Physics 221
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Problem Set 1

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Problem 1 (NL Problem 3.2)

Construct the vectors r' obtained by reflecting vector r in the plane whose unit normal vector is $\hat{n}$. Without any calculations, using only geometric arguments, determine the eigenvalues and eigenvectors of the corresponding transformation matrix $\mathcal{A}$. If $\hat{n} = (n_1, n_2, n_3)$, show that $\mathcal{A}$ has elements $a_{ij} = \delta_{ij} - 2n_i n_j$ and is an improper orthogonal matrix.

[Solution]
Geometrically speaking, the eigenvector, points in the direction in which it is stretched and the eigenvalue is the factor it is stretched. So in this setup we have $+1$ and -1 eigenvalues that corresponds to $(1, 0, 0)$ and $(0, 1, 0)$ eigenvectors for the $+1$ eigenvalue, and $(0, 0, 1)$ for the -1 eigenvalue. To explain better, we have the x and y axis to remain the same so we have the eigenvalue to be $+1$. Meanwhile, the z axis is inverted so its eigenvalue is -1. For the transformation matrix $\mathcal{A}$, consider the figure given below

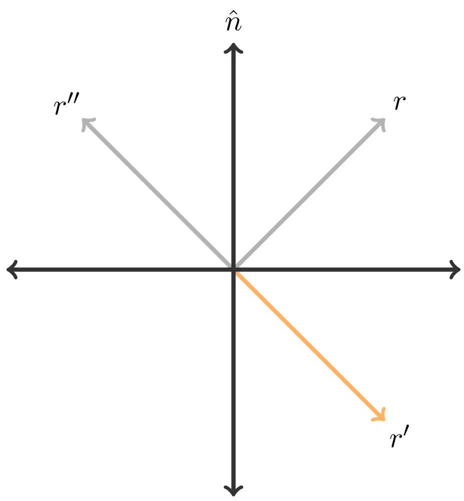

Figure 1: Vectors r, r'' and r' (orange line). Vector r'' is a reflection of r about $\hat{n}$, and vector r' is the reflection of r about the xy-plane.

Given the vector r, we are tasked to find vector r'. But first, I want to compute for r'' and show that the reflection transformation done to r to achieve r'' is the same transformation that

r would experience to achieve r'. Solving for r'', we arrive with the expression

$$r'' = \mathcal{A}r = r - 2(r \cdot \hat{n})\hat{n}. \tag{1}$$

Since we have $r \cdot \hat{n}$ to be the scalar projection of r to $\hat{n}$. It follows that $(r \cdot \hat{n})\hat{n}$ would be the vector projection of r to $\hat{n}$. This can be better visualized by the figure below.

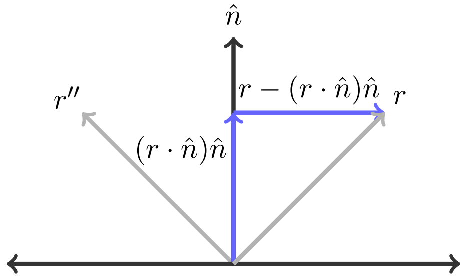

Figure 2: Reflected vectors r'' derived from r. Vector $(r \cdot \hat{n})\hat{n}$ and $r - (r \cdot \hat{n})\hat{n}$ (blue lines) are shown to better visualize how we came up with the value for r''.

Now that we have a better understanding of how reflection transformation works, we can now solve for vector r' as better visualized below.

