Physics 241
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Final Report

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1 Week 1: Linear vector spaces and operators. Wavefunctions and state vectors

1.a Obtain the spectral decomposition of a state vector or wave function with respect to an arbitrary basis set: discretely and continuously labeled elements.

For discretely labeled elements we have the spectral decomposition of a wave function to be

$$\psi(r) = \sum_i c_i u_i(r). \tag{1.a.1}$$

Meanwhile, for continuous labeled elements, we have

$$\psi(r) = \int \bar{\psi}(p) v_p(r) d^3p \tag{1.a.2}$$

where the $v_p(r)$ basis depends on the continuous indices p_x, p_y, and p_z which we write as p.

1.b Use inner products to perform basis transformations

We first introduce a transformation matrix $S_{ik} = \langle u_i | t_k \rangle$ with Hermitian conjugate $(S^\dagger)_{ki} = \langle t_k | u_i \rangle$. Now we look at the transformation of ket to $\langle u_i | \psi \rangle$ from $\langle t_k | \psi \rangle$

$$\begin{aligned} \langle u_i | \psi \rangle &= \langle u_i | \mathbf{I} | \psi \rangle \\ &= \sum_i \langle u_i | t_k \rangle \langle t_k | \psi \rangle \\ &= \sum_i S_{ik} \langle t_k | \psi \rangle \end{aligned} \tag{1.b.1}$$

The same goes for a bra

$$\begin{aligned} \langle \psi | t_k \rangle &= \langle \psi | \mathbf{I} | t_k \rangle \\ &= \sum_i \langle \psi | u_i \rangle \langle u_i | t_k \rangle \\ &= \sum_i \langle \psi | u_i \rangle S_{ik} \end{aligned} \tag{1.b.2}$$

1.c Explain how commuting observables can be used to generate additional quantum numbers to label degenerate states.

Consider an observable A and a basis of $\mathscr{E}$ of eigenvectors $|u_n^i\rangle$ of A. If at least one of the eigenvalues of A are degenerate, we need to introduce another observable B which commutes with A, then we construct an orthonormal basis of eigenvectors common to both observables. If the basis is unique, observables A and B form a complete set of commuting observables. If there still exist several independent vectors that are eigenvectors of A and B for at least one of the pairs a_n, b_n, we add a third observable C that commute with both A and B. We keep adding observables until we get a complete set of commuting observables.

1.d Establish the link between quantum states and elements of a Hilbert space; observables and Hermitian operators.

In Hilbert space, mathematical operators correspond to observable quantities. These operators when acting on a wave function yields another wave function in the same space.
Meanwhile, when the operator A is Hermitian, the eigenvalues are real. Since eigenvalues correspond to measurables, it makes sense that the hermitian operators are observables.

2 Week 2: Postulates of quantum mechanics. Position of quantum mechanics differently.

2.a Review the core postulates of quantum mechanics.

First Postulate: At a fixed time t_0, the state of a physical system is defined by specifying a ket $|\psi(t_0)\rangle$ belonging to the state space $\mathscr{E}$.
Second postulate: Every measurable physical quantity $\mathscr{A}$ is described by an operator A acting on $\mathscr{E}$: this operator is an observable.
Third postulate: The only possible result of the measurement of a physical quantity $\mathscr{A}$ is one of the eigenvalues of the corresponding observable A.
Fourth postulate (case of a discrete non-degenerate spectrum): When the physical quantity $\mathscr{A}$ is measured on a system in the normalized state $|\psi\rangle$, the probability $\mathscr{P}(a_n)$ of obtaining the non-degenerate eigenvalue a_n of the corresponding observable A is:

$$\mathscr{P}(a_n) = |\langle u_n|\psi\rangle|^2 \tag{2.a.1}$$

where $|u_n\rangle$ is the normalized eigenvector of A associated with the eignvalue.
Fourth postulate (case of discrete spectrum): When the physical quantity A is emasured on a system in the normalized state $|\psi\rangle$, the probability $\mathscr{P}(a_n)$ is obtaining the eigenvalue a_n of the corresponding observable A is:

$$\mathscr{P}(a_n) = \sum_{i=1}^{g_n} |\langle u_n^i|\psi\rangle|^2 \tag{2.a.2}$$

where g_n is the degree of degeneracy of a_n and $|u_n^i\rangle (i = 1, 2, ..., g_n)$ is an orthonormal set of

vectors which forms a basis in the in the eigensubspace $\mathscr{E}$ associated with the eigenvalue a_n of A.

