Physics 231
University of the Philippines - Diliman
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Problem Set 1

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1 Problem 1

Use Gauss's theorem to prove the following: (a) Any excess charge placed on a conductor must lie entirely on its surface. (b) A closed hollow conductor shields its interior from the fields due to charges placed outside, but does not shield its exterior due to charges placed inside it. (c) The electric field at the surface of a conductor is normal to the surface and has a magnitude of σ/ϵ_0, where σ is the charge density per unit area on the surface.

(a) We can construct a gaussian surface inside a conductor such that the surface of the gaussian is infinitesimally close to the surface of the conductor.

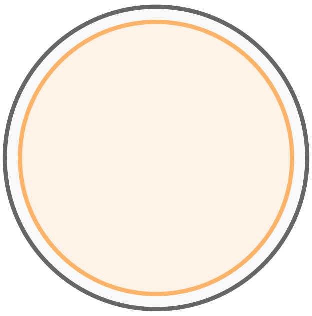

Figure 1: Gaussian surface (orange) inside a conductor (black). The gaussian surface is constructed in such a way that the distance between its boundary and the conductor's surface is practically zero.

Since we wouldn't have an electric field inside a conductor, using Gauss's law, we see that

$$\begin{aligned} \oint_S \vec{E} \cdot da &= \frac{q}{\epsilon_0} \\ (E = 0)A &= \frac{q}{\epsilon_0} \\ q &= 0 \end{aligned} \tag{1}$$

Therefore, the charge would be placed in the conductor's surface. Intuitively we can imagine a charge placed inside a conductor to have a repulsive interaction with the other charges on the system. This results to the charge accelerating inside the conductor until it eventually settles in the surface when the conductor reaches static equilibrium.

(b) We can now then consider a hollow conducting sphere and a charge placed outside of it.

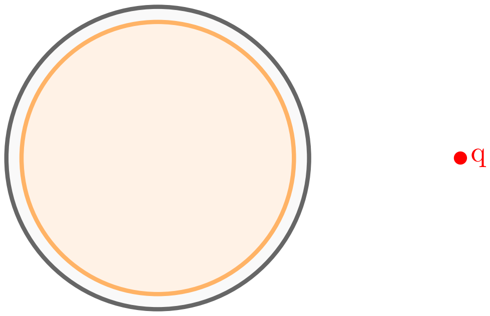

Figure 2: Gaussian surface (orange) inside a hollow conductor (black) and a charge q (red) outside. The gaussian surface is constructed in such a way that the distance between its boundary and the conductor's surface is practically zero.

Since there is no charge inside the gaussian surface, by Gauss's law we have

$$\begin{aligned} \oint_S \vec{E} \cdot da &= 0 \\ E &= 0 \end{aligned} \tag{2}$$

Meanwhile, for a charge placed inside the hollow conductor, we can construct a gaussian surface outside

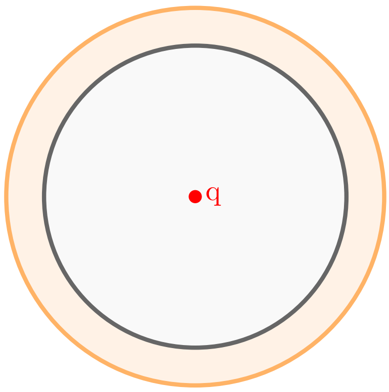

Figure 3: Gaussian surface (orange) outside a hollow conductor (black) and a charge q (red) inside.

Applying Gauss's law we find that the resulting E field would be non zero

$$\begin{aligned} \oint_S \vec{E} \cdot da &= \frac{q}{\epsilon_0} \\ E &\neq 0 \end{aligned} \tag{3}$$

Therefore we can conclude that a closed hollow conductor shields its interior from the fields due to charges placed outside but doesn't shield its exterior from charges placed inside.

