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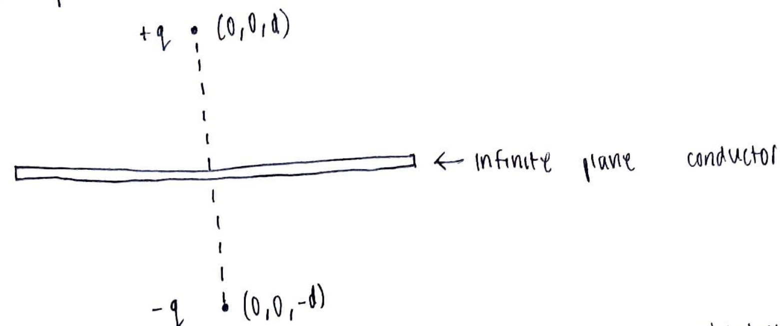
2015 - 01971
Physics 231 PS # 2

1.) a.) Consider the set-up



Therefore, the potential would be the sum of the potentials due to the charges

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

With this we can calculate for the surface charge density given by

$$\sigma = -\epsilon_0 \frac{\partial\Phi}{\partial n}$$

Since our system points to the z - direction, we have

$$\sigma = -\epsilon_0 \left.\frac{\partial\Phi}{\partial z}\right|_{z=0}$$

$$\sigma = -\epsilon_0 \left[\left(\frac{q}{4\pi\epsilon_0}\right)\left(-\frac{2(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{2(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right)\right]_{z=0}$$

$$\sigma = -\epsilon_0\left(\frac{q}{4\pi\epsilon_0}\right)\left(\frac{2d}{(x^2+y^2+d^2)^{3/2}}\right)$$

$$\sigma = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

b.) Using Coulomb's law given by

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

plugging in $q_1 \to q$, $q_2 \to -q$, $r \to 2d$

$$F = -\frac{q^2}{16\pi\epsilon_0 d^2}$$

1.) c.) Integrating $\sigma^2/2\epsilon_0$ over the whole plane, we get the force

$$F = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; \frac{\sigma^2}{2\epsilon_0}$$

Plugging in the surface charge density

$$F = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{2\epsilon_0}\right)\left(\frac{q^2 d^2}{4\pi^2}\right)\left(\frac{1}{(x^2+y^2+d^2)^3}\right) dx\,dy\,dz$$

$$F = \frac{q^2 d^2}{8\pi^2 \epsilon_0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,dz \left(\frac{1}{(x^2+y^2+d^2)^3}\right)$$

this complicated integral can be solved by transforming from cartesian to polar coordinates

$$F = \frac{q^2 d^2}{8\pi^2 \epsilon_0} \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta \int_0^1 dz \; \frac{1}{(r^2+d^2)^3}$$

$$F = \frac{q^2 d^2}{8\pi^2 \epsilon_0} (2\pi)(1) \int_0^{\infty} \frac{r}{(r^2+d^2)^3}\,dr = \frac{q^2 d^2}{4\pi \epsilon_0} \int_0^{\infty} \frac{r\,dr}{(r^2+d^2)^3}$$

Let $u = r^2 \longrightarrow du = 2r\,dr$

$$F = \frac{q^2 d^2}{8\pi \epsilon_0} \int_0^{\infty} \frac{du}{(u+d^2)^3} = \frac{q^2 d^2}{8\pi\epsilon_0}\left[-\frac{1}{2}(u+d^2)^{-2}\right]_0^{\infty}$$

$$F = -\frac{q^2 d^2}{16\pi\epsilon_0}\left(0 - \frac{1}{d^4}\right)$$

$$F = \frac{q^2}{16\pi\epsilon_0 d^2}$$

d.) The work needed to move the charge from d to infinity, we have

$$W = \int_d^{\infty} F(l)\,dl$$

$$W = \int_d^{\infty} \frac{q^2}{16\pi\epsilon_0} \cdot \frac{1}{l^2}\,dl = \frac{q^2}{16\pi\epsilon_0} \int_d^{\infty} \frac{1}{l^2}\,dl$$

