Physics 226
University of the Philippines - Diliman
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Physics 226 - Lecture Notes

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1 Week 1: September 23, 2021

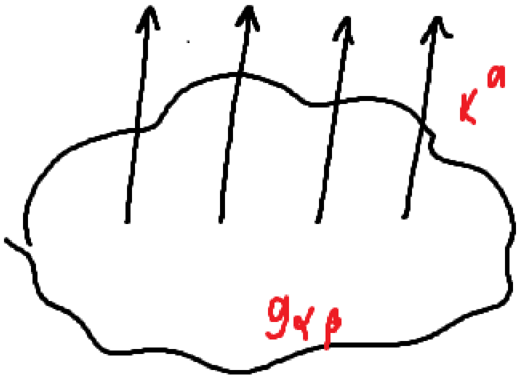
We start with the Schwarzschild black hole system with coordinates, (t, r, θ, ϕ), and metric given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1..1}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. This is a solution because it is a *unique spherically symmetric solution to the vacuum Einstein equation.*
Einstein equation: $G_{ab} = 8\pi T_{ab}$
Vacuum Einstein equation: $G_{ab} = R_{ab} = 0$ (No source/matter field)
Consider the system given in Fig. **??**,

Figure 1: Star system described by the g_{ab} metric and a surrounding $g_{schwarz}$ metric. We match these two metrics similar to the methods in boundary value problems.

Definition: Static - A special case of stationary spacetimes.
Definition: Stationary A spacetime is stationary if it admits a time-like, hypersurface, killing vector k^a such that $g_{ab}k^a k^b < 0$ with metric signature $(-, +, +, +)$.
Suppose we have a manifold



We have the Killing's equation $\mathcal{L}_k g = 0 \Longleftrightarrow \nabla_{(a}k_{b)} = 0$, where $\mathcal{L}_k$ is Lie derivative. We define the symmetrization and antisymmetrization symbol

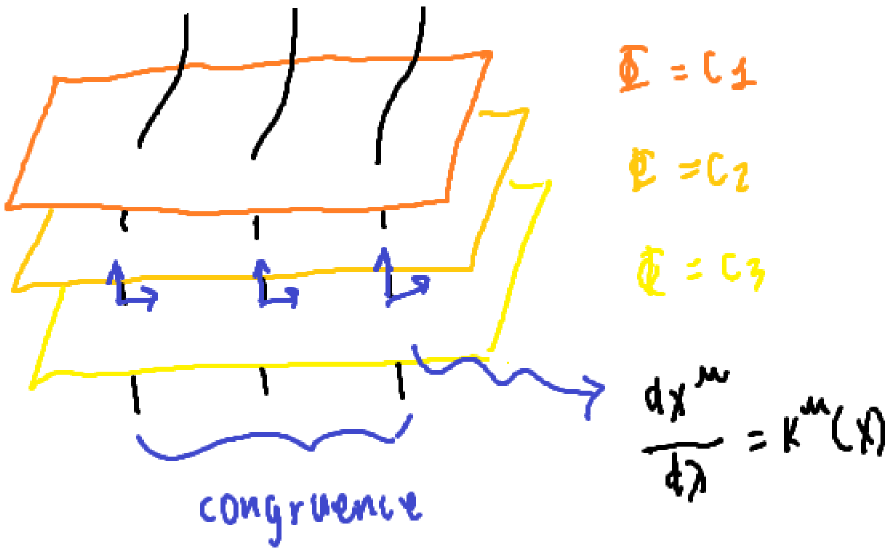
$$\nabla_{(a}k_{b)} = \frac{1}{2}(\nabla_a k_b + \nabla_b k_a) = 0 \tag{1..2}$$

$$\nabla_{[a}k_{b]} = \frac{1}{2}(\nabla_a k_b - \nabla_b k_a) = 0 \tag{1..3}$$

Definition: Hypersurface orthogonal There exists a scalar function Φ and f. If we can write the normal/dual vector k_a in the form of

$$k_a = -f\nabla_a \phi, \tag{1..4}$$

where $\nabla_a \Phi = (\partial_a \Phi)dx^a$, then there exists hypersurface $\Phi(x^\mu) = \text{constant}$.



where $x^\mu(\lambda)$ is $(t(\lambda), x(\lambda), \theta(\lambda), \phi(\lambda))$. The requirement for orthogonality $g_{ab}k^a t^b = 0$ where we say that $k \perp t$.

Consider a hypersurface

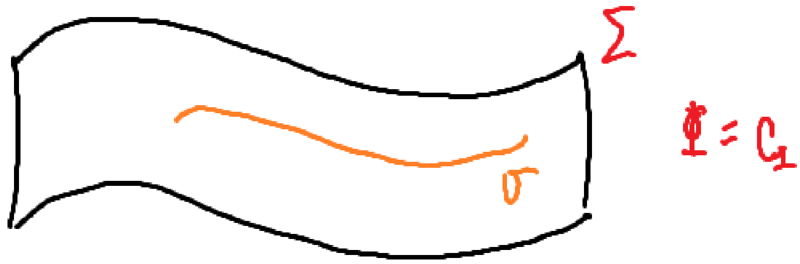


Figure 2: Hypersurface Σ with $\sigma = x^\mu(s)$, where s is a parameter along the curve.

