Linear Mapping and Linear Transformation

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Abstract

This is the note of Linear Mapping and Linear Transformation, maded by Len Fu while his learning progress. The main content is from *Linear Algebra Done Right* and *Linear Algebra Allenby*. It's also the notes from the classes of BIT.

Contents

1 Linear Mapping

To prove a mapping is not linear mapping, you just need to find a counterexample.

1.1 Mapping

Definition 1.1 (Mapping)**.** Suppose X and Y are two non-empty sets. A mapping σ from X to Y, denoted as $\sigma : X \to Y$, is a rule that assigns to each element $x \in X$ exactly one element y in the set Y. The assignment $y = \sigma(x)$ is called the image of a under the mapping σ, and x is called the preimage.

Properties 1.1 (Domain)**.** Every element x in the set X must be mapped to some element in Y.

Properties 1.2 (Uniqueness)**.** For each x in X, there is a unique y in Y such that $\sigma(x) = y$.

The set X is called the domain of the mapping σ, and the set Y is called the codomain. The image of a set, which is the set of all elements in Y that are mapped to by elements in X, denoted by as $Im(\sigma) or \sigma(X)$. , or

$$\sigma(X) = y \in Y | \exists x \in X \ such \ that \ \sigma(x) = y.$$

Definition 1.2 (Injective Mapping)**.** A mapping σ is called an *injective mapping* or an *onto mapping* if for each y in Y, there is a unique x in X such that $\sigma(x) = y$. Formally, for all $x_1, x_2 \in X$, if $\sigma(x_1) = \sigma(x_2)$, then $x_1 = x_2$.

Definition 1.3 (Surjective Mapping)**.** A mapping σ is called a *surjective mapping* or a *onto mapping* if for every y in Y, there exists an x in X such that $\sigma(x) = y$.

Definition 1.4 (Bijective Mapping)**.** A mapping σ is called a *bijective mapping* or a *onto mapping* if it is both injective and surjective.

Definition 1.5 (Product of mappings)**.** Set σ as a mapping from X to Y, and τ as a mapping from Y to Z, then we can define a new mapping $\tau \circ \sigma$ from X to Z by

$$\tau \circ \sigma(x) = \tau(\sigma(x)), \ for \ all \ x \in X.$$

1.2 Linear Mapping

Definition 1.6 (Linear Mapping)**.** Set the V_1 and V_2 as vector spaces on the field F. If a mapping τ from V_1 to V_2 keeps the adding property and the scalar multiplication property, then we say that τ is a *linear mapping* or a *linear transformation.*

$$\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta), \ \sigma(k\alpha) = k\sigma(\alpha), \ for \ any \ \alpha \ and \ \beta \in V_1, k \in \ F.$$

The necessary and sufficient condition for a Linear Mapping is

$$\sigma(k\alpha + l\beta) = k\sigma(\alpha) + l\sigma(\beta).$$

1.3 Identity Mapping

Set V as a vector space on the field F, a mapping

$$\epsilon : V \to V$$

is defined as $\epsilon(\alpha) = \alpha,\ for\ all\ \alpha \in V$.

1.4 Zero Mapping

Set the V_1 and V_2 as vector spaces on the field F. A mapping

$$\tau : V_1 \to V_2$$

is defined as $\tau(0) = 0,\ for\ all\ \alpha \in V_1$.

1.5 Properties

If τ is a linear mapping, then it has follow properties:

Properties 1.3. $\tau(\theta) = \theta,\ \tau(-\alpha) = -\tau(\alpha)$

Properties 1.4. Linear Mappings keep the linear combination and linear coefficents unchanged.

Properties 1.5. Linear Mappings transform the linear relative vector group into another linear relative groups.

1.6 Matrix Reprentation of the Linear Mapping

Definition 1.7. Set σ as a linear mapping from V_1 to V_2, choose a basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ in the V_1 and choose a basis $\beta_1, \beta_2, \cdots, \beta_m$ in the V_2. If the image of the basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ is

$$\begin{cases} \sigma(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \cdots + a_{m1}\beta_m \\ \sigma(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \cdots + a_{m2}\beta_m \\ \cdots \\ \sigma(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \cdots + a_{mn}\beta_m \end{cases}$$

and can be expressed as

$$[\sigma(\alpha_1), \sigma(\alpha_2), \cdots, \sigma(\alpha_n)] = [\beta_1, \beta_2, \cdots, \beta_m]A.$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called the linear mapping matrix of σ under the basis α and β.

