Physics 221
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Problem Set 4

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1 Problem 1 (NL Problem 7.3)

Given the Hamiltonian $H(q_1, q_2, p_1, p_2) = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2$ with a and b constants, prove that

$$F_1 = \frac{p_2 - b q_2}{q_1}, \quad F_2 = q_1 q_2, \quad F_3 = q_1 e^{-t}$$

are constants of motion. Identify a fourth constant of the motion independent of these three constants and, using them, obtain the general solution to the equations of motion - that is, $q_1(t)$, $q_2(t)$, $p_1(t)$, $p_2(t)$ involving four arbitrary constant0s.

[**Solution**] From the Hamiltonian we can solve for Hamilton's equations

$$\begin{aligned} \dot{q_1} &= \frac{\partial H}{\partial p_1} = q_1, \quad \dot{q_2} = \frac{\partial H}{\partial p_2} = -q_2 \\ \dot{p_1} &= -\frac{\partial H}{\partial q_1} = -p_1 + 2a q_1, \quad \dot{p_2} = -\frac{\partial H}{\partial q_2} = p_2 - 2b q_2. \end{aligned} \tag{1.1}$$

To show that F_1, F_2, and F_3 are constants of motion, we can solve for their total time derivative given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial F}{\partial q_2}\frac{\partial q_2}{\partial t} + \frac{\partial F}{\partial p_1}\frac{\partial p_1}{\partial t} + \frac{\partial F}{\partial p_2}\frac{\partial p_2}{\partial t} + \frac{\partial F}{\partial t} \tag{1.2}$$

for $F = F(q_1, q_2, p_1, p_2)$. Now solving for F_1.

$$\begin{aligned} \frac{dF_1}{dt} &= \frac{\partial F_1}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial F_1}{\partial q_2}\frac{\partial q_2}{\partial t} + \frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial t} + \frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial t} + \frac{\partial F_1}{\partial t} \\ &= \left(\frac{-p_2 + b q_2}{q_1^2}\right) q_1 + \left(\frac{b}{q_1}\right) q_2 + \left(\frac{1}{q_1}\right)(p_2 - 2b q_2) \\ &= 0 \end{aligned} \tag{1.3}$$

Doing the same with F_2 and F_3

$$\begin{aligned} \frac{dF_2}{dt} &= \frac{\partial F_2}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial F_2}{\partial q_2}\frac{\partial q_2}{\partial t} + \frac{\partial F_2}{\partial p_1}\frac{\partial p_1}{\partial t} + \frac{\partial F_2}{\partial p_2}\frac{\partial p_2}{\partial t} + \frac{\partial F_2}{\partial t} \\ &= q_2 q_1 + q_1(-q_2) \\ &= 0 \end{aligned} \tag{1.4}$$

and

$$\begin{aligned}\frac{dF_3}{dt} &= \frac{\partial F_3}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial F_3}{\partial q_2}\frac{\partial q_2}{\partial t} + \frac{\partial F_3}{\partial p_1}\frac{\partial p_1}{\partial t} + \frac{\partial F_3}{\partial p_2}\frac{\partial p_2}{\partial t} + \frac{\partial F_3}{\partial t} \\ &= e^{-t}q_1 + (-q_1)e^{-t} \\ &= 0.\end{aligned} \tag{1.5}$$

Since the total time derivatives of F_1, F_2, and F_3 equals zero, we can say that they are a constant of motion of the system.
To solve for a fourth constant of motion, F_4, we can expose the symmetry of the Hamiltonian by making the following substitution: $q_1 \leftrightarrow q_2$, $p_1 \leftrightarrow p_2$, and $a \leftrightarrow b$ on F_4, which yields

$$F_4 = \frac{p_1 - aq_1}{q_2} \tag{1.6}$$

Therefore from

$$F_1 = \frac{p_2 - bq_2}{q_1}, \quad F_2 = q_1q_2, \quad F_3 = q_1e^{-t}, \quad F_4 = \frac{p_1 - aq_1}{q_2} \tag{1.7}$$

we can solve for the quantities q_1, q_2, p_1, and p_2 in terms of t and arbitrary constants F's. We can start by solving for q_1 in terms of F_3

$$q_1 = F_3e^t. \tag{1.8}$$

Then with $q_1(t)$ we can solve for $q_2(t)$

$$q_2(t) = \frac{F_2}{F_3}e^{-t}. \tag{1.9}$$

Then finally we recover $p_1(t)$ and $p_2(t)$

$$p_1(t) = \frac{F_2F_4}{F_3}e^{-t} + aF_3e^t, \quad p_2(t) = F_1F_3e^t + b\frac{F_2}{F_3}e^{-t} \tag{1.10}$$

