1.) a) The first-order energy correction is given by the following equation

$$E_{\pm}^{(1)} = \frac{1}{2}\left[W_{aa} + W_{bb} \pm \sqrt{(W_{aa} - W_{bb})^2 + 4|W_{ab}|^2} \right]$$

where

$$W_{ij} = \int \psi_i(x)^* \, H' \, \psi_j(x)\, dx$$

and we are given the energy eigenstates

$$\psi_n(x) = \frac{1}{\sqrt{L}} e^{2\pi i n x/L}$$

for integer values of n. There is also a perturbing potential given by

$$H' = V_0 e^{-x^2/a^2}$$

with $0 < V_0 \lesssim 2\pi^2\hbar^2/mL^2$ and $a \lesssim L$.

Therefore we need to solve for $W_{n,n}$, $W_{n,-n}$, $W_{-n,n}$, $W_{-n,-n}$.

① solving for $W_{n,n}$

$$W_{n,n} = + \int_{-L/2}^{L/2} \frac{1}{\sqrt{L}} e^{-2\pi i n x/L} \; V_0 e^{-x^2/a^2} \frac{1}{\sqrt{L}} e^{2\pi i n x/L} \, dx$$

$$W_{n,n} = + \frac{V_0}{L} \int_{-L/2}^{L/2} e^{-x^2/a^2} dx$$

since we know how to evaluate gaussian integrals, we know that

$$\int_{-\infty}^{\infty} e^{-x^2/a^2} dx = \int_{-L/2}^{L/2} e^{-x^2/a^2} dx = a\sqrt{\pi}$$

therefore

$$W_{n,n} = + \frac{V_0}{L} a\sqrt{\pi}$$

② Solving for $W_{-n,-n}$

$$W_{-n,-n} = + \int_{-L/2}^{L/2} \frac{1}{\sqrt{L}} e^{2\pi i n x/L} \; V_0 e^{-x^2/a^2} \frac{1}{\sqrt{L}} e^{-2\pi i n x/L} \, dx$$

$$W_{-n,-n} = + \frac{V_0}{L} \int_{-L/2}^{L/2} e^{-x^2/a^2} dx$$

$$W_{-n,-n} = + \frac{V_0}{L} a\sqrt{\pi}$$

③ Solving for $W_{-n,n}$

$$W_{-n,n} = + \int_{-L/2}^{L/2} \frac{1}{\sqrt{L}} e^{2\pi i n x/L} \; V_0 e^{-x^2/a^2} \frac{1}{\sqrt{L}} e^{2\pi i n x/L} \, dx$$

$$W_{-n,n} = + \frac{V_0}{L} \int_{-L/2}^{L/2} e^{4\pi i n x/L - x^2/a^2} dx$$

by completing the square

$$W_{-n,n} = + \frac{V_0}{L} \int_{-L/2}^{L/2} e^{-\frac{1}{a^2}\left(x^2 - \frac{4\pi i n a^2 x}{L} + \frac{4\pi^2 n^2 a^4}{L^2} - \frac{4\pi^2 n^2 a^4}{L^2}\right)} dx$$

$$W_{-n,n} = + \frac{V_0}{L} \int_{-L/2}^{L/2} e^{-\frac{1}{a^2}\left(x - \frac{2\pi i n a^2}{L}\right)^2} e^{\frac{4\pi i n a^2}{L^2}} dx$$

this can be simplified by the gaussian integral

$$W_{-n,n} = + \frac{V_0}{L} a\sqrt{\pi} \, e^{\frac{4\pi^2 n^2 a^2}{L^2}}$$

when $a \ll L$ we have

$$W_{-n,n} = + \frac{V_0}{L} a\sqrt{\pi}$$

1.) a.) (4) solving for $W_{n,-n}$

$$W_{n,-n} = + \int_{-L/2}^{L/2} \frac{1}{\sqrt{L}} e^{-2\pi i n x/L} V_0 e^{-x^2/a^2} \frac{1}{\sqrt{L}} e^{-2\pi i n x/L} dx$$

$$W_{n,-n} = + \frac{V_0}{L} \int_{-L/2}^{L/2} e^{-4\pi i n x/L - x^2/a^2} dx$$

$$W_{n,-n} = + \frac{V_0}{L} \int_{-L/2}^{L/2} e^{-\frac{1}{a^2}\left(x^2 + \frac{4\pi i n a^2 x}{L} - \frac{4\pi^2 n^2 a^4}{L^2} + \frac{4\pi n^2 a^4}{L^2}\right)} dx$$