Theorem 1.1. *If σ is a linear mapping from V_1 to V_2, take a basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ in the V_1, and take a basis $\beta_1, \beta_2, \cdots, \beta_m$ in the V_2, then the linear mapping matrix of σ under the basis α and β is A.*

For every $\alpha \in V$, if the coordinate of α under the basis α is $(x_1, x_2, \cdots, x_n)^T$, then the coordinate of $\sigma(\alpha)$ under the basis β is $(y_1, y_2, \cdots, y_n)^T$. Then

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Proof. Since

$$[\sigma(\alpha_1), \sigma(\alpha_2), \cdots, \sigma(\alpha_n)] = [\beta_1, \beta_2, \cdots, \beta_m]A,$$

$$\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \sigma(\alpha) = [\sigma(\beta_1), \sigma(\beta_2), \cdots, \sigma(\beta_n)] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

and

$$\sigma\alpha = \sigma([\alpha_1, \alpha_2, \cdots, \alpha_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix})$$

$$= [\sigma(\alpha_1, \alpha_2, \cdots, \alpha_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad = [\beta_1, \beta_2, \cdots, \beta_n]A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then we hava

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

□

Theorem 1.2. *Set σ as a linear transformation from V_1 to V_2, $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\alpha_1^{'}, \alpha_2^{'}, \cdots, \alpha_n^{'}$ which*

$$[\alpha_1^{'}, \alpha_2^{'}, \cdots, \alpha_n^{'}] = [\alpha_1, \alpha_2, \cdots, \alpha_n]P,$$

are two basis of V_1, and $[\beta_1^{'}, \beta_2^{'}, \cdots, \beta_n^{'}] = [\beta_1, \beta_2, \cdots, \beta_n]Q$.

If the linear mapping σ under the basis α and β is A, under the basis $\alpha^{'}$ and $\beta^{'}$ is B, then

$$B = Q^{-1}AP.$$

1.7 Isomorphism

Definition 1.8. Set V_1 and V_2 are linear spaces in the field F, if the mapping $\sigma : V_1 \to V_2$ has the follow properties:

1. σ is a bijection.
2. $\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta),\ for\ all\ \alpha, \beta \in V_1$
3. $\sigma(k\alpha) = k\sigma(\alpha),\ for\ all\ k \in F,\ all\ \alpha \in V_1$

Then we say that V_1 and V_2 are isomorphic, and σ is a Isomorphism from V_1 to V_2, we denote $V_1 \cong V_2$.

2 Linear Transformation

2.1 Linear Transformation

Definition 2.1. Set σ as a transformation in linear space V on the field F. If for every $\alpha, \beta \in V,\ k \in F$, that

$$\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta), \sigma(k\alpha) = k\sigma(\alpha)$$

then we say σ is a linear transformation on V.

Example 2.1.1 (Scalar Multiplication)**.** Set $k \in F$, define a transformation in V,

$$\sigma(\alpha) = k\alpha, \alpha \in V$$

It's easy to prove σ is a linear transformation in V, we call it a *scalar multiplication transformation*.

Actualy, if $k = 0$, this is a zero transformation, and if $k = 1$, it is aidentity transformation.

Definition 2.2 (Sum, Multiplication and Scalar multiplication of Transformation)**.** Set σ and τ as two transformations in linear space V, $k \in F$, then the sum of σ *and* τ, is defined as

$$(\sigma + \tau)(\alpha) = \sigma(\alpha) + \sigma(\beta), \forall \alpha \in V$$

the multiplication of σ and τ, is defined as

$$(\sigma \circ \tau)(\alpha) = \sigma(\tau(\alpha)), \forall \alpha \in V$$

the scalar multiplication of σ and k, is defined as

$$(\sigma \circ k)(\alpha) = k\sigma(\alpha), \forall \alpha \in V.$$

Definition 2.3 (Invertible Transformation)**.** Set σ as a transformation in linear space V, if there exists a transformation τ, satisfying that

$$\sigma\tau = \tau\sigma = \epsilon$$

then we say σ is an invertible transformation, and the τ is the inverse transformation of σ, denoted as $\tau = \sigma^{-1}$.

Note. The transformation is invertible if and only if it is bijection and its inverse transformation is unique.

