FOUR-DIMENSIONAL GEOMETRY AND VISUALIZATION

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$$\det(a,b,c,d) + (a \times b)(c \times d) = \begin{vmatrix} a \times c & b \cdot c \\ a \times d & b \cdot d \end{vmatrix} - \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix} + \begin{vmatrix} a \cdot c & b \times c \\ a \cdot d & b \times d \end{vmatrix}$$

$$z \longmapsto e^{(a \times b)(\alpha+\beta)} z e^{-(\bar{a} \times \bar{b})(\alpha-\beta)}$$

1. Introduction

Dimensions of two and four ($\mathbb{R}^2$ and $\mathbb{R}^4$) carry a small miracle found in no other dimension: they each possess a sphere with a continuous group structure—a rule for multiplying points. We are all familiar with the sphere sitting in two dimensions ($\mathbb{S}^1$, the circle), however the sphere in four dimensions ($\mathbb{S}^3$) is known mostly to specialists such as mathematicians and graphics programmers.

Even though our existence is three-dimensional (or appears to be so), we must jump into four dimensions, into this $\mathbb{S}^3$ group, for the most basic description of our three-dimensional life: the description of motion. In the transition from $\mathbb{R}^3$ to $\mathbb{R}^4$ we acquire a wide variety of new phenomenon arising from this continuous group structure. $\mathbb{R}^4$ is symmetrical in many respects, for instance having a uniform treatment of actions and vectors, while $\mathbb{R}^3$ is merely a lopsided subset thereof. It is as if all the good stuff happens in four dimensions but through some cosmic prank we seem to be stuck in three.

This paper is a short introduction to four-dimensional geometry, hitting some main points and essential definitions. It is intended to be widely accessible, perhaps even to a persevering high school student. Aside from the actual mathematics, the main goal is for the reader to understand and visualize four dimensions in much the same manner that one understands and visualizes three, at least to the extent that four dimensions is no longer particularly mysterious.

Quaternions are introduced and discussed first, as they serve as the underlying machinery of four dimensions (they represent the $\mathbb{S}^3$ group). The next sections are stepping stones to Section 7 which at last answers the question, "What *is* quaternion multiplication?" This leads directly to the main result of Section 8, which shows that quaternions indeed describe all motion in $\mathbb{R}^4$. Following this, an attempt is made to demonstrate why it is visually "obvious" the rotation group is parameterized by $\mathbb{S}^3 \times \mathbb{S}^3$. The remaining sections discuss a few other aspects of four-dimensional geometry.

The reasons for hoping this paper is widely accessible are (1) all functions discussed are linear and (2) quaternions provide tremendous shortcuts and generally make things easy. Indeed from a technical standpoint, most of the proofs presented here are either trivial—those carrying out direct computations—or next-to-trivial—those applying typographical rules to coax the quaternion machinery into revealing the desired result. Yet we should be careful not to overlook the meaning behind a given proposition however simple its appearance or its proof.

2. Quaternions

2.1. **Definitions.** The *quaternions* are the set

$$\mathbb{H} = \{a_1 + a_2 i + a_3 j + a_4 k \mid a_1, a_2, a_3, a_3 \text{ are real numbers}\}$$

together with two binary operations, addition and multiplication, defined as follows. Two quaternions $a_1 + a_2 i + a_3 j + a_4 k$ and $b_1 + b_2 i + b_3 j + b_4 k$ are defined to be equal if and only if

$$a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3, \quad a_4 = b_4.$$

Let $a = a_1 + a_2 i + a_3 j + a_4 k$ and $b = b_1 + b_2 i + b_3 j + b_4 k$. Addition in $\mathbb{H}$ is defined component-wise as in polynomials,

$$\begin{aligned} a + b &= (a_1 + a_2 i + a_3 j + a_4 k) + (b_1 + b_2 i + b_3 j + b_4 k) \\ &= (a_1 + b_1) + (a_2 + b_2)i + (a_3 + b_3)j + (a_4 + b_4)k. \end{aligned}$$

Multiplication in $\mathbb{H}$ is defined as polynomial multiplication with the additional rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

Therefore

$$\begin{aligned} ij &= k = -ji, \\ jk &= i = -kj, \\ ki &= j = -ik, \end{aligned}$$

and

$$\begin{aligned} ab &= (a_1 + a_2 i + a_3 j + a_4 k)(b_1 + b_2 i + b_3 j + b_4 k) \\ &= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4) + \\ &\quad (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)i + \\ &\quad (a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2)j + \\ &\quad (a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1)k. \end{aligned}$$

The *absolute value* of a is

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}.$$

The *conjugate* of a is

$$\bar{a} = a_1 - a_2 i - a_3 j - a_4 k.$$

The *dot product* of a and b is

$$a \cdot b = \frac{b\bar{a} + a\bar{b}}{2}.$$

The *cross product* of a and b is

$$a \times b = \frac{b\bar{a} - a\bar{b}}{2}.$$

The *real part* of a is

$$1 \cdot a = a_1.$$

The *pure part* of a is

$$1 \times a = a_2 i + a_3 j + a_4 k.$$

For general quaternions a, b, c, d, define the notation

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

It will be shown that a non-zero quaternion has a unique multiplicative inverse,

$$a^{-1} = \frac{\bar{a}}{|a|^2} \quad \text{for } a \neq 0.$$

If $|a| = 1$ then a is called a *unit quaternion.* A quaternion whose real part is zero is called a *pure quaternion* or a *pure*. A quaternion whose pure part is non-zero is called *non-real*. Two quaternions a, b are called *orthogonal* when $a \cdot b = 0$ and *parallel* when $a \times b = 0$. Two orthogonal unit quaternions are called *orthonormal*. The notation $\mathring{a} = 1 \cdot a$ and $\vec{a} = 1 \times a$ will be used (sparingly) to denote the real part of a and the pure part of a, respectively.

The canonical cross product in three dimensions is a special case of the above-defined quaternionic cross product (when both terms are pure).

Where appropriate, elements of $\mathbb{H}$ and vectors in $\mathbb{R}^4$ will be freely interchangeable (technically, we shall accept that $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$ as a vector space and use the representation provided by $\mathbb{H}$). Unless otherwise noted, all definitions and theorems will take place in the context of $\mathbb{R}^4$. Whenever a, b, c, d appear without qualifiers it is assumed they are general quaternions which are given. By convention u, v are unit pure and lowercase Greek letters are real.

2.2. **Basic Properties.** Quaternions are not in general commutative; one should expect $ab \neq ba$ unless a, b are known to have a particular relationship (Theorem 4).

The identities in Figures 1-4 will be assumed. Figure 5 contains a summary of theorems appearing here. As a general rule, identities will be presented without proof. The reader is encouraged to work out a number of them and to become familiar with the basic forms, since many will used without mention. For example the identity $b\bar{a} = a \cdot b + a \times b$ will be used often.

The dot product always yields a real number and the cross product always yields a pure. For this reason $a \times b$ is often the preferred form of $b\bar{a}$ whenever $a \cdot b = 0$, since $a \times b$ communicates a pure quaternion more concisely than "$b\bar{a}$, and remember it is pure." Likewise $a \cdot b$ is preferred over $b\bar{a}$ when $a \times b = 0$, although its purview is less interesting.

Theorem 1. *a is real if and only if $\bar{a} = a$ if and only if $a^2 = |a|^2$.*

Proof. 0 is real and $\bar{0} = 0$ and $0^2 = |0|^2$. Let $a \neq 0$.
a is real $\iff 1 \times a = 0 \iff a - \bar{a} = 0 \iff a = \bar{a} \iff a^2 = \bar{a}a$. □

Theorem 2. *a is pure if and only if $\bar{a} = -a$ if and only if $a^2 = -|a|^2$.*

Proof. 0 is pure and $\bar{0} = -0$ and $0^2 = -|0|^2$. Let $a \neq 0$.
a is pure $\iff 1 \cdot a = 0 \iff a + \bar{a} = 0 \iff a = -\bar{a} \iff a^2 = -\bar{a}a$. □

Theorem 3. *a^2 is real if and only if a is real or a is pure.*

Proof. a^2 is real $\iff 1 \times a^2 = 0 \iff a^2 - \bar{a}^2 = 0 \iff (a - \bar{a})(a + \bar{a}) = 0 \iff (1 \times a)(1 \cdot a) = 0 \iff [\,1 \times a = 0 \text{ or } 1 \cdot a = 0\,]$. □

$$
\begin{aligned}
a+b &= b+a \\
(a+b)+c &= a+(b+c) \\
(a+b)c &= ac+bc \\
a(b+c) &= ab+ac \\
(ab)c &= a(bc) \\
\bar{\bar{a}} &= a \\
\overline{a+b} &= \bar{a}+\bar{b} \\
\overline{ab} &= \bar{b}\bar{a} \\
|ab| &= |a||b| \\
|a|^2 &= a\bar{a} = \bar{a}a \\
1 &= aa^{-1} = a^{-1}a
\end{aligned}
$$

FIGURE 1. Fundamental identities.

$$
\begin{aligned}
1\cdot a &= \tfrac{1}{2}(a+\bar{a}) \\
1\times a &= \tfrac{1}{2}(a-\bar{a}) \\
a &= 1\cdot a + 1\times a \\
a\cdot a &= |a|^2 = (1\cdot a)^2 + |1\times a|^2 \\
a\times a &= 0 \\
a\cdot b &= b\cdot a = \bar{a}\cdot\bar{b} = 1\cdot b\bar{a} \\
a\times b &= -b\times a = -a(\bar{a}\times\bar{b})a^{-1} = 1\times b\bar{a} \\
b\bar{a} &= a\cdot b + a\times b \\
|ab|^2 &= (a\cdot b)^2 + |a\times b|^2 \\
a\cdot(b+c) &= a\cdot b + a\cdot c \\
a\times(b+c) &= a\times b + a\times c \\
ab\cdot c &= a\cdot c\bar{b} = b\cdot\bar{a}c \\
ab\times c &= a\times c\bar{b} = a(b\times\bar{a}c)a^{-1} \\
a\cdot(b\cdot c) &= \tfrac{1}{2}(ab\cdot c + ac\cdot b) \\
a\cdot(b\times c) &= \tfrac{1}{2}(ab\cdot c - ac\cdot b) \\
a\times(b\cdot c) &= \tfrac{1}{2}(ab\times c + ac\times b) \\
a\times(b\times c) &= \tfrac{1}{2}(ab\times c - ac\times b) \\
(a\cdot b)(c\cdot d) &= \tfrac{1}{2}(ac\cdot bd + ad\cdot bc) \\
(a\times b)\cdot(c\times d) &= \begin{vmatrix} a\cdot c & b\cdot c \\ a\cdot d & b\cdot d \end{vmatrix} + \det(a,b,c,d) \\
(a\times b)\times(c\times d) &= \begin{vmatrix} a\cdot c & b\times c \\ a\cdot d & b\times d \end{vmatrix} + \begin{vmatrix} a\times c & b\cdot c \\ a\times d & b\cdot d \end{vmatrix}
\end{aligned}
$$

FIGURE 2. Dot and cross identities.

$$
\begin{aligned}
a \cdot b &= a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 \\
a \times b &= \left(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} \right) i \; + \\
&\quad \left(\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + \begin{vmatrix} a_4 & b_4 \\ a_2 & b_2 \end{vmatrix} \right) j \; + \\
&\quad \left(\begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \right) k
\end{aligned}
$$

FIGURE 3. Expansion of dot and cross.

$$
\begin{aligned}
\overline{a^{-1}} &= \bar{a}^{-1} \\
ab &= a \cdot \bar{b} - a \times \bar{b} \\
a \cdot \bar{b} &= \bar{a} \cdot b \\
(a+b)^2 &= a^2 + b^2 + ab + ba \\
|a+b|^2 &= |a|^2 + |b|^2 + 2a \cdot b \\
a^2 \cdot b^2 &= (a \cdot b)^2 + (a \times b) \cdot (\bar{a} \times \bar{b}) \\
ab + ba &= a(\bar{a}b + b\bar{a})\bar{a}^{-1} = a(2a \cdot b + a \times b - \bar{a} \times \bar{b})\bar{a}^{-1} = 2a \cdot \bar{b} - a \times \bar{b} + \bar{a} \times b \\
ab - ba &= a(ab - ba)\bar{a}^{-1} = a \times b + \bar{a} \times \bar{b} = -a \times \bar{b} - \bar{a} \times b = \bar{a}\bar{b} - \bar{b}\bar{a} = b\bar{a} - \bar{a}b = \bar{b}a - a\bar{b}
\end{aligned}
$$

FIGURE 4. Other identities.

$$
\begin{array}{rcrclcrcl}
a \text{ is real} & \iff & \bar{a} &=& a & \iff & a^2 &=& |a|^2 \\
a \text{ is pure} & \iff & \bar{a} &=& -a & \iff & a^2 &=& -|a|^2 \\
a \times b = 0 & \iff & a\bar{b} &=& b\bar{a} & \iff & a\bar{b} &=& a \cdot b \\
a \cdot b = 0 & \iff & a\bar{b} &=& -b\bar{a} & \iff & a\bar{b} &=& -a \times b \\
a \times b = 0 \text{ or } a, b \text{ pure} & \iff & a\bar{b} &=& \bar{a}b & \iff & a \times b &=& \bar{a} \times \bar{b} \\
a \cdot b = 0 \text{ and } ab = ba & \iff & a\bar{b} &=& -\bar{a}b & \iff & a^2|b|^2 &=& -|a|^2 b^2 \\
ab = 0 & \iff & a\bar{b} &=& -a \cdot b & \iff & a\bar{b} &=& a \times b \\
\vec{a} \times \vec{b} = 0 & \iff & ab &=& ba & \iff & a \times b &=& -\bar{a} \times \bar{b} \\
ab = 0 \text{ or } [\, a \cdot b = 0 \text{ and } a, b \text{ pure}] & \iff & ab &=& -ba & \iff & ab &=& -ba = a \times b \\
ab = 0 \text{ or } a, b \text{ real} & \iff & ab &=& a \cdot b & \iff & ab &=& ba = a \cdot b \\
ab = 0 \text{ or } [\, a \times b = 0 \text{ and } a, b \text{ pure}] & \iff & ab &=& -a \cdot b & \iff & ab &=& ba = -a \cdot b \\
a = 0 \text{ or } [\, a \cdot b = 0 \text{ and } b \text{ is pure}] & \iff & ab &=& a \times b & \iff & ba &=& -\bar{a} \times \bar{b} \\
ab = 0 \text{ or } [\, a \text{ is pure and } b \text{ is real}] & \iff & ab &=& -a \times b & \iff & ab &=& ba = -a \times b
\end{array}
$$

FIGURE 5. Equivalences.

Theorem 4. $ab = ba$ *iff* $\vec{a}\vec{b} = \vec{b}\vec{a}$ *iff* $\vec{a} \times \vec{b} = 0$ *iff* $a \times b = -\bar{a} \times \bar{b}$ *iff* $a \times \bar{b} = -\bar{a} \times b$. *That is, a and b commute if and only if their pure parts commute if and only if their pure parts are parallel. In particular, real numbers commute with all quaternions.*

Proof. If a is real or b is real then $ab = ba$ directly from the definition of multiplication. For general a and b, $ab = ba \iff (\mathring{a} + \vec{a})(\mathring{b} + \vec{b}) = (\mathring{b} + \vec{b})(\mathring{a} + \vec{a}) \iff \mathring{a}\mathring{b} + \mathring{a}\vec{b} + \vec{a}\mathring{b} + \vec{a}\vec{b} = \mathring{b}\mathring{a} + \mathring{b}\vec{a} + \vec{b}\mathring{a} + \vec{b}\vec{a} \iff \vec{a}\vec{b} = \vec{b}\vec{a} \iff \vec{a} \times \vec{b} = 0$.

The last two statements are identical to $ab = ba$. □

Theorem 5. *Given $a \neq 0$. $ab = ba = 1$ if and only if $b = \bar{a}/|a|^2$.*

Proof. If $b = \bar{a}/|a|^2$ then the relation $ab = ba = 1$ follows trivially from the identity $a\bar{a} = \bar{a}a = |a|^2$ in Figure 1. Conversely, $ab = 1 \iff \bar{a}ab = \bar{a} \iff |a|^2 b = \bar{a} \iff b = \bar{a}/|a|^2$, and likewise $ba = 1 \iff b = \bar{a}/|a|^2$. □

Theorem 6. *$ab \cdot c = a(b \cdot c)$ if and only if $ab \times c = a(b \times c)$ if and only if $bc = 0$ or a is real.*

Proof. $ab \cdot c = a(b \cdot c) \iff c\bar{b}\bar{a} + ab\bar{c} = a(c\bar{b} + b\bar{c}) \iff c\bar{b}\bar{a} = ac\bar{b} \iff c\bar{b}\bar{a} - ab\bar{c} = ac\bar{b} - ab\bar{c} \iff c\bar{b}\bar{a} - ab\bar{c} = a(c\bar{b} - b\bar{c}) \iff ab \times c = a(b \times c)$. Therefore the first two parts of the proposition are equivalent.

Let the notation $\overset{\cdot}{\times}$ represent the dot product or the cross product, in the same way that $\pm$ represents plus or minus. Starting from the common relation above, $c\bar{b}\bar{a} = ac\bar{b} \iff ab\bar{c} = b\bar{c}\bar{a} \implies c\bar{b}\bar{a} \pm b\bar{c}\bar{a} = ac\bar{b} \pm ab\bar{c} \iff (c\bar{b} \pm b\bar{c})\bar{a} = a(c\bar{b} \pm b\bar{c}) \iff (b \overset{\cdot}{\times} c)\bar{a} = a(b \overset{\cdot}{\times} c) \iff (a - \bar{a})(b \overset{\cdot}{\times} c) = 0 \iff (1 \times a)(b \overset{\cdot}{\times} c) = 0 \iff [\, a$ is real or $b \overset{\cdot}{\times} c = 0\,]$. But $b \overset{\cdot}{\times} c = 0 \implies b \cdot c = b \times c = 0 \implies bc = 0$.

Conversely, if a is real then $ab \cdot c = \frac{1}{2}(c\bar{b}\bar{a} + ab\bar{c}) = \frac{1}{2}a(c\bar{b} + b\bar{c}) = a(b \cdot c)$. The proposition holds trivially for $bc = 0$. □

Theorem 7. *$b\bar{a} = -a\bar{b}$ if and only if $b\bar{a} = a \times b$ if and only if $a \cdot b = 0$.*

Proof. $b\bar{a} = -a\bar{b} \iff b\bar{a} + a\bar{b} = 0 \iff a \cdot b = 0 \iff b\bar{a} - a \times b = 0 \iff b\bar{a} = a \times b$. □

Theorem 8. *$b\bar{a} = a\bar{b}$ if and only if $b\bar{a} = a \cdot b$ if and only if $a \times b = 0$.*

Proof. $b\bar{a} = a\bar{b} \iff b\bar{a} - a\bar{b} = 0 \iff a \times b = 0 \iff b\bar{a} - a \cdot b = 0 \iff b\bar{a} = a \cdot b$. □

Theorem 9. *Given $a \neq 0$. b is a scalar multiple of a if and only if $a \times b = 0$ if and only if $b = \left(|a|^{-2} a \cdot b\right) a$.*

Proof. $[\, \gamma a = b$ for some real $\gamma\,] \implies |a|^2\gamma = b\bar{a} \implies |a|^2\gamma = a \cdot b + a \times b \implies a \times b = 0 \iff b\bar{a} = a \cdot b \iff b = |a|^{-2}(a \cdot b)a$. □

Theorem 10. *Given $a \neq 0$. b is a pure multiple of a if and only if $a \cdot b = 0$ if and only if $b = \left(|a|^{-2} a \times b\right) a$ if and only if $b = a\left(-|a|^{-2}\bar{a} \times \bar{b}\right)$.*

Proof. $[\, au = b$ for some pure $u\,] \implies |a|^2 u = \bar{a}b \implies |a|^2 u = \bar{a} \cdot \bar{b} - \bar{a} \times \bar{b} \implies \bar{a} \cdot \bar{b} = 0 \implies a \cdot b = 0$.
$[\, ua = b$ for some pure $u\,] \implies |a|^2 u = b\bar{a} \implies |a|^2 u = a \cdot b + a \times b \implies a \cdot b = 0$.
$a \cdot b = 0 \iff b\bar{a} = a \times b \iff b = |a|^{-2}(a \times b)a \iff b = -|a|^{-2}a(\bar{a} \times \bar{b})$.
(See Figure 2.) □

Theorem 11. *If $a \times b = 0$ then $ab = ba$.*

Proof. $0 \times b = 0$ and $0b = b0$. Let $a \neq 0$.
$a \times b = 0 \iff \bar{a} \times \bar{b} = 0 \implies a \times b = -\bar{a} \times \bar{b} \iff ab = ba$. (Theorem 4.) ☐

Theorem 12. *$ab = a \cdot b$ if and only if $ab = ba = a \cdot b$ if and only if $ab = 0$ or a, b are real.*

Proof. The proposition holds trivially when $a = 0$ or $b = 0$; let a, b be non-zero. Begin from the first part of the proposition.
$ab = a \cdot b \iff a \cdot \bar{b} - a \times \bar{b} = a \cdot b \iff [\, a \cdot \bar{b} = a \cdot b$ and $a \times \bar{b} = 0\,]$.
$a \cdot \bar{b} = a \cdot b \implies a \cdot (b - \bar{b}) = 0 \implies a \cdot \vec{b} = 0 \implies \vec{a} \cdot \vec{b} = 0$.
$a \times \bar{b} = 0 \implies \bar{a} \times b = 0 \implies a \times \bar{b} = -\bar{a} \times b \iff ab = ba \iff \vec{a} \times \vec{b} = 0 \implies -\vec{a} \cdot \vec{b} + \vec{a} \times \vec{b} = 0 \implies \vec{a}\vec{b} = 0 \implies [\, a$ is real or b is real$\,]$.
$[\, a$ is real and $a \times \bar{b} = 0\,] \implies a(1 \times \bar{b}) = 0 \implies 1 \times b = 0 \implies b$ is real.
$[\, b$ is real and $a \times \bar{b} = 0\,] \implies \bar{b}(a \times 1) = 0 \implies 1 \times a = 0 \implies a$ is real.
Therefore $ab = a \cdot b \implies [\, ab = 0$ or a, b are real$\,]$.

