Introduction

In the FX options market, positions often are hedged using delta or both delta and vega. An article in The Journal of Derivatives (Yang et al.) empirically compares delta hedging and delta-vega hedging of vanilla FXO using two models: the standard Black-Scholes (BS) and Stochastic-alpha-beta-rho (SABR). Black-Scholes delta is the implied delta, calculated from Black-Scholes price and market implied volatility; SABR delta is the local delta, calculated from the local stochastic volatility surface (LSV) after calibration to the implied volatility. The article studies 4 commonly traded currency pairs: EURUSD, GBPUSD, USDJPY and AUDUSD.

The motivation for using SABR or other LSV models is to correctly predict the dynamics of volatility when the spot price moves. Since I have access to more exotic over-the-counter currencies from Fenics Market Data, I aim to test how well SABR fits the wider OTC vanilla FXO market and to compare the hedging performance of SABR delta-vega with normal Black-Scholes delta-vega.

Models

I used Hagan's SABR approximation for lognormal volatility.

$$\sigma_{SABR} = \alpha \frac{\left\{1 + \left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24}\nu^2\right)T + \dots\right\}}{(fK)^{\frac{1-\beta}{2}}\left\{1 + \frac{(1-\beta)^2}{24}\log^2\left(\frac{f}{K}\right) + \frac{(1-\beta)^4}{1920}\log^4\left(\frac{f}{K}\right) + \dots\right\}} \times \left(\frac{z}{x(z)}\right) \quad (1)$$

Where:

- α, β, ν, ρ are the SABR parameters
  - α – the volatility of the underlying asset (approximated by $\alpha \approx \frac{\sigma_{ATM}}{f^{1-\beta}}$)
  - β – sensitivity of forward price to movements in spot price
  - ρ – correlation between movement in forward price to movements in volatility of underlying
  - ν – the volatility of the volatility
- $z = \frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}}\log\left(\frac{f}{k}\right)$
- $x(z) = \log\left(\frac{z-\rho+\sqrt{1-2\rho z+z^2}}{1-\rho}\right)$

If z is small, $x(z) \to 0$ which makes $\left(\frac{z}{x(z)}\right)$ undefined. For smiles where $|z| < 10^{-7}$, I used the standard approximation $\frac{z}{x(z)} \to 1$

The Greeks: Delta and Vega

For hedging with Black Scholes,

$$Delta\ \Delta_{BS,Call} = \frac{\delta C}{\delta f} = \mathrm{N}\left(\frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma_{BS}^2}{2}\right)T\right)\right) \tag{2}$$

$$\text{Vega}\ \Lambda_{BS} = \frac{\delta C}{\delta \sigma} = SN'\left(\frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma_{BS}^2}{2}\right)T\right)\right)\sqrt{T} \tag{3}$$

For hedging with SABR,

$$Delta\ \Delta_{SABR} = \frac{\delta C}{\delta f} = \Delta_{BS,Call} + \Lambda_{BS} \times \frac{\delta \sigma_{SABR}}{\delta f} \tag{4}$$

$$\text{Vega}\ \Lambda_{SABR} = \frac{\delta C}{\delta \alpha} = \Lambda_{BS} \times \frac{\delta \sigma_{SABR}}{\delta \alpha} \tag{5}$$

I decided to calculate $\frac{\delta \sigma_{SABR}}{\delta f}$ and $\frac{\delta \sigma_{SABR}}{\delta \alpha}$ numerically from the formula of SABR volatility, using $\frac{\delta \sigma_{SABR}}{\delta f} \approx \frac{\sigma_{SABR}(f+\epsilon) - \sigma_{SABR}(f-\epsilon)}{2\epsilon}\ with\ \epsilon = 10^{-6}$ and my calibrated SABR parameters.

Data

I have EOD data on FX forward, spot, and options from Aug 12th, 2024, to Aug 30th, 2024 (15 trading days). I chose to cover the tenors 3M, 6M, 9M, 1Y, 2Y, and 3Y. Deltas of 5C, 10C, 25C, ATMF, 25P, 10P and 5P were provided, for a total of 7 strikes per maturity. The deltas were converted to strike prices using the Black-Scholes Delta formula before SABR calibration.

	COVERAGE
TENORS	3M, 6M, 9M, 1Y, 2Y, 3Y
STRIKES	5C, 10C, 25C, ATMF, 25P, 10P, 5P
CURRENCIES	AUD, BRL, CAD, CHF, CLP, CNY, CZK, EUR, GBP, HKD, HUF, IDR, ILS, INR, JPY, KRW, MX, MYR, NOK, NZD, PHP, PLN, RUB, SAR, SEK, SGD, THB, TRY, TWD, USD, XAU, ZAR

Methodology

I first calibrated SABR to one FX currency and found the best value of β for my model. After fitting on all currency pairs, I checked for covariance between parameters to see if dimensionality reduction is possible.

Next, I compared hedging performance of BS delta, BS delta-vega, SABR delta, and SABR delta-vega. Backtesting let me compare the prediction error of each model. As SABR parameters are normally recalibrated at the end of every day, I only need to compare predictions from day t-1 to day t.