Fourth Postulate (case of a continuous non-degenerate spectrum): When the physical quantity A is measured on a system in the normalized state $|\psi\rangle$, the probability $\mathrm{d}\mathscr{P}(\alpha)$ of obtained a result included between α and $\alpha + d\alpha$ is equal to:

$$d\mathscr{P}(\alpha) = |\langle v_\alpha|\psi\rangle|^2 d\alpha \tag{2.a.3}$$

where $|v_\alpha\rangle$ si the eigenvector corresponding to the eigenvalue α of the observable A associated with $\mathscr{A}$.

2.b Use an infinitesimal translation operator and its postulated properties to identify momentum as a generator of translations.

We consider an infinitesimal translation operator

$$\mathscr{T}(dx') = 1 - i\mathbf{K} \cdot dx'$$

At first look the $\mathbf{K}$ operator cannot be fathomed to identify with the momentum operator as the dimensions would be all wrong. However, we can set the $\mathbf{K}$ operator to be

$$\mathbf{K} = \frac{\mathbf{p}}{\text{universal constant with the dimension of action}} \tag{2.b.1}$$

to address that problem. Fortunately, this universal constant turns out to be the same as the constant $\hbar$ that appears in the de Broglie's relation. With this we can write the infinitesimal translation operator to be

$$\mathscr{T}(dx') = 1 - i\mathbf{p} \cdot dx'$$

2.c Perform a change of basis to relate the coordinate space wavefunction $\langle \mathbf{x}|\psi\rangle$ and the momentum space wave functions $\langle \mathbf{p}|\psi\rangle$.

We can rewrite the position-space wave function and the momentum-space wave function

$$\langle x'|\alpha\rangle = \int dp' \langle x'|p'\rangle\langle p'|\alpha\rangle \tag{2.c.1}$$

and

$$\langle p'|\alpha\rangle = \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle \tag{2.c.2}$$

that we expanded by a change of basis as

$$\psi_\alpha(x') = \frac{1}{\sqrt{2\pi\hbar}} \int dp' exp\left(\frac{ip'x'}{\hbar}\right) \phi_\alpha(p') \tag{2.c.3}$$

and

$$\phi_\alpha(p') = \frac{1}{\sqrt{2\pi\hbar}} \int dx' exp\left(\frac{-ip'x'}{\hbar}\right) \psi_\alpha(x') \tag{2.c.4}$$

.

This came from the form of

$$\langle x'|x''\rangle = \int dp' \langle x'|p'\rangle\langle p'|x''\rangle \tag{2.c.5}$$

3 Week 3: Time evolution. Heisenberg picture. Coherent states.

3.a Determine the time development of states in the Schrodinger picture.

The Schrodinger picture refers to the formulation wherein the state vectors evolve in time while operators remain constant. To determine the time development of states, we introduce a time-evolution operator $\mathrm{U}(t, t_0)$ such that

$$|\alpha, t_0; t\rangle = \mathscr{U}(t, t_0)|\alpha, t_0\rangle \tag{3.a.1}$$

where the time-evolution has the following properties:
(1) Unitary property

$$\mathscr{U}^\dagger(t, t_0)\mathscr{U}(t, t_0) = 1, \tag{3.a.2}$$

(2) Identity property

$$\lim_{dt\to 0} \mathscr{U}(t_0 + dt, t_0) = 1 \tag{3.a.3}$$

and,
(3) Composition property

$$\mathscr{U}(t_2, t_0) = \mathscr{U}(t_2, t_1)\mathscr{U}(t_1, t_0) \quad (t_2 > t_1 > t_0). \tag{3.a.4}$$

The form of the time-evolution operator for infinitesimal intervals is given by

$$\mathscr{U}(t_0 + dt, t_0) = 1 - \frac{iHdt}{\hbar} \tag{3.a.5}$$

Meanwhile, by using the composition property, we can write the time-evolution operator in the form of a differential equation

$$i\hbar\frac{\partial}{\partial t}\mathscr{U}(t, t_0) = H\mathscr{U}(t, t_0). \tag{3.a.6}$$

Since in the Schrodinger picture the operators are constant in time, we can easily solve this differential equation. This yields to the following form of the time-evolution operator

$$\mathscr{U} = exp\left[\frac{-iH(t - t_0)}{\hbar}\right] \tag{3.a.7}$$

3.b Determine the time development of operators in the Heisenberg picture.

