A FAMILY OF 4-POINT DYADIC HIGH RESOLUTION SUBDIVISION SCHEMES

1. NTRODUCTION

Interpolatory subdivision schemes interpolate a discrete set of data points in a local manner, that is, the value of the interpolation function at a given point depends on a small number of nearby data points. The classical dyadic algorithm introduced by Deslauriers and Dubuc finds the midpoint values by fitting a Lagrange polynomial through the $2N$ closest data points. By repeating this algorithm again and again, each time doubling the number of data points or nodes, we eventually have a dense set of data points and we can determine uniquely a smooth interpolation function.

This type of approach is very convenient to implement in software and the fact that subdivision schemes are strictly local makes them robust. They also inherit their order of approximation from the corresponding Lagrange interpolation scheme. Because interpolatory subdivision schemes relate data points from one scale to the data points at another scale, it is not surprising that they are a key ingredient in the construction of compactly supported wavelets .

Since Deslauriers-Dubuc schemes have important applications, it is tempting to add extra nodes to Deslauriers and Dubuc schemes as an attempt to improve them to get “high resolution” schemes. One might hope to retain the useful properties such as the approximation order and the regularity while making the scheme more local for example. Doubling the number of nodes however is costly (effectively doubling the memory requirements), however, since a dyadic subdivision scheme doubles its memory usage at each step, we can choose to use right away this upcoming extra storage space without any cost. In effect, we can simply make use of the memory we will allocate later in any case. Therefore, we can double (or more) the number of nodes. These new placeholders can then be used to record a first coarse scale guess (using a tetradic filter) which we can later combine with a finer scale interpolation (using a dyadic filter) . As a special case, we may choose to ignore the coarse scale estimate, in which case our approach amounts to a Deslauriers-Dubuc scheme; we can also use this approach to reproduce polynomials of degree 4 by a Richardson extrapolation approach. This paper shows that by summing up the tetradic (coarse) interpolation recorded in placeholders and dyadic (fine) interpolations, we get a range of smooth (C^1) high resolutions schemes reproducing cubic polynomials.

In a general $b-$adic, we propose to extend subdivision schemes of the form $y_{j+1,l} = \sum_{k\in\mathbb{N}} \gamma_{bk-l} y_{j,k}$ by

$$y_{j+1,l} = \sum_{m=1}^{N} \sum_{k\in\mathbb{N}} \gamma^{(m)}_{Nbk+m-1-l} y_{j,Nk+m-1}.$$

In the Fourier domain, this amounts to replacing . $P^{j+1}(z) = \Gamma(z) P^j(z^b)$ by $P^{j+1}(z) = \sum_{i=1}^{M} \Gamma_i(z) P^j\left(e^{2\pi i/b} z^b\right)$.

2. SUBDIVISION SCHEMES

Interpolatory subdivision schemes where first introduced by Deslauriers and Dubuc (quote). Let $b>1$ be an integer, given two integers k, j, the number $x_{j,k} = k/b^j$ is said to be $b-$adic (of depth j). For a fixed j, the $b-$adic numbers form a regularly-spaced set of nodes. Given some corresponding data $\left\{y_{J,k}\right\}_{k\in\mathbb{N}}$ on the dyadic numbers of depth J, we want to build a smooth fonction f such that $f(x_{J,k}) = y_{J,k} \forall k \in \mathbb{N}$. Starting with this initial data $(y_{J,k})$ and using the linear formula

$$y_{j+1,l} = \sum_{k\in\mathbb{N}} \gamma_{bk-l} y_{j,k} \tag{2.1}$$

for some constant array γ, we get values $y_{J,k}$ for any $j>J$ and since $b-$adic numbers form a dense set of the real numbers, there is at most one continuous function such that $f\left(x_{j,k}\right) = y_{j,k}$ for all $k, j > J$.

FIGURE 1.1. Derivatives of 3 fundamental functions for 3 different $4-$point cubic high resolution interpolatory subdivision schemes. The dot-dash curve is the special case where we get the Deslauriers-Dubuc scheme.

