Generating the polynomials to cut out Segre-Veronese varieties

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1 Segre-Veronese varieties

Define $\mathcal{X} = \mathbb{P}V_1 \times \mathbb{P}V_2 \times \cdots \times \mathbb{P}V_k$ where the vector spaces $V_1, \ldots V_k$ have dimensions $n_1, \ldots, n_k \geq 2$ respectively and let $d_1, \ldots d_k$ be positive integers. We can think of it as a subvariety in a projective space by the explicit embedding

$$\mathrm{SV}_{d_1,\ldots,d_k} : \mathbb{P}V_1 \times \mathbb{P}V_2 \times \cdots \times \mathbb{P}V_k \to \mathbb{P}(\mathrm{Sym}^{d_1} V_1 \otimes \cdots \otimes \mathrm{Sym}^{d_k} V_k)$$

given by

$$([e_1], [e_2], \ldots, [e_k]) \mapsto [e_1^{d_1} \otimes e_2^{d_2} \cdots \otimes e_k^{d_k}]$$

$\mathcal{X}$ is called a Segre-Veronese variety. We are interested in enumerating the polynomials $\{f_1, \ldots, f_p\}$ such that the equations $f_i(x) = 0$ for $i \in [p]$ cut out the variety $\mathcal{X}$.

2 Polynomials for Segre varieties

If we start by considering when $d_1 = \cdots = d_k = 1$, we have simply a Segre variety. If we treat this as the k-dimensional array of indeterminants

$$\mathcal{A} = (a_{i_1 \ldots i_k})_{1 \leq i_j \leq n_j, j \in [k]}$$

define the ring $S_\mathcal{A} = \mathbb{C}[\mathcal{A}]$. Then let

$$M_{2\times 2} = \{a_{i_1 \ldots i_\ell \ldots i_k} a_{j_1 \ldots j_\ell \ldots j_k} - a_{i_1 \ldots i_{\ell-1} j_\ell i_{\ell+1} \ldots i_k} a_{j_1 \ldots j_{\ell-1} i_\ell j_{\ell+1} \ldots j_k} : \ell \in [k]\} \subset S_\mathcal{A}$$

be the 2×2 minors of $\mathcal{A}$ about at least one coordinate. Define $I_2(\mathcal{A})$ to be the ideal of $S_\mathcal{A}$ generated by $M_{2\times 2}$. $I_2(\mathcal{A})$ is a prime ideal in $\mathbb{C}[\mathcal{A}]$ and cut out rank-1 tensors in $V_1 \otimes V_2 \otimes \cdots \otimes V_k$. Notice that we can extend this to cut out a Segre-Veronese variety by working with at $k' = d_1 + d_2 + \cdots + d_k$ dimensional array of indeterminants $\mathcal{A}'$ where we identify indeterminants if they share the same orbit in $\mathcal{S}_{d_1} \times \cdots \times \mathcal{S}_{d_k}$ which acts on the index of the indeterminant. In practice, these symmetries simply result in the identification some polynomials in $M_{2\times 2}$. Using some data structure we want to efficiently construct a basis or at very least a spanning set of polynomials.

3 Data structures

Our goal is to find a linearly independent set of polynomials in $I_2(\mathcal{A}')$. These polynomials can be treated simply as vectors in the dual space. Further, due to the simple structure of $M_{2\times 2}$ we know that at least a spanning set can be constructed from vectors f with $|\mathrm{supp}(f)| = 2$ and values at these locations being ± 1.

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We can associate such vectors with directed edges e.g. by defining the head by the index where 1 occurs and the tail by the index where -1 occurs.
To this end first we define an index set I as k-tuple of multisets of size d_j with elements drawn from $[n_j]$,

$$I = \{(M_1, \dots M_k) : M_j \subset [n_j], |M_j| = d_j, j = 1, \dots, k\}.$$

Each multiset gives a basis element in a constituent symmetric space i.e. if $v_j \in \operatorname{Sym}^{d_j} V_j$ then v_{M_j} index a unique coordinate in v_j.
Using the index set we can enumerate a vertex set as ordered tuples