2 Problem 2 (NL Problem 7.5)

A one-degree-of-freedom mechanical system is described by the Lagrangian

$$L(Q, \dot{Q}, t) = \frac{\dot{Q}}{2}\cos^2(\omega t) - \frac{\omega}{2}Q\dot{Q}\sin(2\omega t) - \frac{\omega^2Q^2}{2}\cos(2\omega t)$$

(a) Find the corresponding Hamiltonian. (b) Is this Hamiltonian a constant of motion? (c) Show that the Hamiltonian expressed in terms of the new variable $q = Q\cos(\omega t)$ and its conjugate momentum does not depend explicitly on time. What physical system does this describe?

[Solution]
(a) We can solve for the Hamiltonian from the Lagrangian

$$H(Q, P, t) = \sum \dot{Q}P - L(Q, \dot{Q}, t) \tag{2.1}$$

and the canonical conjugate momentum to $\dot{Q}$ given by $P = \partial L/\partial \dot{Q}$.

$$P = \dot{Q}\cos^2(\omega t) - \frac{\omega}{2} Q \sin(2\omega t) \to \dot{Q} = \frac{P}{\cos^(\omega t)} + \frac{\omega Q \sin(2\omega t)}{2\cos^2(\omega t)} \tag{2.2}$$

The Hamiltonian then has the form

$$\begin{aligned} H &= \dot{Q}^2 \cos^2(\omega t) - \frac{\omega}{2} Q\dot{Q}\sin(2\omega t) - \frac{\dot{Q}^2}{2}\cos^2(\omega t) + \frac{\omega}{2} Q\dot{Q}\sin(2\omega t) + \frac{\omega^2 Q^2}{2}\cos(2\omega t) \\ &= \frac{\dot{Q}^2}{2}\cos^2(\omega t) + \frac{\omega^2 Q^2}{2}\cos(2\omega t) \end{aligned} \tag{2.3}$$

Since the Hamiltonian should depend on Q,P, and t, we need to rewrite the Hamiltonian that we got. Plugging in the $\dot{Q}$ that we solve for from the conjugate momentum

$$\begin{aligned} H(Q,P,t) &= \frac{\omega^2 Q^2}{2}\cos(2\omega t) + \left(\frac{P}{\cos^(\omega t)} + \frac{\omega Q \sin(2\omega t)}{2\cos^2(\omega t)}\right)^2 \frac{\cos^2(\omega)}{2} \\ &= \frac{P^2}{2\cos^2(\omega t)} + \frac{\omega^2 Q^2 \sin^2(2\omega t)}{8\cos^2(\omega t)} + \frac{P\omega Q \sin(2\omega t)}{2\cos^2(\omega t)} + \frac{\omega^2 Q^2}{2}\cos(2\omega t) \\ &= \frac{P^2 \sec^2(\omega t)}{2} + \frac{4\omega^2 Q^2 \sin^2(\omega t)\cos^2(\omega t)}{8\cos^2(\omega t)} + \frac{2P\omega Q \sin(\omega t)\cos(\omega t)}{2\cos^2(\omega t)} + \frac{\omega^2 Q^2}{2}(\cos^2(\omega t) - \sin^2(\omega t)) \\ &= \frac{P^2 \sec^2(\omega t)}{2} + \frac{\omega^2 Q^2}{2}\cos^2(\omega t) + P\omega Q \tan(\omega t) \end{aligned} \tag{2.4}$$

(b) To see if the Hamiltonian is a constant of motion we get its total time derivative

$$\frac{dH}{dt} = \left(\frac{H}{\partial Q}\frac{\partial Q}{\partial t} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial t}\right) + \frac{\partial H}{\partial t}. \tag{2.5}$$

Since $\dot{Q} = \partial H/\partial P$ and $\dot{P} = -\partial H/\partial Q$, we are left with

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial t} \\ &= P^2 \sec^2(\omega t)\tan(\omega t) - \omega^2 Q^2 \cos^2(\omega t)\sec(\omega t) + P\omega Q \sec^2(\omega t) \\ &\neq 0. \end{aligned} \tag{2.6}$$