For the Heisenberg picture, we have the state kets to be constant in time while the operators and observables evolve in time. We can introduce the Heisenberg picture observable $A^{(H)}$

$$A^{(H)}(t) = \mathscr{U}^\dagger(t)A^S\mathscr{U}(t) \tag{3.b.1}$$

where the superscripts H and S refers to the Heisenberg and Schrodinger picture observables. At $t = 0$, the Heisenberg and Schrodinger picture coincides

$$A^{(H)}(0) = A^{(S)} \tag{3.b.2}$$

If $A^{(S)}$ is constant, we have the equations of motion

$$\begin{aligned} \frac{dA^{(H)}}{dt} &= \frac{\partial \mathscr{U}^\dagger}{\partial t} A^{(S)} \mathscr{U} + \mathscr{U}^\dagger A^{(S)} \frac{\partial \mathscr{U}}{\partial t} \\ &= \frac{1}{i\hbar}[A^{(H)}, \mathscr{U}^\dagger H \mathscr{U}] \end{aligned} \tag{3.b.3}$$

which simplifies into the Heisenberg equations of motion

$$\frac{dA^H}{dt} = \frac{1}{i\hbar}[A^{(H)}, H] \tag{3.b.4}$$

3.c Construct the coherent states. Calculate the expectation values of position and momentum in these states.

When expressed as a superposition of energy (or N) eigenstates, we can construct the state $|\lambda\rangle$ as

$$|\lambda\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle \tag{3.c.1}$$

where the distribution of $|f(n)|^2$ with respect to n of the poisson type about some mean value $\bar{n}$:

$$|f(n)|^2 = \left(\frac{\bar{n}^n}{n!} e^{-\bar{n}}\right) \tag{3.c.2}$$

We are given the annihilation operator and the creation operator

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} p\right) \quad \text{and} \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega} p\right). \tag{3.c.3}$$

From these values we can get x and p

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \quad \text{and} \quad pi\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a) \tag{3.c.4}$$

therefore we can solve for the expectation value for $x(t)$

$$\begin{aligned} \langle x(t)\rangle &= \sqrt{\frac{\hbar}{2m\omega}}[\langle\alpha(t)|a^\dagger|\alpha(t)\rangle + \langle\alpha(t)|a|\alpha(t)\rangle] \\ &= \sqrt{\frac{\hbar}{2m\omega}}[\alpha^* e^{i\omega t} + \alpha e^{-i\omega t}]\langle\alpha(t)|\alpha(t)\rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}}[\alpha^* e^{i\omega t} + \alpha e^{-i\omega t}] \end{aligned} \tag{3.c.5}$$

We can clean this up and rewrite

$$\langle x(t)\rangle = \sqrt{\frac{2\hbar}{m\omega}}|\alpha|\cos(\omega t - \theta) \tag{3.c.6}$$

the expectation value for $p(t)$

$$\begin{aligned}\langle p(t)\rangle &= i\sqrt{\frac{m\omega\hbar}{2}}[\langle\alpha(t)|a^\dagger|\alpha(t)\rangle - \langle\alpha(t)|a|\alpha(t)\rangle] \\ &= i\sqrt{\frac{m\omega\hbar}{2}}[\alpha^* e^{i\omega t} - \alpha e^{-i\omega t}]\langle\alpha(t)|\alpha(t)\rangle \\ &= -\sqrt{2m\omega\hbar}|\alpha|\sin(\omega t - \theta)\end{aligned} \tag{3.c.7}$$

4 Week 4: Schrodinger's wave equation, Hamilton-Jacobi formulation. WKB approximation

4.a Identify the classical analogs of the amplitude and phase of a wavefunction in a Hamilton-Jacobi formulation

We can write the wave function as

$$\psi(x,t) = \sqrt{\rho(x,t)}exp\left[\frac{iS(x,t)}{\hbar}\right] \tag{4.a.1}$$

