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Physics 231 PS # 3

1.) a.) Since the charged density at constant velocity is given by

$$\sigma = \frac{k}{\sqrt{R^2 - \rho^2}}$$

the total charge would be

$$Q = \int_0^{2\pi} d\phi \int_0^R \frac{k\, \rho d\rho}{\sqrt{R^2-\rho^2}} = k\,(2\pi) \int_0^R \frac{\rho d\rho}{\sqrt{R^2-\rho^2}}$$

let $u = R^2 - \rho^2$
$du = -2\rho d\rho$

$$Q = -\pi k \int_{R^2}^0 \frac{du}{u^{1/2}} = -2\pi k \left[u^{1/2}\right]_{R^2}^0$$

$$Q = 2\pi k R$$

therefore the charge density is

$$\sigma = \frac{Q}{2\pi R\sqrt{R^2-\rho^2}}$$

then the potential can be evaluated by the integral

$$\Phi(\rho, z) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^R \frac{Q}{2\pi R\sqrt{R^2-\rho^2}} \frac{\rho d\rho}{\sqrt{z^2+\rho^2}}$$

from

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x')}{|x-x'|} d^3x'$$

evaluating the integral

$$\Phi(\rho, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R} (2\pi) \int_0^R \frac{\rho d\rho}{\sqrt{R^2-\rho^2}\sqrt{z^2+\rho^2}}$$

let $u = \sqrt{R^2-\rho^2} \longrightarrow \rho^2 = R^2 - u^2$

$$du = \frac{-\rho}{\sqrt{R^2-\rho^2}} d\rho$$

$$\Phi(z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R} (2\pi) \int_R^0 \frac{du}{\sqrt{z^2+R^2-u^2}}$$

$$\Phi(z) = \frac{Q}{4\pi\epsilon_0 R} \int_0^R \frac{du}{\sqrt{z^2+R^2-u^2}}$$

$$\Phi(z) = \frac{Q}{4\pi\epsilon_0 R} \left[\tan^{-1}\left(\frac{u}{\sqrt{R^2+z^2-u^2}}\right)\right]_0^R$$

$$\Phi(z) = \frac{Q}{4\pi\epsilon_0 R} \tan^{-1}\left(\frac{R}{z}\right)$$

when $\Phi(0) = V$, we have

$$V = \frac{Q}{4\pi\epsilon_0 R}\left(\frac{\pi}{2}\right) = \frac{Q}{8\epsilon_0 R}$$

a.) and we end up with

$$Q = 8\epsilon_0 R V$$

so we can write our potential as

$$\Phi(z) = \frac{2V}{\pi} \tan^{-1}\left(\frac{R}{z}\right)$$

applying taylor series expansion on the $\tan^{-1}$, we get

$$\Phi(z) = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{z}\right)^{2l+1}$$

we can compare this with the general solution to problems w/ azimuthal symmetry

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l \cos\theta$$

at $\theta = 0$ this becomes

$$\Phi(r,0) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right]$$, $A_l \to 0$ since r^l diverges at $r \to \infty$

comparing with our solution, we get a general solution

$$\Phi(r,\theta) = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{r}\right)^{2l+1} P_{2l} \cos\theta$$

b.) For $r < R$, we can use the trigonometric identity

$$\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}(x)$$

or

$$\tan^{-1}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right)$$

therefore the potential turns out to be

$$\Phi(z) = \frac{2V}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{z}{R}\right)\right]$$

applying taylor series expansion

$$\Phi(z) = \frac{2V}{\pi}\left[\frac{\pi}{2} - \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{z}{R}\right)^{2l+1}\right]$$

doing the same process with 1a we then have general solution

$$\Phi(r,\theta) = V - \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{r}{R}\right)^{2l+1} P_{2l+1} \cos\theta$$

where $B_l \to 0$ since $r^{-(l+1)}$ diverges at $r \to 0$.

c.) We have the formula for capacitance

$$C = \frac{Q}{V}$$

we have the total charge density in terms of V $Q = 8\epsilon_0 R V$. Plugging in, we have

$$C = \frac{8\epsilon_0 R V}{V} = 8\epsilon_0 R$$

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2. We can show those 2 forms of solution are equal by deriving those 2 equations using different methods. First, we consider solving this by taking the Green's function method. We have

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' + \frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\frac{\partial \Phi}{\partial x'} - \frac{\partial G}{\partial x'}\Phi \right] da'$$