$$W = \frac{q^2}{16\pi\epsilon_0}\left[-\frac{1}{l}\right]_d^{\infty}$$

$$W = \frac{q^2}{16\pi\epsilon_0 d}$$

1.) e.) The potential energy between the charge and its image

$$U_E = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$$

for $q_1 \to +q$, $q_2 \to -q$, $r \to 2d$

$$U_E = \frac{1}{4\pi\epsilon_0}\frac{-q^2}{2d}$$

$$U_E = -\frac{q^2}{8\pi\epsilon_0 d}$$

f.) For an electron 1 angstrom from the surface, we have

$$W = \frac{(1e)^2}{16\pi(5.526 \times 10^7\ e/Vm)(10^{-10}\ m)}$$

$$W = 3.6\ eV$$

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2.) a.) Given the force from eq (2.6) of Jackson, we have

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1-\frac{a^2}{y^2}\right)^{-2}$$

When we expand this, we expect some factors to cancel

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a^3}{y^3}\right)\left(\frac{y^4}{(y^2-a^2)^2}\right)$$

$$F = \frac{aq^2}{4\pi\epsilon_0}\frac{y}{(y^2-a^2)^2}$$

Solving for the work it takes to move our charge q from r to infinity

$$W = -\int_\infty^r F\cdot dl \quad \text{or} \quad W = \int_r^\infty F\cdot dl$$

or

$$W = \int_r^\infty \frac{aq^2}{4\pi\epsilon_0}\frac{y}{(y^2-a^2)^2}\,dy = \frac{aq^2}{4\pi\epsilon_0}\int_r^\infty \frac{y\,dy}{(y^2-a^2)^2}$$

Let $u = y^2-a^2$, $du = 2y\,dy$

$$W = \left(\frac{aq^2}{4\pi\epsilon_0}\right)\left(\frac{1}{2}\right)\int_{r^2-a^2}^\infty \frac{du}{u^2}$$

$$W = \left(\frac{aq^2}{8\pi\epsilon_0}\right)\left[-\frac{1}{u}\right]_{r^2-a^2}^\infty = \left(\frac{aq^2}{8\pi\epsilon_0}\right)\left[0+\frac{1}{r^2-a^2}\right]$$

$$W = \frac{aq^2}{8\pi\epsilon_0(r^2-a^2)}$$

Taking note that the work can also be expressed as a product of the charge and potential difference, we have

$$W = q\left(\Phi(r>a) - \Phi(r=\infty)\right)$$

We assume that the potential at infinity is zero so we have

$$W = \frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{|\vec{x}-\vec{y}|} - \frac{1}{\left|\frac{y}{a}\vec{x}-\frac{a}{y}\vec{y}\right|}\right]$$

We consider for this case that $\vec{x} = \vec{y} = r$, therefore

$$W = -\frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{\frac{r^2}{a}-a}\right)$$

or

$$W = -\frac{q^2}{4\pi\epsilon_0}\left(\frac{a}{r^2-a^2}\right)$$

The negative sign difference comes since we move the charge to infinty against a force that moves from infinity to r. The 1/2 factor difference comes since we considered both the original charge and its image charge.

2.) b.) Given the force from eq (2.9) of Jackson

$$F = \frac{-1}{4\pi\epsilon_0}\frac{q}{y^2}\left[Q - \frac{qa^3(2y^2-a^2)}{y(y^2-a^2)^2}\right] = \frac{-q}{4\pi\epsilon_0}\left[\frac{Q}{y^2} - \frac{qa^3(2y^2-a^2)}{y^3(y^2-a^2)^2}\right]$$

Solving for the work needed to displace the charge Q into infinity, we have

$$W = -\int_\infty^r F\cdot dl = \int_r^\infty F\cdot dl$$

$$W = -\frac{q}{4\pi\epsilon_0}\int_r^\infty\left[\frac{Q}{y^2} - \frac{qa^3(2y^2-a^2)}{y^3(y^2-a^2)^2}\right]dy$$

$$W = -\frac{q}{4\pi\epsilon_0}\left[\underbrace{Q\int_r^\infty \frac{1}{y^2}dy}_{①} - \underbrace{qa^3\int_r^\infty \frac{(2y^2-a^2)}{y^3(y^2-a^2)^2}dy}_{②}\right]$$

