1.) Jackson 4-2

The potential due to a dipole is given by

$$\Phi_{dipole} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^3} \qquad (1)$$

Meanwhile, the potential due to a charge density is given by

$$\Phi_{\substack{charge\\density}} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}_0|} d\vec{x}' \qquad (2)$$

equating these potentials

$$\frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^3} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d\vec{x}' \qquad (3)$$

$$\frac{\vec{p}\cdot(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^3} = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}_0|} d\vec{x}' \qquad (4)$$

Since we have

$$\frac{\vec{x}-\vec{x}_0}{|\vec{x}-\vec{x}_0|^3} = \nabla\left(\frac{1}{|\vec{x}-\vec{x}_0|}\right) \qquad (5)$$

then we have

$$\vec{p}\cdot\nabla\left(\frac{1}{|\vec{x}-\vec{x}_0|}\right) = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}_0|} d\vec{x}' \qquad (6)$$

Since we know that

$$\int \delta(x-x_0) f(x)\, dx = f(x_0) \qquad (7)$$

we can rewrite eq (6) as

$$\int \delta(\vec{x}'-\vec{x}_0)\, p\cdot\nabla'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d\vec{x}' = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d\vec{x}' \qquad (8)$$

$$\overset{0}{p\cdot\left(\frac{1}{\vec{x}-\vec{x}'}\right)\delta(\vec{x}-\vec{x}_0)\Big|_{-\infty}^{\infty}} - \int p\cdot\left(\frac{1}{\vec{x}-\vec{x}'}\right)\nabla\left(\delta(\vec{x}-\vec{x}_0)\right)d\vec{x}' = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d\vec{x}' \qquad (9)$$

$$-\int p\cdot\left(\frac{1}{\vec{x}-\vec{x}'}\right)\nabla\left(\delta(\vec{x}-\vec{x}_0)\right)d\vec{x}' = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d\vec{x}' \qquad (10)$$

therefore eq (10) simplifies into

$$\rho(\vec{x}') = -p\cdot\nabla\delta(\vec{x}-\vec{x}_0) \qquad (11)$$

rewriting the variables

$$\boxed{\rho_{eff}(\vec{x}) = -p\cdot\nabla\delta(\vec{x}-\vec{x}_0)} \qquad (12)$$

2.1 Jackson 4.5

a) We can expand $\Phi(\vec{x})$ using taylor expansion about the origin

$$\Phi(\vec{x}) = \Phi_i(0) + x_i \cdot \vec{\nabla}\Phi_i(0) + \frac{1}{2}\sum_j\sum_k x_j x_k \frac{\partial^2 \Phi_i(0)}{\partial x_j \partial x_k} + \ldots \quad (1)$$

since we have

$$\vec{E} = -\vec{\nabla}\Phi \quad (2)$$

we'll have

$$\vec{\nabla}\Phi(0) = -\vec{E}(0)$$

and apply this to the last 2 terms

$$\Phi(\vec{x}) = \sum_i \Phi_i(0) - \sum_i x_i E_i(0) - \frac{1}{2}\sum_j\sum_k x_j x_k \frac{\partial E_i(0)}{\partial x_j} + \ldots \quad (3)$$

we can then add a $\frac{1}{6}r^2\vec{\nabla}\cdot\vec{E}(0)$ term since $\vec{\nabla}\cdot\vec{E}=0$. Therefore the potential becomes

$$\Phi(\vec{x}) = \sum_i \Phi_i(0) - \sum_i x_i E_i(0) - \frac{1}{2}\sum_{jk} x_j x_k \frac{\partial E_i(0)}{\partial x_j} + \frac{1}{6}r^2\sum_j\sum_k \frac{\partial E_i(0)}{\partial x_j}\delta_{ij} \quad (4)$$

since we can write $\frac{1}{6}r^2\vec{\nabla}\cdot\vec{E}(0) = \frac{1}{6}r^2\sum_j\sum_k \frac{\partial E_i(0)}{\partial x_j}$. Therefore,