Therefore, the Hamiltonian is not a constant of motion.
(c) Going back to the form of the Lagrangian

$$L(Q,\dot{Q},t) = \frac{\dot{Q}}{2}\cos^2(\omega t) - \frac{\omega}{2} Q\dot{Q}\sin(2\omega t) - \frac{\omega^2 Q^2}{2}\cos(2\omega t) \tag{2.7}$$

where

$$q = Q\cos(\omega t), \quad \dot{q} = \dot{Q}\cos(\omega t) - Q\omega\sin(\omega t). \tag{2.8}$$

Expanding the Lagrangian

$$
\begin{aligned}
L(Q,\dot{Q},t) &= \frac{\dot{Q}}{2}\cos^2(\omega t) + \frac{2\omega Q\dot{Q}\sin(\omega t)\cos(\omega t)}{2} + \frac{\omega^2 Q^2}{2}\sin^2(\omega t) - \frac{\omega^2 Q^2}{2}\cos^2(\omega t) \\
&= \frac{\dot{q}^2}{2} - \frac{\omega^2 q^2}{2}
\end{aligned} \tag{2.9}
$$

Notice that the conjugate momentum to $\dot{q}$ is

$$
p = \frac{\partial L}{\partial dotq} = \dot{q} \tag{2.10}
$$

Solving for the Hamiltonian

$$
\begin{aligned}
H &= p\dot{q} - \frac{\dot{q}^2}{2} + \frac{\omega^2 q^2}{2} \\
&= p^2 - \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \\
&= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2
\end{aligned} \tag{2.11}
$$

which is the Hamiltonian of a simple harmonic oscillator.

3 Problem 3 (NL Problem 7.6)

Consider the n-body problem in the centre-of-mass frame. The Hamiltonian is given by $H = T + V$ where $T = \sum_i |p_i|^2/2m_i$ and

$$
V(r_1, ...r_n) = -\frac{1}{2}\sum_{i,j;i\neq j}\frac{Gm_i m_j}{|r_i - r_j|}
$$

(a) Introduce the quantity $I = \frac{1}{2}\sum_i m_i|r_i|^2$ and prove that $\ddot{I} = E + V$. Hint" show that V is a homogeneous function and apply Euler's theorem from Appendix C. (b) Taking into account that T is always positive, perform two successive integrations to prove that $I(t) \geq I(0) + \dot{I(0)}t + Et^2/2$. Conclude, finally, that if the total energy is positive at least one of the bodies will escape to infinity in the limit $t \to \infty$

[Solution] (a) Starting from the definition I

$$
I = \frac{1}{2}\sum_i m_i|r_i|^2 \tag{3.1}
$$

then taking the first and second derivative

$$
\begin{aligned}
\dot{I} &= \sum_i m_i|r_i||\dot{r_i}| \\
\ddot{I} &= \sum_i m_i|r_i|^2 + \sum_i m_i|r_i||\ddot{r_i}|
\end{aligned} \tag{3.2}
$$

Applying definition C.1 on the potential V

$$V(\lambda r_i, ...\lambda r_n) = -\frac{1}{2}\sum \frac{Gm_i m_j}{\lambda\sqrt{r_i^2 + r_j^2}} = \lambda^{-1}V(r_i, ..., r_j) \tag{3.3}$$

which just shows that V is homogeneous of degree -1. By euler's theorem

$$\sum_{k=1}^{n} r_k \frac{\partial V}{\partial r_k} = -V \tag{3.4}$$

To show that $\ddot{I} = 2T + V$, we can consider the following the momentum and its derivative with respect to t

$$\begin{aligned} p &= m\dot{r} \\ \dot{p} &= m\ddot{r} = F = -\frac{\partial V}{\partial r}. \end{aligned} \tag{3.5}$$

Therefore, $\ddot{I}$ simplifies into

$$\ddot{I} = 2T + V = E + T \tag{3.6}$$

from the definition of T and V and utilizing Euler's theorem as shown above.
(b) From $\ddot{I} = E + V$, we perform two successive integrations to yield

$$\dot{I} = Et + Tt + \dot{I}(0) \tag{3.7}$$

and

$$I = \frac{1}{2}Et^2 + \frac{1}{2}Tt^2 + \dot{I}(0) + I(0). \tag{3.8}$$

With this form we can clearly see that

$$I(t) \geq I(0) + I(\dot{0})t + \frac{1}{2}Et^2 \tag{3.9}$$

because $I(t)$ yields an extra $Tt^2/2$ term, and since T is always positive the inequality holds and only equates when $t = 0$. When we let t approach infinity and keep T as positive, we notice that the moment of inertia, I, also approach infinity, and as a consequence r also approach infinity since $I \propto r^2$. We can also check the force of attraction

$$F = G\frac{m_i m_j}{r^2} = 0 \quad (\text{as r} \to \infty). \tag{3.10}$$

This means that there is no interaction between the objects/bodies and confirms our conclusion that at least one of the bodies fly off to infinity.

