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A New Multiple-relaxation-time Lattice Boltzmann Method for Natural Convection

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Abstract This article is devoted to the study of multiple-relaxation-time (MRT) lattice Boltzmann method with eight-by-eight collision matrix for natural convection flow. In the velocity space, eight speed directions are used and the corresponding incompressible multiple-relaxation-time model with force term is presented. D2Q4 model is for temperature field. The coupled double distribution functions (DDF) overcome artificial compressible effect corresponding to the standard MRT model. The simulations of natural convection flows with $Pr = 0.71$ for air and $Ra = 10^3$–10^9 are carried out and excellent agreements are obtained to demonstrate the numerical accuracy and stability of the proposed model.

Keywords Lattice Boltzmann method · Multi-relaxation-time · Natural convection

1 Introduction

Up to date, the lattice Boltzmann method (LBM) has been considered as a powerful numerical tool for simulating complex flows. This method can be either regarded as an extension of the lattice gas automaton [1–4] or as a special discrete form of the most widely used Boltzmann equation [5, 6]. Unlike traditional numerical methods which directly compute the partial differential equations (PDE) to get the desired physical quantities, the LBM is to solve the discrete velocity Boltzmann equation based on the particle distribution function, which describes the microscopic picture of particle movement from the statistical physics point of view. Then the macroscopic physical quantities, such as density and velocity, are derived with a sum or moment integrations of the distribution function. This background

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brings LBM a number of advantages, such as algorithmic simplicity, fully parallel computation and easy implementation of complex boundary conditions [7]. The Lattice Bhatnagar-Gross-Krook (LBGK) model, the most popular LBM, has not only displayed the remarkable potentials in modeling complex fluid flows [8–12], such as natural convection flow, multiphase fluids, suspensions in fluid, but also showed the strong ability to simulate nonlinear mathematical-physical equations [13–17]. However, these models usually suffer from numerical instability for small viscosity fluids. It is noted that some efforts have been made by some different research groups [18–31] in the past few years to improve the numerical stability, such as Entropic models, TRT models, among which the multi-relaxation-time (MRT) model [25–31] has received particular attentions in recent years due to its distinct numerical stability.

Nowadays, Ahmed Mezrhab et al. [31] simulated the natural convection by MRT model. In their model, the MRT model for velocity field has an artificial compressible effect during the Chapman-Enskog expansion [27] to recover the Boussinesq equations. Here, to overcome the artificial compressible in MRT, we construct the impressible MRT model with eight-by-eight collision matrix for the velocity field and the double distribution function (DDF) is also used in this paper.

The remainder of the article is arranged as follows. In Sect. 2, the incompressible multi-relation-time LB with eight-by-eight collision matrix for natural convection is constructed. In Sect. 3, the numerical simulations of natural convection at different *Ra* numbers are carried on and simulation results demonstrate the numerical accuracy and stability. The article ends with a brief conclusion section.

2 Incompressible MRT LB Model for Natural Convection

For the two dimension eight discrete velocity space, the discrete velocities are:

$$\mathbf{e}_\alpha = \begin{cases} (\cos\phi_\alpha, \sin\phi_\alpha)c & \alpha = 1-4 \\ (\cos\phi_\alpha, \sin\phi_\alpha)\sqrt{2}c & \alpha = 5-8 \end{cases} \tag{1}$$

where $\phi_{1-4} = (\alpha - 1)\pi/2$ and $\phi_{5-8} = (\alpha - 1)\pi/2 + \pi/4$. $c = \delta_x/\delta_t$ is the particle velocity and δ_x and δ_t are the lattice grid spacing and time step respectively. Hereafter we shall use the units of $\delta_x = \delta_t = 1$ such that all the relevant quantities are dimensionless.

The generalized evolutionary equation of the following two-dimensional MRT LB model [30] reads

$$\boldsymbol{f}(\mathbf{x} + \mathbf{e}\delta_t, t + \delta_t) - \boldsymbol{f}(\mathbf{x}, t) = -\mathbf{T}^{-1}\hat{\Lambda}\big(\mathbf{m}(\mathbf{x}, t) - \mathbf{m}^{eq}(\mathbf{x}, t)\big) + \delta t\tilde{\mathbf{F}}, \tag{2}$$

where $\boldsymbol{f}(\mathbf{x}, t) = [f_1(\mathbf{x}, t), \dots, f_8(\mathbf{x}, t)]^T$ is density distribution function for the molecules moving with a discrete velocity at point $\mathbf{x}$ and time t.

