Physics 221
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Problem Set 1

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Problem 1 (NL Problem 1.2)

Consider the system given in Fig. 1 wherein all three masses are equal $m_1 = m_2 = m_3 = m$ and the system is released from rest with $x_2 = 0$ and $x_3 = l$. a.) Determine the equations of motion, b.) Solve the equations of motion to show that

$$x_2(t) = \frac{2mg}{9k}(\cos\omega t - 1) + \frac{1}{6}gt^2 \quad \text{where} \quad \omega = \sqrt{\frac{3k}{2m}}.$$

Prove that $\dot{x_2} > 0$ for all $t > 0$ and conclude that the string always remain taut.

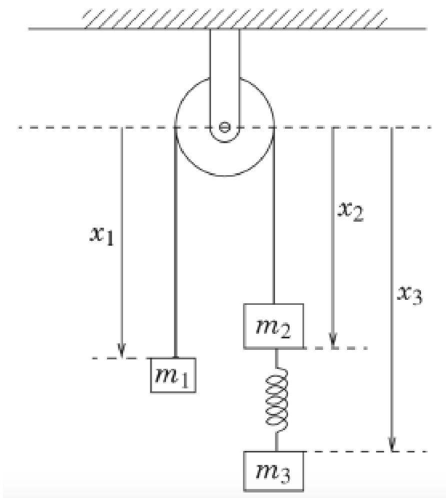

Figure 1: Figure for problem 1

[Solution]
First, we use the constraint $x_1 + x_2 = l_0$ where l_0 is determined by the length the string and the radius of the pulley. This implies that we can only use 2 out of the 3 coordinates. In this solution, we shall use x_3 and x_2. It follows from the above constraint that $\dot{x_1} = -\dot{x_2}$. We shall use this to simplify our equations in terms of our chosen coordinates.
The kinetic energy, T, of the system is given by

$$T = \frac{m}{2}\dot{x_1}^2 + \frac{m}{2}\dot{x_2}^2 + \frac{m}{2}\dot{x_3}^2, \tag{1}$$

which simplifies into

$$T = m\dot{x_2}^2 + \frac{m}{2}\dot{x_3}^2 \tag{2}$$

when we make the substitution $\dot{x_1} = -\dot{x_2}$. Meanwhile, we arrive with the potential energy, V,

$$V = -mgx_1 - mgx_2 - mgx_3 + \frac{k}{2}(x_3 - x_2 - l)^2, \tag{3}$$

when we set the level of the center of the pulley to the the zero potential. We can further simplify the potential by using the constraint $x_2 + x_2 = l_0$. The potential will now be given by,

$$V = -mgl_0 - mgx_3 + \frac{k}{2}(x_3 - x_2 - l)^2. \tag{4}$$

Therefore, the Lagrangian will be given as,

$$\begin{aligned} L &= T - V \\ &= m\dot{x_2}^2 + \frac{m}{2}\dot{x_3}^2 + mgl_0 + mgx_3 - \frac{k}{2}(x_3 - x_2 - l)^2. \end{aligned} \tag{5}$$

The Lagrange's equations are then:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \frac{\partial L}{\partial x_2} = 0 \tag{6}$$

$$\frac{\partial}{\partial t}(2m\dot{x_2}) - k(x_3 - x_2 - l) = 0 \tag{7}$$

$$\boxed{2m\ddot{x_2} - k(x_3 - x_2 - l) = 0}, \tag{8}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_3}}\right) - \frac{\partial L}{\partial x_3} = 0 \tag{9}$$

$$\frac{\partial}{\partial t}(m\dot{x_3}) - mg + k(x_3 - x_2 - l) = 0 \tag{10}$$

$$\boxed{m\ddot{x_3} - mg + k(x_3 - x_2 - l) = 0} \tag{11}$$

To solve these equations we used the `DSolve` command in *Mathematica* and arrived with an expression that simplifies into our desired result,

$$x_2(t) = \frac{2mg}{9k}(\cos\omega t - 1) + \frac{1}{6}gt^2 \quad \text{where} \quad \omega = \sqrt{\frac{3k}{2m}}. \tag{12}$$

