Advanced Calculus (MATH 2023) - MidTerm
February 27th 2002

NAME: ____________________________
Please Print

Id. No.: ____________________________

Note: You are allowed to one sheet (both sides) with handwritten notes and a calculator. You have 50 minutes to write the test.

1. A vector field is given by $\vec{F}(x,y,z) = x^2y\vec{i} + \left(\frac{x^3}{3} + x\right)\vec{j} + \vec{k}$.

   (a) [10 marks] Compute $\bigtriangledown \cdot \vec{F}$ (the divergence). Show your work.
   Answer:$\bigtriangledown \cdot \vec{F} = 2xy$

   (b) [10 marks] Compute $\bigtriangledown \times \vec{F}$ (the curl). Show your work.
   Answer:$\bigtriangledown \times \vec{F} = \vec{k}$

   (c) [20 marks] Compute the line integral $\oint_C \vec{F} \cdot d\vec{r}$ where C is the unit circle centered at 0 on the plane ($x^2 + y^2 = 1$, $z = 0$) using Stoke's theorem. State any assumption you make in integrating.
   Answer: $\oint_C \vec{F} \cdot d\vec{r} = \int_C \bigtriangledown \times \vec{F} dA = \int_C dA = \pi$ assuming counter-clockwise integration.

2. Series and sequences

   (a) [10 marks] Give an example of a monotone sequence that does not converge.
   Answer: 1,2,3,4...

   (b) [10 marks] Does the series $\sum_{k=1}^{\infty} \frac{e^k}{(k^2)!}$converges? Does it converge absolutely? Prove your result.
   Answer: Since $\lim_{k\to\infty} \left| \frac{e^{k+1}}{(k+1)^2!} \frac{k^2!}{e^k} \right| = \lim_{k\to\infty} \left| \frac{e}{(k+1)^2 \ldots (k^2+1)} \right| = 0$ it converges by the ratio test. Since we used the ratio test, it must converge absolutely.

   (c) [10 marks] Show that the series $\sum_{k=1}^{\infty} \frac{(-1)^2}{\sqrt{k}}$converges.
   Answer: It is an alternating series. By AST it must converge since $1/\sqrt{k}$ decreases.

   (d) [10 marks] A convergent series is absolutely convergent. Prove or give a counter-example.
   Answer: Counter-example $1, -1/2, 1/3, -1/4, \ldots$ is $\ln 2$ whereas $1, 1/2, 1/3, 1/4, \ldots$ doesn't converge.

   (e) [10 marks] Let $S_n = \sum_{k=1}^{n} \frac{(-1)^2}{\sqrt{k}}$and $L = \sum_{k=1}^{\infty} \frac{(-1)^2}{\sqrt{k}}$, find N such that $|S_N - L| < \frac{1}{3}$.
   Answer: Since it is an alternating series, we have $|S_N - L| < \frac{1}{\sqrt{N+1}}$ and hence $N = 8$ is a good choice.