The remaining directions are trivial. ☐

Theorem 13. *$ab = -a \cdot b$ if and only if $ab = ba = -a \cdot b$ if and only if $ab = 0$ or a, b are parallel pures.*

Proof. Starting from the first part of the proposition,
$ab = -a \cdot b \iff a \cdot \bar{b} - a \times \bar{b} = -a \cdot b \iff a \cdot (b + \bar{b}) = a \times \bar{b} \iff 2a \cdot \mathring{b} = a \times \bar{b} \iff [\, a \cdot \mathring{b} = 0$ and $a \times \bar{b} = 0\,]$.
$a \cdot \mathring{b} = 0 \implies \mathring{b}(1 \cdot a) = 0 \implies [\, a$ is pure or b is pure$\,]$.
$a = 0 \implies a \times b = 0$.
$[\, a \times \bar{b} = 0$ and a is non-zero pure$\,] \implies \bar{a} \times b = 0 \implies a \times b = 0$.
$[\, a \times \bar{b} = 0$ and b is pure$\,] \implies a \times b = 0$.
Thus $ab = -a \cdot b \implies a \times b = 0$. But $a \times b = 0 \implies ab = ba$, making the first two parts of the proposition equivalent.

To obtain the last part,
$ab = -a \cdot b \iff ab = -a\bar{b} - a \times b \iff a(b + \bar{b}) = -a \times b \iff 2a\mathring{b} = -a \times b$.
$ba = -a \cdot b \iff ba = -b\bar{a} + a \times b \iff b(a + \bar{a}) = a \times b \iff 2\mathring{a}b = a \times b$.
Thus $a\mathring{b} = \mathring{a}b = a \times b = 0$, which is equivalent to the last part of the proposition. ☐

Theorem 14. *$ab = a \times b$ if and only if $ba = -\bar{a} \times \bar{b}$ if and only if $a = 0$ or $[\, a \cdot b = 0$ and b is pure$\,]$.*

Proof. The identity $ab - ba = a \times b + \bar{a} \times \bar{b}$ implies the first two parts of the proposition are equivalent. For the last part,
$ab = a \times b \iff a \cdot \bar{b} - a \times \bar{b} = a \times b \iff a \cdot \bar{b} = a \times (b + \bar{b}) \iff a \cdot \bar{b} = 2a \times \mathring{b} \iff [\, a \cdot \bar{b} = 0$ and $a \times \mathring{b} = 0\,]$.
$a \times \mathring{b} = 0 \implies -\mathring{b}(1 \times a) = 0 \implies [\, a$ is real or b is pure$\,]$.
$[\, a \cdot \bar{b} = 0$ and a is real$\,] \implies \bar{a} \cdot b = 0 \implies a \cdot b = 0 \implies a(1 \cdot b) = 0 \implies [\, a = 0$ or b is pure$\,]$.
$[\, a \cdot \bar{b} = 0$ and b is pure$\,] \implies a \cdot b = 0$.
Therefore $ab = a \times b \implies [\, a = 0$ or $[\, a \cdot b = 0$ and b is pure$\,]\,]$.

Conversely,
$a = 0 \implies ab = 0 = 0 \times b = a \times b$.
$[\, a \cdot b = 0$ and b is pure$\,] \implies ab = -a \cdot b + a \times b = a \times b$. ☐

Theorem 15. *$a\bar{b} = -\bar{a}b$ if and only if $a \cdot b = 0$ and $ab = ba$.*

Proof. $a\bar{b} = -\bar{a}b \iff a \cdot b - a \times b = -\bar{a} \cdot \bar{b} + \bar{a} \times \bar{b} \iff 2a \cdot b = a \times b + \bar{a} \times \bar{b} \iff$ $[\, a \cdot b = 0$ and $a \times b = -\bar{a} \times \bar{b}\,] \iff [\, a \cdot b = 0$ and $ab = ba\,]$. (Theorem 4.) □

Theorem 16. *$a\bar{b} = \bar{a}b$ iff $a \times b = \bar{a} \times \bar{b}$ iff a, b are parallel or a, b are pure.*

Proof. The first two parts of the proposition are identical. To obtain the last part, begin with an identity from Figure 4:
$ab - ba = a \times b + \bar{a} \times \bar{b} \iff ab - ba = 2a \times b \iff ab - ba = b\bar{a} - a\bar{b} \iff$ $a(b + \bar{b}) = b(a + \bar{a}) \iff (1 \cdot b)a = (1 \cdot a)b$, which is equivalent to the last part of the proposition. □

Theorem 17. *$ab = -ba$ iff $ab = -ba = a \times b$ iff $ab = 0$ or a, b are orthogonal pures.*

Proof. The proposition is trivial when $a = 0$ or $b = 0$; let a, b be non-zero. Starting from the first part of the proposition,
$ab + ba = 0 \iff b\bar{a} + \bar{a}b = 0 \iff \bar{b}a + a\bar{b} = 0 \implies (b\bar{a} + a\bar{b}) + (\bar{b}a + \bar{a}b) = 0 \iff$ $a \cdot b + \bar{a} \cdot \bar{b} = 0 \iff a \cdot b = 0$.
$ab + ba = 0 \iff b\bar{a} + \bar{a}b = 0 \implies (ab + \bar{a}b) + (ba + b\bar{a}) = 0 \iff (a + \bar{a})b + b(a + \bar{a}) = 0$ $\iff \mathring{a}b + b\mathring{a} = 0 \iff \mathring{a}b = 0$.
$ab + ba = 0 \iff \bar{b}a + a\bar{b} = 0 \implies (ab + a\bar{b}) + (ba + \bar{b}a) = 0 \iff a(b + \bar{b}) + (b + \bar{b})a = 0$ $\iff a\mathring{b} + \mathring{b}a = 0 \iff a\mathring{b} = 0$.
The relation $\mathring{a}b = a\mathring{b} = a \cdot b = 0$ is equivalent to the last part of the proposition.
The remaining directions are trivial. □

Theorem 18. *$ab = -a \times b$ if and only if $ab = ba = -a \times b$ if and only if $ab = 0$ or $[\, a$ is pure and b is real$\,]$.*

Proof. The proposition is trivially true when $a = 0$ or $b = 0$; let a, b be non-zero. Starting from the first part of the proposition,
$ab = -a \times b \iff a \cdot \bar{b} - a \times \bar{b} = -a \times b \iff [\, a \cdot \bar{b} = 0$ and $a \times \bar{b} = a \times b\,]$.
$a \times \bar{b} = a \times b \implies a \times (b - \bar{b}) = 0 \implies a \times \vec{b} = 0 \implies \vec{b} \cdot (a \times \vec{b}) = 0 \implies$ $\vec{b}a \cdot \vec{b} - \vec{b}^2 \cdot a = 0 \implies -a \cdot \vec{b}^2 - \vec{b}^2 \cdot a = 0 \implies a|\vec{b}|^2 = 0 \implies b$ is real.
$[\, ab = -a \times b$ and b is real$\,] \implies a = 1 \times a \implies a$ is pure.
Therefore $ab = -a \times b \implies [\, ab = 0$ or $[\, a$ is pure and b is real$\,]\,]$.
The remaining directions are trivial. □

Theorem 19. *Given $a \neq 0$. $a \cdot ab = 0$ iff $a \cdot ba = 0$ iff b is pure iff $ab\bar{a}$ is pure.*

Proof. Apply the rule $ab \cdot c = a \cdot c\bar{b} = b \cdot \bar{a}c$ from Figure 2. □

Theorem 20. *$ab = bc$ if and only if $b = 0$ or* $[\, 1 \cdot a = 1 \cdot c$ and $\vec{a}b = b\vec{c}\,]$

Proof. The proposition holds trivially for $b = 0$. Let $b \neq 0$.
$ab = bc \iff a = bcb^{-1} \iff \mathring{a} + \vec{a} = \mathring{c} + b\vec{c}b^{-1} \iff [\, \mathring{a} = \mathring{c}$ and $\vec{a}b = b\vec{c}\,]$.
The last step uses Theorem 19. □

3. Polar Form

Theorem 21. *If $a \neq 0$ and $b \neq 0$ then $a \cdot b = |a||b|\cos\theta$ and $|a \times b| = |a||b|\sin\theta$ where θ is the angle between a and b.*

Proof. The angles of a triangle with vertices $(a, 0, b)$ are respectively equal to those of a triangle with vertices $(|a|^2, 0, b\bar{a})$ because the respective sides are proportional:

$$|a|^2 = |a||a|, \quad |b\bar{a}| = |b||a|, \quad |a\bar{a} - b\bar{a}| = |a-b||a|.$$

In particular, the angle between the vectors a and b is equal to the angle between $|a|^2$ and $b\bar{a}$, call it θ. This is, of course, the angle between 1 and $b\bar{a}$. The identities

$$b\bar{a} = 1 \cdot b\bar{a} + 1 \times b\bar{a}, \quad |b\bar{a}|^2 = (1 \cdot b\bar{a})^2 + |1 \times b\bar{a}|^2$$

geometrically imply $1 \cdot b\bar{a} = |b\bar{a}|\cos\theta$ and $|1 \times b\bar{a}| = |b\bar{a}|\sin\theta$, which are readily transformed into $a \cdot b = |a||b|\cos\theta$ and $|a \times b| = |a||b|\sin\theta$. □

Theorem 22. *A non-real unit quaternion may be expressed as $a = \cos\theta + u\sin\theta$ where $u = \vec{a}/|\vec{a}|$ is a unit pure and θ is the angle between 1 and a.*

Proof. $a = 1 \cdot a + 1 \times a = 1 \cdot a + \frac{1\times a}{|1\times a|}|1 \times a| = \cos\theta + \frac{1\times a}{|1\times a|}\sin\theta$. □

Definition. For unit pure u and real α, θ, define the *polar form*

$$e^{\alpha+u\theta} = e^{\alpha}(\cos\theta + u\sin\theta).$$

For real λ in the open interval $(-\pi, \pi)$ define

$$\left(e^{\alpha+u\lambda}\right)^{\gamma} = e^{(\alpha+u\lambda)\gamma}.$$

The notation $\sqrt{a}$ shall be synonymous with $a^{\frac{1}{2}}$. The form a^{γ} shall remain undefined when a is a negative real number and γ is a not an integer; in particular $\sqrt{-1}$ shall not be defined.

See the appendix for more information on e^a.

Theorem 23. *For unit pure u and real α, θ,*

$$e^{\alpha+u\theta} = e^{\alpha}e^{u\theta}.$$

Proof. $e^{\alpha+u\theta} = e^{\alpha}(\cos\theta + u\sin\theta) = e^{\alpha}\left[e^{0}(\cos\theta + u\sin\theta)\right] = e^{\alpha}e^{0+u\theta} = e^{\alpha}e^{u\theta}$. □

$$\begin{aligned}
e^{a} &= e^{1\cdot a}\left(\cos|\vec{a}| + \frac{\vec{a}}{|\vec{a}|}\sin|\vec{a}|\right) \\
\overline{e^{a}} &= e^{\bar{a}} \\
e^{aba^{-1}} &= ae^{b}a^{-1} \\
e^{a}e^{b} &= e^{a+b} \iff ab = ba \\
e^{a}b &= be^{a} \iff ab = ba \\
e^{a}b &= be^{-a} \iff ab = -ba \\
a \cdot b &= |a||b|\cos\theta \\
|a \times b| &= |a||b|\sin\theta
\end{aligned}$$

Figure 6. Polar identities and relations.

Theorem 24. *$e^a e^b = e^{a+b}$ if and only if $ab = ba$.*

Proof. If a is real or b is real then the proposition is trivial. Let a and b be non-real with $a = \alpha + u\theta$ and $b = \beta + v\gamma$ for unit pure u, v and real $\alpha, \theta, \beta, \gamma$.

$ab = ba \implies u \times v = 0 \implies v = \pm u \implies uv = \pm u^2 = \mp 1 \implies e^{u\theta}e^{v\gamma} = (\cos\theta + u\sin\theta)(\cos\gamma + v\sin\gamma) = (\cos\theta\cos\gamma \mp \sin\theta\sin\gamma) + u(\sin\theta\cos\gamma \pm \cos\theta\sin\gamma) = \cos(\theta \pm \gamma) + u\sin(\theta \pm \gamma) = e^{u(\theta\pm\gamma)} = e^{u\theta \pm u\gamma} = e^{u\theta + v\gamma}$. Therefore $ab = ba \implies e^a e^b = e^{\alpha+u\theta}e^{\beta+v\gamma} = e^{\alpha}e^{u\theta}e^{\beta}e^{v\gamma} = e^{\alpha}e^{\beta}e^{u\theta}e^{v\gamma} = e^{\alpha+\beta}e^{u\theta+v\gamma} = e^{(\alpha+\beta)+(u\theta+v\gamma)} = e^{(\alpha+u\theta)+(\beta+v\gamma)} = e^{a+b}$.

Conversely, if $u\theta = -v\gamma$ then u and v are parallel and consequently $ab = ba$. Otherwise if $u\theta \neq -v\gamma$ then

$$\begin{aligned} &e^a e^b = e^{a+b} \implies e^{u\theta}e^{v\gamma} = e^{u\theta+v\gamma} \implies \\ &\quad(\cos\theta\cos\gamma - u \cdot v\sin\theta\sin\gamma) + (u\sin\theta\cos\gamma + v\cos\theta\sin\gamma + u \times v\sin\theta\sin\gamma) \\ &\qquad\qquad = \cos|u\theta + v\gamma| + \frac{u\theta + v\gamma}{|u\theta + v\gamma|}\sin|u\theta + v\gamma|. \end{aligned}$$

Dotting both sides of this equation by $u \times v$ gives

$$|u \times v|^2 \sin\theta\sin\gamma = 0 \implies u \times v = 0 \implies ab = ba.$$

□

Theorem 25. *If $ab = bc$ then $e^a b = be^c$. If $e^a b = be^c$ with $|\vec{a}|$ and $|\vec{c}|$ both in $[0, \pi]$ then $ab = bc$.*

Proof. If $b = 0$ or a is real or c is real then the proposition holds trivially. Let $b \neq 0$ and let a, c be non-real.

Applying an identity from Figure 6,

$$ab = bc \implies a = bcb^{-1} \implies e^a = e^{bcb^{-1}} \implies e^a = be^c b^{-1} \implies e^a b = be^c.$$

Given $e^a b = be^c$ with $|\vec{a}|$ and $|\vec{c}|$ both in $[0, \pi]$,

$$\begin{aligned} &e^a b = be^c \implies e^{1\cdot a}e^{1\times a}b = be^{1\cdot c}e^{1\times c} \\ &\qquad \implies |e^{1\cdot a}e^{1\times a}b| = |be^{1\cdot c}e^{1\times c}| \implies e^{1\cdot a} = e^{1\cdot c} \implies 1\cdot a = 1\cdot c. \end{aligned}$$

Using this result and the identity from Figure 6 again,

$$\begin{aligned} &e^a b = be^c \implies e^{1\times a}b = be^{1\times c} \implies e^{1\times a} = be^{1\times c}b^{-1} \\ &\quad \implies e^{1\times a} = e^{b(1\times c)b^{-1}} \implies \cos|\vec{a}| + \frac{\vec{a}}{|\vec{a}|}\sin|\vec{a}| = \cos|\vec{c}| + \frac{b\vec{c}\,b^{-1}}{|\vec{c}|}\sin|\vec{c}| \\ &\qquad \implies \cos|\vec{a}| = \cos|\vec{c}| \implies |\vec{a}| = |\vec{c}| \implies \vec{a} = b\vec{c}b^{-1}. \end{aligned}$$

Therefore $1\cdot a = 1\cdot c$ and $\vec{a}b = b\vec{c}$. By Theorem 20, $ab = bc$. □

Theorem 26. *$e^a b = be^a$ if and only if $ab = ba$.*

Proof. Corollary to Theorem 25. □

Theorem 27. *$e^a b = be^{-a}$ if and only if $ab = -ba$.*

Proof. Corollary to Theorem 25. □

4. Determinants and Bases

Definition. The *determinant* of a, b, c, d is defined

$$\det(a,b,c,d) = (a \times b)\cdot(c \times d) - \begin{vmatrix} a\cdot c & b\cdot c \\ a\cdot d & b\cdot d \end{vmatrix}.$$

Theorem 28. *Swapping two elements of a determinant reverses its sign only. For example,* $\det(a,b,c,d) = -\det(b,a,c,d)$.

Proof. The anti-commutative property of the cross product (i.e., $a \times b = -b \times a$) handles the case when a, b are swapped and when c, d are swapped. The remaining cases follow from the identity

$$(a \times b)\cdot(c \times d) + (a \times c)\cdot(b \times d) = \begin{vmatrix} a\cdot c & b\cdot c \\ a\cdot d & b\cdot d \end{vmatrix} + \begin{vmatrix} a\cdot b & c\cdot b \\ a\cdot d & c\cdot d \end{vmatrix}$$

which is

$$\det(a,b,c,d) = -\det(a,c,b,d).$$

As always, identities are punted to the reader. □

Theorem 29.

$$\det(\gamma a,b,c,d) = \gamma \det(a,b,c,d) \tag{1}$$
$$\det(a+r,b,c,d) = \det(a,b,c,d) + \det(r,b,c,d) \tag{2}$$
$$\det(a,a,c,d) = 0 \tag{3}$$
$$\det(ar,br,cr,dr) = \det(ra,rb,rc,rd) = |r|^4 \det(a,b,c,d) \tag{4}$$
$$\det(a,b,c,d) = |a|^{-2}(a \times b)\cdot[(a \times c)\times(a \times d)] \tag{5}$$

Proof. (1) Scalars may be factored out of the dot and cross products (Theorem 6). (2) Apply the distributive rules for dot and cross in Figure 2. (3) Follows from the definition. (4) A result of transfer rules such as $ar \times br = a \times br\bar{r}$. (5) Corollary to the previous relation with $r = \bar{a}$. □

Theorem 30. *The determinant of four pures is zero.*

Proof. The equation

$$(a \times b)\cdot(c \times d) = \begin{vmatrix} a\cdot c & b\cdot c \\ a\cdot d & b\cdot d \end{vmatrix}$$

is an identity when a, b, c, d are pure. □

Exercise. Use the determinant properties and the last identity in Figure 2 to derive the identities

$$a\cdot(b \times c) = b\cdot(c \times a) = c\cdot(a \times b),$$
$$a \times (b \times c) = (a\cdot c)b - (a\cdot b)c$$

when a, b, c are pure. (These identities and the identity in Theorem 30 are special cases of the formula appearing at the very beginning of this paper.)

Theorem 31. *$(a \times b) \cdot (a \times c) = (a \times b) \cdot (b \times c) = 0$ if and only if $a \cdot (a \times b)c = b \cdot (a \times b)c = 0$ if and only if $a \times b = 0$ or $a \cdot c = b \cdot c = 0$.*

Proof. The first two parts of the proposition are identical.

Theorem 29.3 produces the identities (which hold for *all* a, b, c)

$$(a \times b) \cdot (a \times c) = |a|^2(b \cdot c) - (a \cdot b)(a \cdot c),$$
$$(a \times b) \cdot (b \times c) = (a \cdot b)(b \cdot c) - |b|^2(a \cdot c).$$

Clearly the last part of the proposition implies the first part.