Since we have defined value for $\Phi(\vec{x})$, we force that $\partial\Phi(\vec{x})/\partial x' \to 0$. The potential would be

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial G}{\partial x'}\Phi(\vec{x})\, da'$$

Since we have the a hollow sphere there is no charge density. Therefore,

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\oint_S \frac{\partial G}{\partial x'}\Phi(\vec{x})\, da' = -\frac{1}{4\pi}\oint_S \frac{\partial G}{\partial x'}\Phi(\vec{x})\, a^2\, d\Omega'$$

Since at $r=a$ (surface), we have $\Phi = V(\theta,\phi)$, we are left to solve for the directional derivative $\partial G/\partial n'$. Recall that the Green's function is given by

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$$

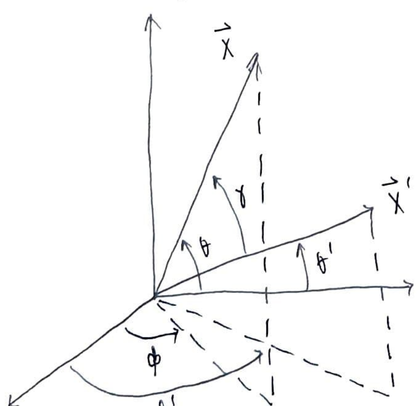
where

$$F(\vec{x},\vec{x}') = -\frac{a}{x'\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right|} = -\frac{1}{x'\left|\frac{\vec{x}}{a} - \frac{a}{x'^2}\vec{x}'\right|}$$

where

$$|\vec{x}-\vec{x}'| = \sqrt{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')}$$

to solve for this consider the figure



$$\vec{x} = x\sin\theta\cos\phi\,\hat{i} + x\sin\theta\sin\phi\,\hat{j} + x\cos\theta\,\hat{k}$$

$$\vec{x}' = x'\sin\theta'\cos\phi'\,\hat{i} + x'\sin\theta'\sin\phi'\,\hat{j} + x'\cos\theta'\,\hat{k}$$

$$\vec{x}-\vec{x}' = (x\sin\theta\cos\phi - x'\sin\theta'\cos\phi')\hat{i} + (x\sin\theta\sin\phi - x'\sin\theta'\sin\phi')\hat{j} + (x\cos\theta - x'\cos\theta')\hat{k}$$

$$|\vec{x}-\vec{x}'| = (x^2 + x'^2 - 2xx'\sin\theta\sin\theta'\cos(\phi-\phi') - 2xx'\cos\theta\cos\theta')^{1/2} = \sqrt{x^2+x'^2-2xx'\cos\gamma}$$

where : $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

then it follows

$$\left|\vec{x} - \frac{a^2}{x'^2}\vec{x}'\right| = \sqrt{\frac{x^2+x'^2-2a^2\vec{x}'\cdot\vec{x}}{x'^2}} = \frac{a}{x'}\sqrt{\frac{x'^2x^2}{a^2} - 2xx'\cos\gamma + a^2}$$

the green's function becomes

$$G(\vec{x},\vec{x}') = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}} - \frac{1}{\sqrt{\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma}}$$

2.) With this the directional derivative

$$\frac{\partial G}{\partial x'}\Big|_{x'=a} = \left[-\frac{1}{2} \frac{2x' - 2x\cos\gamma}{(x^2+x'^2-2xx'\cos\gamma)^{3/2}} + \frac{1}{2} \frac{\frac{2x'}{a^2}x^2 - 2x\cos\gamma}{(\frac{x'^2}{a^2}x^2+a^2-2xx'\cos\gamma)^{3/2}} \right]_{x'=a}$$

$$\frac{\partial G}{\partial y'}\Big|_{x'=a} = \left[-\frac{1}{2} \frac{2a - 2x\cos\gamma}{(x^2+a^2-2ax\cos\gamma)^{3/2}} + \frac{1}{2} \frac{\frac{2x^2}{a} - 2x\cos\gamma}{(x^2+a^2-2ax\cos\gamma)^{3/2}} \right]$$

$$\frac{\partial G}{\partial x'}\Big|_{x'=a} = \frac{x^2-a^2}{a(x^2+a^2-2ax\cos\gamma)^{3/2}}$$

for the internal problem. The potential is then

$$\Phi(\vec{x}) = \frac{a(a^2-x^2)}{4\pi} \oint_S \frac{V(\theta',\phi')}{(x^2+a^2-2xa\cos\gamma)^{3/2}} d\Omega'$$

