Acadia University
Department of Mathematics and Statistics

INTRODUCTORY CALCULUS 1
(MATH 1013)

SECTION 4.2 Solutions

Daniel Lemire, Ph.D.

15th November 2001

4 Applications of Differentiation

4.2 Maximum and Minimum Values

2. **[2 marks]**

   (a) The ***Extreme Value Theorem*** ([Stewart, Section 4.2, Theorem 3])guarantees the existence of absolute maximums and minimums for continuous functions on closed intervals.

   (b) The suggested steps to find the absolute maximum and the absolute minimum of a continuous function f over a closed interval $[a,b]$ are outlined as the ***Closed Interval Method*** ([Stewart, Section 4.2, p. 275]).

      i. Find the values of f at the critical numbers of f in (a,b).
      ii. Find the values of f at the end-points of the interval.
      iii. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

4. **[2 marks]** We will be using definition 2 from Stewart ([Stewart, Section 4.2, Definition 2]). Specifically, we say that f has a local maximum at c if $f(c) \geq f(x)$ when x is a neighborhood of c. We say that f has a local minimum at c if $f(c) \leq f(x)$ when x is a neighborhood of c.

label	description
a	**local maximum** but not an absolute maximum since $f(e) > f(a)$
b	**local mimimum** but not an absolute minimum since $f(t) < f(b)$
c	**local mimimum and minimum** but not an absolute extremum since $f(t) < f(c) < f(e)$
d	**local mimimum** but not an absolute minimum since $f(t) < f(d)$
e	**local maximum** and **absolute maximum**
r	**local mimimum** but not an absolute minimum since $f(t) < f(r)$
s	**local maximum** but not an absolute maximum since $f(e) > f(s)$
t	**local mimimum** and **absolute minimum**

28. **[2 marks]** A critical number is a number c where either $f'(c) = 0$ or else $f'(c) \nexists$. Using the quotient rule, we have

$$\begin{aligned} f'(x) &= \frac{x^2+x+1-(x+1)(2x+1)}{(x^2+x+1)^2} \\ &= \frac{-x^2-2x}{(x^2+x+1)^2}. \end{aligned}$$

Clearly, $f'(c) \nexists$ whenever $x^2+x+1 = 0$ (and actually, it is true for f itself!), but using the quadratic rule we notice that this equation has no real solution since the discriminant is negative ($b^2-4ac = 1-4 = -3$). We conclude that the derivative always exists. However, the derivative is zero whenever $-x^2-2x = 0 \Rightarrow x \in \{0,-2\}$. Therefore, we have found the two critical points: $\{0,-2\}$.

38. **[3 marks]** We first notice that f is continuous and that we are considering a closed interval so that the ***Closed Interval Method*** applies.

   (a) The critical points of $\sqrt{9-x^2}$ are found by taking the derivative which is

   $$\frac{-x}{\sqrt{9-x^2}}.$$

   The derivative doesn't exist whenever $9-x^2 \leq 0$ which happens whenever $x \notin (-3,3)$ but on the proposed interval ($[-1,2]$), it isn't an issue. However, the derivative is 0 when $x = 0$ and thus we have one (and only one) critical point ($x = 0$). The value of the function at the critical point is $f(0) = 3$.

   (b) We have to evaluate the function at the boundary of the closed interval. We readily obtain $f(-1) = 2\sqrt{2}$ and $f(2) = \sqrt{5}$.

   (c) The absolute maximum is 3 and it is reached at $x = 0$. The absolute minimum is $\sqrt{5}$ and it is reached at $x = 2$. The third critical point ($x = -1$) is a local minimum.

42. **[3 marks]** We first notice that f is continuous and that we are considering a closed interval so the ***Closed Interval Method*** applies.

   (a) The critical points of $f(x) = x - 2\cos x$ are found through the derivative which is

   $$f'(x) = 1 + 2\sin x.$$

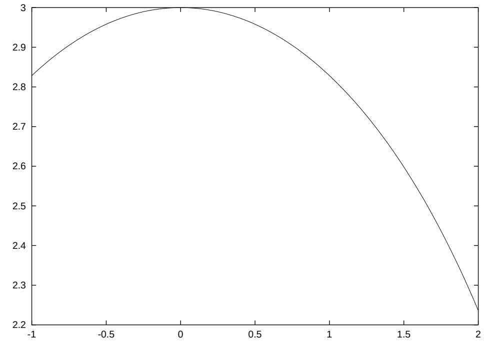

Figure 1: $\sqrt{9-x^2}$

Clearly, the derivative always exist. It is zero whenever $\sin x = \frac{-1}{2}$ or when $x = -\frac{\pi}{6} + 2\pi k$ or $x = -\frac{5\pi}{6} + 2\pi k$ where k is an integer. There is two such critical point in the interval $[-\pi, \pi]$ and they are $-\frac{\pi}{6}$ and $\frac{-5\pi}{6}$. We evaluate the function at these critical point $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - \sqrt{3} \cong -2.26$ and $f\left(-\frac{5\pi}{6}\right) \cong -0.886$.

