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Technical note

Stokes flow in a curved duct – A Ritz method

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A R T I C L E I N F O

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Fully-developed slow viscous flow in a curved duct of arbitrary curvature is solved by an efficient Ritz variational method. For a duct of rectangular cross section the Ritz results agrees well with those obtained by a Fourier–Bessel expansion. The Ritz method is then applied to the elliptic cross section. The fluid properties for Stokes flow in a curved duct are discussed.

1. Introduction

The viscous flow in a curved duct is fundamental in fluid transport. Hundreds of articles on the various aspects of the flow have been reported [1,2]. But practically all of the literature is concerned with the high or moderate Reynolds number flows, which lead to phenomena such as secondary flow and non-uniqueness.

Due to the miniaturization of fluid apparatus, the flows in small curved ducts are becoming important. Small curved vessels are also common in the microcirculation. Typical Reynolds numbers encountered are 10^{-3} or lower, and the Stokes equation is adequate to describe the flow. There are several consequences of very low Reynolds numbers. Firstly secondary flow, of order Reynolds number, is unimportant. Secondly the entrance effects are limited to less than one width, and the fully developed state is rapidly established. Thus the fully developed results can be useful even for short segments of a curved duct.

Even for Stokes flow, the theoretical analysis of the flow in a curved duct is difficult. One can use Dean's orthogonal coordinates [1], but the resulting equation is not separable, and aside from full numerical integration, only perturbations for a slightly curved tube have been done [3–5]. There seems to be few other relevant literature. In this note we shall present a powerful Ritz method for treating Stokes flow in a curved duct of any cross section and the curvature need not be small.

2. Stokes flow in a curved duct of rectangular cross section

Fig. 1 shows the cross section of the curved duct. Let the centroid of the cross section of the curved duct be on an arc of radius R. We normalize all lengths by R, the velocity by $-RG/\mu$, where G is the azimuthal pressure gradient and μ is the fluid viscosity. The fully developed Stokes equation in cylindrical coordinates (r, θ, z) is

$$\upsilon_{rr} + \frac{1}{r}\upsilon_r - \frac{1}{r^2}\upsilon + \upsilon_{zz} = -\frac{1}{r} \tag{1}$$

where υ is the azimuthal velocity. The boundary condition is that $\upsilon = 0$ on the duct wall. The only analysis for Stokes flow in a curved duct seems to be due to Wang [6] where the cross section is a rectangle of $2bR$ by $2aR$ as in Fig. 1. We shall briefly present a simpler Fourier–Bessel solution for the no-slip case, which will be compared to our Ritz results later.

Let

$$\upsilon = \sum_{n=1}^{\infty} \cos(\beta_n z) f_n(r), \quad \beta_n = \left(n - \frac{1}{2}\right)\frac{\pi}{b} \tag{2}$$

which satisfies the boundary conditions at $z = \pm b$. Now expand unity

$$1 = \sum_{n=1}^{\infty} A_n \cos(\beta_n z), \quad A_n = \frac{2(-1)^{n+1}}{b\beta_n} \tag{3}$$

Eq. (1) gives

$$f_n'' + \frac{1}{r}f_n' - \frac{1}{r^2}f_n - \beta_n^2 f_n = -\frac{A_n}{r} \tag{4}$$

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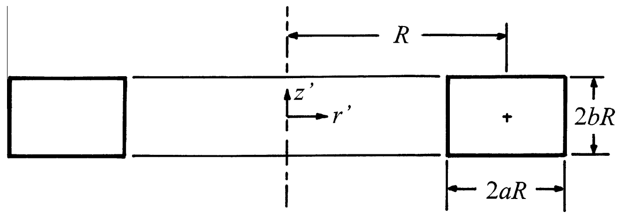

Fig. 1. Cross section of a curved duct and the coordinate system. The dashed line (z-axis) is the symmetry axis about which the cross section is rotated.

