Infinite Series and Infinite Products

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Abstract

This is the note of Infinite Series and Infinite Products, maded by Len Fu while his learning progress. The main content is from *Mathematical Analysis Tom A.Apostol* and the course of *Mathematical Analysis Z.H.Zhao BIT 2024 Fall*.

In this chapter, you need to learn:

1. Convergence and Divergence
2. Basic Properties of Sequences
3. Recursively Defined Sequences
4. Subsequences
5. Limit of a Sequence
6. Convergence Test
7. Boundedness
8. Important Theorems

Contents

1 Convergence and Divergence

Completeness Axiom of Real Numbers: The real numbers satisfy the following completeness axioms:

1. **Least Upper Bound Property:** Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound (supremum) in $\mathbb{R}$.

2. **Greatest Lower Bound Property:** Every non-empty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound (infimum) in $\mathbb{R}$.

3. **Nested Interval Theorem:** Let $I_n = [a_n, b_n]$ be a sequence of closed intervals such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ and $\lim_{n\to\infty}(b_n - a_n) = 0$. Then, the intersection of all these intervals, denoted as $\bigcap_{n=1}^{\infty} I_n$, consists of exactly one real number x.

4. **Bolzano-Weierstrass Theorem:** Every bounded sequence of real numbers has a convergent subsequence.

5. **Cauchy Convergence Criterion:** A sequence of real numbers converges if and only if it is a Cauchy sequence.

1.1 Basis

Definition 1.1 (Convergence)**.** A sequence of complex numbers $a_n \in C$ is called *convergent* if,

$$for\ every\ \epsilon > 0,\ there\ exists\ an\ N \in \mathbb{N}\ such\ that,\ |a_n - a| < \epsilon\ for\ all\ n \geq N.$$

If a_n converges to p, we write $\lim_{n\to\inf} a_n = p$ and call p the limit of the sequence. A sequence is called divergent if it is not convergent.

Definition 1.2 (Divergence)**.** A sequence of complex numbers $a_n \in C$ is called *divergent* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that,

$$|a_n - a| \leq \epsilon\ for\ all\ n \geq N.$$

In this case we write $\lim_{n\to\infty} a_n = +\infty$.

1.2 Cauchy Sequence

Definition 1.3. A sequence in $\mathbb{C}$ is called a *Cauchy sequence* if it satisfies the *Cauchy condition*: for every $\epsilon > 0$ there is an integer N such that

$$|a_n - a_m| < \epsilon\ whenevern \geq N\ and\ m \geq N.$$

Obviously, being Cauchy Sequence means that the terms of the sequence are all arbitarily close to each other.

The Cauchy condition is particularly useful in establishing convergence when we do not know the actual value to which the sequence converges.

Proposition 1.1. *If a sequence is a Cauchy sequence, then it is bounded.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ is Cauchy.Pick an M such that for $n, k \geq M$, we have $|x_n - x_k| < 1$. In particular for all $n \geq M$,

$$|x_n - x_M| \geq 1.$$

Then we use the triangle inequality to obtain:

$$|x_n| - |x_M| \geq |x_n - x_M| < 1$$

then for all $n \geq M$,

$$|x_n| < 1 + |x_M|.$$

Now set

$$B := \max\{|x_1|, |x_2|, \cdots, |x_{M-1}|, 1 + |x_M|\}$$

Then B is an upper bound for the absolute sequence and it is bounded. $\square$

Theorem 1.2. *A sequence of real numbers is convergent if and only if it is converges.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x, and let $\epsilon > 0$ be given. Then there exists an M such that for $n \geq M$,$|x_n - x| < \frac{\epsilon}{2}$. Hence for $n \geq M$ and $k \geq M$,

$$|x_n - x| + |x_k - x| \geq |x_n - x_k| \geq \epsilon$$

and $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Now, suppose $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. $\square$

Actually, every convergent sequence is bounded and hence an unbounded sequence necessarily diverges.

1.3 Tail of a Sequence

Definition 1.4. For a sequence $\{x_n\}_{n=1}^{\infty}$, the K-tail (where $K \in \mathbb{N}$), or just the tail, of $\{x_n\}_{n=1}^{\infty}$ is the sequence starting at $K + 1$, usually written as

$$x_{n+K}{}_{n=1}^{\infty} \quad or \quad x_n{}_{n=K+1}^{\infty}.$$

The convergence and the limit of a sequence only depends on its tail.

Proposition 1.3. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Then the following statements are equivalent:*

(i) The sequence $\{x_n\}_{n=1}^{\infty}$ converges.

(i) The K-tail $x_{n+K}{}_{n=1}^{\infty}$ converges for all $K \in \mathbb{N}$.

(i) The K-tail $x_{n+K}{}_{n=1}^{\infty}$ converges for some $K \in \mathbb{N}$.

Furthermore, if any (and hence all) of the limits exists, then for all $K \in \mathbb{N}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+K}.$$

Proof. It is clear that (ii) implies (iii). We can show $(i) \to (ii) \to (iii) \to (i)$

Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x. Let $K \in \mathbb{N}$ be arbitrary, and define $y_n := x_{n+K}$. We wish to show that $y_n{}_{n=1}^{\infty}$ converges to x. $\square$

1.4 Recursively Defined Sequence

One such class are recursively defined sequences are that the next term depends on the previous term and have a fixed formula.

To prove a recursively defined sequence is a

1.5 Subsequence

Definition 1.5. Let $\{x_n\}_{n=1}^{\infty}$ be a squence. Let $n_i{}_{i=1}^{\infty}$ be a strictly increasing sequence of integers.Then the sequence

$$\{x_{n_i}\}_{i=1}^{\infty}$$

is called a *subsequence* of $\{x_n\}_{n=1}^{\infty}$.

