Infinite Series and Infinite Products

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摘要

This is the note of Infinite Series and Infinite Products, maded by Len Fu while his learning progress. The main content is from *Mathematical Analysis Tom A.Apostol*.

Contents

1 Convergent and Divergent Sequences of Complex Numbers

1.1 Definition of Convergence

A sequence of complex numbers $a_n \in C$ is called *convergent* if,

$$for\ every\ \epsilon > 0,\ there\ exists\ an\ N \in \mathbb{N}\ such\ that,\ |a_n - a| < \epsilon\ for\ all\ n \geq N.$$

If a_n converges to p, we write $\lim_{n \to \inf} a_n = p$ and call p the limit of the sequence. A sequence is called divergent if it is not convergent.

1.2 Cauchy Condition

A sequence in $\mathbb{C}$ is called a *Cauchy sequence* if it satisfies the *Cauchy condition*: for every $\epsilon > 0$ there is an integer N such that

$$|a_n - a_m| < \epsilon\ whenevern \geq N\ and\ m \geq N.$$

The Cauchy condition is particularly useful in establishing convergence when we do not know the actual value to which the sequence converges.

1.3 Bounded and convergent

Every convergent sequence is bounded and hence an unbounded sequence necessarily diverges.

1.4 Subsequence

If a sequence a_n converges to p, then every subsequence a_{k_n} also converges to p.

1.5 Definition of Divergence

A sequence of complex numbers $a_n \in C$ is called *divergent* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that,

$$|a_n - a| \leq \epsilon \; for \; all \; n \geq N.$$

In this case we write $\lim_{n\to\infty} a_n = +\infty$.

If $\lim_{n\to\infty}(-a_n) = \infty$, we write $\lim_{n\to\infty}(a_n) = -\infty$ and say that a_n diverges to $-\infty$.

2 Limit Superior and Limit Inferior of a Real-Valued Sequence

2.1 Definition of inf and sup

Let a_n be a sequence of real numbers. Suppose there is a real number U satisfying the following two conditions:

1. For every $\epsilon > 0$, there exists an integer N such that $n > N$ implies
$$a_n < U + \epsilon.$$
2. Given $\epsilon > 0$ and given $m > 0$, there exists an integer $n > m$ such that
$$a_n > U - \epsilon.$$

Note. Statement (1) means that all terms of the sequence lie to the left of $U + \epsilon$. Statement (2) means that infinite terms of the sequence lie to the right of $U - \epsilon$.Every real sequence has a limit superior and a limit inferior in the extended real number $\mathbb{R}^*$.

Then U is called the *limit superior* of a_n and we write

$$U = \lim_{n\to\infty} \sup a_n$$

. The limit inferior of a_n is defined as follows:

$$\lim_{n\to\infty} inf\ a_n = -\lim_{n\to\infty} sup\ b_n,\ where\ b_n = -a_n\ for\ n = 1, 2, ..., n$$

.

2.1.1 Theorem

Let a_n be a sequence of real numbers. Then we have:

1. $\lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \inf b_n$.
2. The sequence converges if, and only if, $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both finite and equal, in which case $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.
3. The sequence diverges to $+\infty$ if, and only if, $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = +\infty$.
4. The sequence diverges to $-\infty$ if, and only if, $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -\infty$.

Note. A sequence for which $\limsup_{n\to\infty} a_n \neq \liminf_{n\to\infty} a_n$ is said to oscillate.

Proof:

1. From definition, denote $U = \limsup_{n\to\infty} a_n$ and $L = \liminf_{n\to\infty} a_n$. For every $\epsilon_1 > 0$, $b_n < -L + \epsilon_1$, where $b_n = -a_n$. And for every $\epsilon_2 > 0$,

$a_n < U + \epsilon_2$.

$$\begin{aligned} -a_n &< -L + \epsilon_1 \\ a_n &> L - \epsilon_1 \\ a_n &< U + \epsilon_2 \\ L - \epsilon_1 < &a_n < U + \epsilon_2 \\ L &< U + \epsilon_1 + \epsilon_2 \end{aligned}$$

Since ϵ_1 and ϵ_2 is arbitary positive, we have $L \leq U$, that is $\lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \inf b_n$.

2. From 1 we know that