Assignment 1
MATH 3423 - Numerical Methods 2
Acadia University

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1 Requirements

You can do this assignment in teams of two or alone. 5% of the mark will be based on presentation. All required plots should be computer-generated. A title page is required. All computer code must be properly commented and should be computationally efficient. You may use the language you prefer (Matlab, C, Fortran, Java, C++...) as long as you document your work.

2 Numerical Differentiation

2.1 Basic applications

1. For the following problems, approximate the specified derivative.

   (a) Using the forward-difference formula.

   (b) Using the backward-difference formula.

   (c) Using the central-difference formula.

   (d) Using Richardson extrapolation to improve (once) your central-difference result.

   Question 1. Approximate $y'(1.0)$ if $x = [0.8\,0.9\,1.0\,1.1\,1.2]$ and $y = [0.992\,0.999\,1.000\,1.001\,1.008]$.

   Question 2. Approximate $y'(1)$ if $x = [-1\,0\,1\,2\,3]$ and $y = [1/3\,1\,3\,9\,27]$.

2. The altitude of a helicopter at three different instants is found to be $h_1 = 445.98$ at $t_1 = 0.20$, $h_2 = 471.85$ at $t_2 = 0.30$, and $h_3 = 503.46$ at $t_3 = 0.41$. Taylor expanding twice the altitude $h(t)$ of the helicopter about $t = 0.30$, find dh/dt at $t = 0.30$ using all of the available data. (HINT: do not try to plug in a formula... work out the math...)

2.2 Theory

1. Observe that the sum of all the coefficients of the functions values appearing in the numerator of all finite-difference derivatives seen in class is 0. Give at least 4 examples to justify this claim. Prove that it must always be so.

2. Give the order of accuracy ($O(\Delta x)$, $O((\Delta x)^2)$, ...) for the following formulas. Briefly justify your answer (do not guess!).

   (a) $f_i' = \frac{f_{i+1}-f_i}{\Delta x}$

   (b) $f_i' = \frac{f_{i+1}-f_{i-1}}{2\Delta x}$

(c) $f_i'' = \frac{f_{i+2}-2f_{i+1}+f_i}{(\Delta x)^2}$

(d) $f_i'' = \frac{f_{i+1}-2f_i+f_{i-1}}{\Delta x^2}$

(e) $f_i''' = \frac{f_{i+3}-3f_{i+2}-3f_{i+1}-f_i}{\Delta x^3}$

(f) $f_i' = \frac{f_{i+1}-f_i}{2\Delta x}$

3. Recall that $\|f-g\|_{L^\infty(S)} = \max\{|f(x)-g(x)|, x \in S\}$. Give the formulas for two families of real-valued functions f, g with a parameter a (for example $\sin ax$) over the real numbers such that $\|f-g\|_{L^\infty}$ can be made as small as we want (by a change of parameter) while $\|f'-g'\|_{L^\infty} \geq 1$. Suppose now that you have a sampling of two functions f_i, g_i for some x values (x_i), use your example to show that while the values f_i, g_i might be very close, and thus their numerical derivatives, their actual derivatives might be very different.

2.3 Problem

You are working for NASA. The American government has spotted an UFO and you must process the collected data. Write a computer program to estimate the speed as a function of time of the object and its acceleration (one real number per time sample for both speed and acceleration). You must do this work for accuracy orders $O(\Delta x), O((\Delta x)^2)$, and $O((\Delta x)^4)$. You must hand in the computer program you wrote, a brief explanation of the formulas you used and two plots for each order of accuracy: speed vs time and acceleration vs time.

Data (in format $[x, y, z, t]$) : [0.84, 1.71, 1.0], [4.0, 1.81, 5.38, 2.0], [9.0, 0.42, 17.08, 3.0], [16.0, - 3.02, 50.59, 4.0], [25.0, - 4.79, 143.41, 5.0], [36.0, - 1.67, 397.42, 6.0], [49.0, 4.59, 1089.63, 7.0], [64.0, 7.91, 2972.95, 8.0], [81.0, 3.70, 8094.08, 9.0], [100.0, - 5.44, 22016.46, 10.0], [121.0, - 10.99, 59863.14, 11.0], [144.0, - 6.43, 162742.79, 12.0], [169.0, 5.46, 442400.39, 13.0], [196.0, 13.86, 1202590.28, 14.0], [225.0, 9.75, 3269002.37, 15.0],[256.0, - 4.60, 8886094.52, 16.0].

3 Numerical Integration

3.1 Basic Applications

1. Use Gaussian quadrature with $n = 3$ and exact arithmetic to approximate $\int_{-1}^{1} x^4 dx$. Compare your result with the expected value of the integral and discuss the two.

2. Evaluate the integral $\int_0^{2\pi} \cos^2 x dx$ using the following methods with 6 function evaluations (give all of your computations):

   (a) Trapezoidal rule.

   (b) Simpson's rule (1/3)

   (c) Gauss quadrature formula

3.2 Theory

1. Show that the integral given by the trapezoidal rule is the average of the integrals given by the two rectangular rules.

2. Use the definition of the Legendre polynomial $P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}[(1-x^2)^n]$, $n \geq 1$ to find a relationship between $P_n'(x)$ and $P_{n+1}(x)$, and then show the equivalence of the following expressions for the coefficients for Gaussian quadrature: $c_i = \frac{-2}{(n+1)P_n'(x_i)P_{n+1}(x_i)}$ and $c_i = \frac{2(1-x_i^2)}{(n+1)^2 P_{n+1}^2(x_i)}$.
   HINTS:

   (a) The x_i's are the roots of P_n and thus $P_n(x_i) = 0$, however, $P_n'(x_i)$ is not zero in general, nor is $P_{n+1}(x_i)$. (Why?)

(b) You might want to use Leibnitz' rule which says that

$$\frac{d^n}{dx^n}uv = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \frac{d^k}{dx^k} u \frac{d^{n-k}}{dx^{n-k}} u.$$

You don't have to prove Leibnitz' rule, but you should be clever enough to know how to prove it if your life depended on it!

3. Suppose that the quadrature rule $\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$ is exact for all constant functions. What does this imply about the weights w_i or the nodes x_i?

3.3 Problem

Assuming that f is twice differentiable, prove that $\int_a^b f(x)dx = (b-a)f\left(\frac{b-a}{2}\right) + \frac{(b-a)^3}{24} f''(\xi)$ for some $\xi \in [a,b]$. The formula $\int_a^b f(x)dx \cong (b-a)f\left(\frac{b-a}{2}\right)$ is called the Midpoint formula. Write a computer program to implement the Midpoint formula and use it to integrate $\sin x$ between 0 and π using as many function evaluations as you need to properly estimate the integral starting with $2,4,8,16,32,64\ldots$ function evaluations. You must hand in the code of your computer program and a table with your results (function evaluations vs value). Plot your table. Briefly comment on your results (what is the apparent rate of convergence...what do you expect from the theory...).