The moments $\mathbf{m}$ based on the distributions $\mathbf{f}(\mathbf{x}, t)$ are

$$\begin{aligned} \mathbf{m} &= \mathbf{T}\boldsymbol{f} = [m_1, m_2, \dots, m_8]^T, \\ \boldsymbol{f} &= \mathbf{T}^{-1}\boldsymbol{m} = [f_1, f_2, \dots, f_8]^T, \end{aligned} \tag{3}$$

where **T** is a linear transformation which can be constructed from the discrete velocity set via the Gram-Schmidt orthogonalization procedure and defined as

$$\mathbf{T}=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\ -2 & 0 & 2 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 & -1 & -1 \\ 0 & -2 & 0 & 2 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}. \tag{4}$$

With the transformation matrix, we can obtain eight velocity moments given by

$$\mathbf{m}^{eq}=\left[\frac{5}{3}P+\frac{2}{3}u^2, e, u_x, q_x, u_y, q_y, p_{xx}, p_{xy}\right]^T. \tag{5}$$

The collision matrix $\hat{\Lambda}=\mathbf{T}\Lambda\mathbf{T}^{-1}$ in moment space is a diagonal matrix given by

$$\hat{\Lambda}=\mathrm{diag}[\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8]. \tag{6}$$

Λ is an eight-by-eight collision matrix.

Here, the f_i^{eq} are defined as

$$f_\alpha^{eq}=\omega_\alpha\frac{P}{c_s^2}+s_\alpha(\mathbf{u}), \quad \alpha=1-8, \tag{7}$$

where $s_\alpha(\mathbf{u})=\omega_\alpha[3\mathbf{e}_\alpha\cdot\mathbf{u}+4.5(\mathbf{e}_\alpha\cdot\mathbf{u})^2-1.5|\mathbf{u}|^2]$ with the weight coefficient $\omega_{1-4}=1/9$, $\omega_{5-8}=1/36$ and $c_s=1/\sqrt{3}$ is the speed of sound.

So the equilibrium moments m_i^{eq} can be constructed accordingly as

$$\mathbf{m}^{eq}=\left[\frac{5}{3}P+\frac{2}{3}u^2, e^{eq}, u_x, q_x^{eq}, u_y, q_y^{eq}, p_{xx}^{eq}, p_{xy}^{eq}\right]^T, \tag{8}$$

where

$$\begin{aligned} e^{eq}&=\alpha_2 P+\frac{3}{4}\gamma_2\big(u_x^2+u_y^2\big),\\ q_x^{eq}&=\frac{1}{2}c_1 u_x,\\ q_y^{eq}&=\frac{1}{2}c_2 u_y,\\ p_{xx}^{eq}&=\frac{3}{2}\gamma_1\big(u_x^2-u_y^2\big),\\ p_{xy}^{eq}&=\frac{3}{2}\gamma_3(u_x u_y). \end{aligned} \tag{9}$$

To get the correct hydrodynamic equations, the parameters in the above equations are set to be $\alpha_2=-1$, $c_1=c_2=-2$, $\gamma_1=\gamma_3=2/3$, and $\gamma_2=0$.

For the force term, $\tilde{\mathbf{F}}=\mathbf{T}\boldsymbol{F}$, where $F_\alpha=\omega_\alpha\mathbf{c}_\alpha\cdot\boldsymbol{F}/c_s^2$.