The *Mathematica* code can be seen in Fig. 3

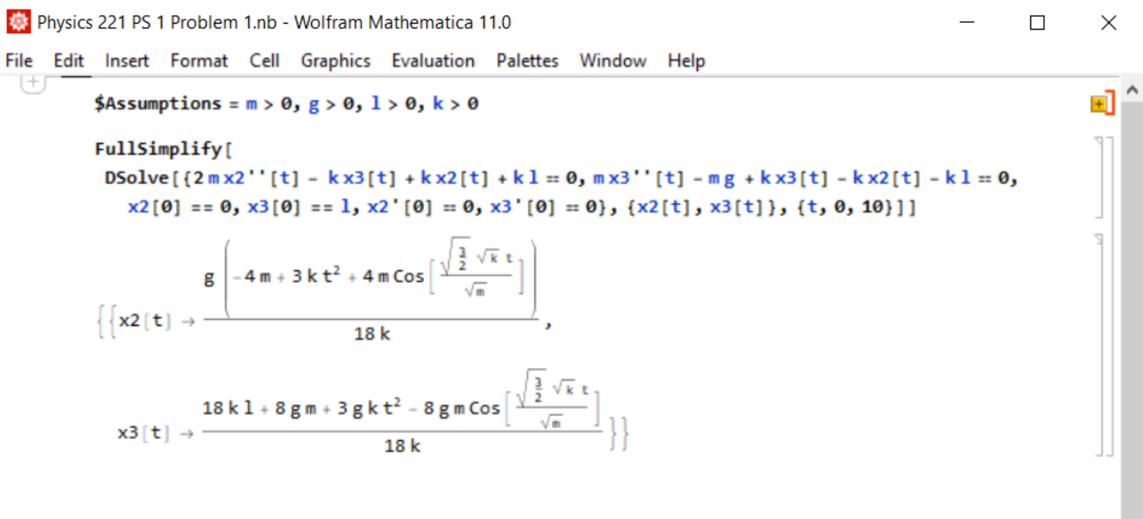

Figure 2: Mathematica Code

where we used the following initial conditions: $x_2(0) = 0$, $x_3(0) = l$ $\dot{x_2}(0) = 0$ and $\dot{x_3}(0) = 0$, since the system initially starts from rest.
To show that $\dot{x_2}$ is greater than zero, we look at the double derivative of x_2 with respect to time.

$$\ddot{x_2}(t) = \frac{g}{3}\left(1 - \cos\omega t\right) \tag{13}$$

We see that the acceleration, $\ddot{x_2}$, is greater than or equal to zero since the possible values for $\cos\omega t$ is in the range $-1 \leq \cos\omega t \leq 1$. There is only a possibility of $\dot{x_2} < 0$ when $\ddot{x_2} < 0$ Therefore the speed wherein m_2 moves downward or $\dot{x_2}$ is always positive after being released from rest. With this in mind, and also considering our constraint introduced earlier, we see that since m_2 is moving downward with the speed $\dot{x_2}(t)$, m_1 should move upward with the same speed. Therefore, the string must always taut.

Problem 2 (NL Problem 1.5)

Consider the so-called swinging Atwood's machine in which M moves only vertically. Using the coordinates indicated in the figure, show that the Lagrangian is given by

$$L = \frac{m+M}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 - gr(M - m\cos\theta)$$

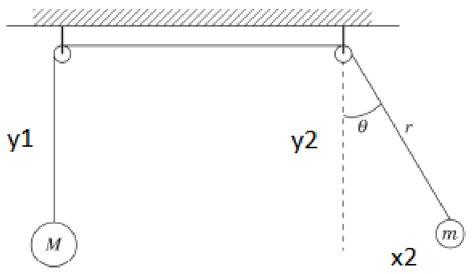

Figure 3: Figure for problem 2

[Solution]

In this problem we use the constraints $y_1 + r = l_0$, wherein l_0 is a constant determined by the length of the string connecting the two masses. It follows from the given constraints that $\dot{y_1} = -\dot{r}$. We shall use these constraints along with the substitution $y_2 = r\cos\theta$ and $x_2 = r\sin\theta$ to simplify our equations in terms of r and θ.