$$\mathcal{V} = \{(S, T) : S, T \in I; S \leq T\}$$

The order we induce lexicographical from the constituent multisets. Each vertex uniquely identifies a basis element in $\operatorname{Sym}^2(V)$ where $V := \mathbb{P}(\operatorname{Sym}^{d_1} V_1 \otimes \cdots \otimes \operatorname{Sym}^{d_k} V_k)$. Our goal is then to use this data structure to determine edges that obey the rules for being in $M_{2\times 2}$ and that will hopefully guarantee independence.
We can then define the the edges set by

$$((S,T),(S',T')) \in \mathcal{E} \iff \exists \ell \in [k] \text{ such that } \operatorname{supp}(S\Delta S') = \operatorname{supp}(T\Delta T') = \{\ell\}, \text{ and } |(S\Delta S')_\ell| = 2, |(T\Delta T')_\ell| = 2$$

We abuse the symmetric difference symbol, Δ, on tuples of multisets to denote an element-wise symmetric difference

$$S\Delta T := (M_1, \dots, M_k)\Delta(N_1, \dots, N_k) = (M_1\Delta N_1, \dots, M_k\Delta N_k)$$

e.g.

$$(\{1^2, 3\}, \{2, 5\}, \{7\})\Delta(\{1, 2, 3\}, \{2, 5\}, \{8\}) = (\{1, 2\}, \emptyset, \{7, 8\})$$

and the support of a tuple of multisets to be the a set containing the locations of non-empty multisets e.g.

$$\operatorname{supp}(\{1, 2\}, \emptyset, \{7, 8\}) = \{1, 3\}$$

This condition can be considered generative a lá the approach for defining $M_{2\times 2}$: pick an index $j \in [k]$ then pick an element from $s \in S_j$ and $t \in T_j$, $s \neq t$, define $S'_j = (S_j \setminus \{s\}) \cup \{t\}$ and $T'_j = (T_j \setminus \{t\}) \cup \{s\}$ as long as $S' \leq T'$. The inequality of s and t prevents self-loops which would correspond to zero polynomials.
However, from this connection rule we can see that if for every $j = 1, 2, \dots, k$ there is some $a_j \in [n_j]$ so that $S_j = T_j = \{a_j^{d_j}\}$ swapping elements between the multisets of S and T has no effect. Further, if there is only one site where $T_j \neq S_j$ but $|T_j\Delta S_j| = 2$ then exchanging the single pair of elements that differ between the sets can only violate the ordering condition i.e. it would be that $S' > T'$ if we assume $S \leq T$. These cases are nuanced but I believe that

$$\sum_{j=1}^{k} |S_j\Delta T_j| > 1$$

is a necessary and perhaps sufficient condition for the node (S, T) to not be isolated.
We will exclude index pairs of these types to construct a restricted vertex set $\mathcal{V}'$. This can be done beforehand or implicitly by identifying vertices that generate no edges.
Next, the complete graph $\mathcal{G} = (\mathcal{V}', \mathcal{E})$ encodes all polynomials in $M_{2\times 2}$; however, we can do better: constructing a spanning forest on $\mathcal{G}$ enforces linear independence between elements so we can generate a basis.
To start, we explain why the structure must be a spanning forest, not a spanning tree. Notice that the generation rule only swap elements between multisets. If we have (Q, R) and $(S, T) \in \mathcal{V}$ then $(Q, R) \sim (S, T)$ only if for every $j \in [k]$, $Q_j \cup R_j = S_j \cup T_j$. We can construct each spanning tree by producing sub-graphs by first partitioning $\mathcal{V}$ using k-tuples of multisets of size $2d_j$ drawn from $[n_j]$,

$$\begin{aligned} C &= \{(M_1, \dots M_k) : M_j \subset [n_j], |M_j| = 2d_j, j = 1, \dots, k\} \\ \mathcal{V}_c &= \{(S, T) \in \mathcal{V} : (S_1 \cup T_1, \dots, S_k \cup T_k) = c\}, \quad c \in C \\ \mathcal{V} &= \bigsqcup_{c\in C} \mathcal{V}_c \end{aligned}$$

With this strategy, we can first enumerate elements of C, determine a splitting to produce a root v_0 for our tree on $\mathcal{V}_c$. For efficiency, we would like to identify rules for swapping that result in $S' \leq T'$, but this would be a later optimization.