$$\Phi(\vec{x}) = \sum_i \Phi_i(0) - \sum_i x_i E_i(0) - \frac{1}{6}\sum_{jk}(3x_j x_k - r^2\delta_{jk})\frac{\partial E_i(0)}{\partial x_j} + \ldots \quad (5)$$

we can then solve for the $\vec{E}$ field

$$E(\vec{x}) = \left[\sum_i E_i(\vec{x}') + \sum_i \frac{\partial}{\partial x_j} x_i E_i(\vec{x}') + \frac{1}{6}\sum_j\sum_k \frac{\partial^2 E_i(\vec{x}')}{\partial x_j \partial x_k}(3x_j x_k - r^2\delta_{jk}) + \ldots\right]_{\vec{x}'=0} \quad (6)$$

taking the force

$$F(\vec{x}) = \left[\sum_i q_i \vec{E}_i(\vec{x}') + \sum_i \hat{x}_i \int \rho(\vec{x}) x_i \frac{\partial}{\partial x_j} E_i(\vec{x}') + \int \rho(\vec{x}) \frac{1}{6}\sum_i\sum_{j,k}\frac{\partial^2 E_i(\vec{x}')}{\partial x_j \partial x_k}(3x_j x_k - r^2\delta_{jk})\hat{x}_i + \ldots\right]_{\vec{x}'=0} \quad (7)$$

since we have the following values

$$\vec{p} = \int x\,\rho(\vec{x}')\,d^3\vec{x}' \quad , \quad Q_{jk} = \int (3x_j x_k - r^2\delta_{jk})\rho(\vec{x}')\,d^3x' \quad (8)$$

we have

$$F(\vec{x}') = q\vec{E}(0) + \nabla\left[\vec{p}\cdot\vec{E}\right]_{\vec{x}'=0} + \nabla\left[\frac{1}{6}\sum_{jk} Q_{jk}\frac{\partial E_j(\vec{x}')}{\partial x_k}\right]_{\vec{x}'=0} + \ldots \quad (9)$$

rewriting the variables used

$$\boxed{F(\vec{x}) = q\vec{E}(0) + \nabla\left[\vec{p}\cdot\vec{E}\right]_{\vec{x}=0} + \nabla\left[\frac{1}{6}\sum_{jk} Q_{jk}\frac{\partial E_j(x)}{\partial x_k}\right]_{\vec{x}=0} + \ldots} \quad (10)$$

we can rewrite this as

$$F(\vec{x}) = -\nabla\left[q\Phi(\vec{x}) - \vec{p}\,\vec{E}(\vec{x}) - \frac{1}{6}\sum_{jk} Q_{jk}\frac{\partial E_j(\vec{x})}{\partial x_k} + \ldots\right]_{\vec{x}=0} \quad (11)$$

when compared to the expansion of the energy

$$W = q\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_i\sum_j Q_{ij}\frac{\partial E_j(0)}{\partial x_i} + \ldots \quad (12)$$

we can see the relation

$$\boxed{W = -\int F\cdot d\vec{x}\,\Big|_{\vec{x}=0}} \quad (13)$$

7 ⓞ

b) The electrostatic torque on a charge distribution ρ as a result of the external field $E^{(0)}$ is

$$\vec{N} = \int \vec{x} \times \left(\rho(\vec{x}) \vec{E}^{(0)}(\vec{x}) \right) d^3\vec{x} \quad (14)$$

evaluating the cross product we have

$$\vec{N} = \int \rho(\vec{x}) \left[(\vec{x}_2 E_3^{(0)} - \vec{x}_3 E_2^{(0)})\hat{i} + (\vec{x}_3 E_1^{(0)} - \vec{x}_1 E_3^{(0)})\hat{j} + (\vec{x}_1 E_2^{(0)} - \vec{x}_2 E_1^{(0)})\hat{k} \right] d^3\vec{x} \quad (15)$$