Solving for ①

$$Q\int_r^\infty \frac{1}{y^2}dy = Q\left[-\frac{1}{y}\right]_r^\infty = \frac{Q}{r}$$

Solving for ②, Let $u = y^2-a^2$

$du = 2y\,dy \longrightarrow y^2 = u+a^2$

therefore,

$$qa^3\int_r^\infty \frac{2y^2-a^2}{y^3(y^2-a^2)^2}dy \longrightarrow \frac{qa^3}{2}\int_{r^2-a^2}^\infty \frac{(2u+a^2)\,du}{(u+a^2)u^2}$$

We can write this in partial fractions

$$\frac{qa^3}{2}\int_{r^2-a^2}^\infty\left[\frac{A}{u+a^2} + \frac{B}{(u+a^2)^2} + \frac{C}{u} + \frac{D}{u^2}\right]du$$

expanding the partial fractions and comparing w/ our integrand

$$Au^2(u+a^2) + Bu^2 + Cu(u+a^2) + D(u+a^2)^2 = 2u+a^2$$

$$Au^3 + Aa^2u^2 + Bu^2 + Cu^2 + Cua^2 + Du^2 + 2Dua^2 + Da^4 = 2u+a^2$$

We get the ff equations

$$\left.\begin{aligned} &Au^3 = 0 \\ &Aa^2u^2 + Bu^2 + Cu^2 + Du^2 = 0 \\ &(Ca^2+2Da^2)u = 2u \\ &Da^4 = a^2 \end{aligned}\right\} \quad \begin{aligned} &A = 0 \\ &B = -1/a^2 \\ &C = 0 \\ &D = 1/a^2 \end{aligned}$$

Going back to our integral, we have

$$\frac{qa^3}{2}\int_{r^2-a^2}^\infty\left(-\frac{1}{a^2}\frac{1}{(u+a^2)^2} + \frac{1}{a^2u^2}\right)du$$

Or

$$\frac{qa}{2}\int_{r^2-a^2}^\infty\left(-\frac{1}{(u+a^2)^2} + \frac{1}{u^2}\right)du$$

2.) b.) which is now easily integrable

$$\frac{qa}{2}\left[\frac{1}{u+a^2} - \frac{1}{u}\right]_{r^2-a^2}^{\infty}$$

evaluating we have

$$\frac{qa}{2}\left[\frac{1}{r^2-a^2} - \frac{1}{r^2}\right]$$

plugging ① and ② back to our equation / integral, we have the work to be

$$W = \frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{2(r^2-a^2)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

Similar to 2a, we can write work as charge multiplied by the potential. From the potential at eq (2.8) we have

$$\Phi(x) = -\frac{1}{4\pi\epsilon_0}\left[\frac{q a}{r^2-a^2} - \frac{q a}{r^2} - \frac{Q}{r}\right]$$

therefore

$$W = -\frac{1}{4\pi\epsilon_0}\left[\frac{q^2 a}{(r^2-a^2)} - \frac{q^2 a}{r^2} - \frac{qQ}{r}\right]$$

Similar to 2a, we get an overall (-) charge that comes as we move against the force. The extra 1/2 factor on the first 2 terms correspond to the contribution due to the original charge as we have an image accompanying our original charge.

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3.) a.) Using the method of images, we can introduce charge q at $(x', y', +z')$ and q' at $(x', y', -z')$. The potential is then

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{4\pi\epsilon_0}\frac{q'}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

at $z=0$, $\Phi(\vec{x})=0$

$$0 = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{(x-x')^2+(y-y')^2+z^2}} + \frac{1}{4\pi\epsilon_0}\frac{q'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}$$

therefore, $q' = -q$. Therefore, we can write the potential as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