The kinetic energy of the system is given by

$$T = \frac{m}{2}M\dot{y_1}^2 + \frac{m}{2}(\dot{x_2}^2 + \dot{y_2}^2). \tag{14}$$

From our substitution above, we know that $\dot{y_2}$ and $\dot{x_2}$ would equal to

$$\dot{y_2} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \quad \text{and} \quad \dot{x_2} = \dot{r}\sin\theta + r\cos\theta\dot{\theta} \tag{15}$$

We can then simplify the kinetic energy, T, into

$$\begin{aligned} T &= \frac{M}{2}\dot{r}^2 + \frac{m}{2}(\dot{r}^2\cos^2\theta + \dot{r}^2\sin^2\theta + \dot{\theta}^2r^2\sin^2\theta + \dot{\theta}^2r^2\cos^2\theta - 2\dot{r}\dot{\theta}\sin\theta\cos\theta + 2\dot{r}\dot{\theta}\sin\theta\cos\theta) \\ &= \frac{M}{2}\dot{r}^2 + \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) \end{aligned} \tag{16}$$

The potential on the other hand is given by

$$\begin{aligned} V &= -Mgy_1 - mgy_2 \\ &= -Mg(l_0 - r) - mgr\cos\theta \\ &= -Mgl_0 - gr(M - m\cos\theta), \end{aligned} \tag{17}$$

which resulted when we make the appropriate substitutions (constraints and values for y_2). The Lagrangian would then be

$$L = \frac{m+M}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2 - gr(M - m\cos\theta) \tag{18}$$

where we disregarded the Mgl_0 constant term as it gets cancelled out and doesn't give any contribution to Lagrange's equation.
Lagrange's equation would then be

$$\boxed{(M+m)\ddot{r} - mr\dot{\theta}^2 + g(M - m\cos\theta) = 0} \tag{19}$$

and

$$\boxed{2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + grm\sin\theta = 0} \tag{20}$$

for r and θ, respectively.

Problem 3 (NL Problem 1.7)

A projectile is fired near the surface of the Earth. Assuming the force of air resistance is proportional to the velocity, obtain the projectile's equation of motion using the dissipation function $\mathcal{F} = \lambda v^2/2$.

[Solution]

In this problem we consider a projectile of mass m moving in the $x-$ and $y-$ direction with a zero potential at the same level as the origin of the projectile. We have the kinetic and potential energies to be

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \tag{21}$$

and

$$V = -mgy \tag{22}$$

Therefore, the Lagrangian would be

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + mgy \tag{23}$$

Given the dissipation function, the equations of motion takes the form

$$\begin{gathered} \frac{d}{dt}(m\dot{x}) + \lambda \frac{d}{d\dot{x}} \left(\frac{\dot{x}^2 + \dot{y}^2}{2} \right) = 0 \\ \boxed{m\ddot{x} + \lambda \dot{x} = 0} \end{gathered} \tag{24}$$

and

$$\begin{gathered} \frac{d}{dt}(m\dot{x}) - mg + \lambda \frac{d}{d\dot{x}} \left(\frac{\dot{x}^2 + \dot{y}^2}{2} \right) = 0 \\ \boxed{m\ddot{y} - mg + \lambda \dot{y} = 0} \end{gathered} \tag{25}$$

Problem 4 (NL Problem 1.8)

Certain dissipative systems admit a Lagrangian formulation that dispenses with the Rayleigh dissipation function. Consider a projectile in the constant gravitational field $\mathbf{g} = -g\hat{y}$ and assume that the force of air resistance is proportional to the velocity. (a) Show that the equations of motion generated by the Lagrangian

$$L = exp\left(\frac{\lambda t}{m}\right)\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2) - mgy\right]$$

coincide with those obtained in the preceding problem, (b) Solve the equations of motion for x and y assuming the projectile is fired from the origin with velocity of magnitude v_0 making angle θ_0 with the horizontal. (c) Eliminate time to get the equation of the trajectory of the projectile.