considering only the torque due to component $\hat{i}$

$$N_1 = \int \rho(\vec{x}) \left(\vec{x}_2 E_3^{(0)} - \vec{x}_3 E_2^{(0)} \right) d^3\vec{x} \quad (16)$$

expanding the E-fields using taylor series expansion

$$E_2^{(0)} = E_2^{(0)}(0) + \sum_j \vec{x}_j \frac{\partial E_2^{(0)}(\vec{x})}{\partial x_j}\Big|_{\vec{x}=0} + \dots \quad (17)$$

$$E_3^{(0)} = E_3^{(0)}(0) + \sum_j \vec{x}_j \frac{\partial E_3^{(0)}(\vec{x})}{\partial x_j}\Big|_{\vec{x}=0} + \dots \quad (18)$$

plugging this into eq (16) we have

$$N_1 = \int \rho(\vec{x}) \left[x_2 \left(E_3^{(0)}(0) + \sum_j x_j \frac{\partial E_3^{(0)}(\vec{x})}{\partial x_j}\Big|_{\vec{x}=0} + \dots \right) - x_3 \left(E_2^{(0)}(0) + \sum_j x_j \frac{\partial E_2^{(0)}(\vec{x})}{\partial x_j}\Big|_{\vec{x}=0} \right) \right] d^3\vec{x}$$

$$\begin{aligned} N_1 = &\int \left(\rho(\vec{x}) \vec{x}_2 E_3^{(0)}(0) - \rho(\vec{x}) \vec{x}_3 E_2^{(0)}(0) \right) d^3\vec{x} \\ &+ \int \left(\rho(\vec{x}) \vec{x}_2 \sum_j x_j \frac{\partial E_3^{(0)}(\vec{x})}{\partial x_j}\Big|_{\vec{x}=0} - \frac{1}{3} r^2 \delta_{2j} \frac{\partial E_3^{(0)}(\vec{x})}{\partial x_j}\Big|_{\vec{x}=0} \right) d^3\vec{x} \\ &+ \int \left(\rho(\vec{x}) \vec{x}_3 \sum_j x_j \frac{\partial E_2^{(0)}(\vec{x})}{\partial x_j}\Big|_{\vec{x}=0} - \frac{1}{3} r^2 \delta_{3j} \frac{\partial E_2^{(0)}(\vec{x})}{\partial x_j}\Big|_{\vec{x}=0} \right) d^3\vec{x} \end{aligned} \quad (19)$$

where we added 2 extra terms since $\nabla \cdot E = 0$ in our system.

Since $Q_{ij} = \int (3x_i x_j - r^2 \delta_{ij}) \rho(\vec{x}) d^3\vec{x}$ and $\vec{p} = \int \vec{x} \rho(\vec{x}) d^3\vec{x}$, we have

$$N_1 = P_2 E_3^{(0)} - P_3 E_2^{(0)} + \frac{1}{3} \sum_j Q_{2j} \frac{\partial E_3^{(0)}}{\partial x_j}\Big|_{\vec{x}=0} - \frac{1}{3} \sum_j Q_{3j} \frac{\partial E_2^{(0)}}{\partial x_j}\Big|_{\vec{x}=0} \quad (20)$$

using $\frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$ from $\nabla \times E = 0$

$$N_1 = P_2 E_3^{(0)} - P_3 E_2^{(0)} + \frac{1}{3} \frac{\partial}{\partial x_3} \sum_j Q_{2j} E_j^{(0)}\Big|_{\vec{x}=0} - \frac{1}{3} \frac{\partial}{\partial x_2} \sum_j Q_{3j} E_j^{(0)}\Big|_{\vec{x}=0} \quad (21)$$

which simplifies into

$$\boxed{N_1 = \left[p \times E^{(0)}(0) \right]_1 + \frac{1}{3} \left[\frac{\partial}{\partial x_3} \left(\sum_j Q_{2j} E_j^{(0)} \right) - \frac{\partial}{\partial x_2} \left(\sum_j Q_{3j} E_j^{(0)} \right) \right]_{\vec{x}=0} + \dots} \quad (22)$$

