LECTURE NOTES ON MULTI-SCALE ANALYSIS
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INTRODUCTION

Typically, one solves **partial differential equations (PDE) by transforming them into ordinary differential equations ODE**. For example, given $u_t + u_{xx} = 0$, we set $u(x,t) = \phi(t)\eta(x)$ and we solve for ϕ and η. The intuition behind this approach is that ordinary differential equations are "easy". Are all ordinary differential equations easy? Well, we know how to solve linear ordinary differential equation, for example, $p(x)y'' + q(x)y' + r(x) = 0$. But what about nonlinear ones such as $p(x)y''y + q(x)(y')^{10} + r(x) = 0$? The answer is that we don't know how to solve these nonlinear equations in general. However, in some cases, a nonlinear equation can be "nearly" linear and we might be interested only in a "coarse" solution. As we shall see, one way to proceed in such cases is to **transform the nonlinear ordinary differential equation into linear partial differentation equations**! The partial differential equations allow us to *zoom in* on the solution.

The concept of mathematical zoom is very powerful for practical applications. Wavelets for example have lead to powerful image compression techniques [4]. Multi-scale methods are routinely used in remote sensing and many other applications [5, 3]. Of course, it is also very useful to solve differential equations [1]. The idea of a mathematical zoom is simple: solve the problem at successively finer scales or respectively, coarser scales.

Often, in solving differential equations, we are only interested in coarse-scale behavior of the solution, but the fine scales affect this behavior so one cannot simply ignore those. Also, sometimes solving for the coarse scale solution and then later for the finer scale solution might be easier. Multi-scale methods approach the same problem on **different scales** simultaneously just like the zoom of a microscope. To do so implies using several variables even if the initial problem only had one variable. As we shall see, this can be quite difficult, but many of the subproblems can be handled easily with some background in PDEs.

1. NOTATION AND REMINDERS

We note the complex conjugate of $z = x + iy \in \mathbb{C}$ by $\bar{z} = x - iy$. By Demoivre's formula, any complex number $z = x + iy$ can be written as $z = re^{i\theta}$ where $r, \theta \in \mathbb{R}$ and $|z| = \sqrt{x^2 + y^2} = r$, $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, $e^{i\theta} - e^{-i\theta} = 2\sin\theta$ and $e^{i\theta} = \cos\theta + i\sin\theta$.

Linear differential equation have the form

$$a_n(t)\frac{d^n y(t)}{dt^n} + \ldots + a_1(t)\frac{dy(t)}{dt} + a_0(t)y(t) = 0$$

and equations that are not of this form are said to be nonlinear.

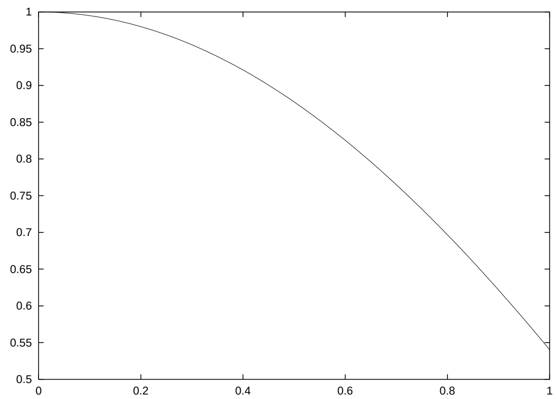

FIGURE 1. $y_0(t) = \cos(t)$

Recall that the binomial theorem states that

$$(x+a)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k a^{n-k}.$$

2. A FIRST EXAMPLE (THIS PART MAY NOT MAKE IT IN THE LECTURE)

The intuition in what follows is that sometimes, an equation can be very close to a known equation which we know how to solve, except that it is slightly different. We cannot ignore the difference per se, however, we can attempt to solve the problem in steps. First, we approximate the answer by simplifying the problem enough so that we know how to solve it, then we solve for the error we are making with our simple solution and so on.