One adaptation was made to the methodology from the paper by Yang et al. As the strike prices are listed in delta for FXOs, an option with delta=0.25 may have a strike price of 86.94 on 12th Aug, but 87.30 on 13th Aug. This is a problem for the backtesting step. For example, if my model's hedge predicts that a 25C option will increase in price by 0.025 from day *t-1* to *t*, I cannot directly compare it with the market price of 25C options on day *t* as they represent different strike prices. The paper by Yang et al. does not provide a solution.

I decided to use a cubic piecewise polynomial to interpolate the real market values. This makes a difference in situations like the USDHKD, where the smile shape was not obvious. This let me calculate the real(observed) change in market price, to compare my models to.

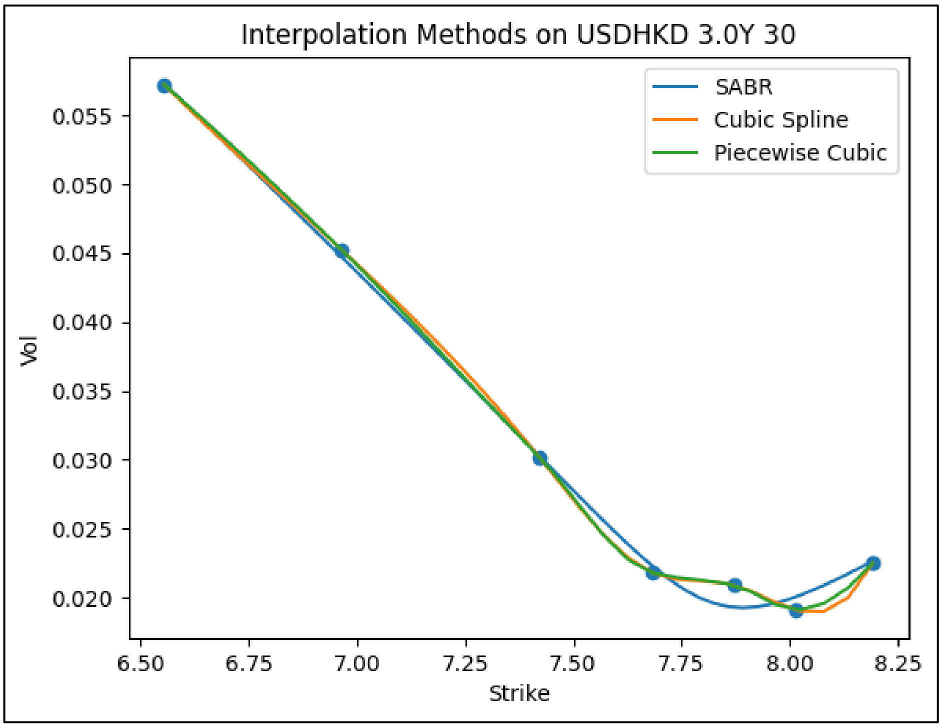


Figure 1 Comparison of Interpolation Methods

SABR Calibration

For prelim testing, I chose the option on EURINR with tenor 1Y.

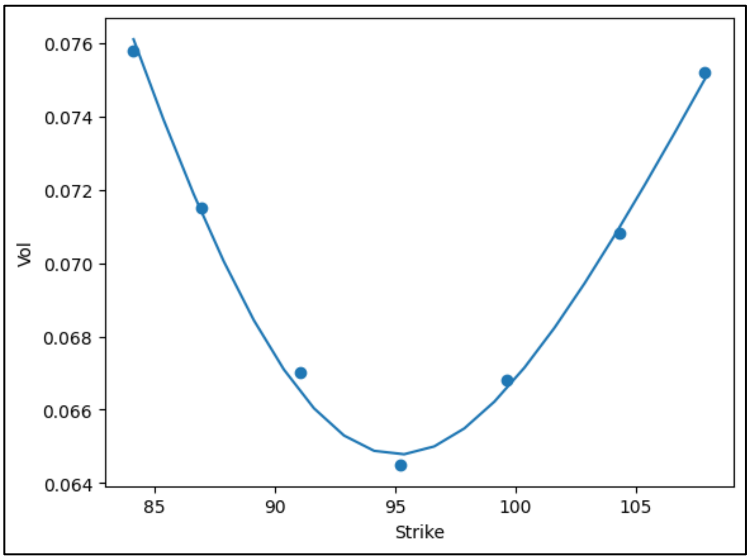


Figure 2 Example Fit on EURINR 1Y option on 12/08/24

For fitting, I used the error function $\sum W_i \times \frac{(\sigma_{SABR} - \sigma_{observed})^2}{\sigma_{SABR}}$ with weights $W_i = 1 - \frac{|K-f|}{f}$ to give more weight to options near ATM.

In practice, β is fixed based on historical data (Zhang). A paper by Chan suggests that a beta of 1.0 best fits the FXO market. After calibrating on my preliminary test for different values of β, I found that 0.75 fit best to my data.