3.) Problem 4.8

A.) To solution to the Laplace solution in polar coordinates

$$\Phi(r,\phi) = a_0 + b_0 \ln r + \sum_{m=1}^{\infty} (a_m r^m + b_m r^{-m})(A_m e^{im\phi} + B_m e^{-im\phi})$$ ①

We want to solve for the potential in 3 regions: a.) for $r<a$, b.) for $a<r<b$, and c.) for $b>r$.
For the $r<a$ roll, we will have a value of infinity at $r=0$. This is why we force the r^{-m} to be zero by making $b_m = 0$.

$$\Phi_{r<a} = a_0 + \sum_{m=1}^{\infty} r^m (A_m e^{im\phi} + B_m e^{-im\phi})$$ ②

The exterior region, $r>b$, we consider the potential given in eq ①. We then use the relation

$$E = -\nabla\Phi \quad \text{or} \quad \Phi = \int -E\,dx = -EX$$ ③

we write X as $r\cos\phi$ since we are in a cylindrical coordinate

$$-Er\cos\phi = a_0 + b_0 \ln r + \sum_{m=1}^{\infty} (a_m r^m + b_m r^{-m})\underbrace{(A_m e^{im\phi} + B_m e^{-im\phi})}_{2\cos\phi \text{ when } A_m = B_m}$$ ④

We can only have terms w/ $\cos\phi$ factor so a_0 and b_0 are forced to be zero. We can also only consider $m=1$ as $m>1$ would yield higher ordered r terms that wouldn't correspond with our LHS. Therefore,

$$-Er\cos\phi = 2a_1 B_1 r\cos\phi \longrightarrow -E = 2a_1 B_1$$ ⑤

Plugging this into our potential

$$\Phi_{r>b} = \left(-\frac{E_0 r}{2B_1} + b_1 r^{-1}\right) 2B_1 \cos\phi = (-E_0 r + b_1 r^{-1})\cos\phi$$ ⑥

note: we used $E = E_0 \hat{z}$

For the middle region $a<r<b$ we have

$$\Phi_{a<r<b} = C_0 + d_0 \ln r + \sum_{m=1}^{\infty} (c_m r^m + d_m r^{-m})(C_m e^{im\phi} + D_m e^{-im\phi})$$ ⑦

We have 2 boundary conditions to consider

$$(\epsilon_2 E_2 - \epsilon_1 E_1)\cdot n = \sigma \quad (8.1)$$

note: $\sigma = 0$; there is no free charge

$$(E_2 - E_1) \times n = 0 \quad (8.2)$$

Imposing (8.1) on the boundary between the middle and outside region

$$\epsilon \frac{\partial \Phi_{a<r<b}}{\partial r} = \epsilon_0 \frac{\partial \Phi_{r>b}}{\partial r}$$ ⑧

$$\epsilon \frac{\partial}{\partial r}\left(C_0 + d_0 \ln r + \sum_{m=1}^{\infty} (c_m r^m + d_m r^{-m})(C_m e^{im\phi} + D_m e^{-im\phi})\right) = \epsilon_0 \frac{\partial}{\partial r}(-E_0 r + b_1 r^{-1})\cos\phi$$ ⑨

simplify and evaluate at $r=b$

$$\frac{\epsilon d_0}{b} + \epsilon \sum_{m=1}^{\infty} (C_m e^{im\phi} + D_m e^{-im\phi})(m c_m b^{m-1} - m d_m b^{-m-1}) = -\epsilon_0 E_0 \cos\phi - \epsilon_0 \frac{b_1}{b^2}\cos\phi$$ ⑩

d_0 is zero bc it doesn't match with the RHS. m should only have a value of $m=1$ as we only need a $\cos\phi$ factor

$$-\left(\epsilon_0 E_0 \cos\phi + \epsilon_0 \frac{b_1}{b^2}\cos\phi\right) = \epsilon C_1 \cos\phi\,(c_1 - d_1 b^{-2})$$ ⑪

