Physics 241
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Problem Set 1

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1 POSITION AND MOMENTUM

1.a What is the action of the momentum operator $\hat{p}$ on the wave function $\langle \mathbf{p}|\psi\rangle$?

[Solution]

Applying the momentum operator $\hat{p}$ to the wave function $\langle \mathbf{p}|\psi\rangle$, and taking note that in the $|\mathbf{p}\rangle$ representation the momentum operator $\hat{p}$ coincides with an operator that just multiplies by p_x, p_y, or p_z. Therefore,

$$\langle \mathbf{p}|\hat{p}|\psi\rangle = \hat{p}\langle \mathbf{p}|\psi\rangle \tag{1}$$

or

$$\begin{aligned} \langle \mathbf{p}|\hat{p}_x|\psi\rangle &= p_x\langle \mathbf{p}|\psi\rangle \\ \langle \mathbf{p}|\hat{p}_y|\psi\rangle &= p_y\langle \mathbf{p}|\psi\rangle \\ \langle \mathbf{p}|\hat{p}_z|\psi\rangle &= p_z\langle \mathbf{p}|\psi\rangle \end{aligned} \tag{2}$$

for a 3 dimensional system.

1.b What is the action of the momentum operator $\hat{p}$ on the wave function $\langle \mathbf{x}|\psi\rangle$?

[Solution]

However, in the $|\mathbf{x}\rangle$ representation, the action of the momentum operator $\hat{p}$ coincides with the differential operator $-\mathbf{i}\hbar\nabla$ applied to the wave function. Therefore the action on the wave function in integral form is given by

$$\langle \mathbf{x}|\hat{p}_x|\psi\rangle = \int d^3p\langle \mathbf{x}|\mathbf{p}\rangle\langle \mathbf{p}|\hat{p}_x|\psi\rangle = (2\pi\hbar)^{-3/2}\int e^{\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}}\hat{p}_x\bar{\psi}(\mathbf{p})d^3p \tag{3}$$

where we used the relation $\langle \mathbf{x}|\mathbf{p}\rangle = (2\pi\hbar)^{-3/2}e^{\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar}}$. We can also write $\hat{p}_x\bar{\psi}(\mathbf{p})$ in terms of its Fourier transform and rewrite the equation above.

$$\langle \mathbf{x}|\hat{p}_x|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle \mathbf{x}|\psi\rangle \tag{4}$$

The same procedure also happens for the $\hat{p}_y$ and $\hat{p}_z$ case. We can then write

$$\begin{aligned}\langle \mathbf{x}|\hat{p}_y|\psi\rangle &= -i\hbar\frac{\partial}{\partial y}\langle \mathbf{x}|\psi\rangle \\ \langle \mathbf{x}|\hat{p}_z|\psi\rangle &= -i\hbar\frac{\partial}{\partial z}\langle \mathbf{x}|\psi\rangle\end{aligned} \tag{5}$$

We can then write the action of the momentum operator $\hat{p}$ as

$$\langle \mathbf{x}|\hat{p}|\psi\rangle = -i\hbar\nabla\langle \mathbf{x}|\psi\rangle \tag{6}$$

2 TRANSLATION

On an arbitrary state $|\psi\rangle$:

2.a How does a finite translation $\mathscr{T}(\mathbf{l})$ affect $\langle \mathbf{x}\rangle_\psi$?

[Solution]

We start by considering an arbitrary position ket $|\alpha\rangle$ and the corresponding translated position ket $|\alpha'\rangle = \mathscr{T}(\mathbf{l})|\alpha\rangle$. We are tasked to find the expectation value of the position under translation

$$\langle \alpha'|x_i|\alpha'\rangle = \langle \alpha|\mathscr{T}(\mathbf{l})^\dagger x_i \mathscr{T}(\mathbf{l})|\alpha\rangle \tag{7}$$

We can evaluate $\mathscr{T}(\mathbf{l})^\dagger x_i \mathscr{T}(\mathbf{l})$ by considering the commutator relation

$$[x_i, \mathscr{T}(\mathbf{l})] = x_i\mathscr{T}(\mathbf{l}) - \mathscr{T}(\mathbf{l})x_i = l_i\mathscr{T}(\mathbf{l}) \tag{8}$$