Given the first part, without loss of generality make a and b unitary (while recognizing the proposition holds for $a = 0$ or $b = 0$). From above the identities, $[\,(a \cdot b)(a \cdot c) = b \cdot c$ and $(a \cdot b)(b \cdot c) = a \cdot c\,] \implies (a \cdot b + 1)(a \cdot c - b \cdot c) = 0 \implies [\,a \cdot b = -1$ or $a \cdot c = b \cdot c\,]$.
$a \cdot b = \pm 1 \implies |ab|^2 = (a \cdot b)^2 + |a \times b|^2 \implies 1 = 1 + |a \times b|^2 \implies a \times b = 0$.
$a \cdot c = b \cdot c \implies (a \cdot b)(a \cdot c) = a \cdot c \implies (a \cdot c)(a \cdot b - 1) = 0 \implies [\,a \cdot c = 0$ or $a \cdot b = 1\,] \implies [\,a \cdot c = b \cdot c = 0$ or $a \times b = 0\,]$. ☐

Theorem 32. *$a \cdot (a \times b) = b \cdot (a \times b) = 0$ if and only if $a \times b = 0$ or a, b are pure.*

Proof. Corollary to Theorem 31 with $c = 1$. ☐

Definition. A *basis* is an ordered four-tuple (a, b, c, d) with $\det(a, b, c, d) \neq 0$. A basis whose determinant is positive is called *positively-oriented*, otherwise it is *negatively-oriented*. A basis whose elements are mutually orthogonal is called an *orthogonal basis*. An orthogonal basis whose elements are unitary is an *orthonormal basis*. Two bases are defined to be equivalent if and only if their respective elements are equivalent.

Theorem 33. *The determinant of an orthogonal basis (a, b, c, d) is*

$$\det(a, b, c, d) = (a \times b) \cdot (c \times d) = -(a \times b)(c \times d) = a\bar{b} \cdot c\bar{d} = -a\bar{b}c\bar{d} = \bar{a}b\bar{c}d$$

with

$$(a \times b) \times (c \times d) = a\bar{b} \times c\bar{d} = 1 \times a\bar{b}c\bar{d} = 1 \times \bar{a}b\bar{c}d = 0.$$

Proof. These are straightforward from the definition except perhaps for $-a\bar{b}c\bar{d} = -a^{-1}a(a\bar{b}c\bar{d}) = -a^{-1}(a\bar{b}c\bar{d})a = -\bar{b}c\bar{d}a = \bar{a}b\bar{c}d$. ☐

Theorem 34. *If (a, b, c, d) is a positively-oriented orthonormal basis then*

$$(a, b, c, d) = (a, ua, c, uc) = (a, -av, c, cv)$$

where $u = a \times b = c \times d$ and $v = \bar{a} \times \bar{b} = -\bar{c} \times \bar{d}$.

Proof. $a \times b = b\bar{a} = b\bar{a}c\bar{d}d\bar{c} = (-a\bar{b}c\bar{d})d\bar{c} = d\bar{c} = c \times d$. The rest is left as exercise. ☐

Theorem 35. *If (a, b, c, d) is a positively-oriented orthonormal basis then*

$$(a, b, c, d) = (a, ua, va, uva)$$

where u, v are orthonormal with $u = a \times b$, $v = a \times c$, $uv = u \times v = a \times d = b \times c$.

Proof. Left as exercise. ☐

Theorem 36. *$(1, i, j, k)$ is a positively-oriented orthonormal basis.*

Proof. Left as exercise (easy). ☐

Theorem 37. *If u, v are unit pure with $u \times v \neq 0$ then*

$$\begin{gathered}(1+uv,\ u-v,\ 1-uv,\ u+v)\,,\\ \big(\sqrt{uv},\ u-v,\ \sqrt{-uv},\ u+v\big)\,,\\ \big(\sqrt{uv},\ u\sqrt{uv},\ \sqrt{-uv},\ u\sqrt{-uv}\big)\,,\end{gathered}$$

are positively-oriented orthogonal bases. The bases are the same up to the element lengths; see Figure 7.

Proof. Checking the identities in Figure 7 is left as an exercise. (Write $\sqrt{uv}$ and $\sqrt{-uv}$ in polar form.) □

Theorem 38. *If a, b are orthonormal with $ab \neq ba$ then*

$$\big(a,\ b,\ a^2+b^2,\ ba-ab\big)$$

is a positively-oriented orthogonal basis. If a, b are also restricted by $ab \neq -ba$ then

$$\big(a^2-b^2,\ ba+ab,\ a^2+b^2,\ ba-ab\big)$$

is a positively-oriented orthogonal basis. See Figure 8.

Proof. The restrictions are sufficient because $ba \pm ab = 0 \iff a \times b \mp \bar{a} \times \bar{b} = 0 \iff b\bar{a} \mp \bar{b}a = 0 \iff a^2 \mp b^2 = 0$. While the relations in Figure 8 may be verified directly, the following parameterization may be of interest:

$$\begin{aligned} u &= a \times b\\ v &= \bar{a} \times \bar{b}\end{aligned}$$

$$\begin{aligned} a^2-b^2 &= a(\overline{1+uv})a\\ ba+ab &= a(u-v)a\\ a^2+b^2 &= a(\overline{1-uv})a\\ ba-ab &= -a(u+v)a.\end{aligned}$$

□

$$\tfrac{1}{2}|1+uv| = \tfrac{1}{2}|u-v| = \sqrt{\tfrac{1}{2}(1-u\cdot v)} = 1\cdot\sqrt{uv} = |1\times\sqrt{-uv}|$$
$$\tfrac{1}{2}|1-uv| = \tfrac{1}{2}|u+v| = \sqrt{\tfrac{1}{2}(1+u\cdot v)} = 1\cdot\sqrt{-uv} = |1\times\sqrt{uv}|$$

$$\begin{aligned}
1+uv &= -u(u-v) &= \quad (u-v)v &= (2\cdot\sqrt{uv})\sqrt{uv}\\
u-v &= u(1+uv) &= -(1+uv)v &= (2\cdot\sqrt{uv})u\sqrt{uv}\\
1-uv &= -u(u+v) &= -(u+v)v &= (2\cdot\sqrt{-uv})\sqrt{-uv}\\
u+v &= u(1-uv) &= \quad (1-uv)v &= (2\cdot\sqrt{-uv})u\sqrt{-uv}
\end{aligned}$$

$$\begin{aligned}
(1+uv)\ \cdot\ (1-uv) &= 0\\
(u-v)\ \cdot\ (u+v) &= 0\\
(1-uv)\ \cdot\ (u\pm v) &= 0\\
(u\pm v)\ \cdot\ (1+uv) &= 0
\end{aligned}$$

$$\det\left(\sqrt{uv},\ \frac{u-v}{2},\ \sqrt{-uv},\ \frac{u+v}{2}\right) = \tfrac{1}{2}|u\times v|$$

FIGURE 7. Identities for unit pure u, v.

$$\tfrac{1}{2}|a^2-b^2| = \tfrac{1}{2}|ba+ab| = \sqrt{\tfrac{1}{2}(1-a^2\cdot b^2)} = 1\cdot\sqrt{(a\times b)(\bar a\times\bar b)}$$
$$\tfrac{1}{2}|a^2+b^2| = \tfrac{1}{2}|ba-ab| = \sqrt{\tfrac{1}{2}(1+a^2\cdot b^2)} = 1\cdot\sqrt{-(a\times b)(\bar a\times\bar b)}$$

$$\begin{aligned}
a^2-b^2 &= -(a\times b)(ba+ab) &= \quad (ba+ab)(\bar a\times\bar b)\\
ba+ab &= \quad (a\times b)(a^2-b^2) &= -(a^2-b^2)(\bar a\times\bar b)\\
a^2+b^2 &= -(a\times b)(ba-ab) &= -(ba-ab)(\bar a\times\bar b)\\
ba-ab &= \quad (a\times b)(a^2+b^2) &= \quad (a^2+b^2)(\bar a\times\bar b)
\end{aligned}$$

$$\begin{aligned}
(a^2-b^2)\ \cdot\ (a^2+b^2) &= 0\\
(ba+ab)\ \cdot\ (ba-ab) &= 0\\
(a^2+b^2)\ \cdot\ (ba\pm ab) &= 0\\
(ba\pm ab)\ \cdot\ (a^2-b^2) &= 0
\end{aligned}$$

$$\det\left(a^2-b^2,\ ba+ab,\ a^2+b^2,\ ba-ab\right) = 4|a^2\times b^2|^2$$

$$\begin{aligned}
a &= -(a\times b)b &= \quad b(\bar a\times\bar b)\\
b &= \quad (a\times b)a &= -a(\bar a\times\bar b)
\end{aligned}$$

$$\begin{aligned}
a\ \cdot\ (a^2+b^2) &= 0\\
a\ \cdot\ (ba-ab) &= 0\\
b\ \cdot\ (a^2+b^2) &= 0\\
b\ \cdot\ (ba-ab) &= 0
\end{aligned}$$

$$\det\left(a,\ b,\ a^2+b^2,\ ba-ab\right) = |a^2+b^2|^2$$

FIGURE 8. Identities for orthonormal a, b.

5. Dependence and Dimension

Definition. A *dependence relation* between the vectors $r_1, r_2, \ldots, r_n$ in $\mathbb{R}^4$ is an equation

$$\gamma_1 r_1 + \gamma_2 r_2 + \cdots + \gamma_n r_n = 0$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are real with at least one non-zero. $r_1, r_2, \ldots, r_n$ are *dependent* if there exists a dependence relation between them, otherwise they are *independent.*

Note if one vector is zero, say $r_1 = 0$, then there is automatically a dependence relation: $1r_1 + 0r_2 + \cdots + 0r_n = 0$.

Theorem 39. *a, b are dependent if and only if*

$$a \times b = 0.$$

Proof. Remove the trivial cases by letting $a \neq 0$ and $b \neq 0$.

Given $\gamma a + \theta b = 0$, without loss of generality let $\theta \neq 0$. Then $\gamma a + \theta b = 0 \implies \gamma a \times a + \theta a \times b = 0 \implies a \times b = 0$.

Conversely, $a \times b = 0 \implies b\bar{a} - a \cdot b = 0 \implies -(a \cdot b)a + |a|^2 b = 0$. This is a dependence relation since $|a|^2 \neq 0$. □

Theorem 40. *a, b, c are dependent if and only if*

$$(a \times b) \times (a \times c) = 0$$

if and only if

$$|a|^2 (b \times c) = \begin{vmatrix} a \cdot b & a \times b \\ a \cdot c & a \times c \end{vmatrix}.$$

Proof. Given $\alpha a + \beta b + \zeta c = 0$, without loss of generality let $\zeta \neq 0$. Then $\alpha a + \beta b + \zeta c = 0 \implies \beta a \times b + \zeta a \times c = 0 \implies (a \times b) \times (a \times c) = 0$.

Making use of Theorem 39, $(a \times b) \times (c \times d) = 0 \implies \gamma a \times b + \theta a \times c = 0 \implies \lambda a + \mu b + \nu a + \xi b = 0$.

The last two statements in the proposition are identical. □

Theorem 41. *a, b, c, d are dependent if and only if*

$$[(a \times b) \times (a \times c)] \times [(a \times c) \times (a \times d)] = 0$$

if and only if

$$[(a \times b) \times (a \times c)] \cdot (a \times d) = 0$$

if and only if

$$(a \times b) \cdot (c \times d) = \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix}$$

if and only if

$$\det(a, b, c, d) = 0.$$

Proof. The first two parts of the proposition may be shown equivalent using the technique in Theorems 39 and 40.

Given $\gamma a + \theta b + \lambda c + \mu d = 0$, without loss of generality let $\mu \neq 0$. Then

$$\begin{aligned} \det(a, b, c, d) &= \det(a, b, c, -\tfrac{\gamma}{\mu} a - \tfrac{\theta}{\mu} b - \tfrac{\lambda}{\mu} c) \\ &= -\det(a, b, c, \tfrac{\gamma}{\mu} a) - \det(a, b, c, \tfrac{\theta}{\mu} b) - \det(a, b, c, \tfrac{\lambda}{\mu} c) = 0. \end{aligned}$$

Given $\det(a,b,c,d)=0$, construct sequentially

$$\begin{aligned}
p &= a,\\
q &= b - |p|^{-2}(b\cdot p)p,\\
r &= c - |p|^{-2}(c\cdot p)p - |q|^{-2}(c\cdot q)q,\\
s &= d - |p|^{-2}(d\cdot p)p - |q|^{-2}(d\cdot q)q - |r|^{-2}(d\cdot r)r,
\end{aligned}$$

under the condition that we stop the moment one of p,q,r is zero, in which case we have a dependence relation between a,b,c,d. If p,q,r are non-zero, it may be checked that p,q,r,s are mutually orthogonal, so that $\det(p,q,r,s)=0 \implies \bar{p}q\bar{r}s=0 \implies s=0$. The equation $s=0$ is (ultimately) a dependence relation between a,b,c,d.

The last three statements in the proposition are equivalent via Theorem 29.5 (while noting the trivial case $a=0$). $\square$

Theorem 42. *Any five vectors in $\mathbb{R}^4$ are dependent.*

Proof. Any four pures are dependent (Theorems 30 and 41). Therefore given $r_1,\ldots,r_5$ in $\mathbb{R}^4$,

$$\begin{gathered}
\gamma_1 r_1\times r_5+\gamma_2 r_2\times r_5+\gamma_3 r_3\times r_5+\gamma_4 r_4\times r_5=0\\
\gamma_1 r_1\bar{r_5}+\gamma_2 r_2\bar{r_5}+\gamma_3 r_3\bar{r_5}+\gamma_4 r_4\bar{r_5}-\\
(\gamma_1 r_1\cdot r_5+\gamma_2 r_2\cdot r_5+\gamma_3 r_3\cdot r_5+\gamma_4 r_4\cdot r_5)=0\\
|r_5|^2\gamma_1 r_1+|r_5|^2\gamma_2 r_2+|r_5|^2\gamma_3 r_3+|r_5|^2\gamma_4 r_4-\\
(\gamma_1 r_1\cdot r_5+\gamma_2 r_2\cdot r_5+\gamma_3 r_3\cdot r_5+\gamma_4 r_4\cdot r_5)r_5=0
\end{gathered}$$

for real $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ with at least one non-zero. If $r_5=0$ then $r_1,\ldots,r_5$ are already dependent. If $r_5\neq 0$ then $|r_5|^2\gamma_m\neq 0$ for some m, making $r_1,\ldots,r_5$ dependent. $\square$

Theorem 43. *ai,aj,ak are orthogonal to a.*

Proof. Left as exercise (easy). $\square$

Theorem 44. *If a,b are orthonormal then*

$$\begin{aligned}
p &= (a\times b)\begin{vmatrix} i & j\\ a & b\end{vmatrix}+\begin{vmatrix} i & j\\ -b & a\end{vmatrix},\\
q &= (a\times b)\begin{vmatrix} j & k\\ a & b\end{vmatrix}+\begin{vmatrix} j & k\\ -b & a\end{vmatrix},\\
r &= (a\times b)\begin{vmatrix} k & i\\ a & b\end{vmatrix}+\begin{vmatrix} k & i\\ -b & a\end{vmatrix},
\end{aligned}$$

are orthogonal to a,b. At least two of p,q,r are non-zero:

$$p=0 \iff a\times b=-k,\qquad q=0\iff a\times b=-i,\qquad r=0\iff a\times b=-j.$$

Proof. Left as exercise. $\square$

Theorem 45. $[(a\times b)\times(a\times c)]\,a = a\left[(\bar{a}\times\bar{b})\times(\bar{a}\times\bar{c})\right]$ *is orthogonal to a,b,c. Furthermore,* $\det(a,b,c,[(a\times b)\times(a\times c)]\,a)=|(a\times b)\times(a\times c)|^2$.

Proof. Left as exercise. $\square$

Theorem 46. *r is orthogonal to four independent vectors in $\mathbb{R}^4$ if and only if $r = 0$.*

Proof. Let r be orthogonal to the independent vectors a, b, c, d.

$$\begin{aligned}|r|^4 \det(a,b,c,d) &= \det(\bar{r}a, \bar{r}b, \bar{r}c, \bar{r}d)\\ &= \det(a\cdot r + a\times r,\ \ b\cdot r + b\times r,\ \ c\cdot r + c\times r,\ \ d\cdot r + d\times r)\\ &= \det(a\times r,\ \ b\times r,\ \ c\times r,\ \ d\times r)\\ &= 0.\end{aligned}$$

Therefore $r = 0$ since $\det(a,b,c,d) \neq 0$.

Conversely, 0 is orthogonal to all vectors; in particular it is orthogonal to, say, the independent vectors $1, i, j, k$. □

Definition. A *subspace* of $\mathbb{R}^4$ is a subset $\mathcal{V}$ of $\mathbb{R}^4$ such that $\gamma x + \theta y$ is in $\mathcal{V}$ for all x, y in $\mathcal{V}$ and all γ, θ in $\mathbb{R}$.

Definition. For a subspace $\mathcal{V}$ of $\mathbb{R}^4$, the *dimension* of $\mathcal{V}$, denoted $\dim \mathcal{V}$, is an integer n such that

(1) there exists n independent vectors in $\mathcal{V}$, and
(2) any $n+1$ vectors in $\mathcal{V}$ are dependent.

The dimension of a given subspace is unique since both conditions cannot logically hold for two different integers.

Example. $\mathbb{R}^4$, the set of pures, and $\{0\}$ are subspaces of $\mathbb{R}^4$ with

$$\begin{aligned}\dim\ \mathbb{R}^4 &= 4 \qquad \text{(Theorems 36 and 42)}\\ \dim\{\, z \text{ in } \mathbb{R}^4 \mid z \text{ is pure}\,\} &= 3 \qquad \text{(Theorems 36 and 30)}\\ \dim\{0\} &= 0.\end{aligned}$$

Definition. The *span* of a finite set of vectors $\{r_1, \ldots, r_n\}$ in $\mathbb{R}^4$ is

$$\operatorname{span}\{r_1, \ldots, r_n\} = \{\, \gamma_1 r_1 + \cdots + \gamma_n r_n \mid \gamma_1, \gamma_2, \ldots, \gamma_n \text{ in } \mathbb{R}\,\}.$$

The *orthogonal span* is

$$\operatorname{span}^{\perp}\{r_1, \ldots, r_n\} = \{\, z \text{ in } \mathbb{R}^4 \mid r_1\cdot z = \cdots = r_n \cdot z = 0\,\}.$$

Theorem 47. $\operatorname{span}\{r_1, \ldots, r_n\}$ *and* $\operatorname{span}^{\perp}\{r_1, \ldots, r_n\}$ *are subspaces of* $\mathbb{R}^4$.

Proof. Left as exercise. □

Theorem 48. $\dim \operatorname{span}\{r_1, r_2\} = 2$ *if and only if* r_1, r_2 *are independent.*

Proof. Given $\dim \operatorname{span}\{r_1, r_2\} = 2$, by definition there exists independent s_1, s_2 in $\operatorname{span}\{r_1, r_2\}$. Let $s_1 = \gamma_1 r_1 + \gamma_2 r_2$ and $s_2 = \theta_1 r_1 + \theta_2 r_2$. Then $s_1 \times s_2 \neq 0 \implies (\gamma_1 r_1 + \gamma_2 r_2) \times (\theta_1 r_1 + \theta_2 r_2) \neq 0 \implies (\gamma_1\theta_2 - \gamma_2\theta_1)(r_1 \times r_2) \neq 0 \implies r_1 \times r_2 \neq 0$.

Any three vectors in $\operatorname{span}\{r_1, r_2\}$ are dependent:

$$\begin{aligned}&[(\gamma_1 r_1 + \gamma_2 r_2) \times (\theta_1 r_1 + \theta_2 r_2)] \times [(\gamma_1 r_1 + \gamma_2 r_2) \times (\lambda_1 r_1 + \lambda_2 r_2)]\\ &\qquad = (\gamma_1\theta_2 - \gamma_2\theta_1)(r_1 \times r_2) \times (\gamma_1\lambda_2 - \gamma_2\lambda_1)(r_1 \times r_2) = 0\end{aligned}$$

for all real $\gamma_1, \gamma_2, \theta_1, \theta_2, \lambda_1, \lambda_2$. □

Theorem 49. *If $r_1, \ldots, r_n$ are mutually orthogonal non-zero vectors then $r_1, \ldots, r_n$ are independent.*

Proof. Left as exercise. $\square$

Theorem 50. $\dim \operatorname{span}^{\perp}\{r_1, r_2\} = 2$ *if and only if r_1, r_2 are independent.*

Proof. Let $\mathcal{V} = \operatorname{span}^{\perp}\{r_1, r_2\}$.

If r_1, r_2 are independent then there are two non-zero vectors s_1, s_2 in $\mathcal{V}$ with s_1, s_2 orthogonal (Theorems 44 and 45), making r_1, r_2, s_1, s_2 independent. But r_1, r_2, s_1, s_2, s_3 are dependent for any non-zero fifth vector s_3 (Theorem 42), and if s_3 is in $\mathcal{V}$ then r_1, r_2, s_3 are independent, forcing s_1, s_2, s_3 to be dependent. Therefore $\dim \mathcal{V} = 2$.

