Acadia University

Department of Mathematics and Statistics

# INTRODUCTORY CALCULUS 1
# (MATH 1013)

# SECTION 4.3 Solutions

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# 4 Applications of Differentiation

## 4.3 Derivatives and the Shapes of Curves

6. **[4 marks]**

   (a) $f$ is increasing on the intervals $(2,4)$, $(6,9]$ because it is where $f' > 0$ (see [Stewart, Section 4.3, Increasing/Decreasing test, p. 282]).

   (b) Given that $f'$ exists everywhere, local max. or local min. will only happen when $f' = 0$ or on the boundary. Firstly, at $x = 0$, we have a local maximum by the ***Second Derivative Test*** ([Stewart, Section 4.3, p. 284]), $f''(0) < 0$ and $f'(0) = 0$. At $x = 2$, we have a local minimum since $f'(2) = 0$ and $f''(2) > 0$. At $x = 4$, we have a local maximum since $f''(4) < 0$ and $f'(4) = 0$. At $x = 6$, we have a local minimum since $f''(6) < 0$ and $f'(6) = 0$. Finally, at $x = 9$, we have a local maximum since $f'(9) > 0$ and it is a boundary point. (We assume that $f''$ is continuous everywhere.)

   (c) $f$ is concave upward when $f'' > 0$ and concave downward when $f'' < 0$ (see [Stewart, Section 4.3, Concavity Test, p. 284]). Looking at the graph of $f'$, we see that $f'$ is increasing (and therefore $f'' > 0$) on the following intervals $(1,3)$, $(5,7)$, and $(8,9]$. It is where $f$ is concave upward. Also, $f'$ is decreasing on the intervals $[0,1)$, $(3,5)$, and $(7,8)$. It is where $f$ is concave downward.

   (d) "A point where a curve changes its direction of concavity is called an inflection point." [Stewart, Section 4.3, p.284] We see that this happens at $x = 3$, $x = 5$, and $x = 7$ according to the intervals of concavity in part (c).

10. **[3 marks]**

(a) Using the quotient rule, derivative is found to be

$$f'(x) = \frac{1-x}{(1+x)^3}.$$

We can now consider two cases separately. Firstly, if $x < -1$, then $1-x > 0$ and $(1+x)^3 < 0$ so that $f'(x) < 0$ and therefore $f$ is decreasing over $(\infty, -1)$. On the other hand, when $x > -1$, we have $(1+x)^3 > 0$ and we have two sub-cases, when $-1 < x < 1$, then $1-x > 0$ and when $x > 1$, $1-x < 0$. Therefore, we can conclude that $f$ is increasing in the interval $(-1,1)$ and decreasing over $(1,\infty)$.

(b) Local min. and max. can only occur when $f' = 0 \Rightarrow x = 1$ of when $f \nexists$ but this only happens at $x = -1$ and this value is not in the domain of $f$. At $x = 1$, the derivative goes from being positive to negative and therefore we have a local maximum by the ***First Derivative Test*** (taking note that $f$ is continuous near $x = 1$).

(c) Again using the quotient rule, we find the second derivative

$$f''(x) = \frac{2x-4}{(x+1)^4}.$$

Since $(x+1)^4$ is always positive, $2x-4$ determines the sign. The second derivative is positive when $x > 2$ (concave upward) and negative when $x < 2$ (concave downward). The inflection point is therefore at $x = 2$.

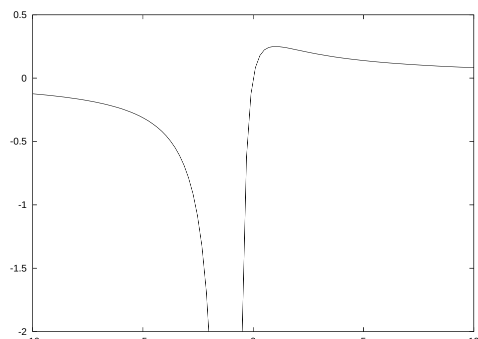


Figure 1: $\frac{x}{(1+x)^2}$

16. **[3 marks]** Since $f$ is a polynomial, $f$, $f'$ and $f''$ are continuous.

(a) Since the derivative exists everywhere, we only need to solve for $f'(x) = 0$ to find the critical numbers. We can use the product rule to evaluate the derivative

$$\begin{aligned} f'(x) &= 3(x-1)^2x^4 + 4(x-1)^3x^3 \\ &= x^3\left(7x^3 - 18x^2 + 15x - 4\right). \end{aligned}$$

We can actually factor $f'(x)$ further (by using Maple for example):

$$f'(x) = x^3(x-1)^2(7x-4).$$

The critical numbers are therefore $x = 0$, $x = 1$, and $x = 4/7$.