We can multiply the above relation by $\mathscr{T}(\mathbf{l})^\dagger$ to yield

$$\begin{aligned}\mathscr{T}(\mathbf{l})^\dagger x_i\mathscr{T}(\mathbf{l}) - \mathscr{T}(\mathbf{l})^\dagger\mathscr{T}(\mathbf{l})x_i &= \mathscr{T}(\mathbf{l})^\dagger\mathscr{T}(\mathbf{l})l_i \\ \mathscr{T}(\mathbf{l})^\dagger x_i\mathscr{T}(\mathbf{l}) &= \mathscr{T}(\mathbf{l})^\dagger\mathscr{T}(\mathbf{l})x_i + \mathscr{T}(\mathbf{l})^\dagger\mathscr{T}(\mathbf{l})l_i\end{aligned} \tag{9}$$

This simplifies into

$$\mathscr{T}(\mathbf{l})^\dagger x_i\mathscr{T}(\mathbf{l}) = x_i + l_i \tag{10}$$

We can then plug this back into our initial equation to yield

$$\langle \alpha'|x_i|\alpha'\rangle = \langle \alpha|x_i + l_i|\alpha\rangle = \langle \alpha|x_i|\alpha\rangle + l_i \tag{11}$$

The expectation value for every component of the $\mathbf{x}$ operator would be

$$\langle \mathbf{x}\rangle_{translated} = \langle \mathbf{x}\rangle_{original} + \mathbf{l} \tag{12}$$

2.b How does a finite translation $\mathscr{T}(\mathbf{l})$ affect $\langle \mathbf{p}\rangle_\psi$?

[Solution]

Similarly, we can follow the same procedure above for the expectation value of the momentum operator under translation.

$$\langle \alpha' | p_i | \alpha' \rangle = \langle \alpha | \mathscr{T}(\mathbf{l})^\dagger p_i \mathscr{T}(\mathbf{l}) | \alpha \rangle \tag{13}$$

To solve this we consider the commutator relation

$$[p_i, \mathscr{T}(\mathbf{l})] = p_i \mathscr{T}(\mathbf{l}) - \mathscr{T}(\mathbf{l}) p_i = 0 \tag{14}$$

Multiplying $\mathscr{T}(\mathbf{l})^\dagger$ to the previous equation we find

$$\mathscr{T}(\mathbf{l})^\dagger p_i \mathscr{T}(\mathbf{l})^\dagger = \mathscr{T}(\mathbf{l})^\dagger \mathscr{T}(\mathbf{l}) p_i = p_i \tag{15}$$

We can then plug this back into our initial equation to yield

$$\langle \alpha' | p_i | \alpha' \rangle = \langle \alpha | p_i | \alpha \rangle = \langle \alpha | p_i | \alpha \rangle \tag{16}$$

The expectation value for every component of the **p** operator would be

$$\langle \mathbf{p} \rangle_{translated} = \langle \mathbf{p} \rangle_{original} \tag{17}$$

3 SPIN PRECESSION

3.a Consider the precession of a spin - $1/2$ system in an external magnetic field $\mathbf{B} = B\hat{z}$. Solve the Heisenberg equations of motion to obtain S(t).

[Solution]

The Heisenberg equations of motion is given by

$$\frac{dA^{(H)}}{dt} = \frac{1}{i\hbar}[A^{(H)}, H] \tag{18}$$

where $H = -\left(\frac{eB}{m_e c}\right) S_z$ when we take B to be a static and uniform magnetic field in the $z-$direction. Solving for S_x, S_y, and S_z, we have,

$$\frac{dS_x}{dt} = \frac{1}{i\hbar}[S_x, H] = \frac{1}{i\hbar}\left(\frac{eB}{m_e c}\right)(S_z S_x - S_x S_z) = \left(\frac{eB}{m_e c}\right) S_y \tag{19}$$

$$\frac{dS_y}{dt} = \frac{1}{i\hbar}[S_y, H] = \frac{1}{i\hbar}\left(\frac{eB}{m_e c}\right)(S_z S_y - S_y S_z) = -\left(\frac{eB}{m_e c}\right) S_x \tag{20}$$

$$\frac{dS_z}{dt} = \frac{1}{i\hbar}[S_z, H] = \frac{1}{i\hbar}\left(\frac{eB}{m_e c}\right)(S_z S_z - S_z S_z) = 0 \tag{21}$$

where we used $[S_z, S_y] = -i\hbar S_x$ and $[S_z, S_x] = i\hbar S_y$. We can write these as

$$\begin{aligned} \frac{d^2 S_x}{dt^2} &= -\left(\frac{eB}{m_e c}\right)^2 S_x \\ \frac{d^2 S_y}{dt^2} &= -\left(\frac{eB}{m_e c}\right)^2 S_y \\ \frac{dS_z}{dt} &= 0 \end{aligned} \tag{22}$$