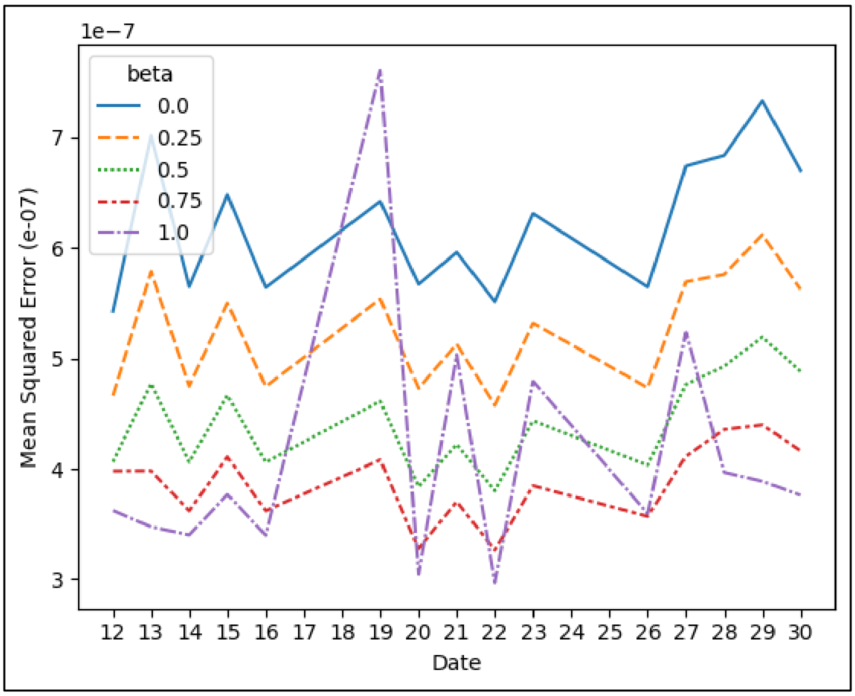


Figure 3 Comparison of EURINR Fit to different values of β

Calibrating for every smile, the mean squared error of each currency shows that SABR fits well to all OTC currencies and fits worst on USDHKD with an mse of 0.000468. As shown in Figure 5, USDHKD and EURPLN did not conform well to the smile shape, which could explain the poorer fit.