$$-\left(\epsilon_0 E_0 \cos\phi + \epsilon_0 \frac{b_1}{b^2}\cos\phi\right) = c_1 \epsilon (1 - d_1 b^{-2})\, 2\cos\phi$$ ⑫

$$c_1 = \frac{-\epsilon_0 (E_0 + b_1 b^{-2})}{2\epsilon(1 - d_1 b^{-2})}$$ ⑬

3.1 Substituting this into eq ⑦ we have

$$\Phi_{a<r<b} = \frac{-\epsilon_0(E_0 + b_1 b^{-2})}{\epsilon(1-d_1 b^{-2})}(r + d_1 r^{-1})\cos\phi \quad (14)$$

Using (8.2) at the outer surface

$$E_{2\phi} = E_{1\phi} \longrightarrow \frac{\partial \Phi_{r>b}}{\partial\phi} = \frac{\partial\Phi_{a<r<b}}{\partial\phi} \quad (15)$$

$$\frac{\partial}{\partial\phi}\left(-E_1 r + b_1 r^{-1}\cos\phi\right) = \frac{\partial}{\partial\phi}\left(c_0 + d_0 \ln r + \sum_{m=1}^{\infty}(C_m r^m + d_m r^{-m})(C_m e^{im\phi} + D_m e^{-im\phi})\right) \quad (16)$$

Simplify and evaluate at $r=b$ (17)

$$-E_0 r + b_1 r^{-1} = \frac{-\epsilon_0(E_0 + b_1 b^{-2})}{\epsilon(1-d_1 b^{-2})}(r + d_1 r^{-1})$$

$$\epsilon(-E_0 b + b_1 b^{-1})(1 - d_1 b^{-2}) = -\epsilon_0(E_0 + b_1 b^{-2})(b + d_1 b^{-2})$$

$$\epsilon(E_0 - b_1 b^{-2}) - d_1 b^{-2}\epsilon(E_0 - b_1 b^{-2}) = \epsilon_0(E_0 + b_1 b^{-2}) + \epsilon_0(E_0 + b_1 b^{-2})d_1 b^{-2}$$

$$d_1\left[b^{-2}E_0(\epsilon+\epsilon_0) - b^{-2} b_1 b^{-2}(\epsilon-\epsilon_0)\right] - \epsilon_0(E_0 - b_1 b^{-2}) - \epsilon_0(E_0 + b_1 b^{-2})$$

$$d_1 = \frac{E_0(\epsilon-\epsilon_0) - b_1 b^{-2}(\epsilon+\epsilon_0)}{b^{-2}[E_0(\epsilon+\epsilon_0) + b_1 b^{-2}(\epsilon-\epsilon_0)]} = \frac{b^2 b^2 E_0(\epsilon-\epsilon_0) - b_1(\epsilon+\epsilon_0)}{b^2 E_0(\epsilon-\epsilon_0) - b_1(\epsilon+\epsilon_0)} \quad (18)$$

The solution in the middle region is

$$\Phi_{a<r<b} = \left[(b_1(\epsilon-\epsilon_0) - E_0 b^2(\epsilon+\epsilon_0))r + b^2(b_1(\epsilon+\epsilon_0) - E_0 b^2(\epsilon-\epsilon_0))r^{-1}\right]\frac{1}{2\epsilon b^2}\cos\phi \quad (19)$$

Imposing the boundary conditions of the inner surface

$$\epsilon\frac{\partial\Phi_{a<r<b}}{\partial r} = \epsilon_0\frac{\partial\Phi_{r<a}}{\partial r}\ ;\ r=a \quad (20)$$

$$\epsilon\left[(b_1(\epsilon-\epsilon_0) - E_0 b^2(\epsilon+\epsilon_0)) - b^2(b_1(\epsilon+\epsilon_0) - E_0 b^2(\epsilon-\epsilon_0)a^{-2})\right]\frac{1}{2\epsilon b^2}\cos\phi = \epsilon_0 A_1\cos\phi$$

