1.) a) Consider that the Hamiltonian and the time reversal operator commutes

$$[H, \Theta] = 0$$

then for every energy eigenstates $|n\rangle$, the time reversed state $\Theta|n\rangle$ share the same energy.

Let $H|n\rangle = E_n|n\rangle$, and from the commutation operation

$$H\Theta|n\rangle - \Theta H|n\rangle = 0$$

$$H\Theta|n\rangle = \Theta H|n\rangle$$

Let $H|n\rangle = E_n|n\rangle$, and from the commutation relation

$$E\Theta|n\rangle = \Theta H|n\rangle$$

Therefore $|n\rangle$ and $\Theta|n\rangle$ share the same energy. If $|n\rangle$ represents the same state as $\Theta|n\rangle$, therefore they differ by a phase factor

$$\Theta|n\rangle = e^{i\delta}|n\rangle$$

applying Θ

$$\Theta^2|n\rangle = \Theta e^{i\delta}|n\rangle$$

$$= e^{-i\delta}\Theta|n\rangle$$

$$= e^{-i\delta}e^{i\delta}|n\rangle$$

$$= |n\rangle$$

this can never happen for any half integer j since

$$\Theta^2|j\text{- half integer}\rangle = -|j\text{ half-integer}\rangle$$

therefore $|n\rangle$ and $\Theta|n\rangle$ must correspond to different states with the same energy. Therefo there is a 2 fold degeneracy.

b.) In an external magnetic field the Hamiltonian would have terms such as

$$S\cdot B,\ p\cdot A + A\cdot p;\quad B = \nabla\times A$$

the spin operator and momentum operator, S and p, are odd under time reversed. With the commutation relation would not be 0.

$$H\Theta \neq \Theta H$$

However for an external electric field we can set the vector potential to 0 so that the Hamiltonian wouldn't be affected so Kramer's degeneracy holds.

2.) We have the perturbation given by

$$V = \frac{e^2}{r} + \frac{e^2}{|\vec r + \vec r_2 - \vec r_1|} - \frac{e^2}{|\vec r + \vec r_2|} - \frac{e^2}{|\vec r - \vec r_1|}$$

expanding the denominators

$$V = \frac{e^2}{r} + \frac{e^2}{(\vec r\cdot\vec r + 2\vec r\cdot(\vec r_2-\vec r_1) + (\vec r_2-\vec r_1)^2)^{1/2}} - \frac{e^2}{(\vec r\cdot\vec r + 2\vec r\cdot\vec r_2 + \vec r_2\cdot\vec r_2)^{1/2}} - \frac{e^2}{(\vec r\cdot\vec r + 2\vec r\cdot\vec r_1 + \vec r_1\cdot\vec r_1)^{1/2}}$$

$$V = \frac{e^2}{r} + \frac{e^2}{\sqrt{r^2 + 2\vec r\cdot(\vec r_2-\vec r_1) + (\vec r_2-\vec r_1)^2}} - \frac{e^2}{\sqrt{r^2 + 2\vec r\cdot\vec r_2 + r_2^2}} - \frac{e^2}{\sqrt{r^2 + 2\vec r\cdot\vec r_1 + r_1^2}}$$

$$V = \frac{e^2}{r}\left[1 + \frac{1}{\sqrt{1 + \frac{2(z_2-z_1)}{r} + \frac{(\vec r_2-\vec r_1)^2}{r^2}}} - \frac{1}{\sqrt{1 + \frac{2z_2}{r} + \frac{r_2^2}{r^2}}} - \frac{1}{\sqrt{1 - \frac{2z_1}{r} + \frac{r_1^2}{r^2}}}\right]$$

using the taylor expansion

$$\frac{1}{\sqrt{1+\epsilon}} = 1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 + O(\epsilon^3)\dots$$

the perturbation would expand as

$$V = \frac{e^2}{r}\Big[1 + \Big(1 - \frac{z_2-z_1}{r} - \frac{(\vec r_2-\vec r_1)^2}{2r^2} + \frac{3}{8}\Big(\frac{2(z_2-z_1)}{r}\Big)^2 + \frac{3}{8}\Big(\frac{(\vec r_2-\vec r_1)^2}{r^2}\Big)^2\Big)$$

$$-\Big(1 - \frac{z_2}{r} - \frac{r_2^2}{2r^2} + \frac{3}{8}\Big(\frac{2z_2}{r} + \frac{r_2^2}{r^2}\Big)^2\Big)$$