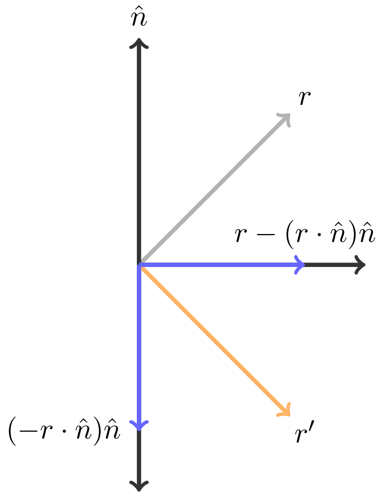

Figure 3:

By vector addition, we then have

$$\begin{aligned} r' &= (-r \cdot \hat{n})\hat{n} + r - (r \cdot \hat{n})\hat{n} \\ r' &= \mathcal{A}r = r - 2(r \cdot \hat{n})\hat{n} \end{aligned} \tag{2}$$

Solving for the transformation matrix $\mathcal{A}$, we can rewrite r' as

$$\begin{aligned} r' = \mathcal{A}r &= r\mathcal{I} - 2r\hat{n}^T\hat{n} \\ &= (\mathcal{I} - 2\hat{n}\hat{n}^T)r \end{aligned} \tag{3}$$

$\mathcal{I}$ is takes the form of the kronecker delta δ_{ij} and $\hat{n}\hat{n}^T$ can be written as $n_i n_j$. Therefore, the transformation matrix $\mathcal{A}$ has elements $\delta_{ij} - 2n_i n_j$. Writing $\mathcal{A}$ in matrix form, we have

$$\mathcal{A} = \begin{pmatrix} 1-2n_1n_1 & -2n_1n_2 & -2n_1n_3 \\ -2n_2n_1 & 1-2n_2n_2 & -2n_2n_3 \\ -2n_3n_1 & -2n_3n_2 & 1-2n_3n_3 \end{pmatrix} \tag{4}$$

Solving for the determinant of $\mathcal{A}$

$$\begin{aligned} |\mathcal{A}| =& (1-2n_1n_1)[(1-2n_2n_2)(1-2n_3n_3) - (-2n_3n_2)(-2n_2n_3)] \\ &- (-2n_1n_2)[(-2n_2n_1)(1-2n_3n_3) - (-2n_3n_1)(-2n_2n_3)] \\ &+ (-2n_1n_3)[(-2n_2n_1)(-2n_3n_2) - (-2n_3n_1)(1-2n_2n_2)] \\ |\mathcal{A}| =& (1-2n_1^2)(1-2n_2^2-2n_3^2) + (2n_1n_2)(2n_2n_1) - (2n_1n_3)(2n_3n_1) \\ |\mathcal{A}| =& 1-2(n_1^2+n_2^2+n_3^2) + 4n_1^2n_2^2 + 4n_1^2n_3^2 - 4n_1^2n_2^2 - 4n_1^2n_3^2 \\ |\mathcal{A}| =& 1-2(n_1^2+n_2^2+n_3^2) = -1 \end{aligned} \tag{5}$$

where we used the formula for the magnitude of a vector $n_1^2 + n_2^2 + n_3^2 = 1$, since $\hat{n}$ is a unit vector. Therefore, we conclude that $\mathcal{A}$ is an improper orthogonal matrix.

Problem 2 (NL Problem 3.6)

> The translational displacement of a body can be represented by a vector and the linear velocity is the time derivative of the position vector. The angular velocity vector, however, is not, in general, the time derivative of an angular displacement vector. In order to prove this, note that if there exists such a vector $\mathbf{\Omega}$, its components Ω_x, Ω_y, Ω_z can be expressed in terms of the Euler angles and must be such that the time derivatives are equal the corresponding components of the angular velocity:
>
> $$\omega_x = \frac{\partial \Omega_x}{\partial \theta}\dot{\theta} + \frac{\partial \Omega_x}{\partial \phi}\dot{\phi} + \frac{\partial \Omega_x}{\partial \psi}\dot{\psi}, \quad \omega_y = \frac{\partial \Omega_y}{\partial \theta}\dot{\theta} + \frac{\partial \Omega_y}{\partial \phi}\dot{\phi} + \frac{\partial \Omega_y}{\partial \psi}\dot{\psi}$$
> $$\omega_z = \frac{\partial \Omega_z}{\partial \theta}\dot{\theta} + \frac{\partial \Omega_z}{\partial \phi}\dot{\phi} + \frac{\partial \Omega_z}{\partial \psi}\dot{\psi}$$
>
> Using (3.86), prove that there can be no vector $(\Omega_x, \Omega_y, \Omega_z)$ such that these equations are satisfied.