Let's consider the nonlinear oscillator equation

$$\frac{d^2y}{dt^2} + y + \varepsilon y^3 = 0,\ y(0) = 1, y'(0) = 0,$$

which most of us don't know how to solve. It could be shown that the solution $y(t)$ of this equation is bounded for all t by $\sqrt{1+\varepsilon/2}$. While you don't know how to solve it, this equation should remind you of another that you know well

$$\frac{d^2y_0}{dt^2} + y_0 = 0,\ y_0(0) = 1, y_0'(0) = 0.$$

Recall that we can solve this equation by setting $y_0(t) = e^{\lambda t}$ so that we get the characteristic equation $\lambda^2 + 1 = 0$ which has solution $\lambda \in \{-i, i\}$. Therefore, we have the general solution $y_0(t) = A\cos(t) + B\sin(t)$ where $A, B \in \mathbb{R}$ must be chosen according to the boundary conditions. We have $1 = y_0(0) = A$ and $0 = y_0'(0) = B$ and thus the unique solution is $y_0(t) = \cos(t)$ (see Fig. 1).

We claim that this is an approximate solution to the nonlinear oscillator equation. To see how close it is, let's substitute y_0 into it...

$$\frac{d^2y_0}{dt^2} + y_0 + \varepsilon y_0^3 = \varepsilon y_0^3 = \varepsilon \cos^3(t),$$

so that, as long as ε is small, it seems we have an approximation to the real thing in that $\varepsilon\cos^3(t) \approx 0$. In fact, if $\varepsilon = 0$ then our solution is exact.

If y_0 is almost a solution, then maybe we can write the solution as $y_0 + something$ where this "something" is a small perturbation. How small should the perturbation be? How about εsmall? Let's write

$$y(t) \approx y_0(t) + \varepsilon y_1(t).$$

We substitute this formula into the full nonlinear oscillator equation to get, after cancellations,

$$\varepsilon\cos^3(t) + \varepsilon\frac{d^2y_1}{dt^2} + \varepsilon y_1 + \varepsilon^3 y_1^3 \approx 0$$

and dividing by ε throughout, we get

$$\cos^3(t) + \frac{d^2y_1}{dt^2} + y_1 + \varepsilon^2 y_1^3 \approx 0.$$

Looking carefully at this equation, we see that we've almost come full circle. Indeed, we are back to having an equation we can solve assuming we drop one term ($\varepsilon^2 y_1^3$). Let's do it, let's drop $\varepsilon^2 y_1^3$... we set

$$\cos^3(t) + \frac{d^2y_1}{dt^2} + y_1 = 0. \tag{1}$$

What about the boundary conditions? Well, because $y_0(0) = 1, y_0'(0) = 0$ and $y(0) = 1, y'(0) = 0$, it makes sense to set $y_1(0) = 0, y_1'(0) = 0$. How do we solve equation 1? This could prove quite lengthy unless you use the identity $\cos^3(t) = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t$. So that the equation becomes

$$\frac{1}{4}\cos 3t + \frac{3}{4}\cos t + \frac{d^2y_1}{dt^2} + y_1 = 0. \tag{2}$$

You should recall that we find the solution by substituting

$$y_1(t) = A\cos(t) + B\sin(t) + C\cos(3t) + Dt\cos t + Et\sin t$$

where $A\cos(t) + B\sin(t)$ is the homogeneous solution whereas $C\cos(3t) + Dt\cos t$ is to be the inhomogeneous solution used to match $\frac{1}{4}\cos 3t + \frac{3}{4}\cos t$. We first check the boundary conditions, $0 = y_1(0) = A + C$ and $0 = y_1'(0) = B + D$. Hence, we can write

$$y_1(t) = A(\cos(t) - \cos(3t)) + B(\sin(t) - t\cos t) + Et\sin t.$$

Substituting this solution into equation 2 gives

$$\frac{1}{4}\cos 3t + \frac{3}{4}\cos t + \frac{d^2y_1}{dt^2} + y_1 = \frac{(32A+1)\cos 3t + 8B\sin(t) + (8E+3)\cos(t)}{4}$$

and setting this equal to zero gives $A = -1/32$, $B = 0$ and $E = -3/8$ so that

$$y_1(t) = \frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t.$$

Putting it all together, we have that

$$y(t) \approx \cos(t) + \varepsilon\left(\frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t\right).$$

Next, we want to see how good a solution $y_0(t) + \varepsilon y_1(t)$ is. Well, remember that we said that we were dropping the $\varepsilon^2 y_1^3$. Thus, we have that our error is now of order ε^2. Remember that $y_0$was accurate up to εy_0^3, thus, we have a more accurate solution for small ε, but we also worked harder.