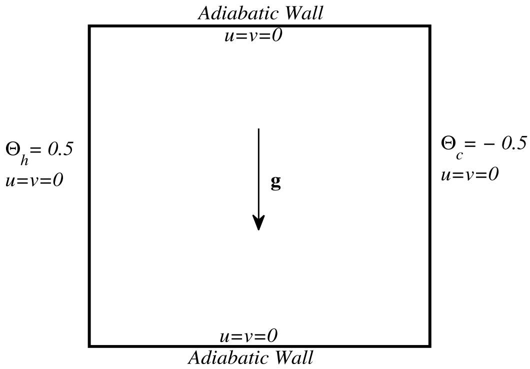

Fig. 1 Problem of natural convection in a square cavity

For the temperature field, the D2Q4 model is used and the evolution function is

$$g_i(\mathbf{x}+\mathbf{c}_i\delta_t, t+\delta_t) - g_i(\mathbf{x}, t) = -\frac{1}{\tau'}\left(g_i(\mathbf{x}, t) - g_i^{eq}(\mathbf{x}, t)\right) \quad i = 1, \ldots, 4, \tag{10}$$

where $g_i(\mathbf{x}, t)$ is the distribution function for temperature at node $\mathbf{x}$ at time t.

The fluid pressure, velocity and temperature are computed from distribution function by

$$\mathbf{u} = \sum_{\alpha=1}^{8} \mathbf{e}_\alpha f_\alpha, \qquad P = \frac{c_s^2}{1-\omega_0}\left[\sum_{\alpha=1}^{8} f_\alpha + s_0(\mathbf{u})\right], \qquad \Theta = \sum_{i=1}^{4} g_i, \tag{11}$$

where $\omega_0 = 4/9$.

Through the C-E expansion, the dimensionless Boussinesq equations can be recovered as

$$\nabla \cdot \mathbf{u} = 0, \tag{12a}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + Pr\Theta \vec{\mathbf{k}}, \tag{12b}$$

$$\frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla \Theta = D \nabla^2 \Theta, \tag{12c}$$

with the kinematic viscosity $\nu = c_s^2(\tau - 1/2)\delta_t$ and $D = \frac{c^2}{2}(\tau' - 1/2)\delta_t$. Here $\vec{\mathbf{k}}$ is the unit vector along the +z-axis, $\upsilon = Pr/\sqrt{Ra}$ and $D = Ra^{-0.5}$ are dimensionless viscosity and thermal diffusivity, respectively, and $Pr = \nu/\alpha_c$ is the Prandtl number.

3 Numerical Simulations of Natural Convection Flow

The natural convection flow in a square cavity has been simulated using the decoupling incompressible MRT model. In this problem, the two side walls are kept at temperature $\Theta_h = 0.5$ and $\Theta_c = -0.5$ respectively and the bottom and top walls are adiabatic which is shown in Fig. 1.

Here υ and α_c are the values of the transport coefficients measured at temperature T_c, H is the height of the cavity and g is the gravity acceleration. In the all simulations, Pr is fixed to 0.71 for the air.