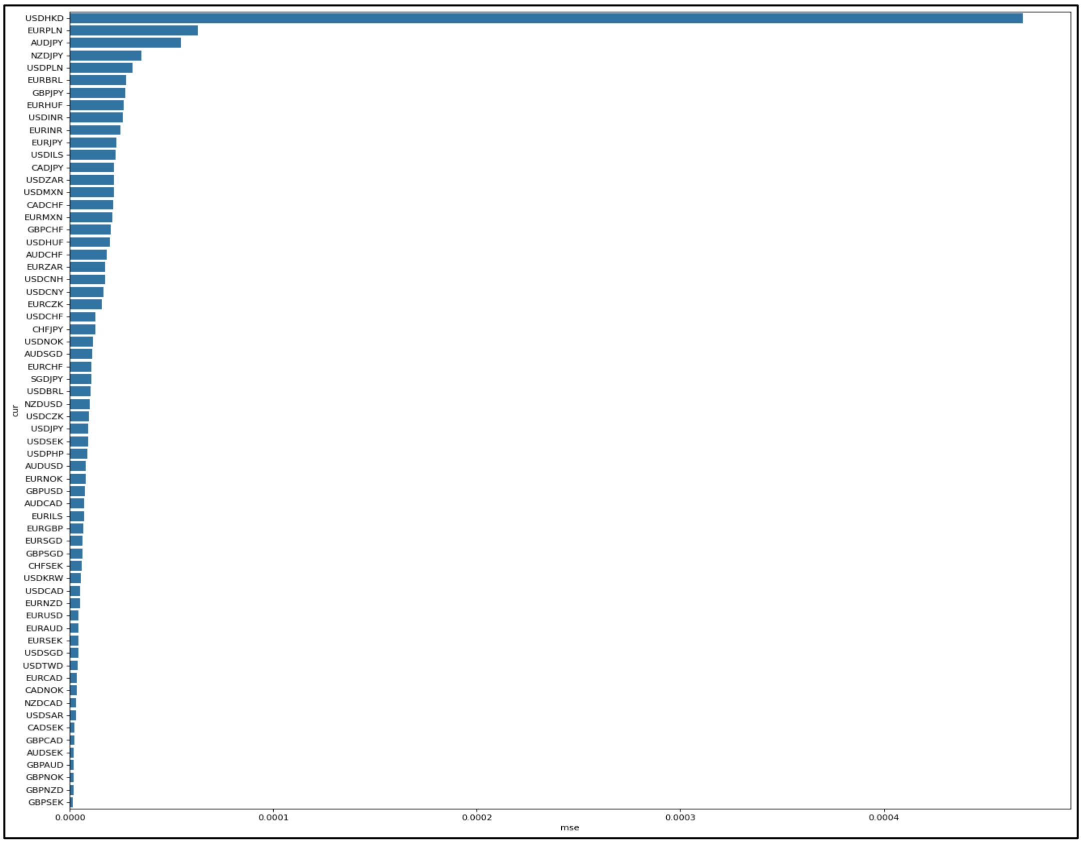


Figure 4 Mean-Squared Error (mse) of SABR calibration on different currencies

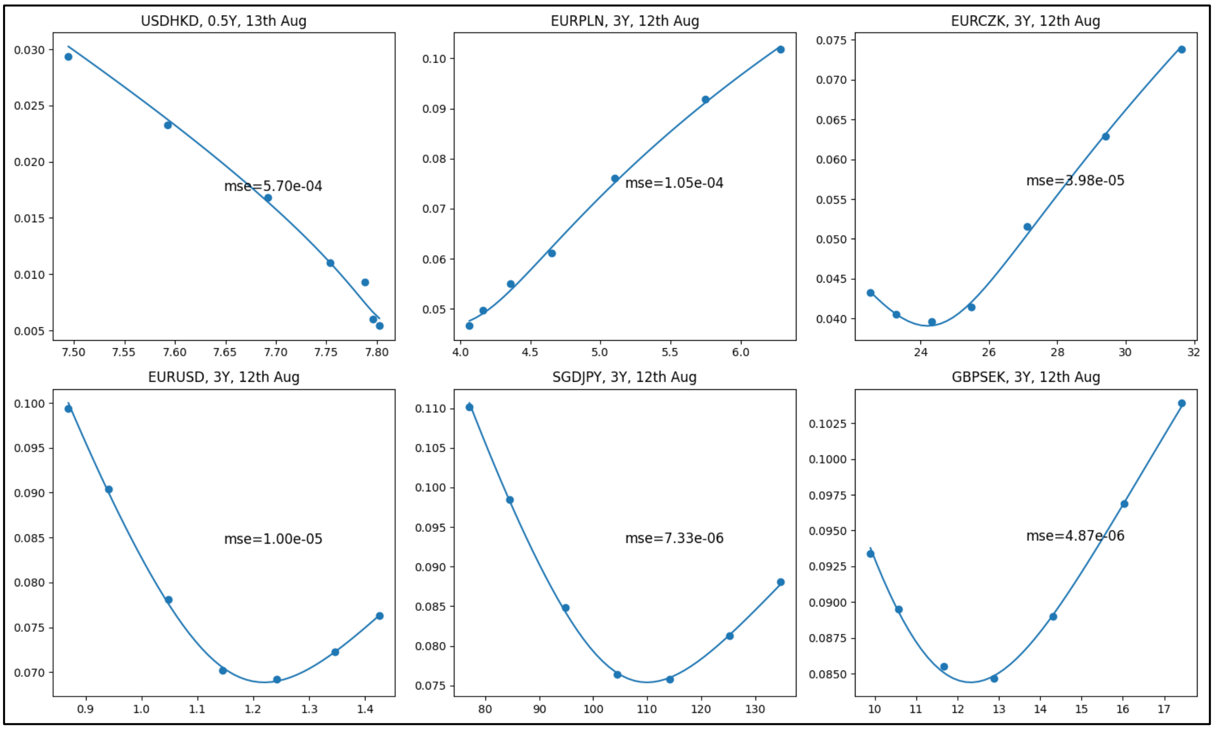


Figure 5 Example fits with varying degrees of Mean-Squared Error (mse)

Parameter Covariance

I fit SABR parameters for 63 currency pairs, 6 tenors and 15 trading dates for a total of 5670 volatility smiles across the FX market. After standardising and scaling to unit variance within each currency, I checked for covariance between all SABR parameters.

Covariance	α	ρ	ν	Forward	Tenor
α					
ρ	-0.046328				
ν	-0.552043	0.037736			
Forward	0.033889	0.293266	-0.024486		
Tenor	0.000000	0.000000	-0.000000	0.000000	

