LPHYS1303 - Spectral methods project

Solitons and Korteweg-De Vries equations

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1 Introduction

The Korteweg–de Vries equation is in the form of **second-order nonlinear partial differential equation** that models wave propagation in shallow water. It is particularly notable for its analytical solutions, which describes waves known as "solitons", stable wave packets that maintain their shape while traveling over long distances.

This report focuses on the interaction between two solitons, a phenomenon where their nonlinear nature becomes clear. Specifically, we aim to analyze and compare the numerical solution of two interacting solitons with the superposition of their individual analytical solutions. This comparison provides insights into how nonlinear interactions influence soliton dynamics and helps evaluate the accuracy of numerical methods in capturing such effects.

2 Mathematical model and numerical method

(This section is largely inspired by the project presentation)

The canonical form of the The Korteweg-de Vries equation is

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{2.1}$$

This equation admits an analytical solution for a single soliton wave.

$$u(x,t) = \frac{c}{2}\cosh^{-2}\left[\frac{\sqrt{c}}{2}(x - ct - a)\right]$$

Equation 2.1 on a periodic domain of size L can be simulated with a split-operator or *split-step Fourier pseudo-spectral method.* Like in the case of the nonlinear Schroedinger equation, the evolution equation is given by the application of two evolution operators, one linear and one nonlinear

$$\frac{\partial u}{\partial t} = \mathscr{L}u + \mathscr{N}u, \quad \text{with } \mathscr{L}u = -\frac{\partial^3 u}{\partial x^3} \text{ and } \mathscr{N}u = -6u\frac{\partial u}{\partial x} = -3\frac{\partial u^2}{\partial x}$$

One can think to create an approximated time stepping scheme by applying first one evolution operator, and then on the output of that apply the other evolution operator.

Contrary to the case of the nonlinear Schroedinger equation, here only the linear dynamics has analytical solution: in spectral space we have

$$\frac{\partial u}{\partial t} = \mathscr{L}u = -\frac{\partial^3 u}{\partial x^3} \quad \rightarrow \quad \hat{u}\_k(t+\Delta t) = e^{i\left(\frac{2\pi}{L}k\right)^3 \Delta t}\hat{u}_k(t)$$

However we can still use the split-step idea. In this case we first perform an update of the linear part in spectral space, and then advance the nonlinear part with a time-stepping scheme of choice. Indicating with $\mathscr{F}$ and $\mathscr{F}^{-1}$ the discrete Fourier transform and its inverse, the update operation from time t to time $t + \Delta t$ is the following:

1. Given $u(x,t)$, we compute the Fourier transform

$$\hat{u}\_k(t) = \mathscr{F}[u(x,t)]$$

2. We advance the linear part by Δt computing the partial update in spectral space

$$\hat{g}\_(t;\Delta t) = e^{i\left(\frac{2\pi}{L}k\right)^3 \Delta t}\hat{u}_k(t)$$

3. We apply the inverse Fourier transform obtaining the partial update in physical space, compute its square, and compute the spatial derivative of that with the spectral method

$$\begin{aligned} g(x,t;\Delta t) &= \mathscr{F}^{-1}\left[\hat{g}_k(t;\Delta t)\right] \\ \frac{\partial g^2(x,t;\Delta t)}{\partial x} &= \mathscr{F}^{-1}\left[i\frac{2\pi}{L}k\mathscr{F}\left[g^2(x,t;\Delta t)\right]\right] \end{aligned}$$

4. We apply Euler forward time stepping scheme:

$$u(x,t+\Delta t) = g(x,t;\Delta t) - 3\frac{\partial g^2(x,t;\Delta t)}{\partial x}\Delta t$$

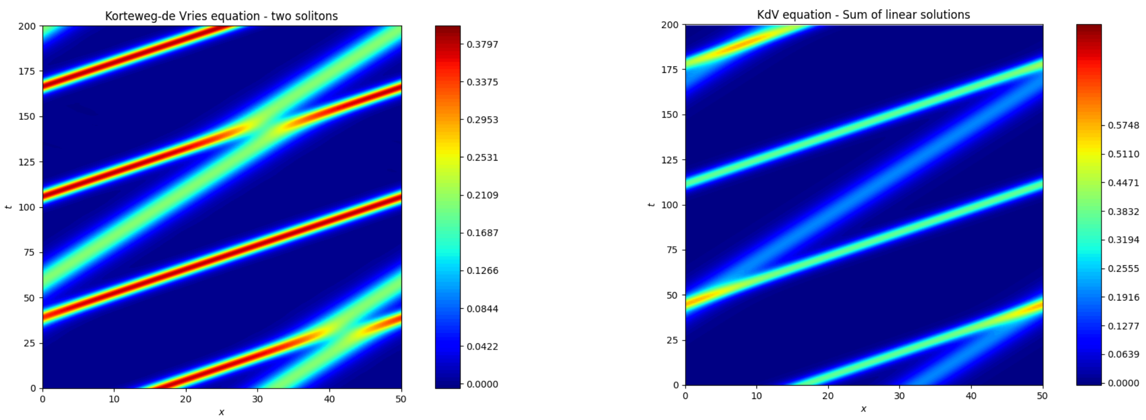
Then we go back to point 1 to repeat the cycle for the next timestep.