A subdivision scheme is interpolatory and will satisfy $f(x_{J,k}) = y_{J,k}$ if $\gamma_{bk} = 0$ except when $k = 0$. We say that a subdivision scheme is stationary if the array γ is constant (doesn't depend on j). An interpolatory subdivision scheme is said to be $2N-$point if $\gamma_l = 0$ for $|l| > Nb$. The interpolation function f computed from a $2N-$point $b-$adic scheme with initial data $y_{0,0} = 1$ and $y_{0,k} = 0$ for all $k \neq 0$ is said to be the fundamental function and has a compact support of $[-(Nb-1)/(b-1), (Nb-1)/(b-1)]$ or $[1-2N, 2N-1]$ when $b = 2$. Hence as N increases the support of the fundamental function increases.

For each $N = 1, 2, 3, \ldots$ there exists a corresponding an interpolatory Deslauriers-Dubuc subdivision scheme and they are built from the midpoint evaluation of Lagrange polynomial of degree $2N-1$ on $2N$ points. For $b = 2$ (dyadic case), the $4-$point Deslauriers-Dubuc scheme can be defined by the array γ^{DD2} given by $\gamma_0^{DD2} = 1, \gamma_1^{DD2} = \gamma_{-1}^{DD2} = -9/16, \gamma_3^{DD2} = \gamma_{-3}^{DD2} = -1/16$ with $\gamma_k^{DD2} = 0$ otherwise; for $b = 4$ (tetradic case), the scheme is defined by γ^{DD4} given by $\gamma_{2k}^{DD4} = \gamma_k^{DD2} \forall k \in \mathbb{N}$, $\gamma_{-1}^{DD4} = \gamma_1^{DD4} = 105/128, \gamma_{-3}^{DD4} = \gamma_{-3}^{DD4} = 35/128, \gamma_{-5}^{DD4} = \gamma_5^{DD4} = -7/128, \gamma_{-7}^{DD4} = \gamma_7^{DD4} = -5/128$, with $\gamma_k^{DD2} = 0$ otherwise. Because $4-$point Deslauriers-Dubuc schemes are derived from cubic Lagrange polynomials, they reproduces cubic polynomials, that is, if the initial data $y_{j,k}$ satisfies $y_{j,k} = p(x_{j,k}) \forall k \in \mathbb{N}$ for some cubic polynomial p then the interpolation function f is this same cubic polynomial $f = p$. Hence the two cases presented above (γ^{DD2} and γ^{DD4}) reproduce cubic polynomials. It can also be shown also that they both give differentiable (C^1) interpolation functions.

We are interested in measuring how well a given subdivsion scheme can approximation functions. One such measure is given by the approximation order of the scheme[8, definition 2]. We say that a subdivision scheme has approximation order p if given given any smooth function $g \in C^p([0,1])$, the interpolation function f satisfying $f(x_{j,k}) = g(x_{j,k}) \forall k \in \mathbb{N}$ is such that $\|f-g\|_{L^\infty([0,1])} \leq C/2^{jp}$ for a constant C independent of j.

For a continuous subdivision scheme reproducing polynomials of degree p, it is sufficient for the scheme to converge to a continous function to have approximation order $p+1$ [8]. Specifically, this means that $4-$point Deslauriers-Dubuc schemes have approximation order 4.

3. High resolution subdivsion schemes

3.1. **Definitions.** An more general alternative to equation 2.1 is given by the linear equation

$$y_{j+1,l} = \sum_{m=1}^{N} \sum_{k \in \mathbb{N}} \gamma_{Nbk+m-1-l}^{(m)} y_{j,Nk+m-1}$$

where $\gamma^{(1)}, \ldots, \gamma^{(N)}$ are constant arrays (independent from j). Because this new formula uses N times the usual number of nodes (see equation 2.1), we say that it is a "high resolution subdivision scheme" however it can still be said to be $b-$adic because the number of nodes is increasing by b. As a special case, when $N = 1$, we have a usual subdivision scheme.