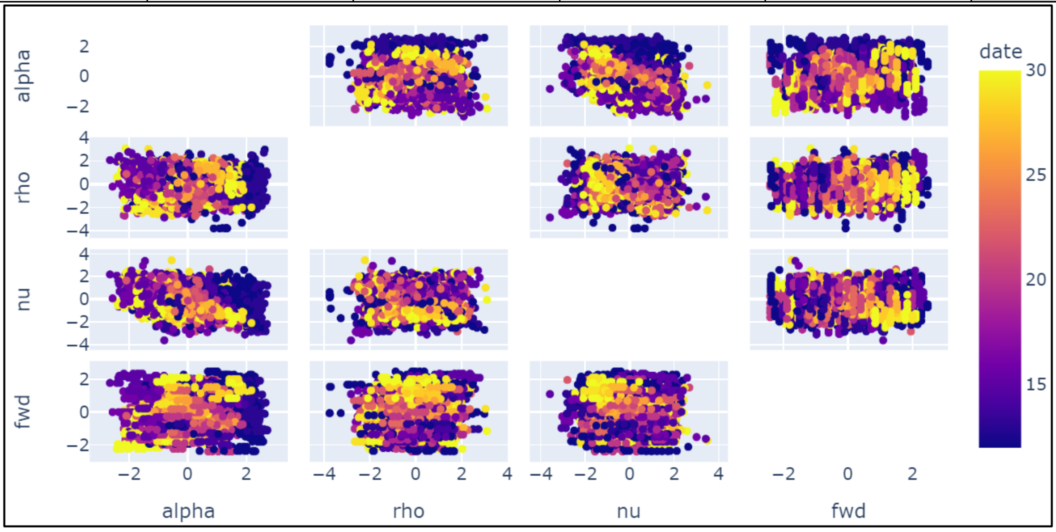
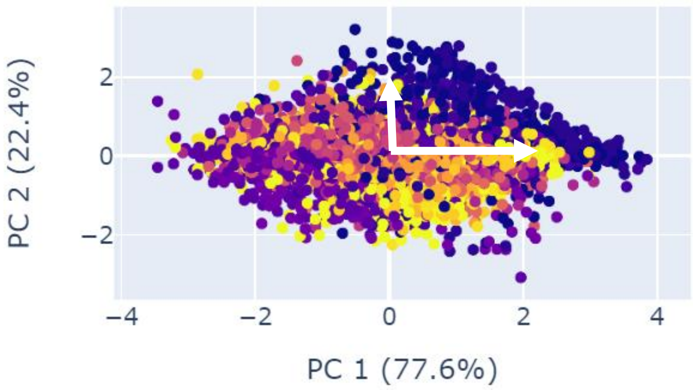


Figure 6 SABR parameter covariance matrix

The only variables with statistically significant covariance were α and ν. I could not find any theoretical relationship or similar results in other papers, or any papers that attempt to reduce the number of parameters further. Applying PCA:



The new PCA components have no significance to price or volatility calculations, and there is no observable trend across trading days. Due to the low correlation between the other parameters, further dimension reduction is probably not possible.

Hedging Performance

Since $\Delta_{BS,Call} = \frac{\delta C}{\delta f}$ and $\Lambda_{BS} = \frac{\delta C}{\delta \sigma}$, the hedge accuracy of an option purchased on day $t-1$ can be backtested on day t by comparing Call price, C. The more accurate the delta and vega, the better the model would have been able to account for the real change in f and σ observed from day $t-1$ to t.

$$\begin{aligned} BS\ Price\ Change = \ & C_t - C_{t-1} = (\Delta f)\frac{\delta C}{\delta f} + (\Delta \sigma)\frac{\delta C}{\delta \sigma} \\ &= (f_t - f_{t-1}) \times \left(\frac{\delta C}{\delta f}\right)_{t-1} \quad + \quad \left(\sigma_{actual}^{t} - \sigma_{actual}^{t-1}\right) \times \left(\frac{\delta C}{\delta \sigma}\right)_{t-1} \\ &= (f_t - f_{t-1}) \times \left(\Delta_{BS,Call}\right)_{t-1} \quad + \quad \left(\boldsymbol{\sigma_{actual}^{t}} - \boldsymbol{\sigma_{actual}^{t-1}}\right) \times (\Lambda_{BS})_{t-1} \end{aligned} \tag{6}$$

The piecewise cubic interpolation is required to obtain σ_{actual}^{t} for the strike price of each option.

Similarly, for the SABR model price change can be calculated the same way. Note that SABR vega measures change due to α, while BS vega measures change due to volatility. Δ_{SABR} and Λ_{SABR} are substituted by equations (4) and (5).

$$\begin{aligned} SABR\ Price\ Change = \ & C_t - C_{t-1} = (\Delta f)\frac{\delta C}{\delta f} + (\Delta \alpha)\frac{\delta C}{\delta \alpha} \\ &= (f_t - f_{t-1}) \times \left(\frac{\delta C}{\delta f}\right)_{t-1} + (\alpha_t - \alpha_{t-1}) \times \left(\frac{\delta C}{\delta \alpha}\right)_{t-1} \\ &= (f_t - f_{t-1}) \times (\Delta_{SABR})_{t-1} + (\alpha_t - \alpha_{t-1}) \times (\Lambda_{SABR})_{t-1} \\ &= (f_t - f_{t-1}) \times \left(\Delta_{BS,Call} + \Lambda_{BS} \times \frac{\delta \sigma_{SABR}}{\delta f}\right)_{t-1} + (\alpha_t - \alpha_{t-1}) \times \left(\Lambda_{BS} \times \frac{\delta \sigma_{SABR}}{\delta \alpha}\right)_{t-1} \\ = (f_t - f_{t-1}) \times & \left(\Delta_{BS,Call}\right)_{t-1} + \left[(\boldsymbol{f_t} - \boldsymbol{f_{t-1}})\frac{\boldsymbol{\delta \sigma_{SABR}}}{\boldsymbol{\delta f}} + (\boldsymbol{\alpha_t} - \boldsymbol{\alpha_{t-1}})\frac{\boldsymbol{\delta \sigma_{SABR}}}{\boldsymbol{\delta \alpha}}\right](\Lambda_{BS})_{t-1} \end{aligned} \tag{7}$$

The two price change functions are nearly identical, with just different ways of calculating the change in volatility as shown in bold. The difference will be examined later. Comparing hedging performance on the preliminary EURINR 1Y case, the SABR model showed nearly identical predictions to the normal BS model when considering both delta and vega.