Since the curves are found in σ, we have $\dfrac{d\Phi}{ds}$.

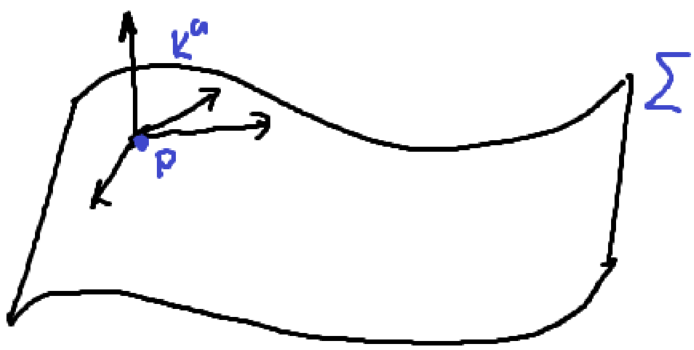
$$\frac{d\Phi}{ds} = \frac{d\Phi(x^\mu(s))}{ds} = \Phi_{,\mu}\frac{dx\mu}{ds} = \Phi_{,\mu}t^\mu = 0 \tag{1..5}$$

where $\Phi_{,\mu}$ = gradient and t^μ = arbitrary tangent vector on σ. Since $k_\mu = -f\Phi_{,\mu}$ from our discussion on the definition of hypersurface orthogonal, then

$$k_\mu t^\mu = 0 \tag{1..6}$$

How do we prove the inverse? How do we show that $k_\mu = -f\Phi$ when we know that $k_\mu t^\mu$?

Proof: Suppose that $k_a t^a = 0$ for all tangent vectors t^a to Σ



$\boxed{\text{Note: } t^a = \text{vector, } t_a = \text{one form}}$

Consider a point p in Σ, there exists 3 independent tangent vectors $t^a_{(i)}$; $i = 1, 2, 3$ that are orthogonal to k^a. From these we can write down 3 conditions

$$t^a_{(i)} k_a = 0, \quad t^\mu_{(i)} k_\mu = 0 \quad \text{(component form)}, \tag{1..7}$$

which results to us having 3 linear equations but 4 unknowns from the 4 components of $k_\mu = (k_0, k_1, k_2, k_3)$. Since we cant solve for these 4 unknowns with only 3 equations we rescale k_μ in such a way that that we can factor out k_0 to decrease the number of unknowns to 3

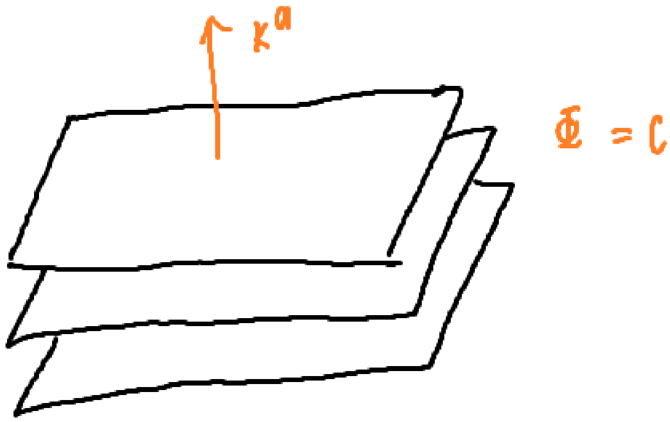
$$k_\mu = k_0(1, k_i). \tag{1..8}$$

Now we verify that $k_\mu t^\mu_{(i)} = 0$ is a solution to these equations. However we can also express $t^\mu_{(i)} \Phi_{,\mu} = 0$. Now since we know these 2 equations solves the same set of equations then we know that k_μ should be proportional to $\Phi_{,\mu}$. We then conclude that

$$k_a = -f \nabla_a \Phi \tag{1..9}$$

for some factor f.

Coordinates for a static spacetime



Choose x^0 such that k^a corresponds to $\dfrac{\partial}{\partial x^0}$. x^0 needs to be the parameter that changes as we move along the integral curve. Meanwhile, $\{x^i\}$ is tangent to Σ and labels the integral curves that we are studying.