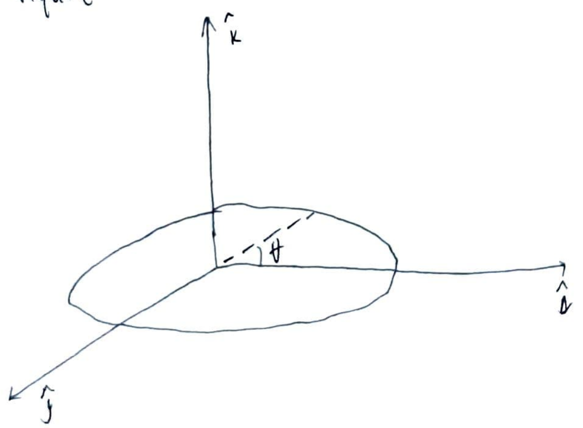
Since the Green's function can be written as

$$G(\vec{x},\vec{x}') = \underbrace{\frac{1}{|\vec{x}-\vec{x}'|}}_{\text{original charge}} + \underbrace{F(\vec{x},\vec{x}')}_{\text{image charge}}$$

It makes sense that the Green's function is

$$G(\vec{x},\vec{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

b.) Consider the figure



we get

$$\vec{\rho} = \rho\cos\theta\,\hat{i} + \rho\sin\theta\,\hat{j} + z\hat{k}$$
$$\vec{\rho}' = \rho'\cos\theta'\,\hat{i} + \rho'\sin\theta'\,\hat{j} + z'\hat{k}$$

therefore

$$\vec{\rho}-\vec{\rho}' = (\rho\cos\theta - \rho'\cos\theta')\hat{i} + (\rho\sin\theta - \rho'\sin\theta')\hat{j} + (z-z')\hat{k}$$

$$|\vec{\rho}-\vec{\rho}'| = (\rho^2\cos^2\theta + \rho'^2\cos^2\theta' - 2\rho\rho'\cos\theta\cos\theta' + \rho^2\sin^2\theta + \rho'^2\sin^2\theta'$$
$$- 2\rho\rho'\sin\theta\sin\theta' + z^2 + z'^2 - 2zz')^{1/2}$$
$$= (\rho^2+\rho'^2 - 2\rho\rho'\cos(\theta-\theta') + (z-z')^2)^{1/2}$$

3.) b.) We remember that the potential is given by

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{\partial\Phi(\vec{x}')}{\partial n'}G(\vec{x},\vec{x}') - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right]da'$$

for dirichlet boundary $\Phi(\vec{x})$ is defined but $G(\vec{x},\vec{x}') = 0$

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}da'$$

we can then plug-in the Green's function

$$G(\vec{x},\vec{x}') = \frac{1}{|\rho-\rho'|} + F(\vec{x},\vec{x}')$$

$$= \frac{1}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2)^{1/2}}$$

$$+ \frac{1}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z+z')^2)^{1/2}}$$

Into the potential and evaluate the normal derivative at $z=0$

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \Phi(\vec{x}')\left(-\frac{\partial}{\partial z'}G(\vec{x},\vec{x}')\Big|_{z=0}\right)da'$$

$$\Phi(\vec{x}) = \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{2z}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{3/2}}da'$$

Let's have the potential inside the surface to be $\Phi = V$. Since we evaluated in cylindrical coordinates, we have

$$\Phi(\vec{x}) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^a \frac{2Vz}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{3/2}}\rho'd\rho'd\phi'$$

or

$$\Phi(\vec{x}) = \frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a \frac{\rho'd\rho'd\phi'}{(\rho^2+\rho'^2-2\rho\rho'\cos\phi'+z^2)^{3/2}}$$

since we have azimuthal symmetry $\phi' = \phi + \phi'$

3.) c.) continuing from 3b, along the axis of the circle ($\rho = 0$), we have

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{(\rho'^2 + z^2)^{3/2}}$$

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} \int_0^{2\pi} d\phi' \int_0^a \frac{\rho' d\rho'}{(\rho'^2 + z^2)^{3/2}}$$

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} (2\pi) \int_0^a \frac{\rho' d\rho'}{(\rho'^2 + z^2)^{3/2}}$$

We then evaluate the remaining integral. Let $u = \rho'^2 + z^2$

$du = 2\rho' d\rho'$

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} (2\pi) \frac{1}{2} \int_{z^2}^{a^2+z^2} \frac{du}{u^{3/2}}$$

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} (2\pi) \frac{1}{2} \left[-\frac{2}{u^{1/2}} \right]_{z^2}^{a^2+z^2}$$

$$\Phi(\vec{x}) = Vz \left[\frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}} \right]$$

$$\Phi(\vec{x}) = V - \frac{Vz}{\sqrt{a^2+z^2}}$$

d.) From the potential

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho' d\rho' d\phi'}{(\rho'^2 + \rho^2 - 2\rho'\rho \cos\phi' + z^2)^{3/2}}$$

we can rewrite this by dividing both numerator and denominator by $(\rho^2+z^2)^{3/2}$ essentially multiplying by 1.