$$W_{n,-n} = + \frac{V_0}{L} \int_{-L/2}^{L/2} e^{-\frac{1}{a^2}\left(x + \frac{2\pi i n a^2}{L}\right)^2} e^{-\frac{4\pi^2 n^2 a^2}{L^2}} dx$$

similar to (3) this can simplify as

$$W_{n,-n} = + \frac{V_0}{L} e^{-4\pi^2 n^2 a^2/L} a\sqrt{\pi}$$

when $a \ll L$ this simplifies into

$$W_{n,-n} = + \frac{V_0}{L} a\sqrt{\pi}$$

The first-order energy correction then becomes

$$E_{\pm}^{(1)} = \frac{1}{2}\left[\frac{2V_0}{L} a\sqrt{\pi} \pm \sqrt{0 + 4\left|\frac{V_0 a\sqrt{\pi}}{L}\right|^2}\right]$$

$$= \frac{1}{2}\left[\frac{2V_0}{L} a\sqrt{\pi} \pm \frac{2V_0 a\sqrt{\pi}}{L}\right]$$

therefore we have

$$\boxed{\begin{aligned} E_{+}^{(1)} &= \frac{2V_0}{L} a\sqrt{\pi} \\ E_{-}^{(1)} &= 0 \end{aligned}}$$

b.) To obtain the linear combination of ψ_n and ψ_{-n} that diagonalizes the perturbation matrix, we first let

$$\psi = \alpha\psi_n + \beta\psi_{-n}$$

we wish to solve for the coefficents α and β for $E_{+}^{(1)}$ and $E_{-}^{(1)}$

(1) For $E_{+}^{(1)}$ we have

$$\int (\alpha\psi_n + \beta\psi_{-n})^* V_0 e^{-x^2/a^2} (\alpha\psi_n + \beta\psi_{-n}) dx = \frac{2V_0 a\sqrt{\pi}}{L}$$

$$\int \left(\alpha\frac{1}{\sqrt{L}} e^{-2\pi i n x/L} + \beta\frac{1}{\sqrt{L}} e^{2\pi i n x/L}\right) V_0 e^{-x^2/a^2} \left(\alpha\frac{1}{\sqrt{L}} e^{2\pi i n x/L} + \beta\frac{1}{\sqrt{L}} e^{-2\pi i n x/L}\right) dx = \frac{2V_0 a\sqrt{\pi}}{L}$$

$$\frac{V_0\alpha^2}{L} a\sqrt{\pi} + \frac{V_0\beta^2}{L} a\sqrt{\pi} + \frac{V_0\alpha\beta}{L} \int e^{-\frac{1}{a^2}\left(x^2 - \frac{4\pi i n x a^2}{L} + \frac{4\pi^2 n^2 a^4}{L^2} - \frac{4\pi^2 n^2 a^4}{L^2}\right)} dx$$

$$+ \frac{V_0\alpha\beta}{L} \int e^{-\frac{1}{a^2}\left(x^2 + \frac{4\pi i n x a^2}{L} - \frac{4\pi^2 n^2 a^4}{L^2} + \frac{4\pi^2 n^2 a^4}{L^2}\right)} dx = \frac{2V_0 a\sqrt{\pi}}{L}$$

this simplifies into

$$V_0\frac{\alpha^2}{L} a\sqrt{\pi} + V_0\frac{\beta^2}{L} a\sqrt{\pi} + \frac{2\alpha\beta}{L} V_0 a\sqrt{\pi} = \frac{2V_0 a\sqrt{\pi}}{L}$$

$$\frac{V_0 a\sqrt{\pi}}{L}(\alpha+\beta)^2 = \frac{2V_0 a\sqrt{\pi}}{L} \longrightarrow \alpha + \beta = \sqrt{2}$$

1.) b.) ① We also consider the normalization condition $|\alpha|^2 + |\beta|^2 = 1$, therefore we find that $\alpha = \beta = \frac{1}{\sqrt{2}}$.

② For $E_-^{(1)}$ we repeat the same procedure and get

$$\alpha + \beta = 0$$

considering the normalization condition, we find that $\alpha = -\beta$ and that $\alpha = \frac{1}{\sqrt{2}}$ while $\beta = -\frac{1}{\sqrt{2}}$.