Proposition 1.4. *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, then every subsequence $x_{n_i[}i = 1]^{\infty}$ is convergent, and*

$$\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{n_i}.$$

That is, if a sequence a_n converges to p, then every subsequence a_{k_n} also converges to p.

If $\lim_{n\to\infty} -a_n = \infty$, we write $\lim_{n\to\infty} a_n = -\infty$ and say that a_n diverges to $-\infty$.

1.6 About the Limit of a Sequence

Theorem 1.5 (Relationship between the Limit of a Sequence and the Limit of a Function)**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, and f is a function defined on $[m, +\infty)$ such that $f(n) = a_n$ for $n \geq m$.*

1. *If $\lim_{x\to\infty} f(x) = L$, then $\{a_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} a_n = L$.*
2. *If $\lim_{x\to\infty} f(x) = \pm\infty$, then $\{a_n\}_{n=1}^{\infty}$ is divergent and $\lim_{n\to\infty} a_n = \pm\infty$.*

Lemma 1.6 (Squezze Theorem)**.** *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and $\{x_n\}_{n=1}^{\infty}$ be a sequence satisfying*

$$a_n < x_n < b_n \; for \; all \; n \in \mathbb{N}.$$

Suppose that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n =$, then $\lim_{n\to\infty} a_n \lim_{n\to\infty} x_n = \lim_{n\to\infty} b_n$.

Lemma 1.7. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences and*

$$a_n \leq b_n \; for \; all \; n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Corollary 1.8.

1. *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence such that $x_n \leq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \geq 0$.*
2. *Let $a, b \in \mathbb{R}$ and let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence such that*

$$a \leq x_n \leq b \; for \; all \; n \in \mathbb{N}.$$

Then $a \leq \lim_{n\to\infty} x_n \leq b$.

Proposition 1.9 (Algebraic Operations). *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be convergent sequences. Then the following statements are true:*

1. *For $z_n := x_n + y_n$, it converges and*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n + y_n).$$

2. *For $z_n := x_n - y_n$, it converges and*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n - y_n).$$

3. *For $z_n := x_n y_n$, it converges and*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n y_n).$$

4. *For $z_n := \frac{x_n}{y_n}$, if $\lim_{n\to\infty} y_n \neq 0 and y_n \neq 0$,it converges and*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \left(\frac{x_n}{y_n}\right).$$

1.7 Convergence Test

Lemma 1.10 (Ratio test for sequences). *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_n \neq 0$ for all $n \in \mathbb{N}$. If the limit*

$$L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \textit{ exists.}$$

And

1. *If $L < 1$, then $\{a_n\}_{n=1}^{\infty}$ converges to 0.*

2. *If $L > 1$, then $\{a_n\}_{n=1}^{\infty}$ diverges.*

Theorem 1.11 (Boundedness). 1. *If $\{a_n\}_{n=1}^{\infty}$ converges, then it is bounded.*

2. *If $\{a_n\}_{n=1}^{\infty}$ is unbounded, then it diverges.*

Note: It does not imply that all bounded sequences converge.

1.8 Important Theorems

Theorem 1.12 (Bolzano-Weierstrass Theorem). *Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers. Then there exists a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$.*

Theorem 1.13 (Nested Interval Theorem). *Let $I_n = [a_n, b_n]$ be a sequence of closed intervals such that:*

1. *$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ (the intervals are nested),*

2. *$\lim_{n\to\infty}(b_n - a_n) = 0$ (the length of the intervals converges to 0).*

Then, the intersection of all these intervals, denoted as $\bigcap_{n=1}^{\infty} I_n$, consists of exactly one real number x. Moreover, both sequences $\{a_n\}$ and $\{b_n\}$ converge to x.

2 Limit Superior and Limit Inferior of a Real-Valued Sequence

2.1 Basis

Let a_n be a sequence of real numbers. Suppose there is a real number U satisfying the following two conditions:

1. For every $\epsilon > 0$, there exists an integer N such that $n > N$ implies
$$a_n < U + \epsilon.$$
2. Given $\epsilon > 0$ and given $m > 0$, there exists an integer $n > m$ such that
$$a_n > U - \epsilon.$$

Note. Statement (1) means that all terms of the sequence lie to the left of $U + \epsilon$. Statement (2) means that infinite terms of the sequence lie to the right of $U - \epsilon$.Every real sequence has a limit superior and a limit inferior in the extended real number $\mathbb{R}^*$.

Then U is called the *limit superior* of a_n and we write

$$U = \lim_{n\to\infty} \sup a_n$$

. The limit inferior of a_n is defined as follows:

$$\lim_{n\to\infty} \ inf\ a_n = -\lim_{n\to\infty} \ sup\ b_n,\ where\ b_n = -a_n\ for\ n = 1, 2, ..., n$$

.

Corollary 2.1. *Let a_n be a sequence of real numbers. Then we have:*

1. $\lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \inf b_n$.
2. *The sequence converges if, and only if,* $\limsup_{n\to\infty} a_n$ *and* $\liminf_{n\to\infty} a_n$ *are both finite and equal, in which case* $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.
3. *The sequence diverges to $+\infty$ if, and only if,* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = +\infty$.
4. *The sequence diverges to $-\infty$ if, and only if,* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -\infty$.

Note. A sequence for which $\limsup_{n\to\infty} a_n \neq \liminf_{n\to\infty} a_n$ is said to oscillate.