4 Problem 4 (NL Problem 7.7)

Show that in terms of the polar coordinates r, ϕ and their conjugate momenta, Eq. (7.32) takes the form

$$2\langle T\rangle = -\langle V\rangle - \left\langle \frac{p_\phi^2}{mr^2}\right\rangle$$

Comparing this result to (7.41), show that $\langle p_\phi^2/mr^2\rangle = 0$. Using the fact that for elliptical orbits p_ϕ is a non-zero constant of motion, conclude that $\langle 1/r^2\rangle = 0$. Argue that this is impossible. Examine carefully the conditions for the validity of the virial theorem in the present case and explain why its use in polar ccoordinates lead to an absurdity.

[Solution]
From eqn. (7.15), we recover the Hamiltonian in the orbital plane

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{k}{r} \quad \text{where } V = -\frac{k}{r} \tag{4.1}$$

By using the virial theorem, we get

$$\begin{aligned} \left\langle p_r\left(\frac{p_r}{m}\right) + p_\phi\left(\frac{p_\phi}{mr^2}\right)\right\rangle &= \left\langle r\left(\frac{k}{r^2}\right)\right\rangle - \left\langle r\frac{p_\phi^2}{mr^3}\right\rangle \\ \left\langle \frac{p_r^2}{m} + \frac{p_\phi^2}{mr^2}\right\rangle &= \left\langle \frac{k}{r}\right\rangle - \left\langle \frac{p_\phi^2}{mr^2}\right\rangle \end{aligned} \tag{4.2}$$

and by considering the value for T and V, the above equation further simplifies into

$$2\langle T\rangle = \langle -V\rangle - \left\langle \frac{p_\phi^2}{mr^2}\right\rangle \tag{4.3}$$

Comparing this with eqn. (7.41), we see that

$$\left\langle \frac{p_\phi^2}{mr^2}\right\rangle = 0. \tag{4.4}$$

Since for elliptic orbits, p_ϕ is a non-zero constant of motion, we conclude that $1/r^2 = 0$ or $r \to \infty$, which is impossible because if we consider the centripetal force between the sun and a planet an infinite distance away

$$F = G\frac{m_1 m_2}{r^2} = 0 \tag{4.5}$$

we see that the force is zero. This just implies that the planet we are considering in this system doesn't revolve around the sun as there is no force and, therefore, no interaction between the two bodies. Also, if we go back to the requirements of validity of the virial theorem, we see that coordinates q and p should be bounded. In this example, since $r \to \infty$, we see that r is not bounded. Therefore, the use of polar coordinates leads to an absurdity.

5 Problem 5 (NL Problem 7.12)

Construct the Legendre transformation that takes from $H(q,p,t)$ and $Y(p,\dot{p},t)$ and derive the equations of motion in terms of Y. Apply this approach to the particular case in which $H = p^2/2 + \omega^2 q^2/2$ and comment on the results obtained.

[Solution]
We have the Legendre transform in the form of

$$g = f - wy \tag{5.1}$$

where g is our new function, f is our original function, $w = \partial f/\partial y$ and y is the variable we want to switch.
Following the this to derive $Y(p,\dot{p},t)$ from the Hamiltonian, we get

$$\begin{aligned} Y(p,\dot{p},t) &= H(q,p,t) - \sum_i \frac{\partial H(q,p,t)}{\partial q_i} q_i \\ &= H(q,p,t) + \sum_i \dot{p}q \end{aligned} \tag{5.2}$$

where we used $\dot{p} = -\partial H/\partial q$ and $\dot{q} = \partial H/\partial p$. Taking the equations of motion

$$\begin{aligned} &\frac{\partial Y}{\partial p} = \frac{\partial H}{\partial p} + \frac{\partial H}{\partial q}\frac{\partial q}{\partial p} + \dot{p}\frac{\partial q}{\partial p} = \dot{q} \\ &\frac{\partial Y}{\partial \dot{p}} = q. \end{aligned} \tag{5.3}$$