If r_1, r_2 are dependent then both may be zero, in which case $\dim \mathcal{V} = 4 \neq 2$ (Theorems 42 and 46). If one is non-zero, say $r_1 \neq 0$, then $r_1 i, r_1 j, r_1 k$ are in $\mathcal{V}$ (Theorem 43). $r_1 i, r_1 j, r_1 k$ are mutually orthogonal and hence independent, therefore $\dim \mathcal{V} \neq 2$. $\square$

Theorem 51. *If (a, b, c, d) is a basis then for any z in $\mathbb{R}^4$ there exist unique reals z_1, z_2, z_3, z_4 such that*

$$z = z_1 a + z_2 b + z_3 c + z_4 d.$$

Proof. The proposition is trivial for $z = 0$; let $z \neq 0$.

From Theorem 42 we have a dependence relation $\theta z = \gamma_1 a + \gamma_2 b + \gamma_3 c + \gamma_4 d$. Our goal is to show $\theta \neq 0$. If $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$ then $\theta \neq 0$ by the definition of dependence. If one of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ is non-zero then $\gamma_1 a + \gamma_2 b + \gamma_3 c + \gamma_4 d \neq 0$, forcing $\theta \neq 0$. Thus $z_m = \gamma_m / \theta$ for $m = 1, 2, 3, 4$.

The uniqueness of z_1, z_2, z_3, z_4 is left as an exercise.

$\square$

Theorem 52. *If (a, b, c, d) is an orthogonal basis then for any z in $\mathbb{R}^4$,*

$$z = \frac{a \cdot z}{|a|^2} a + \frac{b \cdot z}{|b|^2} b + \frac{c \cdot z}{|c|^2} c + \frac{d \cdot z}{|d|^2} d.$$

Proof. Left as exercise (easy). $\square$

Definition. Given a basis (a, b, c, d) and some z in $\mathbb{R}^4$, to *resolve* z in (a, b, c, d) means to write $z = z_1 a + z_2 b + z_3 c + z_4 d$ for real z_1, z_2, z_3, z_4.

Theorem 53. *The center of $\mathbb{H}$ is $\mathbb{R}$, i.e., $\mathbb{R} = \{p \text{ in } \mathbb{H} \mid pz = zp \text{ for all } z \text{ in } \mathbb{H}\}$.*

Proof. If p is real then it commutes with all quaternions and belongs to the center. If p is non-real, there is a non-zero u orthogonal to 1 and $\vec{p}$ (Theorem 44). Then $\vec{p} \times u = \vec{p} u \neq 0 \implies pu \neq up$ (Theorems 17 and 4). $\square$

6. Planes and Oriented Planes

Definition. For our purposes, a *plane* will be defined as a 2-dimensional subspace of $\mathbb{R}^4$. Two planes $\mathcal{P}$ and $\mathcal{Q}$ are *orthogonal* when $\mathcal{P} \cap \mathcal{Q} = \{0\}$.

Theorem 54. $\operatorname{span}\{a, b\} \cap \operatorname{span}^{\perp}\{a, b\} = \{0\}$.

Proof. By definition $\operatorname{span}\{a, b\} \cap \operatorname{span}^{\perp}\{a, b\}$ is non-empty, at least containing 0. Let z belong to both $\operatorname{span}\{a, b\}$ and $\operatorname{span}^{\perp}\{a, b\}$. Then
$z = \alpha a + \beta b \implies z \cdot z = \alpha a \cdot z + \beta b \cdot z \implies |z|^2 = 0 \implies z = 0.$ □

Theorem 55. *If u, v are unit pure then the following two sets are orthogonal planes:*

$$\{\, z \text{ in } \mathbb{R}^4 \mid uz + zv = 0 \,\} = \begin{cases} \operatorname{span}^{\perp}\{1, u\} & \text{if } u = v \\ \operatorname{span}\{1, u\} & \text{if } u = \text{-}v \\ \operatorname{span}\{\sqrt{uv},\ u - v\} & \text{otherwise } (u \times v \neq 0) \end{cases}$$

$$\{\, z \text{ in } \mathbb{R}^4 \mid uz - zv = 0 \,\} = \begin{cases} \operatorname{span}\{1, u\} & \text{if } u = v \\ \operatorname{span}^{\perp}\{1, u\} & \text{if } u = \text{-}v \\ \operatorname{span}^{\perp}\{\sqrt{uv},\ u - v\} & \text{otherwise } (u \times v \neq 0) \end{cases}$$

Proof. Theorems 4 and 17 handle the cases $u = \pm v$. Given $u \times v \neq 0$ and some z in $\mathbb{R}^4$, resolve z into the basis from Theorem 37,

$$z = z_1(1 + uv) + z_2(u - v) + z_3(1 - uv) + z_4(u + v).$$

The equation $uz + zv = 0$ becomes

$$\begin{aligned} u\,[z_1(1 + uv) + z_2(u - v) + z_3(1 - uv) + z_4(u + v)] +& \\ [z_1(1 + uv) + z_2(u - v) + z_3(1 - uv) + z_4(u + v)]\,v &= 0 \\ z_3(2u + 2v) + z_4(-2 + 2uv) &= 0 \\ z_3(u + v) &= z_4(1 - uv). \end{aligned}$$

Since $(u + v) \cdot (1 - uv) = 0$ this forces $z_3 = z_4 = 0$ while z_1 and z_2 remain free variables.

Solutions to $uz - zv = 0$ follow similarly by substituting $-v$ for v. □

Theorem 56. *If $\mathcal{P}$ is a plane in $\mathbb{R}^4$ then there exist unit pures u, v such that*

$$\mathcal{P} = \{\, z \text{ in } \mathbb{R}^4 \mid uz + zv = 0 \,\}.$$

Namely, if $\mathcal{P} = \operatorname{span}\{a, b\}$ for orthonormal a, b then $u = a \times b$ and $v = \bar{a} \times \bar{b}$.

Proof. Let $z = \gamma a + \theta b$. Then $uz + zv = b\bar{a}(\gamma a + \theta b) + (\gamma a + \theta b)\bar{b}a = \gamma b + \theta b\bar{a}b + \gamma a\bar{b}a + \theta a = \gamma b - \theta a - \gamma b + \theta a = 0$.

Given some z such that $(a \times b)z + z(\bar{a} \times \bar{b}) = 0$, create an orthonormal basis (a, b, c, d) and then write $z = z_1a + z_2b + z_3c + z_4d$. In similar manner to Theorem 55, $(a \times b)z + z(\bar{a} \times \bar{b}) = 0$ directly implies $z_3 = z_4 = 0$. □

Theorem 57. *If $\operatorname{span}\{a, b\}$ contains c, d then $(a \times b) \times (c \times d) = 0$.*

Proof. Let $c = \gamma a + \theta b$ and $d = \lambda a + \mu b$ for some real $\gamma, \theta, \lambda, \mu$.
$(a \times b) \times (c \times d) = (a \times b) \times [(\gamma a + \theta b) \times (\lambda a + \mu b)] = (a \times b) \times (\gamma\mu a \times b + \theta\lambda b \times a) = (\gamma\mu - \theta\lambda)(a \times b) \times (a \times b) = 0$.

Note $(a \times b) \cdot (c \times d) = \gamma\mu - \theta\lambda$ is the area determinant in the plane under this condition. □

Theorem 58. *If* $\mathrm{span}^{\perp}\{a,b\}$ *contains* c,d *then* $(a\times b)\times(c\times d)=0$.

Proof. See the last identity in Figure 2. Note $(a\times b)\cdot(c\times d)=\det(a,b,c,d)$ under this condition. □

Definition. The *pair determinant* of (a,b) and (c,d) is

$$(a\times b)\cdot(c\times d).$$

If $(a\times b)\cdot(c\times d)>0$ then (a,b) and (c,d) together are *positively-oriented*. If $(a\times b)\cdot(c\times d)<0$ then together they are *negatively-oriented*. Alternatively, one may say the pairs have the *same orientation* or *opposite orientation*, respectively.

Definition. An *oriented plane* is an ordered pair of non-parallel vectors $a\lozenge b$ together with an associated plane. Two oriented planes $a\lozenge b$ and $c\lozenge d$ are defined to be equivalent ("=") if and only if their associated planes are equivalent and $(a,b),(c,d)$ are positively-oriented.

Let $a\wedge b$ be an oriented plane associated with $\mathrm{span}\{a,b\}$ and let $a\vee b$ be an oriented plane associated with $\mathrm{span}^{\perp}\{a,b\}$. These will be our only examples of oriented planes, so that $a\lozenge b$ is necessarily either $a\wedge b$ or $a\vee b$.

Theorem 59. *"=" for oriented planes is an equivalence relation.*

Proof. Given two oriented planes, whether their associated planes are equal is delegated to the equivalence relation between sets. When the sets are equal, it remains to prove the orientation requirement.

Given $a\lozenge b$, the reflexive property holds since by definition $a\times b\neq 0$, $(a\times b)\cdot(a\times b)=|a\times b|^2>0 \implies a\lozenge b=a\lozenge b$.

The symmetric property follows from the commutation of the dot product, $a\lozenge b=c\lozenge d \implies (a\times b)\cdot(c\times d)>0 \implies (c\times d)\cdot(a\times b)>0 \implies c\lozenge d=a\lozenge b$.

Lastly we must show transitivity. We shall require the identity

$$[(a\times b)\cdot(c\times d)]\,[(c\times d)\cdot(r\times s)]=\tfrac{1}{2}\left[|c\times d|^2(a\times b)\cdot(r\times s)+(a\times b)(c\times d)\cdot(c\times d)(r\times s)\right].$$

(Expand the left side using the distributive-like rule for dot products in Figure 2.) Given $a\lozenge b=c\lozenge d$, from Theorems 57 and 58 we have $(a\times b)\times(c\times d)=0$ which simplifies this identity to $[(a\times b)\cdot(c\times d)]\,[(c\times d)\cdot(r\times s)]=|c\times d|^2(a\times b)\cdot(r\times s)$. Finally we are ready to complete the proof:

$$\begin{aligned}
&[\,a\lozenge b=c\lozenge d \text{ and } c\lozenge d=r\lozenge s\,]\\
&\quad\implies [\,(a\times b)\cdot(c\times d)>0 \text{ and } (c\times d)\cdot(r\times s)>0\,]\\
&\quad\implies [(a\times b)\cdot(c\times d)]\,[(c\times d)\cdot(r\times s)]>0\\
&\quad\implies |c\times d|^2(a\times b)\cdot(r\times s)>0\\
&\quad\implies (a\times b)\cdot(r\times s)>0\\
&\quad\implies a\lozenge b=r\lozenge s.
\end{aligned}$$

□

Theorem 60.

$$\begin{aligned}
a\wedge b&=b\wedge(-a)=(-b)\wedge a=(-a)\wedge(-b),\\
a\vee b&=b\vee(-a)=(-b)\vee a=(-a)\vee(-b).
\end{aligned}$$

Proof. This is the anti-commutative property of the cross product. □

Theorem 61. *If* (a, b, c, d) *is an orthogonal basis then*

$$\begin{aligned} a \wedge b &= c \vee d & a \vee b &= c \wedge d \\ a \wedge c &= d \vee b & a \vee c &= d \wedge b \\ a \wedge d &= b \vee c & a \vee d &= b \wedge c. \end{aligned}$$

Proof. Apply the element-swapping rule for determinants. □

7. The Meaning of Quaternion Multiplication

Definition. A *linear transformation* of $\mathbb{R}^4$ is a map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$T(\gamma x) = \gamma T(x),$$
$$T(x + y) = T(x) + T(y)$$

for all γ in $\mathbb{R}$ and all x, y in $\mathbb{R}^4$.

Definition. Given $T : \mathbb{R}^4 \to \mathbb{R}^4$, a set of vectors $\mathcal{V}$ is *invariant under* T when $T(z)$ is in $\mathcal{V}$ for all z in $\mathcal{V}$. The set $\mathcal{V}$ is *fixed under* T when $T(z) = z$ for all z in $\mathcal{V}$.

A quaternion may be interpreted either as a vector in $\mathbb{R}^4$ or as a linear transformation of $\mathbb{R}^4$. Specifically, the product ab may be interpreted in any of the following ways:

(1) a is a linear transformation acting on the vector b, or
(2) b is a linear transformation acting on the vector a, or
(3) ab is the composition of the linear transformations a and b.

Call these, respectively, *the left action of* a, *the right action of* b, and *the action* ab.

We shall now investigate what a quaternion action does to vectors on which it operates. Given unit pure u and real θ, consider the left action of $e^{u\theta}$,

$$z \longmapsto e^{u\theta} z.$$

There is a unit pure v which is orthogonal to 1 and u (Theorem 44). Use Theorem 45 to generate a fourth basis element $[(1 \times u) \times (1 \times v)]\,1 = uv$. This could also be obtained by recognizing that multiplication of orthogonal pures is the same as the cross product, $u \times v = uv$. We now have a positively-oriented orthonormal basis $(1, u, v, uv)$. Resolve z into this basis,

$$z = z_1 + z_2 u + z_3 v + z_4 uv.$$

The product $e^{u\theta} z$ expands to

$$\begin{aligned} e^{u\theta} z &= (\cos\theta + u\sin\theta)(z_1 + z_2 u + z_3 v + z_4 uv) \\ &= (z_1\cos\theta - z_2\sin\theta)\ + \\ &\quad (z_1\sin\theta + z_2\cos\theta)u\ + \\ &\quad (z_3\cos\theta - z_4\sin\theta)v\ + \\ &\quad (z_3\sin\theta + z_4\cos\theta)uv. \end{aligned}$$

Observe that $1 \wedge u$ and $v \wedge uv = 1 \vee u$ are invariant planes under $z \longmapsto e^{u\theta} z$. The right action $z \longmapsto z e^{u\theta}$ has the same invariant planes,

$$\begin{aligned} z e^{u\theta} &= (z_1 + z_2 u + z_3 v + z_4 uv)(\cos\theta + u\sin\theta) \\ &= (z_1\cos\theta - z_2\sin\theta)\ + \\ &\quad (z_1\sin\theta + z_2\cos\theta)u\ + \\ &\quad (z_3\cos\theta + z_4\sin\theta)v\ + \\ &\quad (-z_3\sin\theta + z_4\cos\theta)uv. \end{aligned}$$

Unless this is already familiar, the reader should verify the transformation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a geometric rotation in the plane by angle θ.

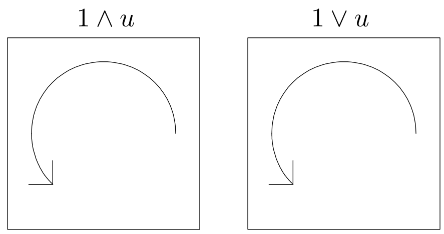

FIGURE 9. $z \longmapsto e^{u\theta} z$

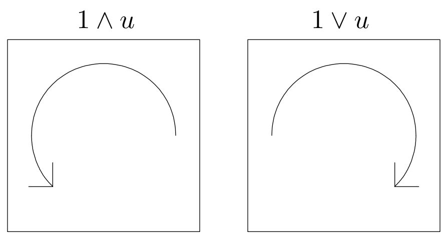

FIGURE 10. $z \longmapsto z e^{u\theta}$

The above expansions of $ze^{u\theta}$ and $e^{u\theta}z$ imply the following related theorems. When we say an oriented plane $a \Diamond b$ is rotated by θ, we mean the plane is invariant with p rotated toward q by angle θ for some (p, q) lying in the plane with $(a, b), (p, q)$ positively-oriented.

Theorem 62. *For unit pure u and real θ, the transformation*

$$z \longmapsto e^{u\theta} z$$

rotates $1 \wedge u$ by angle θ and $1 \vee u$ by angle θ.

Theorem 63. *For unit pure u and real θ, the transformation*

$$z \longmapsto z e^{u\theta}$$

rotates $1 \wedge u$ by angle θ and $1 \vee u$ by angle $-\theta$.

Theorem 64. *For unit pure u and real θ, the transformation*

$$z \longmapsto e^{u\theta} z e^{u\theta}$$

rotates $1 \wedge u$ by angle 2θ and leaves $1 \vee u$ fixed.

Theorem 65. *For unit pure u and real θ, the transformation*

$$z \longmapsto e^{u\theta} z e^{-u\theta}$$

rotates $1 \vee u$ by angle 2θ and leaves $1 \wedge u$ fixed. (In the subspace of pures, this is a rotation around axis u.)

Theorem 66. *For unit pure u and real θ, γ, the transformation*

$$z \longmapsto e^{u\theta} z e^{u\gamma}$$

rotates $1 \wedge u$ by angle $\theta + \gamma$ and $1 \vee u$ by angle $\theta - \gamma$.

$z \longmapsto e^{u\theta} z e^{-v\gamma}$		
condition	rotate by $\theta + \gamma$	rotate by $\theta - \gamma$
$u = v$	$1 \vee u$	$1 \wedge u$
$u = -v$	$1 \wedge u$	$1 \vee u$
$u \times v \neq 0$	$\sqrt{uv} \wedge (u - v)$	$\sqrt{uv} \vee (u - v)$
condition	alternate forms	
$u \times v \neq 0$	$\sqrt{-uv} \vee (u + v)$	$\sqrt{-uv} \wedge (u + v)$
	plane equations	
	$uz + zv = 0$	$uz - zv = 0$

Figure 11. Theorems 37, 55, 67.

Theorem 67. *For unit pure u, v and real θ, γ, the transformation*

$$z \longmapsto e^{u\theta} z e^{-v\gamma}$$

rotates $a \wedge ua$ by angle $\theta + \gamma$ and $a \vee ua$ by angle $\theta - \gamma$ where $ua + av = 0$, $a \neq 0$.

Proof. Pick a non-zero a such that $ua + av = 0$ (Theorem 55). Substituting $v = -a^{-1}ua$ and using an identity from Figure 6 gives

$$e^{u\theta} z e^{-v\gamma} = e^{u\theta} z e^{a^{-1}ua\gamma} = e^{u\theta} z a^{-1} e^{u\gamma} a = A \circ R \circ A^{-1}(z)$$

where

$$\begin{aligned} A(z) &= za, \\ R(z) &= e^{u\theta} z e^{u\gamma}. \end{aligned}$$

Invoking A as a change-of-basis we have that $z \longmapsto e^{u\theta} z e^{-v\gamma}$ behaves exactly as $z \longmapsto e^{u\theta} z e^{u\gamma}$ in Theorem 66 except the rotation planes are $a \wedge ua$ and $a \vee ua$ respectively. □

Remark. Theorem 67 may also be proved by using a basis aligned with the invariant planes. Theorem 66 handles the cases $u = \pm v$. For $u \times v \neq 0$, one may verify that

$$\begin{aligned} u\, e^{u\theta}(1 + uv)e^{-v\gamma} \;&+\; e^{u\theta}(1 + uv)e^{-v\gamma}\, v = 0, \\ u\, e^{u\theta}(u - v)e^{-v\gamma} \;&+\; e^{u\theta}(u - v)e^{-v\gamma}\, v = 0, \\ u\, e^{u\theta}(1 - uv)e^{-v\gamma} \;&-\; e^{u\theta}(1 - uv)e^{-v\gamma}\, v = 0, \\ u\, e^{u\theta}(u + v)e^{-v\gamma} \;&-\; e^{u\theta}(u + v)e^{-v\gamma}\, v = 0. \end{aligned}$$

These are solutions to the invariant plane equations of the transformation. The behavior of any vector may be described in terms of these solutions, which form an orthogonal basis of $\mathbb{R}^4$ (Theorem 37).

$z \longmapsto e^{\frac{1}{2}(a\times b)(\alpha+\beta)} z e^{-\frac{1}{2}(\bar{a}\times\bar{b})(\alpha-\beta)}$		
	rotate by α	rotate by β
	$a \wedge b$	$a \vee b$
condition	alternate forms	
$ab \neq ba$	$(a^2+b^2)\vee(ba-ab)$	$(a^2+b^2)\wedge(ba-ab)$
$ab \neq -ba$	$(a^2-b^2)\wedge(ba+ab)$	$(a^2-b^2)\vee(ba+ab)$
	plane equations	
	$(a\times b)z + z(\bar{a}\times\bar{b}) = 0$	$(a\times b)z - z(\bar{a}\times\bar{b}) = 0$
(alt form)	$\bar{a}z\bar{a} + \bar{b}z\bar{b} = 0$	$\bar{a}z\bar{a} - \bar{b}z\bar{b} = 0$

FIGURE 12. Theorems 38, 55, 68.

Theorem 68. *For orthonormal a, b and real numbers α, β, the transformation*

$$z \longmapsto e^{\frac{1}{2}(a\times b)(\alpha+\beta)} z e^{-\frac{1}{2}(\bar{a}\times\bar{b})(\alpha-\beta)}$$

rotates $a \wedge b$ by angle α and $a \vee b$ by angle β.