From this we see that the amplitude in classical mechanics is just the square root of the probability density in wave mechanics. Substituting this into the time-dependent wave equation yields

$$\begin{aligned}&-\left(\frac{\hbar^2}{2m}\right)\times\left[\nabla^2\sqrt{\rho} + \frac{2i}{\hbar}(\nabla\sqrt{\rho})(\nabla S) - \frac{1}{\hbar^2}\sqrt{\rho}|\nabla S|^2 + \frac{i}{\hbar}\sqrt{\rho}\nabla^2 S\right] + \rho V \\ &= i\hbar\left[\frac{\partial\sqrt{\rho}}{\partial t} + \frac{i}{\hbar}\sqrt{\rho}\frac{\partial S}{\partial t}\right]\end{aligned} \tag{4.a.2}$$

In the classical limit $\hbar \to 0$, we can simplify this complicated equation into

$$\frac{1}{2m}|\nabla S(x,t)|^2 + V(x) + \frac{\partial S(x,t)}{\partial t} = 0 \tag{4.a.3}$$

which is just the Hamilton-Jacobi equation. Therefore we conclude that $S(x,t)$ refers to the Hamilton's principal function.

4.b Set up the WKB connection formulas to be able to approximate the energy eigenvalues in slowly-varying potentials.

We start with the WKB solution

$$\psi(x,t) = \frac{constant}{[E - V(x)]^{1/4}} \times exp\left[\pm\frac{i}{\hbar}\int dx'\sqrt{2m(E - V(x'))} - \frac{iEt}{\hbar}\right] \tag{4.b.1}$$

which is valid in the classically allowed region where $E - V(x) > 0$, for the $E - V(x) < 0$, we

modify this solution

$$\psi(x,t) = \frac{constant}{[E-V(x)]^{1/4}} exp\left[\pm\frac{1}{\hbar}\int dx'\sqrt{2m(E-V(x'))} - \frac{iEt}{\hbar}\right]. \tag{4.b.2}$$

For a potential well with turning points at x_1 and x_2. For this example we expect the wave function solution must be in the form of Eqn. (4.b.2) in the region where $x < x_1$ (region I) and $x > x_2$ (region III), and be in the form of Eqn. (4.b.1) in the region $x_1 < x < x_2$ (region II). For region I to II, we have the wave function to be

$$\psi(x,t) = \frac{2}{[E-V(x)]^{1/4}} \cos\left[\frac{1}{\hbar}\int_{x_1}^{x} dx'\sqrt{2m[E-V(x')] - \frac{\pi}{4}}\right] \tag{4.b.3}$$

Meanwhile, for region III to region II

$$\psi(x,t) = \frac{2}{[E-V(x)]^{1/4}} \cos\left[-\frac{1}{\hbar}\int_{x}^{x_2} dx'\sqrt{2m[E-V(x')] + \frac{\pi}{4}}\right] \tag{4.b.4}$$

These two wave equations overlapping in region II implies that the cosine in our solutions must differ by an integer multiple of π. We then obtain a consistency condition that we can use to solve for our energy eigenvalues

$$\int_{x_1}^{x_2} dx\sqrt{2m(E-V(x))} = \left(n+\frac{1}{2}\right)\pi\hbar \quad n = 0,1,2,3,... \tag{4.b.5}$$

We can then solve the integral then manipulate the resulting equation to yield an equation that corresponds to the energy eigenvalues. In this case, our energy eigenvalues would depend on the form of the slowly varying potential, V.

5 Week 5: Probset 1 deadline

6 Week 6: Feynman path integral

6.a Recognize the propagator as a Green's function and integral kernel

We start with the time-evolution of an initial ket

$$|\alpha, t_0; t\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha, t_0\rangle exp\left[\frac{-iE_{a'}(t-t_0)}{\hbar}\right] \tag{6.a.1}$$

multiplying both sides with $\langle x|$.we have

$$\langle x|\alpha, t_0; t\rangle = \sum_{a'} \langle x|a'\rangle\langle a'|\alpha, t_0\rangle exp\left[\frac{-iE_{a'}(t-t_0)}{\hbar}\right] \tag{6.a.2}$$

Taking note that

$$\langle a'|\alpha, t_0\rangle = \int d^3x'\langle a'|x'\rangle\langle x'|\alpha, t_0\rangle \tag{6.a.3}$$

Therefore, to yield the final wave function from the initial wave function, we need to to introduce

some kind of integral operator

$$\psi(x'',t) = \int d^3x' K(x'',t;x',t_0)\psi(x',t_0) \tag{6.a.4}$$

where the kernel of the integral operator is known as the propagator given by

$$\begin{aligned} K(x'',t,x',t_0) &= \sum_{a'} \langle x''|a'\rangle\langle a'|x'\rangle exp\left[\frac{-iE_{a'}(t-t_0)}{\hbar}\right] \\ &= \langle x''|exp\left[\frac{-iE_{a'}(t-t_0)}{\hbar}\right]|x'\rangle \end{aligned} \tag{6.a.5}$$

When you compare this to the scalar potential expression in Jackson, we can see some sort of similarity between the wave function and the potential.