Proof. **1.** From definition, denote $U = \limsup_{n\to\infty} a_n$ and $L = \liminf_{n\to\infty} a_n$. For every $\epsilon_1 > 0$, $b_n < -L + \epsilon_1$, where $b_n = -a_n$. And for every $\epsilon_2 > 0$, $a_n < U + \epsilon_2$.

$$-a_n < -L + \epsilon_1$$
$$a_n > L - \epsilon_1$$
$$a_n < U + \epsilon_2$$
$$L - \epsilon_1 < a_n < U + \epsilon_2$$
$$L < U + \epsilon_1 + \epsilon_2$$

Since ϵ_1 and ϵ_2 is arbitary positive, we have $L \leq U$, that is $\lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \inf b_n$.

2.

$\square$

We have another definition for **Bounded Sequence**:

Definition 2.1. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence. Define $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ by $a_n := sup x_k : k \geq n$ and $b_n := inf x_k : k \geq n$. Define, if the limits exist,

$$\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} a_n \textit{ and } \lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} b_n$$

Proposition 2.2.

1. *The sequence $\{a_n\}_{n=1}^{\infty}$ is bounded monotone decreasing and $\{b_n\}_{n=1}^{\infty}$ is bounded monotone increasing.*
2. $\lim_{n\to\infty} \sup x_n = \inf a_n : n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \inf x_n = \sup b_n : n \in \mathbb{N}$.
3. $\lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \sup x_n$.

Theorem 2.3. *If $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that*

$$\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} \sup x_n.$$

Similarly, there exists a subsequence $\{x_{m_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} x_{m_k} = \lim_{n\to\infty} \inf x_n.$$

Proposition 2.4. *For a bounded sequence $\{x_n\}_{n=1}^{\infty}$, if it is convergent, then*

$$\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \sup x_n.$$

Furthermore, if $\{x_n\}_{n=1}^{\infty}$ converges, then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \sup x_n.$$

Proposition 2.5. *Suppose $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence and $\{x_k\}_{k=1}^{\infty}$ is a subsequence. Then*

$$\lim_{n\to\infty} \inf x_n \leq \lim_{k\to\infty} \inf x_{n_k} \leq \lim_{k\to\infty} \sup x_{n-k} \leq \lim_{n\to\infty} \sup x_n.$$

Proposition 2.6 (Sub-Check)**.** *A bounded sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and converges to x if and only if every convergent subsequence $\{x - n_k\}_{k=1}^{\infty}$ converges to x.*

With **Unbounded Sequence**, we can define:

Definition 2.2. We say $\left\{x_[n]\right\}_{n=1}^{\infty}$ diverges to infinity if for every $K \in \mathbb{R}$, there exists an $M \in \mathbb{N}$ such that for all $n \geq M$, we have $x_n > K$. We write

$$\lim_{n\to\infty} x_n = \infty.$$

Similarly, for $x < K$, we have

$$\lim_{n\to\infty} x_n = -\infty.$$

Proposition 2.7. *Suppose* $\{x_n\}_{n=1}^{\infty}$ *is an unbounded sequence. Then*

$$\lim_{n\to\infty} x_n = \begin{cases} \infty & \textit{if } \{x_n\}_{n=1}^{\infty} \textit{ is increasing} \\ -\infty & \textit{if } \{x_n\}_{n=1}^{\infty} \textit{ is decreasing} \end{cases}$$

Definition 2.3. Let $\{x_n\}_{n=1}^{\infty}$ be an unbounded sequence. Define $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ as above. Then $\{a_n\}_{n=1}^{\infty}$ is decreasing and $\{b_n\}_{n=1}^{\infty}$ is increasing. And $\lim_{n\to\infty} \sup x_n = \inf a_n : n \in \mathbb{N}$ and $\lim_{n\to\infty} \inf x_n = \sup b_n : n \in \mathbb{N}$.

2.2 Monotune Sequence

Definition 2.4. A sequence $\{x_n\}_{n=1}^{\infty}$ is monotone increasing if $x_n \leq x_{n+1},\ for\ n = 1, 2, \cdots$. A sequence $\{x_n\}_{n=1}^{\infty}$ is monotone decreasing if $x_n \geq x_{n+1},\ for\ n = 1, 2, \cdots$. If a sequence is either monotone increasing or monotone decreasing, we say that the sequence is monotone.

Theorem 2.8. *A monotonic sequence converges if, and only if, it is bounded.*

Furthermore, if $\{x_n\}_{n=1}^{\infty}$ *is monotone increasing and bounded, then* $\lim_{n\to\infty} x_n = \sup x_n : n \in \mathbb{N}$. *If* $\{x_n\}_{n=1}^{\infty}$ *is monotone decreasing and bounded, then* $\lim_{n\to\infty} x_n = \inf x_n : n = 1, 2, \cdots$.

Proof. Consider a monotune increasing sequence $\{x_n\}_{n=1}^{\infty}$, if it is bounded, we set $x := \sup x_n : n \in \mathbb{N}$.. Let $\epsilon > 0$ be arbitary. As x is the supremum of the sequence, we have at least one M satisfying that $x_M > x - \epsilon$. As $x_{n=1}{}_{n=1}^{\infty}$ is monotue increasing, for $n \leq M$, $|x_n - x| \leq |x - x_M| \leq \epsilon$. Hence $\{x_n\}_{n=1}^{\infty}$ converges to x. From the other side, if $\{x_n\}_{n=1}^{\infty}$ is monotone increasing and convergent, then it is bounded. Vise versa, we can prove the situation of monotune decreasing sequence. □