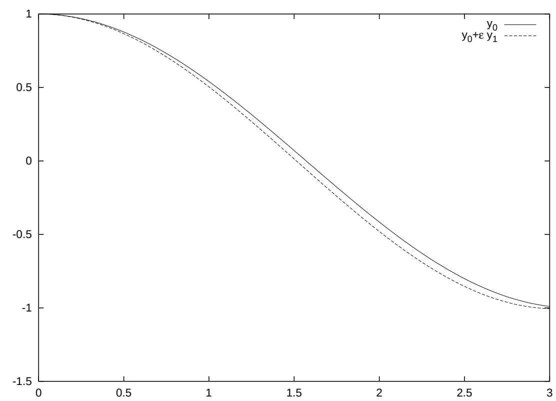

FIGURE 2. $y_0(t) = \cos(t)$ and $y_0(t) + \varepsilon y_1(t) = \cos(t) + \varepsilon\left(\frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t\right)$ with $\varepsilon = 1/10$

We've realized our goal: we've solved a very difficult problem incremental by first getting a coarse solution and then a somewhat finer one... One could prove that if we were to continue this process, we would eventually converge to a solution.

2.1. **Alternative route.** There is a quicker way to solve the problem because we don't need to solve for y_0 first and then y_1 . **First substitute** $y_0(t)+\varepsilon y_1(t)$ **for** $y(t)$ **in the differential equation** and expand out the equation as

$$\begin{aligned} 0 = \frac{d^2y}{dt^2} + y + \varepsilon y^3 \quad &\approx \quad \frac{d^2y_0}{dt^2} + y_0 + \varepsilon y_0^3 + \varepsilon\frac{d^2y_1}{dt^2} + \varepsilon y_1 + \varepsilon^4 y_1^3 \\ &= \quad \approx \frac{d^2y_0}{dt^2} + y_0 + \varepsilon\left(y_0^3 + \frac{d^2y_1}{dt^2} + y_1\right) + \varepsilon^4 y_1^3 \end{aligned}$$

Next collect like powers of ε dropping higher powers to get the equations

$$\frac{d^2y_0}{dt^2} + y_0 = 0$$

and

$$y_0^3 + \frac{d^2y_1}{dt^2} + y_1 = 0.$$

And finally, solve these equations one at a time to get $y_0(t) = \cos(t)$ and $y_1(t) = \frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t$.

Exercise 1. Given $\frac{d^2y}{dt^2} + 2(1+\varepsilon\cos t)y = 0$ solve for y_0 and give the equation for y_1 using the approach above with $y(t) \approx y_0(t) + \varepsilon y_1(t)$. The boundary conditions are not given.

2.2. **Problems?** *We've been hasty so far. We've been collecting terms of like powers, dropping terms at will... are we sure we won't have any problems?*

Well, we may have realized our goal, but we've stumbled on a problem: $|y_0(t)|$ is bounded by 1 whereas $|y_1(t)|$ grows with t so that $\varepsilon^2 y_1^3$ might be much larger than εy_0^3. This means that our solution will only be good enough for t small ($t \ll 1/\varepsilon$) so that given

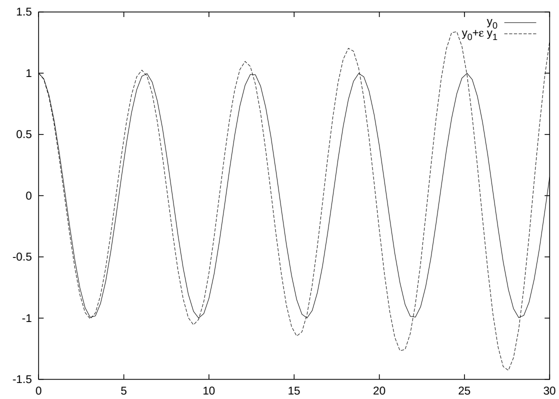

FIGURE 3. $y_0(t) = \cos(t)$ and $y_0(t) + \varepsilon y_1(t) = \cos(t) + \varepsilon\left(\frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t\right)$ for $\varepsilon = 1/10$

$\varepsilon = 1/10$, our solution is only valid for $t \ll 10$. In Fig. 3, we see the effect because $y_0(t)$ and $y_0(t) + \varepsilon y_1(t)$ come apart when t is of order 10 or more.