Proof. This is a simple reparameterization of Theorem 67 with

$$u = a \times b \qquad \theta = \tfrac{1}{2}(\alpha+\beta)$$
$$v = \bar{a} \times \bar{b} \qquad \gamma = \tfrac{1}{2}(\alpha-\beta).$$

$ua + av = (a\times b)a + a(\bar{a}\times\bar{b}) = b\bar{a}a + a(-\bar{a}b) = b - b = 0$ so the same "a" applies in Theorem 67. ☐

Exercise. Reduce the transformation in Theorem 68 to the transformation in Theorem 65 in two distinct ways.

Example. Let us interpret the product ij as the left action of i on j. To make this clear, we write

$$ij = (e^{i\pi/2})j.$$

The left action i rotates both $1\wedge i$ and $1 \vee i = j \wedge k$ by $\frac{\pi}{2}$. The vector j is unaffected by the rotation of $1 \wedge i$, but is rotated by $\frac{\pi}{2}$ toward k by the rotation of $j \wedge k$, bringing it to k.

Now viewing ij as the right action of j on i, we write

$$ij = i(e^{j\pi/2}).$$

The right action j rotates $1 \wedge j$ by $\frac{\pi}{2}$ and $1 \vee j$ by $-\frac{\pi}{2}$. But is $1 \vee j$ equal to $i \wedge k$ or $k \wedge i$? We test the pair determinant $(1\times j)\cdot(k \times i) = j \cdot j = 1 > 0$ and conclude $1 \vee j = k \wedge i = i \wedge -k$. Therefore i is rotated toward $-k$ by $-\frac{\pi}{2}$, arriving at k.

Example. For unit pure u, v, the equation $vz - zu = 0$, written as $zuz^{-1} = v$, may be interpreted as a rotation which brings u to v. The set of solutions for z is described by Theorem 55 (note the change of variables). But suppose we did not already know the solutions and we wanted to concoct some z which satisfies the equation.

There are three cases to consider: $u = v$, $u = -v$, and $u \times v \neq 0$. If $u = v$ then $z = 1$. If $u = -v$ then z may be any unit pure which is orthogonal to u. This is the rotation of u around axis z by angle π (see Theorem 65),

$$e^{z\pi/2} u e^{-z\pi/2} = zuz^{-1} = zu(-z) = (-uz)(-z) = uz^2 = -u = v.$$

For $u \times v \neq 0$, let θ be the angle between u and v. Rotate u around axis $w = (u \times v)/|u \times v|$ by angle θ. Since $u \cdot w = 0$,

$$\begin{aligned} e^{\frac{1}{2}w\theta} u e^{-\frac{1}{2}w\theta} &= e^{\frac{1}{2}w\theta} e^{\frac{1}{2}w\theta} u = e^{w\theta} u = (\cos\theta + w \sin\theta) u \\ &= \left(u \cdot v + \frac{u \times v}{|u \times v|} |u \times v| \right) u = (u \cdot v + u \times v) u = (-vu) u = -vu^2 = v. \end{aligned}$$

Therefore $z = e^{\frac{1}{2}w\theta} = \sqrt{-vu}$.

When $u \times v \neq 0$, another method of bringing u to v is to construct a "flip" rotation: place the axis on the angle bisector between u and v and rotate by π. This is precisely the solution $z = u + v$, which is the other direction in the plane of solutions span$\{\sqrt{-vu},\ u + v\}$ given by Theorem 55.

Exercise. Given non-parallel unit quaternions a, b, let θ be the angle between them. For real τ, the *slerp* function ("spherical linear interpolation")

$$\operatorname{slerp}(a, b, \tau) = \frac{a \sin[(1-\tau)\theta] + b \sin \tau\theta}{\sin\theta}$$

produces a unit quaternion rotated from a toward b by angle $\tau\theta$. Derive this formula directly from Theorem 68. (Show $a \times b = a \times [b - (a \cdot b)a]$ and $|a \times b| = |b - (a \cdot b)a|$. Also use Theorem 25.)

Remark. Consider Theorem 68 in isolation without the context of this section. It gives a formula whose input is an orthonormal pair along with two angles and whose output is a specific rotation of $\mathbb{R}^4$. But there exist several different choices of these inputs which describe the same rotation. It might be interesting to know if any of these choices produce a different form. For example we could have chosen a different orthonormal pair (a', b') lying in the (a, b)-plane:

$$z \longmapsto e^{\frac{1}{2}(a' \times b')(\alpha+\beta)} z e^{-\frac{1}{2}(\overline{a'} \times \overline{b'})(\alpha-\beta)}$$

with

$$\begin{aligned} a' &= e^{a \times b \lambda} a e^{-\bar{a} \times \bar{b} \lambda}, \\ b' &= e^{a \times b \lambda} b e^{-\bar{a} \times \bar{b} \lambda} \end{aligned}$$

for some real λ. But

$$\begin{aligned} b'\overline{a'} &= e^{b\bar{a}\lambda} b e^{-\bar{b}a\lambda} e^{\bar{b}a\lambda} \bar{a} e^{-b\bar{a}\lambda} = e^{b\bar{a}\lambda} b\bar{a} e^{-b\bar{a}\lambda} = e^{b\bar{a}\lambda} e^{-b\bar{a}\lambda} b\bar{a} = b\bar{a}, \\ \overline{b'}a' &= e^{\bar{b}a\lambda} \bar{b} e^{-b\bar{a}\lambda} e^{b\bar{a}\lambda} a e^{-\bar{b}a\lambda} = e^{\bar{b}a\lambda} \bar{b}a e^{-\bar{b}a\lambda} = e^{\bar{b}a\lambda} e^{-\bar{b}a\lambda} \bar{b}a = \bar{b}a \end{aligned}$$

and the result is unchanged.

Next, we could have chosen an orthonormal pair in the complement plane and swapped the angles. Given a positively-oriented orthonormal basis (a, b, c, d), we could have rotated $c \wedge d$ by angle β and $c \vee d$ by angle α:

$$z \longmapsto e^{\frac{1}{2}(c \times d)(\beta+\alpha)} z e^{-\frac{1}{2}(\bar{c} \times \bar{d})(\beta-\alpha)}.$$

But from Theorem 34, $a \times b = c \times d$ and $\bar{a} \times \bar{b} = -\bar{c} \times \bar{d}$, giving the same form again.

Finally, we could have chosen angles of any multiple of 2π. The equations

$$\begin{cases} \alpha' + \beta' = \alpha + \beta + 2\pi m \\ \alpha' - \beta' = \alpha - \beta + 2\pi n \end{cases}$$

admit two possibilities mod 2π,

$$\begin{cases} \alpha' = \alpha \\ \beta' = \beta \end{cases}$$

and

$$\begin{cases} \alpha' = \alpha + \pi \\ \beta' = \beta + \pi. \end{cases}$$

The second solution produces

$$z \longmapsto \left(-e^{\frac{1}{2}(a\times b)(\alpha+\beta)}\right) z \left(-e^{-\frac{1}{2}(\bar{a}\times\bar{b})(\alpha-\beta)}\right).$$

The transformation is the same, however the pair of quaternions is different. The next section describes this result in more detail.

8. The Quaternion Representation of SO_4

Definition. A *linear orthogonal transformation* of $\mathbb{R}^4$ is a linear transformation which preserves length, belonging to the set

$$GO_4 = \{T : \mathbb{R}^4 \to \mathbb{R}^4 \mid T \text{ is linear and } |T(z)| = |z| \text{ for all } z \text{ in } \mathbb{R}^4\}.$$

Theorem 69. *T is linear orthogonal if and only if T is linear and preserves the dot product.*

Proof. $|T(x+y)| = |x+y| \iff |T(x)+T(y)| = |x+y| \iff T(x)\cdot T(x) + T(y)\cdot T(y) + 2T(x)\cdot T(y) = x\cdot x + y\cdot y + 2x\cdot y \iff T(x)\cdot T(y) = x\cdot y$. □

Theorem 70. *A linear orthogonal transformation preserves angles.*

Proof. $T(x)\cdot T(y) = x\cdot y \implies |T(x)||T(y)|\cos\theta_1 = |x||y|\cos\theta_2 \implies \cos\theta_1 = \cos\theta_2 \implies \theta_1 = \theta_2$ since we may impose that angles be in $[0,\pi]$. □

Definition. Given a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$, define

$$\det T = \det\left(T(1), T(i), T(j), T(k)\right).$$

If $\det T > 0$ then T is called *orientation-preserving*. If $\det T < 0$ then T is called *orientation-reversing*.

Theorem 71. *If T is a linear orthogonal transformation then* $\det T = \pm 1$.

Proof. $T(1), T(i), T(j), T(k)$ are mutually orthogonal since $1, i, j, k$ are mutually orthogonal and T preserves angles. Thus $|\det T| = |\det\left(T(1), T(i), T(j), T(k)\right)| = |\overline{T(1)}T(i)\overline{T(j)}T(k)| = |T(1)||T(i)||T(j)||T(k)| = |1||i||j||k| = 1$. □

Definition. The set of orientation-preserving transformations in GO_4 is

$$SO_4 = \{T \text{ in } GO_4 \mid \det T = 1\}.$$

Theorem 72. *For unit quaternions p, q, the transformation $z \longmapsto pzq$ is in SO_4.*

Proof. Let $T(z) = pzq$. T is clearly linear by the properties of quaternions. $|T(z)| = |pzq| = |p||z||q| = |z|$. $\det T = \det\left(pq, piq, pjq, pkq\right) = \overline{pq}piq\overline{pjq}pkq = (\bar{q}\bar{p})(piq)(-\bar{q}j\bar{p})(pkq) = -\bar{q}(ijk)q = \bar{q}q = 1$. □

Theorem 73. *A given element of SO_4 may be expressed as $z \longmapsto pzq$ for some unit quaternions p, q.*

Proof. For an element T of SO_4 we construct a rotation $R(z) = pzq$ whose action on the basis $(1, i, j, k)$ matches that of T, making $T(z) = T(z_1 + z_2 i + z_3 j + z_4 k) = z_1 T(1) + z_2 T(i) + z_3 T(j) + z_4 T(k) = z_1 R(1) + z_2 R(i) + z_3 R(j) + z_4 R(k) = R(z_1 + z_2 i + z_3 j + z_4 k) = R(z)$.

Using the results of the previous section, construct a rotation A which brings 1 to $T(1)$, a rotation B which brings $A(i)$ to $T(i)$ and fixes $T(1)$, and a rotation C which brings $B\circ A(j)$ to $T(j)$ and fixes $T(1)$ and $T(i)$. We now have $R(z) = C\circ B\circ A(z)$.

One such construction is $R(z) = srz\bar{r}\bar{s}T(1)$ where r satisfies $ri\bar{r} = T(1)\times T(i)$ and s satisfies $srj\bar{r}\bar{s} = T(1)\times T(j)$ with $\vec{s}\times[T(1)\times T(i)] = 0$. It is left as an exercise to show $R(1) = T(1), R(i) = T(i), R(j) = T(j), R(k) = T(k)$. □

Theorem 74. $SO_4 = \{z \longmapsto pzq \mid p\bar{p} = q\bar{q} = 1\}$.

Proof. Theorem 72 states pzq-type transformations are a subset of SO_4 and Theorem 73 states SO_4 is a subset of pzq-type transformations. □

Definition. The *unit n-sphere* is

$$\mathbb{S}^n = \{\, z \text{ in } \mathbb{R}^{n+1} \mid |z| = 1 \,\}.$$

The following conventions apply wherever quaternions are used:

$$\begin{aligned}
\mathbb{S}^3 &= \{\, z \text{ in } \mathbb{R}^4 \mid z\bar{z} = 1 \,\},\\
\mathbb{S}^2 &= \{\, z \text{ in } \mathbb{R}^4 \mid z\bar{z} = 1 \text{ and } 1 \cdot z = 0 \,\},\\
\mathbb{S}^1 &= \{\, z \text{ in } \mathbb{R}^4 \mid z\bar{z} = 1 \text{ and } j \cdot z = k \cdot z = 0 \,\}.
\end{aligned}$$

Theorem 75. *The map*

$$\Phi : \mathbb{S}^3 \times \mathbb{S}^3 \to SO_4,$$
$$\Phi(p, q) = (z \longmapsto pz\bar{q})$$

is 2:1 with $\Phi(p, q) = \Phi(-p, -q)$.

Proof. Clearly Φ is surjective from Theorem 74. Let

$$\Phi(r, s) = \Phi(p, q).$$

Then

$$\begin{aligned}
(\Phi(r, s))\,(1) &= (\Phi(p, q))\,(1)\\
r\bar{s} &= p\bar{q}
\end{aligned}$$

and

$$\begin{aligned}
(\Phi(r, q\bar{p}r))\,(z) &= (\Phi(p, q))\,(z)\\
rz\bar{r}p\bar{q} &= pz\bar{q}\\
z\bar{r}p &= \bar{r}pz.
\end{aligned}$$

We have that $\bar{r}p$ commutes with all z in $\mathbb{H}$, which by Theorem 53 means $\bar{r}p$ is real. $|\bar{r}p| = |\bar{r}||p| = 1$ thus $\bar{r}p = \pm 1$. Either $r = p$ or $r = -p$, so either

$$(r, s) = (r, q\bar{p}r) = (p, q\bar{p}p) = (p, q)$$

or

$$(r, s) = (r, q\bar{p}r) = (-p, q\bar{p}(-p)) = (-p, -q).$$

□

Theorem 76. *SO_4 is a group under function composition.*

Proof. Let $T(z) = pzq$ with $p\bar{p} = q\bar{q} = 1$ and claim that $T^{-1}(z) = \bar{p}z\bar{q}$. Then $T^{-1} \circ T(z) = T^{-1}(pzq) = \bar{p}pzq\bar{q} = z$ and $T \circ T^{-1}(z) = T(\bar{p}z\bar{q}) = p\bar{p}z\bar{q}q = z$. The remaining requirements of a group may be easily checked. □

Theorem 77. *GO_4 is a group with subgroup SO_4 of index* 2.

$$\begin{aligned}
GO_4 &= SO_4 \cup SO_4 \circ (z \longmapsto \bar{z})\\
&= \{\, z \longmapsto pzq \mid p\bar{p} = q\bar{q} = 1 \,\} \cup \{\, z \longmapsto p\bar{z}q \mid p\bar{p} = q\bar{q} = 1 \,\}.
\end{aligned}$$

Proof. Let $H(z) = \bar{z}$ and note $H^{-1} = H$. Given T in GO_4, we have $\det T = \pm 1$. If $\det T = 1$ then T is in SO_4. If $\det T = -1$ then compute $\det(T \circ H) = \det(T(1), T(-i), T(-j), T(-k)) = -\det(T(1), T(i), T(j), T(k)) = -\det T = 1$. Therefore $T \circ H$ is in SO_4 and consequently T is in $SO_4 \circ H$. Thus T has the form $T(z) = p\bar{z}q$ for unit p, q. It may be verified that $T^{-1}(z) = q\bar{z}p$, from which it follows that GO_4 a group. □

9. SO_3, SO_2 and Projective Space

Definition. $SO_3 = \{T \text{ in } SO_4 \mid T(1) = 1\}$.

Theorem 78. $SO_3 = \{z \longmapsto pz\bar{p} \mid p\bar{p} = 1\}$.

Proof. Let $T(z) = pz\bar{q}$ with $p\bar{p} = q\bar{q} = 1$ and $T(1) = 1$. But $T(1) = p\bar{q} = 1$, therefore $p = q$. □

Theorem 79. *The map*

$$f : \mathbb{S}^3 \to SO_3,$$
$$f(p) = (z \longmapsto pz\bar{p})$$

is 2:1 with $f(p) = f(-p)$.

Proof. Note $\Phi(p, \bar{p}) = f(p)$ from Theorem 75. Theorem 78 states

$$SO_3 = \{\, f(p) \mid p \text{ in } \mathbb{S}^3 \,\}$$

making f surjective. Again drawing from Theorem 75,

$$f(r) = f(p) \implies \Phi(r, \bar{r}) = \Phi(p, \bar{p}) \implies [\, r = p \text{ or } r = -p \,].$$

□

Definition. $\mathbb{S}^n$ with antipodal points identified is *real projective space*,

$$\mathbb{RP}^n = \{\, \{z, -z\} \mid z \text{ in } \mathbb{S}^n \,\}.$$

For our purposes, an element p in $\mathbb{RP}^n$ will be represented by a point in $\mathbb{S}^n$ together with the understanding that p is not distinguished from $-p$. Each statement involving p must remain the same when p is replaced with $-p$, else the statement is not well-defined. In effect, p will be shorthand for $\{p, -p\}$.

Theorem 80. *The map*

$$f : \mathbb{RP}^3 \to SO_3,$$
$$f(p) = (z \longmapsto pz\bar{p})$$

is bijective.

Proof. Follows trivially from Theorem 79. □

Definition. $SO_2 = \{T \text{ in } SO_4 \mid T(j) = j, T(k) = k\}$.

Theorem 81. $SO_2 = \{\, z_1 + z_2 i + z_3 j + z_4 k \longmapsto s(z_1 + z_2 i) + z_3 j + z_4 k \mid s \text{ in } \mathbb{S}^1 \,\}$.

Proof. Let $T(z) = pz\bar{q}$ with $T(j) = j, T(k) = k$. Then $i = jk = -j\bar{k} = -T(j)\overline{T(k)}$ $= -pj\bar{q}\overline{pk\bar{q}} = pj\bar{q}qk\bar{p} = pjk\bar{p} = pi\bar{p}$. Therefore $ip = pi$ and similarly $iq = qi$. Thus p and q have the forms $p = e^{i\alpha}$ and $q = e^{i\beta}$. $T(z_1 + z_2 i + z_3 j + z_4 k) = T(z_1 + z_2 i) + T(z_3 j + z_4 k) = e^{i\alpha}(z_1 + z_2 i)e^{-i\beta} + z_3 j + z_4 k = e^{i(\alpha-\beta)}(z_1 + z_2 i) + z_3 j + z_4 k$. □

Theorem 82. *The map*

$$f : \mathbb{S}^1 \to SO_2,$$
$$f(s) = [z_1 + z_2 i + z_3 j + z_4 k \longmapsto s(z_1 + z_2 i) + z_3 j + z_4 k]$$

is bijective.

Proof. (Trivial.) □

$$
\begin{tikzcd}
\mathbb{S}^0 \arrow[d, "2:1"] & GO_1 \arrow[d, "2:1"] \\
\{1\} \arrow[r, equal] & SO_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}^3 \arrow[d, "2:1"] & GO_3 \arrow[d, "2:1"] \\
\mathbb{RP}^3 \arrow[r, equal] & SO_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{S}^1 \times \mathbb{S}^0 \arrow[d, "2:1"] & GO_2 \arrow[d, "2:1"] \\
\mathbb{S}^1 \arrow[r, equal] & SO_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S}^3 \times \mathbb{S}^3 \arrow[d, "2:1"] & GO_4 \arrow[d, "2:1"] \\
\mathbb{P}(\mathbb{S}^3 \times \mathbb{S}^3) \arrow[r, equal] & SO_4
\end{tikzcd}
$$

FIGURE 13. Surjective and bijective maps relating spaces and orthogonal groups.

10. Longitude and Latitude on $\mathbb{S}^3$: A Visual Approach

Definition. The map $\mathbb{R}^n \to \mathbb{R}^{n-1}$ defined by stripping a coordinate from $\mathbb{R}^n$ is called an *orthogonal projection*. (A more general definition of orthogonal projection will be presented later.)

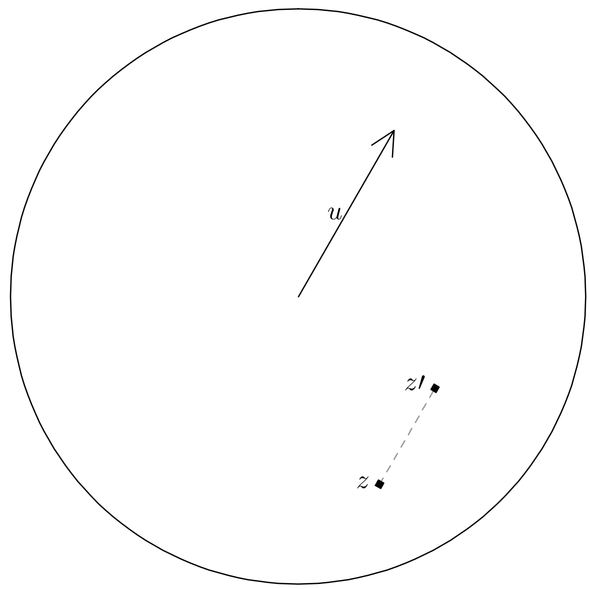

Figure 14. Orthogonal projection of $\mathbb{S}^2$ into $\mathbb{R}^2$.

