Acadia University

Department of Mathematics and Statistics

Topics from Advanced Calculus

(MATH 2023)

Assignment 3 - solutions

(was due March 18th 2002)

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For a lot of these problems, the answer was in the back of the book. This was intentional. Marking scheme should reflect the fact that students had the answers. Students who simply give the answer without showing work when the result is clearly not trivial get 0.

1 Basic Concepts

1.14 [3 marks] - (a) order 2 (b) y is unknown (c) x is the variable (raw answer is enough)
1.15 [3 marks] - (a) order 4 (b) y is unknown (c) x is the variable (raw answer is enough)
1.16 [3 marks] - (a) order 2 (b) s is unknown (c) t is the variable (raw answer is enough)
1.17 [3 marks] - (a) order 4 (b) y is unknown (c) x is the variable (raw answer is enough)
1.24 [10 marks] - d,e (student must show work proving the result - simply substitute in d.e.)
1.28 [10 marks] - d only (student must show work proving the result- simply substitute in d.e.)

2 Classification of First-Order Differential Equations

2.15 [2 marks] - $y' = \frac{y^2}{x}$ (simple algebra)
2.25 [2 marks] - $y' = -1$ ("divide" by dx: abuse the notation)
2.29 [5 marks] - homogeneous (student must prove homo. property with $x \to xt$ and $y \to yt$), nonlinear, Bernouilli (student must justify that $x/y^2 = xy^{-2}$), separable, not exact (student must show work starting at $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$).
2.34 [5 marks] - not homogeneous (student must justify with with $x \to xt$ and $y \to yt$), nonlinear, Bernouilli, non separable, not exact (student must show work starting at $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$).

2.35 [5 marks] - not homogeneous (student must justify with with $x \to xt$ and $y \to yt$), linear, not Bernouilli, non separable, exact (student must show computation for exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$).

3 Separable First-Order Differential Equations

3.30 [5 marks] - Integrating throughout, we get $\frac{t^3}{3} + t + \frac{y^3}{3} + \frac{y^2}{2} = C$.
3.37 [5 marks] - Rewrite as $\frac{dy}{y^2} = dx$ then integrate $-1/y = x + C$.
3.39 [10 marks] - Rewrite as $\frac{dx}{x} = \frac{dt}{t}$, integrate to get $\ln|x| = \ln|t| + C$, get $|x| = e^C|t|$ or $x = kt$. (This problem had to be done carefully without skipping a step.)
3.42 [10 marks] - Integrate throughout to get $\frac{x^3}{3} + x + \ln|y| = C$, get $\ln|y| = C - \frac{x^3}{3} - x$ or $|y| = e^C e^{-\frac{x^3}{3} - x}$ or $y = ke^{-\frac{x^3}{3} - x}$. Now, $y(-1) = 1$ implies $1 = ke^{\frac{1^3}{3}+1}$ or $k = e^{-4/3}$ finally, you get $y(x) = e^{-4/3 - \frac{x^3}{3} - x}$.
3.49 [2 marks] - Not homogeneous (student must justify with with $x \to xt$ and $y \to yt$).
3.52 [10 marks] - Homogeneous (not necessary to justify) so use $y = vx$ and $y' = v + xv'$ to get $v + xv' = \frac{vx}{x + \sqrt{vx^2}} = \frac{vx}{x + |x|\sqrt{v}}$ hence for $x \neq 0$, $(v + xv')(1 + sign(x)\sqrt{v}) = v$ or $sign(x)v^{3/2} + xv' + |x|\sqrt{v}v' = 0$ or $v^{3/2} + |x|v' + x\sqrt{v}v' = 0$. For simplicity, assume $x > 0$, then $\frac{dx}{x} + \frac{1+\sqrt{v}}{v^{3/2}}dv = 0$ hence $\ln|x| + \ln|v| - \frac{2}{\sqrt{v}} = C$ or $\ln|xv| - \frac{2}{\sqrt{v}} = C$ or $\frac{1}{\sqrt{x}}\ln|y| - \frac{2}{\sqrt{y}} = C$.
3.53 [2 marks] - Not homogeneous (student must justify with with $x \to xt$ and $y \to yt$).
3.54 [10 marks] - Homogeneous. $xv' + v = \frac{x^4 + 3v^2x^4 + v^4x^4}{x^4 v} = \frac{1 + 3v^2 + v^4}{v}$ or $\frac{vdv}{1 + 2v^2 + v^4} = \frac{dx}{x}$ or $\frac{-1}{2v^2+2} = \ln|x| + C$ or $\frac{-x^2}{2y^2 + 2x^2} = \ln|kx|$. The book solves for y^2 at this point.