The general solution is

$$f_n(r) = \frac{A_n}{\beta_n^2 r} + C_{1n}K_1(\beta_n r) + C_{2n}I_1(\beta_n r) \tag{5}$$

Here K_1 and I_1 are modified Bessel functions. The boundary conditions at $r = 1 \pm a$ are

$$f_n(1-a) = 0, \quad f_n(1+a) = 0 \tag{6}$$

giving

$$\begin{aligned} C_{1n} &= A_n\{(1+a)I_1[(1+a)\beta_n] - (1-a)I_1[(1-a)\beta_n]\}/D_n \\ C_{2n} &= -A_n\{(1+a)K_1[(1+a)\beta_n] - (1-a)K_1[(1-a)\beta_n]\}/D_n \\ D_n &= (1-a^2)\beta_n^2\{I_1[(1-a)\beta_n]K_1[(1+a)\beta_n] \\ &\quad - I_1[(1+a)\beta_n]K_1[(1-a)\beta_n]\} \end{aligned} \tag{7}$$

The flow rate, normalized by R^3G/μ, is then

$$Q = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\beta_n} S_n \tag{8}$$

where

$$\begin{aligned} S_n &= \int_{1-a}^{1+a} f_n(r)dr \\ &= \beta_n\{2A_n\beta_n(\tanh^{-1}a) - C_2\langle I_0[(1-a)\beta_n] - I_0[(1+a)\beta_n]\rangle \\ &\quad + C_1\langle K_0[(1-a)\beta_n] - K_0[(1+a)\beta_n]\rangle\} \end{aligned} \tag{9}$$

The average or mean velocity is

$$V = \frac{Q}{4ab} \tag{10}$$

If the width b is infinite (a slit), the z dependence is absent and the form of the solution is different. Eq. (1) gives

$$\upsilon = -\frac{1}{2} r \ln r + C_3 r + C_4 \frac{1}{r} \tag{11}$$

The boundary conditions give

$$C_3 = -\frac{(1-a)^2}{8a} \ln\left(\frac{1-a}{1+a}\right) + \frac{1}{2}\ln(1+a) \tag{12}$$

$$C_4 = \frac{(1-a^2)^2}{8a} \ln\left(\frac{1-a}{1+a}\right) \tag{13}$$

The average velocity is

$$V = \frac{1}{2a}\int_{1-a}^{1+a} \upsilon\, dr = \frac{1}{16a^2}\left\{4a^2 - (1-a^2)^2\left[\ln\left(\frac{1-a}{1+a}\right)\right]^2\right\} \tag{14}$$

3. The Ritz method

We present the Ritz method which can be applied to any cross section. The Ritz or Rayleigh–Ritz variational method [7,8] has been used extensively in vibration of plates and membranes, but not as often in fluid mechanics, After some work, we find Eq. (1) is equivalent to minimizing the following integral over the cross sectional area.

$$J = \iint (r\upsilon_r^2 + \upsilon^2/r + r\upsilon_z^2 - 2\upsilon)dzdr \tag{15}$$

This can be verified by using Euler's condition for minimizing a functional [7]. Let $s = r - 1$ and let

$$g(s,z) = 0 \tag{16}$$

describe the tube wall. Let the solution be represented by the series

$$\upsilon = \sum_{n=1}^{\infty} c_i \varphi_i(s,z) \tag{17}$$

where c_i are coefficients to be determined, and φ_i is a complete set of functions which satisfy the boundary conditions. The necessary condition for minimal J is

$$\frac{\partial J}{\partial c_i} = 0 \tag{18}$$

which can be shown to be equivalent to

$$\sum A_{ij}c_j = \sum B_i \tag{19}$$

where

$$A_{ij} = \iint [(s+1)(\varphi_{is}\varphi_{js} + \varphi_{iz}\varphi_{jz}) + \frac{1}{s+1}\varphi_i\varphi_j]\,dzds \tag{20}$$

$$B_i = \iint \varphi_i\, dzds \tag{21}$$

Then the linear algebraic Eq. (19) is inverted for the coefficients c_i. The flow rate is simply

$$Q = \iint \upsilon\, dzds = \sum c_i B_i \tag{22}$$

and the average velocity is

$$V = \frac{Q}{\iint dzds} \tag{23}$$

We illustrate by computing the Stokes flow through the curved rectangular duct studied previously. The boundary is bounded by

$$g = (z^2 - b^2)(s^2 - a^2) = 0 \tag{24}$$

Consider the set of polynomials

$$\{\varphi_i\} = g(z,s)\{1, s, s^2, z^2, s^3, sz^2, s^4, s^2z^2, z^4, s^5, s^3z^2, sz^4, \ldots\} \tag{25}$$

where due to symmetry, only the even powers of z are used. The number of terms can be taken as 4, 6, 9, 12, 16, etc., retaining the highest homogeneous powers. Eqs. (20) and (21) are evaluated by integrating analytically with respect to y then numerically with respect to x (Mathematica adaptive recursion library program with a relative error of 10^{-8}). The coefficients are then found by Eq. (19). Table 1 shows a comparison of the two methods. Both are accurate and efficient.

