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Abstract

The main aim of this paper is to develop a fast algorithm for solving problems arising from image restoration. Abstract abstract.

1 Introduction

Linear systems with Toeplitz and Toeplitz-related coefficient matrices arise in many different applications. While many efficient algorithms have been developed for solving problems with Toeplitz structure, a few emerging applications lead to Toeplitz-related problems for which the available algorithms are not directly applicable.

In this paper, we consider the preconditioned iterative method for weighted Toeplitz regularized least squares problems

$$\min_{x\in\mathbb{R}^n} \|Bx-b\|_2^2, \tag{1}$$

where the rectangular coefficient matrix B and the right-hand side b are of the form

$$B = \left[\begin{array}{c} \Xi K \\ \sqrt{\mu} I \end{array}\right] \quad \text{and} \quad b = \left[\begin{array}{c} \Xi f \\ 0 \end{array}\right].$$

Here $K \in \mathbb{R}^{m\times n}$ $(m \geq n)$ is a full-rank Toeplitz matrix, $\Xi \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix (as a weighting matrix), I is the identity matrix, $f \in \mathbb{R}^m$ is a given right-hand side, and $\mu > 0$ is a regularization parameter.

2 A New Preconditioner

We propose to study the following HSS preconditioner:

$$M_\alpha = \frac{1}{2}\Sigma^{-1}(\Sigma + H)(\Sigma + S), \tag{2}$$

where

$$\Sigma = \begin{bmatrix} \alpha I & 0 \\ 0 & \mu I \end{bmatrix} \quad \text{with} \quad \alpha > 0.$$

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We note that A has the following decomposition:

$$A = M_\alpha - N_\alpha,$$

where

$$N_\alpha = \frac{1}{2}\Sigma^{-1}(\Sigma - H)(\Sigma - S).$$

It implies that $M_\alpha^{-1}A = I - M_\alpha^{-1}N_\alpha$, and therefore we have

$$\lambda(M_\alpha^{-1}A) = 1 - \lambda(M_\alpha^{-1}N_\alpha).$$

Here $\lambda(\cdot)$ denotes the spectrum of a matrix. We further rewrite the matrix $M_\alpha^{-1}N_\alpha$ as follows:

$$\begin{aligned} M_\alpha^{-1}N_\alpha &= \left[\frac{1}{2}\Sigma^{-1}(\Sigma + H)(\Sigma + S)\right]^{-1}\left[\frac{1}{2}\Sigma^{-1}(\Sigma - H)(\Sigma - S)\right] \\ &= (\Sigma + S)^{-1}(\Sigma + H)^{-1}(\Sigma - H)(\Sigma - S) \\ &= \begin{bmatrix} \alpha I & K \\ -K^T & \mu I \end{bmatrix}^{-1} \begin{bmatrix} \alpha I + W & 0 \\ 0 & 2\mu I \end{bmatrix}^{-1} \begin{bmatrix} \alpha I - W & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha I & -K \\ K^T & \mu I \end{bmatrix}. \end{aligned}$$

By mean of the Schur complement, we have

$$\begin{bmatrix} \alpha I & K \\ -K^T & \mu I \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\alpha}I - \frac{1}{\alpha^2}K\Phi^{-1}K^T & -\frac{1}{\alpha}K\Phi^{-1} \\ \frac{1}{\alpha}\Phi^{-1}K^T & \Phi^{-1} \end{bmatrix},$$

where

$$\Phi = \mu I + \frac{1}{\alpha}K^T K.$$

Theorem 2.1. *Let $K \in \mathbb{R}^{m\times n}$ with $m \ge n$. Then $M_\alpha^{-1}A$ has*

(1) an eigenvalue at 1 with multiplicity n, and

(2) m eigenvalues of the form $1 - \lambda_j$ $(j = 1, 2, \ldots, m)$ where λ_j is the eigenvalue of the matrix

$$G = FE \quad \textit{with} \quad F = I - \frac{2}{\alpha}K\Phi^{-1}K^T.$$

Proof. It follows from (**??**) that we can rewrite $M_\alpha^{-1}N_\alpha$ into

$$M_\alpha^{-1}N_\alpha = \begin{bmatrix} I & -\frac{1}{\alpha}K \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} (I - \frac{2}{\alpha}K\Phi^{-1}K^T)E & 0 \\ \Phi^{-1}K^T E & 0 \end{bmatrix} \begin{bmatrix} I & -\frac{1}{\alpha}K \\ 0 & I \end{bmatrix}.$$

As the matrix $(I - \frac{2}{\alpha}K\Phi^{-1}K^T)E$ is of $m \times m$, it follows that $M_\alpha^{-1}N_\alpha$ has m eigenvalues which are same as that of $(I - \frac{2}{\alpha}K\Phi^{-1}K^T)E$ and an eigenvalues at 0 with multiplicity n. Since $M_\alpha^{-1}A = I - M_\alpha^{-1}N_\alpha$, the result follows. □

References

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