[Solution]
From equation (3.86) in NL, we have

$$\begin{aligned} &\omega_x = \cos\phi\dot{\theta} + \sin\theta\cos\phi\dot{\psi}, \quad \omega_y = \sin\phi\dot{\theta} - \sin\theta\cos\phi\dot{\psi} \\ &\omega_z = \dot{\phi} + \cos\theta\dot{\psi} \end{aligned} \tag{6}$$

Comparing the corresponding components of the angular velocity with the set of equations given above, we get some sort of a complicated collection of partial derivatives.

$$\begin{aligned}
&\frac{\partial \Omega_x}{\partial \theta} = \cos\phi, \quad \frac{\partial \Omega_x}{\partial \phi} = 0, \quad \frac{\partial \Omega_x}{\partial \psi} = \sin\theta\cos\phi \\
&\frac{\partial \Omega_y}{\partial \theta} = \sin\phi, \quad \frac{\partial \Omega_y}{\partial \phi} = 0, \quad \frac{\partial \Omega_y}{\partial \psi} = -\sin\theta\cos\phi \\
&\frac{\partial \Omega_z}{\partial \theta} = 0, \quad \frac{\partial \Omega_z}{\partial \phi} = 1, \quad \frac{\partial \Omega_z}{\partial \psi} = \cos\theta
\end{aligned} \tag{7}$$

Solving for Ω_x, Ω_y, and Ω_z, we can expose the inconsistency because

$$\begin{aligned}
&\theta\cos\phi \neq \phi_0 \neq \psi\sin\theta\cos\phi \quad (\text{Solving for } \Omega_x) \\
&\theta\sin\phi \neq \phi_0 \neq -\psi\sin\theta\cos\phi \quad (\text{Solving for } \Omega_y) \\
&\theta_0 \neq \phi \neq \psi\cos\theta \quad (\text{Solving for } \Omega_z)
\end{aligned} \tag{8}$$

where θ_0 and ϕ_0 are just some constant values. Therefore, we conclude that there is no vector $(\Omega_x, \Omega_y, \Omega_z)$ such that the given equations are satisfied.

Problem 3 (NL Problem 3.9)

> A particle is fired vertically upward from the surface of the Earth with initial velocity v_0, reaches its maximum height and returns to the ground. Show that the Coriolis deflection when it hits the ground has the opposite sense and is four times bigger than the deviation for a particle dropped from the same maximum height.

[Solution]

The Coriolis force is given by

$$F_c = 2m\mathbf{v} \times \omega. \tag{9}$$

If the x-axis is pointing north, we have the angular velocity to be

$$\boldsymbol{\omega} = \omega\cos\lambda\hat{x} + \omega\sin\lambda\hat{z}. \tag{10}$$

For a particle initially fired upwards from the surface of the Earth, we have the Coriolis force to follow the equation

$$F_c = 2mv_z\hat{z} \times (\omega\cos\lambda\hat{x} + \omega\sin\lambda\hat{z}) = 2m\omega v_z\cos\lambda\hat{y} \tag{11}$$

after evaluating the cross product. Therefore, the acceleration in the y-direction is

$$\ddot{y}(t) = 2\omega v_z\cos\lambda. \tag{12}$$

Looking closely at the vertical direction of the movement of the particle, it is intuitive that $z(t) = z_0 + v_{0,z}t - \frac{1}{2}gt^2$ and $v_z(t) = v_{0,z} - gt$. Solving for the y displacement and plugging in these values, we have

$$y(t) = y_0 + v_{0,y}t + 2\omega\cos\lambda\left(\frac{1}{2}v_{0,z}t^2 - \frac{1}{6}gt^3\right). \tag{13}$$