We would have the general solution for these differential equation

$$\begin{aligned} S_x &= C_1 \cos\left(\frac{eB}{m_e c} t\right) + C_2 \sin\left(\frac{eB}{m_e c} t\right) \\ S_y &= C_3 \cos\left(\frac{eB}{m_e c} t\right) + C_4 \sin\left(\frac{eB}{m_e c} t\right) \\ S_z &= C_5 \end{aligned} \tag{23}$$

Imposing initial conditions $S_i(t = 0) = S_i(0)$ and $\dot{S}_i(t = 0) = \dot{S}_i(0)$, we have

$$\begin{aligned} S_x(t) &= S_x(0) \cos\left(\frac{eB}{m_e c} t\right) + \dot{S}_x(0) \sin\left(\frac{eB}{m_e c} t\right) \\ S_y(t) &= S_y(0) \cos\left(\frac{eB}{m_e c} t\right) + \dot{S}_y(0) \sin\left(\frac{eB}{m_e c} t\right) \\ S_z(t) &= S_z(0) \end{aligned} \tag{24}$$

3.b Let the initial state be an eigenstate of S_x with eignenvalue $\hbar/2$. Show that the expectation value $\langle \mathbf{S(t)} \rangle$ precesses about the z-axis.

[Solution]

Consider $\mathbf{S} = \hbar/2\boldsymbol{\sigma}$ where σ_x, σ_y, and σ_z are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{25}$$

and the fact that $\langle +|+\rangle = \langle -|-\rangle = 1$ while $\langle +|-\rangle = \langle -|+\rangle = 0$. We can solve for the expectation value $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$

$$\begin{aligned} \langle S_x \rangle &= \frac{\hbar}{4} \langle \psi^* | \sigma_x | \psi \rangle \\ &= \frac{\hbar}{4} \left(e^{-\frac{i\omega t}{2}} \langle +| + e^{\frac{i\omega t}{2}} \langle -| \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(e^{-\frac{i\omega t}{2}} |-\rangle + e^{\frac{i\omega t}{2}} |+\rangle \right) \\ &= \frac{\hbar}{4} \left(e^{-\frac{i\omega t}{2}} \langle +| + e^{\frac{i\omega t}{2}} \langle -| \right) \left(e^{-\frac{i\omega t}{2}} |+\rangle + e^{\frac{i\omega t}{2}} |-\rangle \right) \\ &= \frac{\hbar}{4} \left(e^{-i\omega t} + e^{i\omega t} \right) \end{aligned} \tag{26}$$

$$
\begin{aligned}
\langle S_y\rangle &= \frac{\hbar}{4}\langle\psi^*|\sigma_y|\psi\rangle \\
&= \frac{\hbar}{4}\left(e^{-\frac{i\omega t}{2}}\langle+| + e^{\frac{i\omega t}{2}}\langle-|\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\left(e^{-\frac{i\omega t}{2}}|-\rangle + e^{\frac{i\omega t}{2}}|+\rangle\right) \\
&= \frac{\hbar}{4}\left(e^{-\frac{i\omega t}{2}}\langle+| + e^{\frac{i\omega t}{2}}\langle-|\right)\left(ie^{\frac{i\omega t}{2}}|-\rangle - ie^{-\frac{i\omega t}{2}}|+\rangle\right) \\
&= \frac{\hbar}{4}\left(ie^{-i\omega t} - ie^{i\omega t}\right)
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\langle S_z\rangle &= \frac{\hbar}{4}\langle\psi^*|\sigma_z|\psi\rangle \\
&= \frac{\hbar}{4}\left(e^{-\frac{i\omega t}{2}}\langle+| + e^{\frac{i\omega t}{2}}\langle-|\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(e^{-\frac{i\omega t}{2}}|-\rangle + e^{\frac{i\omega t}{2}}|+\rangle\right) \\
&= \frac{\hbar}{4}\left(e^{-\frac{i\omega t}{2}}\langle+| + e^{\frac{i\omega t}{2}}\langle-|\right)\left(e^{\frac{i\omega t}{2}}|+\rangle - e^{\frac{-i\omega t}{2}}|-\rangle\right) \\
&= \frac{\hbar}{4}(1-1) = 0
\end{aligned}
\tag{28}
$$