4 Exact First-Order Differential Equations

4.24 [10 marks] - Yes, it is exact, solve $\frac{\partial g}{\partial x} = y + 2xy^3$ and $\frac{\partial g}{\partial y} = 1 + 3x^2y^2 + x$ to get $g(x,y) = xy + x^2y^3 + y = c$.
4.32 [10 marks] - Yes, it is exact, solve $\frac{\partial g}{\partial t} = -2y/t^3$ and $\frac{\partial g}{\partial y} = 1/t^2$ to get $g(t,y) = y/t^2 = c$
4.33 [2 marks] - Not exact, demonstrate using $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.
4.36 [10 marks] - Yes, it is exact, solve $\frac{\partial g}{\partial t} = t^2 - x$ and $\frac{\partial g}{\partial x} = -t$ to get $g(t,x) = -xt - t^3/3 = c$ or $x = t^2/3 - c/t$.
4.37 [10 marks] - Yes, it is exact, solve $\frac{\partial g}{\partial t} = t^2 + x^2$ and $\frac{\partial g}{\partial x} = 2tx - x$ to get $g(t,x) = t^3/3 + tx^2 - x^2/2 = c$ or$2t^3 + 6tx^2 - 3x^2 = c'$.
4.38 [10 marks] - Yes, it is exact, solve $\frac{\partial g}{\partial t} = 2xe^{2t}$ and $\frac{\partial g}{\partial x} = 1 + e^{2t}$ to get $g(t,x) = x + xe^{2t} = c$ or $x = \frac{k}{1+e^{2t}}$.
4.45 [20 marks] - Using the integrating factor $I(x,y) = \frac{1}{(xy)^2}$, the differential form becomes exact and we get $g(x,y) = \frac{x^4}{3} - \frac{1}{xy} = c$ or $\frac{1}{y} = \frac{1}{3}x^4 - cx$.
4.47 [20 marks] - Using the integrating factor $I(x,y) = e^{-y^2}$, the differential form becomes exact and we get ,$g(x,y) = xe^{-y^2} = c$ or $y^2 = \ln|kx|$.
4.51 [20 marks] - Using the integrating factor $I(x,y) = \frac{1}{(xy)^2}$, the differential form becomes exact

and we get $g(x,y) = y + \ln|x| + \ln|y| = c$ or $\ln|xy| = c - y$.
4.65 [5 marks] - We had $x = \frac{k}{1+e^{2t}}$, solve for k to get $x(t) = \frac{-2(1+e^2)}{1+e^{2t}}$.

5 Linear First-Order Differential Equations

5.22 [5 marks] - Separable, simply integrate throughout $y = e^{0.01x}$.
5.33 [15 marks] - Integrating factor is e^{-7x} so we have $\frac{d}{dx}\left(e^{-7x}y\right) = e^{-7x}\sin 2x$ use integration by parts to get $y = ce^{7x} - \frac{2}{53}\cos 2x - \frac{7}{53}\sin 2x$. (Student must do the integration by parts in full!)
5.51 [10 marks] - The integrating factor is e^{3x^2} and so the general solution is $y(x) = ce^{-3x^2}$ solve for c to get $y = e^{-3(x^2-\pi^2)}$.