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')}{|x-x'|}d^3x' \tag{6.a.6}$$

where $1/|x-x'|$ is a green's function. We can think of potential in EM as some sort of propagation through space, meanwhile the final wave function is some sort of propagation through time. Therefore, the kernel of the integral operator is a green's function.

6.b Write down the propagator as Feynman path integral

We can write the propagator as

$$\begin{aligned} K(x'',t;x',t_0) &= \sum_{a'} \langle x''|exp\left(\frac{-iHt}{\hbar}\right)|a'\rangle\langle a'|exp\left(\frac{iHt_0}{\hbar}\right)|x'\rangle \\ &= \langle x'',t|x',t_0\rangle \end{aligned} \tag{6.b.1}$$

Considering the form of the Feynman's path integral

$$\langle x_N,t_N|x_1,t_1\rangle = \int_{x_1}^{t_N} \mathscr{D}[x(t)]exp\left[i\int_{t_1}^{t_N} dt\frac{L_{classical(x,\dot{x})}}{\hbar}\right] \tag{6.b.2}$$

Comparing these 2 equations we see that

$$\langle x'',t|x',0\rangle = \int \langle x|x'\rangle exp\left[\frac{iS(x)}{\hbar}\right] \tag{6.b.3}$$

where we have the integral of the Lagrangian to be the action of along the path followed by our physical system. The $\mathscr{D}$ is the sum of probability density of states $\langle x|x'\rangle$ that the system has over time.

6.c Provide examples where a path integral formulation is convenient

The path integral formulation is commonly used in statistical mechanics and thermodynamics, stochastics, and quantum mechanics as it is useful in answering questions that can be decomposed as a collection of paths, steps or systems. In quantum mechanics, we can see its use in the time evolution of quantum systems. As such, it is also used a lot in quantum field theory

topics. In statistical mechanics, the path integral formulation is convenient as we can decompose brownian motion or random walk as a sum of paths or steps. In thermodynamics, we can use this formulation to calculate the temperature an object as it interacts with one system to another.

7 Week 7: Potentials and gauge transformation

7.a Describe how the scalar and vector potentials enter the Schrodinger equation for quantum systems in the presence of electromagnetic fields.

We can look at the Schrodinger equation with a solution $|\alpha, t_0; t\rangle$ under the presence of the vector potential A

$$\left[\frac{(p - eA/c)^2}{2m} + e\phi\right] |\alpha, t_0; t\rangle = i\hbar \frac{\partial}{\partial t} |\alpha, t_0; t\rangle \tag{7.a.1}$$

when we solve for the solution of the Schrodinger equation under the influence of $\tilde{A} = A + \nabla\Lambda$, we get the following equation

$$\left[\frac{(p - eA/c - e\nabla/c)^2}{2m} + e\phi\right] |\alpha, \widetilde{t_0}; t\rangle = i\hbar \frac{\partial}{\partial t} |\alpha, \widetilde{t_0}; t\rangle \tag{7.a.2}$$

which leads us to this solution

$$|\alpha, \widetilde{t_0}; t\rangle = exp\left(\frac{ie\Lambda}{\hbar c}\right) |\alpha, t_0; t\rangle. \tag{7.a.3}$$

This solution implies that

$$\tilde{\psi}(x', t) = exp\left[\frac{ie\Lambda(x', t)}{\hbar c}\right] \psi(x', t) \tag{7.a.4}$$

This is a solution to the modified Schrodinger equation for $\tilde{A} = A + \nabla\Lambda$ when we assert that

$$exp\left(\frac{-ie\Lambda}{\hbar c}\right)\left(p - \frac{eA}{c} - \frac{e\nabla\Lambda}{\hbar c}\right) exp\left(\frac{ie\Lambda}{\hbar c}\right) = \left(p - \frac{eA}{c}\right)^2 \tag{7.a.5}$$

When we verify this by plugging into the Schrodinger equation, we see that the probability density ρ remains unchanged while the phase S is modified $S \to S + e\Lambda/c$. This means that the probability flux given by

$$j = \frac{\rho}{m}\left(\nabla S - \frac{eA}{c}\right) \tag{7.a.6}$$

remains invariant.