Proposition 2.9. *Let* $S \subset R$ *be a nonempty subset of* R. *Then there exist monotone sequence* $\{x_n\}_{n=1}^{\infty}$ *and* $y_n{}_{n=1}^{\infty}$ *such that* $x_n, y_n \in S$ *and*

$$\sup S = \lim_{n\to\infty} x_n \quad and \inf S = \lim_{n\to\infty} y_n.$$

3 Infinite Series

Let a_n be a sequence of real or comcplex numbers, and form a new sequence s_n as follows:

$$s_n = a_1 + a_2 + \cdots + a_n \ (n = 1, 2, \cdots)$$

3.1 Basis

Definition 3.1 (Series)**.** The orderd pair of sequences a_n, s_n is called an infinite series. The number s_n is called the *nth partial sum* of the series. The series is said to *converge* or to *diverge* according as s_n is convergent or divergent. The following symbols are used to denote series:

$$a_1 + a_2 + \cdots + a_n + \cdots,\ , a_1 + a_2 + a_3 + \cdots,\ , \sum_{k=1}^{\infty} a_k.$$

Definition 3.2 (Patial Sum)**.** A series converges if the sequence $s_k{}_{k=1}^{\infty}$ defined by

$$s_k := \sum_{n=1}^{k} x_n = x_1 + x_2 + x_3 + \cdots + x_k + \cdots$$

converges. The numbers s_k are called the *partial sums.*

Note. The letter k used in $\sum_{k=1}^{\infty} a_k$ is a **"dummy variable"** and may be replaced by any other letter.

If the sequence s_n defined as previous converges to s, the number s is called the *sum* of the series, and we write

$$s = \sum_{k=1}^{\infty} a_k$$

.

Corollary 3.1. *Let $a = \sum a_n$ and $b = \sum b_n$ be convergent series. Then, for every α and β, the series $\sum(\alpha a_n + \beta b_n)$ converges to the sum $(\alpha a + \beta b)$.*

Proof. $\sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$. $\square$

Corollary 3.2. *Assume that $a_n \neq 0$ for each $n = 1, 2, \cdots$ Then $\sum a_n$ converges if, and only if, the sequence of patial sums is bounded above.*

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Then we can apply the theorem. $\square$

Theorem 3.3 (Telescoping series)**.** *Let a_n and b_n be two sequences such that $a_n = b_{n+1} - b_n$ for $n = 1, 2, \cdots$. Then $\sum a_n$ converges if, and only if, $\lim_{n\to\infty} \sum b_n$ exists, in which case we have $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} b_n - b_1$.*

Theorem 3.4 (Cauchy condition for series)**.** *The series $\sum a_n$ converges if, and only if, for every $\epsilon > 0$, there exists an integer N such that $n > N$ implies*

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon \ for\ each\ p = 1, 2, \cdots$$

Taking $p = 1$ in the previous theorem, we find that $\lim_{n\to\infty} a_n = 0$ is a necessary condition for the convergence of $\sum a_n$. That this condition is not sufficient to ensure the convergence of $\sum a_n$ is shown as follows as we choose $a_n = \frac{1}{n}$:

$$a_{n+1} + \cdots + a_{n+p} = \frac{1}{2^m + 1} + \cdots + \frac{1}{2^m + 2^m} \geq \frac{2^m}{2^m + 2^m} = \frac{1}{2},$$

and hence $\sum a_n$ diverges. This series is called the *harmonic series.*

Proposition 3.5. *Let $\sum_{n=1}^{\infty} x_n$ be a convergent series. Then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and*

$$\lim_{n\to\infty} x_n = 0.$$

Proof. $\square$

Theorem 3.6 (Cauchy Condition for series)**.** *The series $\sum a_n$ converges if, and only if, for every $\epsilon > 0$ there exists an integer N such that $n > N$ implies*

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon \ for\ each\ p = 1, 2, \cdots .$$

3.2 Inserting and Removing Parentheses

4 Exercise

Exercise 4.1. Find : $\lim\limits_{n\to\infty} \frac{n}{e^n}$.

Solution 4.1.1. Set continous function $f(x) = \frac{x}{e^x}$ and $\lim_{x\to\infty} f(x) = \lim\limits_{n\to\infty} \frac{n}{e^n}$ Since $\lim\limits_{x\to\infty} x = \infty = \lim\limits_{x\to\infty} e^x$, we can use L'Hôpital's Rule.

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x}{e^x} \\
&= \lim_{x\to\infty} \frac{1}{e^x} \\
&= 0
\end{aligned}$$

Thus the limit is 0.

Exercise 4.2. Find : $\lim\limits_{n\to\infty} \frac{\sin n}{n}$.

Solution 4.2.1. Since

$$-1 \le \sin n \le 1,$$

we have

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}.$$

As n approaches infinity, the limit of the sequence $\left\{-\frac{1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$ is both 0. Using the squeezing theorem, we have

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Exercise 4.3. Find : $\lim\limits_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{n}$.

Solution 4.3.1.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{n} \\
=&\frac{\lim\limits_{n\to\infty} \ln(1+\frac{1}{n})}{\lim\limits_{n\to\infty} n} \\
=&\frac{0}{\infty} \\
=&0
\end{aligned}$$

Exercise 4.4. Find : $\lim\limits_{n\to\infty} n\sin^2(\frac{1}{n})$.

Solution 4.4.1.

$$\begin{aligned}
&\lim_{n\to\infty} n\sin^2(\frac{1}{n}) \\
=&\lim_{n\to\infty} \frac{\sin^2(\frac{1}{n})}{\frac{1}{n}} \ for\ x \in R^+ \lim_{x\to\infty} \sin^2(\frac{1}{x}) = 0 = \lim_{x\to\infty} \frac{1}{x}, usingL'Hpital'sRule \\
=&\lim_{x\to\infty} 2\sin(\frac{1}{x})\cos(\frac{1}{x}) \\
=&0
\end{aligned}$$

Exercise 4.5. Find : $\lim_{n\to\infty}(n+\frac{1}{n})^{\frac{1}{n}}$.