3 Visualisation of the solution

The two-soliton system behaves as if the solitons were moving independently, except during their interactions, where they appear to collide and influence each other. We can also see that the higher-amplitude soliton has a greater speed than the lower one.

If instead of explicitly solving numerically the nonlinear equation for the entire initial condition, we solved it by using the sum of the two analytical solutions for each soliton, the "bumping" no longer occurs. They don't interact anymore, passing right through each other, but are still moving at the same speed.

Figures 2 and 3 can help to visually understand this, and animations of the time evolutions can be found here : soliton.gif -



(a) Numerical solution

(b) Sum of the linear solutions

Figure 1

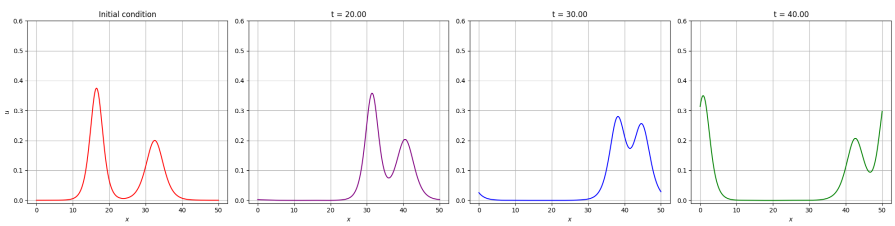


Figure 2: Numerical solution

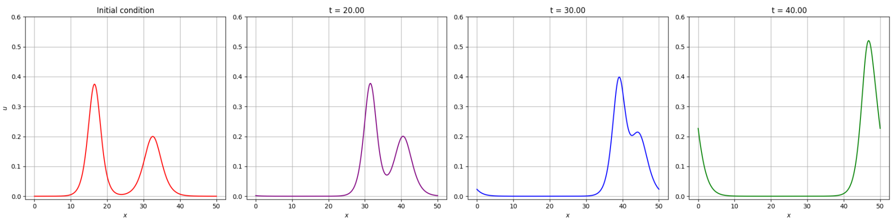
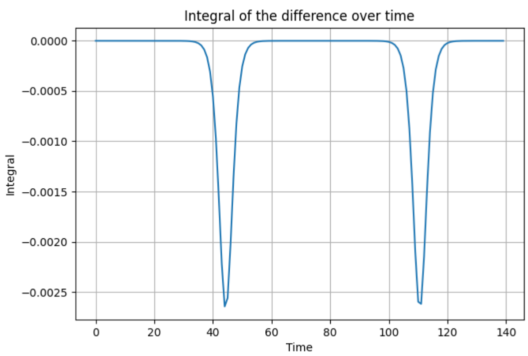


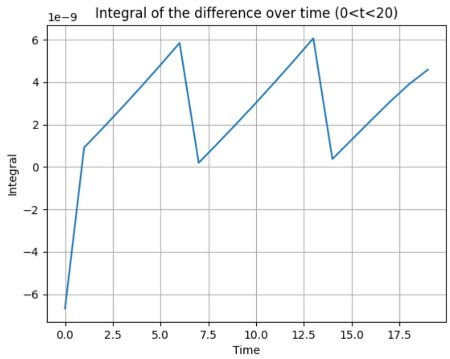
Figure 3: Linear solution

4 Numerical errors

For the case of one soliton travelling alone, we can estimate the error of our numerical scheme, by knowing the analytical solution. The plots in figure 4 show the mean error at each time step between the linear and analytical solution. We can see that when the soliton is not travelling at the border of our domain, the error is pretty small (10^{-9}), but when going from one side to another, the error gets bigger (10^{-3}), yet still negligible.



(a) Error with respect to time



(b) Error with respect to time (only 20s)

Figure 4

If we consider the case of two solitons, the error becomes much greater because of the interacting effect, as shown in figure 5. The analytical and numerical are not in phase anymore, and that explains this high value in the error of the sum of the analytical solution. If the phase difference "accidentally" becomes slower, the error will again be smaller. There is a beginning of that at the end of the figure 5.

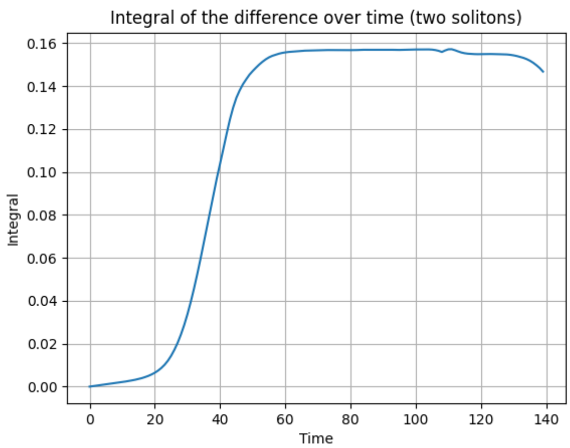


Figure 5: Error for the two-soliton case

Github repository

For further details on the codes used and graphics produced for this project, see our Github repository.