Acadia University

Department of Mathematics and Statistics

INTRODUCTORY CALCULUS 1
(MATH 1013)

ASSIGNMENT 1 Solutions

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Contents

2 Limits and Derivatives

2.1 The Tangent and Velocity Problems

2. It is always useful to begin by plotting the points we are given. You can do it very easily using Maple[1] (see Fig. 1).

   (a) We want to compute the slope of the secant line passing through $(36, 2530)$ and $(42, 2948)$. Recall that the slope of the line passing through (x_1, y_1) and (x_1, y_1) is given my

   $$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

   In the present case, we therefore have

   $$m = \frac{2530 - 2948}{36 - 42} \cong 69.7.$$

[1] Many different plotting tools can be used including Mathematica, gnuplot, Matlab, Mathcad, EasyCalc (for Palm Pilot or Visor users). While you'll learn to use Maple in this course, you can use any tool that suits your needs. Some of them are free (gnuplot and EasyCalc for example).

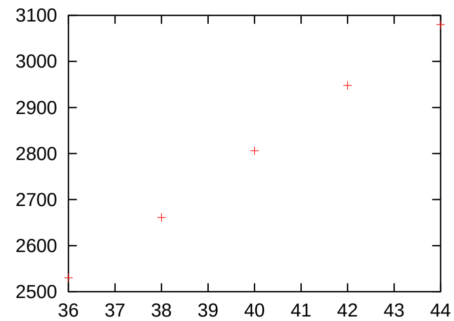

Figure 1: Heartbeats vs minutes

(b) We have to compute the slope of the secant line passing through $(38,2661)$ and $(42,2948)$ which is

$$m = \frac{2661-2948}{38-42} = 71.75.$$

(c) We have to compute the slope of the secant line passing through $(40,2806)$ and $(42,2948)$ which is

$$m = \frac{2806-2948}{40-42} = 71.$$

(d) We have to compute the slope of the secant line passing through $(42,2948)$ and $(44,3080)$ which is

$$m = \frac{2948-3080}{42-44} = 66.$$

If we want to approximate the slope of the tangent line at 42, we can look at the two closest secant lines (c and d) and expect the slope of the tangent line to be somewhere between 66 and 71. We can use the average of the two values $\frac{66+71}{2} \cong 68.5$ as a good estimation for the heart rate in beats per minute at time $t = 72$.

4. (a) We have have to compute the slope of the line passing through $(2,\ln 2)$ and $(x,\ln x)$ for some values of x. The general formula for the slope is going to be

$$m = \frac{\ln 2 - \ln x}{2-x}.$$

Note that while the question in Steward require 6 decimal places values for the slope, we give only 3 decimal places below.

i. For $x = 1.5$, we have

$$m = \frac{\ln 2 - \ln 1.5}{2-1.5} \cong 0.575.$$

ii. For $x = 0.9$, we have

$$m = \frac{\ln 2 - \ln 0.9}{2-0.9} \cong 0.726.$$

iii. For $x = 1.99$, we have

$$m = \frac{\ln 2 - \ln 1.99}{2-1.99} \cong 0.501.$$

iv. For $x = 1.999$, we have

$$m = \frac{\ln 2 - \ln 1.999}{2 - 1.999} \cong 0.5.$$

v. For $x = 2.5$, we have

$$m = \frac{\ln 2 - \ln 2.5}{2 - 2.5} \cong 0.446.$$

vi. For $x = 2.1$, we have

$$m = \frac{\ln 2 - \ln 2.1}{2 - 2.1} \cong 0.488.$$

vii. For $x = 2.01$, we have

$$m = \frac{\ln 2 - \ln 2.01}{2 - 2.01} \cong 0.499.$$

viii. For $x = 2.001$, we have

$$m = \frac{\ln 2 - \ln 1.5}{2 - 1.5} \cong 0.5.$$

(b) To guess the value of the slope, we look at the secant lines we get with $x = 1.999$ and $x = 2.001$, they both have a slope of ≈ 0.5 and therefore, $m = 0.5$ is our best guess.