Theorem 2.1. *Invertible transformation's inverse transformation is also a linear transformation.*

Proof. Set σ as a invertible linear transformation, σ^{-1} is its inverse transformation. Choose $\alpha_1, \alpha_2 \in V, k \in F$, and let

$$\sigma^{-1}(\alpha_1) = \beta_1, \sigma^{-1}(\alpha_2) = \beta_2,$$

then

$$\begin{aligned} &\sigma^{-1}(\alpha_1 + \alpha_2) \\ &\qquad = \sigma^{-1}(\sigma(\beta_1 + \beta_2)) \qquad = \beta_1 + \beta_2 \end{aligned}$$

and

$$\sigma^{-1}(k\alpha_1) = k\beta_1 = k\sigma(\alpha_1).$$

So σ^{-1} is also linear. □

Definition 2.4. Set σ as a linear transformation from V_n to V_n, choose a basis $\alpha_1, \alpha_2, \cdots, \alpha_n$, if the image of this basis under the transformation is

$$\begin{cases} \sigma(\alpha_1) = a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n \\ \sigma(\alpha_2) = a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n \\ \vdots \\ \sigma(\alpha_n) = a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n \end{cases}$$

and can be expressed as

$$[\sigma(\alpha_1), \sigma(\alpha_2), \cdots, \sigma(\alpha_n)] = [\alpha_1, \alpha_2, \cdots, \alpha_n]A$$

where

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}$$

is called the linear transformation matrix of σ under the basis α.

Note. After the basis is setted, the lineaer transformation and the matrix reprentation is unique.

If σ is a linear transformation in the vector space F^n, and

$$\sigma(\alpha) = A\alpha, \ \alpha \in F^n$$

then A is the matrix reprentation of σ in F^n under the basis $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$.

Theorem 2.2. *Set $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a basis of the linear vector V_n on the field F, and two martrix reprentations of linear transformation σ and τ is A and B, then*

1. *$\sigma + \tau$ under the basis is $A + B$.*

2. *$\sigma\tau$ under the basis is AB.*

3. *$\sigma(k\alpha)$ under the basis is kA.*

4. *σ is invertible if and only if A is invertible, and σ^{-1} under the basis is A^{-1}.*

Theorem 2.3 (Matrix Transformation under the basis transformation)**.** *In the vector space V_n, we choose two basis*

$$\alpha_1, \alpha_2, \cdots, \alpha_n; \beta_1, \beta_2, \cdots, \beta_n,$$

and the transition matrix from the basis α to the basis β is P. Assume that the matrix reprentation of linear transformation σ under the two basis is A and B, then

$$B = P^{-1}AP.$$

Proof. For

$$\begin{aligned}
(\beta_1, \beta_2, \cdots, \beta_n) &= (\alpha_1, \alpha_2, \cdots, \alpha_n)P \\
\sigma[\alpha_1, \alpha_2, \cdots, \alpha_n] &= [\alpha_1, \alpha_2, \cdots, \alpha_n]A \\
\sigma[\beta_1, \beta_2, \cdots, \beta_n] &= [\beta_1, \beta_2, \cdots, \beta_n]B \\
\textit{then} \\
&[\beta_1, \beta_2, \cdots, \beta_n]B \\
&= \sigma[\beta_1, \beta_2, \cdots, \beta_n] \\
&= \sigma[(\alpha_1, \alpha_2, \cdots, \alpha_n)P] \\
&= \sigma[\alpha_1, \alpha_2, \cdots, \alpha_n]P \\
&= [\alpha_1, \alpha_2, \cdots, \alpha_n]AP \\
&= [\beta_1, \beta_2, \cdots, \beta_n]P^{-1}AP
\end{aligned}$$

Since $\beta_1, \beta_2, \cdots, \beta_n$ is linear irrelative, thus

$$B = P^{-1}AP.$$

□

2.2 Eigenvalue and Eigenvector of the Linear Transformation

Definition 2.5. Set σ as a linear transformation in the vector space V, if for a number $\lambda \in F$, there exists a **nonvector** $\alpha \in V$, that

$$\sigma(\alpha) = \lambda\alpha.$$

then we say λ is an *eigenvalue* of linear transformation σ, and α is an *eigenvector* of σ.

Assume σ is a linear transformation in the vector space V, and λ is an eigenvalue of σ, then α is an eigenvector of σ with eigenvalue λ, then $\sigma(\alpha) = \lambda\alpha$. Choose a basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ of V, and set

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

then

$$\sigma(\alpha) = \lambda\alpha = \lambda x_1\alpha_1 + \lambda x_2\alpha_2 + \cdots + \lambda x_n\alpha_n$$

and

$$\sigma(\alpha) = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} (\lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix})$$

then

$$\lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where A is the matrix reprentation of σ under the basis $\alpha_1, \alpha_2, \cdots, \alpha_n$. Let $X = (x_1, x_2, \cdots, x_n)^T$, then

$$AX = \lambda X.$$

Thus λ is the eigenvalue of the matrix A, and the X is the eigenvector of A with eigenbvalue λ.