(c) consider a surface distribution of a conductor and construct a gaussian surface over it

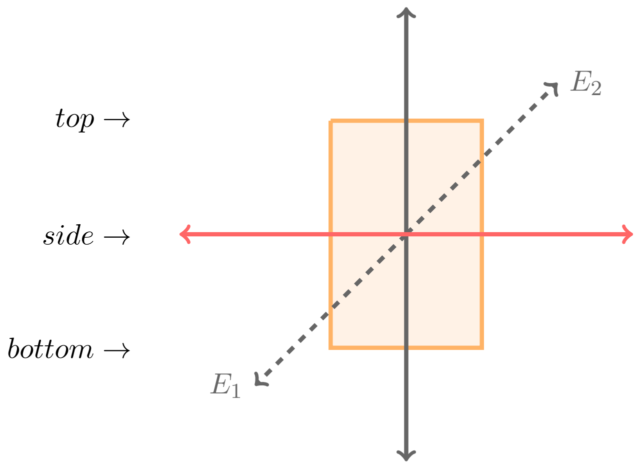

Figure 4: Gaussian surface (orange) outside a hollow conductor (black) and a charge q (red) inside.

Applying Gauss law over the surface

$$\begin{aligned}\int \vec{E} \cdot d\vec{a} &= \frac{Q_{enc}}{\epsilon_0} \\ \int \vec{E} \cdot d\vec{a} &= \int \vec{E} \cdot d\vec{a}_{top} + \int \vec{E} \cdot d\vec{a}_{side} + \int \vec{E} \cdot d\vec{a}_{bottom}\end{aligned} \tag{4}$$

Take the limit as the top and bottom surfaces approach the surface charge

$$\begin{aligned}&\int \vec{E} \cdot d\vec{a}_{side} = 0 \\ &\int \vec{E} \cdot d\vec{a}_{top} = \vec{E}_2 \cdot d\vec{a} = \vec{E}_2 \cdot \hat{n} da = \vec{E_2}^{perp} da \\ &\int \vec{E} \cdot d\vec{a}_{bottom} = -\vec{E}_1 \cdot d\vec{a} = -\vec{E}_1 \cdot \hat{n} da = -\vec{E_1}^{perp} da\end{aligned} \tag{5}$$

where $E_{1^{perp}}$ is the component of $\vec{E_1}$ normal to the surface and E_2^{perp} is the component of $\vec{E_2}$ normal to the surface. We also have $Q_{enc} = \sigma(\hat{x})da$ where σ is the surface charge density. Therefore Gauss's law simplifies into

$$\int \vec{E} \cdot d\vec{a} = E_2^{perp} da - E_1^{perp} da = \frac{\sigma(\hat{x})da}{\epsilon_0} \tag{6}$$

Therefore, we have shown that the electric field at the surface of the conductor is normal to the surface and has a magnitude of σ/ϵ_0.

2 Problem 2

Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three dimensional charge densities $\rho(\mathbf{x})$. (a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R. (b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius b. (c) In cylindrical coordinates, a charge Q spread uniformly over a flat circular disk of negligible thickness and radius R.

(a) In spherical coordinates, we have the following values for x, y, z

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{7}$$

Solving for the scale factors, we have

$$\begin{aligned} h_r^2 &= \left[\left(\frac{\partial}{\partial r} r\sin\theta\cos\phi\right)^2 + \left(\frac{\partial}{\partial r} r\sin\theta\sin\phi\right)^2 + \left(\frac{\partial}{\partial r}\cos\theta\right)^2\right] \\ h_r^2 &= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1 \end{aligned} \tag{8}$$

$$\begin{aligned} h_\theta^2 &= \left[\left(\frac{\partial}{\partial \theta} r\sin\theta\cos\phi\right)^2 + \left(\frac{\partial}{\partial \theta} r\sin\theta\sin\phi\right)^2 + \left(\frac{\partial}{\partial \theta}\cos\theta\right)^2\right] \\ h_\theta^2 &= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2 \end{aligned} \tag{9}$$

and

$$\begin{aligned} h_\phi^2 &= \left[\left(\frac{\partial}{\partial \phi} r\sin\theta\cos\phi\phi\right)^2 + \left(\frac{\partial}{\partial \phi} r\sin\theta\sin\phi\right)^2 + \left(\frac{\partial}{\partial \phi}\cos\theta\right)^2\right] \\ h_\phi^2 &= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta \end{aligned} \tag{10}$$

Therefore,

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta \tag{11}$$

Since in spherical coordinates the θ and ϕ coordinates are symmetric

$$r^2\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta = r^2(2\pi)(2\pi) = 4\pi r^2 \tag{12}$$

