CS 344: Design and Analysis of Computer Algorithms

Instructor: Kangning Wang

Homework 5: Due on **May 3, 2025**

Acknowledgment: Our team members are Alexander Lin (al1655), Pranav Tikkawar (pt422), and Ivan Zheng (iz72) .

Problem 1

There are n children forming a circle. The i-th child currently has a_i apples. Let $\bar{a} = \frac{\sum_{i=1}^{n} a_i}{n}$ be the average number of apples that a child has, and we assume that it is an integer.

In one move, we can take one apple from a child and give it to an *adjacent* child. What is the minimum number of moves we have to make to make every child have exactly $\bar{a}$ apples? Model this problem by a flow network with costs, and therefore show that this problem can be solved in polynomial time.

Solution. Modeling a flow network, each child is a node in the network with nodes 1, 2,…,n. Each node i has a supply (if $a_i > \bar{a}$) or demand (if $a_i < \bar{a}$). Supply $s_i = a_i - \bar{a}$:

- If $s_i > 0$: node i has s_i apples to give away.
- If $s_i < 0$: node i needs $|s_i|$ apples.

For each pair of adjacent children (i and (i mod n) + 1), created two directed edges:

1. From i to (i mod n) + 1
2. From (i mod n) + 1 to i

Each edge has:

- Capacity: infinite (or at least large enough for all apples)
- Cost: 1 per apple (because one move = moving one apple to an adjacent node)

Sending flow along an edge represents moving apples from one child to an adjacent child. The cost of the flow is the total number of moves. This is a minimum cost flow problem, which can be solved in polynomial time using algorithms like the Successive Shortest Path algorithm or the Cycle-Canceling algorithm. The minimum cost flow will give us the minimum number of moves needed to balance the apples among the children.

Problem 2

Explicitly construct a bijection between each of the following pairs of sets and therefore show that they have the same cardinality.

1. Natural numbers and perfect squares.
2. $(0, 1)$ and $(1, +\infty)$.
3. $(-1, 1)$ and $\mathbb{R}$.
4. $[0, 1]$ and $[0, 1)$.

Solution.

1. We map every natural number n to its square n^2. This is injective, since each natural number has a unique square, and it is surjective, as for each perfect square $p = q^2$, we can take its square root and get the natural number $|q|$ which maps to it. Therefore this mapping is bijective, and the two sets have the same cardinality.
2. For every number n in the set $(1, +\infty)$, we map it to its multiplicative inverse $\frac{1}{n}$. This is injective, as every n maps to a unique number, and surjective, as for every number $p \in (0, 1)$, we can find its corresponding number $n \in (1, +\infty)$ by doing $n = \frac{1}{p}$. Thus the mapping is bijective.
3. Similarly to the last problem, for every number p in the subset $(-1, 1) \setminus 0$ of the first set $(-1, 1)$, we map it to its multiplicative inverse $\frac{1}{p}$ in the subset $\mathbb{R} \setminus 0$ of the second set $\mathbb{R}$. We additionally map 0 in the first set to 0 in the second set. This is injective, as each n maps to its unique multiplicative inverse, except 0, which does not have a multiplicative inverse. However, we have mapped 0 in the first set to 0 in the second set, and since no number has multiplicative inverse 0, 0 uniquely maps to 0, satisfying injectiveness. This is surjective, as for every $q \in (R)$ we can find its corresponding element in the first set by performing $\frac{1}{q}$. Again, the exception is 0, which we know maps to 0. Therefore we have a bijection.
4.

Problem 3

Review the halting problem, and prove that the following problem is also undecidable: Given a Turing machine, decide whether it always correctly decides the set cover problem. (Note that the given Turing machine may not halt on a valid input, in which case it does not correctly decide the set cover problem.)

Solution. Here is my solution.

Problem 4

Prove that the following problem is `NP`-complete: Given n positive integers that sum to $2W$, decide whether we can partition the integers into two parts with equal sum (W each).

Solution. Here is my solution.

Problem 5

Prove that the following problem is `NP`-complete: Given a weighted directed graph with n vertices and m edges, decide whether the graph has a simple cycle with sum of weights equal to 344. The weight of each edge must be an integer, but can be positive, negative, or zero.

Solution. Here is my solution.