We can then get the component of the metric

$$g_{00} = g(\partial_0, \partial_0) < 0 \quad \text{(because time-like curve)} \tag{1..10}$$

and

$$g_{oi} = g(\partial_0, \partial_i) = g_{ab} \left(\frac{\partial}{\partial x_0}\right)^a \left(\frac{\partial}{\partial x_i}\right)^b = 0 \tag{1..11}$$

Therefore, we can choose a line element with coordinates that follow the following equation

$$ds^2 = g_{00}(x^\mu)(dx^0)^2 + g_{ij}(x^\mu)dx^i dx^j = g_{\mu\nu}dx^\mu dx^\nu \tag{1..12}$$

2 Week 2: September 28 and September 30, 2021

In choosing $\left(\frac{\partial}{\partial t}\right)^a = K$, the components of the killing flow is $K = (1, 0, 0, 0)$.

Note: We can also set $\left(\frac{\partial}{\partial \lambda}\right)^a = K$, which will lead to our killing flow to admit the following components: $((\partial\lambda/\partial t)^{-1}, 0, 0, 0)$.

How does this choice help us?

We have,

$$g_{00} = g(\partial_0, \partial_0) \neq 0 \tag{2..1}$$

$$g_{ij} = g(\partial_i, \partial_j) \tag{2..2}$$

$$g_{0i} = g(\partial_0, \partial_i) = 0 \tag{2..3}$$

We can then evaluate the Lie derivative of a $(0, 2)$ tensor

$$\mathscr{L}_k g = 0 = \cancelto{\partial_t g_{\alpha\beta}}{k^\mu \partial_\mu g_{\alpha\beta}} + \cancelto{0}{g_{\alpha\mu}k^\mu_{,\beta}} + \cancelto{0}{g_{\mu\beta}k^\mu_{,\alpha}}. \tag{2..4}$$

Therefore,

$$\frac{\partial g_{\alpha\beta}}{\partial t} = 0 \tag{2..5}$$

and conclude that the metric components don't necessarily depend on time.

Note: $\mathscr{L}_k g = 0$ **(static)**

Our metric then simplify into

$$ds^2 = g_{00}(x^i)dt^2 + g_{ij}(x^k)dx^i dx^j, \tag{2..6}$$

which is our general ansatz for static spacetimes.

Aside from beign stationary, the Schwarzschild metric is also under spatial symmetry. We will write

$$x^i = r, \theta, \phi. \tag{2..7}$$

This allows our metric to be written in this form

$$ds^2 = A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2 \tag{2..8}$$

where $d\Omega^2 = d\theta^2 + \sin\theta d\phi^2$.

We can simplify this metric by setting

$$C(r) = R^2 \tag{2..9}$$

$$C'dr = 2RdR \tag{2..10}$$

$$dr = \frac{2RdR}{C'} \tag{2..11}$$

therefore the metric is given by

$$ds^2 = F(R) + G(R)dR^2 + R^2 d\Omega^2 \longrightarrow \text{standard form for static spherically symmetric metric} \tag{2..12}$$

where the coefficients are defined as

$$F(R) = A(r(R)) \tag{2..13}$$

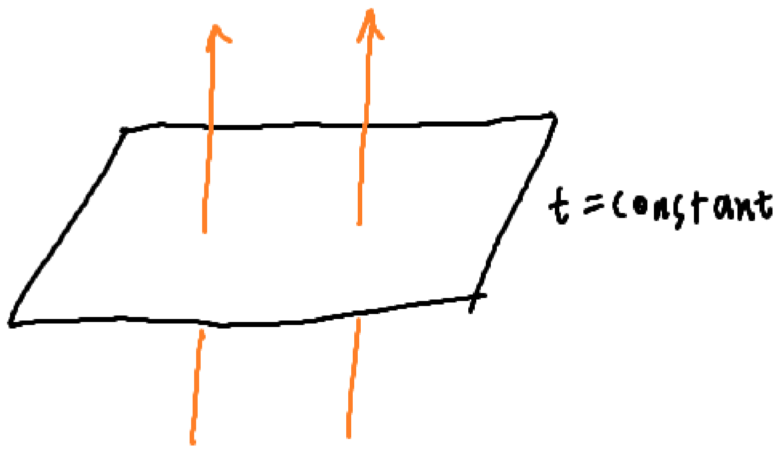
$$G(R) = \frac{4r^2}{(C')^2} B(r(R)) \tag{2..14}$$

Renaming the labels

$$ds^2 = F(r)dt^2 + G(r)dr^2 + r^2 d\Omega^2 \tag{2..15}$$

We consider

$$K^\alpha = \frac{\partial}{\partial t} \quad \text{and} \quad \mathscr{L}_k g = 0 \tag{2..16}$$



Let us look at the $t =$ constant, and $\theta, \phi =$ constant surface. In this case we would have $dl^2 = G(r)dr^2$ or $l = \int G(r)dr$.

