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Physics 221 Probset 2

1. From eq (2.59) of NL

$$S[\bar{x}] - S[x] = \int_{t_1}^{t_2} \left[\frac{m}{2}\dot{\eta}^2 - \frac{1}{2}V''(x+\xi)\eta^2 \right] dt$$

If $X_{ph}(t)$ is the solution of the equations of motion and $x(t) = X_{ph}(t) + \eta(t)$ w/ $\eta(0) = \eta(T) = 0$, we have

$$S[\bar{x}] - S[X_{phi}] = \int_0^T \left[\frac{m}{2}\dot{\eta}^2 - \frac{1}{2}V''(x+\xi)\eta^2 \right] dt$$

Since the potential for a harmonic oscillator is

$$V = \frac{1}{2}k(x+\xi)^2 \longrightarrow V'' = k$$

where: $\omega = \sqrt{k/m} \longrightarrow k = \omega^2 m$. Therefore

$$S[\bar{x}] = S[X_{phi}] + \int_0^T \left[\frac{m}{2}\dot{\eta}^2 - \frac{m}{2}\omega^2\eta^2 \right] dt = S[X_{phi}] + \frac{m}{2}\int_0^T [\dot{\eta}^2 - \omega^2\eta^2] dt$$

We can expand $\eta(t)$ in the fourier series

$$\eta(t) = \sum_{n=1}^{\infty} C_n \sin(n\pi t/T)$$

note: we can do this substitution because $\eta(0) = \eta(T) = 0$.
Substituting this series expansion, we have

$$S[\bar{x}] = S[X_{phi}] + \sum_{n=1}^{\infty} \frac{m}{2}\int_0^T \left[C_n^2 \left(\frac{n\pi}{T}\right)^2 \cos^2\left(\frac{n\pi t}{T}\right) - \omega^2 C_n^2 \sin^2\left(\frac{n\pi t}{T}\right) \right] dt$$

$$= S[X_{phi}] + \sum_{n=1}^{\infty} \frac{m}{2} C_n^2 \int_0^T \left[\left(\frac{n\pi}{T}\right)^2 \cos^2\left(\frac{n\pi t}{T}\right) - \omega^2 \sin^2\left(\frac{n\pi t}{T}\right) \right] dt$$

$$= S[x_{phi}] + \frac{m}{2}\sum_{n=1}^{\infty} C_n^2 \left[\int_0^T \left(\frac{n\pi}{T}\right)^2 \cos^2\left(\frac{n\pi t}{T}\right) dt - \omega^2 \int_0^T \sin^2\left(\frac{n\pi t}{T}\right) \right] dt$$

Note that:

$$\int_0^T \cos^2\left(\frac{n\pi t}{T}\right) dt = \left[\frac{1}{2}x + \frac{1}{4}\sin\left(\frac{2n\pi t}{T}\right) \right]_0^T = \frac{T}{2}$$

$$\int_0^T \sin^2\left(\frac{n\pi t}{T}\right) dt = \left[\frac{1}{2}x - \frac{1}{4}\sin\left(\frac{2n\pi t}{T}\right) \right]_0^T = \frac{T}{2}$$

Therefore,

$$S[\bar{x}] = S[X_{phi}] + \frac{mT}{4}\sum_{n=1}^{\infty} C_n^2 \left(\frac{n^2\pi^2}{T^2} - \omega^2 \right)$$

From the above equation, the action seems to be at a minimum when the summation term is 0. Then,

$$n^2\pi^2/T^2 - \omega^2 \geq 0$$

Also notice that it is at its minimum when $n = 1$.

$$\pi^2/T^2 - \omega^2 \geq 0$$

$$\pi^2/\omega^2 \geq T^2$$

or

$$\pi/\omega \geq T$$

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3. We have the Lagrangian for the charged particle in an EM field

$$L = \frac{1}{2}mv^2 - e\phi + \frac{e}{c}v \cdot A$$

We can define a new Lagrangian

$$\mathcal{L} = \frac{d}{dt}\left(\frac{d}{d\dot{q}_k} L\right) - \frac{d}{dq_k} L$$