10.1. **The Orthogonal Projection of $\mathbb{S}^2$.** We begin by investigating the orthogonal projection of a 2-sphere in $\mathbb{R}^3$ into $\mathbb{R}^2$. Let the vectors i, j, k in $\mathbb{R}^3$ be oriented such that i is pointing to the right, j is pointing up, and k is pointing directly out of the page. The projection will be along k, stripping away the k-coordinate.

Consider a unit 2-sphere centered at the origin which is allowed to rotate freely. Let z be a point on the sphere with positive k-coordinate. Imagine grabbing the closest portion of the sphere (at k) and pushing a small amount in a direction u lying in the (i, j)-plane. The orthogonal projection makes it appear as though z has been translated in the direction of u (see Figure 14). Although z is moving in a circle in (i, j, k)-space, the orthogonal projection collapses one dimension of this circle, resulting in the appearance of straight-line motion. Note if z had begun with a negative k-coordinate then its initial motion would have been in the direction of $-u$.

Definition. The *unit n-ball* is $\mathbb{B}^n = \{z \text{ in } \mathbb{R}^n \mid |z| < 1\}$.

The orthogonal projection of a 2-sphere in $\mathbb{R}^3$ into $\mathbb{R}^2$ results in a doubly-covered 2-ball together with a singly-covered boundary. Points with opposite k-coordinates are projected onto the same point inside the 2-ball. Rotations of the (k, u)-plane where u lies in (i, j), call them *longitudinal rotations*, translate points in the direction of u and $-u$ inside the 2-ball. Rotations of the (i, j)-plane, call them *latitudinal* *rotations*, are ordinary rotations in $\mathbb{R}^2$; the (i,j)-plane is unaffected by the projection. A longitudinal rotation has an invariant circle of longitude—the circle in the chosen direction u which intersects both poles k and $-k$. A latitudinal rotation leaves the equatorial circle of latitude invariant—the circle enclosing the 2-ball.

Definition. Given z in $1 \wedge i$, let $\arg z$ be the (positive) angle from 1 to z with $0 \leq \arg z < 2\pi$; let $\arg 0 = 0$.

Definition. $\mathbb{S}^1_{i,j} = \{z \text{ in } \mathbb{S}^3 \mid 1 \cdot z = k \cdot z = 0\}$.

Definition. *Longitudinal rotations* of SO_3 are defined

$$\Upsilon^\star : \mathbb{S}^1_{i,j} \times \mathbb{RP}^1 \to SO_3,$$
$$\Upsilon^\star(u,s) = (z \longmapsto e^{ku \arg s} z e^{-ku \arg s}).$$

Latitudinal rotations of SO_3 are defined

$$\Theta^\star : \mathbb{RP}^1 \to SO_3,$$
$$\Theta^\star(t) = (z \longmapsto e^{k \arg t} z e^{-k \arg t}).$$

The *longitudinal/latitudinal parameterization* of SO_3 is defined

$$\Psi^\star : \mathbb{S}^1_{i,j} \times \mathbb{RP}^1 \times \mathbb{RP}^1 \to SO_3,$$
$$\Psi^\star(u,s,t) = \Upsilon^\star(u,s) \circ \Theta^\star(t) = (z \longmapsto e^{ku \arg s} e^{k \arg t} z e^{-k \arg t} e^{-ku \arg s}).$$

Theorem 83. *$\Psi^\star$ is surjective.*

Proof. Left as exercise. □

$\Psi^\star$ is clearly not bijective since $\Psi^\star(u,1,1) = \Psi^\star(v,1,1)$ for any u, v in $\mathbb{S}^1_{i,j}$. Such is the price of separating SO_3 into longitudinal and latitudinal parameters. The non-bijectivity of $\Psi^\star$ is related to the "bunching up" of longitudinal lines near the (arbitrarily chosen) poles on a sphere.

Another method of parameterizing SO_3 is via $\mathbb{S}^1 \times \mathbb{RP}^2$—an angle taken from $\mathbb{S}^1$ and an axis taken from $\mathbb{RP}^2$. This parameterization is also not bijective since the identity is represented by an angle of zero with any choice of axis.

10.2. **The Orthogonal Projection of $\mathbb{S}^3$.** We now examine the orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^3$ of a unit 3-sphere in $\mathbb{R}^4$, which results in a doubly-covered 3-ball together with a singly-covered boundary. From $\mathbb{R}^4$ project along 1, stripping away the real coordinate, to (i,j,k)-space with the axes oriented as before.

This time let z be a point on the 3-sphere with positive real coordinate. Given a direction u in (i,j,k)-space, rotating the $(1,u)$-plane a small amount appears to translate z in the direction of u (see Figure 15). This is the motion obtained by grabbing the closest portion of the 3-sphere and pushing in the direction of u. "Closest" means at the center of the 3-ball, where points with the highest (and lowest) real coordinates are located; the portion grabbed is a three-dimensional patch of the 3-sphere, and the direction pushed is any direction in (i,j,k)-space. Since one dimension of the rotation plane is projected away (the real dimension), z moves in a straight line within the viewing 3-ball. This is all in direct analogy to rotating the 2-sphere described previously.

"Push"-type motions, call them *longitudinal rotations*, translate points back and fourth within the 3-ball. A given longitudinal rotation leaves invariant a circle of longitude passing through the poles -1 and 1, which because of the projection

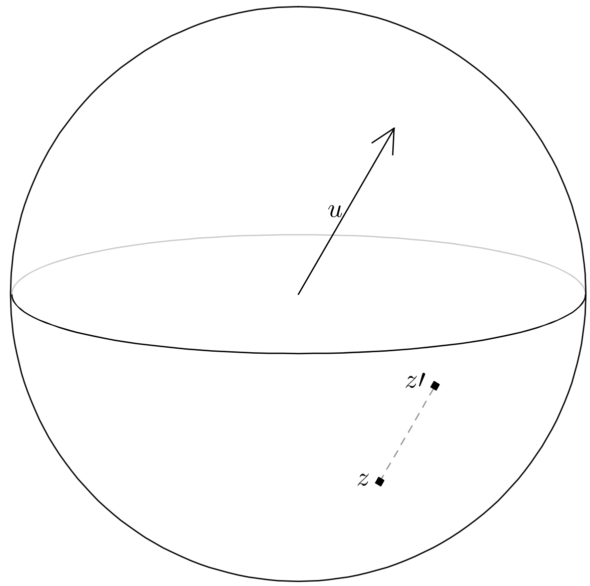

FIGURE 15. Orthogonal projection of $\mathbb{S}^3$ into $\mathbb{R}^3$.

appears as a straight line in the direction of u passing through the origin in $\mathbb{R}^3$. The familiar rotations of SO_3, call them *latitudinal rotations*, leave the equatorial 2-sphere invariant—the boundary of the 3-ball consisting of pure quaternions.

In the following definitions, let $p = e^{u\theta}$ where u is defined as above and θ is half the rotation angle (or the "push" amount in the case of longitudinal rotations).

Definition. *Longitudinal rotations* of SO_4 are defined

$$\Upsilon : \mathbb{RP}^3 \to SO_4,$$
$$\Upsilon(p) = (z \longmapsto pzp).$$

Latitudinal rotations of SO_4 are defined

$$\Theta : \mathbb{RP}^3 \to SO_4,$$
$$\Theta(p) = (z \longmapsto pz\bar{p}).$$

The *longitudinal/latitudinal parameterization* of SO_4 is defined

$$\Psi : \mathbb{RP}^3 \times \mathbb{RP}^3 \to SO_4,$$
$$\Psi(p,q) = \Upsilon(p) \circ \Theta(q) = (z \longmapsto pqz\bar{q}p).$$

Theorem 84. Ψ *is surjective.*

Proof. Given an element of SO_4 such as $(z \longmapsto azb)$, we wish to find p, q in $\mathbb{RP}^3$ such that $\Psi(p,q) = (z \longmapsto azb)$. If $ab \neq -1$ then

$$\Psi((ab)^{\frac{1}{2}}, (ab)^{-\frac{1}{2}}a) = (z \longmapsto azb).$$

Otherwise if $ab = -1$ then

$$\Psi(u, ua) = (z \longmapsto azb)$$

for any unit pure u. Therefore Ψ is surjective. $\square$

Definition. *Left Hopf rotations* of SO_4 are defined

$$\Lambda(p) = (z \longmapsto pz),$$
$$\Lambda : \mathbb{S}^3 \to SO_4.$$

Right Hopf rotations of SO_4 are defined

$$\Gamma : \mathbb{S}^3 \to SO_4,$$
$$\Gamma(p) = (z \longmapsto z\bar{p}).$$

The *Hopf parameterization* of SO_4 is defined

$$\Phi : \mathbb{S}^3 \times \mathbb{S}^3 \to SO_4,$$
$$\Phi(p, q) = \Lambda(p) \circ \Gamma(q) = (z \longmapsto pz\bar{q}).$$

Ψ provides an intuitive parameterization of SO_4 via longitudinal and latitudinal motions, however Ψ is not smooth like Φ. That is, Ψ suddenly becomes many-to-1 whenever it maps to elements of the form $(z \longmapsto -az\bar{a})$. The non-bijective nature of Ψ is analogous to that of $\Psi^\star$.

Note Φ has the property that its factors Λ and Γ *commute*; this is not true in general for Υ and Θ which comprise Ψ. Thus in switching from Φ to Ψ we gain a more human-understandable description of SO_4 but we lose commutation of the two factors.

11. Tori and Fibers

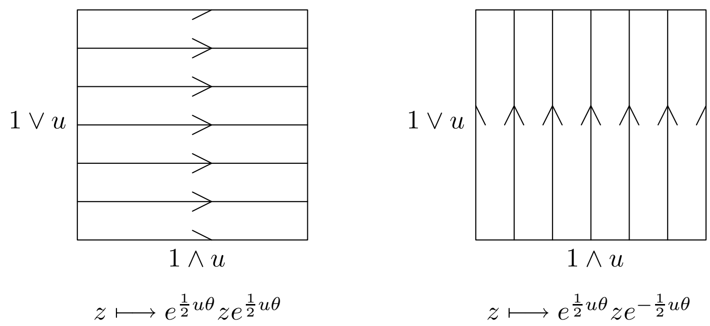

Figure 16. Longitudinal/latitudinal parameterization.

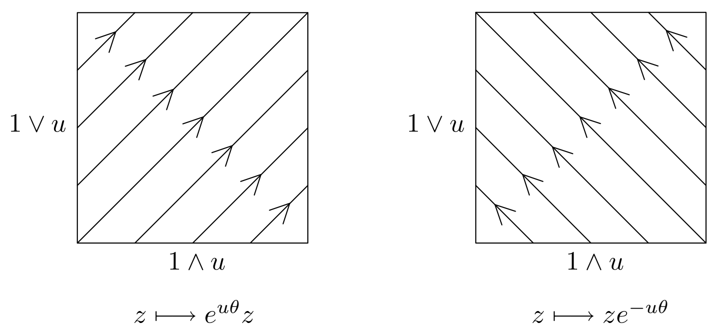

Figure 17. Hopf parameterization.

The slope of the arrowed lines in Figures 16 and 17 represent the relative motion of the two invariant planes under the action shown. For example the vertical lines on the right in Figure 16 indicate $1 \wedge u$ does not move at all. For a given square each side is matched with its opposite side, forming a round trip of 2π in either direction.

For example rotating each plane by π (which is the antipodal map $z \longmapsto -z$) means traveling diagonally from, say, the lower-left corner to the center of a square. This is exactly what Hopf motions do (Figure 17). In Figure 16 it is clear that both horizontal and vertical actions (longitudinal and latitudinal rotations) are required in order to travel diagonally (to form the antipodal map).

The definitions and theorems in this section will give a more precise meaning to Figures 16 and 17.

Theorem 85. *For orthonormal pures u, v,*

$$\mathbb{S}^3 = \{ e^{u\psi} \cos\tau + e^{u\xi} v \sin\tau \mid \psi, \xi, \tau \text{ in } \mathbb{R} \}.$$

Proof. Given z in $\mathbb{S}^3$, resolve z into the orthonormal basis $(1, u, v, uv)$,

$$z = z_1 + z_2u + z_3v + z_4uv.$$

Then

$$z = e^{u\psi} \cos\tau + e^{u\xi} v \sin\tau$$

where

$$\begin{aligned}\psi &= \arg(z_1 + z_2 i),\\ \xi &= \arg(z_3 + z_4 i),\\ \tau &= \arg\left(\sqrt{z_1^2 + z_2^2} + i\sqrt{z_3^2 + z_4^2}\right).\end{aligned}$$

It may be easily checked that $|e^{u\psi}\cos\tau + e^{u\xi}v\sin\tau|^2 = 1$ for all ψ, ξ, τ in $\mathbb{R}$. □

Definition. For orthonormal pures u, v and real τ, define

$$\mathcal{C}_{u,v,\tau} = \{e^{u\psi}\cos\tau + e^{u\xi}v\sin\tau \mid \psi, \xi \text{ in } \mathbb{R}\}.$$

$\mathcal{C}_{u,v,\tau}$ is called a *Clifford torus.*

Theorem 86. *For the interval $A = [0, \frac{\pi}{2}]$,*

$$\bigcup_{\tau \text{ in } A} \mathcal{C}_{u,v,\tau} = \mathbb{S}^3 \quad \text{and} \quad \bigcap_{\tau \text{ in } A} \mathcal{C}_{u,v,\tau} = \emptyset.$$

Proof. Left as exercise. □

Theorem 87. *$\mathcal{C}_{u,v,\tau}$ is invariant under the transformation*

$$z \longmapsto e^{u\theta} z e^{u\gamma}$$

for all real θ, γ.

Proof. Left as exercise. □

Definition. For orthonormal pures u, v and real τ, λ, define

$$\begin{aligned}\textit{left Hopf fiber: } \mathcal{L}_{u,v,\tau,\lambda} &= \{e^{u(\phi-\lambda)}\cos\tau + e^{u(\phi+\lambda)}v\sin\tau \mid \phi \text{ in } \mathbb{R}\},\\ \textit{right Hopf fiber: } \mathcal{R}_{u,v,\tau,\lambda} &= \{e^{-u(\phi+\lambda)}\cos\tau + e^{u(\phi-\lambda)}v\sin\tau \mid \phi \text{ in } \mathbb{R}\},\\ \textit{longitudinal fiber: } \mathcal{G}_{u,v,\tau,\lambda} &= \{e^{u\phi}\cos\tau + e^{u\lambda}v\sin\tau \mid \phi \text{ in } \mathbb{R}\},\\ \textit{latitudinal fiber: } \mathcal{H}_{u,v,\tau,\lambda} &= \{e^{-u\lambda}\cos\tau + e^{u\phi}v\sin\tau \mid \phi \text{ in } \mathbb{R}\}.\end{aligned}$$

Theorem 88. *Given $\tau \neq \frac{n\pi}{2}$ and the intervals $A = [0, \pi)$ and $B = (-\pi, \pi]$,*

$$\bigcup_{\lambda \text{ in } A} \mathcal{L}_{u,v,\tau,\lambda} = \bigcup_{\lambda \text{ in } A} \mathcal{R}_{u,v,\tau,\lambda} = \bigcup_{\lambda \text{ in } B} \mathcal{G}_{u,v,\tau,\lambda} = \bigcup_{\lambda \text{ in } B} \mathcal{H}_{u,v,\tau,\lambda} = \mathcal{C}_{u,v,\tau},$$

$$\bigcap_{\lambda \text{ in } A} \mathcal{L}_{u,v,\tau,\lambda} = \bigcap_{\lambda \text{ in } A} \mathcal{R}_{u,v,\tau,\lambda} = \bigcap_{\lambda \text{ in } B} \mathcal{G}_{u,v,\tau,\lambda} = \bigcap_{\lambda \text{ in } B} \mathcal{H}_{u,v,\tau,\lambda} = \emptyset.$$

Proof. Left as exercise. □

Theorem 89.

$$\begin{aligned}
&\mathcal{L}_{u,v,\tau,\lambda} \textit{ is invariant under } z \longmapsto e^{u\theta}z,\\
&\mathcal{R}_{u,v,\tau,\lambda} \textit{ is invariant under } z \longmapsto ze^{-u\theta},\\
&\mathcal{G}_{u,v,\tau,\lambda} \textit{ is invariant under } z \longmapsto e^{u\theta}ze^{u\theta},\\
&\mathcal{H}_{u,v,\tau,\lambda} \textit{ is invariant under } z \longmapsto e^{u\theta}ze^{-u\theta}.
\end{aligned}$$

Proof. Left as exercise. □

The arrowed lines in Figures 16 and 17, which were introduced as abstractly representing the longitudinal, latitudinal, and Hopf rotations, may now be interpreted more directly as the invariant fibers lying on a torus under each of these motions. Each square in Figures 16 and 17 represents $\mathcal{C}_{u,v,\tau}$ for $\tau \neq \frac{n\pi}{2}$, however if we wish the squares to have the same scale on the horizontal and vertices axes then τ must be a diagonal angle such as $\frac{\pi}{4}$.

Incidentally we now have a final answer to the question, "What *is* quaternion multiplication?" Left multiplication is a rotation which leaves the left Hopf fibers invariant and right multiplication is a rotation which leaves the right Hopf fibers invariant. Furthermore, left multiplication permutes the right Hopf fibers and right multiplication permutes the left Hopf fibers.

The definitions in this section use the basis $(1, u, v, uv)$ as a matter of convenience. A general orthonormal basis is just one multiplication away from the form $(1, u, v, uv)$ (Theorem 35), so there is little to be lost in choosing a basis aligned with 1. After all, 1 is naturally distinguished from all other vectors and its uniqueness has already been used in defining "longitudinal" and "latitudinal," whose names themselves imply a distinguished vector, i.e., a choice of poles. The remaining Clifford tori which were not explicitly defined (those not aligned with 1) all have the form $\mathcal{C}_{u,v,\tau}a$ for some respective a.

Figures 16 and 17 suggest that left/right Hopf rotations are not so different from longitudinal/latitudinal rotations. We could obtain Hopf motions from longitudinal/latitudinal motions and vice versa if we could somehow "turn" one motion 45 degrees toward the other. Indeed, Υ, Θ, Λ, Γ and $\mathcal{G}$, $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$ have the following generalization.

Definition. For orthonormal pures u, v and real $\tau, \lambda, \alpha, \beta$, define the *knot fiber*

$$\mathcal{K}_{u,v,\tau,\lambda,\alpha,\beta} = \{\, e^{u(\alpha\phi-\beta\lambda)}\cos\tau + e^{u(\beta\phi+\alpha\lambda)}v\sin\tau \;\mid\; \phi \text{ in } \mathbb{R}\}$$

and the *knot rotation*

$$\begin{gathered}
\Omega : \mathbb{S}^2 \times \mathbb{R} \times \mathbb{R} \to SO_4,\\
\Omega(u,\alpha,\beta) = \left[z \longmapsto e^{\frac{1}{2}u(\alpha+\beta)}ze^{\frac{1}{2}u(\alpha-\beta)}\right].
\end{gathered}$$

Theorem 90. *$\mathcal{K}_{u,v,\tau,\lambda,\alpha,\beta}$ is invariant under $\Omega(u, \alpha\theta, \beta\theta)$ for all real θ.*

Proof. Left as exercise. □

The special cases are

$$
\begin{aligned}
&\mathcal{L}_{u,v,\tau,\lambda} = \mathcal{K}_{u,v,\tau,\lambda,1,1} && \Lambda(e^{u\theta}) = \Omega(u,\theta,\theta) && \text{(left Hopf)}\\
&\mathcal{R}_{u,v,\tau,\lambda} = \mathcal{K}_{u,v,\tau,\lambda,-1,1} && \Gamma(e^{u\theta}) = \Omega(u,-\theta,\theta) && \text{(right Hopf)}\\
&\mathcal{G}_{u,v,\tau,\lambda} = \mathcal{K}_{u,v,\tau,\lambda,1,0} && \Upsilon(e^{\frac{1}{2}u\theta}) = \Omega(u,\theta,0) && \text{(longitudinal)}\\
&\mathcal{H}_{u,v,\tau,\lambda} = \mathcal{K}_{u,v,\tau,\lambda,0,1} && \Theta(e^{\frac{1}{2}u\theta}) = \Omega(u,0,\theta) && \text{(latitudinal)}
\end{aligned}
$$

It should first be noted that $\mathcal{G}$, $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$ are distinguished because they are circles while the other $\mathcal{K}$ are not. $\mathcal{L}$ and $\mathcal{R}$ are especially distinguished because they are *great* circles, having a radius of 1. $\mathcal{K}_{u,v,\tau,\lambda,\alpha,\beta}$ and $\mathcal{K}_{u,v,\tau,\lambda,-\beta,\alpha}$ intersect at right angles on a square torus ($\tau = \frac{\pi}{4}$), as stated in the following theorem. $\Omega(u,\alpha,\beta)$ and $\Omega(u,-\beta,\alpha)$ are likewise "orthogonal" for a certain definition thereof.