If we now $t, r =$ constant surfaces, the line element along the surface would reduce to

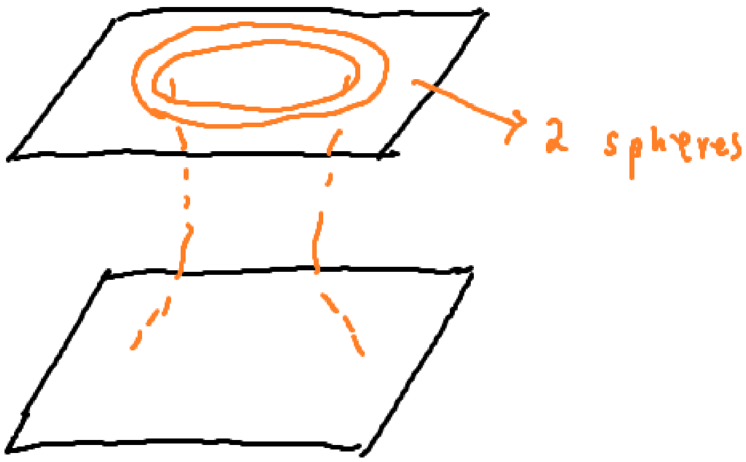
$$ds^2 = r^2 d\Omega^2. \tag{2..17}$$

Just like in euclidean and schwarzschild space, the area would still be $A = 4\pi r^2$. In fact,

this is true for all metrics that follow the form

$$ds^2 = F(r)dt^2 + G(r)dr^2 + r^2 d\Omega^2 \tag{2..18}$$

where $r = \sqrt{\frac{A}{4\pi}}$ is the areal radius and is the geometric meaning of r.



Important: Geometry supplies the meaning of the coordinates.
Going back to the schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2..19}$$

where M is the schwarzschild mass.
Next consider the geodesics around schwarzschild
The angular frequency will depend on r

$$\left(\frac{d\phi}{dt}\right)^2 = \Omega^2 = \frac{M}{r^2} \quad \text{(Kepler's law)} \tag{2..20}$$

Remember, our metric becomes singular at $r = 0, 2M$. To check this if this is caused by a bad choice of coordinates, since metrics aren't measureable, we check the Riemann curvature or the $2nd$ derivative of the metric. We set $g = \eta$ and the Christoffel symbols

as $\partial g = \Gamma = 0$. This is the principle of equivalence.
The non-zero components of the riemann tensors are

$$R^t_{rtr} = -\frac{2M}{r^3}\left(1 - \frac{2M}{r}\right)^{-1} \tag{2..21}$$

$$R^t_{\theta t\theta} = \frac{1}{\sin^2\theta} R^t_{\phi t\phi} = \frac{M}{r^5} \tag{2..22}$$

$$R^r_{\theta r\theta} = \frac{1}{\sin^2\theta} R^r_{\phi r\phi} = -\frac{M}{r^5} \tag{2..23}$$

$$R^\theta_{\phi\theta\phi} = \frac{2M}{r^5}\sin^2\theta \tag{2..24}$$

We have 20 components of the riemann tensor but for the Schwarzschild case, only 4 of them are non-zero.
Curvature invariants: To avoid dependence on coordinates we look at scalars. But how do we find these scalars? We get the contractions/combining curvature components.

$$\text{Contraction of Ricci:} \quad R^{\mu\nu}R_{\mu\nu} = 0 \tag{2..25}$$

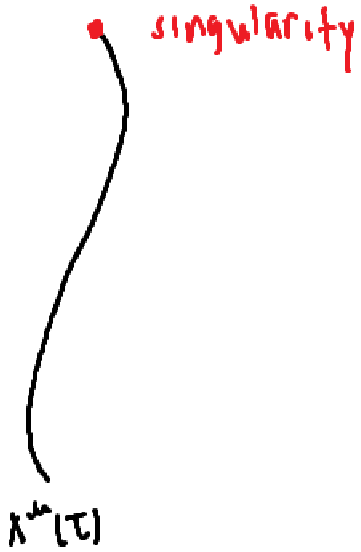
$$\text{Contraction of Riemann:} \quad R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{48M^2}{R^6} = K_1 \tag{2..26}$$

$$\epsilon_{\mu\nu}\rho^\sigma R^{\mu\nu\alpha\beta}R_{\rho\sigma\alpha\beta} \tag{2..27}$$

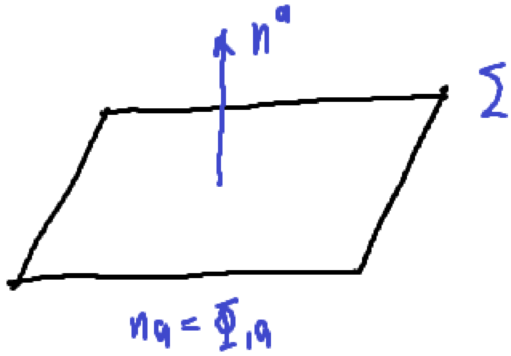
$$R^{\mu\nu\alpha\beta;\sigma}R_{\mu\nu\alpha\beta;\sigma} \tag{2..28}$$

Let us look at the Kretschmann scalar, $K_1 = \frac{48M^2}{R^6}$, which is completely independent of coordinates. Notice that it is now not singular at $r = 2M$. From our curvature invariant computation we see that the "singularity" at $r = 2M$ is just an apparent/coordinate singularity while the $r = 0$ singularity is a curvature singularity.