Substituting the Lagrangian

$$\mathcal{L} = \frac{d}{dt}\left(\frac{d}{d\dot{q}_k}\left(\frac{1}{2}mv^2 - e\phi + \frac{e}{c}v \cdot A\right)\right) - \frac{d}{dq_k}\left(\frac{1}{2}mv^2 - e\phi + \frac{e}{c}v \cdot A\right)$$

under the gauge transform

$$A \rightarrow A' = A + \nabla\Lambda$$
$$\phi \rightarrow \phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}$$

the Lagrangian of $\mathcal{L}'$ becomes

$$\mathcal{L}' = \frac{d}{dt}\left(\frac{d}{d\dot{q}_k}\left(\frac{1}{2}mv^2 - e\left(\phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) + \frac{e}{c}v \cdot (A + \nabla\Lambda)\right)\right)$$
$$- \frac{d}{dq_k}\left(\frac{1}{2}mv^2 - e\left(\phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right) + \frac{e}{c}(v \cdot (A + \nabla\Lambda))\right)$$

$$\mathcal{L}' = \frac{d}{dt}\left(\frac{d}{d\dot{q}_k}\left(\frac{1}{2}mv^2 - e\phi + \frac{e}{c}v \cdot A\right)\right) - \frac{d}{dq_k}\left(\frac{1}{2}mv^2 - e\phi + \frac{e}{c}v \cdot A\right)$$
$$+ \frac{e}{c}\left(\frac{d}{dt}\left(\frac{d}{d\dot{q}_k}\left(\frac{\partial \Lambda}{\partial t} + v \cdot \nabla\Lambda\right)\right) - \frac{d}{dq_k}\left(\frac{\partial \Lambda}{\partial t} + v \cdot \nabla\Lambda\right)\right)$$

$$\mathcal{L}' = \mathcal{L} + \frac{e}{c}\left(\frac{d}{dt}\left(\frac{d}{d\dot{q}_k}\left(\underbrace{\cancel{\frac{\partial \Lambda}{\partial t}}}_{0,\ \text{no } \dot{q}_k \text{ dependence}} + v \cdot \nabla\Lambda\right)\right) - \frac{d}{dq_k}\left(\frac{\partial \Lambda}{\partial t} + v \cdot \nabla\Lambda\right)\right)$$

$$\mathcal{L}' = \mathcal{L} + \frac{e}{c}\left(\frac{d}{dt}\nabla\Lambda - \frac{d}{dq_k}\left(\frac{\partial \Lambda}{\partial t} + v \cdot \nabla\Lambda\right)\right)$$

we were able to simplify this because $\dot{q}_k = v$. Using the definition of total derivatives and chain rul

$$\mathcal{L}' = \mathcal{L} + \frac{e}{c}\left[\frac{\partial}{\partial t}\frac{\partial}{\partial q_k}\Lambda + \frac{\partial}{\partial q_k}\frac{\partial}{\partial q_k}\Lambda\,\dot{q}_k - \frac{\partial}{\partial q_k}\frac{\partial}{\partial t}\Lambda - \dot{q}_k\frac{\partial}{\partial q_k}\frac{\partial}{\partial q_k}\Lambda\right]$$

These terms just cancel out. Therefore,

$$\mathcal{L}' = \mathcal{L}$$

Meaning there's no difference in the Lagrangian and EOM under the gauge transform.