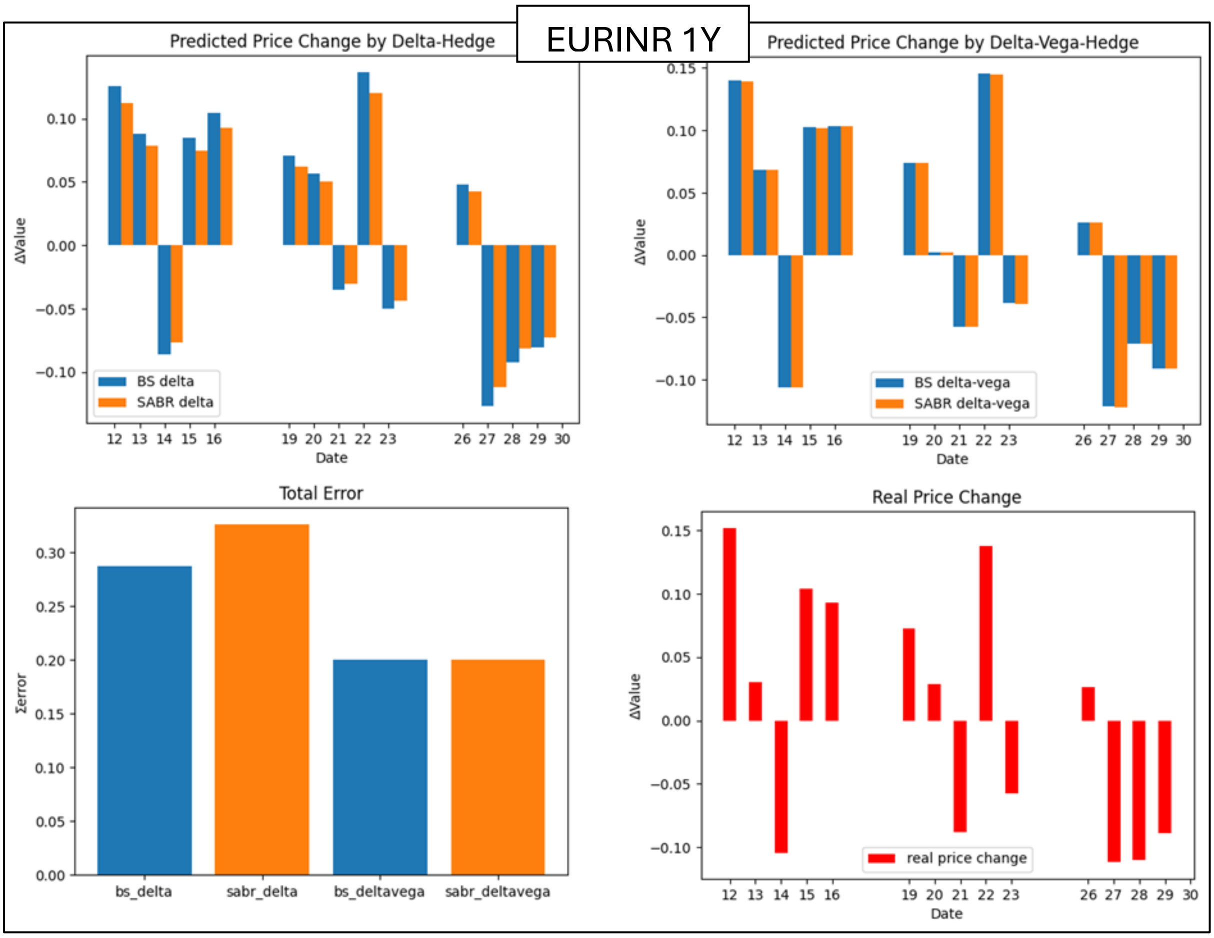


Figure 7 Hedging Performance on EURINR 1Y Option

Repeating with all currency pairs, tenors and strikes, SABR delta-vega hedging performed better on ITM options but performs worse than BS on OTM options, shown in Figure 8. The BS and SABR have nearly identical delta-vega predictions with r=0.998, and are both correlated to real price movement at r=0.948 and 0.947 respectively.