7.b Explain how the Aharonov-Bohm effect can distinguish situations in which a quantum particle confined to an annular cylinder is subject to the vector potential of a magnetic field within the hollow or not.

Let us consider a hollow cylindrical tube and let a quantum particle be contained inside its walls. Introduce a magnetic field enclosed inside the cylindrical tube such that the magnetic field is exclusively just in the hole in the middle of the tube and that this magnetic field wouldn't move

into the walls of the tube. We might think that since the magnetic field wouldn't meet the particle in the walls of the cylinder that there wouldn't be a change in the energy spectrum. But we are neglecting the fact that the vector potential doesn't vanish inside the walls of the tube. Using Stoke's theorem we have an equation that relates how much vector potential is need to construct the magnetic field

$$A = \frac{B\rho^2}{2\rho}\hat{\phi} \tag{7.b.1}$$

When plugged in to solve the Schrodinger equation, we need to make the replacement $\nabla \rightarrow \nabla - (ie/\hbar c)A$ or simply

$$\frac{\partial}{\partial\phi} \rightarrow \frac{\partial}{\partial\phi} - \frac{ie}{\hbar c}\frac{B\rho^2}{2} \tag{7.b.2}$$

in cylindrical coordinates. This leads to an observable change in energy spectrum. This setup is the bound-state version of the Aharonov-Bohm effect. To conclude, using the Aharonov-Bohm effect, we will see a observable change in energy for a quantum particle inside the walls of a cylindrical tube subjected to the vector potential of a magnetic field in the hollow region of the tube.

8 Week 8: Angular momentum and tensor operators

8.a Identify advanced experimental situations that utilize angular momentum eigenstates

Neutron interferometry compares an unrotated state and rotated state to detect the minus sign predicted from the board work of a 2π rotation on a general ket. The setup of this experiment consists of a monoenergetic beam of thermal neutrons that is split into 2 beams - one beam passes through a region without magnetic fields, while the other beam passes through a beam induced with a magnetic field. The 2 beams will then meet at a common interference region. After the experiment proceeds, we notice that the neutron state ket of the beam that passes through the path induced with the magnetic field will exhibit a phase change which is dependent on the time spent in the magnetic field region. Additionally, we expect that angular momentum eigenstates are used in experiments regarding molecular, atomic, and nuclear spectroscopy.

8.b Construct angular momentum operators as generators of rotations

Consider a rotation by some angle ϕ about an axis. Let's say that we rotate the system about the z-axis. From an initial ket $\alpha\rangle$, the ket after rotation is given by

$$|\alpha\rangle_R = \mathscr{D}_z(\phi)|\alpha\rangle \tag{8.b.1}$$

where $\mathscr{D}_z = exp(-iS_z\phi/\hbar)$. Under rotation we see that the expectation value

$$\begin{aligned}\langle S_x\rangle &= exp\left(\frac{iS_z\phi}{\hbar}\right)S_x exp\left(\frac{iS_z\phi}{\hbar}\right)\\ &= \left(\frac{\hbar}{2}\right)exp\left(\frac{iS_z\phi}{\hbar}\right)[(|+\rangle\langle -|)+(|-\rangle\langle +|)]exp\left(\frac{-iS_z\phi}{\hbar}\right)\\ &= \left(\frac{\hbar}{2}\right)(e^{i\phi/2}|+\rangle\langle -|e^{i\phi/2}+e^{-i\phi/2}|-\rangle\langle +|e^{-i\phi/2})\end{aligned} \tag{8.b.2}$$

