The wavelength of synchrotron light emitted from an undulator

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Abstract: This text describes the basic function of an undulator and the properties of the emitted synchrotron radiation. A brief introduction to how electromagnetic radiation is generated is made. Additionally the dependence of several parameters for the wavelength of emitted synchrotron radiation from an undulator is examined. The dependence of the emitted wavelength is shown to depend on the energy of the electrons, the angular deviation from the central beam and the magnetic field strength. An analysis of the brilliance of the emitted radiation's dependence of the emittance is also made.

Contents

1 Introduction

This text describes the basic function of an undulator and how different parameters of the undulator affects the emitted synchrotron radiation when the undulator is used as an insertion device. More specifically parameters such as the magnetic grating period, the energy of the electrons, the field strength of the static magnetic field and the angular deviation are examined. Additionally it is explained why a low emittance of a radiation source is equal to a relatively high brilliance of the emitted light. All expressions and equations derived in this text follows from the book by Rubensson [1].

2 The undulator

An undulator is an insertion device used for generation of synchrotron radiation. Properties of the emitted radiation include low divergence of the beam, high brilliance and short bandwidth. The undulator is a periodic structure of permanent magnets see figure 1 where the undulator period λ_u is defined as the length of one north and one south magnet. The gap between the opposing rails of magnets is adjustable to fine accuracy, hence the magnetic field strength in the path of the electrons is tuneable. Electrons are emitted from a linear accelerator at high energies and are forced to travel through the undulator. From Maxwell's equations we know that electron's travelling in a perpendicular homogenous magnetic field are subdued to central motion. The periodic switching of the magnetic poles as the electrons travel through the undulator yields a superposition of central motions with alternating acceleration which means that the resulting motion is an oscillation of the electrons. The electron's have an acceleration in the entire oscillation. This means that they emit radiation throughout the motion.

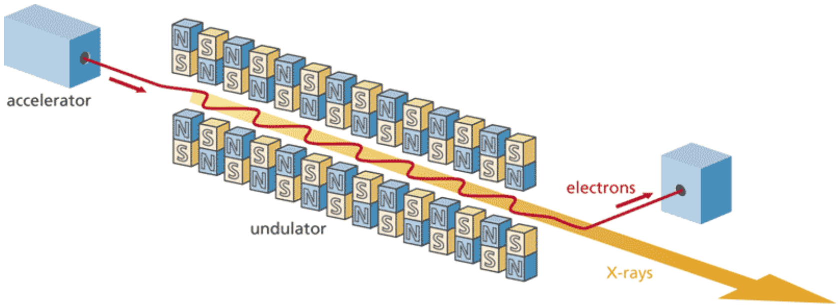

Figure 1. The basic set up of an undulator, https://www.psi.ch/swissfel/how-it-works

Electromagnetic radiation is generated by accelerated charges. This was first discovered in the 19th century by Henrich Hertz (1857-1894). Figure 2 shows how an electromagnetic pulse is generated. At $t = 0$ the electromagnetic field lines from the electron are isotropic and the magnitude of the electromagnetic field decrease proportional to the inverse square of the distance to the electron.

$$\overline{E} = \frac{e}{4\pi\varepsilon_0} \frac{\hat{r}}{\overline{r}^2}$$

Between $t = 0$ and $t = t_1$ the electron is accelerated to velocity v. Lorentz transformation tells us that this acceleration gives rise to a skewing of the field lines. According to Maxwell's equation the field lines must be continuous. Combining these to constraints it is possible two see the electromagnetic pulse generated from the acceleration of the electron.