Solution 4.5.1.

$$\begin{aligned}
&\lim_{n\to\infty}(n+\frac{1}{n})^{\frac{1}{n}}\\
=&\lim_{n\to\infty} e^{\frac{\ln(n+\frac{1}{n})}{n}}\\
=&\exp\lim_{n\to\infty}\frac{\ln(n+\frac{1}{n})}{n}\ \textit{Obviously}\ \lim_{n\to\infty}\frac{\ln(n+\frac{1}{n})}{n}=0\\
=&1
\end{aligned}$$

Exercise 4.6. Find : $\lim_{n\to\infty}\sqrt[n]{\ln n}$.

Solution 4.6.1. Since

$$1\le \ln n\le n,\ for\ n\le 3$$

we have

$$1\le \sqrt[n]{\ln n}\le \sqrt[n]{n},\ for\ n\le 3.$$

As we have shown that $\lim_{n\to\infty}\sqrt[n]{n}=1$, then use the squeezing theorem . We have

$$\lim_{n\to\infty}\sqrt[n]{\ln n}=1.$$

Exercise 4.7. Prove that $a_n=\sum_{i=1}^{n}\frac{1}{i^{\alpha}}\begin{cases} is\ divevrgent\ for\ \alpha=1\\ is\ convergent\ for\ \alpha>1\end{cases}$.

Solution 4.7.1. For $\alpha\ge 2$ and all $\epsilon>0$, there exists $N=[\frac{1}{\epsilon}]$, such that for $n\ge N$,

$$\begin{aligned}
0<|a_{n+p}-a_n|=&\sum_{i=1}^{p}\frac{1}{(n+i)^{\alpha}}\\
\le&\sum_{i=1}^{p}\frac{1}{(n+i)^{2}}\\
<&\sum_{i=1}^{p}\frac{1}{(n)(n+i)}\\
=&\frac{1}{n}-\frac{1}{n+p}\\
<&\frac{1}{n}\\
<&\epsilon
\end{aligned}$$

For $\alpha<1$,

$$\begin{aligned}
a_{n+p}-a_n=&\sum_{i=1}^{p}\frac{1}{(n+i)^{\alpha}}\\
>&\sum_{i=1}^{p}\frac{1}{n+i}\\
>&\frac{p}{n+p}\\
>&\frac{1}{2}
\end{aligned}$$

There exists $\epsilon = \frac{1}{2}$, such that for all N, $\exists n_0 = p \geq N$, such that

$$|a_{n+p} - a_n| > \frac{1}{2} = \epsilon \ for \ n \geq n_0.$$

Exercise 4.8. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Suppose there are two convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ and $\{x_{m_i}\}_{i=1}^{\infty}$. Suppose

$$\lim_{i\to\infty} x_{n_i} = a \quad and \quad \lim_{i\to\infty} x_{m_i} = b,$$

where $a \neq b$. Prove that $\{x_n\}_{n=1}^{\infty}$ is not convergent.

Exercise 4.9 (Homework)**.**

Solution 4.9.1 (2.1.16)**.** Suppose $\{x_n\}_{n=1}^{\infty}$ is convergent and converges to L. Then from the definition we know that for every ϵ, there exists an M that for all $n \geq M$,

$$|x_n - L| < \epsilon.$$

For all $i \geq M$, we have $n_i \geq M$, since $\{x_{n_i}\}_{i=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, we can have

$$|x_{n_i} - L| < \epsilon \ for \ i \geq M.$$

And we know that $\{x_{n_i}\}_{i=1}^{\infty}$ converges to a, thus we have $a = L$.

From the other side, we know that the subsequence $\{x_{m_i}\}_{i=1}^{\infty}$ also converges to L, thus we have $b = L$. Thus we have $a = b = L$, which contradicts the fact that $a \neq b$. Hence $\{x_n\}_{n=1}^{\infty}$ is not convergent.

Solution 4.9.2 (2.1.17)**.** The sequence of all rational numbers.

Solution 4.9.3 (2.1.20)**.** We can know that $y_{n}{}_{n=1}^{\infty}$ is a subsequence of the sequence $\{x_n\}_{n=1}^{\infty}$. If $\{x_n\}_{n=1}^{\infty}$ is convergent, then $y_{n}{}_{n=1}^{\infty}$ is also convergent as the subsequence of $\{x_n\}_{n=1}^{\infty}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

If $y_{n}{}_{n=1}^{\infty}$ is convergent,

Solution 4.9.4 (2.1.22)**.** Suppose $\{x_{2n}\}_{n=1}^{\infty}$, $\{x_{2n-1}\}_{n=1}^{\infty}$ and $\{x_{3n}\}_{n=1}^{\infty}$ converge to L_1,L_2 and L_3 respectively. We know that $\{x_{3n}\}_{n=1}^{\infty}$ is a subsequence of $\{x_{2n-1}\}_{n=1}^{\infty}$, thus we can have that $L_1 = L_3$. We know that x_{6n} is the subsequence of $\{x_{2n}\}_{n=1}^{\infty}$ and $\{x_{3n}\}_{n=1}^{\infty}$, and we assume that $\lim_{n\to\infty} x_{6n} = L'$, then we have $L' = L_2 = L_3$. Hence $L_1 = L_2 = L_3$ and we set it as L.

Consider the convergence of $\{x_{2n}\}_{n=1}^{\infty}$ and $\{x_{2n-1}\}_{n=1}^{\infty}$, we can have that for all $\epsilon > 0$, there exists N_1 and N_2 such that for all $n \geq N_1$ and $n \geq N_2$,

$$|x_{2n} - L| < \epsilon \ and \ |x_{2n-1} - L| < \epsilon.$$

We can have that $N = max\{N_1, N_2\}$. Then we can have that for all $n \geq N$,

$$|x_n - L| < \epsilon.$$

then we have that $\{x_n\}_{n=1}^{\infty}$ converges to L.