In what follows, we set $b = N = 2$ and consider the equation

$$y_{j+1,l} = \sum_{k \in \mathbb{N}} \gamma_{4k-l}^{(1)} y_{j,2k} + \gamma_{4k+1-l}^{(2)} y_{j,2k+1} \tag{3.1}$$

with

$$\gamma_k^{(1)} = \gamma_k^{DD4} - \alpha \frac{\left((-1)^k + 1\right)}{2} \gamma_{k/2}^{DD2} + \alpha \delta_{k,0} \forall k \in \mathbb{N} \tag{3.2}$$

$$\gamma_0^{(2)} = \alpha, \gamma_k^{(2)} = 0\, otherwise \tag{3.3}$$

for some parameter $\alpha \in \mathbb{R}$. As we will see, this choice is made so that the scheme can reproduce cubic polynomials for all α. The odd terms $\{y_{j,2k+1}\}_{k \in \mathbb{N}}$ will be referred as placeholders because their assigned value will change in general. In the simplest case, $\alpha = 0$, equation 3.1becomes

$$y_{j+1,l} = \sum_{k \in \mathbb{N}} \gamma_{4k-l}^{DD4} y_{j,2k}. \tag{3.4}$$

Because $\gamma^{(2)} = 0$ in this case, we can see that the placeholders are effectively ignored. Indeed, we observe that this last equation discards odd terms at each step: $y_{j+1,l}$ depends only on terms of the form $y_{j,2k}$ (even terms) and not at all on the odd terms $y_{j,2k+1}$. Hence, we can replace equation 3.4 by

$$y_{j+1,2l} = \sum_{k \in \mathbb{N}} \gamma_{4k-2l}^{DD4} y_{j,2k}$$

but because $\gamma_{2k}^{DD4}=\gamma_{k}^{DD2}$, this last equation becomes $y_{j+1,2l}=\gamma_{2k-l}^{DD2}y_{j,2k}$ and if we define $\widetilde{y}_{j,k}=y_{j,2k}$ then

$$\widetilde{y}_{j+1,2l}=\sum_{k\in\mathbb{N}}\gamma_{2k-l}^{DD2}\widetilde{y}_{j,2k} \tag{3.5}$$

which we recognize as the cubic Deslauriers-Dubuc scheme.

Proposition 3.1. *For $\alpha=0$, the high resolution scheme given by equations 3.1, 3.2, and 3.3 is equivalent to the $4-$point dyadic Deslauriers-Dubuc subdivision scheme (discarding the odd nodes or placeholders in the first iteration).*

In general, since $\gamma_{2k}^{DD4}=\gamma_{k}^{DD2}$, we can rewrite equation 3.1 for even and odd terms. Firstly, setting $l=2s$ (l even), we have

$$\begin{aligned} y_{j+1,2s} &= \textstyle\sum_{k\in\mathbb{N}} \quad \gamma_{4k-2s}^{(1)}y_{j,2k}+\gamma_{4k+1-2s}^{(2)}y_{j,2k+1}\\ &= \textstyle\sum_{k\in\mathbb{N}} \quad \left(\gamma_{4k-2s}^{DD4}-\alpha\gamma_{2k-s}^{DD2}+\alpha\delta_{4k,2s}\right)y_{j,2k}+\alpha\delta_{4k+1,s}y_{j,2k+1}\\ &= \textstyle\sum_{k\in\mathbb{N}} \quad \left((1-\alpha)\gamma_{2k-s}^{DD2}+\alpha\delta_{2k,s}\right)y_{j,2k}+\delta_{2k+1,s+1}\alpha y_{j,2k+1}\end{aligned}$$

so that when s is even ($l=2s=4r$), we have the interpolatory condition

$$y_{j+1,4r}=y_{j,2r} \tag{3.6}$$

otherwise, when s is odd $(l=2s=4r+2)$

$$y_{j+1,4r+2}=\alpha y_{j,2r+1}+(1-\alpha)\sum_{k\in\mathbb{N}}\gamma_{2k-2r-1}^{DD2}y_{j,2k}. \tag{3.7}$$