Using the relation $\cos(\omega t) = \frac{e^{i\omega t}+e^{-i\omega t}}{2}$ and $\sin(\omega t) = \frac{e^{i\omega t}-e^{-i\omega t}}{2i}$, we can simplify the expectation value for S_x and S_y

$$
\begin{aligned}
\langle S_x\rangle &= \frac{\hbar}{2}\cos(\omega t) \\
\langle S_y\rangle &= \frac{\hbar}{2}\sin(\omega t) \\
\langle S_z\rangle &= 0
\end{aligned}
\tag{29}
$$

Therefore we have shown that the expectation precesses about the $z-$ axis.

4 COHERENT STATE

The energy eigenkets of the harmonic oscillator $|n\rangle$ are stationary states under the harmonic oscillator Hamiltonian. In one dimension:

4.a Construct the state $|\lambda\rangle$ that satisfies $a|\lambda\rangle = \lambda|\lambda\rangle$ in the basis of harmonic oscillator energy eigenkets $|n\rangle$.

[Solution]

When expressed as a superposition of energy (or N) eigenstates, we can construct the state $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle \tag{30}$$

where the distribution of $|f(n)|^2$ with respect to n of the poisson type about some mean value $\bar{n}$:

$$|f(n)|^2 = \left(\frac{\bar{n}^n}{n!}e^{-\bar{n}}\right) \tag{31}$$

4.b Show that the coherent state $|\lambda\rangle$ has expectation values $\langle x(t)\rangle$ and $\langle p(t)\rangle$ that have the same time development as a classical oscillator.

[Solution]

We have the Heisenberg equations of motion given by

$$\frac{dA^{(H)}}{dt} = \frac{1}{i\hbar}[A^{(H)}, H] \tag{32}$$

and the identities

$$[a, H] = \hbar\omega a \quad \text{and} \quad [a^\dagger, H] = -\hbar\omega a^\dagger \tag{33}$$

The Heisenberg equations of motion therefore becomes

$$\frac{da}{dt} = -i\omega a \quad \text{and} \quad \frac{da^\dagger}{dt} = i\omega a^\dagger \tag{34}$$

Solving these equations we get

$$a(t) = a(0)e^{-i\omega t} \quad \text{and} \quad a^\dagger(t) = a^\dagger(0)e^{i\omega t} \tag{35}$$

From the definition of the annihilation and creation operator

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \quad \text{and} \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) \tag{36}$$

We can get the following values for $a(t)$ and $a^\dagger(t)$

$$a(t) = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(x(0)e^{-i\omega t} + \frac{ip(0)}{m\omega}e^{-i\omega t}\right) \tag{37}$$

and

$$a^\dagger(t) = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right) = \sqrt{\frac{m\omega}{2\hbar}}\left(x(0)e^{i\omega t} - \frac{ip(0)}{m\omega}e^{i\omega t}\right) \tag{38}$$

or simply put

$$x + \frac{ip}{m\omega} = x(0)e^{-i\omega t} + \frac{ip(0)}{m\omega}e^{-i\omega t} \quad \text{and} \quad x - \frac{ip}{m\omega} = x(0)e^{i\omega t} - \frac{ip(0)}{m\omega}e^{i\omega t} \tag{39}$$

Solving for x and p

$$x(t) = x(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t) \quad \text{and} \quad p(t) = p(0)\cos(\omega t) - m\omega x(0)\sin(\omega t) \tag{40}$$

It is now clear to see that $\langle x(t)\rangle$ and $\langle p(t)\rangle$ oscillate just like in classical mechanics .

4.c Evaluate the position-momentum uncertainty relation for this state at arbitrary time.