Vise versa, if λ is the eigenvalue of the matrix A, and X is the eigenvector of A with eigenbvalue λ, then set $\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$, If $\alpha \in V$, and $\alpha \neq \theta$.

$$\begin{aligned}
\lambda\alpha &= \lambda([\alpha_1, \alpha_2, \cdots, \alpha_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}) = [\alpha_1, \alpha_2, \cdots, \alpha_n](\lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}) \\
&= [\alpha_1, \alpha_2, \cdots, \alpha_n](\lambda X) = [\alpha_1, \alpha_2, \cdots, \alpha_n](AX) \\
&= ([\alpha_1, \alpha_2, \cdots, \alpha_n]A)X = \sigma([\alpha_1, \alpha_2, \cdots, \alpha_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}) \\
&= \sigma(\alpha)
\end{aligned}$$

then λ is the eigenvalue of σ with eigenvector α.

3 Exercise

Exercise 3.1. Set $D : R[x]_{n+1} \to R[x]_n$ as *Derivative Map*, you should find the matrix reprentation of D under the basis $1, x, x^2, \cdots, x_n$ and $1, x, x^2, \cdots, x_{n-1}$.

Solution 3.1.1. Set $f_1 = 1, f_2 = x, \cdots, f_{n+1} = x^n$, then

$$D(f_1) = 0, D(f_2) = 1, D(f_3) = 2x, \cdots, D(f_{n+1}) = nx^{n-1}.$$

$$\begin{cases} D(f_1) = 0f_1 + 0f_2 + 0f_3 + \cdots + 0f_{n-1} \\ D(f_2) = 1f_1 + 0f_2 + 0f_3 + \cdots + 0f_{n-1} \\ D(f_3) = 0f_1 + 2f_2 + 0f_3 + \cdots + 0f_{n-1} \\ \vdots \\ D(f_{n+1}) = 0f_1 + 0f_2 + \cdots + nf_{n-1} \end{cases}$$

$$\begin{aligned} &[D(f_1), D(f_2), D(f_3), \cdots, D(f_{n+1})] \\ &= [0, 1, 2x, \cdots, nx^{n-1}] \\ &= [1, x, \cdots, x^{n-1}] \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix} \end{aligned}$$

Thus, the matrix representation of D under the basis $1, x, x^2, \cdots, x^n$ and basis $1, x, x^2, \cdots, x^{n-1}$ is

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

Exercise 3.2. In R^3, we form a mapping $\sigma : R^3 \to R^3$ by $\sigma[(x_1, x_2, x_3)] = (x_3, 0, x_2 - 2x_1),\ (x_1, x_2, x_3) \in R$.

1. Prove σ is a linear mapping.
2. Find the matrix representation of σ under the basis $(1, 0, 0), (1, 1, 0), (1, 1, 1)$.

Solution 3.2.1. Choose any (x_1, x_2, x_3) and $(y_1, y_2, y_3) \in R^3$, $k \in R$, Since

$$\begin{aligned}
\sigma[(x_1, x_2, x_3) + (y_1, y_2, y_3)] &= \sigma[(x_1 + y_1, x_2 + y_2, x_3 + y_3)] \\
&= (x_3 + y_3, 0, x_2 + y_2 - 2(x_1 + y_1)) \\
&= (x_3, 0, x_2 - 2x_1) + (y_3, 0, y_2 - 2y_1) \\
&= \sigma[(x_1, x_2, x_3)] + \sigma[(y_1, y_2, y_3)] \\
\sigma[k(x_1, x_2, x_3)] &= \sigma[(kx_1, kx_2, kx_3)] \\
&= (kx_3, 0, kx_2 - 2kx_1) \\
&= k(x_3, 0, x_2 - 2x_1) \\
&= k\sigma[(x_1, x_2, x_3)]
\end{aligned}$$

Solution 3.2.2. Choose the natrual basis of R^3, $(1,0,0), (0,1,0), (0,0,1)$.

$$\begin{aligned}
\sigma[(1,0,0)] &= (0,0,-2) \\
\sigma[(0,1,0)] &= (0,0,1) \\
\sigma[(0,0,1)] &= (1,0,0)
\end{aligned}$$

and

$$\begin{cases}
(0,0,-2) & = a_{11}(1,0,0) + a_{12}(1,1,0) + a_{13}(1,1,1) \\
(0,0,1) & = a_{21}(1,0,0) + a_{22}(1,1,0) + a_{23}(1,1,1) \\
(1,0,0) & = a_{31}(1,0,0) + a_{32}(1,1,0) + a_{33}(1,1,1)
\end{cases}$$

$$\begin{cases}
a_{11} = 0, \ a_{12} = 2, \ a_{13} = -2 \\
a_{21} = 0, \ a_{22} = -1, \ a_{23} = 1 \\
a_{31} = 1, \ a_{32} = 0, \ a_{33} = 0
\end{cases}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

That is the answer.