We can also change $x \to r$ to finally get

$$\Phi(\vec{x}) = \frac{a(a^2-r^2)}{4\pi} \oint_S \frac{V(\theta',\phi')}{(r^2+a^2-2ra\cos\gamma)^{3/2}} d\Omega'$$

We can also solve through Laplace eq by starting with the potential in terms of spherical harmonics

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (A_{l,m} r^l + B_{l,m} r^{-(l+1)}) Y_{lm}(\theta,\phi)$$

Since its region of validity is inside the sphere (including $r \to 0$), $B_{l,m}$ must be 0 since at $r \to 0$, the 2nd term will diverge.

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m} r^l Y_{lm}(\theta,\phi)$$

applying the boundary condition at the surface $r=a$

$$V(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m} a^l Y_{lm}(\theta,\phi)$$

We can multiply

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{l'm'}(\theta,\phi) V(\theta,\phi) \sin\theta \, d\theta d\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{l,m} a^l \int_0^{2\pi}\int_0^{\pi} Y^*_{l',m'}(\theta,\phi) Y_{lm}(\theta,\phi) \sin\theta d\theta d\phi$$

Considering the normalization and orthogonality condition

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{l',m'}(\theta',\phi') Y_{l,m}(\theta,\phi) \sin\theta \, d\theta d\phi = \delta_{l'l}\delta_{m'm}$$

We get the value for $A_{l,m}$

$$A_{l,m} = a^{-l} \int_0^{2\pi}\int_0^{\pi} Y^*_{l,m'}(\theta,\phi) V(\theta,\phi) \sin\theta \, d\theta \, d\phi$$

therefore,

$$\Phi(\vec{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} \left(\frac{r}{a}\right)^l Y_{lm}(\theta,\phi) \quad ; \quad A_{lm} = \int d\Omega' \, Y^*_{lm}(\theta',\phi') V(\theta',\phi')$$

by plugging $A_{l,m}$ into the potential

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3.) a.) Laplace's equation in cylindrical coordinates yield

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{r}\frac{\partial \Phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Since we're dealing with a disc fitted inside and infinite plain we have symmetry wrt to ϕ. Therefore, the above equation simplifies into

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

using the anzatz solution $\Phi(r,z) = R(r)\,Z(z)$. Therefore,

$$Z(z)\frac{\partial^2 R(r)}{\partial r^2} + \frac{Z(z)}{r}\frac{\partial R(r)}{\partial r} + R(r)\frac{\partial^2 Z(z)}{\partial z} = 0$$

dividing both sides by $\Phi(r,z)$.

$$\underbrace{\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{R(r)r}\frac{\partial R(r)}{\partial r}}_{\text{dependent only on } r} + \underbrace{\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z}}_{\text{dependent only on } z} = 0$$

we can let the first 2 terms equal $-\lambda^2$

$$\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = \lambda^2 \longrightarrow \frac{\partial^2 Z(z)}{\partial z^2} = \lambda^2 Z(z)$$

therefore

$$Z(z) = e^{-\lambda z}$$

with these the Laplace's equation can simplify into

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \lambda^2 Z(z) = 0$$

we can make the substitution $\eta = \lambda r$ so we can rewrite our equation in such a way that it is in the form of a Bessel eqn.

$$\frac{\partial^2 R(\eta)}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial R(\eta)}{\partial \eta} + R = 0$$

which has the solution $R(r) = J_0(\lambda r)$. Now that we have the solution for $R(r)$ and $Z(z)$ we can solve for $\Phi(r,\phi,z)$

$$\Phi(r,\phi,z) = \sum_{\lambda} A_\lambda e^{-\lambda z} J_0(\lambda r)$$

where the summation comes into play because we want the solution to be the linear combination of all solutions with coefficients A_λ. This can be written as

$$\Phi(r,\phi,z) = \int_0^\infty A(\lambda)\, e^{-\lambda z} J_0(\lambda r)\, d\lambda$$

Where we wrote the coefficients A_λ as a function $A(\lambda)$.