Secondly, if l is odd ($l=2s+1$), we have

$$y_{j+1,2s+1} = \sum_{k\in\mathbb{N}}\gamma_{4k-2s-1}^{(1)}y_{j,2k}+\gamma_{4k-2s-1}^{(2)}y_{j,2k+1} \tag{3.8}$$

$$= \sum_{k\in\mathbb{N}}\gamma_{4k-2s-1}^{DD4}y_{j,2k}. \tag{3.9}$$

Equations 3.6, 3.7, and 3.9 can be used to describe the chosen high resolution schemes: while qquation 3.6 is the interpolatory condition, equation 3.9 fills the placeholders with tetradic (coarse scale) interpolated values whereas equation 3.7 combines the value stored in the placeholder with the newly available interpolated value (fine scale) given by the summation term which we recognize from the dyadic Deslauriers-Dubuc interpolation.

3.2. **Reproduced polynomials.** Assume that for some j, $y_{j,k}=p_3\left(x_{j,k}\right)$ for some cubic polynomial p_3. We then have that

$$\sum_{k\in\mathbb{N}}\gamma_{2k-2r-1}^{DD2}y_{j,2k}=y_{j,2r+1}=p_3\left(x_{j,2r+1}\right)$$

and thus equation 3.7 becomes $y_{j+1,4r+2}=p_3\left(x_{j,2r+1}\right)$. Moreover, equation 3.9 implies $y_{j+1,2s+1}=p_3\left(x_{j+1,2s+1}\right)$. We can conclude that $y_{j+1,k}=p_3\left(x_{j+1,k}\right)$ if $y_{j,k}=p_3\left(x_{j,k}\right)$. For practical implementations of a high subdivision scheme, it is necessary to first apply a one-step subdivision scheme since in general, we don't have placeholder values precomputed. This can be achieved with a dyadic Deslauriers-Dubuc filter. Let $\left\{y_{j,k}\right\}_k$ be some initial data. As a first step, we apply equation

$$y_{j+1,2l}=\sum_{k\in\mathbb{N}}\gamma_{2k-l}^{DD2}y_{j,2k} \tag{3.10}$$

followed by equation 3.1 with $j=j+1$, and so on. By induction, we get the following lemma.

Lemma 3.2. *The high resolution scheme given by equations 3.1, 3.2, and 3.3 (using 3.10 as an initialization step) reproduces cubic polynomials.*

We can also get a stronger result by choosing α. Let $p_4(x)=a_4x^4+p_3(x)$ where p_3 is some cubic polynomial. Suppose that for some j, $y_{j,2k}=p_4\left(x_{j,2k}\right)$ (and accordingly $y_{j-1,2k}=p_4\left(x_{j-1,2k}\right)$). Substituting equation 3.9 into 3.7, we get

$$y_{j+1,4r+2} = \alpha y_{j,2r+1}+(1-\alpha)\sum_{k\in\mathbb{N}}\gamma_{2k-2r-1}^{DD2}y_{j,2k} \tag{3.11}$$

$$= \alpha\sum_{k\in\mathbb{N}}\gamma_{4k-2r-1}^{DD4}y_{j-1,2k}+(1-\alpha)\sum_{k\in\mathbb{N}}\gamma_{2k-2r-1}^{DD2}y_{j,2k}. \tag{3.12}$$

We can compute each of the sums explicitely

$$\begin{aligned}\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-2r-1}y_{j,2k} &= p_3\left(x_{j,2r+1}\right)+a_4\sum_{k\in\mathbb{N}}\gamma^{DD2}_{2k-2r-1}\left(x_{j,2k}\right)^4\\ &= p_4\left(x_{j,2r+1}\right)-\frac{9a_4}{16\times 2^{4j}}\end{aligned}$$

and

$$\begin{aligned}\sum_{k\in\mathbb{N}}\gamma^{DD4}_{4k-2r-1}y_{j-1,2k} &= p_3\left(x_{j,2r+1}\right)+a_4\sum_{k\in\mathbb{N}}\gamma^{DD4}_{4k-2s-1}\left(x_{j-1,2k}\right)^4\\ &= p_4\left(x_{j,2r+1}\right)-\frac{105a_4}{16\times 2^{4j}}.\end{aligned}$$