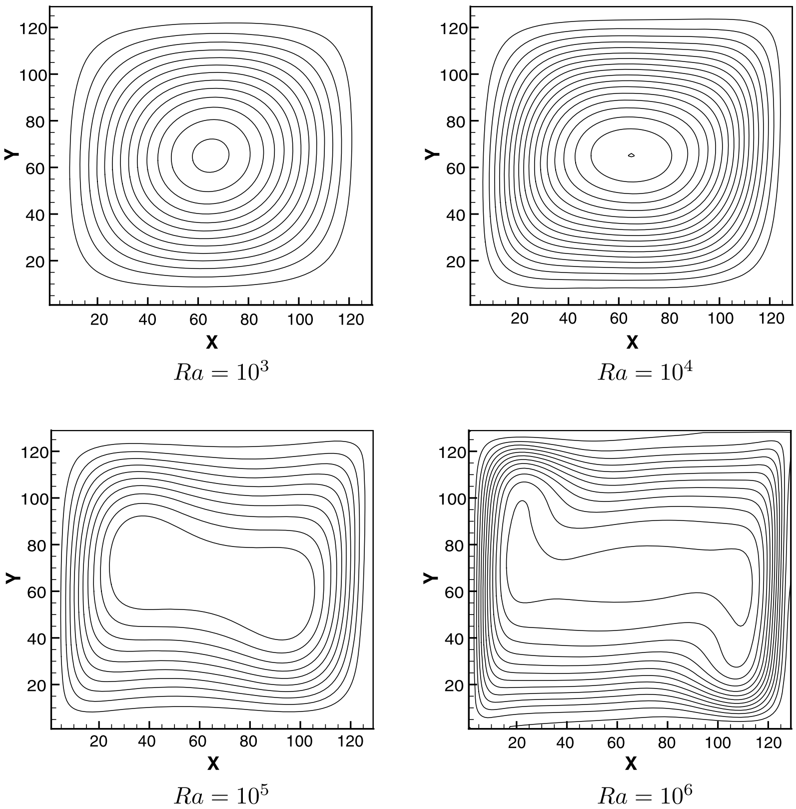

Fig. 2 Streamlines at different *Ra* numbers

For the adiabatic boundary conditions at the bottom and the top walls, the temperature at the boundary node can be obtained by discretizing the macroscopic boundary condition, $\partial\Theta/\partial y = 0$, using a finite-difference scheme. Furthermore it is easy to use the nonequilibrium extrapolation method [33] to treat all the boundary condition. Our simulations are carried on 128×128 lattice with $Ra = 10^3$–10^6, and 256×256 for $Ra = 10^7$–10^8.

The streamlines and isothermal lines are shown in Figs. 2, 3, 4 and 5. From the figures, it is shown that a central vortex appears as the typical features of the flow when the *Ra* number is low. As the *Ra* number increases, the vortex tends to become elliptic and breaks up into two vortices at $Ra = 10^5$. All of these observations are in good agreement with the results reported in previous studies [31, 32, 34].

For a quantitative comparison, the average Nu number defined

$$Nu = -\frac{1}{\Theta_h - \Theta_c} \int_0^H \left(\frac{\partial \Theta}{\partial x}\right)_{wall} dy. \tag{13}$$

And the comparisons of the average *Nu* number are listed in Table 1. From the table, it can be seen our simulation results are agreement with others.

To demonstrate the numerical stability, we take the simulation at $Ra = 10^9$ on the lattice 256×256. The isothermal contours and streamlines are shown in Fig. 6 and the average