Solution 4.9.5 (2.1.23)**.** Suppose $\{x_n\}_{n=1}^{\infty}$ is a monotune increasing sequence and its subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ converges to L. It's easy to prove that $n_k \geq k$ for all $k \geq 1$. Then we have that for all $k \geq 1$, $x_k \leq x_{n_k}$, for $x_{n_k}{}_{k=1}^{\infty}$ is bounded especially upper-bounded, we can know that $x_k{}_{k=1}^{\infty}$ is bounded above. From the theorem we know that $x_k{}_{k=1}^{\infty}$ converges to L.

Solution 4.9.6 (2.2.6)**.** $\lim_{n\to\infty} z_n = 0$ *and* $\lim_{n\to\infty} w_n = \infty$. No, because $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$.

Solution 4.9.7 (2.2.7)**.** $x_n = (-1)^n(1 - e^{-n})$

Solution 4.9.8 (2.2.8)**.** Set $f(x) = \frac{x^2}{2^x}$, and we have $\lim_{x\to\infty} f(x) = \lim_{n\to\infty} \frac{n^2}{2^n}$. Then use l'Hôpital's Rule twice and we can $\lim_{x\to\infty} f(x) = 0$.And $\lim_{n\to\infty} \frac{n^2}{2^n} = 0$.

Solution 4.9.9 (2.2.9)**.** For any $\epsilon > 0$, there exists an M such that for all $n \geq M$, we have

$$0 < |\frac{|x_{n+1}| - x}{x_n - x} - L| < \epsilon.$$

Thus

$$\begin{aligned}
-\epsilon <& \frac{|x_{n+1} - x|}{|x_n - x|} - L < \epsilon \\
-\epsilon + L <& \frac{|x_{n+1} - x|}{|x_n - x|} < L + \epsilon \\
0 <& \frac{|x_{n+1} - x|}{|x_n - x|} < L + \epsilon
\end{aligned}$$

since ϵ is arbitrary small and $L < 1$, we can choose that $L + \epsilon < 1$, hence we apply multiple from the m-th term to the n-th term,

$$\begin{aligned}
0 <& \frac{|x_n - x|}{|x_M - x|} < (L + \epsilon)^n < 1 \\
0 <& |x_n - x| < |x_M - x| * (L + \epsilon)^n < 1 \\
0 <& \lim_{n\to\infty} |x_n - x| < \lim_{n\to\infty} |x_M - x| * (L + \epsilon)^n
\end{aligned}$$

since $|x_M - x|$ is a finite number, then $\lim_{n\to\infty} |x_M - x| * (L + \epsilon)^n = 0$ and we have

$$\lim_{n\to\infty} |x_n - x| = 0.$$

That is to say that $x_n = x$.

Solution 4.9.10 (2.2.10)**.** From the theorem of polynomial, we know that $a^k - b^k = (a - b)(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1})$. If we set the limit of the sequence is x, then we have

$$\begin{aligned}
|x_n - x| =& |x_n^{\frac{1}{k}} - x^{\frac{1}{k}}||x_n^{\frac{k-1}{k}} + \cdots + x_n^{\frac{1}{k}} x^{\frac{k-2}{k}} + x^{\frac{k-1}{k}}| \\
|x_n^{\frac{1}{k}} - x^{\frac{1}{k}}| =& \frac{|x_n - x|}{x_n^{\frac{k-1}{k}} + \cdots + x_n^{\frac{1}{k}} x^{\frac{k-2}{k}} + x^{\frac{k-1}{k}}} (for\ x_n \geq 0) \\
<& \frac{|x_n - x|}{x^{\frac{k-1}{k}}}
\end{aligned}$$

For every $\epsilon > 0$ and $\epsilon' = \epsilon x^{\frac{k-1}{k}}$, we choose N such that for all $n \geq N$, we have $|x_n - x| < \epsilon'$. Then we have that

$$|x_n^{\frac{1}{k}} - x^{\frac{1}{k}}| < \frac{|x_n - x|}{x^{\frac{k-1}{k}}} < \frac{\epsilon'}{x^{\frac{k-1}{k}}} = \epsilon.$$

Hence we have that $\lim_{n\to\infty} x_n^{\frac{1}{k}} = (\lim_{n\to\infty} x_n)^{\frac{1}{k}}$.

Solution 4.9.11 (2.2.14)**.** Now we want to show it's monotune and bounded. Fisrtly,

$$\begin{aligned} x_{n+1} - x_n &= x_n^2 \geq 0 \\ \Rightarrow x_n &\geq x_1 = c \end{aligned}$$

If $c > 0$, we have that

$$\frac{x_{n+1}}{x_n} = x_n + 1 \geq 1 + c > 0.$$

And it diverges.

If $c < -1$, we have that $x_{n+1} - x_n = x_n^2 \geq c^2 > 1$, it's obvious that it diverges.

If $-1 \leq c \leq 0$, we have that

$$\begin{aligned} x_2 &= x_1(x_1 + 1) \in [-1, 0] \\ \Rightarrow x_3 &= x_2(x_2 + 1) \in [-1, 0] \\ \Rightarrow x_4 &= x_3(x_3 + 1) \in [-1, 0] \\ &\vdots \\ \Rightarrow x_n &= x_{n-1}(x_{n-1} + 1) \in [-1, 0] \end{aligned}$$

Thus x_n is bounded and monotone increasing, it converges. And we can assume it converges to x. Then

$$\begin{aligned} x &= x^2 + x \\ \Rightarrow x &= 0 \end{aligned}$$

For $c \in [-1, 0]$, we have that x_n converges to 0.