$$\left[(b_1(\epsilon-\epsilon_0) - E_0 b^2(\epsilon+\epsilon_0)) - b^2(b_1(\epsilon+\epsilon_0) - E_0 b^2(\epsilon-\epsilon_0)a^{-2})\right]\frac{1}{2b^2} = \epsilon_0 A \quad (21)$$

and

$$\frac{\partial\Phi_{a<r<b}}{\partial\phi} = \frac{\partial\Phi_{a>r}}{\partial r}\ ;\ r=a \quad (22)$$

$$-\left[(b_1(\epsilon-\epsilon_0) - E_0 b^2(\epsilon+\epsilon_0))a + b^2(b_1(\epsilon+\epsilon_0) - E_0 b^2(\epsilon-\epsilon_0))a^{-1})\right]\frac{1}{2\epsilon b^2}\sin\phi$$

$$= -A_1 a\sin\phi$$

$$\left[b_1(\epsilon-\epsilon_0) - E_0 b^2(\epsilon+\epsilon_0)a + b^2(b_1(\epsilon+\epsilon_0) - E_0 b^2(\epsilon-\epsilon_0))a^{-1}\right]\frac{1}{2\epsilon a b^2} = A_1 \quad (23)$$

Solving the systems of equations formed by eq (21) and (23)

$$b_1 = \frac{E_0 b^2(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} \quad (24)$$

$$A_1 = \frac{-4b^2 E_0\epsilon\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} \quad (25)$$

3) 3.) Since we were able to solve for the coefficients we get the potential

$$\Phi(r,\phi) = \begin{cases} \dfrac{-4b^2\epsilon\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 r\cos\phi, & r<a \\ \dfrac{-2ab^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left((\epsilon+\epsilon_0)\dfrac{r}{a} + (\epsilon-\epsilon_0)\dfrac{a}{r}\right)E_0\cos\phi, & a<r<b \\ \left(-r + \dfrac{(b^2-a^2)(\epsilon^2-\epsilon_0)^2}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\dfrac{b^2}{r}\right)E_0\cos\phi \end{cases} \quad (26)$$

solving for the E-field

$$E = -\nabla\Phi = -\left(\frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\hat{\phi}\right) \quad (27)$$

$$E_{r<a} = -\left[\frac{-4b^2\epsilon_0\epsilon}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0\cos\phi\,\hat{r} + \frac{1}{r}\frac{4b^2\epsilon\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0 r\sin\phi\,\hat{\phi}\right]$$

$$E_{r>a} = \left(\frac{4b^2\epsilon\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0\right)(\cos\phi\,\hat{r} - \sin\hat{\phi}) = \frac{4b^2\epsilon_0\epsilon\,E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\hat{i} \quad (28)$$

$$E_{b>r>a} = -\left[\frac{-2ab^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left((\epsilon+\epsilon_0)\frac{1}{a} - (\epsilon-\epsilon_0)\frac{a}{r^2}\right)E\cos\phi + \frac{2ab^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left((\epsilon+\epsilon_0)\frac{1}{a} + (\epsilon-\epsilon_0)\frac{a}{r^2}\right)E_0\sin\phi\right]$$

$$E_{b>r>a} = \left[\frac{2ab^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left((\epsilon+\epsilon_0)\frac{1}{a} - (\epsilon-\epsilon_0)\frac{a}{r^2}\right)E_0\cos\phi - \frac{2ab^2}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left((\epsilon+\epsilon_0)\frac{1}{a} + (\epsilon-\epsilon_0)\frac{a}{r^2}\right)E_0\sin\phi\right]$$

$$E_{b>r>a} = \frac{2ab^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0\left(\frac{\epsilon+\epsilon_0}{a}\right)(\cos\phi - \sin\phi) - \frac{2ab^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0(\epsilon-\epsilon_0)\frac{a}{r^2}(\cos\phi - \sin\phi + 2\sin\phi)$$