[Solution]

We can consider the fact that $a|\lambda\rangle = \lambda|\lambda\rangle$ and $\langle\lambda|a^\dagger = \langle\lambda|\lambda^*$ to write the expectations of x and p in the following form

$$\langle\lambda|x|\lambda\rangle = \sqrt{\frac{\hbar}{2m\omega}}[\lambda + \lambda^*] \quad \text{and} \quad \langle\lambda|p|\lambda\rangle = i\sqrt{\frac{m\hbar\omega}{2}}[\lambda^* - \lambda] \tag{41}$$

We can also solve for the expectation values of x^2 and p^2

$$x^2 = \frac{\hbar}{2m\omega}[a^2 + (a^\dagger)^2 + 2aa^\dagger + 1] \quad \text{and} \quad p^2 = -\frac{mh\omega}{2}[(a^\dagger)^2 + a^2 - 2aa^\dagger - 1] \tag{42}$$

which leads to our desired expectation value

$$\langle\lambda|x^2|\lambda\rangle = \frac{\hbar}{2m\omega}[\lambda^2 + (\lambda*)^2 + 2\lambda\lambda^* + 1] \quad \text{and} \quad \langle\lambda|p^2|\lambda\rangle = -\frac{mh\omega}{2}[(\lambda^*)^2 + \lambda^2 - 2\lambda\lambda^* - 1] \tag{43}$$

where we used $a^\dagger a = aa^\dagger + 1$ and $\langle\lambda|(a^\dagger)^2|\lambda\rangle = (\lambda^*)^2$. We can now then evaluate the position-momentum uncertainty relation

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega} \quad \text{and} \quad (\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{m\hbar\omega}{2} \tag{44}$$

Therefore,

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{45}$$

5 PROBABILITY FLUX AND PHASE

Let $\psi(\mathbf{x}, t)$ be a wavefunction

5.a Prove that the probability flux $\mathbf{j} = (\hbar/m)\mathscr{I}[\psi^*\nabla\psi]$ satisfies the continuity equation

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\mathbf{j}$$

with $\rho = |\psi|^2$.

[Solution]

Consider the given probability density $\rho = \psi^*\psi = |\psi|^2$. We can write this in the form of a time derivative of a volume integral such that

$$\frac{\partial}{\partial t}\int \rho dV = \frac{\partial}{\partial t}\int (\psi^*\psi)dV \tag{46}$$

Differentiating the RHS under the integral sign

$$\frac{\partial}{\partial t}\int \rho dV = \int \left(\frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}\right) \tag{47}$$

Remember from the Schrodinger's wave equation we have

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} \tag{48}$$

Solving for $\partial\psi/\partial t$ and $\partial\psi^*/\partial t$, we have

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\right] \tag{49}$$

and

$$\frac{\partial\psi^*}{\partial t} = \frac{i}{\hbar}\left[-\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*\right] \tag{50}$$

Plugging both of these into the time derivative of the volume integral of the probability density

$$\begin{aligned}\frac{\partial}{\partial t}\int\rho dV &= \int\left[\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi*\right)\psi - \frac{i}{\hbar}\psi^*\left(-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\right)\right]dV \\ &= \int\left[-\frac{i\hbar}{2m}\psi\nabla^2\psi * + \frac{i}{\hbar}V\psi^*\psi + \frac{i\hbar}{2m}\psi^*\nabla^2\psi - \frac{i}{\hbar}\psi^*V\psi\right]dV\end{aligned} \tag{51}$$

This simplifies into

$$\frac{\partial}{\partial t}\int\rho dV = \int -\frac{i\hbar}{2m}(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi)dV \tag{52}$$

Notice that $\nabla\cdot(\psi\nabla\psi^* - \psi^*\nabla\psi) = \nabla\psi\nabla\psi^* + \psi\nabla^2\psi^* - \nabla\psi^*\nabla\psi - \psi^*\nabla^2\psi = \psi\nabla^2\psi^* - \psi^*\nabla^2\psi$. Therefore,

$$\frac{\partial}{\partial t}\int\rho dV = -\int\nabla\cdot\left[\frac{i\hbar}{2m}(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi)\right] \tag{53}$$

Up to a constant $\frac{i\hbar}{2m}(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi)$ is just the probability flux $\mathbf{j}$. Therefore, we have that

$$\frac{\partial}{\partial t}\int\rho dV = -\int\nabla\mathbf{j}dV \tag{54}$$

or

$$\frac{\partial}{\partial t}\rho = -\nabla\mathbf{j} \tag{55}$$

5.b Write the wavefunction as the product of an amplitude and complex phase factor

$$\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}\times e^{\mathbf{i}S(\mathbf{x},t)/\hbar}$$

to show that $\mathbf{j} = \rho\nabla S/m$

[Solution]