where we used the substitution for $\dot{q}$ and $\dot{p}$ given above.
Now that we have the form for $Y(p,\dot{p},t)$, we can then plug in the given Hamiltonian

$$H(q,p,t) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \tag{5.4}$$

Therefore

$$Y(q,p,t) = \frac{p^2}{2} - \frac{\omega^2 q^2}{2} \tag{5.5}$$

Since Y should be with respect to $\dot{p}$ and not q, we need to rewrite the above equation by using

$$\dot{p} = -\frac{\partial H}{\partial q} = -\omega^2 q \rightarrow q = -\frac{\dot{p}}{\omega^2}. \tag{5.6}$$

Therefore the Yamiltonian (as what I like to call it) is recast in the form

$$Y(p,\dot{p},t) = \frac{p^2}{2} - \frac{\dot{p}^2}{2\omega^2} \tag{5.7}$$

Note: Note that this form of the Yamiltonian describes a Lagrangian that is similar to the form of the Lagrangian for a simple harmonic oscillator for different variables. In fact the given Hamiltonian is similar to the Hamiltonian of a simple harmonic oscillator. We then conclude that Yamiltonian is just the Lagrangian recast in different variables.

6 Problem 6 (NL Problem 7.21)

A one-degree-of-freedom mechanical system obeys the following equations of motion: $\dot{q} = pf(q)$, $\dot{p} = g(q,p)$. What restrictions must be imposed on the functions f and g in order that this system be Hamiltonian? If the system is Hamiltonian, what is the general form of Hamilton's function H?

[Solution]

Starting off with the form of the Hamiltonian $H(q,p,t)$ and the Lagrangian $Y(p,\dot{p},t)$ in the form of the Lagrangian $L(q,\dot{q},t)$ and the Hamiltonian, respectively.

$$H(q,p,t) = \sum_i \dot{q}_i p_i - L(q,\dot{q},t) \quad \text{and} \quad Y(p,\dot{p},t) = \sum_i \dot{p}_i q_i + H(q,p,t). \tag{6.1}$$

Solving for Hamiltonian we get

$$H = \frac{1}{2}\sum_i \dot{q}_i p_i - \frac{1}{2}\sum_i \dot{p}_i q_i. \tag{6.2}$$

Since we are given values for $\dot{q}$ and $\dot{p}$, we can rewrite the Hamiltonian as

$$H = \frac{1}{2}\sum_i f(q)p_i^2 - \frac{1}{2}\sum_i g(q,p)q_i. \tag{6.3}$$

To check the restrictions on the functions f and q for the system to be Hamiltonian, we check Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q} \quad \text{and} \quad \dot{q} = \frac{\partial H}{\partial p}. \tag{6.4}$$

From the recast Hamiltonian, we see the Hamilton's equations

$$\begin{aligned} pf(q) &= pf(q) - \frac{1}{2}\frac{\partial g(q,p)}{\partial p}q \\ g(q,p) &= \frac{1}{2}\frac{\partial g(q,p)}{\partial p}q + \frac{1}{2}g(q,p) - \frac{1}{2}p^2\frac{\partial f(q)}{\partial q}. \end{aligned} \tag{6.5}$$

From the first of the two equations, we see that

$$\frac{1}{2}\frac{\partial g(q,p)}{\partial p}q = 0. \tag{6.6}$$

Solving the differential equation to get the form of $g(q,p)$, we see that the function g should be a function dependent only on q

$$g(q,p) = G(q). \tag{6.7}$$

With this result we can proceed to solve the second hamilton's equation. By plugging in $g(q,p) =$

$G(q)$

$$
\begin{aligned}
G(q) &= \frac{1}{2}G(q) - \frac{1}{2}p^2\frac{\partial f(q)}{\partial q} \\
G(q) &= -p^2\frac{\partial f(q)}{\partial q}.
\end{aligned}
\tag{6.8}
$$

This leads to an inconsistency since we established earlier that $G(q)$ only depends on q. Also $f(q)$ is only dependent on q_i. So $f(q)$ should be equal to zero. Therefore, the requirements for this system to be a Hamiltonian are

$$
g(q,p) = G(q) \quad \text{and} \quad f(q) = 0 \tag{6.9}
$$

Now, the most general form of the Hamiltonian is

$$
H = -\frac{G(q)q}{2} \tag{6.10}
$$