(b) We evaluate the function at the boundary of the interval: $f(-\pi) = -\pi + 2 \cong -1.1$ and $f(\pi) = \pi + 2 \cong 5.1$.

(c) We see that the absolute maximum is reached at $f(\pi) = \pi + 2$ and the absolute minimum at $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - \sqrt{3}$. $f(-\pi) = -\pi + 2$ is a local minimum whereas $f\left(-\frac{5\pi}{6}\right)$ is a local maximum.

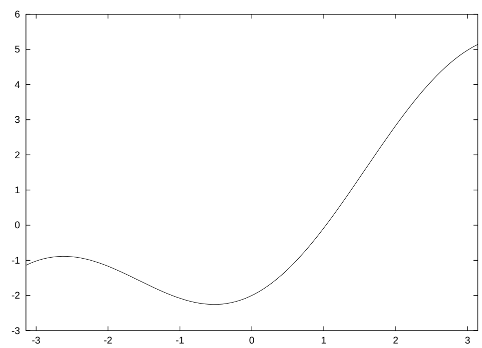

Figure 2: $x - 2\cos x$

52. **[2 marks]** The independent variable is θ. We can take the derivative either using the chain rule or else, using the quotient rule. We get

$$F'(\theta) = \frac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}.$$

Setting $F'(\theta_{min}) = 0$, we get $\sin\theta_{min} - \mu\cos\theta_{min} = 0$ or $\mu = \tan\theta_{min}$ (choosing θ_{min} so $0 \le \theta_{min} \le \frac{\pi}{2}$). To verify it is a minimum, we use the ***Closed Interval Method*** since f

is continuous and we have a closed interval.

$$\begin{aligned} F(\theta_{min}) &= \frac{\mu W}{\mu \sin\theta_{min} + \cos\theta_{min}} \\ &= \frac{\mu W}{\tan\theta_{min}\sin\theta_{min} + \cos\theta_{min}} \\ &= \frac{\mu W \cos\theta_{min}}{\sin^2\theta_{min} + \cos^2\theta_{min}} \\ &= \mu W \cos\theta_{min}. \end{aligned}$$

Whereas we have $F(0) = \mu W$ and $F\left(\frac{\pi}{2}\right) = W$. Clearly, $F(\theta_{min})$ is minimal.
Alternative: We take the second derivative (using the quotient rule)

$$F''(\theta) = \frac{\mu W \left(2(\sin\theta - \mu\cos\theta)^2(\mu\sin\theta + \cos\theta) + (\mu\sin\theta + \cos\theta)^3\right)}{(\mu\sin\theta + \cos\theta)^4}.$$

Since $\sin\theta_{min} - \mu\cos\theta_{min} = 0$,

$$\begin{aligned} F''(\theta_{min}) &= \frac{\mu W(\mu\sin\theta_{min} + \cos\theta_{min})^3}{(\mu\sin\theta_{min} + \cos\theta_{min})^4} \\ &= \frac{\mu W}{\tan\theta\sin\theta_{min} + \cos\theta_{min}} \\ &= \frac{\mu W\cos\theta_{min}}{\sin^2\theta_{min} + \cos^2\theta_{min}} \\ &= \mu W\cos\theta_{min}. \end{aligned}$$

Since both $\tan\theta_{min} = \mu$ and $\cos\theta_{min}$ are positive (recall that $0 \le \theta_{min} \le \frac{\pi}{2}$) and so is the weight W, we have $F''(\theta_{min}) > 0$ and therefore, $\theta = \theta_{min}$ ($\mu = \tan\theta_{min}$) is a local minimum[1].

References

[Stewart] James Stewart, *Calculus: Concepts and Contexts* (Second Edition), Brooks/Cole, 2001.

[1] We could also simplify further and write $F''(\theta_{min}) = W\sin\theta_{min}$.