3.1 a) Since we have the disc to have fixed potential, we have a boundary condition at $z=0$ or at the surface of the disc

$$V(r) = \Phi(r, z=0)$$

or

$$V(r) = \int_0^\infty A(\lambda)\, J_0(\lambda r)\, d\lambda$$

multiplying both sides of the equation by $r\, J_0(\lambda' r)$ and integrating wrt to r.

$$\int_0^\infty V(r)\, r\, J_0(\lambda' r)\, dr = \int_0^\infty \int_0^\infty A(\lambda)\, r\, J_0(\lambda r)\, J_0(\lambda' r)\, d\lambda\, dr$$

the potential is fixed at the disc so we can rewrite this equation as

$$V \int_0^a r\, J_0(\lambda' r)\, dr = \int_0^\infty A(\lambda) \underbrace{\int_0^\infty r\, J_0(\lambda r)\, J_0(\lambda' r)\, dr}_{\text{can be simplified by Jackson 3.108}}\, d\lambda$$

$$V \int_0^a r\, J_0(\lambda' r)\, dr = \int_0^\infty A(\lambda)\, \frac{\delta(\lambda' - \lambda)}{\lambda}\, d\lambda$$

the RHS can be evaluated by the property of dirac deltas

$$V \int_0^a r\, J_0(\lambda' r)\, dr = \frac{A(\lambda')}{\lambda'}$$

therefore $A(\lambda)$ can be written as

$$A(\lambda') = V \lambda' \int_0^a r\, J_0(\lambda' r)\, dr$$

or

$$A(\lambda) = V\lambda \int_0^a r\, J_0(\lambda r)\, dr = V\lambda\, J_1(\lambda a)$$

Therefore,

$$\Phi(r, \phi, z) = V \int_0^\infty \int_0^a \lambda r' J_0(\lambda r')\, dr'\, e^{-\lambda z}\, J_1(\lambda r)\, d\lambda$$

using the recursion formula

$$\int x^n J_{n-1}(x)\, dx = x^n J_n(x) + C \quad \text{for } n \geq 1$$

we have

$$\int_0^a r' J_0(\lambda r')\, dr' = \frac{a}{\lambda} J_1(\lambda a) \quad \text{where } x = \lambda r' \text{ in this case}$$

Therefore,

$$\Phi(r, \phi, z) = V \int_0^\infty \lambda \frac{a}{\lambda} J_1(\lambda a)\, e^{-\lambda z}\, J_0(\lambda r)\, d\lambda$$

$$\Phi(r, \phi, z) = Va \int_0^\infty J_1(\lambda a)\, J_0(\lambda r)\, e^{-\lambda z}\, d\lambda$$

3) b) Coming from the potential above

$$\Phi(r,z) = V\int_0^\infty \int_0^a \lambda r' J_0(\lambda r')\,dr'\, e^{-\lambda z} J_0(\lambda r)\,d\lambda$$

we can simplify this by using the identity

$$\frac{1}{\sqrt{r^2+z^2}} = \int_0^\infty e^{-\lambda z} J_0(\lambda r)\,d\lambda$$

or

$$\frac{1}{\sqrt{r'^2+z^2}} = \int_0^\infty e^{-\lambda z} J_0(\lambda r')\,d\lambda$$

Also notice that the potential can be rewritten as

$$\Phi(r=0,z) = V\int_0^\infty \int_0^a \lambda r' J_0(\lambda r')\,dr'\, e^{-\lambda z}\,d\lambda$$

when $r=0$. We set this since we are solving for the potential directly on top the center of the disc.

Notice that the potential can be rewritten as

$$\Phi(z) = V\int_0^a r'\left(-\frac{d}{dz}\int_0^\infty J_0(\lambda r')\, e^{-\lambda z}\,d\lambda\right) dr'$$

We can then use our above identity plugging it directly to our potential.

$$\Phi(z) = V\int_0^a -\frac{d}{dz}\left(\frac{1}{\sqrt{r'^2+z^2}}\right) r'\,dr'$$

evaluating the derivative and simplifying

$$\Phi(z) = V\int_0^a -\left(-\frac{1}{2}\frac{2z}{(r'^2+z^2)^{3/2}}\right) r'\,dr'$$

$$\Phi(z) = V\int_0^a \frac{r'z}{(r'^2+z^2)^{3/2}}\,dr'$$

$$\Phi(z) = Vz\int_0^a \frac{r'}{(r'^2+z^2)^{3/2}}\,dr'$$

let $u = r'^2 + z^2$

$du = 2r'\,dr$

$$\Phi(z) = \frac{Vz}{2}\int_{z^2}^{a^2+z^2} \frac{du}{u^{3/2}} = \frac{Vz}{2}\left[-2\frac{1}{u^{1/2}}\right]_{z^2}^{a^2+z^2}$$