Theorem 91. *For orthonormal pures u, v, let*

$$
\begin{gathered}
f : \mathbb{R}^4 \to \mathbb{S}^3,\\
f(\alpha,\beta,\lambda,\phi) = \tfrac{1}{\sqrt{2}} e^{u(\alpha\phi-\beta\lambda)} + \tfrac{1}{\sqrt{2}} e^{u(\beta\phi+\alpha\lambda)} v.
\end{gathered}
$$

Then

$$\frac{\partial f}{\partial \lambda} \cdot \frac{\partial f}{\partial \phi} = 0$$

and

$$f(\alpha,\beta,\lambda,\phi) = f(-\beta,\alpha,-\phi,\lambda).$$

Proof. Left as exercise. (Notice the dot product "$\cdot$" between the two partial derivatives.) □

So what does a general $\mathcal{K}$ look like? This has more to do with the properties of a torus than with four-dimensional geometry, per se. If α, β are relatively prime integers then $\mathcal{K}$ is a knot which winds α times around the torus in the longitudinal direction and β times around in the latitudinal direction. If β/α is irrational then $\mathcal{K}$ is a forever-winding curve which passes arbitrarily close to every point on the torus [3, p33-34].

12. Orthogonal Projection

Definition. For unit quaternion r, the *orthogonal projection along* r in $\mathbb{R}^4$ is the transformation $P_r : \mathbb{R}^4 \to \mathbb{R}^4$,

$$P_r(z) = z - (r \cdot z)r.$$

Theorem 92. $P_r(\mathbb{R}^4) = \operatorname{span}^\perp\{r\}$.

Proof. Given z in $\mathbb{R}^4$ then $r \cdot P_r(z) = r \cdot [z - (r \cdot z)r] = r \cdot z - r \cdot z = 0$. Given z in $\operatorname{span}^\perp\{r\}$ then $z = z - 0r = z - (r \cdot z)r = P_r(z)$. $\square$

Theorem 93. *All orthogonal projections are similar to each other by rotation.*

Proof. $P_r(z) = z - (r \cdot z)r = (z\bar{r} - r \cdot z)r = (z\bar{r} - 1 \cdot z\bar{r})r = (1 \times z\bar{r})r = R \circ P_1 \circ R^{-1}(z)$ where $R(z) = zr$ and $P_1(z) = z - 1 \cdot z = 1 \times z$. Every orthogonal projection is therefore similar to P_1, and it follows that every orthogonal projection is similar to every other one. $\square$

Theorem 94. *Given two orthogonal planes in $\mathbb{R}^4$, the orthogonal projection P_r produces either an orthogonal line and plane in $\mathbb{R}^3$ (when a plane contains r) or two orthogonal planes in $\mathbb{R}^3$ (otherwise).*

Proof. Since P_r is similar to P_1 by a rotation, and rotations preserve angles, it suffices to prove the theorem for P_1. Let the two planes be $\operatorname{span}\{a, b\}$ and $\operatorname{span}^\perp\{a, b\} = \operatorname{span}\{c, d\}$ with (a, b, c, d) an orthonormal basis.

If $\operatorname{span}\{a, b\}$ contains 1 then $\lambda a + \mu b = 1$ for some real λ and μ. $P_1(\lambda a + \mu b) = P_1(1) = 0 = \lambda\vec{a} + \mu\vec{b}$ thus $\vec{a}$ and $\vec{b}$ are parallel and $P_1(\operatorname{span}\{a, b\}) = \operatorname{span}\{\vec{a}, \vec{b}\}$ is a line. $(\lambda a + \mu b) \cdot c = 1 \cdot c = 0$ so c is pure and likewise d is pure. Therefore $\vec{a} \cdot \vec{c} = a \cdot c = 0$ and $\vec{a} \cdot \vec{d} = a \cdot d = 0$.

If $\operatorname{span}\{a, b\}$ does not contain 1 then for all real λ and μ we have $\lambda a + \mu b \neq 1$ $\implies P_1(\lambda a + \mu b) \neq P_1(1) \implies \lambda\vec{a} + \mu\vec{b} \neq 0$ making $\vec{a}$ and $\vec{b}$ non-parallel. Likewise $\vec{c}$ and $\vec{d}$ are non-parallel when $\operatorname{span}\{c, d\}$ does not contain 1. We wish to show $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ is equal to zero. The determinant of four pures is zero, leaving (from the definition of determinant)

$$\begin{aligned}
(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \\
&= (a \cdot c - \mathring{a}\mathring{c})(b \cdot d - \mathring{b}\mathring{d}) - (a \cdot d - \mathring{a}\mathring{d})(b \cdot c - \mathring{b}\mathring{c}) \\
&= (-\mathring{a}\mathring{c})(-\mathring{b}\mathring{d}) - (-\mathring{a}\mathring{d})(-\mathring{b}\mathring{c}) \\
&= 0.
\end{aligned}$$

$\square$

In other words, when we view the orthogonal projection of two orthogonal planes in $\mathbb{R}^4$, *they will always be orthogonal planes as we see them in* $\mathbb{R}^3$ for the non-degenerate case or an orthogonal line and plane for the degenerate case.

Figures 18 and 19 show the non-degenerate and degenerate cases of the orthogonal projection of two circles (centered at the origin) lying in orthogonal planes in $\mathbb{R}^4$. The ellipses in Figure 18 are really ellipses and not circles which appear as ellipses on the page. (If you expected them to be linked then you are thinking of the stereographic projection.) These circles in $\mathbb{R}^4$ rotate *independently* of each other, a property which has no direct analogy in SO_3 (however GO_3 has the independent actions of a rotation and a reflection over the invariant plane of the rotation).

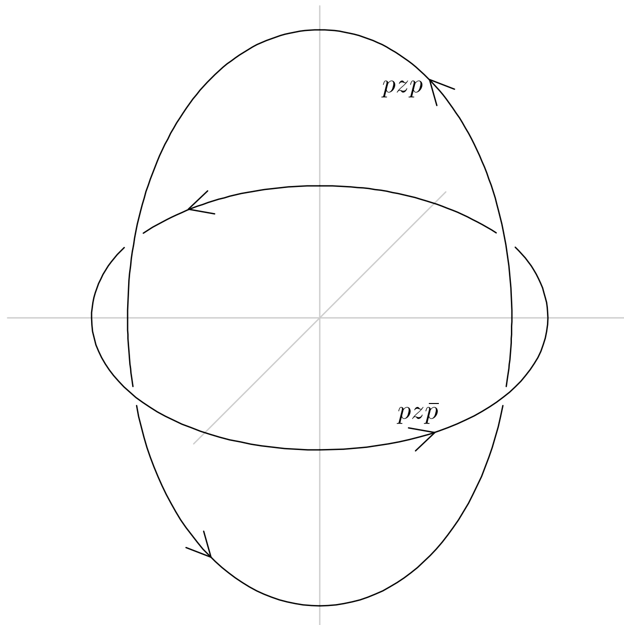

FIGURE 18. Orthogonal projection of two circles lying in orthogonal planes (non-degenerate case).

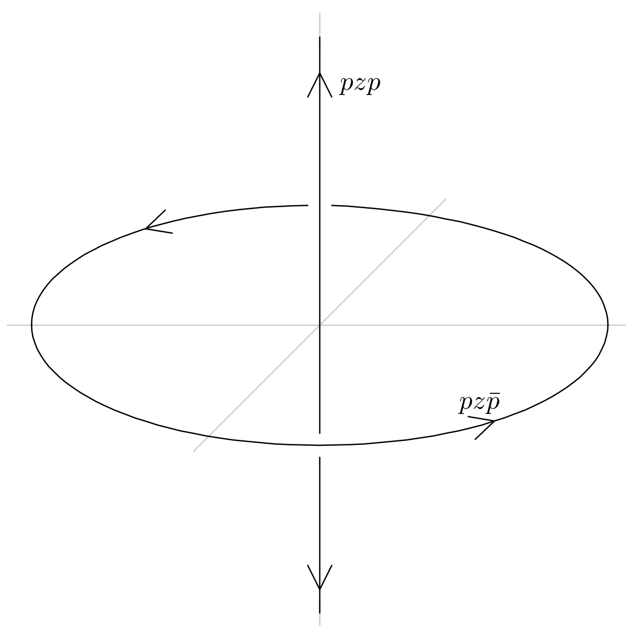

FIGURE 19. Orthogonal projection of two circles lying in orthogonal planes (degenerate case).

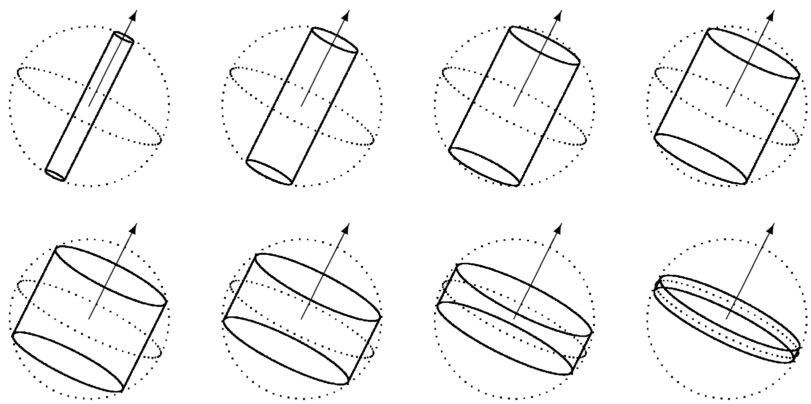

FIGURE 20. Orthogonal projection of $\mathcal{C}_{u,v,\tau}$ for various τ. The central vector points in the direction of u.

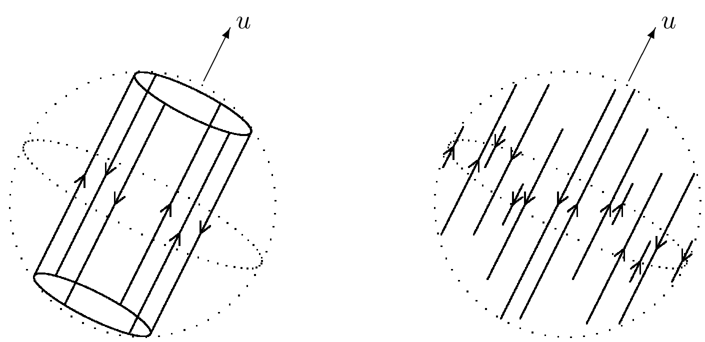

FIGURE 21. Invariant longitudinal fibers under a longitudinal motion $z \longmapsto e^{u\theta} z e^{u\theta}$.

For the orthogonal projection along 1, a Clifford torus $\mathcal{C}_{u,v,\tau}$ appears as a doubly-covered cylinder with singly-covered edges lying on the boundary of the 3-ball (Figure 20). The τ-family of tori begins with a line—a cylinder having no width and maximal height—and ends with a circle—a cylinder having no height and maximal width; Figure 19 depicts these two extremes.

Longitudinal fibers appear as line segments running along the height of a torus, connecting the edges (Figure 21). Latitudinal fibers are circles running parallel to the edges (Figure 22).

Hopf fibers are diagonal ellipses on these cylinders, connecting antipodal points on opposite edges, and are almost like the trace of a coin wobbling to rest on a smooth surface (Figure 23) except that a coin is (usually) not elliptical. Each fiber is tangent to the equatorial 2-sphere (i.e., the boundary of the 3-ball) at exactly two antipodal points, except for one left fiber and one right fiber which lie on the equatorial 2-sphere itself ($\tau = \frac{\pi}{2}$).

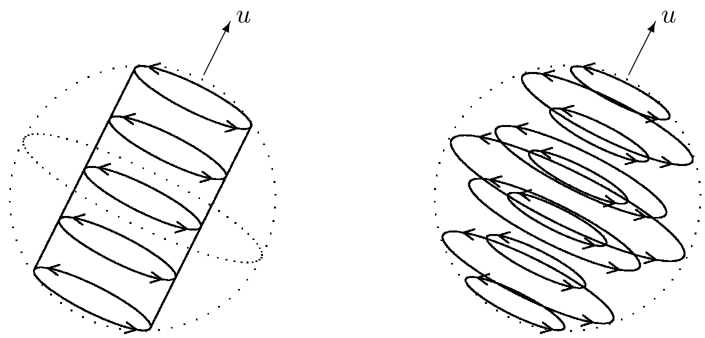

FIGURE 22. Invariant latitudinal fibers under a latitudinal motion $z \longmapsto e^{u\theta} z e^{-u\theta}$. The fixed plane $1 \wedge u$ is projected to the axis u.

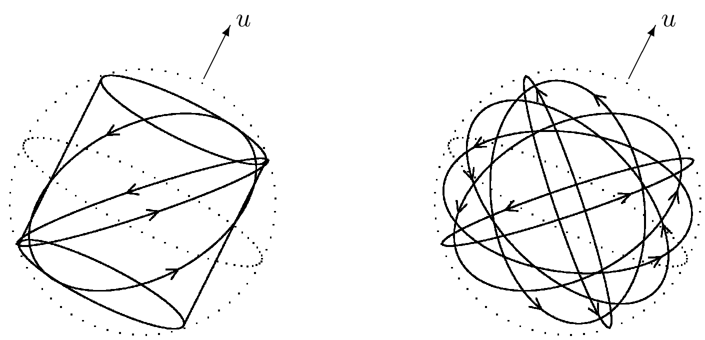

FIGURE 23. Invariant Hopf fibers on $\mathcal{C}_{u,v,\tau}$ under a Hopf motion $z \longmapsto e^{u\theta} z$.

Figures 20-23 are degenerate-type projections since the line of projection (the real line) shares a plane with one of the generating circles. The orthogonal projection of $\mathcal{C}_{u,v,\tau}$ along an arbitrary line resembles Figure 24, which may be imagined as the bifurcation of Figure 22's doubly-covered cylinder into separate trunks while the singly-covered edges remain fixed. The smaller circles in Figure 24 (which are really ellipses) all *face the same direction* in $\mathbb{R}^3$, just as the arrowed circles in Figure 22 face the same direction.

As Theorem 93 points out, there is a similarity between projecting $\mathcal{C}_{u,v,\tau}$ along a and projecting a rotated version of $\mathcal{C}_{u,v,\tau}$ along 1, namely $P_a(\mathcal{C}_{u,v,\tau}) = aP_1(\bar{a}\mathcal{C}_{u,v,\tau})$.

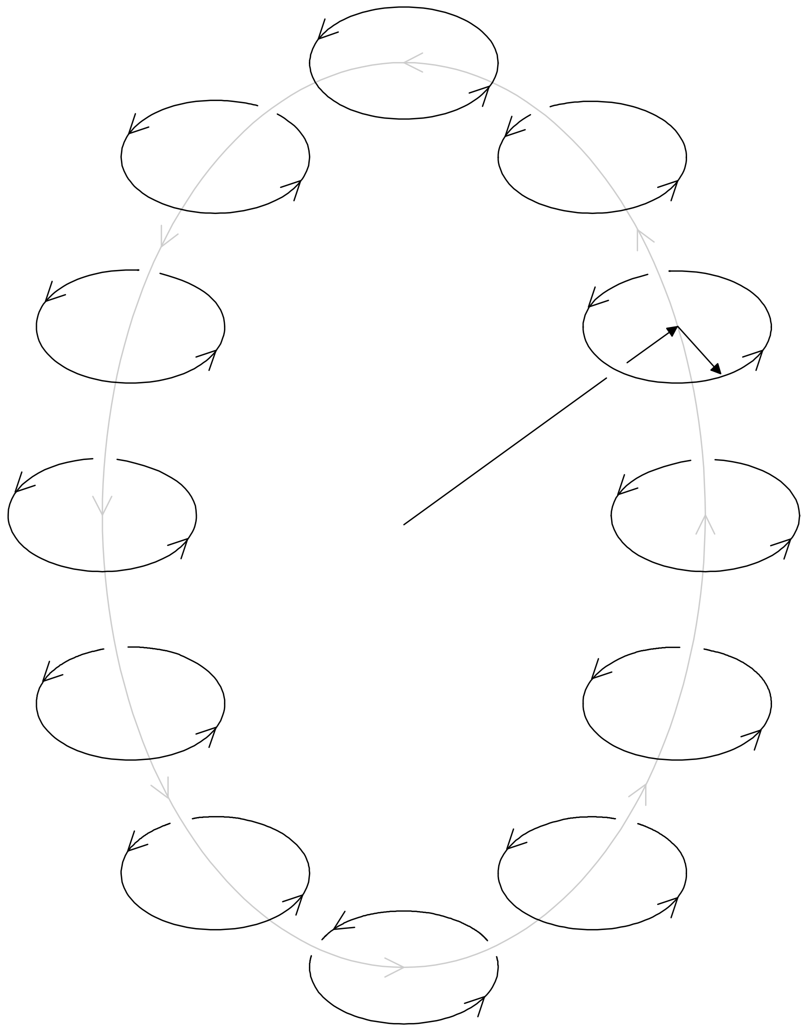

FIGURE 24. Orthogonal projection, non-degenerate case.

13. Möbius Transformations

Definition. Let $\infty = 1/0$, called the *point at infinity*, and let $\overline{\mathbb{R}^4} = \mathbb{R}^4 \cup \{\infty\}$. The group of *Möbius transformations* is

$$\mathcal{M} = \left\{ f : \overline{\mathbb{R}^4} \to \overline{\mathbb{R}^4},\ f(z) = (az+b)(cz+d)^{-1} \ \Big|\ \begin{matrix} a,b,c,d \text{ in } \mathbb{H} \text{ with} \\ |ad|^2 + |bc|^2 \neq 2a\bar{c} \cdot b\bar{d} \end{matrix} \right\}.$$

Joining ∞ with $\mathbb{R}^4$ is called *one point compactification* of $\mathbb{R}^4$. With the addition of this point, lines become topologically equivalent to circles and planes become topologically equivalent to spheres. Möbius transformations continuously transform circles to lines and lines to circles, effectively removing the distinction between the two. Spheres and planes are likewise blended into one concept, with Möbius transformations continuously bringing one to the other.

Möbius transformations are *conformal*, meaning they locally preserve angles: in a small neighborhood containing the intersection of two circles, the image of those two circles have same angle between them.

13.1. Representation in $PGL_2(\mathbb{H})$.

For matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with quaternion entries a, b, c, d define the following.

The *adjoint* of A is

$$A^H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^H = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}.$$

The *determinant* of A is

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The *norm* of A is

$$\|A\| = \det AA^H = \det A^H A = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix} = |ad|^2 + |bc|^2 - 2a\bar{c} \cdot b\bar{d}.$$

If $\|A\| \neq 0$ then A has a unique multiplicative inverse

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}^{-1} \begin{pmatrix} |d|^2\bar{a} - \bar{c}d\bar{b} & |b|^2\bar{c} - \bar{a}b\bar{d} \\ |c|^2\bar{b} - \bar{d}c\bar{a} & |a|^2\bar{d} - \bar{b}a\bar{c} \end{pmatrix}$$

such that

$$AA^{-1} = A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The *general linear group* of these matrices is

$$GL_2(\mathbb{H}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \Big|\ a,b,c,d \text{ in } \mathbb{H} \text{ with } \begin{Vmatrix} a & b \\ c & d \end{Vmatrix} \neq 0 \right\}.$$

The corresponding projective group is

$$PGL_2(\mathbb{H}) = \{ A \text{ in } GL_2(\mathbb{H}) \mid A \text{ is identified with } \gamma A \text{ for all non-zero } \gamma \text{ in } \mathbb{R} \}.$$

Two elements A, B in $GL_2(\mathbb{H})$ are equivalent in $PGL_2(\mathbb{H})$ if and only if there exists a non-zero real γ such that $A = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} B$.

Projective $\mathbb{H}$*-space* is

$$\mathbb{HP} = \{\, z \text{ in } \mathbb{H}^2 - \{(0,0)\} \mid z \text{ is identified with } zr \text{ for all non-zero } r \text{ in } \mathbb{H}\}.$$

Two elements $(a,b),(c,d)$ in $\mathbb{HP}$ are equivalent if and only if there exists a non-zero r in $\mathbb{H}$ such that $a = cr$ and $b = dr$.