3. HERE COME THE PDES!

What we did so far certainly works, but it has drawbacks. Can we improve our solution $\cos(t)$ without assuming that t is small?

Now, we'll use a multi-scale analysis!

Remark 1. There are lecture notes available on the web giving a similar review of this problem in different words [2].

Our physical intuition is that the nonlinear oscillator has really two different "time scales": t and $\tau = \varepsilon t$. The new variable, τ defines a long (longer) time scale than t because it takes a large change in t to significantly change τ. Another way to put it is that τ sees the world in slow motion whereas t would be the normal (fast-paced) time. Thus, we want to look at the problem both in slow motion and in normal time... hence the word "multi-scale".

So, instead of writing $y(t) \approx y_0(t) + \varepsilon y_1(t)$, we write $y(t) \approx Y_0(t,\tau) + \varepsilon Y_1(t,\tau)$ because we have two variables. At this point, you might be totally confused... *What does $Y_0(t,\tau)$ mean? Are t and τ* ***independent variables****?*

No, **they are not** independent, $\tau = \varepsilon t$ and so

$$\frac{d\tau}{dt} = \varepsilon.$$

So then, $Y_0(t,\tau)$ really only depends on t, what's the point? Well, actually, you could say that $Y_0(t,\tau)$ depends on both t and ε and that's a valuable idea as we shall see.

Exercise 2. Given $f(t,\tau)$, prove that

$$\frac{d}{dt} f(t,\tau) = \frac{\partial f}{\partial t} + \varepsilon \frac{\partial f}{\partial \tau}$$

and

$$\frac{d^2}{dt^2} f(t,\tau) = \frac{\partial^2 f}{\partial t^2} + 2\varepsilon \frac{\partial^2 f}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2 f}{\partial \tau^2}.$$

We have that for $i = 0, 1$,

$$\frac{dY_i(t,\tau)}{dt} = \frac{\partial Y_i(t,\tau)}{\partial t} + \varepsilon \frac{\partial Y_i(t,\tau)}{\partial \tau}$$

and therefore

$$\begin{aligned}
\frac{dy(t)}{dt} &\approx \frac{dY_0(t,\tau)}{dt} + \varepsilon \frac{dY_1(t,\tau)}{dt} \\
&= \frac{\partial Y_0(t,\tau)}{\partial t} + \varepsilon \frac{\partial Y_0(t,\tau)}{\partial \tau} + \varepsilon \frac{\partial Y_1(t,\tau)}{\partial t} + \varepsilon^2 \frac{\partial Y_1(t,\tau)}{\partial \tau} \\
&= \frac{\partial Y_0(t,\tau)}{\partial t} + \varepsilon \left(\frac{\partial Y_0(t,\tau)}{\partial \tau} + \frac{\partial Y_1(t,\tau)}{\partial t} \right) + \varepsilon^2 \frac{\partial Y_1(t,\tau)}{\partial \tau}
\end{aligned}$$

but the nonlinear oscillator equation has a second derivative, so we need to differentiate again