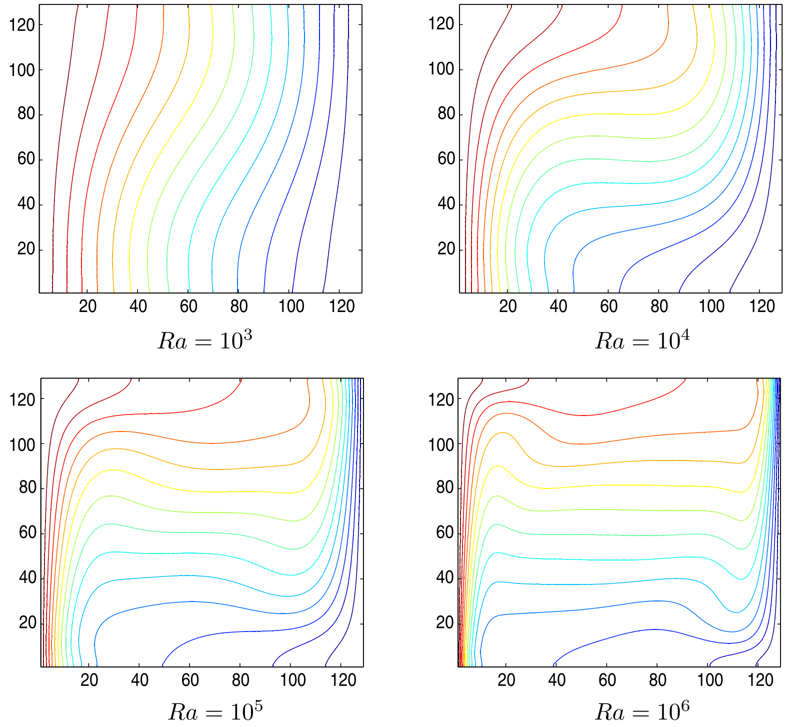

Fig. 3 Isothermal contours at different *Ra* numbers

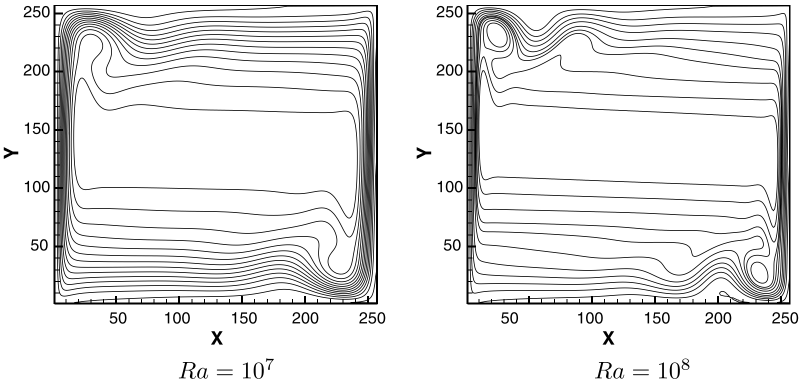

Fig. 4 Streamlines at different *Ra* numbers

Nu is 49.11. However, when *Ra* is up to 10^9, LBGK model performed on 256×256 is divergent.

The MRT model with eight-by-eight collision matrix is of better numerical stability and its computation time is not expensive. We take the same problem with $Ra = 10{,}000$ and grid size 256×256, the computation time of our model is about 12 % more than LBGK model.

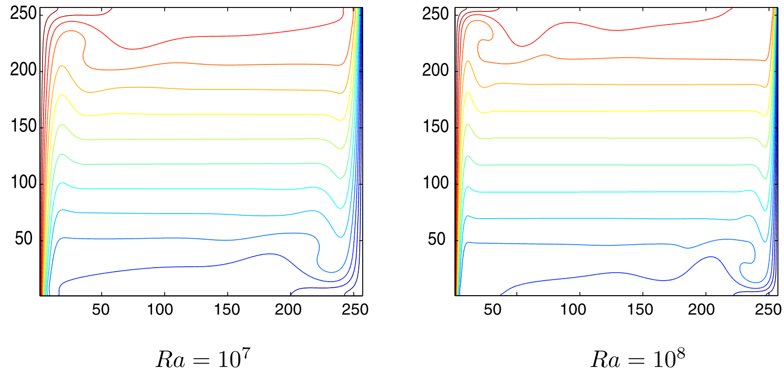

Fig. 5 Isothermal contours at different Ra numbers

Table 1 Comparison of the average Nu numbers with previous works

Ra	Our results	Ref. [32]	Ref. [34]	Ref. [31]
10^3	1.108	1.116	1.121	1.112
10^4	2.252	2.244	2.286	2.241
10^5	4.596	4.541	4.546	4.519
10^6	8.822	8.816	8.652	8.817
10^7	16.424	No data	16.79	16.510
10^8	29.094	No data	30.506	30.033

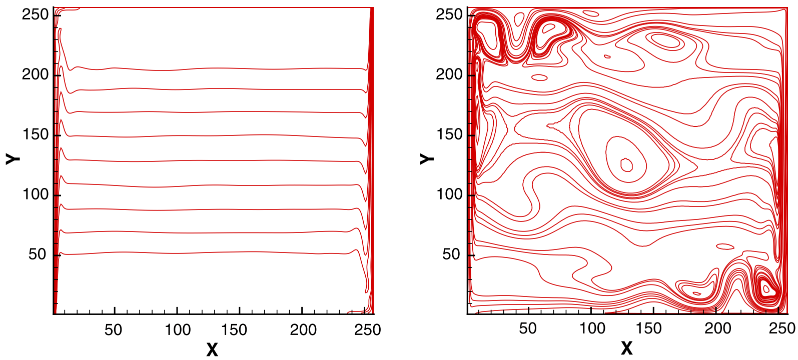

Fig. 6 Isothermal contours and streamlines at $Ra = 10^9$

4 Conclusion

In this article, we have constructed an incompressible multiple-relaxation-time LB model eight-by-eight collision matrix to simulate the natural convection in a square cavity. The advantage of the model is to overcome the artificial compressible comparing to other MRT model and eight speed velocity used can make the model have high computational quality. The two-dimensional natural convection are simulated for different Ra number. The good agreement with other previous results shows the numerical accuracy and stability of the presented model in this paper. Finally, the proposed method can be extended to the thermo-hydrodynamics, such as Rayleigh-Bernard convection flow [35].

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