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MATHEMATICS 2023 - INFINITE SERIES I
ACADIA UNIVERSITY

1. EXERCISES

1. [15 marks] Find the first 5 terms of the Taylor series for $f(x) = \ln(x)$ expanded about the point $x = e$. From these terms predict what the n^{th} term will be and use the ratio test to determine the open interval on which the series converges.
   Solution: $f(x) = 1 + \frac{x-e}{e} - \frac{(x-e)^2}{2e^2} + \frac{(x-e)^3}{3e^3} - \frac{(x-e)^4}{4e^4} + \ldots$ (5 marks) hence the n^{th} term will be $\frac{(-1)^{n+1}(x-e)^n}{ne^n}$ (for $n > 0$) (5 marks). Ratio test gives $\lim_{n\to\infty}\left|\frac{(x-e)^{n+1}}{(n+1)e^{n+1}}\frac{ne^n}{(x-e)^n}\right| = \frac{|x-e|}{e}\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{|x-e|}{e}$ hence the series will converge for $x \in (0, 2e)$ (5 makrs).
2. [6 marks] Suppose we approximate $e^x \cong 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} = P_5(x)$
   (a) Use the Taylor remainder term to bound the error $|e^x - P_5(x)|$ in this estimate if we restrict $-2 \le x \le 2$.
   Solution:$|e^x - P_5(x)| \le \frac{e^2 2^6}{6!}$
   (b) If we take $x = 1$, then we have $e \cong 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!} = \frac{163}{60}$. Use the Taylor remainder term to bound the error in this approximation.
   Solution: $|e^x - P_5(x)| \le \frac{e^1 1^6}{6!} \cong 0.003775$
   (c) Use your calculator to calculate the error in $e \cong \frac{163}{60}$ and compare it with the bound found in part (b).
   Solution: $e - \frac{163}{60} \cong 0.001615 \le \frac{e^1 1^6}{6!}$
3. [5 marks] To find the Taylor series for $\arctan(x)$ expanded about the point $x = 0$ you could proceed to differentiate $\arctan(x)$many times, but the derivatives quickly become cumbersome. (Try a few and see!) An alternate way to find this series is to note that
   $$\int_0^x \frac{1}{1+z^2}dz = \arctan(x).$$
   But we can expand $\frac{1}{1+z^2}$in a geometric series
   $$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots$$
   Now integrate this series from 0 to x term by term, to arrive at the series for $\arctan(x)$.
   Solution:$\arctan(x) = \int_0^x \frac{1}{1+z^2}dz = \int_0^x 1 - z^2 + z^4 - z^6 + \cdots dz = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$
4. [15 marks] Using only the definition, prove that a Cauchy sequence is bounded.
   Solution: (Warning: must show that the sequence is bounded from above and from below!!! Take off 5 marks if student forgets.) Given any $\varepsilon > 0$, there exist N such that $\left|a_{N+1} - a_j\right| < \varepsilon$ for any $j > N$ hence all $a_j < a_{N+1} + \varepsilon$ and $a_j > a_{N+1} - \varepsilon$. Therefore, we have the upper bound $\max\{a_1, a_2, a_3, \ldots, a_N, a_{N+1} + \varepsilon\}$ and the lower bound $\min\{a_1, a_2, a_3, \ldots, a_N, a_{N+1} - \varepsilon\}$ and this works no matter which ε we chose.
5. [30 marks - 6 marks for each part] Use the ratio test to determine the radius of convergence and open interval on which the following series converge:

Solution: We give R, the interval of convergence and the radius of convergence. For full marks, student must provide all three. Giving x instead of $|x|$ for R is a major mistake.

(a) $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \cdots$.
Solution: $R = \frac{|x|}{2}$ and $(-2, 2)$. Radius of convergence is 2.

(b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} \cdots$.
Solution: $R = |x|$ and $(-1, 1)$. Radius of convergence is 1.

(c) $\sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = 1 + \frac{3x}{1!} + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots$.
Solution: $R = 0$ and $(-\infty, \infty)$. Radius of convergence is ∞.

(d) $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{2n^3} = \frac{(2x-1)}{2 \cdot 1^3} + \frac{(2x-1)^2}{2 \cdot 2^3} + \frac{(2x-1)^3}{2 \cdot 3^3} + \frac{(2x-1)^4}{2 \cdot 4^3} \cdots$.
Solution: $R = |2x - 1|$ and $(0, 1)$. Radius of convergence is $1/2$.

(e) $\sum_{n=0}^{\infty} \frac{n! x^n}{2^n} = 1 + \frac{1! x}{2} + \frac{2! x^2}{2^2} + \frac{3! x^3}{2^3} + \cdots$.
Solution: $R \to \infty$ except when $x = 0$. Radius of convergence is 0.

6. [10 marks] Consider the series 4 - 4/3 + 4/5 - 4/7 + 4/9 + . . . = $\sum\limits_{n=0}^{\infty} \frac{(-1)^n 4}{2n+1}$. Show that the conditions of the alternating series test are satisfied and hence the series converges. It is known that this series converges to the value of π. Suppose we wanted to approximate π by using the n^{th} partial sum of the series. If we wanted the error to be less than 0.001, how many terms of the series do we need to use?
Solution: It is an alternating series because the signs alternate and $\frac{4}{2n+1}$ is decreasing as n increases (5 marks). AST estimate gives $\sum_{n=0}^{N-1} \frac{(-1)^n 4}{2n+1}$ is $\frac{4}{2N+1}$ close to π hence we need $\frac{4}{2N+1} \leq 0.001$ or $N \geq \frac{3999}{2}$ so a good answer is 2000 (5 marks). (However $N = 1999$ is not a good answer according to AST because it allows an error of 0.00100025 which is bigger than the threshold.)

7. [10 marks] The function *si(x)* = $\int\limits_0^x \frac{\sin(t)}{t} dt$ is called the "sine integral" function. Since the anti-derivative of $\frac{\sin(t)}{t}$ is not expressible in terms of elementary functions, we can't use the fundamental theorem of integral calculus to evaluate the definite integral. To find the Taylor series for $si(x)$ proceed as follows:

(a) [2 marks] Using the series for $\sin(t)$, divide it by t to get the series for $\frac{\sin(t)}{t}$.
Solution: $\frac{\sin(t)}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{4!} - \ldots$

(b) [2 marks] Integrate the series for $\frac{\sin(t)}{t}$ to evaluate the definite integral in the definition of $si(x)$.
Solution: $\int_0^x \frac{\sin(t)}{t} dt = \int_0^x 1 - \frac{t^2}{3!} + \frac{t^4}{4!} - \ldots dt = x - \frac{x^3}{3 \times 3!} + \frac{x^5}{5 \times 5!} - \ldots$

(c) [6 marks] What is the radius of convergence of the series for *si(x)*?
Solution: Using the ratio test, we get $R = 0$ and hence the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.