$$\begin{aligned}
\frac{d^2y(t)}{dt^2} &\approx \frac{d}{dt}\frac{\partial Y_0(t,\tau)}{\partial t} + \varepsilon \left(\frac{d}{dt}\frac{\partial Y_0(t,\tau)}{\partial \tau} + \frac{d}{dt}\frac{\partial Y_1(t,\tau)}{\partial t} \right) + \varepsilon^2 \frac{d}{dt}\frac{\partial Y_1(t,\tau)}{\partial \tau} \\
&= \frac{\partial^2 Y_0(t,\tau)}{\partial^2 t} + \varepsilon \frac{\partial^2 Y_0(t,\tau)}{\partial \tau \partial t} + \varepsilon \left(\frac{\partial^2 Y_0(t,\tau)}{\partial \tau \partial t} + \varepsilon \frac{\partial^2 Y_0(t,\tau)}{\partial \tau^2} + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} + \varepsilon \frac{\partial^2 Y_1(t,\tau)}{\partial \tau \partial t} \right) \\
&\quad + \varepsilon^2 \frac{\partial^2 Y_1(t,\tau)}{\partial t \partial \tau} + \varepsilon^3 \frac{\partial^2 Y_1(t,\tau)}{\partial \tau^2} \\
&= \frac{\partial^2 Y_0(t,\tau)}{\partial^2 t} + \varepsilon \left(2\frac{\partial^2 Y_0(t,\tau)}{\partial \tau \partial t} + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} \right) + \varepsilon^2 \left(\text{some stuff}\right).
\end{aligned}$$

We can now substitute this into the nonlinear oscillation equation to get

$$\begin{aligned}
&\frac{\partial^2 Y_0(t,\tau)}{\partial^2 t} + \varepsilon \left(2\frac{\partial^2 Y_0(t,\tau)}{\partial \tau \partial t} + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} \right) + \varepsilon^2 \left(\text{some stuff}\right) \\
&\qquad + Y_0(t,\tau) + \varepsilon Y_1(t,\tau) + \varepsilon \left(Y_0(t,\tau) + \varepsilon Y_1(t,\tau) \right)^3 \quad \approx \quad 0.
\end{aligned}$$

Next, you can convince yourself that

$$\left(Y_0(t,\tau) + \varepsilon Y_1(t,\tau) \right)^3 = Y_0^3(t,\tau) + \varepsilon \left(\text{something}\right)$$

by using the binomial theorem or brute force. Therefore, substituting this last equation back into the nonlinear oscillation gives us

$$\frac{\partial^2 Y_0(t,\tau)}{\partial^2 t} + \varepsilon \left(2\frac{\partial^2 Y_0(t,\tau)}{\partial \tau \partial t} + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} \right) + Y_0(t,\tau) + \varepsilon Y_1(t,\tau) + \varepsilon Y_0^3(t,\tau) + \varepsilon^2 \left(\text{some stuff}\right) \approx 0.$$

We collect like powers of ε to get

$$\frac{\partial^2 Y_0(t,\tau)}{\partial^2 t} + Y_0(t,\tau) + \varepsilon \left(Y_1(t,\tau) + Y_0^3(t,\tau) + 2\frac{\partial^2 Y_0(t,\tau)}{\partial \tau \partial t} + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} \right) + \varepsilon^2 \left(\text{some stuff}\right) \approx 0$$

and we have the corresponding equations

$$\frac{\partial^2 Y_0(t,\tau)}{\partial^2 t} + Y_0(t,\tau) = 0 \tag{3}$$

and

$$Y_1(t,\tau) + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} + Y_0^3(t,\tau) + 2\frac{\partial^2 Y_0(t,\tau)}{\partial \tau \partial t} = 0. \tag{4}$$

We can solve equation 3 easily by separation of variables to get

$$Y_0(t,\tau) = A(\tau)e^{it} + B(\tau)e^{-it}$$

and because we want $Y_0(t,\tau)$ to be real, we need to require that $Y_0(t,\tau) = \overline{Y}_0(t,\tau)$ and so $A(\tau) = \overline{B}(\tau)$ or equivalently $\overline{A}(\tau) = B(\tau)$. Thus, we write

$$Y_0(t,\tau) = A(\tau)e^{it} + \overline{A}(\tau)e^{-it}.$$

This is not yet a complete solution because we don't know what $A(\tau)$ is. We can then substitute this solution into equation 4 to get

$$Y_1(t,\tau) + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} + \left(A(\tau)e^{it} + \overline{A}(\tau)e^{-it}\right)^3 + 2iA'(\tau)e^{it} - 2i\overline{A}'(\tau)e^{-it} = 0.$$