We can also use our values for z-displacement and velocity to solve for (1) the time it takes for the particle to reach its peak, and (2) the time it takes to return to the surface of the Earth. We find that

$$\begin{aligned} t &= \frac{v_{0,z}}{g} \quad \text{with} \quad z_{peak} = \frac{v_{0,z}^2}{2g} \quad \text{(Time it takes to reach the peak)} \\ t &= \frac{2v_{0,z}}{g} \quad \text{(Time it takes to return to the surface)}, \end{aligned} \tag{14}$$

which we solve by solving for the critical value of time then plugging in to the displacement equation for the maximum height, and by solving for the time t for the particle to return to the surface. Therefore, we have the total horizontal deflection to be

$$\Delta y = 2\omega\cos\lambda\left(\frac{1}{2}v_{0,z}\left(\frac{2v_{0,z}}{g}\right)^2 - \frac{1}{6}g\left(\frac{2v_{0,z}}{g}\right)^3\right) = \frac{4}{3}\omega\cos\lambda\frac{v_{0,z}^3}{g^2} \tag{15}$$

which can be written in terms of the maximum height z_{peak}

$$\Delta y = \frac{8\sqrt{2}}{3}\omega\cos\lambda z_{peak}\sqrt{\frac{z_{peak}}{g}} \tag{16}$$

wherein we plugged in the the initial values $y_0 = 0$ and $v_{0,y} = 0$ along with the time it takes to return to the surface. Meanwhile, for a particle dropped from the same height z_{peak}, we have the time to reach the surface of the Earth to be $t = \sqrt{2z_{peak}/g}$. Its equation of motion for y is given by

$$y(t) = y_0 + v_{0,y}t - \frac{1}{3}\omega\cos\lambda gt^3. \tag{17}$$

Notice that this is just Eqn. (13) that has $v_{0,z}$ set to zero. Again, this is because of the fact that there wouldn't be any initial z velocity because the particle is just dropped from z_{peak} to the surface of the Earth. We can then plug in the time needed for the particle to reach the surface with initial conditions $y_0 = 0$ and $v_{0,y} = 0$, to solve for the horizontal deflection of the particle.

$$\Delta y = -\frac{1}{3}\omega\cos\lambda g\left(\frac{2z_{peak}}{g}\right)^{3/2} = -\frac{2\sqrt{2}}{3}\omega\cos\lambda z_{peak}\sqrt{\frac{z_{peak}}{g}}. \tag{18}$$

Comparing Eqns. (16) and (18), we find that the Coriolis deflection for the particle from the surface has the opposite sense and is 4 times in magnitude of the Coriolis deflection for the particle dropped from z_{peak} reaching the surface of the Earth.

Problem 4 (NL Problem 3.11)

At $t = 0$ a projectile is fired horizontally near the surface of the Earth with speed v. (a) Neglecting gravity, show that, to a good approximation, the horizontal deviation of the projectile due to the Coriolis force is

$$d_H = \omega v|\sin\lambda|t^2$$

where λ is the latitude. (b) Show that the projectile is deflected to the right in the northern hemisphere and to the left in the southern hemisphere. (c) To the same degree with approximation, express d_H in terms of the projectile's distance D from the firing point at time t. (d) How would the inclusion of the influence of gravity affect the result? (e) What changes if the projectile is fired at an angle above the horizontal? (f) It is told that during a World War I naval battle in the south Atlantic British shells missed German ships by about 90 meters to the left because the British gunsights had been corrected to a latitude of 50 deg north instead of 50 deg south where the battle took place. Conclude that the British shells would have missed the targets by twice the deflection calculated above. (g) Assuming the tale is true and $v = 700m/s$, bow distant must the German ships have been?