With these solutions we can finally solve for $\psi_+ = \frac{\psi_n + \psi_{-n}}{\sqrt{2}}$ and $\psi_- = \frac{\psi_n - \psi_{-n}}{\sqrt{2}}$.

We have

$$\psi_+ = \frac{\psi_n + \psi_{-n}}{\sqrt{2}} = \frac{1}{\sqrt{2L}} e^{2\pi i n x/L} + \frac{1}{\sqrt{2L}} e^{-2\pi i n x/L} = \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi n x}{L}\right)$$

and

$$\psi_- = \frac{\psi_n - \psi_{-n}}{\sqrt{2}} = \frac{1}{\sqrt{2L}} e^{2\pi i n x/L} - \frac{1}{\sqrt{2L}} e^{-2\pi i n x/L} = \sqrt{\frac{2}{L}}\, i \sin\left(\frac{2\pi n x}{L}\right)$$

or

$$\boxed{\begin{aligned} \frac{\psi_n + \psi_{-n}}{\sqrt{2}} &= \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi n x}{L}\right) \\ \frac{\psi_n - \psi_{-n}}{\sqrt{2}} &= \sqrt{\frac{2}{L}}\, i \sin\left(\frac{2\pi n x}{L}\right) \end{aligned}}$$

Physics 242 Probset 2

2) We consider a gaussian waveform

$$\psi = Ae^{-(r/a)^2} \quad , \quad A = \text{normalization constant}$$

where we have

$$E = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

where we have the hamiltonian to be

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{r}$$

evaluating we have

$$E = \frac{\langle\psi|-\frac{\hbar^2}{2m}\nabla^2|\psi\rangle - \langle\psi|\frac{e^2}{r}|\psi\rangle}{\langle\psi|\psi\rangle}$$

① solving for the kinetic part

$$\langle\psi|T|\psi\rangle = -\frac{\hbar^2}{2m}A^2\int_0^\infty\int_0^{2\pi}\int_0^\pi e^{-(r/a)^2}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}e^{-(r/a)^2}\right)r^2\sin\theta\, dr\, d\theta\, d\phi$$

evaluating the θ and ϕ parts

$$\langle\psi|T|\psi\rangle = -\frac{\hbar^2}{2m}A^2 4\pi\int_0^\infty e^{-(r/a)^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{2r}{a^2}\right)e^{-(r/a)^2}\right]dr$$

$$= -\frac{\hbar^2}{2m}A^2 4\pi\int_0^\infty e^{-(r/a)^2}\frac{\partial}{\partial r}\left(-\frac{2r^3}{a^2}e^{-(r/a)^2}\right)dr$$

$$= -\frac{\hbar^2}{2m}A^2 4\pi\int_0^\infty e^{-(r/a)^2}\left[-\frac{2}{a^2}(3r^2)e^{-(r/a)^2} - \frac{2}{a^2}r^3\left(-\frac{2r}{a^2}\right)e^{-(r/a)^2}\right]dr$$

$$= -\frac{\hbar^2}{2m}A^2 4\pi\int_0^\infty e^{-2(r/a)^2}\left[-\frac{6}{a^2}r^2 + \frac{4r^4}{a^4}\right]dr$$

$$= -\frac{\hbar^2}{2m}A^2 4\pi\left[\int_0^\infty -\frac{6}{a^2}r^2e^{-2(r/a)^2}dr + \int_0^\infty\frac{4r^4}{a^4}e^{-2(r/a)^2}dr\right]$$

$$= -\frac{\hbar^2}{2m}A^2 4\pi\left[-3\int_0^\infty\frac{2}{a^2}r^2e^{-2(r/a)^2}dr + \int_0^\infty\frac{4r^4}{a^4}e^{-2(r/a)^2}dr\right]$$

we now have 2 integrals that resemble gaussian integrals that have been differentiated under the integral sign (feynman trick)

$$\int_0^\infty x^2e^{-x^2}dx = \sqrt{\pi}/4$$

$$\int_0^\infty x^4e^{-x^2}dx = 3\sqrt{\pi}/8$$

for both terms we let $x = \frac{\sqrt{2}r}{a}$, $dx = \frac{\sqrt{2}}{a}dr \rightarrow dr = \frac{a}{\sqrt{2}}dx$