We are not going to solve this equation as it is clearly too long and difficult! However, we will try to go some insight out of it as to what $A(\tau)$ should be. Again, using the binomial theorem, we see that

$$\left(A(\tau)e^{it} + \overline{A}(\tau)e^{-it}\right)^3 = A^3(\tau)e^{3it} + \overline{A}^3(\tau)e^{-3it} + 3\overline{A}(\tau)A^2(\tau)e^{it} + 3\overline{A}^2(\tau)A(\tau)e^{-it}.$$

Substituting this back into our equation gives

$$\begin{aligned} Y_1(t,\tau) + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} + A^3(\tau)e^{3it} + \overline{A}^3(\tau)e^{-3it} & \\ + \left(3\overline{A}(\tau)A^2(\tau) + 2iA'(\tau)\right)e^{it} + \left(3\overline{A}^2(\tau)A(\tau) - 2i\overline{A}'(\tau)\right)e^{-it} &= 0. \end{aligned}$$

Writing $\alpha(\tau) = 3\overline{A}(\tau)A^2(\tau) + 2iA'(\tau)$, we have

$$Y_1(t,\tau) + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} + A^3(\tau)e^{3it} + \overline{A}^3(\tau)e^{-3it} + \alpha(\tau)e^{it} + \overline{\alpha}(\tau)e^{-it} = 0.$$

If $\alpha(\tau) \neq 0$, then we must solve this differential equation by separation of variables using

$$Y_1(t,\tau) = B_1(\tau)e^{it} + B_2(\tau)e^{-it} + B_3(\tau)e^{3it} + B_4(\tau)e^{-3it} + B_5(\tau)te^{it} + B_6(\tau)te^{-it}.$$

Exercise 3. In the above, why do we need the terms $B_5(\tau)te^{it}$ and $B_6(\tau)te^{-it}$ for $\alpha(\tau) \neq 0$?

Because the terms $B_5(\tau)te^{it} + B_6(\tau)te^{-it}$ grow large for t large and we want to avoid this[1], we choose $\alpha(\tau) = 0$ or

$$3\overline{A}(\tau)A^2(\tau) + 2iA'(\tau) = 0.$$

Write $A(\tau) = R(\tau)e^{i\omega(\tau)}$ where both R and ω are real functions so that

$$A'(\tau) = R'(\tau)e^{i\omega(\tau)} + R(\tau)i\omega'(\tau)e^{i\omega(\tau)}.$$

Therefore, $\alpha(\tau)$ is given by

$$3R^3(\tau)e^{i\omega(\tau)} + 2iR'(\tau)e^{i\omega(\tau)} - 2R(\tau)\omega'(\tau)e^{i\omega(\tau)} = 0$$

or

$$3R^3(\tau) + 2iR'(\tau) - 2R(\tau)\omega'(\tau) = 0.$$

Because $z = x + iy = 0$ implies $x = 0$ and $y = 0$ that if a complex number is zero, both its real and imaginary parts must be zero as well, we have two equations

$$R'(\tau) = 0$$

and

$$3R^3(\tau) - 2R(\tau)\omega'(\tau) = 0.$$

[1] Avoiding large terms (also called secular terms) is the key ingredient in this recipe.

The first differential equation tells use that R must be a constant and let us assume $R \neq 0$, the second one becomes a simple equation $3R^2 - 2\omega'(\tau) = 0$ with $\omega(\tau) = \frac{3R^2}{2}\tau + K$ where K is some constant. Therefore $A(\tau) = Re^{i\left(\frac{3R^2}{2}\tau + K\right)}$ and while we haven't solved for Y_1, we now have Y_0! Indeed, we have

$$Y_0(t,\tau) = Re^{i\left(\frac{3R^2}{2}\tau + K + t\right)} + Re^{-i\left(\frac{3R^2}{2}\tau + K + t\right)}$$

or

$$Y_0(t,\tau) = 2R\cos\left(\frac{3R^2}{2}\tau + K + t\right).$$

All that is left to do is to enforce the boundary conditions

$$Y_0(0,0) = 2R\cos(K) = 1$$

and

$$0 = \frac{dY_0(0,0)}{dt} = \frac{\partial Y_0(0,0)}{\partial t} + \varepsilon\frac{\partial Y_0(0,0)}{\partial \tau} = -2R\sin(K) - 3R^3\varepsilon\sin(K).$$