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4a. We can plug-in the infinitesimal transformation $x \to x' = x + \epsilon\beta$, $y \to y' = y - \epsilon\alpha$ into the Lagrangian

$$L = \frac{m}{2}\left(\left(\frac{d}{dt}(x+\epsilon\beta)\right)^2 + \left(\frac{d}{dt}(y-\epsilon\alpha)\right)^2\right) - (\alpha x + \epsilon\alpha\beta + \beta y - \epsilon\alpha\beta)$$

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - (\alpha x + \beta y)$$

Notice that we arrive with our original Lagrangian. Therefore, the Lagrangian is invariant under the infinitesimal transform we introduced. Applying Noether's theorem given by

$$C = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}(\dot{q}_i X - \Psi_i) - LX$$

where $q_1 = x$, $\Psi_1 = \beta$ and $q_2 = y$, $\Psi_2 = -\alpha$, we have

$$C = \frac{\partial L}{\partial \dot{x}}(0 - \beta) + \frac{\partial L}{\partial \dot{y}}(0 + \alpha) - 0$$

$$C = m\dot{x}(-\beta) + m\dot{y}(\alpha)$$

which is proportional to

$$A = \beta\dot{x} - \alpha\dot{y}$$

Therefore A is a constant of motion.

4b. Introducing new generalised coordinates

$$\bar{x} = \alpha x + \beta y \quad , \quad \bar{y} = \beta x - \alpha y$$

Our Lagrangian turns into

$$L = \frac{m}{2(\alpha^2 + \beta^2)}(\dot{\bar{x}}^2 + \dot{\bar{y}}^2) - \bar{x}$$

since

$$\dot{\bar{x}}^2 = \alpha^2\dot{x}^2 + 2\alpha\beta\dot{x}\dot{y} + \beta^2\dot{y}^2$$

$$\dot{\bar{y}}^2 = \beta^2\dot{x}^2 - 2\alpha\beta\dot{x}\dot{y} + \alpha^2\dot{y}^2$$

Notice that $\bar{y}$ does not appear in the new form of the Lagrangian. Therefore $\bar{y}$ is the cyclic coordinate. The momentum conjugate to the cyclic coordinate $\bar{y}$ is then

$$p_{\bar{y}} = \frac{\partial L}{\partial \dot{\bar{y}}} = \frac{m}{\alpha^2 + \beta^2}\dot{\bar{y}}$$

remember that $\dot{\bar{y}} = (\beta\dot{x} - \alpha\dot{y})$

$$p_{\bar{y}} = \frac{m}{\alpha^2 + \beta^2}(\beta\dot{x} - \alpha\dot{y})$$

which is proportional to A in 4a. If you think of this geometrically, the infinitesimal translational translation is made such that $\bar{y}$ is a cyclic coordinate.

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5a. To show that the Laplace - Runge - Lenz vector is a constant of motion, we need to show that $A(t) = 0$.

$$\frac{dA}{dt} = \frac{d}{dt}(\vec{p} \times \vec{l}) - mk\frac{d}{dt}\hat{r}$$

$$= \frac{d\vec{p}}{dt} \times \vec{l} + \frac{d\vec{l}}{dt} \times \vec{p} - mk\frac{d}{dt}\hat{r}$$

force

0, because l is constant for central force problems

Therefore, we can rewrite this into

$$\frac{dA}{dt} = -\frac{k}{r^2}\hat{r} \times m(\vec{r} \times \dot{\vec{r}}) - mk\frac{d}{dt}\hat{r}$$

force displacement velocity

We can rewrite this as

$$\frac{dA}{dt} = -\frac{mk}{r^2}(\dot{r}\,\vec{r} - r\,\dot{\vec{r}}) - mk\frac{d}{dt}\hat{r} = -mk\left(\frac{\dot{r}\vec{r}}{r^2} - \frac{\dot{\vec{r}}}{r}\right) - mk\frac{d}{dt}\hat{r}$$

notice that

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{\dot{\vec{r}}}{r} - \frac{\vec{r}}{r^2}\dot{r}$$

therefore,

$$\frac{dA}{dt} = mk\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) - mk\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = 0$$

Therefore, A is a constant of motion.