$$-\Big(1 + \frac{z_1}{r} - \frac{r_1^2}{2r^2} + \frac{3}{8}\Big(-\frac{2z_1}{r} + \frac{r_1^2}{r}\Big)^2\Big)\Big]$$

we would only keep terms that don't exceed $1/r^3$ in its overall contribution

$$V = \frac{e^2}{r}\Big[1 + 1 - \frac{(z_2-z_1)}{r} - \frac{(\vec r_2-\vec r_1)^2}{2r^2} + \frac{3}{2}\frac{(z_2-z_1)}{r^2} - 1 + \frac{z_2}{r}$$

$$+ \frac{r_2^2}{2r^2} - \frac{3}{2}\frac{z_2^2}{r^2} - 1 - \frac{z_1}{r} + \frac{r_1^2}{2r^2} - \frac{3}{2}\frac{z_1^2}{r^2}\Big]$$

$$V = \frac{e^2}{r}\Big[\underbrace{\Big(\frac{-z_2 + z_1 + z_2 - z_1}{r}\Big)}_{0}$$

$$+ \Big(\frac{-r_2^2 + 2\vec r_2\cdot\vec r_1 - r_1^2 + 3z_2^2 - 6z_1z_2 + 3z_1^2 + r_2^2 - 3z_2^2 + r_1^2 - 3z_1^2}{2r^2}\Big)\Big]$$

$$V = \frac{e^2}{r}\Big(\frac{2\vec r_1\cdot\vec r_2 - 6z_1z_2}{2r^2}\Big) = \frac{e^2}{r^3}(x_1x_2 + y_1y_2 + z_1z_2 - 3z_1z_2)$$

$$V = \frac{e^2}{r^3}(x_1x_2 + y_1y_2 - 2z_1z_2)$$

the first order perturbation is 0 while the 2nd order perturbation is non-vanishing

$$E_0^2 = \sum_{k\neq 0}\frac{|\langle k^{(0)}|V|0^{(0)}\rangle|^2}{E_0 - E_k}$$

this simplifies into

$$\boxed{E_0^2 = \frac{e^4}{r^6}\sum_{k\neq 0}\frac{|\langle k^{(0)}|x_1x_2 + y_1y_2 - 2z_1z_2|0^{(0)}\rangle|^2}{E_0 - E_k}}$$

Therefore there is a r^{-6} attractive potential between 2 atoms in their ground state. ∎

3.) a.) Consider the Hamiltonian of SHO

$$H_0 = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 x^2$$

with perturbation

$$H' = fx$$

the Hamiltonian would then by

$$H = H_0 + H'$$

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 x^2 + fx$$

the first order perturbation theory

$$E_n' = \langle \psi_n^0 | H' | \psi_n^0 \rangle$$

$$E_n' = f \langle \psi_n^0 | x | \psi_n^0 \rangle$$

$$\boxed{E_n' = 0}$$

the second order perturbation theory is

$$E_n^2 = \sum_{n' \neq n} \frac{|\langle \psi_{n'} | fx | \psi_n \rangle|^2}{E_n - E_{n'}}$$

$$= \sum_{n' \neq n} \frac{f^2 |\langle \psi_{n'} | x | \psi_n \rangle|^2}{E_n - E_{n'}}$$

$$= f^2 \frac{\hbar}{2m\omega} \sum_{n' \neq n} \frac{(\sqrt{n}\langle n' | n-1 \rangle + \sqrt{n+1}\langle n' | n+1 \rangle)^2}{E_n - E_{n'}}$$

the only non zero contribution is from $n' = n+1$, $n' = n-1$. Therefore,

$$E_n^2 = f^2 \frac{\hbar}{2m\omega}\left(-\frac{n+1}{\hbar\omega} + \frac{n}{\hbar\omega}\right) = f^2\left(\frac{\hbar}{2m\omega}\right)\left(\frac{1}{\hbar\omega}\right)$$

$$\boxed{E_0^2 = \frac{f^2}{2m\omega^2}}$$

b.) We have the Hamiltonian

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 x^2 + fx$$

we can rewrite this as

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2\left(x^2 + \frac{2f}{m\omega^2}x\right)$$

Completing the square inside the parenthesis

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2\left(\left(x + \frac{f}{m\omega^2}\right)^2 - \left(\frac{f}{m\omega^2}\right)^2\right)$$

letting $x \to x + \frac{f}{m\omega^2}$, we have

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2\left(x^2 - \left(\frac{f}{m\omega^2}\right)^2\right)$$