Pearson R	**Actual Price Change**	**BS delta-vega**	**SABR delta-vega**
Actual Price Change			
BS delta-vega	0.994795		
SABR delta-vega	0.994703	0.999789	

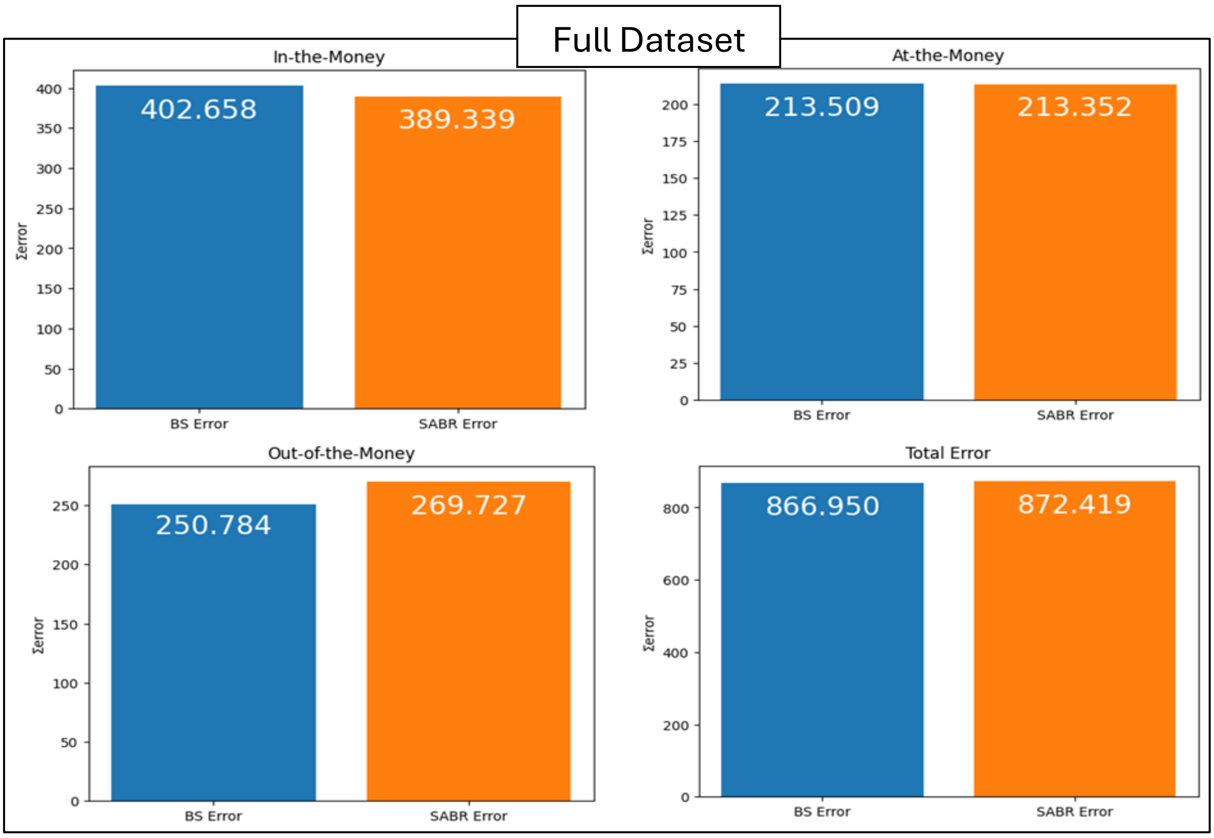


Figure 8 Delta-Hedging on All Currencies at ATM,OTM,ITM

Residuals Analysis

The residuals for both models show a strong linear relationship, shown in Figure 9, suggesting that the models are very similar. Figure 10 shows the residuals plot of the linear regression of the real price change against both models' predictions. As the residuals of SABR model are not of statistical significance compared to BS, there is minimal performance gain to using SABR for hedging purposes only.