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{1}{(\rho^2+z^2)} \frac{\rho' d\rho' d\phi'}{\left(1 + \frac{\rho'^2 - 2\rho'\rho\cos\phi'}{\rho^2+z^2}\right)^{3/2}}$$

using binomial expansion, we have

$$(1+x)^n = \binom{n}{0} 1^n x^0 + \binom{n}{1} 1^{n-1} x^1 + \binom{n}{2} 1^{n-2} x^2 + \ldots$$

$$= 1 + nx + \frac{n(n-1)}{2} x^2 + O(3)$$

applying this to $\left(1 + \frac{\rho'^2 - 2\rho'\rho\cos\phi'}{\rho^2+z^2}\right)^{-3/2}$, we have

$$\Phi(\vec{x}) = \frac{Vz}{2\pi} \frac{1}{(\rho^2+z^2)^{3/2}} \int_0^a \int_0^{2\pi} \left[1 - \frac{3}{2}\left(\frac{\rho'^2 - 2\rho'\rho\cos\phi'}{\rho^2+z^2}\right) + \frac{15}{8}\left(\frac{\rho'^2 - 2\rho'\rho\cos\phi'}{\rho^2+z^2}\right) + \ldots \right] \rho' d\rho' d\phi'$$

3d.) Solving for :

(1) $\int_0^a \int_0^{2\pi} \rho' d\rho' d\phi' = \int_0^{2\pi} d\phi' \int_0^a \rho' d\rho' = (2\pi - 0)\left(\frac{1}{2}a^2 - 0\right) = \underline{a^2\pi},$

(2) $-\frac{3}{2}\int_0^a\int_0^{2\pi}\left(\frac{1}{\rho^2+z^2}\right)\left(\rho'^2 - 2\rho\rho'\cos\phi\right)\rho' d\rho' d\phi$

$$= \frac{-3}{2(\rho^2+z^2)}\int_0^{2\pi}\int_0^a\left(\rho'^2-2\rho\rho'\cos\phi\right)\rho' d\rho' d\phi' = -\frac{3}{2(\rho^2+z^2)}\int_0^{2\pi}\left[\frac14\rho'^4 - \frac23\rho\rho'^3\cos\phi'\right]_0^a d\phi'$$

$$= \frac{-3}{2(\rho^2+z^2)}\int_0^{2\pi}\left(\frac14 a^4 - \frac23\rho a^3\cos\phi'\right)d\phi' = \frac{-3}{2(\rho^2+z^2)}\left[\frac14 a^4\phi' - \frac23\rho a^3\sin\phi'\right]_0^{2\pi}$$

$$= \frac{-3}{2(\rho^2+z^2)}\left[\frac24 a^4\pi\right] = \underline{\frac{-3a^4\pi}{4(\rho^2+z^2)}}$$

(3) $\frac{15}{8}\frac{1}{(\rho^2+z^2)^2}\int_0^a\int_0^{2\pi}\left(\rho'^4+4\rho'^2\rho^2\cos^2\phi' - 4\rho'^3\rho\cos\phi'\right)d\phi'\rho' d\rho$

$$= \frac{15}{8}\frac{1}{(\rho^2+z^2)^2}\int_0^a\left[\rho'^4\phi - 4\rho'^3\rho\sin\phi' + 2\rho'^2\rho^2\phi + \rho'^2\rho^2\sin 2\phi\right]_0^{2\pi}\rho' d\rho'$$

making the substitution $\cos 2\phi = 2\cos^2\phi - 1 \longrightarrow \cos^2\phi = \frac{\cos 2\phi + 1}{2}$.