$$\langle\psi|T|\psi\rangle = -\frac{\hbar^2}{2m}A^2 4\pi\left[-\frac{3a}{\sqrt{2}}\frac{\sqrt{\pi}}{4} + \frac{a}{\sqrt{2}}\frac{3\sqrt{\pi}}{8}\right]$$

$$= -\frac{\hbar^2}{2m}A^2 4\pi\left[-\frac{6a\sqrt{\pi}}{8\sqrt{2}} + \frac{3a\sqrt{\pi}}{8\sqrt{2}}\right]$$

$$= -\frac{\hbar^2}{2m}A^2 4\pi\left(-\frac{3a\sqrt{\pi}}{8\sqrt{2}}\right) = \frac{\hbar^2}{2m}A^2 4\pi\left(\frac{3a\sqrt{\pi}}{8\sqrt{2}}\right)$$

2) a.) ② For the potential part we have

$$\langle \psi | V | \psi \rangle = A^2 e^2 \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{-e^{-2(r/a)^2}}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= -A^2 e^2 4\pi \int_0^\infty r e^{-2(r/a)^2} dr$$

we also have a corresponding identity to this

$$\int_0^\infty x e^{-x^2} dx = \frac{1}{2}$$

multiply and divide by $\left(\frac{\sqrt{2}}{a} \cdot \frac{a}{\sqrt{2}}\right)$ and we'll have

$$\langle \psi | V | \psi \rangle = -A^2 e^2 4\pi \frac{a}{\sqrt{2}} \int_0^\infty \frac{\sqrt{2} r}{a} e^{-2(r/a)^2} dr$$

Let $x = \frac{\sqrt{2} r}{a}$, $dx = \frac{\sqrt{2} dr}{a} \longrightarrow \frac{a}{\sqrt{2}} dx = dr$. Therefore

$$\langle \psi | V | \psi \rangle = -A^2 e^2 4\pi \left(\frac{a}{\sqrt{2}}\right)^2 \int_0^\infty x e^{-x^2} dx$$

using the identity, we have

$$\langle \psi | V | \psi \rangle = -A^2 e^2 4\pi \left(\frac{a}{\sqrt{2}}\right)^2 \frac{1}{2} = -A^2 e^2 4\pi \frac{a^2}{2} \frac{1}{2} = -A^2 e^2 4\pi \frac{a^2}{4}$$

③ computing for $\langle \psi | \psi \rangle$

$$\langle \psi | \psi \rangle = \int_0^\infty \int_0^{2\pi} \int_0^\pi A^2 e^{-2(r/a)^2} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 4\pi \int_0^\infty A^2 r^2 e^{-2(r/a)^2} dr$$

multiplying and dividing by $\left(\frac{2}{a^2} \cdot \frac{a^2}{2}\right)$

$$\langle \psi | \psi \rangle = 4\pi A^2 \int_0^\infty \frac{2r^2}{a^2} \cdot \frac{a^2}{2} e^{-2(r/a)^2} dr$$

Let $x = \frac{\sqrt{2} r}{a}$, $dx = \frac{\sqrt{2} dr}{a} \rightarrow dr = \frac{a}{\sqrt{2}} dx$

$$\langle \psi | \psi \rangle = 4\pi A^2 \frac{a^2}{2} \frac{a}{\sqrt{2}} \int_0^\infty x^2 e^{-x^2} dx = 4\pi A^2 \frac{a^3}{2\sqrt{2}} \frac{\sqrt{\pi}}{4}$$

$$= 4\pi A^2 \frac{a^3 \sqrt{\pi}}{8\sqrt{2}}$$

Therefore,

$$E = \frac{\frac{\hbar^2}{2m} A^2 4\pi \left(\frac{3a\sqrt{\pi}}{8\sqrt{2}}\right) - A^2 e^2 4\pi \frac{a^2}{4}}{\frac{4\pi A^2 a^3 \sqrt{\pi}}{8\sqrt{2}}} = \frac{3\hbar^2}{2a^2 m} - \frac{2\sqrt{2} e^2}{a\sqrt{\pi}}$$

We let $a_0 = \frac{\hbar^2}{2me^2}$ be the Bohr radius. Therefore E can be rewritten as

$$E = \frac{3a_0}{a} e^2 - \frac{2\sqrt{2} e^2}{a\sqrt{\pi}}$$

solving for a_{min} we have

$$E' = -\frac{6a_0 e^2}{a^3} + \frac{2\sqrt{2} e^2}{a^2\sqrt{\pi}} = 0 \longrightarrow a_{min} = 3a_0\sqrt{\frac{\pi}{2}}$$