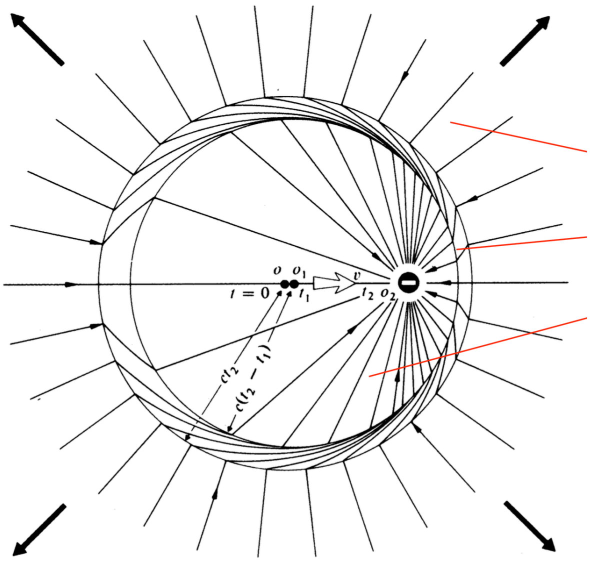

Figure 2. Course material, Waves and optics, UU-1FA522.,

In the undulator the electrons are constantly accelerated. This means that instead of giving rise to electro magnetic pulses, the electrons in the undulator give rise to electromagnetic waves. The electrons in the undulator are travelling at extreme velocities, $v \approx 0.9999999\ c$. This means that to get an accurate description of the physics of the undulator we must include special relativity in our description. First of the electromagnetic radiation generated from an electron is approximated as dipole radiation, this can be seen in figure 3. Under the Lorentz transformation the Dipole radiation is transformed to a cone like structure with an opening angle as a function of the velocity of the electron, $\theta = \pm\frac{1}{\gamma(v)}$. Since the velocity is extremely close to c, gamma is large which means that the opening angle is small.

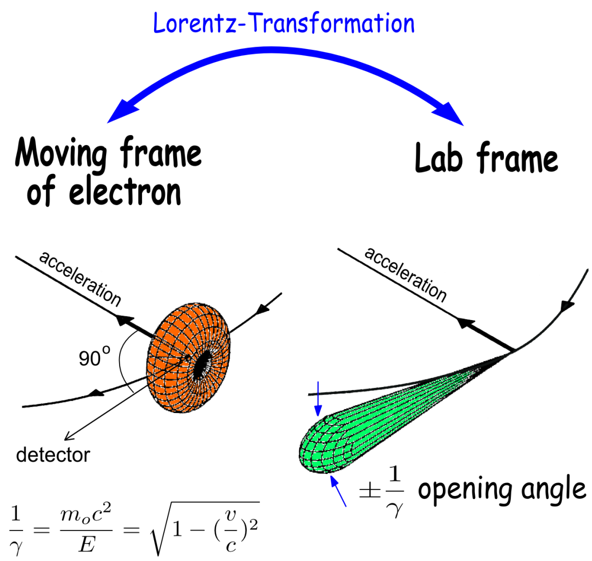

Figure 3. http://photon-science.desy.de/research/studentsteaching/primers/synchrotron_radiation/index_eng.html

3 The emitted wavelength

The emitted wavelength from an undulator is an interesting quantity for those who want to use synchrotron radiation in experiments. The emitted wavelength depends primarily on the period of the magnetic structure, the field strength of the magnetic field, the energy of the electron and the angular deviation. Consider an electron travelling with velocity v where v is very close to the speed of light, c. The electron emits a photon perpendicular to it's path and after the time Δt the electron have travelled a distance $v \cdot \Delta t$. From the assumption that the speed of light is constant in all reference frames makes it possible to define the proper time of the photon as Δt_0. Constructing a diagram over the traveled distance of the electron, the photon in the electron's reference frame, and the photon in the photon's reference frame shows us that the motion follows a right-angled triangle. Pythagoras theorem of the same triangle gives.

$$(c\Delta t)^2 = (v\Delta t)^2 + (c\Delta t_0)^2 \tag{3.1}$$

$$c\Delta t \sin\theta = c\Delta t_0 \Leftrightarrow sin\theta = \frac{\Delta t_0}{\Delta t} \tag{3.2}$$

The second order Maclaurin approximation of $\sin\theta$ is given by.

$$\sin\theta = \theta \tag{3.3}$$

Inserting the second order Maclaurin approximation of $\sin\theta$ for $\theta < \varepsilon$ gives.

$$\theta \approx \frac{\Delta t_0}{\Delta t} \tag{3.4}$$

From equation (3.1) the relation between the proper time and the observer time can be derived.

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \Delta t_0 \gamma \tag{3.5}$$

Using the total relativistic energy of the electron, $E = \gamma m_0 c^2$, gives the magnitude of γ.

$$\gamma = \frac{E}{m_0 c^2} = \frac{\text{Gev-range}}{511\text{keV}} \rightarrow \gamma \in [1000, 5000] \tag{3.6}$$

Writing the ratio between the observed time and the proper time gives the divergence of the synchrotron beam.

$$\theta \approx \frac{\Delta t_0}{\Delta t} = \frac{1}{\gamma} \tag{3.7}$$

At high velocities β approaches one. This means that we can write γ in the following way.

$$\gamma^2 = \frac{1}{1-\beta^2} = \frac{1}{(1-\beta)(1+\beta)} \approx \frac{1}{2(1-\beta)} \tag{3.8}$$

The electron are forced to oscillate in the undulator by the alternating static magnetic field generated from the permanent magnets. A period of the oscillation is given by the length of one south and one north magnet in the propagation direction. This means that the classical undulator frequency is given by.

$$f_{Classical} = \frac{v}{\lambda_u} \tag{3.9}$$

We have to include Lorentz invariance in our description of the emitted wavelength. According to special relativity the electron sees the magnetic structure length contracted giving the contracted magnetic period as.