Hence, by substituting these two results (fine and coarse scales interpolation) and setting $\alpha=-3/35$, we get

$$y_{j+1,4r+2}=p_4\left(x_{j,2r+1}\right)=p_4\left(x_{j+1,4r+2}\right)$$

since for $\alpha=-3/35$, $105\alpha+9\alpha=0$. Therefore, the scheme reproduces polynomials of degree 4. Of course, this result assumes that we initialize the data so that $y_{j,2k+1}=p_4\left(x_{j,2k+1}\right)-\frac{105a_4}{16\times 2^{4j}}$ and $y_{j,2k}=p_4\left(x_{j,2k}\right)$ for all $k\in\mathbb{N}$. We can get this result naturally by having $y_{j-1,k}=p_4\left(x_{j,k}\right)$ and applying first the high resolution scheme (equation 3.1) with $\alpha=1$ so that equations 3.6 and 3.7 will guarantee $y_{j,2k}=p_4\left(x_{j,k}\right)$ whereas equation 3.9 will initialize the placeholders properly.

Proposition 3.3. *For any given j, if $y_{j-1,k}=p_4\left(x_{j-1,k}\right)$ where p_4 is a polynomial of degree 4 then applying the high resolution scheme given by equations 3.1, 3.2, and 3.3 first with $\alpha=1$ as an initialization step and then with $\alpha=-3/35$ will guarantee that $y_{j',2k}=p_4\left(x_{j',2k}\right)$ for $\forall k\in\mathbb{N}$ and all $j'\geq j-1$.*

3.3. **Sufficient conditions for regularity.** To study the regularity of high resolution schemes, it is convenient rewrite formula 3.1 in terms of (trigonometric) polynomials. Given some data $y_{j,k}$, define $P^j(z)=\sum_{k\in\mathbb{N}}y_{j,k}z^k$. If $P_2(z)=\sum_{k\in\mathbb{N}}\gamma^{DD2}_k z^k$, then equation 3.10, can be rewritten $P^{j+1}(z)=P_2(z)P^j(z^2)$. Similarly, if $P_4(z)=\sum_{k\in\mathbb{N}}\gamma^{DD4}_k z^k$, then the tetradic subdivision scheme is given by $P^{j+1}(z)=P_4(z)P^j(z^2)$. It can be shown that we can rewrite the general equation for high resolution subdivision schemes as

$$P^{j+1}(z)=\sum_{i=1}^{M}\Gamma_i(z)P^j\left(e^{2\pi i/b}z^b\right).$$

where Γ_i must be Laurent polynomials. Using symbols, equation 3.1 becomes

$$\begin{aligned}P^{j+1}(z) &= \left\{P_4(z)-\alpha P_2\left(z^2\right)+\alpha\right\}\left(\frac{P^j\left(z^2\right)+P^j\left(-z^2\right)}{2}\right)+\alpha\left(\frac{P^j\left(z^2\right)-P^j\left(-z^2\right)}{2}\right) && (3.13)\\ &= \left\{\frac{P_4(z)-\alpha P_2\left(z^2\right)}{2}+\alpha\right\}P^j\left(z^2\right)+\frac{P_4(z)-\alpha P_2\left(z^2\right)}{2}P^j\left(-z^2\right) && (3.14)\\ &= \Gamma_1(z)P^j\left(z^2\right)+\Gamma_2(z)P^j\left(-z^2\right) && (3.15)\end{aligned}$$

Because $\gamma^{DD4}_{2k}=\gamma^{DD2}_k\,\forall k\in\mathbb{N}$, we observe that

$$P_4(z)-\alpha P_2\left(z^2\right)=\frac{P_4(z)-P_4(-z)}{2}+(1-\alpha)\frac{P_4(z)+P_4(-z)}{2}$$

and thus

$$\begin{aligned}\Gamma_1(z) &= \Gamma_2(z)+\alpha\\ \Gamma_2(z) &= \frac{P_4(z)-P_4(-z)}{4}+(1-\alpha)\frac{P_4(z)+P_4(-z)}{4}.\end{aligned}$$

