Numerical Methods 2 (MATH 3423) - Final
Acadia University
Spring 2002

Instructor: Daniel Lemire, Ph.D.

Due April 23rd 2002 - 4pm (Math & Stat. Secretary)

Name: ______________________________

Student Number: ______________________

The exam is due on April 23rd 2002 at the Mathematics and Statistics secretary before 4pm (Huggins Science Hall, Room 130). While you can use all available documentation to write this exam including textbooks, web sites and lecture notes, you are to write this exam on your own. Specifically, you are not allowed to discuss this exam with other students or professors. While you can use a calculator or a computer, they are not required. The exam is designed to take about 3 hours of your time. You must show your work.

Question 1. ____ / 15
Question 2. ____ / 20
Question 3. ____ / 25
Question 4. ____ / 20
Question 5. ____ / 20

____ /100

1. [15 marks] (Numerical Derivative) An elevator is moving with a variable velocity v. Estimate the velocity at time $t = 0$ if the following data is available (give one velocity per case) using **Taylor series expansions**.

   (a) The elevator is at height $p = 10$ m at $t = -1$ s and at height $p = 15$ m at $t = 1$ s.

   (b) The elevator is at height $p = 10$ m at $t = -1$ s and at height $p = 15$ m at $t = 2$ s.

(c) The elevator is at height $p = 10$ m at $t = -1$ s, at height $p = 12$ m at $t = -0.1$, and at height $p = 15$ m at $t = 2$ s.

2. [20 marks] (Numerical Integration) Suppose we want to compute integrals like $I(f) = \int_0^1 f(x)e^x dx$ numerically using an approach similar to Gaussian Quadrature.

   (a) Solve for c_1, x_1, c_2, and x_2 in the quadrature formula $I(f) = \int_0^1 f(x)e^x dx = c_1 f(x_1) + c_2 f(x_2)$ assuming that the quadrature formula must be exact for polynomials of order 3 (example $f(x) = x^3$).

(b) Using the quadrature formula in part (a) (with the same c_1, x_1, c_2, and x_2), show how you could integrate numerically $\int_{-1}^{0} g(x)e^x dx$ for a given g. **Prove** that your solution is exact for polynomials of order 3 (example $g(x) = x^3$).

(c) Find three polynomials $p_0(x)$, $p_1(x)$, and $p_2(x)$ of degree 0, 1 and 2 respectively such that $\int_0^1 p_i(x)p_j(x)e^x dx = 0$ whenever $i \neq j$.

(d) Explain how you could compute c_1, x_1, c_2, and x_2 in the quadrature formula $I(f) = \int_0^1 f(x)e^x dx = c_1 f(x_1) + c_2 f(x_2)$ (see part a) using the polynomials in part (c). Specifically, **explain** how the roots of p_2 relate to x_1 and x_2.

3. [25 marks] (Interpolation)

(a) Using Newton's pyramidal approach and showing all of your work, find the Newton polynomial interpolating the points $(x_1, y_1) = (1, 2)$, $(x_2, y_2) = (4, -1)$, and $(x_3, y_3) = (-2, 1)$.

(b) Find the Lagrange polynomial interpolating the points $(x_1, y_1) = (1, 2)$, $(x_2, y_2) = (4, -1)$, and $(x_3, y_3) = (-2, 1)$.

(c) **Compare** your answers in part (a) and (b). **Explain**.

(d) Explain how you can efficiently add new data nodes to an existing Newton polynomial as long as you still have the Newton's pyramid. That is, explain how you are able to add new nodes **without recomputing Newton's pyramid entirely** each time.

(e) Using the approach described in part (d), modify the polynomial in part (a) to interpolate the points $(x_1, y_1) = (1, 2)$, $(x_2, y_2) = (4, -1)$, $(x_3, y_3) = (-2, 1)$, $(x_4, y_4) = (6, 12)$, and $(x_5, y_5) = (9, 1)$.

4. [20 marks] (Numerical Solutions to ODEs)

   (a) Solve $y' = x^2 + y$ over the interval $[0, 1]$ using Euler's method with initial condition $y(0) = 0$ and a step size of $1/2$.

   (b) What is the order of accuracy of Euler's method in this case? Derive your answer from Taylor's theorem.

   (c) Is Euler method stable when solving $y' = x^2 + y$ over the interval $[0, 1]$ with initial condition $y(0) = 0$? Explain.

(d) Solve $y'' = x^2 + y$ over the interval $[0,1]$ with boundary conditions $y(0) = 0$ and $y(1) = 0$ using a step size of $1/2$ by shooting (hint: the differential equation is linear).

5. [20 marks] (Finite-Difference Methods) Consider the linear differential equation $y'' - y' + y = 0$ with boundary values $y(0) = y(1) = 0$.

   (a) Write the finite difference approximation to the differential equation using central-difference formulas.

   (b) Assuming a step size of $1/4$ write the finite difference system.

   (c) Write the system as a matrix equation.

(d) Rewrite the system with general boundary values $y(0) = a, y(1) = b$.