Therefore, we have the charge density

$$\rho(r) = \frac{Q}{4\pi R^2}\delta(r-R) \tag{13}$$

(b) In cylindrical coordinates, we have

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned} \tag{14}$$

Solving for the scale factors, we have

$$\begin{aligned} h_r^2 &= \left[\left(\frac{\partial}{\partial r} r\cos\phi\right)^2 + \left(\frac{\partial}{\partial r} r\sin\phi\right)^2 + \left(\frac{\partial}{\partial r} z\right)^2\right] \\ h_r^2 &= \cos^2\phi + \sin^2\phi = 1 \end{aligned} \tag{15}$$

$$\begin{aligned} h_\phi^2 &= \left[\left(\frac{\partial}{\partial \phi} r\cos\phi\right)^2 + \left(\frac{\partial}{\partial \phi} r\sin\phi\right)^2 + \left(\frac{\partial}{\partial \phi} z\right)^2\right] \\ h_\phi^2 &= r^2\sin^2\phi + r^2\cos^2\phi = r^2 \end{aligned} \tag{16}$$

and

$$\begin{aligned} h_z^2 &= \left[\left(\frac{\partial}{\partial \theta} r\cos\phi\right)^2 + \left(\frac{\partial}{\partial \theta} r\sin\phi\right)^2 + \left(\frac{\partial}{\partial \theta} z\right)^2\right] \\ h_z^2 &= 1 \end{aligned} \tag{17}$$

In this system, we have the z and ϕ coordinates to be symmetric. Therefore,

$$r\int_0^1 dz \int_0^{2\pi} d\phi = 2\pi r \tag{18}$$

For a cylindrical surface of radius b, we have the charge density

$$\rho(r) = \frac{\lambda}{2\pi b}\delta(r-b) \tag{19}$$

(c) Since this system is also in cylindrical coordinates, we are free to use the same scale factors from 2b. The only difference is that we are considering a circular disk for this case. Therefore the denominator of the diract delta differs.

$$z\int_0^r r dr \int_0^{2\pi} d\phi = r^2\pi \tag{20}$$

Since we have negligible thickness, we can neglect z. Therefore the charge distribution for this system with radius R is

$$\rho(r) = \begin{cases} \frac{Q}{\pi R^2}\delta(z), & \text{if } r < R \\ 0, & \text{if} \mathring{>} R \end{cases} \tag{21}$$

3 Problem 3

Two long, cylindrical conductors of radii a_1 and a_2 are parallel and separated by a distance d, which is large compared with their radius. Show that the capacitance per unit length is given approximately by

$$C = \pi\epsilon_0 \left(\mathbf{ln}\frac{d}{a}\right)^{-1}$$

where a is the geometrical mean of the two radii.

Construct the first cylindrical conductor at $r = 0$ and the second cylindrical conductor at $r = d$. Give them a corresponding charge of $+Q$ and $-Q$, respectively. Now using Gauss' law we solve for the electric fields about the 2 conductors.

$$\begin{aligned} \int E_1 da &= \frac{Q}{\epsilon_0} \rightarrow E_1(2\pi r L) = \frac{Q}{\epsilon_0} \\ E_1 &= \frac{Q}{2\pi r L\epsilon_0} \end{aligned} \tag{22}$$

and

$$\begin{aligned} \int E_2 da &= -\frac{Q}{\epsilon_0} \rightarrow E_2(2\pi(r-d)L) = -\frac{Q}{\epsilon_0} \\ E_2 &= \frac{Q}{2\pi(d-r)L} \end{aligned} \tag{23}$$

where we wrote the denominator of E_2 with a $d - r$ factor to cancel out the overall negative sign. Therefore we have the Electric field about the 2 conductors to be

$$E = \frac{Q}{2\pi r L\epsilon_0} + \frac{Q}{2\pi(d-r)L\epsilon_0} \tag{24}$$

Solving for the electric potential ΔV.

$$\begin{aligned} \Delta V &= \int_{a_2}^{d-a_2} \frac{Q}{2\pi r L\epsilon_0} + \frac{Q}{2\pi(d-r)L\epsilon_0} dr = \frac{Q}{2\pi\epsilon_0 L}\int_{a_2}^{d-a_2}\left(\frac{1}{r} + \frac{1}{d-r}\right) dr \\ \Delta V &= \frac{Q}{2\pi r L\epsilon_0}\ln\left(\frac{(d-a_2)(d-a_1)}{a_1a_2}\right) = \frac{Q}{2\pi r L\epsilon_0}\ln\left(\frac{d^2-(a_2+a_1)d+a_1a_2}{a_1a_2}\right) \end{aligned} \tag{25}$$