When $\alpha=0$, $\Gamma_1(z)=\Gamma_2(z)=\frac{P_4(z)}{2}$ and equation 3.1 becomes

$$P^{j+1}(z)=P_4(z)\left(\frac{P^j\left(z^2\right)+P^j\left(-z^2\right)}{2}\right)$$

and it can be shown to be equivalent to the Deslauriers-Dubuc dyadic scheme by writing

$$\begin{aligned}\frac{P^{j+1}(z)+P^{j+1}(-z)}{2} &= \left(\frac{P_4(z)-P_4(-z)}{2}\right)\left(\frac{P^j\left(z^2\right)+P^j\left(-z^2\right)}{2}\right)\\ &= P_2\left(z^2\right)\left(\frac{P^j\left(z^2\right)+P^j\left(-z^2\right)}{2}\right).\end{aligned}$$

which gives an alternate proof for proposition 3.1.

In order to study the regularity and stability of the chosen high resolution schemes, we need to find corresponding schemes for the (forward) finite differences

$$\frac{dy_{j,k}}{dx_{j,k}}=\frac{y_{j,k+1}-y_{j,k}}{1/2^j}$$

and of the correspondin higher order finite differences which can be defined recursively $d^{(n)}y_{j,k}=d\left(d^{(n-1)}y_{j,k}\right)/(dx_{j,k})^n$. Let

$$\begin{aligned}H_1^j(z) &= \sum_{k\in\mathbb{N}}2^j\left(y_{j,k+1}-y_{j,k}\right)z^k\\ &= \sum_{k\in\mathbb{N}}2^jy_{j,k}z^{k-1}-\sum_{k\in\mathbb{N}}2^jy_{j,k}z^k\\ &= 2(1/z-1)P^j(z)=2(1-z)P^j(z)/z,\end{aligned}$$

then $P^j\left(z^2\right)=z^22^jH_1^j\left(z^2\right)/(1-z^2)$, $P^j\left(-z^2\right)=-z^22^jH_1^j\left(-z^2\right)/(1+z^2)$, and $P^{j+1}(z)=z2^{j+1}H_1^{j+1}(z)/(1-z)$. Substituting these equations into 3.15 gives

$$H_1^{j+1}(z)=\frac{2z(1-z)}{(1-z^2)}\Gamma_1(z)H_1^j\left(z^2\right)-\frac{2z(1-z)}{(1+z^2)}\Gamma_2(z)H_1^j\left(-z^2\right).$$

The higher order finite differences are given by

$$H_n^j(z)=\frac{2(1-z)}{z}H_{n-1}^j(z)=\left(\frac{2(1-z)}{z}\right)^nP^j(z)$$

where we let $H_0(z)=P(z)$ and it can be seen that higher order finite differences are related through

$$H_n^{j+1}(z)=\left(\frac{2z}{1+z}\right)^n\Gamma_1(z)H_n^j\left(z^2\right)+\left(\frac{-2z(1-z)}{1+z^2}\right)^n\Gamma_2(z)H_n^j\left(-z^2\right). \tag{3.16}$$

H_n is said to be equivalent to a high resolution subdivision scheme if $\Gamma_1(z)/(1+z)$ and $\Gamma_2(z)/(1+z^2)$ are Laurent polynomials. However, we have

$$P_4(z)=\frac{-(1+z)^4\left(1+z^2\right)^4\left(5z^2-12z+5\right)}{128z^7}$$

and therefore $\Gamma_2(z)/(1+z^2)^n$ is a Laurent polynomial for n $=1,2,3,4$. On the other hand, we can also show that $\Gamma_1(z)/(1+z)^n$ is a Laurent polynomial for $n=1,2,3,4$. Therefore, H_n is the symbol of a high resolution subdivision scheme if $n=1,2,3,4$. While H_n is equivalent to the subdivision scheme of $d^{(n)}y_{j,k}/(dx_{j,k})^n=2^{jn}d^{(n)}y_{j,k}$, we can define dH_n^j based on $d\left(2^{j(n-1)}d^{(n-1)}y_{j,k}\right)$ with, for example, $dH_1^j(z)=(1-z)P^j(z)/z$ and more generally

$$dH_{n-1}^j(z)=\frac{(1-z)}{z}H_{n-1}^j(z)=\frac{2^{j(n-1)}(1-z)^n}{z^n}P^j(z). \tag{3.17}$$

