Physics 241
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Problem Set 2

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1 TIME EVOLUTION OF A GAUSSIAN WAVEPACKET

A particle is initially in the ground state of a harmonic oscillator potential. It then evolves freely in time (the potential is turned off).

1.a Use the free-particle propagator to determine the time evolution of its wavefunction.

[Solution]

We have the normalized ground-state wave function of the harmonic oscillator to be

$$\langle x'|0\rangle = \frac{1}{\pi^{\frac{1}{4}}\sqrt{x_0}} exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right] \tag{1}$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ sets the length scale of the oscillator. We also have the free-particle propagator given by

$$K(x'', t; x', t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right]. \tag{2}$$

To determine the time evolution of the corresponding wave function from (x', t_0) to (x'', t), we solve for the integral

$$\psi(x'', t) = \int d^3x' K(x'', t; x', t_0)\psi(x', t_0). \tag{3}$$

Plugging in the values for the ground-state wave function and the free-particle propagator, we have.

$$\psi(x'', t) = \int d^3x' \sqrt{\frac{m}{2\pi^{3/2} x_0 i\hbar(t-t_0)}} exp\left[\frac{im(x''-x')^2}{2\hbar(t-t_0)}\right] exp\left[-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2\right]. \tag{4}$$

We can then combine the 2 exponentials and group the like terms to prepare for the completing the square method

$$\psi(x'', t) = \int d^3x' \sqrt{\frac{m}{2\pi^{3/2} x_0 i\hbar(t-t_0)}} exp\left[\frac{imx_0^2 - h(t-t_0)}{2\hbar(t-t_0)x_0^2}\left(x'^2 - \frac{2imx_0^2 x'' x'}{imx_0^2 - \hbar(t-t_0)} + \frac{imx_0^2 x''^2}{imx_0^2 - \hbar(t-t_0)}\right)\right] \tag{5}$$

We can then make the following substitution

$$
\begin{aligned}
A &= \frac{imx_0^2 - \hbar(t-t_0)}{2\hbar(t-t_0)x_0^2} \quad , \quad B = \frac{2imx_0^2 x''}{imx_0^2 - \hbar(t-t_0)} \\
C &= \frac{imx_0^2 x''^2}{imx_0^2 - \hbar(t-t_0)}
\end{aligned} \tag{6}
$$

to make the integral look like

$$
\begin{aligned}
\psi(x'',t) &= \sqrt{\frac{m}{2\pi^{3/2}x_0 i\hbar(t-t_0)}} \int d^3x' exp\left[A(x' - \frac{B}{2})^2 + A\left(C - \frac{B^2}{4}\right)\right] \\
&= \sqrt{\frac{m}{2\pi^{3/2}x_0 i\hbar(t-t_0)}} exp\left[A\left(C - \frac{B^2}{4}\right)\right] \int d^3x' exp\left[A(x' - \frac{B}{2})^2\right].
\end{aligned} \tag{7}
$$

Notice that the exponential inside the integral is in the form of the gaussian integral given by

$$
\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \tag{8}
$$

when we make the substitution $y = x' - B/2$ and $dy = dx'$, with $a = -A$. Therefore, we have

$$
\psi(x'',t) = \sqrt{\frac{m}{2\pi^{3/2}x_0 i\hbar(t-t_0)}} exp\left[A\left(C - \frac{B^2}{4}\right)\right]\left(\sqrt{\frac{\pi}{-A}}\right). \tag{9}
$$

Putting back the values for A, B, and C

$$
\begin{aligned}
\psi(x'',t) =& \sqrt{\frac{m}{2\pi^{1/2}x_0 i\hbar(t-t_0)}} \sqrt{\frac{-2\hbar(t-t_0)x_0^2}{imx_0^2 - \hbar(t-t_0)}} \\
& exp\left[\frac{imx_0^2 - \hbar(t-t_0)}{2\hbar(t-t_0)x_0^2}\left(\frac{imx_0^2 x''^2}{imx_0^2 - \hbar(t-t_0)} + \frac{m^2x_0^4 x''^2}{(imx_0^2 - \hbar(t-t_0))^2}\right)\right]
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\psi(x'',t) =& \sqrt{\frac{m}{2\pi^{1/2}x_0 i\hbar(t-t_0)}} \sqrt{\frac{-2\hbar(t-t_0)x_0^2}{imx_0^2 - \hbar(t-t_0)}} \\
& exp\left[\frac{imx''^2}{2\hbar(t-t_0)} + \frac{m^2x_0^2 x''^2}{(2\hbar(t-t_0))(imx_0^2 - \hbar(t-t_0))}\right].
\end{aligned} \tag{11}
$$

Rationalizing both the square root and exponential,

$$
\begin{aligned}
\psi(x'',t) =& \sqrt{\frac{m}{2\pi^{1/2}x_0 i\hbar(t-t_0)}} \sqrt{\frac{2i\hbar m x_0^4(t-t_0) + 2\hbar^2(t-t_0)^2x_0^2}{m^2x_0^4 + \hbar^2(t-t_0)^2}} \\
& exp\left[\frac{-imx''^2}{2(imx_0^2 - \hbar(t-t_0))}\right] \\
\psi(x'',t) =& \sqrt{\frac{m^2x_0^3 - i\hbar(t-t_0)x_0 m}{\pi^{1/2}(m^2x_0^4 + \hbar^2(t-t_0)^2)}} exp\left[\frac{-m^2x_0^2x''^2 + i\hbar m x''^2(t-t_0)}{2(m^2x_0^4 + \hbar^2(t-t_0)^2)}\right]
\end{aligned} \tag{12}
$$

1.b Determine how the probability density spreads in time.