When $d \gg a_1, a_2$, we have

$$\Delta V = \frac{Q}{2\pi r L\epsilon_0}\ln\left(\frac{d^2}{a_1a_2}\right) = \frac{Q}{\pi r L\epsilon_0}\ln\left(\frac{d}{\sqrt{a_1a_2}}\right) \tag{26}$$

when we take out the power of 2. Using the formula for capacitance per unit length, we have

$$C = \frac{Q}{L\Delta V} = \frac{Q}{L\frac{Q}{\pi r L\epsilon_0}\ln\left(\frac{d}{\sqrt{a_1a_2}}\right)} = \frac{\epsilon_0\pi}{\ln\left(\frac{d}{\sqrt{a_1a_2}}\right)} = \pi\epsilon_0\ln\left(\frac{d}{\sqrt{a_1a_2}}\right)^{-1} \tag{27}$$

4 Problem 4

Consider the configuration of conductors of Problem 1.17, with all conductors except S_1 held at zero potential. (a) Show that the potential $\Phi(\mathbf{x})$ anywhere in the volume V and on any of the surfaces S_i can be written as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\oint_{S_1} \sigma_1(\mathbf{x}')G(\mathbf{x},\mathbf{x}'))dz'$$

where $\sigma_1(\mathbf{x}')$ is the surface charge density on S_1 and $G(\mathbf{x},\mathbf{x}')$ is the Green's function potential for a point charge in the presence of all the surfaces that are held at zero potential. Show that the electrostatic energy is

$$W = \frac{1}{8\pi\epsilon_0}\oint_{S_1} da \oint_{S_1} da' \sigma_1(\mathbf{x})G(\mathbf{x},\mathbf{a}')\sigma_1(\mathbf{x}')$$

We have the general form of the potential to be

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x'})G(\vec{x},\vec{x'})d^3x' + \frac{1}{4\pi}\oint_S \left[G(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x'})\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right] da' \tag{28}$$

We can use Dirchlet boundary conditions with $G(\vec{x},\vec{x}') = 0$ and considering that $\Phi(\vec{x}) = 0$ for any other surface to reduce the form of the potential to be

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x'})G_D(\vec{x},\vec{x'})d^3x' \tag{29}$$

where we introduce the Green's function $G_D(\vec{x},\vec{x'})$. Since the only charge distribution comes from S_1. The contribution to the potential due to a unit charge at $\vec{x'}$ can be written as

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V 4\pi\epsilon_0\delta(\vec{x'})G(\vec{x},\vec{x'})d^3x' \tag{30}$$

where $G(\vec{x},\vec{x}')$ is the Green's function potential for a point charge under the presence of all the zero potential surfaces. The surface charge density for S_1 is given by

$$\sigma_1(\vec{x'}) = 4\pi\epsilon_0\delta(\vec{x'}). \tag{31}$$

We can then solve for the potential inside the volume V by plugging the value for σ_1

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\oint_{S_1} \sigma_1(\vec{x'})G(\vec{x},\vec{x'})d^3x' \tag{32}$$

Plugging this equation into

$$W = \frac{1}{2} \int \rho(\vec{x}) \Phi(\vec{x}) d^3 x \tag{33}$$

to solve for the electrostatic energy, we have

$$W = \frac{1}{2} \left(\frac{1}{4\pi\epsilon_0} \right) \int \rho(\vec{x}) d^3 x \oint_{S_1} \sigma_1(\vec{x'}) G(\vec{x}, \vec{x'}) d^3 x' \tag{34}$$

Since the charge can only be found in surface S_1

$$\begin{aligned} W &= \frac{1}{2} \left(\frac{1}{4\pi\epsilon_0} \right) \oint_{S_1} \sigma_1(\vec{x}) da \int_{S_1} \sigma_1(\vec{x'}) G(\vec{x}, \vec{x'}) da' \\ &= \frac{1}{8\pi\epsilon_0} \oint_{S_1} da \oint_{S_1} da' \sigma(\vec{x}) \sigma(\vec{x'}) G(\vec{x}, \vec{x'}) \end{aligned} \tag{35}$$

where we rewrite $d^3 x \to da$ and $d^3 x' \to da'$.