[**Solution**]

(a) Given the Lagrangian

$$L = exp\left(\frac{\lambda t}{m}\right)\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2) - mgy\right] \tag{26}$$

The Lagrange's equation given by

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{27}$$

takes the form

$$\begin{aligned}
\frac{d}{dt}\left(m\dot{x}e^{\frac{\lambda t}{m}}\right) &= 0 \\
m\ddot{x}e^{\frac{\lambda t}{m}} + \frac{\lambda}{m}m\dot{x}e^{\frac{\lambda t}{m}} &= 0 \\
m\ddot{x}e^{\frac{\lambda t}{m}} + \lambda\dot{x}e^{\frac{\lambda t}{m}} &= 0 \\
\boxed{e^{\frac{\lambda t}{m}}(m\ddot{x}+\lambda\dot{x}) = 0} &\quad \text{or} \\
\boxed{(m\ddot{x}+\lambda\dot{x}) = 0}
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
\frac{d}{dt}\left(m\dot{y}e^{\frac{\lambda t}{m}}\right) - mge^{\frac{\lambda t}{m}} &= 0 \\
m\ddot{y}e^{\frac{\lambda t}{m}} + \frac{\lambda}{m}m\dot{y}e^{\frac{\lambda t}{m}} - mge^{\frac{\lambda t}{m}} &= 0 \\
m\ddot{y}e^{\frac{\lambda t}{m}} + \lambda\dot{y}e^{\frac{\lambda t}{m}} - mge^{\frac{\lambda t}{m}} &= 0 \\
\boxed{e^{\frac{\lambda t}{m}}(m\ddot{y}+\lambda\dot{y}-mg) = 0} &\quad \text{or} \\
\boxed{(m\ddot{y}+\lambda\dot{y}-mg) = 0.}
\end{aligned} \tag{29}$$

We can eliminate the $e^{\frac{\lambda t}{m}}$ factor from our equations of motion by dividing $e^{\frac{\lambda t}{m}}$ on both sides of the equation. Notice that the resulting equations coincide with those of the previous problem.
(b) Using the `DSolve` command in *Mathematica*, we arrived with a value for our $x(t)$ and $y(t)$.

$$x = \frac{mv_0 \cos\theta_0}{\lambda}\left(1 - e^{-\frac{\lambda t}{m}}\right) \tag{30}$$

and

$$y = \frac{m}{\lambda^2}\left(-mg + \lambda g t + v_0\lambda \sin\theta_0 + (mg - v_0\lambda\sin\theta_0)e^{-\frac{\lambda t}{m}}\right) \tag{31}$$

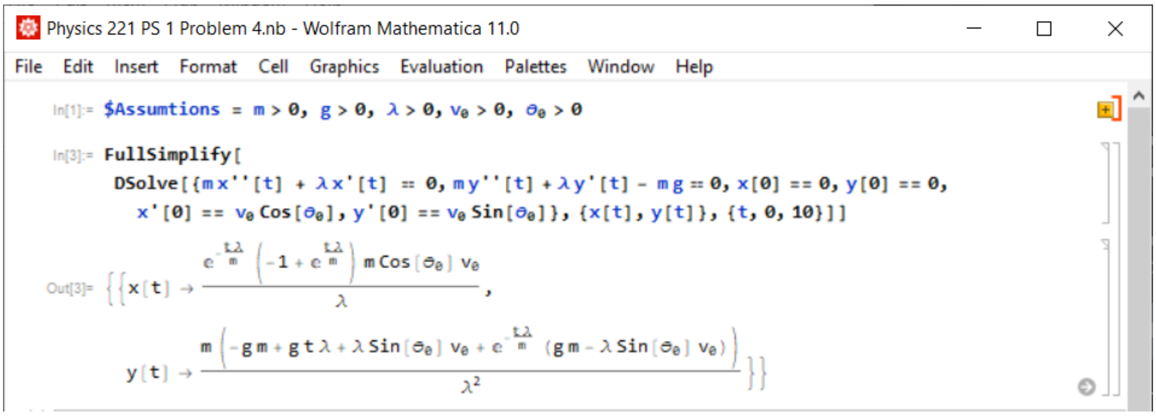

Figure 4: Mathematica Code

(c) We can solve for t from Eq. 30

$$t = -\frac{m}{\lambda}\ln\left(1 - \frac{\lambda x}{mv_0\cos\theta_0)}\right) \tag{32}$$

and substitute this into Eq. 31.