[Solution]

Solving for the probability density given by

$$\rho(x'',t) = \psi(x'',t)^* \psi(x'',t) \tag{13}$$

Substituting our value for $\psi(x'',t)$ and its conjugate $\psi(x'',t)$

$$\begin{aligned}\rho(x'',t) =& \sqrt{\frac{m^2x_0^3 - i\hbar(t-t_0)x_0 m}{\pi^{1/2}(m^2x_0^4+\hbar^2(t-t_0)^2)}}\sqrt{\frac{m^2x_0^3 + i\hbar(t-t_0)x_0 m}{\pi^{1/2}(m^2x_0^4+\hbar^2(t-t_0)^2)}} \\ & exp\left[\frac{-m^2x_0^2x''^2 + i\hbar m x''^2(t-t_0)}{2(m^2x_0^4+\hbar^2(t-t_0)^2)}\right] exp\left[\frac{-m^2x_0^2x''^2 - i\hbar m x''^2(t-t_0)}{2(m^2x_0^4+\hbar^2(t-t_0)^2)}\right]\end{aligned} \tag{14}$$

Simplifying

$$\rho(x'',t) = \sqrt{\frac{1}{\pi}}\frac{m x_0}{\sqrt{m^2x_0^4+\hbar^2(t-t_0)^2}} exp\left[\frac{-m^2x_0^2x''^2}{m^2x_0^4+\hbar^2(t-t_0)^2}\right] \tag{15}$$

$$\rho(x'',t) = \frac{D}{\sqrt{\pi}} exp\left[-(Dx'')^2\right] \quad \text{where} \quad D = \frac{m x_0}{\sqrt{m^2x_0^4+\hbar^2(t-t_0)^2}} \tag{16}$$

2 AHARANOV-BOHM EFFECT

Describe how the Aharanov-Bohm effect can be used to measure magnetic flux through a tight solenoid by an interferometer experiment.

[Solution]

One experimental proof of the Aharanov-Bohm effect consists of an infinite solenoid that has been applied with an electric charge. This would mean that there would be a magnetic flux inside the solenoid as a consequence of the introduction of charge inside the system. Meanwhile, outside the solenoid there would be no magnetic field but there would be a vector potential. If we then introduce 2 beams of electrons, one passing through one side, and the second on another then record the resulting interference pattern, we would observe a shift in the interference pattern. According to the Aharanov-Bohm effect, the shift in the interference pattern is directly proportional to the magnetic flux inside the solenoid.

3 DENSITY OPERATOR

3.a Explain why all wavefunctions describe pure states.

[Solution]

We can consider a general example of a wave function, commonly described as a mixed state, in such a way that if there is a N total systems and it is divided in such a way that N_i systems are in the state $|n_i\rangle$, where in $\sum_i N_i = N$. The probability w_i is then given by

$$w_i = \frac{N_i}{N} \quad \text{where} \quad \sum_i w_i = 1 \tag{17}$$

In general, we can then write any wave function as a sum of pure state density matrices

$$\rho^{mixed} = \sum_i w_i \rho_i^{pure} = \sum_i w_i |n_i\rangle\langle n_i|. \tag{18}$$

We can then conclude that all wave functions describe pure states.

3.b Write down the matrix elements of a density operator ρ in the eigenbasis $|n\rangle$. Explain why the eigenvalue spectrum of a density operator always corresponds to a probability distribution.

[Solution]

The matrix elements of a density operator ρ in the eigenbasis $|n\rangle$ is given by

$$\langle b''|\rho|b'\rangle = \sum_i w_i \langle b''|n_i\rangle\langle n_i|b'\rangle \tag{19}$$

To prove that the eigenvalue spectrum of a density operator corresponds to a probability distribution, we want to compute for Trace of ρ

$$\begin{aligned}
Tr(\rho) &= \sum_{b'} \langle b'|\rho|b'\rangle \\
&= \sum_{b'} \langle b'| \sum_i w_i |n_i\rangle\langle n_i||b'\rangle \\
&= \sum_i w_i \langle n_i|n_i\rangle \\
&= \sum_i w_i \\
&= 1
\end{aligned} \tag{20}$$

when we use the normalization condition $\sum_i w_i = 1$ which we discussed above. Since it sums up to 1, then we can say that the eigenvalue spectrum of a density operator corresponds to a probability distribution.