$$E_{b>r>a} = \frac{2ab^2\epsilon_0 E_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left[(\epsilon+\epsilon_0)\hat{i} - (\epsilon-\epsilon_0)\frac{a^2}{r^2}(\hat{i} + 2\hat{\phi}\sin\phi)\right] \quad (29)$$

$$E_{r>b} = -\left[\left(-1 - \frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{r^2}\right)E_0\cos\phi - \frac{1}{r}\left(-r + \frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{r}\right)E_0\sin\phi\right]$$

$$E_{r>b} = E_0(\cos\phi - \sin\phi) + \frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{r^2}E\cos\phi - \frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{r^2}E_0\sin\phi$$

$$E_{r>b} = E_0\hat{i} + \frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0\frac{b^2}{r^2}(\cos\phi - \sin\phi + 2\sin\phi)$$

$$E_{r>b} = E_0\hat{i} + \frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}E_0\frac{b^2}{r^2}(\hat{i} + 2\hat{\phi}\sin\phi) \quad (30)$$

) The Electric field would be

$$
E = \begin{cases} \left(\dfrac{4b^2 \epsilon \epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 \right) \hat{i} & , r < a \\[2ex] \dfrac{2b^2 \epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 \left[(\epsilon+\epsilon_0)\hat{i} - (\epsilon-\epsilon_0) \dfrac{a^2}{r^2} (\hat{i} + 2\hat{\phi} \sin\phi) \right] & , a < r < b \\[2ex] E_0 \hat{i} + \dfrac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2} E_0 \dfrac{b^2}{r^2} (\hat{i} + 2\hat{\phi} \sin\phi) & , r > b \end{cases}
$$

(3)

c) For a solid dielectric cylinder in a uniform field, we let $a \to 0$, we would have

$$
E = \begin{cases} \dfrac{2\epsilon_0}{(\epsilon+\epsilon_0)} E_0 \hat{i} & , r < b \\[2ex] E_0 \hat{i} + \dfrac{(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)} \dfrac{b^2}{r^2} E_0 (\hat{i} + 2\hat{\phi} \sin\phi) & , r > b \end{cases}
$$

(32)

For a cylindrical cavity in a uniform dielectric, we let $b \to \infty$. We have

$$
E = \begin{cases} \dfrac{4\epsilon\epsilon_0}{(\epsilon+\epsilon_0)^2} E_0 \hat{i} & , r < a \\[2ex] \dfrac{2\epsilon_0}{(\epsilon+\epsilon_0)^2} E_0 \left[(\epsilon+\epsilon_0)\hat{i} - (\epsilon-\epsilon_0) \dfrac{a^2}{r^2} (\hat{i} + 2\hat{\phi} \sin\phi) \right] & , r > a \end{cases}
$$

(33)

1.) Jackson 4.13

Solving the Laplace eq in polar coordinates

$$\nabla^2 \Phi = 0$$ (1)

We then have the form for the potential

$$\Phi(r,\phi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} a_n r^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n r^{-n} \sin(n\phi + \beta_n)$$ (2)

Since we have an infinitely long cylinder, we don't consider the z-dimension. We also have a symmetry about the ϕ-dimension so all terms with ϕ dependence would be negligible. We are left with

$$\Phi(r) = a_0 + b_0 \ln r$$ (3)

We are given a potential difference between the cylinders. Let us assume that

$$\Phi(a) = 0 \quad \text{and} \quad \Phi(b) = V$$ (4)

Imposing the boundary conditions

$$\Phi(r=a) = a_0 + b_0 \ln a = 0$$

$$\rightarrow a_0 = -b_0 \ln a$$ (5)

the potential becomes

$$\Phi(r) = -b_0 \ln a + b_0 \ln r$$

$$= b_0 \ln\left(\frac{r}{a}\right)$$ (6)