5b. To show that A lies on the plane of the orbit, we can take its dot product in terms of $\vec{l}$.

$$A\cdot\vec{l} = [\vec{p}\times\vec{l} - mk\frac{\vec{r}}{r}] \times \vec{l}$$

$$= (\vec{p}\times\vec{l})\times\vec{l} - mk\frac{\vec{r}}{r}\times(\vec{r}\times\vec{p})$$

perpendicular to $\vec{l}$ perpendicular to $\vec{l}$

Therefore,

$$A\cdot\vec{l} = 0$$

which implies that A lies in the plane of motion.

5c. We take the scalar product of A with $\hat{r}$

$$A\cdot\vec{r} = Ar\cos\theta = r\cdot(\vec{p}\times\vec{l}) - mkr$$

We are given that $\vec{r}\cdot(\vec{p}\times\vec{l}) = \vec{l}\cdot\vec{l} = l^2$. Therefore

$$A\cdot\vec{r} = Ar\cos\theta = l^2 - mkr$$

We can then rearrange this to solve for $1/r$

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \frac{A}{mk}\cos\theta\right)$$

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5d. comparing our result from 5c with the kepler problem result given by

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + e\cos\theta\right)$$

↳ kepler problem

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \frac{A}{mk}\cos\theta\right)$$

↳ result from 5c

We notice that

$$e = \frac{A}{mk}$$

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6a. Similar to #5, we take the time derivative of Q and show that it is 0.

$$\frac{dQ}{dt} = \frac{d}{dt}\left(m(\vec{r}\times\vec{v})\right) - \frac{d}{dt}\left(\frac{eg}{c}\hat{r}\right)$$

$$= m(\vec{v}\times\vec{v}) + m(\vec{r}\times\vec{a}) - \frac{eg}{c}\left(\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)\right)$$

$$= m\underset{0}{\cancel{(\vec{v}\times\vec{v})}} + \underset{\frac{eg}{c}\vec{r}\times(\vec{v}\times\hat{r})}{\underset{\downarrow}{m(\vec{r}\times\vec{a})}} - \frac{eg}{c}\left(\frac{\vec{v}}{r} - \frac{\vec{r}}{r^2}\frac{dr}{dt}\right)$$

Therefore, this simplifies into

$$\frac{dQ}{dt} = \frac{eg}{c}\left[\frac{1}{r^3}\left(r^2\vec{v} - \vec{r}(\vec{r}\cdot\vec{v})\right) - \frac{\vec{v}}{r} + \frac{\vec{r}}{r^2}\frac{d\sqrt{\vec{r}\cdot\vec{r}}}{dt}\right]$$

$$= \frac{eg}{c}\left[-\frac{1}{r^3}\vec{r}(\vec{r}\cdot\vec{v}) + \frac{\vec{r}}{r^3}(\vec{r}\cdot\vec{v})\right]$$

$$= 0$$

We can then conclude that Q is a constant of motion.

6b. Consider that

$$\hat{Q}\cdot\hat{\phi} = 0, \quad \text{where } \hat{Q} = Q\hat{r}$$

picking the z-axis parallel to Q, we have

$$Q\hat{z}\cdot\hat{\phi} = 0$$

Evaluating $\hat{Q}\cdot\hat{\phi}$, we have

$$\hat{Q}\cdot\hat{\phi} = m(\vec{r}\times\vec{v})\cdot\hat{\phi} - \underset{0}{\cancel{\frac{eg}{c}\hat{r}\cdot\hat{\phi}}} = m(\vec{r}\times\vec{v})\cdot\hat{\phi}$$

considering that the system is in spherical coordinates, we have

$$\hat{Q}\cdot\hat{\phi} = m\left[(r\hat{r})\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi})\right]\cdot\hat{\phi}$$

$$= mr^2\dot{\theta} = 0$$

For arbitrary r, we then have

$$\dot{\theta} = 0$$

Therefore, θ is a constant of motion.