Since the exponentials can be written as $e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$, our expectation value becomes

$$\begin{aligned}\langle S_x\rangle &= \frac{\hbar}{2}[(|+\rangle\langle -|+|-\rangle\langle +|)\cos\phi + i\sin\phi(|+\rangle\langle -|-|-\rangle\langle +|)]\\ &= \langle S_x\rangle\cos\phi - \langle S_y\rangle\sin\phi\end{aligned} \tag{8.b.3}$$

where $S_x = \frac{\hbar}{2}[(|+\rangle\langle -|)+(|-\rangle\langle +|)]$, $S_y = \frac{i\hbar}{2}[(-|+\rangle\langle -|)+(|-\rangle\langle +|)]$, and $S_z = \frac{\hbar}{2}[(|+\rangle\langle +|)+(|-\rangle\langle -|)]$.
Following this procedure we come up with

$$\begin{aligned}\langle S_y\rangle &= \langle S_y\rangle\cos\phi + \langle S_x\rangle\sin\phi\\ \langle S_z\rangle &= \langle S_z\rangle\end{aligned} \tag{8.b.4}$$

This shows that the rotation operator when applied to the state ket does rotate the expectation value of S around the z-axis by angle ϕ. This mean that the expection value of the spin operator behaves as thought it were a classical vector under rotation:

$$\langle S_k\rangle \to \sum_l R_{kl}\langle S_l\rangle \tag{8.b.5}$$

or in general

$$\langle J_k\rangle \to \sum_l R_{kl}\langle J_l\rangle \tag{8.b.6}$$

where J_k are the generators of rotations satisfying the angular momentum commutation relations.

8.c Familiarize oneself with the notation used for vector and tensor operations. Use formalism to write angular momentum operators as spherical tensor operators.

Tensors in the context of rotations would correspond to the three-dimensional or three-directional shift in the system, while the vector corresponds to the one-dimensional shift. We can therefore right tensor operators as

$$R(\alpha,\beta,\gamma) = R_x(\alpha)R_y(\beta)R_z(\gamma) \tag{8.c.1}$$

in the context of rotations where the matrices on the RHS refer to fixed angle rotations. Which we can write as

$$\mathscr{D}(\alpha,\beta,\gamma) = \mathscr{D}_x(\alpha)\mathscr{D}_y(\beta)\mathscr{D}_z(\gamma) \tag{8.c.2}$$

which describes a product of of rotation operators in ket space of spin 1/2 system. Take note that these equations all describe system transformations in cartesian coordinates. In spherical coordinates, for a spherical tensor operator of rank k with $2k+1$ components we have

$$\mathscr{D}^\dagger(R)T_q^{(k)}\mathscr{D}(R) = \sum_{q'=-k}^{k} \mathscr{D}_{q'q}^{(k)*}(R)T_{q'}^{(k)} \quad \text{or} \quad \mathscr{D}^\dagger(R)T_q^{(k)}\mathscr{D}(R) = \sum_{q'=-k}^{k} \mathscr{D}_{q'q}^{(k)}(R)T_{q'}^{(k)} \tag{8.c.3}$$

where $T_q^{(k)}$ can be written as the spherical harmonic $Y_{l=k}^{m=q}(V)$ characterized by both θ and ϕ.

8.d Illustrate use cases for the projection formula

We introduce the Projection theorem derived from the Wigner-Eckart theorem

$$\langle \alpha', jm'|V_q|\alpha, jm\rangle = \frac{\langle \alpha', jm|J\cdot V|\alpha, jm\rangle}{\hbar^2 j(j+1)}\langle jm'|J_q|jm\rangle \tag{8.d.1}$$

where we choose

$$J_{\pm 1} = \mp\frac{1}{\sqrt{2}}(J_x \pm iJ_y) = \mp\frac{1}{\sqrt{2}}J_\pm, \quad J_0 = J_z \tag{8.d.2}$$

We can use this to derive the Lande g-factor. We start with the hamiltonian due to an external magnetic field

$$H = H_0 + V \quad V = -\vec{\mu}\cdot\vec{B} \tag{8.d.3}$$

If we assume the magnetic field B to be small, we can use perturbation theory to calculate for the energy shift. The first order correction to the energy levels is given by

$$\Delta E = \langle\psi|V|\psi\rangle, \quad |\psi\rangle = |\psi\rangle = |jm_j ls\rangle \tag{8.d.4}$$

This will lead to

$$\Delta E = -\vec{B}\langle jm_j ls|\vec{\mu}|jm_j ls\rangle \tag{8.d.5}$$

We can then use the the projection theorem to evaluate for ΔE

$$\Delta E = \mu_B g_j m_j B_z \tag{8.d.6}$$

9 Week 9: Eigenvalues and eigenstates

9.a Construct AM eigenkets and derive the corresponding eigenvalues using ladder operators