Imposing the boundary conditions at $r=b$

$$\Phi(r=b) = b_0 \ln\left(\frac{b}{a}\right) = V$$

$$\rightarrow b_0 = \frac{V}{\ln\left(\frac{b}{a}\right)}$$ (7)

therefore the potential becomes

$$\Phi(r) = V \frac{\ln\left(\frac{r}{a}\right)}{\ln\left(\frac{b}{a}\right)}$$ (8)

We can compute the electric field from the potential

$$E = -\nabla \Phi$$ (9)

Substituting eq (8)

$$E = -\frac{V}{\ln\left(\frac{b}{a}\right)} \cdot \frac{a}{r} \cdot \frac{1}{a}\,\hat{r}$$

$$E = -\frac{V}{r\ln\left(\frac{b}{a}\right)}\,\hat{r}$$ (10)

4) The potential energy before submerging the cylinders in water is given as W_0. After submerging we get a new potential energy W_{final}. Therefore, we have

$$\Delta W = W_{final} - W_0 \tag{11}$$

Since we have

$$W = \frac{1}{2}\int E \cdot D \, d^3\vec{x} \tag{12}$$

We then have

$$\Delta W = \frac{1}{2}\int E_{final} \cdot D_{final} \, d^3\vec{x} - \frac{1}{2}\int E_0 \cdot D_0 \, d^3\vec{x} \tag{13}$$

$$\Delta W = \left[\frac{1}{2}(L-h)\epsilon_0 \int E_{air}^2 \, d^3\vec{x} + \frac{1}{2}\epsilon_0 (1+\chi_e) h \int E_{liquid}^2 \, d^3\vec{x} \right] - \left[\frac{1}{2} L \epsilon_0 \int E_{air}^2 \, d^3\vec{x} \right] \tag{14}$$

The electric field would remain the same regardless of the region

$$\Delta W = \left[\frac{1}{2}(L-h)\int E^2 d^3\vec{x} + \frac{1}{2}\epsilon_0(1+\chi_e)h\int E^2 d^3\vec{x} \right] \tag{15}$$

$$-\left[\frac{1}{2} L\epsilon \int E^2 d^3\vec{x}\right] \tag{16}$$

$$= \frac{1}{2}\epsilon_0 h \chi_e \int E^2 d^2\vec{x}$$

evaluate the integral

$$\Delta W = \frac{1}{2}\epsilon_0 h \chi_e \int_0^{2\pi} d\phi \int_a^b E^2 \, r \, dr \tag{17}$$

$$= \epsilon_0 h \chi_e \pi \int_0^b E^2 \, r \, dr$$

$$= \epsilon_0 h \chi_e \pi \int_a^b \left(-\frac{V}{r \ln(b/a)}\right)^2 r \, dr$$

$$= \epsilon_0 h \chi_e \pi \left(\frac{V}{\ln\left(\frac{b}{a}\right)}\right)^2 \int_a^b \frac{1}{r} dr$$

$$= \epsilon_0 h \chi_e \pi \left(\frac{V}{\ln\left(\frac{b}{a}\right)}\right)^2 \ln\left(\frac{b}{a}\right) = \frac{\epsilon_0 h \chi_e \pi V^2}{\ln\left(\frac{b}{a}\right)} \tag{18}$$

We also consider the potential energy due to particles

$$\Delta W = mgh = \underset{\text{density}}{\rho}\, \underbrace{hA}_{\text{volume}}\, gh \qquad \text{since } \frac{m}{V} = \text{density} = \rho \tag{19}$$

The area in the donut region is

$$A_{donut} = \pi(b^2 - a^2) \tag{20}$$

therefore,

$$\Delta W = \rho h^2 \pi (b^2 - a^2) g \tag{21}$$

comparing to our result in eq (18)

$$\frac{\epsilon_0 h \chi_e \pi V^2}{\ln\left(\frac{b}{a}\right)} = \rho\pi(b^2 - a^2) h^2 g \tag{22}$$

solving for χ_e

$$\boxed{\chi_e = \frac{\rho(b^2 - a^2) h g}{V^2 \epsilon_0} \ln\left(\frac{b}{a}\right)} \tag{23}$$