$$y = \frac{m}{\lambda^2}\left(-mg + v_0\lambda\sin\theta_0 - mg\ln\left(1 - \frac{\lambda x}{mv_0\cos\theta_0}\right) + (mg - v_0\lambda\sin\theta_0)\left(1 - \frac{\lambda x}{mv_0\cos\theta_0}\right)\right)$$
$$\boxed{y = \frac{m}{\lambda^2}\left[(v_0\lambda\sin\theta_0 - mg)\frac{\lambda x}{mv_0\cos\theta_0} - mg\ln\left(1 - \frac{\lambda x}{mv_0\cos\theta_0)}\right)\right]} \tag{33}$$

to eliminate the t dependence of y and get the equation for the trajectory of the projectile.

Problem 5 (NL Problem 1.10)

In Weber's electrodynamics the force between two charge particles in motion is directed along the line connecting them and has a magnitude of

$$F = \frac{q_1 q_2}{r^2}\left[1 + \frac{\ddot{r}r}{c^2} - \frac{\dot{r}}{2c^2}\right]$$

where r denotes the distance between the particles and c is the speed of light in vacuum. Find the generalized potential $U(r, \dot{r})$ associated with this force. Set up the Lagrangian and Lagrange's equations for a charge in the presence of another charge held fixed at the origin of the coordinate system.

[Solution]

Given

$$F = \frac{q_1 q_2}{r^2}\left[1 + \frac{\ddot{r}r}{c^2} - \frac{\dot{r}^2}{2c^2}\right] = \frac{1}{r^2} + \frac{\ddot{r}}{rc^2} - \frac{\dot{r}}{2r^2c^2} \tag{34}$$

we notice that the 2nd and 3rd term follow the chain rule

$$\frac{d}{dr}\left(\frac{1}{r}\left(\frac{dr}{dt}\right)^2\right) = -\frac{1}{r^2}\left(\frac{dr}{dt}\right)^2 + 2\frac{1}{r}\frac{d^2r}{dt^2} \tag{35}$$

where

$$\frac{d\left(\frac{dr}{dt}\right)^2}{dr} = 2\frac{d^2r}{dt^2}. \tag{36}$$

Therefore, the expression for the force simplifies into

$$F = q_1 q_2\left(\frac{1}{r^2} + \frac{1}{2c^2}\frac{d\left(\frac{1}{r}\dot{r}^2\right)}{dr}\right). \tag{37}$$

Since we know that the force can be defined as the negative gradient of the potential ($F = -\nabla U$), we arrive with

$$U = -\int F dr. \tag{38}$$

Solving for U, we have

$$\begin{aligned} U &= -q_1 q_2\left[\int \frac{1}{r^2}dr + \frac{1}{2c^2}\int \frac{d\left(\frac{1}{r}\dot{r}^2\right)}{dr}dr\right] \\ U &= \frac{q_1 q_2}{r}\left[1 - \frac{\dot{r}^2}{2c^2}\right] \end{aligned} \tag{39}$$

Therefore, the Lagrangian is given by

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - \frac{q_1 q_2}{r}\left(1 - \frac{\dot{r}^2}{2c^2}\right) \tag{40}$$

Therefore the Lagrange equations are,

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0\\
&\frac{d}{dt}\left(m\dot{r}+\frac{q_1q_2\dot{r}}{2rc^2}\right)-\left(mr\dot{\theta}^2+mr\sin^2\theta\dot{\phi}^2+\frac{q_1q_2}{r^2}\left(1-\frac{\dot{r}^2}{2c^2}\right)\right)=0\\
&m\ddot{r}+\frac{q_1q_2\ddot{r}}{2rc^2}-\frac{q_1q_2\dot{r}^2}{2r^2c^2}-\left(mr\dot{\theta}^2+mr\sin^2\theta\dot{\phi}^2+\frac{q_1q_2}{r^2}\left(1-\frac{\dot{r}^2}{2c^2}\right)\right)=0\\
&\boxed{m\ddot{r}+\frac{q_1q_2\ddot{r}}{2rc^2}-\left(mr\dot{\theta}^2+mr\sin^2\theta\dot{\phi}^2+\frac{q_1q_2}{r^2}\right)=0,}
\end{aligned}\tag{41}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0\\
&\frac{\partial}{\partial t}\left(mr^2\dot{\theta}\right)-mr^2\sin\theta\cos\theta\dot{\phi}^2=0\\
&\boxed{mr^2\ddot{\theta}+2mr\dot{r}\dot{\theta}-mr^2\sin\theta\cos\theta\dot{\phi}^2=0}
\end{aligned}\tag{42}$$

$$\begin{aligned}
&\frac{d}{dt}\left(mr^2\sin^2\theta\dot{\phi}\right)=0\\
&\boxed{mr^2\sin^2\theta\dot{\phi}=\text{constant}}
\end{aligned}\tag{43}$$

