\log_2 5} = 3 \times 5 = 15$

    (b) $e^{3 \ln 2} = e^{\ln(2^3)} = 2^3 = 8$

42. The logarithm has $(0, \infty)$ for its domain (it isn't defined elsewhere, at least in this course). Therefore, in $\ln(4-x^2)$, we need to have $4 - x^2 \geq 0$ or

$$x^2 < 4. \tag{9}$$

At this point, one must be careful. We may think of taking the square root on both sides of equation 9 (getting $x < 2$) and indeed, it is correct to do so... as long as you realize that if $a^2 = b$ then so does $(-a)^2 = b$! That is, you have to consider negative values as well... Therefore, we have

$$x < 2$$

and

$$x > -2.$$

We conclude that the domain of $\ln(4-x^2)$ is $(-2,2)$.
As for the range of the function... we first have to look at the range of $4-x^2$ over $(-2,2)$. Clearly, the polynomial is at its maximum when $x = 0$ and so its range has 4 as an upper bound. On the other hand, it has 0 as its lower bound and therefore the range of $4-x^2$ over $(-2,2)$ is $(0,4]$. Since the logarithm is a strictly increasing function, we can conclude that the range of $\ln(4-x^2)$ is $(\ln 0, \ln 4]$ or $(-\infty, \ln 4]$ or approximately $(-\infty, 1.386]$ (see Fig. 6).

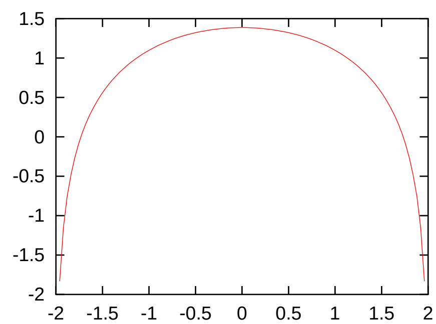

Figure 6: $f(x) = (1+e^x)/(1-e^x)$

52. (a) Starting from $\ln\ln x = 1$, we write $e^1 = e^{\ln\ln x}$ but since $e^{\ln w} = w$, we have $e = \ln x$ and finally, we write

$$e^e = e^{\ln x} = x$$

so that $x = e^e$.

(b) Starting from $e^{ax} = Ce^{bx}$, we divide both sides of the equation by e^{bx} (which we know to be a non-zero positive number! why?) to get

$$e^{(a-b)x} = C.$$

We can then simply take the logarithm of both sides of the equation to get

$$(a-b)x = \ln C$$

and therefore $x = \frac{\ln C}{a-b}$ because $a \neq b$.

58. (a) We have to solve for t in

$$Q_0\left(1-e^{-t/a}\right) = Q.$$

We begin by dividing by Q_0 both sides of the equation and after some algebra, we get

$$e^{-t/a} = 1 - \frac{Q}{Q_0}.$$

Taking the logarithm, we get

$$\frac{-t}{a} = \ln\left(1 - \frac{Q}{Q_0}\right)$$

or

$$t = -a\ln\left(1 - \frac{Q}{Q_0}\right)$$

or

$$t = a\ln\frac{Q_0}{Q_0 - Q} \tag{10}$$

because $\ln c/b = -\ln b/c$. It should be noted that if $Q_0 = Q$ (at time $t = 0$), then we have the logarithm of 0 or of ∞ which is not very good! We must therefore require that $Q < Q_0$ in which case

$$\frac{Q_0}{Q_0 - Q} > 1$$

and therefore $t > 0$ (because $\ln(1) = 0$). When $Q = 0$, we have $t = 0$. What does equation 10 mean? Well, it gives us time as a function of the charge Q. When the charge is at its minimum (0), then $t = 0$ and we know that we just started recharging... as Q increases (getting closer to Q_0), we can compute the corresponding number of seconds we waited.

(b) Setting $Q = 0.9Q_0$ and substituting in equation 10, we have

$$\begin{aligned} t &= a\ln\frac{Q_0}{Q_0 - 0.9Q_0} \\ &= a\ln\frac{Q_0}{0.1Q_0} \\ &= a\ln\frac{1}{0.1} \\ &= a\ln 10 \end{aligned}$$

and because $a = 2$, we have $t = 2\ln 10 \cong 4.6$ seconds. Bonus: how long for 99% of the charge? We substitute again to get $t = 2\ln\frac{1}{0.01} \cong 9.2$ seconds... how long for 99.9% of the charge? We have $t = 2\ln\frac{1}{0.001} \cong 13.8$ seconds... What is happening? The logarithm grows very slowly! We can get very, very close to full charge without waiting all that long!

References

[Stewart] James Stewart, *Calculus: Concepts and Contexts* (Second Edition), Brooks/Cole, 2001.