Theorem 95. *$PGL_2(\mathbb{H})$ is isomorphic to $\mathcal{M}$ via*

$$\Xi : PGL_2(\mathbb{H}) \to \mathcal{M},$$
$$\Xi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left[z \longmapsto (az+b)(cz+d)^{-1} \right].$$

Proof. This is an algebraic identity which ties matrix composition in $PGL_2(\mathbb{H})$ to function composition in $\mathcal{M}$ with the same terms. (Left as exercise.) □

Theorem 96. *The map $f : \mathbb{HP} \to \overline{\mathbb{R}^4}$, $f(x,y) = xy^{-1}$ is bijective.*

Proof. For all non-zero x, x' in $\mathbb{H}$, $(x,0) = (xx^{-1},0) = (1,0) = (x',0)$ and $f(x,0) = x0^{-1} = \infty = x'0^{-1} = f(x',0)$. Similarly for all non-zero y, y' in $\mathbb{H}$, $(0,y) = (0,yy^{-1}) = (0,1) = (0,y')$ and $f(0,y) = 0y^{-1} = 0 = 0y'^{-1} = f(0,y')$.

Let x, x', y, y' be non-zero. $f(x,y) = f(x',y') \implies xy^{-1} = x'y'^{-1} \implies x'^{-1}x = y'^{-1}y \implies \frac{|x|}{|x'|} = \frac{|y|}{|y'|} \implies [\, x = \pm\gamma x' \text{ and } y = \pm\gamma y' \,]$ for real γ. If these coefficients have the same sign then $(x,y) = (\gamma x, \gamma y) = (-\gamma x, -\gamma y) = (x',y')$. Otherwise if they have opposite sign then $xy^{-1} = x'y'^{-1} \implies -x'y'^{-1} = x'y'^{-1} \implies x'y'^{-1} = 0 \implies x' = 0$, contrary to our supposition. Therefore $(x,y) = (x',y')$.

Given z in $\overline{\mathbb{R}^4}$, if $z = \infty$ then $f(1,0) = \infty = z$, otherwise $f(z,1) = z$. □

Theorem 97. *Let*

$$\mathcal{M}' = \{T : \mathbb{HP} \to \mathbb{HP},\ T(z) = Az \mid A \text{ in } PGL_2(\mathbb{H})\}.$$

Using the correspondence in Theorem 96, $\mathcal{M}'$ is isomorphic to $\mathcal{M}$ via

$$\Xi : \mathcal{M}' \to \mathcal{M},$$
$$\Xi \left[z \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} z \right] = \left[z \longmapsto (az+b)(cz+d)^{-1} \right].$$

Proof. Follows directly from Theorems 95 and 96. □

13.2. **Frames and actions.** An element of a group may be interpreted as a *position* or as an *action*. In particular, $ab = c$ has the following interpretations:

(1) the left action a brings b to c, or
(2) the right action b brings a to c, or
(3) the actions a and b combine to form the action c.

When the group represents transformations of space, the word *frame* is often used instead of *position*.

The expression rar^{-1} may be interpreted as the action a with r as a *frame of reference* for the action. Read from right to left it says, "bring r to the identity frame, perform the action, then return to r." rar^{-1} is essentially the action a from a different location—a different frame of reference.

Writing $ra = (rar^{-1})r$ says the right action of a on r is *the action a within the frame of reference of r*. On the other hand, $ar = (1a1^{-1})r$ says the left action of a on r is *the action a within the frame of reference of the identity.*

In linear algebra, a and rar^{-1} are called *similar*; in group theory they are called *conjugate* (not to be confused with the conjugate operation for quaternions).

13.3. **Möbius frames and actions.** A Möbius transformation may be interpreted as a frame which holds a local origin, infinity, orientation (rotational state), and scaling factor. The local origin is the image of $(0, 1)$ and the local infinity is the image of $(1, 0)$. Thus the local origin of the frame $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is at (b, d) and the local infinity is at (a, c).

There are four basic actions.

Translation: The action

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$$

moves the local origin from g to $g' = g + r$.

Inversion: The action

$$\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$$

moves the local infinity from h to h' per the relation ${h'}^{-1} = h^{-1} + r$.

Rotation: The action

$$\begin{pmatrix} e^{\frac{1}{2}(r\times s)(\alpha+\beta)} & 0 \\ 0 & e^{\frac{1}{2}(\bar{r}\times\bar{s})(\alpha-\beta)} \end{pmatrix}$$

rotates the local 2-spheres $r \wedge s$ and $r \vee s$ by angles α and β respectively, given r, s are orthonormal (Theorem 68).

Scaling: For real $\gamma \neq 0$, the action

$$\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}$$

is an expansion ($|\gamma| > 1$) or a contraction ($|\gamma| < 1$) by a factor of γ.

The actions described above pertain to the frame of reference. For example, a translation applied on the left (i.e., in the frame of reference of the identity) moves $(0, 1)$ and leaves $(1, 0)$ fixed. A translation applied on the right of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ moves the local origin (b, d) and leaves the local infinity (a, c) fixed. For a given frame, the local 2-sphere $r \wedge s$ is the image of $r \wedge s$ in standard coordinates, just as the local origin is the image of $(0, 1)$.

Translation fixes infinity and moves the origin, inversion fixes the origin and moves infinity, rotation fixes both infinity and the origin, and scaling fixes both infinity and the origin. A rotation permutes the circles connecting the origin to infinity and leaves invariant the concentric spheres centered at the origin (which are also centered at infinity). Scaling does the converse, leaving invariant the connecting circles and permuting the concentric spheres.

There are 4 parameters involved for translation (each coordinate of r), 4 parameters for inversion, 6 parameters for rotation ($\mathbb{S}^3 \times \mathbb{S}^3$), and 1 parameter for scaling, for 15 parameters in all. These actions are continuous and generate $\mathcal{M}$.

Example. Suppose we wish to bring the (i, j)-plane to the unit 2-sphere in (i, j, k)-space. First apply an inversion which moves the local infinity to $2k$, then translate by $-k$, bringing the local origin from 0 to $-k$ and the local infinity from $2k$ to k. These actions are applied on the left, starting from the identity frame:

$$\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (2k)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -k \\ -\frac{1}{2}k & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2k \\ -k & 2 \end{pmatrix}.$$

The associated function is

$$f : \overline{\mathbb{R}^4} \to \overline{\mathbb{R}^4},$$
$$f(z) = (z - 2k)(-kz + 2)^{-1}.$$

It may be checked that

$$f(\mathrm{span}\{i, j\} \cup \{\infty\}) = \mathbb{S}^2.$$

From this frame we may move the local origin to another place on $\mathbb{S}^2$ by performing right-action translations in the (i, j)-plane. For example,

$$\begin{pmatrix} 1 & -2k \\ -k & 2 \end{pmatrix} \begin{pmatrix} 1 & i + j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & i + j - 2k \\ -k & 2 + i - j \end{pmatrix},$$
$$g(z) = (z + i + j - 2k)(-kz + 2 + i - j)^{-1},$$
$$g(\mathrm{span}\{i, j\} \cup \{\infty\}) = \mathbb{S}^2.$$

Similarly, the local infinity may be moved around on $\mathbb{S}^2$ by performing right-action inversions in the (i, j)-plane.

14. Stereographic Projection

The previous section gave an example of a Möbius transformation which mapped a plane to a sphere. The inverse of this type of transformation is called *stereographic projection*,

$$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

Interpreting the matrices as left actions on $\mathbb{S}^3$, the first action translates the center of the sphere to -1, the second action scales the sphere by half so that 0 and -1 are now antipodal points on the sphere (poles), and the last action, an inversion, maps $-1 \mapsto \infty$. The representation in $\overline{\mathbb{R}^4}$ is

$$f(z) = (1-z)(1+z)^{-1}$$

from which it is clear by inspection that $1 \mapsto 0$ and $-1 \mapsto \infty$. The result is that $\mathbb{S}^3$ is mapped fully onto $\mathbb{R}^3$, the subspace of pures: $f(\mathbb{S}^3) = \{\infty\} \cup \operatorname{span}^{\perp}\{1\}$.

The following figures are the stereographic counterparts to Figures 20-23. The two projections deceive in different ways. The orthogonal projection shrinks lengths (possibly to zero) and, being doubly-covered, shows false intersections. The stereographic projection does not give false intersections but at the cost of a most extreme exaggeration of length: points near -1 are sent far away toward infinity.

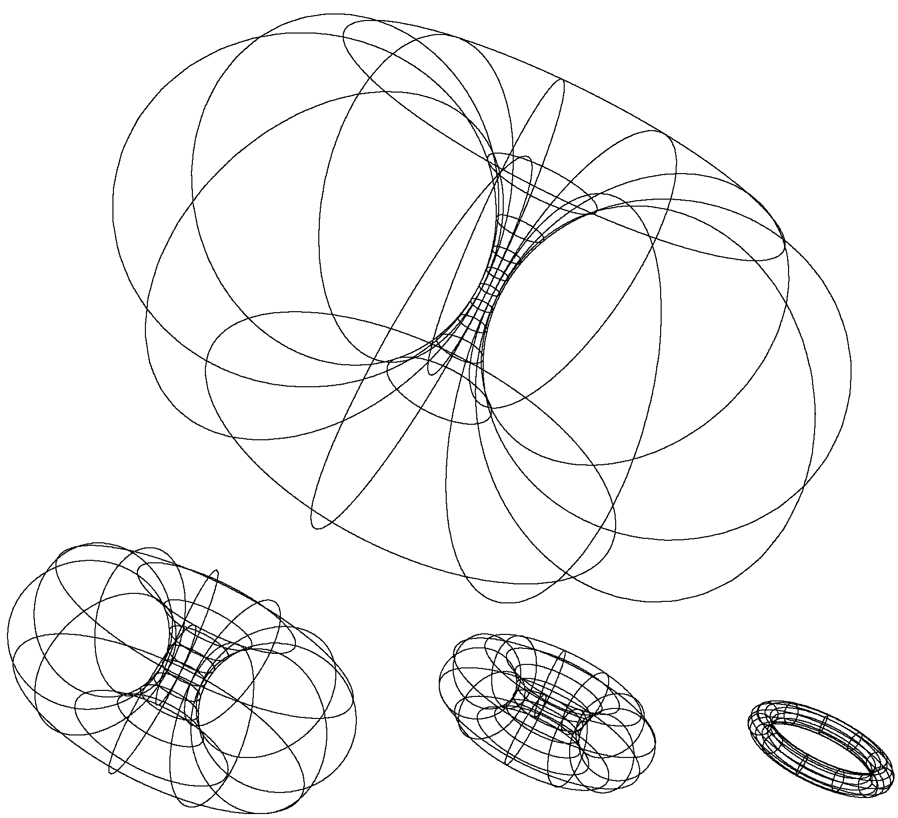

Figure 25. Stereographic projection of the torus $\mathcal{C}_{u,v,\tau}$ for $\tau = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{3\pi}{9}, \frac{4\pi}{9}$. Each torus has been drawn to the same scale.

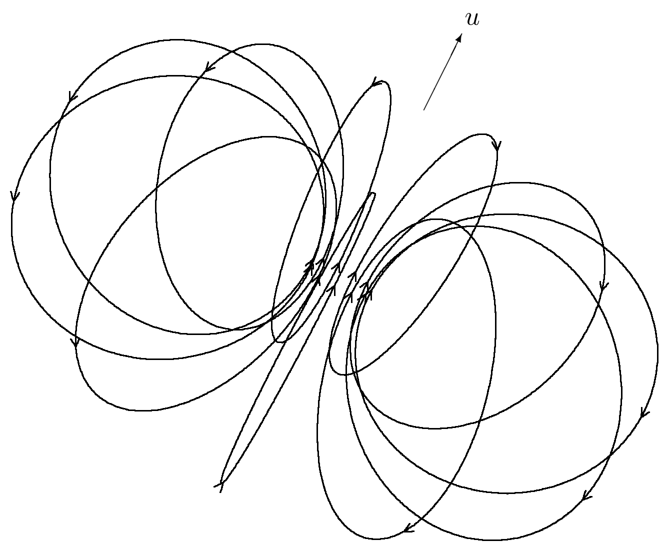

FIGURE 26. Invariant longitudinal fibers under a longitudinal motion $z \longmapsto e^{u\theta} z e^{u\theta}$.

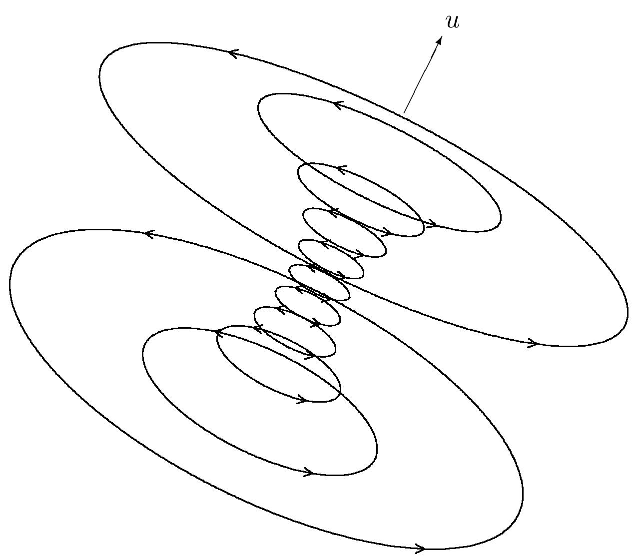

FIGURE 27. Invariant latitudinal fibers under a latitudinal motion $z \longmapsto e^{u\theta} z e^{-u\theta}$.

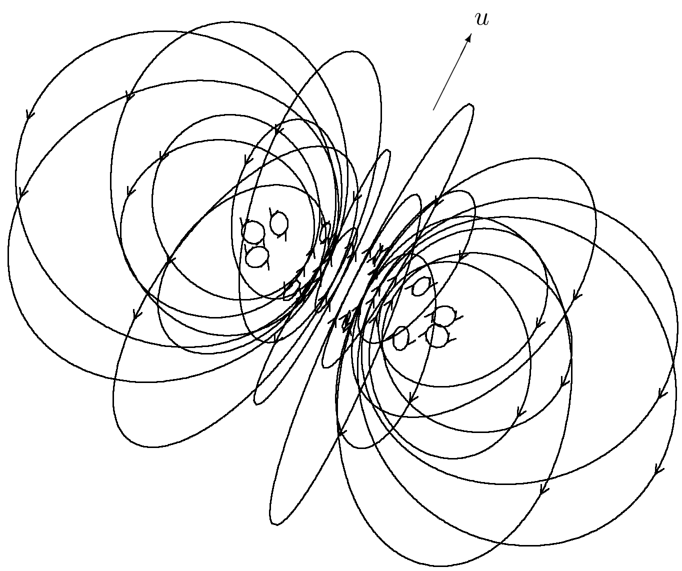

FIGURE 28. Invariant longitudinal fibers under a longitudinal motion $z \longmapsto e^{u\theta} z e^{u\theta}$ for various τ.

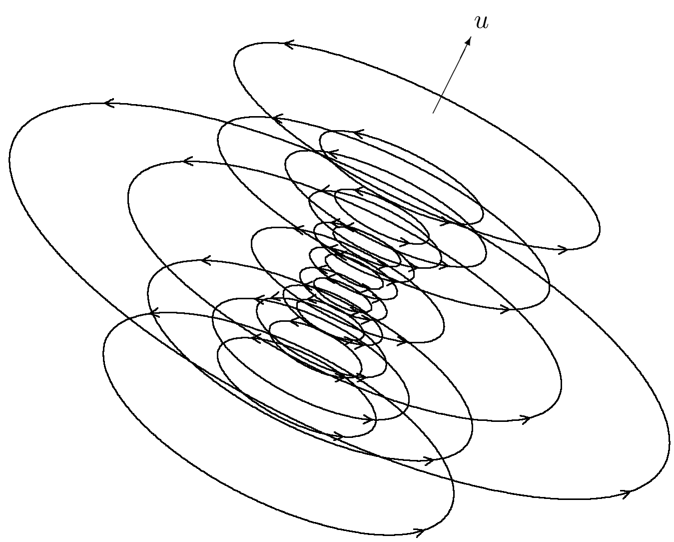

FIGURE 29. Invariant latitudinal fibers under a latitudinal motion $z \longmapsto e^{u\theta} z e^{-u\theta}$ for various τ.

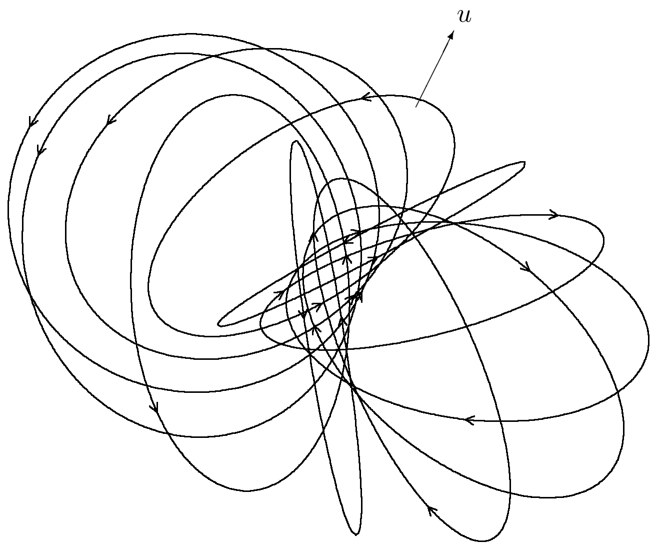

FIGURE 30. Invariant left Hopf fibers under a left Hopf motion $z \longmapsto e^{u\theta} z$.

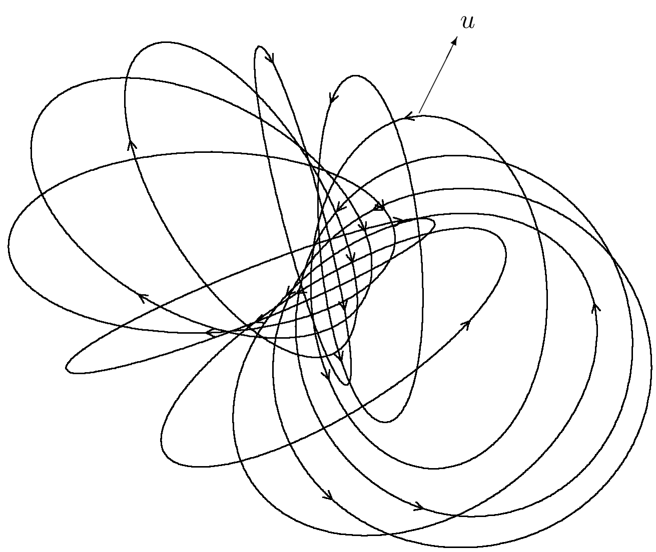

FIGURE 31. Invariant right Hopf fibers under a right Hopf motion $z \longmapsto ze^{-u\theta}$.

Appendix: The Exponential

The polar form e^a defined in Section 3 served as mere notational convenience. Had it not been introduced, we would have simply dealt with the sin and cos terms directly, with no change in meaning.

As it turns out, e^a (called the *exponential*) is greatly more important than a mere notational tool. It is normally defined by a convergent power series and belongs to a canonical family of such series. Each of the following series converges for all quaternions a, except $\ln(1+a)$ which converges for all $|a|<1$.

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots$$

$$\cos a = 1 - \frac{a^2}{2!} + \frac{a^4}{4!} - \frac{a^6}{6!} + \cdots$$

$$\sin a = a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} + \cdots$$

$$\cosh a = 1 + \frac{a^2}{2!} + \frac{a^4}{4!} + \frac{a^6}{6!} + \cdots$$

$$\sinh a = a + \frac{a^3}{3!} + \frac{a^5}{5!} + \frac{a^7}{7!} + \cdots$$

$$\ln(a+1) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \cdots \qquad \text{for } |a| < 1$$

Recall our shorthand $\mathring{a} = 1 \cdot a$ and $\vec{a} = 1 \times a$. Let a be a non-real quaternion ($\vec{a} \neq 0$) and let $u = \vec{a}/|\vec{a}|$. The following closed forms may be deduced directly from their counterparts for complex numbers.

$$e^a = e^{1\cdot a}\left(\cos|\vec{a}| + u \sin|\vec{a}|\right)$$

$$\cos a = (\cos\mathring{a})\cosh|\vec{a}| - u(\sin\mathring{a})\sinh|\vec{a}| = \frac{e^{ua} + e^{-ua}}{2}$$

$$\sin a = (\sin\mathring{a})\cosh|\vec{a}| + u(\cos\mathring{a})\sinh|\vec{a}| = \frac{e^{ua} - e^{-ua}}{2}u^{-1}$$

$$\cosh a = (\cosh\mathring{a})\cos|\vec{a}| + u(\sinh\mathring{a})\sin|\vec{a}| = \frac{e^{a} + e^{-a}}{2}$$

$$\sinh a = (\sinh\mathring{a})\cos|\vec{a}| + u(\cosh\mathring{a})\sin|\vec{a}| = \frac{e^{a} - e^{-a}}{2}$$

$$\ln a = \ln|a| + u\cos^{-1}\frac{\mathring{a}}{|a|}$$

For a given a, these functions produce quaternions which commute with each other (since their pure parts are parallel). As a consequence, $\tan a = (\sin a)(\cos a)^{-1}$ and $\tanh a = (\sinh a)(\cosh a)^{-1}$ are well-defined.

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