3.) b.) We have the relation

$$\langle \psi | H | \psi \rangle = \langle \psi | E_n | \psi \rangle$$

therefore,

$$\langle \psi | \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}m\omega^2 x^2 - \frac{1}{2}m\omega^2\left(\frac{f}{m\omega^2}\right)^2 | \psi \rangle = \langle \psi | E_n | \psi \rangle$$

rewriting

$$\langle \psi | E_n | \psi \rangle = \frac{1}{2m}\langle \psi | p^2 | \psi \rangle + \frac{1}{2}m\omega^2 \langle \psi | x^2 | \psi \rangle - \frac{1}{2m\omega^2}\langle \psi | f^2 | \psi \rangle$$

since we know that

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$

$$p = i\sqrt{\frac{m\hbar\omega}{2}}\,(-a + a^\dagger)$$

we have

$$\langle \psi | E_n | \psi \rangle = -\frac{1}{2m}\left(\frac{m\hbar\omega}{2}\right)\langle \psi | -a^\dagger a + a^\dagger a^\dagger - a a^\dagger + aa | \psi \rangle$$

$$+ \frac{1}{2m\omega^2}\left(\frac{\hbar}{2m\omega}\right)\langle \psi | aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger | \psi \rangle$$

$$- \frac{f^2}{2m\omega^2}\langle \psi | \psi \rangle$$

where the creation and annihilation operator operates as

$$a^\dagger | n \rangle = \sqrt{n+1}\,| n \rangle \quad ; \quad a | n \rangle = \sqrt{n}\,| n-1 \rangle$$

therefore $\langle \psi | E_n | \psi \rangle$ simplifies into

$$-\frac{\hbar\omega}{4}\left[-n - (n+1)\right] + \frac{\hbar\omega}{4}\left[n + n + 1\right] - \frac{f^2}{2m\omega^2} = E_n$$

$$\frac{\hbar\omega}{4}(2n+1) + \frac{\hbar\omega}{4}(2n+1) - \frac{f^2}{2m\omega^2} = E_n$$

$$E_n = \frac{\hbar\omega(2n+1)}{2} - \frac{f^2}{2m\omega^2}$$

for the ground state energy we consider the value when $n = 0$

$$\boxed{E_0 = \frac{\hbar\omega}{2} - \frac{f^2}{2m\omega^2}}$$

4.) For the first order correction of the ground state energy, we have

$$E_n^1 = \langle \psi_n^0 | H | \psi_n^0 \rangle$$

for the infinite square well we have

$$\psi_n^0(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

therefore

$$E_n^1 = \langle \psi_n^0(x) | \alpha \delta(x - \tfrac{a}{2}) | \psi_n^0(x) \rangle$$

$$E_n^1 = \frac{2}{a}\alpha \int_0^\infty \sin^2\left(\frac{n\pi x}{a}\right) \delta(x - \tfrac{a}{2})\, dx$$

$$E_n^1 = \frac{2}{a}\alpha \sin^2\left(\frac{n\pi}{2}\right)$$

for the ground state energy for an infinite square well we have

$$\boxed{E_1^1 = \frac{2}{a}\alpha}$$

For the second order correction we have

$$E_n^2 = \sum_{n \neq l} \frac{|\langle \psi_l^0(x) | H | \psi_n^0(x) \rangle|^2}{E_n - E_l}$$

where

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}$$

therefore

$$E_n^2 = \sum_{n \neq l} \frac{\left| \int_0^\infty \sqrt{\frac{2}{a}} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \alpha \delta(x - \frac{a}{2})\, dx \right|^2}{\frac{\hbar^2 \pi^2}{2ma^2}(n^2 - l^2)}$$

$$E_n^2 = \sum_{n \neq l} \frac{\frac{4}{a^2}\alpha^2 \sin^2\left(\frac{n\pi}{2}\right) \sin\left(\frac{l\pi}{2}\right)}{\frac{\hbar^2 \pi^2 (n^2 - l^2)}{2ma^2}}$$

$$E_n^2 = \sum_{n \neq l} \frac{8m\alpha^2}{\hbar^2 \pi^2 (n^2 - l^2)} \sin^2\left(\frac{n\pi}{2}\right) \sin^2\left(\frac{l\pi}{2}\right)$$

Again, since we are looking for groundstate corrections

$$\boxed{E_0^2 = \sum_{l \neq 1} \frac{8m\alpha^2}{\hbar^2 \pi^2 (1 - l^2)} \sin^2\left(\frac{l\pi}{2}\right) \rightarrow \begin{cases} 0, & l = \text{even} \\ \dfrac{8m\alpha^2}{\hbar^2 \pi^2 (1 - l^2)}, & l = \text{odd} \end{cases}}$$