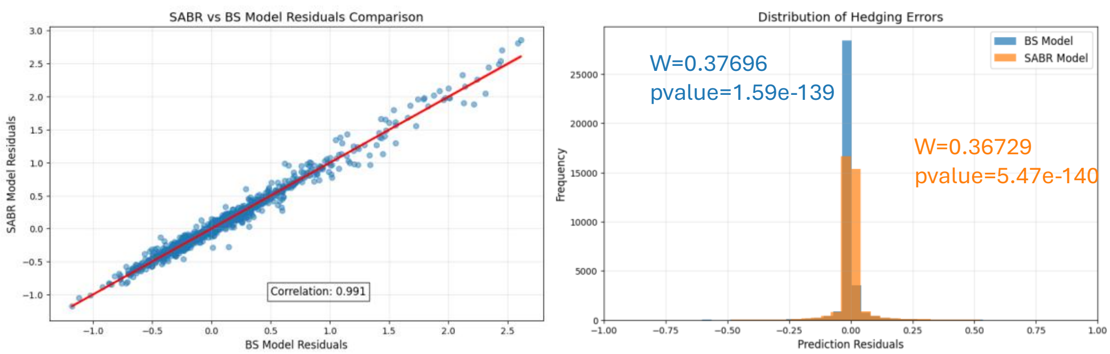


Figure 9 Residuals Comparison

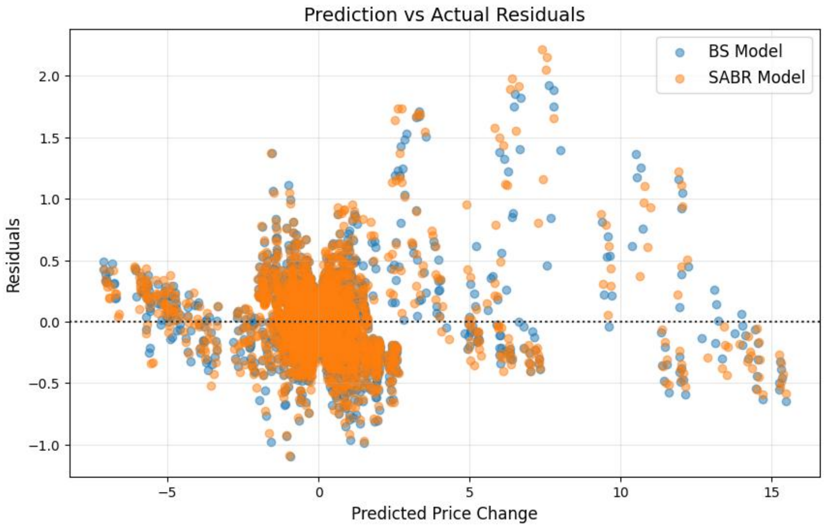


Figure 10 Residuals plot of Price Change against Both Models' Predictions

Vega Component Comparison

As the only difference between both price change predictions are the bolded terms in equations (6) and (7), which are the coefficients of vega, I tested the impact of this component in both models. In BS, the vega component accounted for only 27.9% of the predicted price change, while in SABR it contributed 28.9%. This low impact and high correlation of r=0.962 explains the similarity of both models' predictions in the previous section.

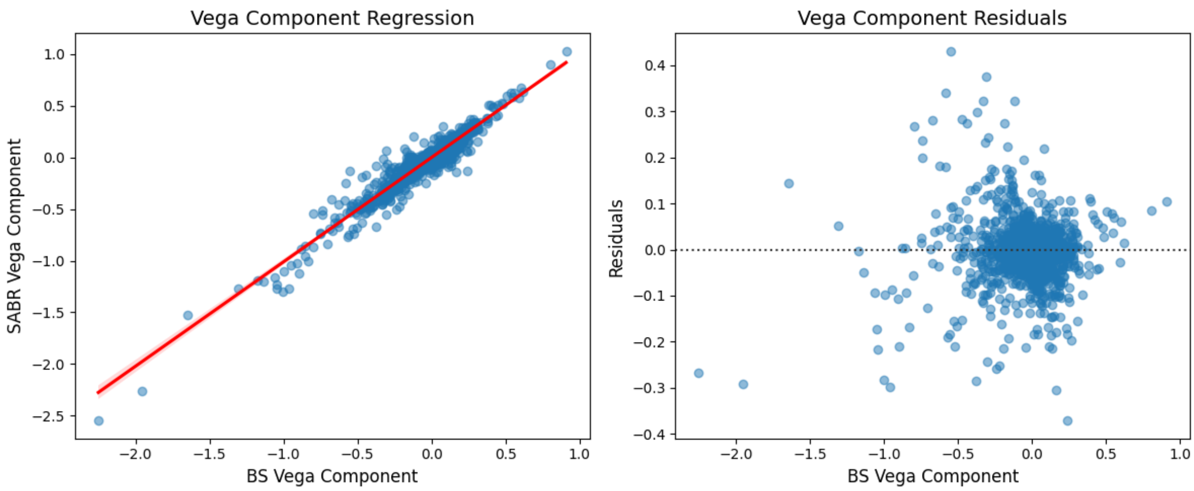


Figure 11 Linear Regression of the Vega (bold) components

Conclusion

This test fits the discrete volatility points to a smooth line (piecewise for cubic, and smoother for SABR) and shows that both methods can be used for hedging and for interpolating volatilities for intermediate strikes. Despite SABR's good fit on volatility smiles across all currencies, there is no significant benefit to using the SABR model over standard BS delta and vega for hedging in the forex market.

Bibliography

Hagan, Patrick, et al. *MANAGING SMILE RISK*. journal-article, www.next-finance.net/IMG/pdf/pdf_SABR.pdf.

Hagan, Patrick, and Andrew Lesniewski. “Bartlett’s Delta in the SABR Model.” *SSRN Electronic Journal*, Jan. 2017, https://doi.org/10.2139/ssrn.2950749.

Yang, Yifan, et al. “Stochastic Alpha-Beta-Rho hedging for foreign exchange options: Is it worth the effort?” *The Journal of Derivatives*, vol. 23, no. 2, Nov. 2015, pp. 76–89. https://doi.org/10.3905/jod.2015.23.2.076.

Zhang, Nan. “Properties of the SABR model.” *Uppsala Universitet*, Jan. 2011, uu.diva-portal.org/smash/get/diva2:430537/FULLTEXT01.pdf.