Definition: Singularity - "belong to the manifold", reachable, geodesics can reach the offending point at finite affine parameter (geodesic completeness).
If the range of λ spans the entire $(-\infty, \infty)$ then the geodesics is complete. If any of your geodesics are incomplete then we have a singularity.



Definition: Hypersurfaces - Consider a hypersurface Σ, $\Phi(x^\mu) = 0$, let $p \in \Sigma$ and n^a be perpedicular to Σ at p.



Any t^a (tangent vector) is perpendicular to n^a $x^\alpha(\lambda) t^\alpha = \dfrac{dx^\alpha}{d\lambda}$ and $n_\alpha t^\alpha = \dfrac{d\Phi}{d\lambda} = 0$. Go to a local inertial frame at p. We have the local minkowski spacetime to be

$$ds^2 = -dt^2 + dx^i dx_i. \tag{2..29}$$

Rotate the spatial axis so that it only has one spatial coordiate such as

$$n_a = (n^0, n^1, 0, 0), \tag{2..30}$$

where the tangent vector has coordinates

$$t^a = (t^0, t^1, t^2, t^3). \tag{2..31}$$

Therefore,

$$n_a t^a = -n^0 t^0 + n^1 t^1 = 0 \longrightarrow \frac{t^0}{t^1} = \frac{n^1}{n^0} \tag{2..32}$$

We can rewrite

$$t^a = \left(t^1\frac{n^1}{n^0}, t^1, t^2, t^3\right) = \frac{t^1}{n^0}(n^1, n^0, a, b). \tag{2..33}$$

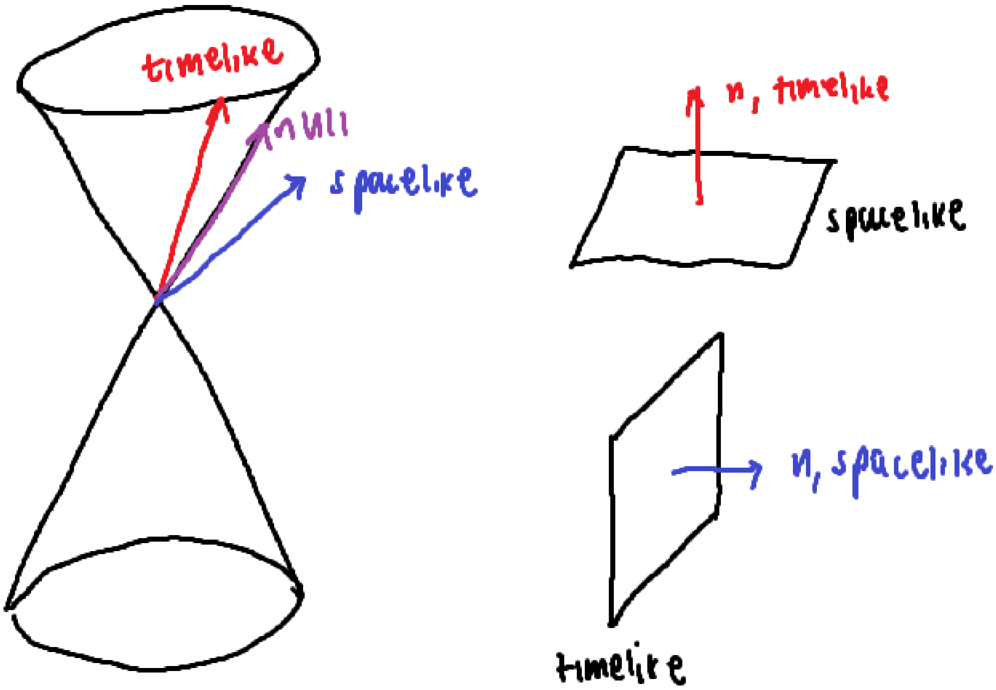
Therefore, the tangent vector can be rewritten as

$$t^a = \Lambda(n^1, n^0, a, b) \tag{2..34}$$

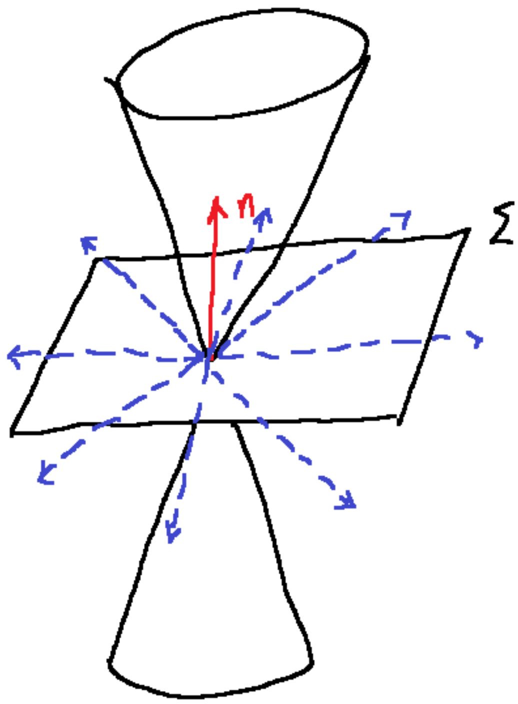
while the normal vector normal to Σ can be rewritten as