$$\lambda_u' = \frac{\lambda_u}{\gamma} \tag{3.10}$$

The relativistic doppler shift measured by an observer in the same linear trajectory as the electron is given by.

$$f' = f\frac{1 - \frac{v}{c}\cos\pi}{\sqrt{1-\frac{v^2}{c^2}}} = f\frac{1+\beta}{\sqrt{1-\beta^2}} = \frac{f}{\gamma(1-\beta)} \tag{3.11}$$

The frequency of the synchrotron radiation in the rest-frame of the electron as the electron approaches the speed of light is given by.

$$f_0 \approx \frac{c}{\lambda_u'} = \frac{c\gamma}{\lambda_u} \tag{3.12}$$

To get the frequency measured by the laboratory the relativistic doppler shift is applied giving.

$$f_0' = \frac{f_0}{\gamma(1-\beta)} = \frac{c}{\lambda_u(1-\beta)} \approx \frac{2c\gamma^2}{\lambda_u} \tag{3.13}$$

The frequency of the synchrotron radiation in the rest-frame of the electron as the electron approaches the speed of light is given by.

$$f_0 \approx \frac{c}{\lambda_u'} = \frac{c\gamma}{\lambda_u} \tag{3.14}$$

To get the frequency measured by the laboratory the relativistic doppler shift is applied giving.

$$f_0' = \frac{f_0}{\gamma(1-\beta)} = \frac{c}{\lambda_u(1-\beta)} \approx \frac{2c\gamma^2}{\lambda_u} \tag{3.15}$$

The measured frequency is equal to the classically dopplershifted radiation. We have obtained our first result that the relationship between the emitted wavelength and the magnetic grating period and the energy of the electrons looks as follows.

$$\lambda = \frac{\lambda_u}{2\gamma^2} \tag{3.16}$$

This relation tells us that for a longer magnetic grating period we will get a higher wavelength. This is equivalent to a lower emitted power. A small electron energy corresponds to a long emitted wavelength which in turn corresponds to a lower emitted power.

3.1 The angular dependence

We now have to ask ourselves the question if it is possible to get an even more accurate description than what is given by (3.16). To do this we consider the case when the observer is outside of the the optical axis. If the observer is outside of the linear trajectory of the electron, according to relativistic doppler shift, the frequency of the emitted synchrotron radiation is given by.

$$f = \frac{c}{\lambda_u(1-\beta\cos\theta)} \tag{3.17}$$

The Maclaurin expansion of $\cos\theta$ for $\theta < \varepsilon$ is given by.

$$\cos\theta = 1 - \frac{\theta^2}{2} + \mathcal{O}(\theta^4) \tag{3.18}$$

Inserting the second order approximation of $\cos\theta$ into the analytical expression of the frequency gives.

$$f = \frac{c}{\lambda_u\left(1-\beta\left(1-\frac{\theta^2}{2}\right)\right)} = \frac{\frac{c}{\lambda_u(1-\beta)}}{1+\frac{\beta\theta^2}{2(1-\beta)}} \approx \frac{\frac{2\gamma^2 c}{\lambda_u}}{1+\frac{2\gamma^2\beta\theta^2}{2}} \tag{3.19}$$

$$\approx \frac{2\gamma^2 c}{\lambda_u(1+\gamma^2\theta^2)} \tag{3.20}$$

This gives the angular dependent observer wavelength as.

$$\lambda \approx \frac{\lambda_u}{2\gamma^2}(1+\gamma^2\theta^2) \tag{3.21}$$

This relation holds for $\theta < \varepsilon$. From (3.21) we can see that for a high angular deviation the wavelength increase proportional to θ^2. This means that the wavelength gets rapidly redshifted when the observer is outside of the optical axis. Once again we ask ourselves if it is possible to increase the accuracy of the model. It turns out that the degeneracy can be lifted further by considering the wavelength's dependence of the magnetic field.

3.2 The dependence of the static magnetic field

Consider an electron moving in the z-direction and oscillating in the xz-plane. According to special relativity the following holds.

$$\gamma = \frac{1}{\sqrt{1-\frac{\overline{v}^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v_x^2+v_z^2}{c^2}}} \tag{3.22}$$

Isolating v_z gives.

$$\frac{v_z^2}{c^2} = 1 - \frac{1}{\gamma^2} - \frac{v_x^2}{c^2} \tag{3.23}$$

The magnetic field is perpendicular to the electric field and the propagation direction of the electromagnetic wave. The period of the magnetic field is determined by the period of the permanent magnets.

$$B_y = B_0 \cos\frac{2\pi z}{\lambda_u} \tag{3.24}$$

The Lorentz force is given by.