Exercise 3.3. In the matrix space $R^{2\times 2}$, we set a linear transformation $\sigma : \sigma(X) = AX, \ X \in R^{2\times 2}$ where

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

Find the representation of σ under the natural basis in $R^{2\times 2}$.

Solution 3.3.1. Set

$$I_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, I_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, I_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, I_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{cases} \sigma(I_{11}) = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix} \\ \sigma(I_{12}) = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix} \\ \sigma(I_{21}) = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \\ \sigma(I_{22}) = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} \end{cases}$$

Thus,

$$\begin{cases} \sigma(I_{11}) = 1I_{11} + 0I_{12} + 4I_{21} + 0I_{22} \\ \sigma(I_{12}) = 0I_{11} + 1I_{12} + 0I_{21} + 4I_{22} \\ \sigma(I_{21}) = 2I_{11} + 0I_{12} + 3I_{21} + 0I_{22} \\ \sigma(I_{22}) = 0I_{11} + 2I_{12} + 0I_{21} + 3I_{22} \end{cases}$$

then

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 4 & 0 & 3 & 0 \\ 0 & 4 & 0 & 3 \end{bmatrix}$$

is the answer.

Exercise 3.4. We know that a linear transformation σ's matrix reprentation under the basis $\alpha_1, \alpha_2, \alpha_3$ is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Find the matrix reprentation of σ under the basis $\alpha_3, \alpha_2, \alpha_1$

Solution 3.4.1. Since

$$[\alpha_3, \alpha_2, \alpha_1] = [\alpha_1, \alpha_2, \alpha_3] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

then from the theorem 2.3,

$$
\begin{aligned}
B &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}
\end{aligned}
$$

is the answer.

Exercise 3.5. Set the linear transformation σ under the basis $\alpha_1, \alpha_2, \alpha_3$, and its matrix reprentation is

$$
A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.
$$

Find the eigenvalue of σ and its eigenvector.

Solution 3.5.1. Solve the eigenvalue of the A,

$$
|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} = \begin{vmatrix} \lambda - 5 & -2 & -2 \\ \lambda - 5 & \lambda - 1 & -2 \\ \lambda - 5 & -2 & \lambda - 1 \end{vmatrix} = (\lambda - 5) \begin{vmatrix} 1 & -2 & -2 \\ 1 & \lambda - 1 & -2 \\ 1 & -2 & \lambda - 1 \end{vmatrix} = (\lambda - 5) \begin{vmatrix} 1 & 0 & 0 \\ 1 & \lambda + 1 & 0 \\ 1 & 0 & \lambda + 1 \end{vmatrix} = (\lambda +
$$

Thus, the eigenvalue of σ is $\lambda_1 = \lambda_2 = -1$ or $\lambda_3 = 5$. For $\lambda = -1$,

$$
\begin{cases} -2x_1 - 2x_2 - 2x_3 = 0 \\ -2x_1 - 2x_2 - 2x_3 = 0 \\ -2x_1 - 2x_2 - 2x_3 = 0 \end{cases}
$$

then two solutions of it is $X_1 = (1, 0, -1)^T$ and $X_2 = (0, 1, -1)$. For $\lambda = 5$,

$$
\begin{cases} 4x_1 - 2x_2 - 2x_3 = 0 \\ -2x_1 + 4x_2 - 2x_3 = 0 \\ -2x_1 - 2x_2 + 4x_3 = 0 \end{cases}
$$

then one solution of it is $X_3 = (1, 1, 1)^T$.

For eigenvectors with eigenvalue $\lambda = -1$,

$$
\beta_1 = [\alpha_1, \alpha_2, \alpha_3] X_1 = \alpha_1 - \alpha_3
$$

$$
\beta_2 = [\alpha_1, \alpha_2, \alpha_3] X_2 = \alpha_2 - \alpha_3
$$

then the eigenvector with eigenvalue -1 is

$$k_1\beta_1 + k_2\beta_2.$$

For eigenvalue with eigenvalue $\lambda = 5$,

$$\beta_3 = [\alpha_1, \alpha_2, \alpha_3]X_3 = \alpha_1 + \alpha_2 + \alpha_3$$

then

$$k_3\beta_3$$

is the eigenvector with eigenvalue 5.

4 Conclusion

1. Judge whether it's a linear transformation.

2. Find the matrix representation of a linear transformation under the specific basis.

3. Find the matrix representation of a linear transformation under the different basis.