$$= \frac{15}{8}\frac{1}{(\rho^2+z^2)^2}\int_0^a\left[2\pi\rho'^4 + 4\rho'^2\rho^2\pi\right]\rho' d\rho' = \frac{15}{8}\frac{1}{(\rho^2+z^2)^2}\int_0^a\left(2\pi\rho'^5 + 4\rho'^3\rho^2\pi\right)d\rho'$$

$$= \frac{15}{8}\frac{1}{(\rho^2+z^2)}\left[\frac{\pi}{3}\rho'^6 + \rho'^4\rho^2\pi\right]_0^a = \underline{\frac{5}{8}\frac{1}{(\rho^2+z^2)^2}\left(a^6\pi + 3a^4\rho^2\pi\right)}$$

Therefore, the potential is

$$\Phi(\vec{x}) = \frac{Vza^2}{2(\rho^2+z^2)^{3/2}}\left[1 - \frac34\frac{a^2}{\rho^2+z^2} + \frac58\frac{(a^4+3a^2\rho^2)}{(\rho^2+z^2)^2} + \dots\right]$$

along the axis $\rho = 0$

$$\Phi(\vec{x}) = \frac{Vza^2}{2z^3}\left[1 - \frac34\frac{a^2}{z^2} + \frac58\frac{a^4}{z^4} + \dots\right]$$

$$\Phi(\vec{x}) = \frac{Va^2}{2z^2}\left[1 - \frac34\frac{a^2}{z^2} + \frac58\frac{a^4}{z^4} + \dots\right] = V\left[\frac{a^2}{2z^2} - \frac38\frac{a^4}{z^4} + \frac{5}{16}\frac{a^6}{z^6}\right]$$

$$\Phi(\vec{x}) = V\left[1 - \left(1 - \frac{a^2}{2z^2} + \frac38\frac{a^4}{z^4} - \frac{5}{16}\frac{a^6}{z^6} + \dots\right)\right]$$

3.) d.) Taking note that $(x+1)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \dots$, we have

$$\Phi(\vec{x}) = V\left[1 - \left(1 + \frac{a^2}{z^2}\right)^{-1/2}\right] = V\left[1 - \left(\frac{z^2+a^2}{z^2}\right)^{-1/2}\right]$$

$$\Phi(\vec{x}) = V\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]$$

$$\Phi(\vec{x}) = V - \frac{Vz}{\sqrt{a^2+z^2}}$$

which matches w/ 3c.

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1.) a.) we recall the definition of the Green's function

$$(\partial_{x'}^2 + \partial_{y'}^2)\, G(\vec{x}, \vec{x}') = -4\pi\, \delta(\vec{x} - \vec{x}')\, \delta(\vec{y} - \vec{y}')$$

we can also expand this as a Fourier sine series to follow the boundary conditions at $x' = 0$ and $x' = 1$.

$$G(x, y; x', y') = \sum_{n=1}^{\infty} f_n(x, y; y') \sin(n\pi x')$$

Substituting this to the equation above

$$\sum_{n=1}^{\infty} (n\pi^2)(-\sin(n\pi x'))\, f_n(x, y; y') + \frac{\partial^2 f_n(x, y; y')}{\partial y^2} \sin(n\pi x') = -4\pi\, \delta(\vec{x}' - \vec{x})\, \delta(\vec{y}' - \vec{y})$$

or simply

$$\sum_{n=1}^{\infty} (\partial_{y'}^2 - (n\pi)^2)\, f_n(x, y; y') \sin(n\pi x') = -4\pi\, \delta(\vec{x}' - \vec{x})\, \delta(\vec{y}' - \vec{y})$$

using the completeness relation for the sine series

$$\sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi x') = \frac{1}{2} \delta(\vec{x}' - \vec{x})$$

we have

$$\sum_{n=1}^{\infty} (\partial_{y'}^2 - (n\pi)^2)\, f_n(x, y; y') \sin(n\pi x') = -8\pi \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi x')\, \delta(\vec{y}' - \vec{y})$$

comparing factors we can write

$$f_n(x, y; y') = 2\, g_n(y, y') \sin(n\pi x)$$

where we put a 2 factor for convenience. Therefore,

$$G(x, y; x', y') = 2 \sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x')$$

and plugging in to our above eqn

$$2 \sum_{n=1}^{\infty} (\partial_{y'}^2 - (n\pi)^2)\, g_n(y, y') \sin(n\pi x') \sin(n\pi x)$$

$$= -8\pi\, \delta(\vec{y} - \vec{y}') \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi x')$$

therefore, we see that

$$(\partial_{y'}^2 - (n\pi)^2)\, g_n(y, y') = -4\pi\, \delta(\vec{y}' - \vec{y})$$