(c) The tangent line will have a slope of 0.5 and therefore, its equation is going to be

$$y = 0.5x + b \tag{1}$$

and it remains to solve for b. We know that the tangent line must pass through $(2, \ln 2)$ and so we substitute these values in equation 1. Solving for b, we get

$$b = \ln 2 - 0.5 \times 2 = \ln 2 - 1 \cong -0.306.$$

We therefore have the equation

$$y = 0.5x - \ln 2 + 1.$$

(d) Before sketching, we may compute the actual equations for the second lines. Choosing $x = 1.5$ and $x = 2.5$ from part a, we have to solve for b in the following equations

$$0.575 \times 2 + b = \ln 2$$

and

$$0.446 \times 2 + b = \ln 2.$$

It is, of course, very easy. We get $b \cong -0.457$ and $b \cong -0.198$ respectively. For sketch, see Fig. 2

6. We plot the height as a function in time (see Fig. 3).

(a) Computing the average velocity is very similar to computing the slope of a secant line as we shall see.

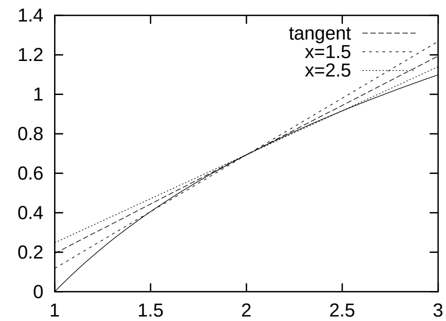

Figure 2: The tangent line and two secant lines at $x = 2$ for the curve $y = \ln x$.

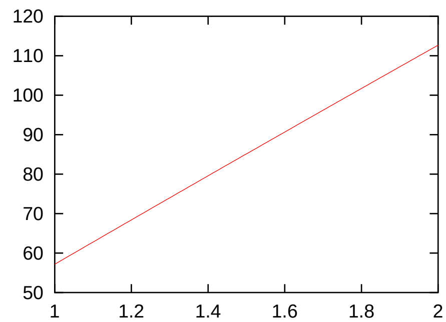

Figure 3: $h = 58t - 0.83t^2$

i. The average velocity is given by

$$\frac{\left(58\times 1 - 0.83\times 1^2\right) - \left(58\times 2 - 0.83\times 2^2\right)}{1-2} = 55.51.$$

ii. The average velocity is given by

$$\frac{\left(58\times 1 - 0.83\times 1^2\right) - \left(58\times 1.5 - 0.83\times 1.5^2\right)}{1-1.5} = 55.925.$$

iii. The average velocity is given by

$$\frac{\left(58\times 1 - 0.83\times 1^2\right) - \left(58\times 1.1 - 0.83\times 1.1^2\right)}{1-1.1} = 56.257.$$

iv. The average velocity is given by

$$\frac{\left(58\times 1 - 0.83\times 1^2\right) - \left(58\times 2 - 0.83\times 1.01^2\right)}{1-1.01} = 56.3317.$$

v. The average velocity is given by

$$\frac{\left(58\times 1 - 0.83\times 1^2\right) - \left(58\times 1.001 - 0.83\times 1.001^2\right)}{1-1.001} = 56.33917.$$

(b) We can estimate from the previous results that the velocity at $t = 1$ is going to be 56.34.

2.2 The Limit of a Function

4. (a) We have a continuous function at $x = 0$ and the limit from both sides will tend to 3.

   (b) If we come from the left, the limit will tend to 4 even though it might not agree with the limit from the right or the value of the function at $x = 3$.

   (c) This time, the limit tends to 2 even though it doesn't agree with the limit from the left and the value of the function at $x = 3$.

   (d) From our previous comments, the limits from the left and from the right don't agree so that the limit doesn't exist (see [Stewart, section 2.2, equation 3])

   (e) Even though, it might seem strange, the value of $f(3)$ is simply 3.

6. We sketch the graph of the function f using our favorite software package (in this case, we are using gnuplot, see Fig. 4). For the graph, it is clear that $\lim_{x\to-1} f(x)$ and $\lim_{x\to1} f(x)$ do not exist. Other than than, all limits will exist so that the set we are looking for is $\mathbb{R} - \{-1,1\}$.

