CS 344: Design and Analysis of Computer Algorithms

Instructor: Kangning Wang

Homework 5: Due on **May 3, 2025**

Acknowledgment: **Our team members are Alexander Lin (al1655), Pranav Tikkawar (pt422), and Ivan Zheng (iz72)** .

Problem 1

There are n children forming a circle. The i-th child currently has a_i apples. Let $\bar{a} = \frac{\sum_{i=1}^{n} a_i}{n}$ be the average number of apples that a child has, and we assume that it is an integer.

In one move, we can take one apple from a child and give it to an *adjacent* child. What is the minimum number of moves we have to make to make every child have exactly $\bar{a}$ apples? Model this problem by a flow network with costs, and therefore show that this problem can be solved in polynomial time.

Solution. Here is my solution.

Problem 2

Explicitly construct a bijection between each of the following pairs of sets and therefore show that they have the same cardinality.

1. Natural numbers and perfect squares.
2. $(0, 1)$ and $(1, +\infty)$.
3. $(-1, 1)$ and $\mathbb{R}$.
4. $[0, 1]$ and $[0, 1)$.

Solution.

1. We map every natural number n to its square n^2. This is injective, since each natural number has a unique square, and it is surjective, as for each perfect square $p = q^2$, we can take its square root and get the natural number $|q|$ which maps to it. Therefore this mapping is bijective, and the two sets have the same cardinality.
2. For every number n in the set $(1, +\infty)$, we map it to its multiplicative inverse $\frac{1}{n}$. This is injective, as every n maps to a unique number, and surjective, as for every number $p \in$

$(0, 1)$, we can find its corresponding number $n \in (1, +\infty)$ by doing $n = \frac{1}{p}$. Thus the mapping is bijective.

3. Similarly to the last problem, for every number p in the subset $(-1, 1) \setminus 0$ of the first set $(-1, 1)$, we map it to its multiplicative inverse $\frac{1}{p}$ in the subset $\mathbb{R} \setminus 0$ of the second set $\mathbb{R}$. We additionally map 0 in the first set to 0 in the second set. This is injective, as each n maps to its unique multiplicative inverse, except 0, which does not have a multiplicative inverse. However, we have mapped 0 in the first set to 0 in the second set, and since no number has multiplicative inverse 0, 0 uniquely maps to 0, satisfying injectiveness. This is surjective, as for every $q \in (R)$ we can find its corresponding element in the first set by performing $\frac{1}{q}$. Again, the exception is 0, which we know maps to 0. Therefore we have a bijection.
4.

Problem 3

Review the halting problem, and prove that the following problem is also undecidable: Given a Turing machine, decide whether it always correctly decides the set cover problem. (Note that the given Turing machine may not halt on a valid input, in which case it does not correctly decide the set cover problem.)

Solution. Here is my solution.

Problem 4

Prove that the following problem is `NP`-complete: Given n positive integers that sum to $2W$, decide whether we can partition the integers into two parts with equal sum (W each).

Solution. Here is my solution.

Problem 5

Prove that the following problem is `NP`-complete: Given a weighted directed graph with n vertices and m edges, decide whether the graph has a simple cycle with sum of weights equal to 344. The weight of each edge must be an integer, but can be positive, negative, or zero.

Solution. Here is my solution.