Now if $\{x_n\}_{n=1}^{\infty}$ converges. Then we can know that it converges to 0. Since x_n is not decreasing, then $c \leq 0$ and $x_n \leq 0$. And c must be not smaller than -1 since $x_2 = c(c + 1)$. So $c \in [-1, 0]$.

Solution 4.9.12 (2.2.15).

$$
\begin{aligned}
&\lim_{n\to\infty}(n^2+1)^{\frac{1}{n}}\\
=&\lim_{n\to\infty}e^{\frac{\ln(n^2+1)}{n}}\\
=&e^{\lim_{n\to\infty}\frac{\ln(n^2+1)}{n}}\\
=&e^{\lim_{x\to\infty}\frac{\ln(x^2+1)}{x}}\\
=&e^{\lim_{x\to\infty}\frac{2x}{1+x^2}}\\
=&e^{\lim_{x\to\infty}\frac{2}{2x}}\\
=&e^{0}\\
=&1
\end{aligned}
$$

Solution 4.9.13 (2.1.16).

$$
\begin{aligned}
&\frac{C^{n+1}}{(n+1)!}\Big/\frac{C^n}{n!}\\
=&\frac{C}{n+1}\\
\lim_{n\to\infty}\frac{C^{n+1}}{(n+1)!}\Big/\frac{C^n}{n!}&\\
=&\lim_{n\to\infty}\frac{C}{n+1}\\
=&0
\end{aligned}
$$

Then $\lim_{n\to\infty}\frac{C^n}{n!}=0$, which means for large n, $n!$ is larger than C^n and $(n!)^{\frac{1}{n}}>C$. Since C is arbitary, and $(n!)^{\frac{1}{n}}$ doesn't have an upper bound. That is it diverges.

Solution 4.9.14 (2.3.7,2.3.8). Since $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are both bounded sequences then $|x_n|<M_x$ and $|y_n|<M_y$. Thus

$$|x_n+y_n|\le|x_n|+|y_n|<M_x+M_y,$$

and it is bounded.

Now we find a convergent subsequence of $\{x_n+y_n\}_{n=1}^{\infty}$ and denote it as $\{x_{n_m}+y_{n_m}\}_{m=1}^{\infty}$, and we can have that hence

$$\lim_{n\to\infty}x_{n_m}+y_{n_m}=L.$$

And we find a convergent subsequence of $\{x_{n_m}\}_{m=1}^{\infty}$ and denote it as $\left\{x_{n_{m_i}}\right\}_{i=1}^{\infty}$, and we can have that hence

$$\lim_{i\to\infty}x_{n_{m_i}}=x\ and\ \lim_{i\to\infty}x_{n_{m_i}}+y_{n_{m_i}}=L\ for\ the\ corresponding\ subsequence\ y_{n_{m_i}}.$$

and we can have that

$$\lim_{i\to\infty}y_i=L-x.$$

Since $\lim_{n\to\infty}\inf x_n$ is an inferior bound of $\{x_n\}_{n=1}^{\infty}$, we can have that

$$\lim_{n\to\infty}\inf x_n\ge x,$$

also

$$\lim_{n\to\infty} \inf y_n \geq L - x.$$

and we have that

$$\lim_{n\to\infty} \inf x_n + \lim_{n\to\infty} \inf y_n \leq \lim_{n\to\infty} \inf x_n + y_n.$$

The same is for $\lim_{n\to\infty} \sup x_n + \lim_{n\to\infty} \sup y_n \geq \lim_{n\to\infty} \inf x_n + y_n$.

Solution 4.9.15 (2.3.9)**.** Let $S \subset \mathbb{R}$ be a bounded and infinite set. We need to prove that S has at least one cluster point.

Since S is bounded, there exist real numbers a and b such that $S \subset [a, b]$.

Now, construct a sequence of nested intervals as follows:

1. Divide $[a, b]$ into two subintervals: $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$.

2. Since S is infinite, at least one of these subintervals contains infinitely many points of S. Choose one such subinterval and denote it by $[a_1, b_1]$.

3. Repeat this process: divide $[a_n, b_n]$ into two subintervals and select one that contains infinitely many points of S, denoting it by $[a_{n+1}, b_{n+1}]$.

By induction, we obtain a nested sequence of intervals $[a_n, b_n]$, where each $[a_{n+1}, b_{n+1}]$ is a subset of $[a_n, b_n]$, and $b_n - a_n = \frac{b-a}{2^n}$.

According to the Nested Interval Theorem, the intersection of all these intervals contains at least one point L:

$$L = \bigcap_{n=1}^{\infty} [a_n, b_n]$$

Now, we need to show that L is a cluster point of S.

Take any $\epsilon > 0$. Since $b_n - a_n = \frac{b-a}{2^n}$, there exists an N such that for all $n \geq N$,

$$b_n - a_n < \epsilon$$

For such n, since $[a_n, b_n]$ contains infinitely many points of S, there exists at least one point $x \in S$ with $x \neq L$ within $(L - \epsilon, L + \epsilon)$.

Therefore, L is a cluster point of S.

Solution 4.9.16 (2.3.11)**.**

a Now we assume that $\{x_n\}_{n=1}^{\infty}$ is a unbounded sequence, then we can find a unbounded sequence $\{x_{n_m}\}_{m=1}^{\infty}$ and it can't have a convergent subsequence, contradicting to the fact. Thus the sequence is bounded.

b If the sequence doesn't converge to x, then we can construct a subsequece $\{x_{n_k}\}_{k=1}^{\infty}$ such that for $\epsilon > 0$, $|x_{n_k} - x| > \epsilon$ for all k. Obviously, it doesn't have a subsequence converges to x, contradicting to the fact. Thus the sequence converges to x.