Plugging this back to E to get E_{min}, we have

$$E = 3a_0 e^2 \left(\frac{2}{9a_0^2 \pi}\right) - \frac{2\sqrt{2}\, e^2}{\sqrt{\pi}} \cdot \frac{2}{3a_0\sqrt{\pi}}$$

$$= \frac{2e^2}{3\pi a_0} - \frac{4e^2}{3a_0\pi} = \frac{-2e^2}{3a_0\pi}$$

Comparing this with the value $E_0 = -\frac{e^2}{2a_0}$ from sakurai we have

$$\frac{E_{min}}{E_0} = \frac{-2e^2}{3a_0\pi} \cdot \frac{2a_0}{-e^2} = \frac{4}{3\pi} = \boxed{0.42 \approx -0.42\ R_y}$$

with an error of

$$\left|\frac{0.42\, E_0 - E_0}{E_0}\right| = 0.58 = 58\%$$

b) We can get the standard deviation of our gaussian by comparing it to the gaussian probability density function

$$f(x) = A e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

comparing our wavefunction of

$$\psi(r) = A e^{-\frac{r^2}{a^2}}$$

we have the standard deviation, σ, to be

$$\sigma = \frac{a\sqrt{2}}{2}$$

we have $a_{min} = 3a_0\sqrt{\frac{\pi}{2}}$, therefore

$$\sigma = \frac{3a_0\sqrt{\pi}}{2}$$

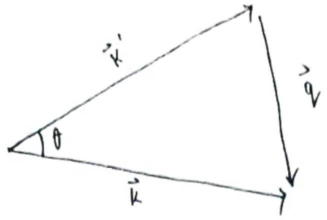
comparing this with the Bohr radius, a_0, we have

$$\boxed{\frac{\sigma}{a_0} = \frac{3\sqrt{\pi}}{2} \approx 2.7}$$

3.) a.) We are given the first - order born amplitude

$$f(k,k') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x' \, e^{i(k-k')\cdot x'} V(x')$$

where $q = |k-k'|$ is the scattering through an angle θ



We then get the following result by squaring $q = |k-k'|$

$$q^2 = \left(\sqrt{k^2 + k'^2 - 2\vec{k}\cdot\vec{k}'}\right)^2$$

$$q^2 = k^2 + k'^2 - 2\vec{k}\cdot\vec{k}' = k^2 + k'^2 - 2kk'\cos\theta$$

when $k \approx k'$ this can be approximated as

$$q^2 \approx 2k^2 - 2k^2\cos\theta \approx 2k^2(1-\cos\theta)$$

recalling the half - angle identity

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \longrightarrow \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$2\sin^2\frac{\theta}{2} = 1-\cos\theta$$

therefore

$$q^2 \approx 4k^2\sin^2\frac{\theta}{2}$$

$$q \approx 2k\sin\frac{\theta}{2}$$

going back to the born amplitude

$$f(k,k') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x' \, e^{iqx'\cos\theta} V(x')$$

we can then write this in terms of θ

$$f(\theta) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int e^{iqr\cos\theta} V(r)\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

Since $\int d\phi = 2\pi$, we have

$$f(\theta) = -\frac{1}{2}\frac{2m}{\hbar^2}\iint e^{iqr\cos\theta} V(r)\, r^2\sin\theta\, dr\, d\theta$$

evaluating over the theta component, we let

$$y = iqr\cos\theta$$

$$dy = -iqr\sin\theta\, d\theta$$

therefore,

$$-\frac{1}{2m}\frac{2m}{\hbar^2}\iint e^{y} V(r)\frac{r^2\sin\theta}{-iqr\sin\theta}dy$$

$$\frac{1}{2}\frac{2m}{\hbar^2}\iint e^{y} V(r)\frac{r}{iq}dy = \frac{1}{2}\frac{2m}{\hbar^2}\int\left[\frac{rV(r)\, e^{iqr\cos\theta}}{iq}\right]_0^{\pi} dr$$

$$= \frac{1}{2}\frac{2m}{\hbar^2}\int\frac{rV(r)}{iq}\left[e^{-iqr} - e^{iqr}\right]dr$$