Problem 6 (NL Problem 1.14)

The system depicted below is such that the strings are inextensible and its mass, as well as that of the pulleys, is negligible. Mass m_1 moves on a frictionless horizontal table whereas m_2 moves only vertically. Show that, up to a constant, the Lagrangian for the system in terms of coordinate x us

$$L = \frac{m_1(h^2+x^2)+m_2x^2}{h^2+x^2}\dot{x}^2 - mg\sqrt{h^2+x^2}$$

Set up Lagrange's equation for x.

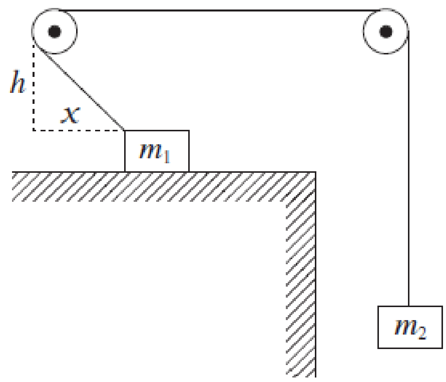

Figure 5: Figure for problem 6

[Solution]

For this problem we have the constraint to be $y+\sqrt{x^2+h^2}=l_0$, where l_0 is determined by the length of the string and the radius of the pulley. It follows that $\dot{y}=-(x^2+y^2)^{-1/2}x\dot{x}$. With these constraints, we can simplify our equations of motion from having 3 coordinates into 2 coordinates. We then set-up our Lagrangian. Our kinetic energy would be

$$\begin{aligned} T &= \frac{m_1}{2}\dot{x}^2+\frac{m_2}{2}\dot{y}^2 \\ T &= \frac{m_1}{2}\dot{x}^2+\frac{m_2}{2}\frac{x^2\dot{x}^2}{x^2+h^2} = \frac{m_1(x^2+h^2)+m_2x^2}{2(h^2+x^2)}\dot{x}^2 \end{aligned} \tag{44}$$

We didn't consider the $\dot{h}^2$ term as h has a constant value. Meanwhile, the potential energy would be

$$V = -m_2gy \tag{45}$$

Making the substitution $y=-\sqrt{x^2+h^2}+l_0$, we have

$$V = m_2g\sqrt{x^2+h^2}-m_2gl_0 \tag{46}$$

In the overall Lagrangian we can omit the m_2gl_0 term as it does not give an overall contribution to the equations of motion. It just vanishes in the process. The Lagrangian would then be

$$L = T - V = \frac{m_1(x^2+h^2)+m_2x^2}{2(h^2+x^2)}\dot{x}^2 - m_2g\sqrt{x^2+h^2} \tag{47}$$