(b) We first need to evaluate the second derivative. It is most convenient to do so when we use the form $f'(x) = 7x^6 - 18x^5 + 15x^4 - 4x^3$

$$f''(x) = 42x^5 - 90x^4 + 60x^3 - 12x^2.$$

We can then evaluate $f''$ at the critical numbers and notice that $f''$ is a continuous function. Firstly, $f''(0) = 0$ so the ***Second Derivative Test*** is inconclusive. We also have $f''(1) = 0$ and the ***Second Derivative Test*** is again inconclusive. Finally, we have $f''(4/7) = 576/2401 > 0$ and so $f$ has a local minimum at $4/7$.

(c) At $x = 0$, the sign of the derivative will be go from being positive to being negative since both $(0-1)^2 > 1$ and $(7\times 0 - 4) < 0$ . This tells us that we have a local maximum at $x = 0$ by the ***First Derivative Test***. At $x = 1$, the sign will not change (the sign will remain positive), and therefore, according to the ***First Derivative Test***, $f$ has no local maximum and no local minimum at $x = 1$. At $x = 4/7$, the sign goes from being negative to being positive since $(4/7)^3 > 0$ and $(4/7-1)^2 > 0$ and so, we have a local minimum.

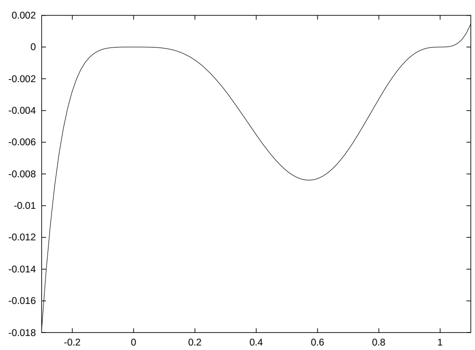


Figure 2: $x^4(x-1)^3$

42. **[2 marks]** First of all, assuming $f(2)$ is a maximum, we need $f'(2) = 0$ since $f$ is differentiable. Using the product rule and then the chain rule, we have

$$f'(x) = (2bx^2 + 1)ae^{bx^2}.$$

Setting this equal to 0, we get $2bx^2 + 1 = 0$ or $x^2 = \frac{-1}{2b}$. Since we want $x = 2$ as a solution of $f'(x) = 0$, we need $x^2 = 4 = \frac{-1}{2b}$ and so, $b = -1/8$. With this choice, we'll have $f'(2) = 0$. We need to make sure that $f(2) = 1 \Rightarrow 2ae^{-1/2} = 1$ and so $a = \sqrt{e}/2$. Since $f'(x)$ shows only two critical numbers ($-2$ and 2), and since $\lim_{x\to\pm\infty} f(x) = 0$, $f(2) = 1$, and $f(-2) = -1$, we can safely conclude that $x = 2$, $f(2) = 1$ is the maximum of $f$.

48. **[2 marks]** In 10 minutes or $1/6$ hours, the car went from 30 mi/h to 50 mi/h. The average acceleration was

$$\frac{50\,\text{mi/h} - 30\,\text{mi/h}}{1/6\,\text{h}} = 20 \times 6\,\text{mi/h}^2 = 120\,\text{mi/h}^2.$$

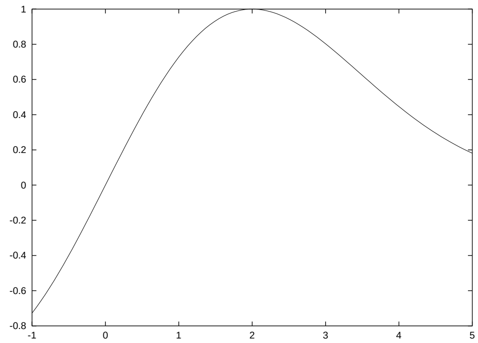


Figure 3: $\frac{\sqrt{e}}{2}e^{\frac{-x^2}{8}}$

Taking $f$ to be the velocity of the car and $f'$ to be the acceleration (assume $f$ to be differentiable), then the ***Mean Value Theorem*** [Stewart, Section 4.3, p. 281] says that at some time $t$ between $2:00$ and $2:10$, $f'$ must be equal to its average or, in other words, $f'(t) = 120\,\mathrm{mi/h}^2$ .

# References

[Stewart] James Stewart, *Calculus: Concepts and Contexts* (Second Edition), Brooks/Cole, 2001.