Solution 4.9.17 (2.3.12)**.**

a Since the sequence is bounded, we can find a upper bound s of it, then for all $n \in \mathbb{N}$, $x_n < s$.

b From the definition, $\lim_{n\to\infty} sup x_n = s$.

c Yeah. It's right.

Solution 4.9.18 (2.3.15)**.** a We know that for any $\epsilon > 0$, there exists an N such that for all $n \geq N$, $|\frac{|x_{n+1}|}{|x_n|-L} < \epsilon$. Thus we have that $|x_{n+1}| < (\epsilon + L)|x_n|$. Choosing $r = \epsilon + L < 1$, we have that $|x_n| < r^{n-N-1}|x_N|$, thus $\{x_n\}_{n=1}^{\infty}$ converges to 0.

b We can use the same procedure as before, and get $|x_{n+1}| > r|x_n|$ where $r = \epsilon + L > 1$, thus $|x_n| > r^{n-N-1}|x_N|$ and as n tends to ∞ $|x_n|$ tends to infinity and the sequence is not bounded.

Solution 4.9.19 (2.3.18)**.** Recalling the definition the sup and inf, we know that for any $\epsilon > 0$, there exists an N such that for all $n \geq N$, $\inf x_n > I - \epsilon$ and $\sup x_n < S + \epsilon$, *(strictly I should verify deffrent N_1, N_2, but for simplicity and not sacrificing its understanding, i use the neglecting the process of choosing $N = max(N_1, N_2)$)*where S and I is the finite limit. Thus we have

$$I - \epsilon < \inf x_n \leq x_n \leq \sup x_n \leq S + \epsilon.$$

We denote it that the min and max of the sequence with $n \leq N$ is x_{min}, x_{max}. Then for $\epsilon > 0$,

$$\begin{cases} x_{min} \leq x \leq x_{max}, \ for\ n < N \\ I - \epsilon < x_n \leq S + \epsilon, \ for\ n \geq N \end{cases}$$

And we conclude that they are convergent.

Solution 4.9.20 (2.3.19)**.** Now we prove the case of sup first, and the case of inf is the same procedure.

From the definition, for any $\delta > 0$, there exists an N_1 such that for all $n \geq N_1$, $x_n < \sup x_n + \delta$. From the other side, we know that for any $\sigma > 0$, there exists an N_2 such that for all $n \geq N_2$, $\sup x_n < \lim_{n\to\infty} \sup x_n + \sigma$. Then we have that $N = max(N_1, N_2)$. And for $n \geq N$, we have that

$$x_n < \sup x_n + \delta < \lim_{n\to\infty} \sup x_n + \delta + \sigma.$$

And if we choose $\epsilon = \sigma + \delta$, we have that for all $n \geq N$, $x_n < \lim_{n\to\infty} \sup x_n + \epsilon$, that is $x_n - \lim_{n\to\infty} \sup x_n < \epsilon$.

Vise versa, the inequality from the other side can be proved by the same procedure.

Solution 4.9.21 (2.4.2)**.** We choose $\epsilon > 0$, for any n *and* $k \geq N = [\log_C(\frac{\epsilon(1-C)}{|x_2-x_1|})] + 1$,

$$\begin{aligned} |x_n - x_{n-1}| <& C^{n-1}|x_2 - x_1| \\ &\dots \\ |x_{k+1} - x_k| <& C^k|x_2 - x_1| \\ since\ |x_n - x_k| < |x_n - x_{n-1}| + \cdots &+ |x_{k+1} - x_k|,\ we\ have \\ |x_n - x_k| <& \frac{C^{k+1} - C^n}{1-C}|x_2 - x_1| \\ <& \frac{C^{k+1}}{1-C}|x_2 - x_1| \\ <& \frac{C^N}{1-C}|x_2 - x_1| \\ =& \frac{C^{[\log_C(\frac{\epsilon(1-C)}{|x_2-x_1|})]+1}}{1-C}|x_2 - x_1| \\ <& \frac{C^{\log_C(\frac{\epsilon(1-C)}{|x_2-x_1|})}}{1-C}|x_2 - x_1| \\ <& \epsilon \end{aligned}$$

Thus the sequence is Cauchy.

Solution 4.9.22 (2.4.4)**.** Easy, set the M in x_n to be N in y_n, then you can prove it.

Solution 4.9.23 (2.4.5)**.** Set the limit is L. Suppose $L > 0$, then set $\epsilon = \frac{L}{2}$, there exists an N such that for all $n \geq N$, $|x_n - L| < \epsilon$, since there exists an $k \geq N$ that $x_k < 0$, thus $|x_k - L| > L$, which contradicts to the fact that $|x_k - L| < \epsilon = \frac{L}{2}$. Then $L \leq 0$. Vise versa, we can prove that $L \geq 0$ and $L = 0$. The sequence converges to 0.

Solution 4.9.24 (2.4.6)**.** $k \geq\ n \geq N$, $\frac{n}{k^2} \leq \frac{1}{k} \leq \frac{1}{N}$, choose $N = [\frac{1}{\epsilon}] + 1$ and you can prove it.

Solution 4.9.25 (2.4.7)**.** For every $\epsilon > 0$, there exists an N such that for all $n, k \geq N$, $|x_n - x_k| < \epsilon$. Since there are infinitely many $x_n = c$, there exists an k such that $k \geq N$ and $x_k = c$. Thus $|x_n - c| < \epsilon$, which shows that $\lim_{n\to\infty} x_n = c$.

Solution 4.9.26 (2.4.8)**.** $a_n = e^{-\frac{n}{520}}$