$$n_a = (n_0, n_1, 0, 0) \tag{2..35}$$

Definition: Σ_p is spacelike, timelike, or null at p depending on the normal n. If n^a is timelike, then Σ is spacelike at p. If n^a is spacelike, then Σ is timelike at p. Lastly, if n^a is null, then Σ is null at p.

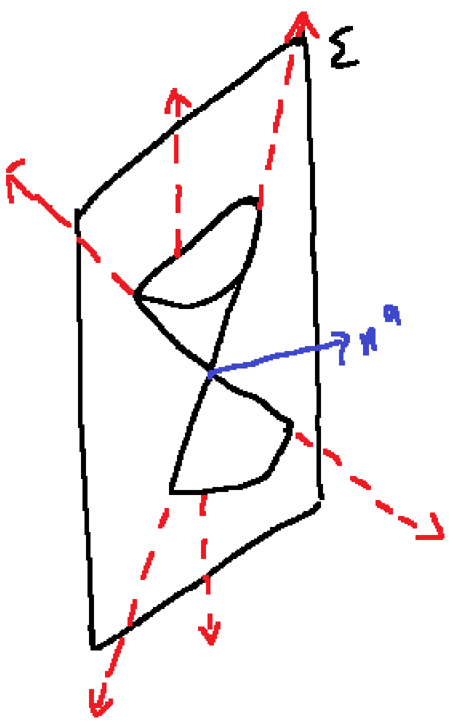


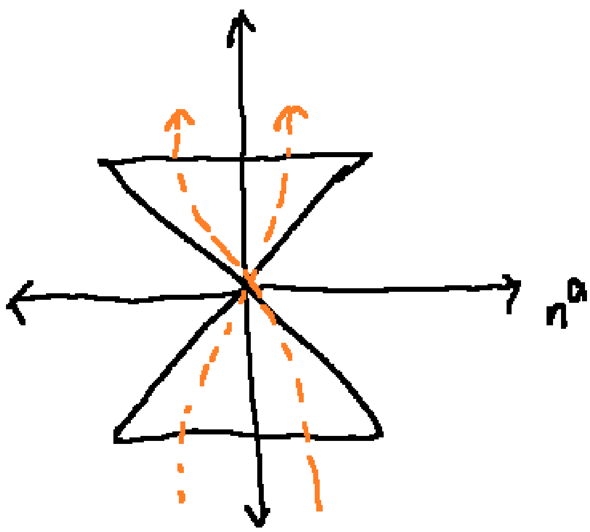
Cases: 1: $n^a n_a < 0$ (Σ is spacelike) then $t^a t_a = \Lambda^2(-n_a n^a + (a^2+b^2)\Phi) > 0$. We conclude that t^a is spacelike. Therefore, all curves passing through t must be spacelike on t.



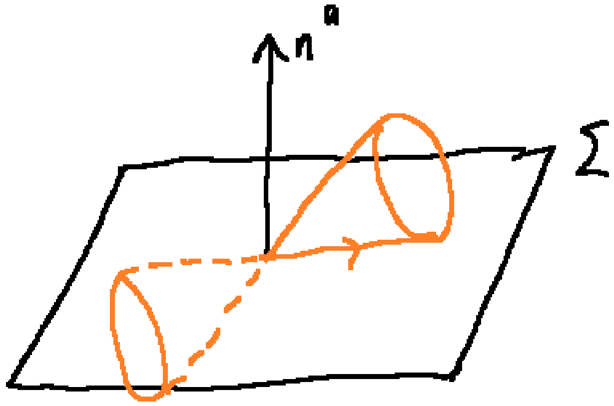
Therefore, if Σ is spacelike then Σ can be crossed in 1 direction by causal curves.

2: $n^a n_a > 0$ (Σ is timelike). If this is the case then t^a can be spaccelike, timelike, or null. We conclude from the figure below that Σ can be crossed in both directions by causal curves.





3: $n^a n_a = 0$ Then $t^a t_a \geq 0$ where $t^a t_a = 0$ if $a = 0, b = 0$.



From the figure above we know that Σ can only crossed in 1 direction.

