Eric Malm
MA 198
Homework #1
09/03/2004

Rudin 3.5, Saff and Snider 1.5.11, 1.7.5ad, Dummit and Foote 1.1.25, Logan 1.8.6

Rudin 3.5 For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Suppose that $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \neq \infty - \infty$, so that this sum is determinate. Define

$$A_n = \sup_{k\geq n} a_n, \quad B_n = \sup_{k\geq n} b_n, \quad \text{and} \quad C_n = \sup_{k\geq n} (a_n + b_n).$$

We first show that $C_n \leq A_n + B_n$ for all n. For k and n such that $k \geq n$, we have that $a_k \leq A_n$ and $b_k \leq B_n$. Then $a_k + b_k \leq A_n + B_n$ for all $k \geq n$, so $C_n = \sup_{k\geq n} (a_n + b_n) \leq A_n + B_n$. Thus, using the alternate definition of the lim sup, we have

$$\begin{aligned}\limsup_{n\to\infty} (a_n + b_n) &= \lim_{n\to\infty} C_n \\ &\leq \lim_{n\to\infty} (A_n + B_n) = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.\end{aligned}$$

■

SS 1.5.11 Solve the equation $(z+1)^5 = z^5$.

Taking fifth roots of the equation yields

$$z + 1 = ze^{ik\frac{2\pi}{5}},$$

where $k \in \mathbb{Z}$. We note that $k = 0$ (and all other multiples of 5) yields $z + 1 = z$, which reduces to $1 = 0$, an inconsistent equation. Isolating z, we therefore have the solutions

$$\boxed{z = \frac{1}{e^{ik\frac{2\pi}{5}} - 1},}$$

with four unique solutions obtained using $k = 1, 2, 3, 4$. We expect 4 unique solutions because $(z+1)^5 - z^5$ is a fourth-degree polynomial. ■

SS 1.7.5ad Describe the projections on the Riemann sphere of the following sets in the complex plane:

(a) the right half-plane $\{z \mid \operatorname{Re} z > 0\}$,

(d) the set $\{z \mid |z| > 3\}$.

(a) The right half-plane corresponds to the right open hemisphere $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 > 0, x_1^2 + x_2^2 + x_3^2 = 1\}$. ∎

(d) Since $|z| > 3$, $|z|^2 + 1 > 10$, so

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} = 1 - \frac{2}{|z|^2 + 1} > 1 - \frac{2}{10} = \frac{4}{5}.$$

Thus, the set $\{z \mid |z| > 3\}$ corresponds to the dome of the Riemann sphere above the plane $x_3 = 4/5$. Hand sketches of these projections are shown below:

(a) (d)

(to be drawn in later)

∎

DF 1.1.25 Prove that if $x^2 = 1$ for all $x \in G$ then G is abelian.

Suppose $x^2 = 1$ for all $x \in G$. Then $x = x^2x^{-1} = 1x^{-1} = x^{-1}$ for all $x \in G$. For two arbitrary elements a and b of G,

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba,$$

so a and b commute. Since a and b are arbitrary, $ab = ba$ for all a and $b \in G$, and G is abelian. ∎

Logan 1.8.6 This exercise illustrates an important numerical procedure for solving Laplace's equation on a reactangle. Consider Laplace's equation on the rectangle $D : 0 < x < 4, 0 < y < 3$ with boundary conditions given on the bottom and top by $u(x,0) = 0$, $u(x,3) = 0$ for $0 \le x \le 4$ and on the sides by $u(0,y) = 2y(3-y)$, $u(4,y) = 0$ for $0 \le y \le 3$. Apply the average value property (1.45) with $h = 1$ at each of the six lattice points $(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)$ inside D to obtain a system of six equations for the six unknown temperatures on these lattice points. Solve the system to approximate the steady temperature distribution and plot the approximate surface using a software package.

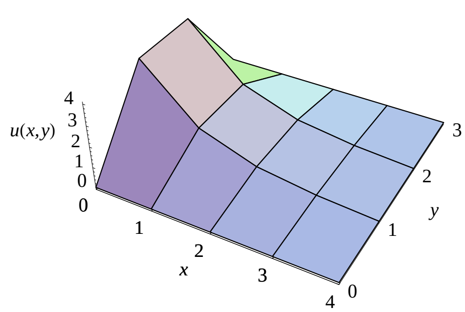
Applying this average value property with $h = 1$ yields the following linear system:

$$\begin{aligned}
u(1,1) &= \frac{1}{4}(u(1,0) + u(1,2) + u(0,1) + u(2,1)) = \frac{1}{4}(4 + u(1,2) + u(2,1)),\\
u(1,2) &= \frac{1}{4}(u(1,1) + u(1,3) + u(0,2) + u(2,2)) = \frac{1}{4}(4 + u(1,1) + u(2,2)),\\
u(2,1) &= \frac{1}{4}(u(2,0) + u(2,2) + u(1,1) + u(3,1)) = \frac{1}{4}(u(1,1) + u(2,2) + u(3,1)),\\
u(2,2) &= \frac{1}{4}(u(2,1) + u(2,3) + u(1,2) + u(3,2)) = \frac{1}{4}(u(2,1) + u(2,3) + u(3,2)),\\
u(3,1) &= \frac{1}{4}(u(3,0) + u(3,2) + u(2,1) + u(4,1)) = \frac{1}{4}(u(3,2) + u(2,1)),\\
u(3,2) &= \frac{1}{4}(u(3,1) + u(3,3) + u(2,2) + u(4,2)) = \frac{1}{4}(u(3,1) + u(2,2)),
\end{aligned}$$

which we solve in *Mathematica* 5.0 to obtain

$$\begin{pmatrix} u(1,1)\\ u(1,2)\\ u(2,1)\\ u(2,2)\\ u(3,1)\\ u(3,2) \end{pmatrix} = \begin{pmatrix} \frac{32}{21}\\ \frac{32}{21}\\ \frac{4}{7}\\ \frac{4}{7}\\ \frac{4}{21}\\ \frac{4}{21} \end{pmatrix}.$$

Plotting these lattice point values yields the following approximate temperature surface:



■