The results for other dimensions are given in Table 2. Both methods agree within 0.1%. The $b = \infty$ results are from Eq. (14). Typical constant velocity lines are shown in Fig. 2.

Table 1
Typical convergence for Bessel function solution and Ritz solution for a rectangular duct ($a = b = 0.5$).

Terms used	4	6	9	12	16	20
Eq. (10)	0.03541	0.03543	0.03544	0.03544	0.03544	0.03544
Eq. (22)	0.03532	0.03537	0.03543	0.03543	0.03544	0.03544

Table 2
The average velocity for a rectangular duct. a, b are normalized with respect to R.

b/a	0.2	0.4	0.6	0.8
0.2	0.005632	0.009384	0.01144	0.01362
0.4	0.009125	0.02262	0.03260	0.04084
0.6	0.01050	0.03117	0.05116	0.06837
0.8	0.01119	0.03621	0.06445	0.09080
1	0.01160	0.03936	0.07361	0.1077
2	0.01232	0.04574	0.09332	0.1473
5	0.01293	0.04957	0.1053	0.1723
∞	0.01326	0.05213	0.1133	0.1889

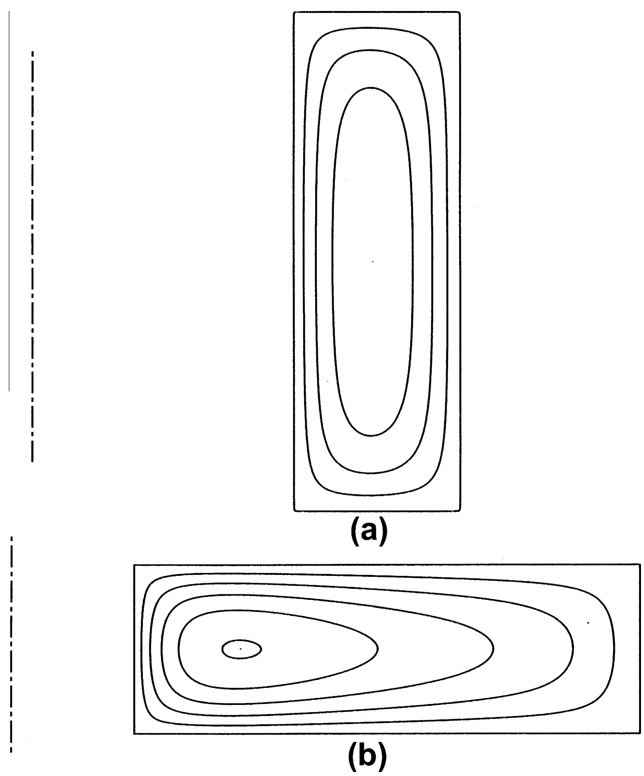

Fig. 2. Constant velocity lines for a rectangular duct. Dash-dot line is the z axis. (a) $a = 0.2$, $b = 0.6$ From outside: $\upsilon = 0$, 0.005, 0.01, 0.015 Maximum $\upsilon = 0.0195$ at $r = 0.987$. The average velocity is $V = 0.01050$ (b) $a = 0.6$, $b = 0.2$. From outside: $\upsilon = 0$, 0.005, 0.01, 0.015, 0.02, 0.025. Maximum $\upsilon = 0.0253$ at $r = 0.651$. The average velocity is $V = 0.01144$.

Next we use the Ritz method to study the Stokes flow in a curved elliptic duct. Elliptic cross sections occur in deformed circular tubes under external pressure, bending, or internal plaque [9]. The cross section is described by

$$g = \left(\frac{s}{a}\right)^2 + \left(\frac{z}{b}\right)^2 - 1 = 0 \tag{26}$$

The sequence Eq. (25) is used and the integration is over the elliptic area. The cross section is circular when $a = b$, for which perturbations for small centerline curvature exists. At zero Reynolds number we find [3,5]

$$V = \frac{Q}{\pi a^2} = \frac{a^2}{8}\left(1 + \frac{a^2}{48} + O(a^4)\right) \tag{27}$$

Table 3 shows the results. For the circular tube, the results from the perturbation formula gradually lose accuracy as the ratio of cross sectional radius to centerline radius increases. The $b = \infty$ (slit) results are from Eq. (14). It seems only for very large aspect ratios would the elliptic results approach those of the slit.