$$\Phi(z) = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

3.) c.) At a perpendicular distance of z above the edge of the disc, we solve the potential by considering

$$\Phi(r,\phi,z) = Va\int_0^\infty J_1(\lambda a)J_0(\lambda r)e^{-\lambda z}\,d\lambda$$

at the edge of the disc we have $r=a$

$$\Phi(a,z) = Va\int_0^\infty J_1(\lambda a)\,J_0(\lambda a)\,e^{-\lambda z}\,d\lambda$$

using a table of integrals we have

$$\int_0^\infty e^{-px}J_1(ax)J_0(bx)\,dx = -\frac{pk}{2\pi a^2}K(k) + \frac{1}{2a} \quad ; \quad a=b$$

applying to our potential, we have $p=z$, $a=b=a$, $x=\lambda$. Therefore

$$\int_0^\infty e^{-z\lambda}J_1(a\lambda)\,J_0(a\lambda)\,d\lambda = -\frac{z\lambda}{2\pi a^2}K(k) + \frac{1}{2a}$$

plugging into our potential

$$\Phi(a,z) = Va\left(\frac{1}{2a} - \frac{z\lambda}{2\pi a^2}K(k)\right)$$

$$\Phi(a,z) = \frac{V}{2}\left(1 - \frac{z\lambda}{\pi a}K(k)\right)$$

where k is originally

$$k^2 = \frac{4ab}{p^2+(a+b)^2}$$

since we have $p=z$, $a=b=a$, $x=\lambda$, we have

$$k^2 = \frac{4a^2}{z^2+4a^2} \Rightarrow k = \frac{2a}{\sqrt{z^2+4a^2}}$$

and $K(k)$ is the complete elliptical integral of the first kind

$$K(k) \cong \int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-k^2\sin\alpha}}$$

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4.) a) We have Green's reciprocation theorem states that

$$\int_V \rho' \Phi \, d^3x + \int_S \sigma' \Phi \, da = \int_V \rho \Phi' \, d^3x + \int_S \sigma \Phi' \, da$$

For this problem we consider the given system for problem 3.18 and 3.19. Let us say the the unprimed variables are from 3.18 and primed are for 3.19.

For 3.18 we have a conducting plane at $z=0$ held at zero potential and another conducting plane at $z=L$ with a disc insert on it. The volume charge density is then

$$\rho(r,z) = 0$$

and the potentials would be

$$\Phi(r,z) = \begin{cases} 0 & , \text{ at } z=0 \\ 0 & , \text{ at } z=L, \; r>a \\ V & , \text{ at } z=L, \; r<a \\ V\int_0^\infty d\lambda \, J_1(a\lambda) J_0(\lambda r) \, a \dfrac{\sinh(\lambda z)}{\sinh(\lambda L)} & , \text{ at } 0<z<L \end{cases}$$

with an unknown surface charge density.

Mean while, for 3.19, we are given a charge q between two infinite conducting planes held at zero potential. Therefore, the volume charge density is

$$\rho'(r,z) = q\,\delta(r-0)\,\delta(z-z_0) = q\,\delta(r)\,\delta(z-z_0)$$

and the potential would be

$$\Phi'(r,z) = \begin{cases} 0 & , \text{ at } z=0,L \\ \text{unknown}, & 0<z<L \end{cases}$$

with the surface charge density set to be unknown.

We can then plug this in to Green's reciprocation theorem

$$qV\int_0^\infty d\lambda \, a \, J_1(a\lambda) \frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}$$

$$+ V\int_{z=L,\, r<a} \sigma'(r,z)\, da = 0$$

rewriting, we have

$$\int_{z=L,\, r<a} \sigma'(r,z)\, da = -q\int_0^\infty d\lambda \, a \, J_1(a\lambda) \frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}$$