Replacing H_{n-1} by dH_{n-1} in equation 3.16, we find

$$2dH_{n-1}^{j+1}(z)=\left(\frac{2z}{1+z}\right)^n\Gamma_1(z)dH_{n-1}^j\left(z^2\right)+\left(\frac{-2z(1-z)}{1+z^2}\right)^n\Gamma_2(z)dH_{n-1}^j\left(-z^2\right). \tag{3.18}$$

dH_{n-1} is said to be equivalent to a high resolution subdivision scheme if $\Gamma_1(z)/(1+z)^n$ and $\Gamma_2(z)/\left(1+z^2\right)^n$ are Laurent polynomials or if H_n is equivalent to a high resolution subdivision scheme which is true for $n=1,2,3,4$.

Using results from Dyn [6], we have the following theorem.

Theorem 3.4. *(Dyn) If dH_n as in equations3.17 and 3.18 is equivalent to a high resolution subdivision scheme converging uniformly to zero for all initial data, then the corresponding scheme P as in equation 3.15 is C^n, that is, all interpolation functions f are C^n*

Proof. See theorems 4.2 and 4.4 in [6] as they apply to this high resolution subdivision schemes as well. □

A sufficient condition for $y_{j,k} \to 0$ uniformly as $j \to \infty$ is that $\max_{l=0,1}\{\sum_{k\in\mathbb{N}} |\gamma_{2k-l}|\} < 1$ when $y_{j+1,l} = \sum_{k\in\mathbb{N}} \gamma_{2k-l} y_{j,k}$. Indeed, let

$$I_{j,l} = \left[2^{j+1} - 2 + 2^j(l-1) + l, 2^j(l+1) - 2^{j+1} + 2\right]$$

writing

$$M_{j,l} = \max\left\{\left|y_{j,k}\right| : k \in I_{j,l}\right\}$$

then $M_{j+1} \leq \max_{l=0,1}\{\sum_{k\in\mathbb{N}} |\gamma_{2k-l}|\} M_j$.

Theorem 3.5. *For $-7 < 32\alpha < 6$, the high resolution subdivision scheme given by equation3.15 are C^1.*

Proof. We need to consider dH_1 given by

$$\frac{dH_1^{j+1}(z)}{2} = \left(\frac{z}{1+z}\right)^2 \Gamma_1(z) dH_1^j\left(z^2\right) + \left(\frac{-z(1-z)}{1+z^2}\right)^2 \Gamma_2(z) dH_1^j\left(-z^2\right).$$

Let

$$dh1(z) = \left(\frac{z}{1+z}\right)^2 \Gamma_1(z) + \left(\frac{-z(1-z)}{1+z^2}\right)^2 \Gamma_2(z),$$

to show that the high resolution subdivision scheme corresponding to dH_1 converges uniformly to zero, it is enough to show that the sum of the absolute values of the coefficients of the odds and even terms, $dh1_{even}(z) = \frac{dh1(z)+dh1(-z)}{2}$ and $dh1_{odd}(z) = \frac{dh1(z)-dh1(-z)}{2}$, are smaller that $1/2$. This is effectively equivalent to requiring that $2^j\left|\left(y_{j,k+2} - y_{j,k+1}\right) - \left(y_{j,k+1} - y_{j,k}\right)\right| = \lambda^j M \to 0 \forall k \in \mathbb{N}$ as $j \to \infty$ $(j > J)$ where $0 \leq \lambda < 1$ and M can be chosen to be $2^J \max\left\{\left|\left(y_{J,k+2} - y_{J,k+1}\right) - \left(y_{J,k+1} - y_{J,k}\right)\right|\right\}$. □

Proposition: cubic order of approx. for any α

Proposition: For α, the fundamental function of the high resolution scheme has values of amplitude <fds for $|x| > 1$ whereas the cubic Deslauriers-Dubuc scheme reaches a maximum of .

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