Table 3
Average velocity for the elliptic duct. The values in the parentheses ($b = a$) are from Eq. (27).

b/a	0.2	0.4	0.6	0.8
0.2	0.005004 (0.005004)	0.008139	0.09492	0.01057
0.4	0.007981	0.02006 (0.02007)	0.02845	0.03449
0.6	0.008969	0.02754	0.04523 (0.04534)	0.05965
0.8	0.009375	0.03167	0.05703	0.08025 (0.08107)
1	0.009576	0.03404	0.06487	0.09559
2	0.009857	0.03780	0.07946	0.1285
5	0.009939	0.03901	0.0848	0.1424
∞	0.01326	0.05213	0.1133	0.1889

It is obvious if the dimensions a or b increase, the cross sectional area increases. Tables 1 and 2 show the average velocity also increases. For the same area, an increase in a promotes more flow than an increase in b. This is due to the fact that the velocity is asymmetric for a curved tube. As shown in Fig 2, there is more flow in the region closer to the z-axis, where the azimuthal pressure drop per arc length is larger.

For a straight elliptic duct of width $2aR$ and height $2bR$ the dimensional flow rate is [11]

$$Q' = -\frac{\pi G R^3}{4\mu}\left(\frac{a^3 b^3}{a^2 + b^2}\right) \tag{28}$$

For the average velocity we divided by the area πabR^2. The normalize average velocity is then

$$V_0 = \frac{a^2}{4}\left(\frac{\lambda^2}{1 + \lambda^2}\right) \tag{29}$$

where $\lambda = b/a$ is the aspect ratio. Fig. 3 shows the relative difference between V and V_0, versus a for various aspect ratios. We see that as $a \to 0$ the average velocity (or flow rate) of the curved duct approaches that of the straight duct. Note that the curved circular duct ($\lambda = 1$) has a larger flow than a straight duct, as predicted by Eq. (27). Also, for the same cross section, the flow is larger for a curved duct if the aspect ratio $\lambda < 1.14$ and is smaller otherwise. For $\lambda = 1.14$ the elliptic duct has (almost) the same flow (or resistance) for any axial curvature.

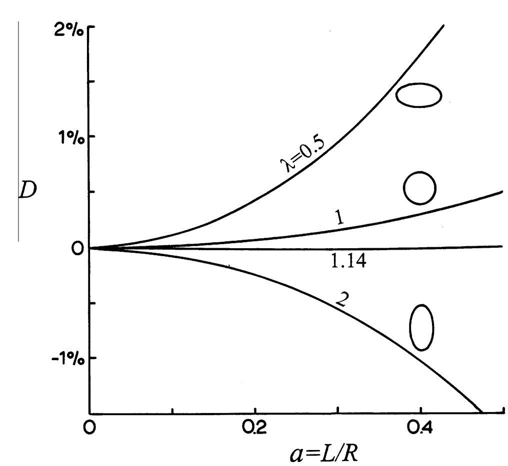

Fig. 3. Relative average velocity percentage difference of a curved elliptic duct to that of a straight duct $D = (V - V_0)/V_0$ versus half width to radius of curvature $a = L/R$ for various constant aspect ratio $\lambda = b/a$.

4. Discussions

The only analytic solution for Stokes flow in a curved duct is the duct with rectangular cross section. Our semi-analytic Ritz method is much more versatile, since it not only can solve the rectangular and elliptic cross sections illustrated in this note, but also many other cross sectional shapes. For example, the flow through a tube with super-elliptic cross section [10], which, due to the small radii of curvature at the corners causing difficulties for direct numerical integration, can be successfully computed by the Ritz method.

A comparison of the Ritz method with the finite element method (FEM) has been done in [12] for the plate equation, and will not be repeated here. Their conclusion that the Ritz method is more advantageous than FEM also applies to our problem. The reasons are: (1) It does not require mesh generation as in FEM (2) It uses a smaller memory space, about the square root of that used in FEM (3) It does not require discretization of curved boundaries (4) It decreases computational effort by implicitly imbedding the boundary conditions into the Ritz functions.

In conclusion, we have developed and verified an efficient Ritz method which can be applied to Stokes flow in highly curved ducts.

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