The Lagrange's equation for x is

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{m_1(x^2+h^2)+m_2x^2}{(h^2+x^2)}\dot{x}\right) - \frac{d}{dx}\left(\frac{m_1(x^2+h^2)+m_2x^2}{2(h^2+x^2)}\dot{x}^2\right) + \frac{m_2gx}{\sqrt{x^2+h^2}} = 0\\
&\frac{d}{d\dot{x}}\left(\frac{m_1(x^2+h^2)+m_2x^2}{(h^2+x^2)}\dot{x}\right)\frac{d\dot{x}}{dt} + \frac{d}{dx}\left(\frac{m_1(x^2+h^2)+m_2x^2}{h^2+x^2}\dot{x}\right)\frac{dx}{dt} - \frac{d}{dx}\left(\frac{m_1(x^2+h^2)+m_2x^2}{2(h^2+x^2)}\dot{x}^2\right)\\
&+\frac{m_2gx}{\sqrt{x^2+h^2}} = 0\\
&\frac{d}{d\dot{x}}\left(\frac{m_1(x^2+h^2)+m_2x^2}{(h^2+x^2)}\dot{x}\right)\ddot{x} + \frac{d}{dx}\left(\frac{m_1(x^2+h^2)+m_2x^2}{h^2+x^2}\right)\dot{x}^2 - \frac{d}{dx}\left(\frac{m_1(x^2+h^2)+m_2x^2}{2(h^2+x^2)}\right)\dot{x}^2\\
&+\frac{m_2gx}{\sqrt{x^2+h^2}} = 0\\
&\boxed{\frac{m_1(x^2+h^2)+m_2x^2}{(h^2+x^2)}\ddot{x} + \frac{m_2xh^2}{(x^2+h^2)^2}\dot{x}^2 + \frac{m_2gx}{\sqrt{x^2+h^2}} = 0}
\end{aligned} \tag{48}$$

Problem 7 (NL 1.17)

Show that the Lagrangian

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right) - \frac{eg}{c}\dot{\phi}\cos\theta$$

describes a charged particle in the magnetic field $\mathbf{B} = g\mathbf{r}/r^3$ of a magnetic monopole and find Lagrange's equation.

[**Solution**]
We start by checking that the vector potential for a magnetic monopole in spherical coordinates has components $A_r = A_\theta = 0$ and $A_\phi = g(-cos\theta)/r\sin\theta$. [Note: The hint given in the probset is $A_r = A_\theta = 0$ and $A_\phi = g(1 - cos\theta)/r\sin\theta$ but we will use the components given above as it would just result to the same value for the magnetic field.]
The curl of A for spherical coordinates is given by:

$$\nabla \times A = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right) \tag{49}$$

Since the r and θ components are zero, this simplifies into

$$\begin{aligned}\nabla \times A &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{g}{r}\cos\theta\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{g\cos\theta}{\sin\theta}\right)\hat{\theta} \\ &= \frac{g}{r^2}\hat{r} = \frac{g\mathbf{r}}{r^3}\end{aligned} \tag{50}$$

Using the form of the generalized potential form Eq (1.142) of NL, we have the generalized potential for this configuration to be

$$U = e\phi - \frac{e}{c}\mathbf{v}\cdot\mathbf{A} \tag{51}$$

where $\phi = 0$ and $\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}$.
Evaluating the generalized potential,

$$\begin{aligned}U &= -\frac{e}{c}\left((\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi})\cdot\left(\frac{-g\cos\theta}{r\sin\theta}\right)\right) \\ &= \frac{eg}{c}\dot{\phi}\cos\theta\end{aligned} \tag{52}$$

Therefore, our Lagrangian would be

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - \frac{eg}{c}\dot{\phi}\cos\theta \tag{53}$$

for spherical coordinates.

Problem 8 (NL Problem 1.20)

In Kepler's Problem, show that for elliptic orbits the relation between r and t can be put in the form

$$t = \sqrt{\frac{m}{2}} \int_{r_{min}}^{r} \frac{dr}{\sqrt{E + \frac{\kappa}{r} - \frac{l^2}{2mr^2}}} = \sqrt{\frac{m}{2\kappa}} \int_{r_{min}}^{r} \frac{rdr}{\sqrt{r - \frac{r^2}{2a} - \frac{a(1-e^2)}{2}}}$$

where a is the semi-major axis, e is the eccentricity and at $t = 0$ th eplanet passes the perihelion. Show that, in terms of the angle ψ known as eccentric anomaly and defined by $r = a(1 - e\cos\psi)$ one has

$$\omega t = \psi - e \sin\psi$$

with $\omega = \sqrt{\kappa/ma^3}$. This last transcendental equation, which implicitly determines ψ as a function of t, is known as Kepler's equation.