[Solution]

(a) Similar to number 3, the Coriolis force is given by

$$F_c = 2m\mathbf{v} \times \omega. \tag{19}$$

If the x-axis is pointing south and the y-axis is pointing to the east, we have the angular velocity to be

$$\boldsymbol{\omega} = -\omega\cos\lambda\hat{x} + \omega\sin\lambda\hat{z}. \tag{20}$$

Considering that the projectile is fired horizontally (x-axis) and that the trajectory is influenced by gravity, we then have $\mathbf{v} = \mp v_x\hat{x} - gt\hat{z}$, where v_x is negative if the projectile is fired to the northern hemisphere and positive when fired to the southern hemisphere. Then the Coriolis force simplifies to

$$F_c = 2m(\pm v_x\hat{x} - gt\hat{z}) \times (-\omega\cos\lambda\hat{x} + \omega\sin\lambda\hat{z}) = 2m\omega(\pm v_x\sin\lambda + gt\cos\lambda)\hat{y} \tag{21}$$

Therefore, the acceleration of the deflection in the y direction is

$$\ddot{y}(t) = 2\omega(\pm v_x\sin\lambda + gt\cos\lambda)\hat{y} \tag{22}$$

Solving for the horizontal deflection

$$d_H = y(t) = \omega\left(\pm v\sin\lambda t^2 + \frac{1}{3}g\cos\lambda t^3\right) \tag{23}$$

where the initial values $y_0 = 0$ and $v_{0,y} = 0$, and the speed $v_x = v$ has been plugged in.

Neglecting gravity we get the desired result

$$d_H = v\omega|\sin\lambda|t^2 \tag{24}$$

(b) When the projectile is set to move in the negative x-direction, we have the angle λ to have a positive value with respect to the equator if evaluating in the northern hemisphere. Meanwhile, when the projectile is fired in the southern hemisphere, we have the angle λ to have a negative value with respect to the equator. Therefore,

$$\begin{aligned} d_H &= v\omega\sin(\lambda)t^2 = v\omega\sin\lambda t^2 \quad \text{(for projectiles fired in the NH)} \\ d_H &= v\omega\sin(-\lambda)t^2 = -v\omega\sin\lambda t^2 \quad \text{(for projectiles fired in the SH)} \end{aligned} \tag{25}$$

Since we set our positive y axis pointing to the east, we have projectiles fired to the NH to be deflected to the east (right direction) and projectiles fired to the SH to be deflected to the west (left direction).

(c) We can say that the distance travelled D can be written as $D = D(t) = v_x t$ when gravity and air resistance is neglected. Therefore, the transverse deflection due to the Coriolis force can be written as

$$d_H = \frac{D(t)^2}{v}\omega|\sin\lambda| \quad \text{where} \quad D(t) = vt \tag{26}$$

(d) Considering gravity we have the deflection displacement to be

$$d_H = y(t) = v\omega\sin\lambda t^2 + \frac{1}{3}g\omega\cos\lambda t^3 \tag{27}$$

Near the surface of the Earth $v \approx gt$, so it follows that $vt^2 > (1/3)gt^3$. So for a projectile moving northward in the NH, we have the angle λ to be positive. Therefore, d_H remains positive. This means that the projectile will move from the west to the east. Meanwhile, for a projectile moving northward in the SH, we have λ to be negative. Therefore the $\sin\lambda$ term dominates and results in a negative value for d_H. This means that the projectile will move form the east to the west.