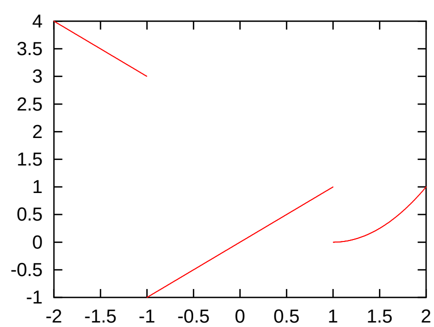

Figure 4: The discontinuous function $f(x)$.

16. (a) We plot $\frac{6^x-2^x}{x}$ near $x = 0$ using our favorite plotting software (see Fig. 5). Clearly we have that the limit must be closed to 1.10.

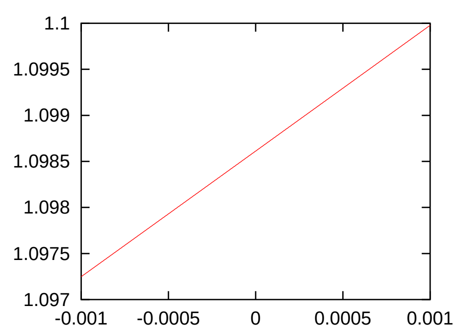

Figure 5: The function $\frac{6^x-2^x}{x}$ near $x = 0$.

   (b) Evaluating $\frac{6^x-2^x}{x}$ at $x = 0.0001$, we get ≈ 1.098 and at $x = -0.0001$, we get ≈ 1.098 and therefore 1.10 is clearly a good choice.

18. (a) We use our calculator to get

$$\frac{\tan 1 - 1}{1^3} \cong 0.5574,$$
$$\frac{\tan 0.5 - 0.5}{0.5^3} \cong 0.3704,$$
$$\frac{\tan 0.1 - 0.1}{0.1^3} \cong 0.3347,$$
$$\frac{\tan 0.05 - 0.05}{0.05^3} \cong 0.3337,$$
$$\frac{\tan 0.01 - 0.01}{0.01^3} \cong 0.3333,$$
$$\frac{\tan 0.005 - 0.005}{0.005^3} \cong 0.3333.$$

(b) Since values tend to go to $1/3$ when x goes down to 0, we must assume that the limit will tend to 0.

(c) Using a calculator, we get 0 for values around 1×10^{-8}[2]. Looking closely at the problem, we see that $\tan x - x$ has value 0 on the calculator for $x = 1 \times 10^{-8}$, whereas $x^3 = 1 \times 10^{-24}$. Why do we get $\tan x - x = 0$ instead of $\sim x^3/3 \cong 3 \times 10^{-25}$? First of all, what is $\tan x$ when $x = 1 \times 10^{-8}$? Simply 1×10^{-8}, of course, which explain why we get $\tan x - x = 0$ in the first place! We might expect $\tan x = 1 \times 10^{-8} + 3 \times 10^{-25}$ instead... but wait! Can your calculator manage a number like $1 \times 10^{-8} + 3 \times 10^{-25}$. Of course not! For most calculator, 3×10^{-25} is just too small with respect to 1×10^{-8} and it will just be discarded. Therefore, we are still confident about the value of our limit being $1/3$ and that the zero values come from numerical error.

(d) We use our favorite plotting package... See Fig. 6 and 7. We can see on Fig. 7 how the graphing tool has problems with values very close to 0 since the curve doesn't seem to be as smooth as one would expect and it eventually goes to 0. Notice that scientists often get around these problems by changing the units they are using.

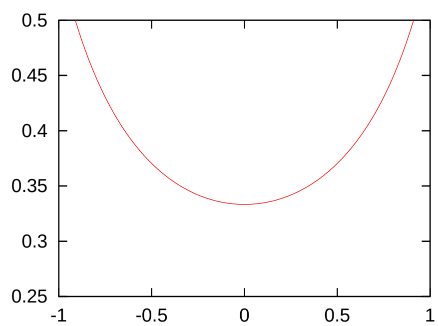

Figure 6: The function $\frac{\tan x - x}{x^3}$.

[2] Smarter calculators may still print a non-zero value, but no matter what, you'll eventually get small enough values for your calculator to print 0. Just try smaller values for x.