Let us then consider a constant-r hypersurface of schwarzschild

$$\Phi(x^\alpha) = r - \text{constant} = 0 \tag{2..36}$$

then

$$n_a n^a = g_{ab} n_a n_b = g^{rr} \left(\frac{d\Phi}{dr}\right)^2 = g^{rr} = 1 - \frac{2M}{r} \tag{2..37}$$

Since

$$n_a = \Phi_{,a} = \left(0, \frac{d\Phi}{dr}, 0, 0\right) \tag{2..38}$$

therefore when

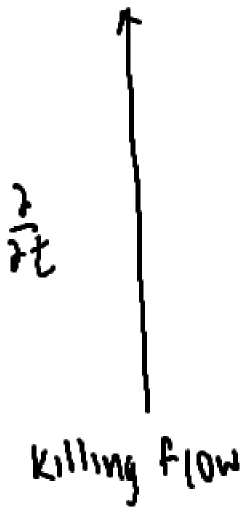
$$r > 2M, \quad \Sigma \text{ is timelike} \tag{2..39}$$

$$r = 2M, \quad \Sigma \text{ is null} \tag{2..40}$$

$$r < 2M, \quad \Sigma \text{ is spacelike} \tag{2..41}$$

Notice that the transition in the normal vector coincides with the transition of the killing vector from timelike to spacelike.

Consider the killing flow in the timelike direction



Killing vector: $k^a = (1, 0, 0, 0)$

$$k^a k_a = g_{ab} = k^a k^b = g^{tt} \tag{2..42}$$
$$= -\left(1 - \frac{2M}{r}\right) \tag{2..43}$$

When $r > 2M$, $k^a k_a < 0$. Therefore k^a is timelike. Likewise, when $r = 0$, when $r < 2M$, $k^a k_a < 0$. Therefore, the $r = 0$ singularity is a singularity in time.

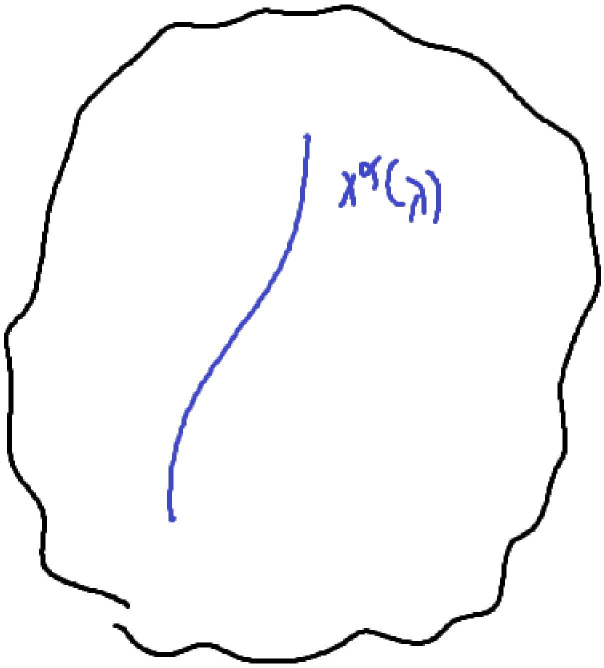
3 Week 3: October 5

Singularities on Schwarzschild:

1. $r = 2M \longrightarrow$ null hypersurface
2. $r = 0 \longrightarrow$ spacelike hypersurface or t=constant

Singularities $\longrightarrow$ check curvature invariants

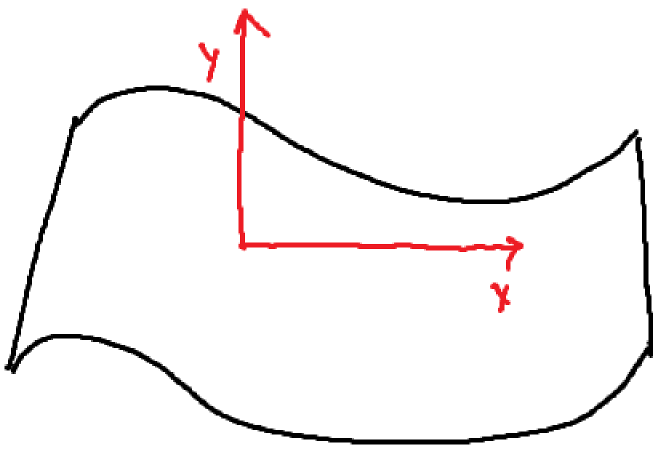
Geodesic completeness



Spacetime is geodesically incomplete if you find a geodesic where the geodesic stops at some fine λ.

Consider a 2 dimensional spatial manifold

$$ds^2 = \frac{1}{x^2 + y^2}(dx^2 + dy^2) \tag{3..1}$$



We seem to have a singularity at $(x, y) = (0, 0)$. If we use the following coordinate transform

$$x' = \frac{x}{x^2 + y^2} \quad \text{and} \quad y' = \frac{y}{x^2 + y^2} \tag{3..2}$$

which yields

$$ds^2 = (dx')^2 + (dy')^2 \longrightarrow \text{flat space} \tag{3..3}$$

The singularities are pushed to $(x' = \infty, y' = \infty)$.