[Solution]

We start with the total energy E as given by Eqn, (1.162) in NL.

$$E = \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) \tag{54}$$

To show the relationship of r with t, we first try to isolate the $\dot{r}^2$ term of the equation.

$$\begin{aligned} \dot{r}^2 &= \frac{2}{m}\left(E - \frac{l^2}{2mr^2} - V(r)\right) \\ \dot{r} &= \sqrt{\frac{2}{m}}\sqrt{E - \frac{l^2}{2mr^2} - V(r)} \end{aligned} \tag{55}$$

Since we know that $\dot{r} = dr/dt$, we will have a first order differential equation. We can then isolate dt.

$$dt = \sqrt{\frac{m}{2}} \frac{dr}{\sqrt{E - \frac{l^2}{2mr^2} - V(r)}} \tag{56}$$

Taking the integral of the LHS from t_{min} to t and the integral of the RHS from r_{min} to r

$$t - t_{min} = \sqrt{\frac{m}{2}} \int_{r_{min}}^{r} \frac{dr}{\sqrt{E - \frac{l^2}{2mr^2} - V(r)}} \tag{57}$$

We can say that our system starts from time $t = 0$ such that $t_{min} = t_0 = 0$. We also have the potential $V(r) = -\frac{\kappa}{r}$. Therefore our equation simplifies into,

$$t = \sqrt{\frac{m}{2}} \int_{r_{min}}^{r} \frac{dr}{\sqrt{E - \frac{l^2}{2mr^2} + \frac{\kappa}{r}}}. \tag{58}$$

From Eqn. (1.169) in NL we have the eccentricity e given by $e = \sqrt{1 + 2El^2/m\kappa^2}$ and $l^2/m = a\kappa(1 - e^2)$ that we derived from $p = l^2/mk$ and $a = p/(1 - e^2)$ given in NL. Eqn. (1.169) can then be manipulated to show the value of the total energy E.

$$E = -\frac{\kappa}{2a}. \tag{59}$$

We can also make similar substitutions to the other terms in the denominator in Eqn.58 and get

$$t = \sqrt{\frac{m}{2}} \int_{r_{min}}^{r} \frac{dr}{\sqrt{-\frac{\kappa}{2a} - \frac{a\kappa(1-e^2)}{2r^2} + \frac{\kappa}{r}}} = \sqrt{\frac{m}{2}} \int_{r_{min}}^{r} \frac{dr}{\sqrt{-\frac{\kappa r^2}{2ar^2} - \frac{a^2\kappa(1-e^2)}{2ar^2} + \frac{2ar\kappa}{r^2}}} \tag{60}$$

We can now factor out $\sqrt{\kappa/r^2}$ and make the necessary cancellations in the denominator to yield.

$$t = \int_{r_{min}}^{r} \frac{r}{\sqrt{\kappa}} \frac{dr}{\sqrt{-\frac{r^2}{2a} - \frac{a(1-e^2)}{2} + r}} \tag{61}$$

It is important to note that we were able to get this relationship between r and t since the total energy is conserved for any time t. Next we make the substitution $r = a(1 - e\cos\psi)$. It follows that $dr = ae\sin\psi d\psi$. The limits of integration also change accordingly.

$$t = \sqrt{\frac{m}{2\kappa}} \int_{\psi_{min}}^{\psi} \frac{a^2 e\sin\psi(1-\cos\psi)d\psi)}{\sqrt{a(1-e\cos\psi) - \frac{a^2(1-e\cos\psi)^2}{2a} - \frac{a(1-e^2)}{2}}} \tag{62}$$

We expand our denominator and make necessay cancellations.

$$t = \sqrt{\frac{m}{\kappa}} \int_{\psi_{min}}^{\psi} \frac{a^2 e\sin\psi(1-\cos\psi)d\psi)}{\sqrt{ae^2 - ae^2\cos^2\psi}} \tag{63}$$

Since we know from trigonometry that $cos^2\theta + \sin^2\theta = 1$, we can simplify our expression further. In this next step we also proceed with cancelling the $\sqrt{a}e\sin\psi$ from the numerator and denominator.

$$t = \sqrt{\frac{ma^3}{\kappa}} \int_{\psi_{min}}^{\psi} (1 - e\cos\psi)d\psi \tag{64}$$

We can transfer the constants into the LHS and evaluate the integral and get

$$\omega t = \psi - e\sin\psi \quad \text{where} \quad \omega = \sqrt{\frac{\kappa}{ma^3}}. \tag{65}$$

We set $\psi_{min} = 0$ since this value for ψ gives the minimum value for r.