(e) The velocity vector changes to

$$\mathbf{v} = v\cos\theta\hat{x} + (v\sin\theta - gt)\hat{z} \tag{28}$$

The Coriolis force then changes into

$$\begin{aligned} F_c &= 2m(\mp v\cos\theta\hat{x} + (v\sin\theta - gt)\hat{z}) \times (-\omega\cos\lambda\hat{x} + \omega\sin\lambda\hat{z}) \\ &= 2m[\pm v\omega\sin\lambda\cos\theta - \omega\cos\lambda(v\sin\theta - gt)]\hat{y} \end{aligned} \tag{29}$$

Therefore the transverse deflection changes as well. We have

$$\ddot{y}(t) = 2[\pm v\omega\sin\lambda\cos\theta - \omega\cos\lambda(v\sin\theta - gt)] \tag{30}$$

Therefore, we have

$$d_H = y(t) = \pm\frac{1}{2}\omega v\sin\lambda\cos\theta t^2 - \frac{1}{2}\omega v\cos\lambda\sin\theta + \frac{1}{3}\omega g\cos\lambda t^3 \tag{31}$$

(f) We could take the magnitude of the ratio between the resulting d_H values for the $\lambda = -50\,\text{deg}$ and $\lambda = +50\,\text{deg}$ angles. In this calculation we again consider the approximation $v \approx gt$ for projectiles near the surface of the Earth

$$\begin{aligned}\frac{|d_H(50\,\text{deg}\,N)|}{|d_H(50\,\text{deg}\,S)|} &= \frac{|v\omega\sin(50\,\text{deg})t^2 + \frac{1}{3}v\omega\cos(50\,\text{deg})t^2|}{|v\omega\sin(-50\,\text{deg})t^2 + \frac{1}{3}v\omega\cos(-50\,\text{deg})t^2|} \\ &= \frac{|\sin(50\,\text{deg}) + \frac{1}{3}\cos(50\,\text{deg})|}{|\sin(-50\,\text{deg}) + \frac{1}{3}\cos(-50\,\text{deg})|} \approx 2\end{aligned} \tag{32}$$

Therefore the deflection is almost twice in magnitude of the calculation originally prescribed in the question.

(g) For this question, we consider the equation

$$90 = v\omega\sin\lambda t^2 + \frac{1}{3}g\omega\cos\lambda t^3 \tag{33}$$

Once again taking advantage of the approximation $v \approx gt$ and writing d_H in terms of the distance travelled D, we end up with

$$90\text{m} = \frac{D^2}{v}\omega\sin\lambda + \frac{D^2}{3v}\omega\cos\lambda \tag{34}$$

Solving for D we have

$$D = \sqrt{\frac{v90\text{m}}{\omega(\frac{\sin\lambda+\cos\lambda}{3})}} \tag{35}$$

Plugging the following values $7.3 \times 10^{-5} rad/s$, $\lambda = 50\,\text{deg}$, and $v = 700\text{m/s}$, we end up with

$$D = 3919.82\text{m} \tag{36}$$

which we evaluated using Mathematica as shown in the figure below. Note that we had to convert radians to degrees which we have shown with the additional conversion factor in the denominator.

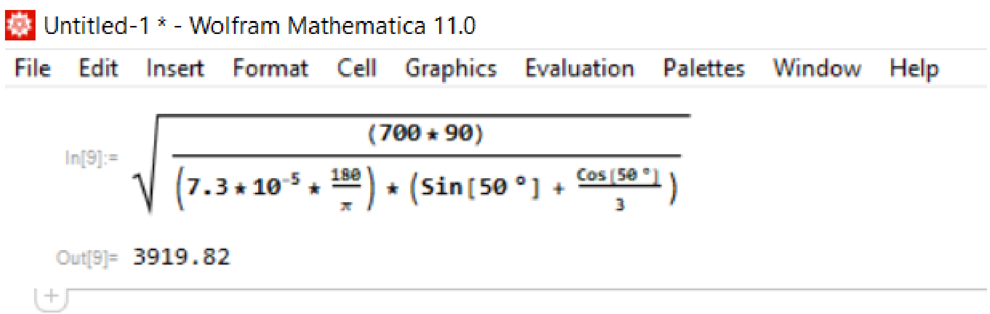

Figure 4: Mathematica computation of 4(g)