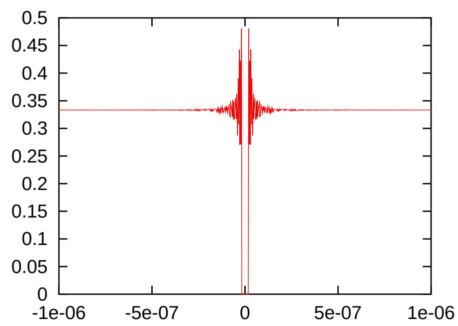

Figure 7: The function $\frac{\tan x - x}{x^3}$ very near $x = 0$. Notice how we stress the floating point lower limit of our graphing tool as x tends to 0.

2.3 Calculating Limits Using the Limit Laws

6. We have that

$$
\begin{aligned}
\lim_{u \to -2} \sqrt{u^4 + 3u + 6} &= \sqrt{\lim_{u \to -2} u^4 + 3u + 6} && (2) \\
&= \sqrt{\lim_{u \to -2} u^4 + \lim_{u \to -2} 3u + \lim_{u \to -2} 6} && (3) \\
&= \sqrt{\left(\lim_{u \to -2} u\right)^4 + 3 \lim_{u \to -2} u + \lim_{u \to -2} 6} && (4) \\
&= \sqrt{(-2)^4 + 3 \times -2 + 6} && (5) \\
&= \sqrt{16} = 4 && (6)
\end{aligned}
$$

by using respectively the Root Law (equation 2), Law 1 (equation 3), the Power Law and Law 3 (equation 4) and Laws 7 and 8 (equation 5).

8. (a) It is not true when $x = 2$ because the right hand side isn't defined (division by 0).

   (b) The limit of a function at $x = 2$ doesn't depend on the value of the function at $x = 2$ itself, but only on the values nearby. Since the equality presented in question a is true everywhere but at $x = 2$, this new equation is correct.

10. We have that $\lim_{x \to -4} x^2 + 3x - 4 = 0$ and therefore, we cannot use the quotient rule. Notice however that

$$\frac{x^2 + 5x + 4}{x^2 + 3x - 4} = \frac{(x+4)(x+1)}{(x+4)(x-1)}$$

and so, for $x \neq -4$, we can consider

$$f(x) = \frac{(x+1)}{(x-1)} \qquad (7)$$

instead. Now, it is much easier because we can use the quotient rule.

$$\begin{aligned}
\lim_{x\to-4}\frac{(x+1)}{(x-1)} &= \frac{\lim_{x\to-4}(x+1)}{\lim_{x\to-4}(x-1)} \\
&= \frac{\lim_{x\to-4}x+\lim_{x\to-4}1}{\lim_{x\to-4}x-\lim_{x\to-4}1} \\
&= \frac{-4+1}{-4-1}=\frac{3}{5}.
\end{aligned}$$

12. We must first simplify the function for $x \neq 1$,

$$\frac{x^3-1}{x^2-1}=\frac{(x-1)\left(x^2+x+1\right)}{(x-1)(x+1)}=\frac{x^2+x+1}{x+1}$$

and now, we can use the Quotient Law,

$$\begin{aligned}
\lim_{x\to1}\frac{x^3-1}{x^2-1} &= \lim_{x\to1}\frac{x^2+x+1}{x+1} \\
&= \frac{\lim_{x\to1}x^2+x+1}{\lim_{x\to1}x+1} \\
&= \frac{3}{2}.
\end{aligned}$$

26. We use the Squeeze Theorem (see [Stewart, Theorem 3 in section 2.3]). We have that

$$\lim_{x\to1}3x \le \lim_{x\to1}f(x) \le \lim_{x\to1}x^3+2$$

and therefore

$$3 \le \lim_{x\to1}f(x) \le 3$$

which implies

$$\lim_{x\to1}f(x)=3.$$

30. On the one hand, we have

$$\lim_{x\to2^-}\frac{|x-2|}{x-2}=\lim_{x\to2^-}\frac{2-x}{x-2}=-1$$

and on the other, we have

$$\lim_{x\to2^+}\frac{|x-2|}{x-2}=\lim_{x\to2^+}\frac{x-2}{x-2}=1.$$

Clearly, the limit doesn't exist!

References

[Stewart] James Stewart, *Calculus: Concepts and Contexts* (Second Edition), Brooks/Cole, 2001.