Geodesic completeness

1. coordinate invariant description of singularities
2. **Defn:** a spacetime is geodesically complete if every timelike or null geodesic $x^\alpha(\lambda)$ can be extended to arbitrary large values of the affine parameter

Curve γ from some interval $\subset \mathscr{R}$ mapping to some manifold. If the interval is the same as $\mathscr{R}$ then we have a geodesically complete system.

If there exists at least 1 inextendible geodesic then the spacetime is geodesically incomplete and has a singularity where it isnt extendible.

For an affine parameter λ, we have an example

$$(x'(\lambda), y'(\lambda)) = (\lambda, 0) \longrightarrow \text{complete geodesic} \tag{3..4}$$

Notice that when we rewrite the above equation in terms of $(x(\lambda), y(\lambda)) = \left(\frac{1}{\lambda}, 0\right)$, λ needs to approach infinity fo there to be a singularity.

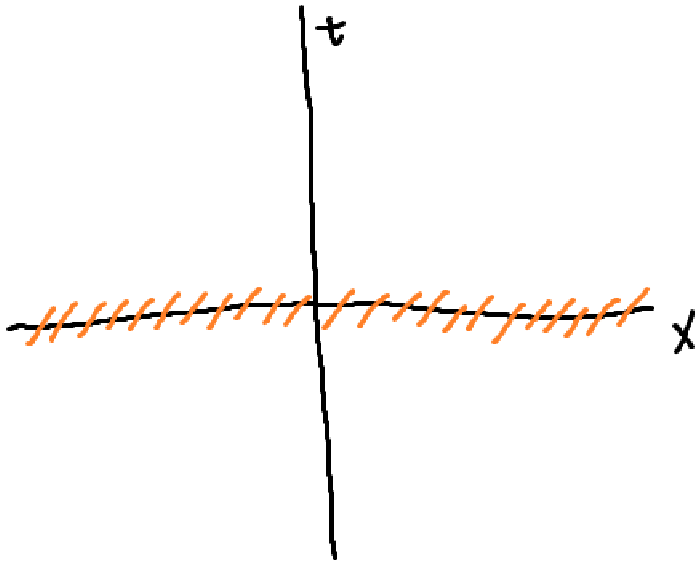
Consider another example

$$ds^2 = -\frac{1}{t^4} dt^2 + dx^2 \tag{3..5}$$

with where the range of t and x are $0 < t < \infty$ and $-\infty < x < \infty$. Notice that this system has a singularity at $t = 0$. We can make the transformation $t \longrightarrow t' = \dfrac{1}{t}$ and yield

$$ds^2 = -(dt')^2 + dx^2 \tag{3..6}$$

which is just 2D minkowski.



Consider another example

$$ds^2 = -x^2 dt^2 + dx2 \tag{3..7}$$

where the range of t and x spans $-\infty < t < \infty$ while x spans $0 < x < \infty$. When $x = 0$ the determinant of $g_{\mu\nu}$ is zero.

Is this a real singularity/coordinate singularity? We can check that the riemann curvature is 0.

Is $x = 0$ "reachable"? Consider $x^\mu(\tau)$ to be a timelike geodesic. We hae a killing vector

$$k^a = \left(\frac{\partial}{\partial t}\right)^a = (1, 0) \tag{3..8}$$

The contraction between this and the 4-velocity is

$$g_{ab}k^a k^b = \text{constant} \tag{3..9}$$

Let

$$-E = g_{ab}k^b u^b = -x^2 \left(\frac{dt}{d\tau}\right) \tag{3..10}$$

where $u^b = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}\right)$.
Therefore,

$$\frac{dt}{d\tau} = \frac{E}{x^2} \tag{3..11}$$

We also have the constraint for timelike vector

$$g_{ab}u^a u^b = -1 \quad \text{normalization condition} \tag{3..12}$$

$$-x^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = -1 \tag{3..13}$$

We can then get

$$\left(\frac{dx}{d\tau}\right)^2 = \frac{E^2}{x^2} - 1 \longrightarrow \frac{dx}{d\tau} = \pm\sqrt{\frac{E^2}{x^2} - 1} \tag{3..14}$$

Let us choose the in-going one $\frac{dx}{d\tau} = -\sqrt{\frac{E^2}{x^2} - 1}$. Let $x(0) = x_0$.

$$\frac{d\tau}{dx} = -\frac{1}{\sqrt{\frac{E^2}{x^2} - 1}} \tag{3..15}$$

$$\tau(x=0) - \tau(x_0) = -\int_{x_0}^{0} \frac{x dx}{\sqrt{E^2 - x^2}} \tag{3..16}$$

$$\Delta\tau = E - \sqrt{E^2 - x_0^2} \tag{3..17}$$

There is a finite proper time that reaches $x = 0$. Therefore, this system is geometrically incomplete since $x = 0$, which is a singularity, is reachable for some finite time.