The last equation gives us $-2 - 3R^2 = 0$ or $\sin(K) = 0$ hence we must choose $K = 0$ and thus, the other equation gives us $R = 1/2$. We then have

$$y(t) \approx Y_0(t,\tau) = \cos\left(\frac{3}{8}\tau + t\right)$$

or

$$y(t) \approx \cos\left(\left(\frac{3\varepsilon}{8} + 1\right)t\right)$$

because $\tau = \varepsilon t$. Essentially, our new solution is identical to the previous one $\cos(t)$ for small t's but because of a phase difference, they eventually differ quite a bit(see Fig. 4). As the next exercise shows, $\cos\left(\left(\frac{3\varepsilon}{8} + 1\right)t\right)$ is much better than $\cos(t)$ as a solution.

Exercise 4. Using $y(t) = \cos\left(\left(\frac{3\varepsilon}{8} + 1\right)t\right)$ show that

$$\frac{d^2y}{dt^2} + y + \varepsilon y^3 = \varepsilon\cos^3\left(\left(\frac{3\varepsilon}{8} + 1\right)t\right) - \frac{3}{4}\varepsilon\cos\left(\left(\frac{3\varepsilon}{8} + 1\right)t\right) + \varepsilon^2(\text{something})$$

and

$$\left|\cos^3\left(\left(\frac{3\varepsilon}{8} + 1\right)t\right) - \frac{3}{4}\varepsilon\cos\left(\left(\frac{3\varepsilon}{8} + 1\right)t\right)\right| \leq 1/4.$$

Compare with the result we got with $y(t) = \cos(t)$, that is

$$\frac{d^2y}{dt^2} + y + \varepsilon y^3 = \varepsilon\cos^3(t).$$

To summarize the multi-scale method: substitute $Y_0(t,\tau) + \varepsilon Y_1(t,\tau)$ for $y(t)$ in the differential equation, expand and collect the powers of ε and solve the first PDE, and then substitute your (partial) solution into the second PDE, then requiring that there be no secular terms, finish solving for the first order (Y_0) term.

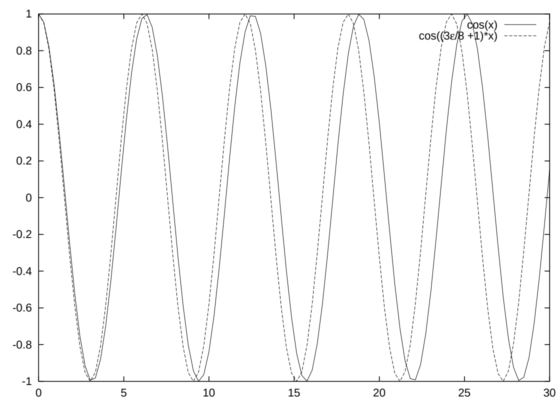

FIGURE 4. Comparison between the first order solution $y_0(t) = \cos(t)$ and the multi-scale solution $y(t) \approx \cos\left(\left(\frac{3\varepsilon}{8}+1\right)t\right)$ for $\varepsilon = 1/10$

4. PROBLEMS

In the following problems, use the formulas

$$y(t) \approx Y_0(t,\tau) + \varepsilon Y_1(t,\tau),$$

$$\frac{dy}{dt} \approx \frac{\partial Y_0(t,\tau)}{\partial t} + \varepsilon\left(\frac{\partial Y_0(t,\tau)}{\partial \tau} + \frac{\partial Y_1(t,\tau)}{\partial t}\right) + \varepsilon^2 \frac{\partial Y_1(t,\tau)}{\partial \tau},$$

and

$$\frac{d^2y}{dt^2} \approx \frac{\partial^2 Y_0(t,\tau)}{\partial^2 t} + \varepsilon\left(2\frac{\partial^2 Y_0(t,\tau)}{\partial\tau\partial t} + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2}\right).$$

Exercise 5. Solve $\frac{d^2y}{dt^2} + y + \varepsilon\left(\frac{dy}{dt}\right)^2 = 0$, $y(0) = 0, y'(0) = 1$, using a multiscale approach.