A.) a.) Notice that the RHS follows the form of

$$\Phi(r,z) = V\int_0^\infty d\lambda\, J_1(\lambda)\, J_0(\lambda r/a)\, \sinh(\lambda z/a)/\sinh(\lambda L/a)$$

for $r=0$ and $z=z_0$. Therefore our equation reduces into

$$\int_{z=L,\, r<a} \sigma'(r,z)\, da = -\frac{q}{V}\Phi(0,z_0)$$

therefore, the total induced charge for the plane at $z=L$ inside a disc with radius r is

$$Q_L(a) = -\frac{q}{V}\Phi(0,z_0)$$

b.) Considering the equation

$$\int \sigma'(r,z)\, da = -q\int_0^\infty d\lambda\, a J_1(a\lambda)\frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}$$

and evaluating the integral on the LHS

$$\int_0^{2\pi} d\theta \int_0^a \sigma'(r,z)\, r\,dr = -q\int_0^\infty d\lambda\, a J_1(a\lambda)\frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}$$

$$2\pi\int_0^a \sigma'(r,z)\, r\,dr = -q\int_0^\infty d\lambda\, a J_1(a\lambda)\frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}$$

we can proceed with differentiating both sides wrt $r=a$

$$2\pi a\,\sigma'(a,L) = -q\int_0^\infty d\lambda\, \frac{\partial}{\partial a}\left(a J_1(a\lambda)\right)\frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}$$

using the identity

$$\frac{\partial}{\partial z}\left(z^\nu J_\nu(z)\right) = z^\nu J_{\nu-1}(z)$$

we can simplify our equation into

$$2\pi a\,\sigma'(a,L) = -q\int_0^\infty d\lambda\, a\lambda\, J_0(a\lambda)\frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}$$

simplifying, we have

$$\sigma'(a,L) = -\frac{q}{2\pi}\int_0^\infty d\lambda\, \lambda J_0(a\lambda)\frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}$$

c.) at $a=0$, we have

$$\sigma'(0,L) = -\frac{q}{2\pi}\int_0^\infty \lambda\frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}\, d\lambda$$

using the integral identity

$$\int_0^\infty \frac{\sinh ax}{\sinh bx}\, dx = \frac{\pi}{2b}\tan\left(\frac{a\pi}{2b}\right)$$

a) (c) we then have

$$\int_0^\infty \frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}\,d\lambda = \frac{\pi}{2L}\tan\left(\frac{z_0\pi}{2L}\right)$$

to evaluate $\int_0^\infty \lambda \frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}\,d\lambda$ we can use feynman's trick under the integral sign.

Differentiate $2k$ times wrt to z_0

$$\frac{\partial^{2k}}{\partial z_0^{2k}}\int_0^\infty \frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}\,d\lambda = \frac{\partial^{2k}}{\partial z_0^{2k}}\left(\frac{\pi}{2L}\tan\left(\frac{z_0\pi}{2L}\right)\right)$$

$$\int_0^\infty \frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}\lambda^{2k}\,d\lambda = \frac{\partial^{2k}}{\partial z_0^{2k}}\left(\frac{\pi}{2L}\tan\left(\frac{z_0\pi}{2L}\right)\right)$$

when $k=1/2$, we have

$$\int_0^\infty \frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}\lambda\,d\lambda = \frac{\partial}{\partial z}\left(\frac{\pi}{2L}\tan\left(\frac{z_0\pi}{2L}\right)\right)$$

$$\int_0^\infty \frac{\sinh(\lambda z_0)}{\sinh(\lambda L)}\lambda\,d\lambda = \left(\frac{\pi}{2L}\right)^2\sec^2\left(\frac{z_0\pi}{2L}\right)$$

plugging this into our surface charge density, we have

$$\sigma'(0,L) = -\frac{q}{2\pi}\left(\frac{\pi}{2L}\right)^2\sec^2\left(\frac{z_0\pi}{2L}\right)$$

or

$$\sigma'(0,L) = \frac{-q\pi}{8L^2}\sec^2\left(\frac{\pi z_0}{2L}\right)$$

using the series expansion

$$\sec^2\left(\frac{\pi y}{2}\right) = \frac{4}{\pi^2}\sum_{k=1}^{\infty}\left[\frac{1}{(2k-1-x)^2}+\frac{1}{(2k-1+x)^2}\right]$$

or

$$\sec^2\left(\frac{\pi x}{2}\right) = \frac{4}{\pi^2}\sum_{n>0,\,odd}^{\infty}\left[\frac{1}{(n-x)^2}+\frac{1}{(n+x)^2}\right]$$

we have

$$\sigma'(0,L) = -\frac{q}{2\pi L^2}\sum_{n>0,\,odd}^{\infty}\left[(n-z_0/L)^{-2}+(n+z_0/L)^{-2}\right]$$