# Acadia University

Department of Mathematics and Statistics

# INTRODUCTORY CALCULUS 1
# (MATH 1013)

# SECTION 4.5 Solutions

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31st October 2001

## 4 Indeterminate Forms and L'Hospital Rule

### 4.5 Derivatives and the Shapes of Curves

8. **[1 marks]** Since $\lim_{x\to 0}\sin x = 0$ and $\lim_{x\to 0} x+\tan x = 0$, l'Hospital's rule apply

$$\lim_{x\to 0}\frac{x+\tan x}{\sin x} = \lim_{x\to 0}\frac{1+\sec^2 x}{\cos x} = \frac{\lim_{x\to 0} 1+\sec^2 x}{\lim_{x\to 0}\cos x} = \frac{2}{1} = 2.$$

10. **[1 marks]** Since $\lim_{x\to\pi} x = \pi$, l'Hospital's rule doesn't apply

$$\lim_{x\to\pi}\frac{\tan x}{x} = \frac{\lim_{x\to\pi}\tan x}{\lim_{x\to\pi} x} = \frac{0}{\pi} = 0.$$

14. **[2 marks]** Since $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^3 = \infty$, l'Hospital's rule apply

$$\lim_{x\to\infty}\frac{e^x}{x^3} = \lim_{x\to\infty}\frac{e^x}{3x^2}.$$

At this point, we still have a problem since $\lim_{x\to\infty} 3x^2 = \infty$, so we need to apply l'Hospital's rule

$$\lim_{x\to\infty}\frac{e^x}{3x^2} = \lim_{x\to\infty}\frac{e^x}{6x}.$$

Again, we need to apply l'Hospital's rule since $\lim_{x\to\infty} 6x = \infty$,

$$\lim_{x\to\infty}\frac{e^x}{6x} = \lim_{x\to\infty}\frac{e^x}{6}.$$

$\frac{1}{6}$ is a constant and so

$$\lim_{x\to\infty}\frac{e^x}{6} = \frac{1}{6}\lim_{x\to\infty} e^x = \infty.$$

24. **[2 marks]** We have $\lim_{x\to(\frac{\pi}{2})^-}\cos 3x = \cos\frac{3\pi}{2} = 0$ and $\lim_{x\to(\frac{\pi}{2})^-}\cos 7x = \cos\frac{7\pi}{2} = 0$, so l'Hospital's rule apply

$$\begin{aligned}\lim_{x\to(\frac{\pi}{2})^-}\sec 7x\cos 3x &= \lim_{x\to(\frac{\pi}{2})^-}\frac{\cos 3x}{\cos 7x}\\ &= \lim_{x\to(\frac{\pi}{2})^-}\frac{3\sin 3x}{7\sin 7x}.\end{aligned}$$

And we have $\lim_{x\to(\frac{\pi}{2})^-}\sin 3x = \sin\frac{3\pi}{2} = -1$ and $\lim_{x\to(\frac{\pi}{2})^-}\sin 7x = \sin\frac{7\pi}{2} = -1$ so l'Hospital's rule no longer applies

$$\lim_{x\to(\frac{\pi}{2})^-}\frac{3\sin 3x}{7\sin 7x} = \frac{\lim_{x\to(\frac{\pi}{2})^-}3\sin 3x}{\lim_{x\to(\frac{\pi}{2})^-}7\sin 7x} = \frac{-3}{-7} = \frac{3}{7}.$$

34. **[2 marks]** We are going to assume $a \neq 0$ since in such a case, the limit is equal to 1. We have

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty}e^{bx\ln\left(1+\frac{a}{x}\right)}. \tag{1}$$

Looking only at the exponent

$$\lim_{x\to\infty}bx\ln\left(1+\frac{a}{x}\right) = \lim_{x\to\infty}b\frac{\ln\left(1+\frac{a}{x}\right)}{1/x} = b\lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{1/x}. \tag{2}$$

We can see that $\lim_{x\to\infty}\ln\left(1+\frac{a}{x}\right) = \ln(1) = 0$ and $\lim_{x\to\infty}1/x = 0$, so l'Hospital's rule applies

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{1/x} &= \lim_{x\to\infty}\frac{\frac{\frac{-a}{x^2}}{1+\frac{a}{x}}}{-1/x^2}\\ &= \lim_{x\to\infty}\frac{-a}{x^2\left(1+\frac{a}{x}\right)\left(\frac{-1}{x^2}\right)}\\ &= \lim_{x\to\infty}\frac{a}{1+\frac{a}{x}}\\ &= \frac{a}{1+\lim_{x\to\infty}\frac{a}{x}} = a.\end{aligned}$$

Going back to equation 2

$$\lim_{x\to\infty}bx\ln\left(1+\frac{a}{x}\right) = ab.$$

Therefore[1], by equation 1,

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}.$$

52. **[1 marks]** Since $\lim_{x\to\infty}\ln x = \infty$ and $\lim_{x\to\infty}x^p = \infty$, l'Hospital's rule apply. Therefore

$$\lim_{x\to\infty}\frac{\ln x}{x^p} = \lim_{x\to\infty}\frac{1/x}{px^{p-1}} = \lim_{x\to\infty}\frac{1}{px^p} = p\lim_{x\to\infty}\frac{1}{x^p} = 0$$

assuming $p > 0$.

[1]Compare this next equation with equation 6 on page 3 in [Stewart, section 3.7].

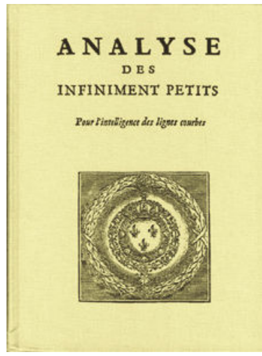


Figure 1: Cover page of “L’analyse des infiniment petits du Marquis de l’Hospital” (1696) where L’Hospital’s Rule first appeared.

## References

[Stewart] James Stewart, *Calculus: Concepts and Contexts* (Second Edition), Brooks/Cole, 2001.