4.) a.) Since G vanishes at $y' = 0$ and $y' = 1$, we have

$$2 \sum_{n=1}^{\infty} g_n(y,1) \sin(n\pi x) \sin(n\pi x') = 0$$

and

$$2 \sum_{n=1}^{\infty} g_n(y,0) \sin(n\pi x) \sin(n\pi x') = 0$$

we force that

$$g_n(y,1) = 0 \quad \text{and} \quad g_n(y,0) = 0$$

to follow the boundary conditions.

b.) Solving for

$$(\partial^2_{y'} - n^2\pi^2)\, g_n(y,y') = 0$$

Or

$$\frac{\partial^2}{\partial y'^2} g_n(y,y') + (n^2\pi^2)\, g_n(y,y') = 0$$

We have

$$g_n(y,y') = \begin{cases} g_1 = A \sinh(n\pi y') + B \cosh(n\pi y') & , y' < y \\ g_2 = C \sinh(n\pi y') + D \cosh(n\pi y') & , y' > y \end{cases}$$

imposing boundary conditions on the 2 regions, we have

$$A \sinh(\cancel{n\pi(0)}^{0}) + B = 0 \longrightarrow B = 0$$

and

$$C \sinh(n\pi) + D \cos(n\pi) = 0 \longrightarrow D = -C \tanh(n\pi)$$

to solve for A and C we impose the continuity and jump conditions

$$g_1 = g_2 \quad ; \quad \partial_{y'} g_1 = \partial_{y'} g_2 - 4\pi \quad \text{at} \quad y = y'$$

We then have

$$A \sinh(n\pi y') - C \sinh(n\pi y') - C \tanh(n\pi) \cosh(n\pi y') = 0$$

and

$$n\pi A \cosh(n\pi y') - n\pi C \cosh(n\pi y') + n\pi C \tanh(n\pi) \sin(n\pi y') = 4\pi$$

or

$$A \cosh(n\pi y') - C\cosh(n\pi y') + C \tanh(n\pi) \sin(n\pi y') = 4/n$$

solving for A and C, we have

$$A = -\frac{4}{n \sinh(n\pi)} \left(\cosh(n\pi) \sinh(n\pi y) - \sinh(n\pi) \cosh(n\pi y)\right)$$

$$C = -\frac{4}{n \sinh(n\pi)} \cosh(n\pi) \sinh(n\pi y)$$

1.) b.) we then have

$$g_1(y,y') = \frac{4}{n\sinh(n\pi)} \sinh(n\pi y')\left[\sinh(n\pi)\cosh(n\pi y) - \cosh(n\pi)\sinh(n\pi y)\right]$$

$$g_2(y,y') = \frac{4}{n\sinh(n\pi)} \sinh(n\pi y)\left[\sinh(n\pi)\cosh(n\pi y') - \cosh(n\pi)\sinh(n\pi y')\right]$$

using the sum identity

$$\sinh(n\pi - n\pi y) = \sinh(n\pi)\cosh(n\pi y) - \cosh(n\pi)\sinh(n\pi y)$$

and the definitions

$$y_< = \min(y,y') \quad \text{and} \quad y_> = \max(y,y')$$

we have

$$g_n(y,y') = \frac{4}{n\sinh(n\pi)} \sinh(n\pi y_<)\sinh(n\pi(1-y_>))$$

we have the Green's function as

$$G(x,y;x',y') = \sum_n \frac{8}{n\sinh(n\pi)} \sin(n\pi x)\sin(n\pi x')\sinh(n\pi y_<)\sinh(n\pi(1-y_>))$$