We first denote the eigenvalues of J^2 and J_z by a and b

$$\begin{aligned} J^2|a,b\rangle &= a|a,b\rangle \\ J_z|a,b\rangle &= b|a,b\rangle \end{aligned} \tag{9.a.1}$$

To find for the values of a and b, we work with the ladder operators given by

$$J_{\pm} \equiv J_x \pm i_y \tag{9.a.2}$$

wherein they satisfy the commutation relations

$$[J_+, J_-] = 2\hbar J_z \tag{9.a.3}$$

and

$$[J_z, J_{\pm}] = \pm\hbar J_{\pm}. \tag{9.a.4}$$

We also note that

$$[J^2, J_{\pm}] = 0 \tag{9.a.5}$$

We now check how J_z acts on $J_{\pm}|a, b\rangle$

$$\begin{aligned} J_z(J_{\pm}|a,b\rangle) &= ([J_z, J_{\pm}] + J_{\pm}J_z)|a,b\rangle \\ &= (b \pm \hbar)(J_{\pm}|a,b\rangle) \end{aligned} \tag{9.a.6}$$

we then notice that when we apply $J_{\pm}$ to a J_z eigenket the resulting would still be a J_z eigenket except that it's eigenvalue now increases or decreases by a unit of $\hbar$ but does not change the eigenvalue of J^2

$$J^2(J_{\pm}|a,b\rangle) = J_{\pm}J^2|a,b\rangle = a(J_{\pm}|a,b\rangle). \tag{9.a.7}$$

In summary

$$J_{\pm}|a,b\rangle = c_{\pm}|a, b \pm \hbar\rangle. \tag{9.a.8}$$

where $a = \hbar^2 j(j+1)$ and $b = m\hbar$ where $j = n/2$ and $m = -j, -j+1, ..., j-1, j$, and n is just any integer. It is important to note that we computed for the upper and lower limit of b to compute for the value of a and b.

10 Week 10: Pure and mixed states

10.a Use the density operator formulation to distinguish pure from mixed states

A pure state is specified $w_i = 1$ for some state $|\alpha^{(i)}\rangle$ with $i = n$ so the corresponding density operator is written as

$$\rho = |\alpha^{(n)}\rangle\langle\alpha^{(n)}| \tag{10.a.1}$$

with no summation. The density operator for a pure ensemble is given by

$$\rho^2 = \rho \tag{10.a.2}$$

Thus for a pure ensemble, we have

$$Tr(\rho^2) = 1 \tag{10.a.3}$$

and when diagonalized, the density matrix for a pure ensemble have only one non-zero element

which corresponds to its eigenvalues. Meanwhile, for a mixed state, we have

$$0 < Tr(\rho^2) < 1 \tag{10.a.4}$$

11 Week 11: Problem Set 2 deadline

12 Week 12: Quantum correlations

12.a Explain how quantum correlations are measured in simple entangled systems.

We can imagine a system of two spin 1/2 particles in a spin-singlet state. Observer A can measure the S_x and S_z of particle A while observer B does the same for particle B. First, if observer A measures S_z on particle A and observer B measures S_x on particle B, we say that there is random correlation or there is no real connection between this measurement. Meanwhile, if observer A measures S_x on particle A and observer B does the same for particle B, we can say that there is 100% between the two measurements and you just have to take the opposite of the sign of one of the measurements. Lastly, if one observer makes no measurement, the measurement of the other would yield a purely random result. However, this violates Einstein's locality principle that states that system S_1 should be independent of measurements in system S_2 and vice versa, when they are spatially separated from each other. To over rule this we consider the simple model conceived by E.P.Wigner. It states that for a large number of spin 1/2 particles, we have a fraction of them to have these properties: (1) If S_z is measured, we have a plus sign with certainty, and (2) If S_x is measured, we have a minus sign with certainty. As an example, we have measurements on composite spin-singlet systems which follows our new model

$$\begin{aligned} &\text{particle 1} \quad \text{particle 2} \\ &(z+, x-) \leftrightarrow (z-, x+) \\ &(z+, x+) \leftrightarrow (z-, x-) \\ &(z-, x+) \leftrightarrow (z+, x-) \\ &(z-, x-) \leftrightarrow (z+, x+) \end{aligned} \tag{12.a.1}$$