Solution: Keeping only the first two powers of ε, we have

$$\frac{\partial^2 Y_0(t,\tau)}{\partial^2 t} + \varepsilon\left(2\frac{\partial^2 Y_0(t,\tau)}{\partial\tau\partial t} + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2}\right) + Y_0 + \varepsilon Y_1 + \varepsilon\left(\frac{\partial Y_0}{\partial t}\right)^2 \approx 0.$$

Collecting the like powers of ε, we have first $\frac{\partial^2 Y_0(t,\tau)}{\partial^2 t} + Y_0 = 0$ and then $2\frac{\partial^2 Y_0(t,\tau)}{\partial\tau\partial t} + \frac{\partial^2 Y_1(t,\tau)}{\partial t^2} + Y_1 + \left(\frac{\partial Y_0}{\partial t}\right)^2 = 0$. We solve the first PDE to get $Y_0 = A(\tau)e^{it} + \overline{A}(\tau)e^{-it}$. Hence, the second differential equation becomes

$$\frac{\partial^2 Y_1}{\partial t^2} + Y_1 + 2iA'(\tau)e^{it} - 2i\overline{A}'(\tau)e^{-it} - A^2(\tau)e^{2it} - \overline{A}^2(\tau)e^{-2it} + 2\left|A(\tau)\right|^2 = 0.$$

To avoid the secular terms (terms in the solution of Y_1 growing big), we need $A'(\tau) = 0$ or $A(\tau) = Re^{i\eta}$ and thus, $Y_0 = Re^{i(t+\eta)} + \overline{R}e^{-i(t+\eta)} = 2C\cos(t+\eta) = 2C\cos(t+\eta)$ where $C = \frac{R+\overline{R}}{2}$ and η is a constant. Because $y(0) = 0, y'(0) = 1$, we set $\eta = \pi/2$ and $C = 1/2$ so that $Y_0 = \sin(t)$ is our solution.

Exercise 6. (Damped Oscillator) Given $y'' + y + \varepsilon(y')^3 = 0$, find the leading order term (Y_0) using multi-scale analysis.

Exercise 7. Given $y'' + y + \varepsilon y' y^2 = 0$, find the leading order term (Y_0) using multi-scale analysis.

Exercise 8. Given $y'' + y + \varepsilon y'^2 y = 0$, find the leading order term (Y_0) using multi-scale analysis.

Solution: Keeping only leading terms in ε, we have

$$\frac{\partial^2 Y_0}{\partial^2 t} + Y_0 + \varepsilon \left(Y_1 + 2\frac{\partial^2 Y_0}{\partial\tau\partial t} + \frac{\partial^2 Y_1}{\partial t^2} + \frac{\partial Y_0}{\partial t}^2 Y_0 \right) = 0.$$

Substituting $Y_0 = A(\tau)e^{it} + \overline{A}(\tau)e^{-it}$ into

$$Y_1 + \frac{\partial^2 Y_1}{\partial t^2} + 2\frac{\partial^2 Y_0}{\partial\tau\partial t} + \frac{\partial Y_0}{\partial t}^2 Y_0 = 0,$$

we have

$$Y_1 + \frac{\partial^2 Y_1}{\partial t^2} + 2(A'(\tau)ie^{it} - i\overline{A}'(\tau)e^{-it}) + (iA(\tau)e^{it} - i\overline{A}(\tau)e^{-it})^2(A(\tau)e^{it} + \overline{A}(\tau)e^{-it}) = 0.$$

Hence, we should set $2iA' + A^2\overline{A} = 0$ (why?). Write $A = Re^{i\omega}$ where both ω and R are real functions of τ, then $2iR' - 2\omega' R + R^3 = 0$ so that $R' = 0$ (R is a constant!) and $\omega' = R^2/2$ or $\omega(\tau) = R^2\tau/2 + K$. We have $A = Re^{iR^2\tau/2+iK}$ and therefore, $Y_0 = Re^{iR^2\tau/2+it+iK} + Re^{-iR^2\tau/2-it-iK} = 2R\cos(R^2\tau/2 + t + K)$. The parameters R and K would need to be set using initial conditions.

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