and we end up with

$$f(\theta) = -\frac{2m}{\hbar^2}\int_0^{\infty}\frac{rV(r)}{q}\sin(qr)\, dr$$

3.) a.) Now we consider a potential given by the yukawa potential

$$V(r) = \frac{V_0 e^{-\mu r}}{\mu r} \frac{1}{4\pi \epsilon_0}$$

substituting this into $f(\theta)$ we have

$$f(\theta) = \left(\frac{1}{4\pi\epsilon_0}\right)\left(-\frac{2m}{\hbar^2}\right)\int_0^\infty \frac{V_0 e^{-\mu r}}{q\mu}\left(e^{iqr} - e^{-iqr}\right)dr$$

$$f(\theta) = \left(\frac{1}{4\pi\epsilon_0}\right)\left(-\frac{2m}{\hbar^2}\right)\frac{V_0}{q\mu}\int_0^\infty \left(e^{-\mu r + iqr} - e^{-\mu r - iqr}\right)dr$$

$$= \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{-2m}{\hbar^2}\right)\frac{V_0}{q\mu}\left[\frac{1}{-\mu + iq} e^{-\mu r + iqr} - \frac{1}{-\mu - iq} e^{-\mu r - iqr}\right]_0^\infty$$

$$= \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{-2m}{\hbar^2}\right)\frac{V_0}{q\mu}\left[-\frac{1}{-\mu + iq} + \frac{1}{-\mu - iq}\right]$$

$$= \left(\frac{1}{4\pi\epsilon_0}\right)\left(\frac{-2m}{\hbar^2}\right)\frac{V_0}{q\mu}\left[\frac{\mu + iq - \mu + iq}{\mu^2 + q^2}\right]$$

$$= -\left(\frac{2mV_0}{\hbar^2 \mu}\right)\left(\frac{i}{\mu^2 + q^2}\right)\frac{1}{4\pi\epsilon_0}$$

we take the imaginary part which yields

$$f(\theta) = -\left(\frac{2mV_0}{\hbar^2\mu}\right)\frac{1}{\mu^2 + q^2}\left(\frac{1}{4\pi\epsilon_0}\right)$$

Since we have the form of the differential cross section to be

$$\frac{d\sigma}{d\Omega} = |f(k, k')|^2 = |f(\theta)|^2$$

In our case, we have

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{2mV_0}{\hbar^2\mu}\right)^2\left(\frac{1}{\mu^2 + q^2}\right)^2$$

since we know that $q^2 = 4k^2 \sin^2\left(\frac{\theta}{2}\right)$, we find that

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\hbar^2\mu}\right)^2\left(\frac{1}{4k^2\sin^2\left(\frac{\theta}{2}\right) + \mu^2}\right)^2\left(\frac{1}{4\pi\epsilon_0}\right)^2$$

when we want the yukawa potential to reduce to the coulomb potential we let

$$\mu \to 0$$

$$\frac{V_0}{\mu} \to q_1 q_2$$

therefore the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{2m}{\hbar^2}\right)^2 (q_1 q_2)^2 \frac{1}{16k^4\sin^4(\theta/2)}$$

and we have the total energy to be

$$E = \frac{\hbar^2 k^2}{2m}$$

therefore,

$$\boxed{\frac{d\sigma}{d\Omega} = \left[\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)}\right]^2}$$

we evaluate for σ

$$\sigma = \int \left(\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)} \right)^2 \sin\theta \, d\theta \, d\phi$$

$$\sigma = 2\pi \left(\frac{q_1 q_2}{8\pi\epsilon_0 E} \right)^2 \int_0^{\pi} \frac{\sin\theta}{\sin^4(\theta/2)} d\theta$$

using the small angle approximation $\sin x \approx x$, we have

$$\sigma = 2\pi \left(\frac{q_1 q_2}{8\pi\epsilon_0} \right)^2 16 \int_0^{\pi} \frac{1}{\theta^3} d\theta$$

$$\sigma = 2\pi \left(\frac{q_1 q_2}{4\pi\epsilon_0} \right) 8 \left[-\frac{1}{\theta^2} \right]_0^{\pi} \longrightarrow \infty$$

this is obtained because the integral doesn't converge.

The total cross section is infinite because the coulomb force has an infinite range. so take for example particles about a nucleus. Whatever distance this particle is to the nucleus, it still experiences some force and will be scattered with a non-zero angle.