$$\overline{F} = q(\overline{E} + \overline{v} \times \overline{B}) \tag{3.25}$$

The velocity in the z-direction dominates. This gives rise to an acceleration in the x-direction.

$$v_z \gg v_x \Rightarrow F_x \gg F_z \tag{3.26}$$

It is now possible to formulate Newton's second law using relativistic momentum.

$$F_x = \gamma m_0 \frac{\partial v_x}{\partial t} \Leftrightarrow e v_z B_y = \gamma m_0 \frac{\partial v_x}{\partial t} \tag{3.27}$$

$$\Leftrightarrow e \frac{\partial z}{\partial t} B_y = \gamma m_0 \frac{\partial v_x}{\partial t} \tag{3.28}$$

Integrating equation (3.28) gives.

$$\gamma m_0 v_x = e B_0 \int \left(\cos \frac{2\pi z}{\lambda_u} \frac{\partial z}{\partial t} \right) dt = e B_0 \int \left(\cos \frac{2\pi z}{\lambda_u} \right) dz \tag{3.29}$$

$$= \frac{\lambda_u e B_0}{2\pi} \sin \frac{2\pi z}{\lambda_u} \tag{3.30}$$

$$v_x = \frac{Kc}{\gamma} \sin \frac{2\pi z}{\lambda u}, \quad K \equiv \frac{e B_0 \lambda_u}{2\pi m_0 c} \tag{3.31}$$

Where K is the magnetic deflection parameter. Combining equation (3.23) and (3.31) gives.

$$\frac{v_z^2}{c^2} = 1 - \frac{1}{\gamma^2} - \frac{K^2}{\gamma^2} \sin^2 \frac{2\pi z}{\lambda_u} \tag{3.32}$$

The mean value of the variation of the $\sin^2$-function is given by.

$$\int_0^{2\pi} \frac{\sin^2 z}{2\pi} \, dz = \frac{1}{2} \tag{3.33}$$

The expectation value of equation (3.32) is given by.

$$\langle \frac{v_z^2}{c^2} \rangle = 1 - \frac{1}{\gamma^2} - \frac{K^2}{2\gamma^2} = 1 - \frac{1}{\gamma^{*2}} \tag{3.34}$$

Where γ^* is the effective axial relativistic factor.

$$\gamma^* = \frac{\gamma}{\sqrt{1 + \frac{K^2}{2}}} \tag{3.35}$$

Inserting the effective axial relativistic factor in equation (3.21) gives the undulator equation.

$$\lambda = \frac{\lambda_u}{2\gamma^{*2}} \left(1 + \gamma^{*2}\theta^2\right) = \frac{\lambda_u}{2\gamma^2} \left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right) \tag{3.36}$$

In convenient units the wavelength can be written as.

$$\lambda(nm) = \frac{1.306 \lambda_u(cm)}{E(GeV)} \left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right) \tag{3.37}$$

$$K \approx 0.9337 B_0(T) \lambda_u(cm) \tag{3.38}$$

The undulator equation (3.37) holds for $v_z \gg v_x$, $v_z \approx c$ and $\theta < \varepsilon$. We observe that large field strength of the static magnetic field corresponds to a redshift of the emitted radiation.

4 Emittance

The emittance of a radiation source is defined as.

$$\varepsilon_x = \sigma_x \sigma'_x \tag{4.1}$$

Where σ_x is the extension of the source of the emitted light. σ'_x is defined as the divergence of the emitted light. Convenient units for σ_x and σ'_x respectively is mm and $mrad$. This means that the unit of the emittance is $mmsr$. The emittance is small if the extension of the source is small and the divergence of the emitted radiation is small. The brilliance of a radiation source is defined as.

$$B = \frac{n_{ph}}{\Delta t \cdot \varepsilon_x \cdot \varepsilon_y \cdot \frac{\Delta\omega}{\omega}} \tag{4.2}$$

n_{ph} is the number of photon's emitted during a measurement that last for Δt. ε_x and ε_y is the emittance in the axises perpendicular to the propagation direction of the central beam. $\frac{\Delta\omega}{\omega}$ is the bandwidth of the light. From the definition of the brilliance (4.2) it is obvious that a small emittance of the source is equivalent to high brilliance.

5 Conclusions

The undulator is an insertion device for generation of high brilliance synchrotron radiation. The emitted wavelength of the undulator is described by the undulator equation (3.37). Large field strength of the static magnetic field, angular deviation from the central beam, a short magnetic grating period and a low electron energy corresponds to a redshift of emitted synchrotron radiation. The brilliance of any radiation source is inversely proportional the the emittance of the source of the radiation.

References

[1] J-E. Rubensson, "Synchrotron Radiation: An everyday application of special relativity". Morgan & Claypool Publishers, ISBN: 978-1-6817-4115-4, (2016).