From the previous question we have the probability flux given by

$$\mathbf{j} = (\hbar/m)\mathscr{I}[\psi^*\nabla\psi] \tag{56}$$

We can directly plug in the given wave function and evaluate for some component of x

$$\mathbf{j} = (\hbar/m)\mathscr{I}\left[\sqrt{\rho(\mathbf{x},t)}\times e^{-\mathbf{i}S(\mathbf{x},t)/\hbar}\frac{\partial}{\partial x_i}(\sqrt{\rho(\mathbf{x},t)}\times e^{\mathbf{i}S(\mathbf{x},t)/\hbar})\right] \tag{57}$$

Using the chain rule on the derivative, we arrive with

$$\mathbf{j} = \frac{\hbar}{m}\mathscr{I}\left[\sqrt{\rho(\mathbf{x},t)}e^{-\mathbf{i}S(\mathbf{x},t)/\hbar}\left(\frac{\partial}{\partial x_i}\left(\sqrt{\rho(\mathbf{x},t)}\right)e^{\mathbf{i}S(\mathbf{x},t)/\hbar} + \frac{i}{\hbar}\sqrt{\rho(\mathbf{x},t)}\frac{\partial S(\mathbf{x},t)}{\partial x_i}e^{\mathbf{i}S(\mathbf{x},t)/\hbar}\right)\right] \tag{58}$$

I left the first partial derivative unevaluated because we can make our lives easier by only considering the imaginary term in the RHS of the equation. The equation above can then simplify into

$$\mathbf{j} = \left(\frac{\hbar}{m}\right)\left(\frac{1}{\hbar}\right)\sqrt{\rho(\mathbf{x},t)}\sqrt{\rho(\mathbf{x},t)}\frac{\partial S(\mathbf{x},t)}{\partial x_i}e^{-\mathbf{i}S(\mathbf{x},t)/\hbar}e^{\mathbf{i}S(\mathbf{x},t)/\hbar} \tag{59}$$

Summing over all components of x_i and cancelling some factors we arrive with

$$\mathbf{j} = \rho\nabla S/m \tag{60}$$

6 WKB APROXIMATION

Describe the limit of validity of the WKB approximation in terms of the two length scales: spatial variation of the wavefunction and spatial variation of the potential

[Solution]

For a slowly varying potential V, we have the constraint that V does not change much on the scale of a wavelength $\lambda \ll L$ where L is some characteristic scale length of the potential. We can write the wave function as

$$\psi(\mathbf{x}) = \mathbf{A}(\mathbf{x})e^{i\mathbf{S}(\mathbf{x})/\hbar} \tag{61}$$

where $\mathbf{A}(\mathbf{x})$ and $S(\mathbf{x})$ are functions of $\mathbf{x}$ that we need to determine. The given solution $\psi(\mathbf{x})$ looks like a plane wave if examined in a small region compared to L. We can then let $\mathbf{x}_0$ be a fixed point in space wherein $\mathbf{x} = \mathbf{x}_0 + \zeta$, where $|\zeta| \ll L$. Expanding the wave function about $\mathbf{x}_0$ we have

$$\psi(\mathbf{x}_0 + \zeta) \approx \mathbf{A}(\mathbf{x}_0)e^{i\mathbf{S}(\mathbf{x}_0)/\hbar}e^{i\zeta\cdot\nabla\mathbf{S}/\hbar} \tag{62}$$

wherein we have the momentum $\mathbf{p}(\mathbf{x}) = \nabla\mathbf{S}(\mathbf{x})$. Therefore, we can use the WKB approximation for slowly varying potentials.

Meanwhile for the spatial variations of the wave function, the WKB approximation can be used everywhere except for a close region around the immediate vicinity of a classical turning point, where $E \approx V$. In this region $\lambda \to \infty$, and V wouldn't vary slowly as compared to the small variations in other regions. However, it is the boundary conditions at the turning points that determine the allowed energies. We then try to "splice" the WKB solutions at different regions near the turning points together, introducing a "patching" wave function that bridges the gap around the turning point. Solving the Schrodinger's equation using this "splicing" method leads us to the Airy's equations and its solution – the Airy functions – and eventually the connection formula that "joins" the WKB solutions at either side of the turning point. Therefore we can conclude that the WKB solutions are valid for wave functions with slowly varying wavelengths.