Real Numbers

Len Fu

11.29.2024

Abstract

Contents

1 Basic Properties

2 The Set of Real Numbers

2.1

2.2 Archimedean Property

Theorem 2.1. *1. (Archimedean Property) If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that*

$$nx > y.$$

2. ($\mathbb{Q}$ is dense in $\mathbb{R}$) If $x, y \in \mathbb{R}$ and $x < y$, then there exists an $r \in \mathbb{Q}$ such that

$$x < r < y.$$

Proof. Consider (i), for every real number $t := \frac{y}{x}$

Consider (ii), first assume $x \leq 0$, and $y - x > 0$, then there exists an $n \in \mathbb{N}$ such that $n(y-x) > 1$, and $y - x > \frac{1}{n}$. And there has a least integer $m > nx$, divide through by n we get $x < \frac{m}{n}$.

If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \leq nx$. If $m = 1$, $m - 1 = 0 \leq nx$. That is to say $nx \geq m - 1$.

Then $y > x + \frac{1}{n} \geq \frac{m}{n} > x$, that is $\mathbb{Q}$ is dense in $\mathbb{R}$. □

2.3 Inf and Sup

Proposition 2.1. *Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \geq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and* $\sup A \geq \inf B$.

Proof. □

Proposition 2.2.

Definition 2.1. Let $A \subset \mathbb{R}$ be a set.

1. If A is empty, then $\sup A := -\infty$.
2. If A is empty, then $\inf A := \infty$.
3. If A is not bounded above, then $\sup A := \infty$.
4. If A is not bounded below, then $\inf A := -\infty$.

And $\mathbb{R}^* = \mathbb{R} \bigcup \infty, -\infty$ is defined as **the set of Extended Real Numbers**

But we must leave $\infty - \infty, 0 \cdot \pm\infty,\ and\ \frac{\pm\infty}{\pm\infty}$ as undefined.

2.4 Absolute Value and Bounded Functions

Proposition 2.3 (Triangle Inquality)**.** *Let $x, y \in \mathbb{R}$ and $x > 0$, then $|x + y| \leq |x| + |y|$.*

Corollary 2.1.1. *Let $x, y \in \mathbb{R}$. (i) (reverse triangle inequality) $||x| - |y|| \leq |x - y|$. (ii) $|x - y| \leq |x| + |y|$.*

Definition 2.2 (Bounded Functions)**.** Suppose $f : D \to \mathbb{R}$ is a function. We say f is **bounded** if there exists a constant $M \in \mathbb{R}$ such that $|f(x)| \leq M$ whenever $x \in D$.

Proposition 2.4. *If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions and $f(x) \leq g(x)$ for all $x \in D$ then*

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x) \ and\ \inf_{x \in D} f(x) \leq \sup_{x \in D} g(x).$$

2.5 Intervals and the size of $\mathbb{R}$

Proposition 2.5. *A set $I \subset \mathbb{R}$ is an interval if and only if I contains at least 2 points and for all $a, c \in I$ and $a < b < c$, we have $b \in I$.*

2.6 Decimal Representation of the Reals

We represent rational numbers with positive integer M, K and digts $d_K d_{K-1} \cdots d_1 d_0 d_{-1} \cdots d_{-M+1} d_M$ such that

$$x = d_K 10^K + d_{K-1} 10^{K-1} + \cdots + d_1 10^1 + d_0 10^0 + d_{-1} 10^{-1} + \cdots + d_{-M+1} 10^{-M+1} + d_M 10^{-M}$$

and call D_n the **truncation of x to n decimal digits**.

However for irrarional numbers, we can not represent them in this way. And for some infinite curcilation, we can not represent them in this way either.

For every real number $x \in (0, 1]$, we define

$$x = \sup_{n \in \mathbb{N}} D_n = \sup_{n \in \mathbb{N}} \left(\frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} \right).$$

Proposition 2.6. *(i) Every infinite sequece of digts $0.d_1 d_2 \cdots$ represents a unique real number $x \in (0, 1]$, and*

$$D_n \le x \le D_n + \frac{1}{10^n} \ for\ all\ n \in \mathbb{N}.$$

(ii) For every real number $x \in (0, 1]$, there exists an infinite sequence of digts $0.d_1 d_2 \cdots$ that represents x. There exists a unique representation such that

$$D_n < x \le D_n + \frac{1}{10^n} \ for\ all\ n \in \mathbb{N}.$$

Proposition 2.7. *If $x \in (0, 1]$ is a rational number and $x = 0.d_1 d_2 \cdots$, then the decimal digits eventually start repeating. That is, there are positive integers N and P, such that for all $n \ge N$, $d_n = d_{n+P}$.*

3 Exercise

Solution 3.0.1 (1.1.2). Since A is a subset of ordered set S, we suppose the number of its elements is n and denote it as A_n. Using the induction, we have:
Base Case: If $n = 1$, the only element is both the infimum and supremum of A_1, and A_1 is bounded.
Induction Step: Assume the hypothesis holds for $n = k$, then we can find a smallest and a largest element a_k and b_k in A, then we insert an element x of S into A_k and regard it as A_{k+1}. Since S is ordered, either $x > a_k$ or $a_k > x$ and there must have a smallest element in A_{k+1}, furthermore it is the infimum of A_{k+1} and in A_{k+1}. Similarly, we can find the largest element as the supremum of A_{k+1}. And obviously A_{k+1} is bounded.
Conclusion: By the principle of induction, we have shown that for any $n \in \mathbb{N}$, A_n is bounded. That is for every nonempty subset of ordered set, it is bounded with infimum and supremum within it.

Solution 3.0.2 (1.1.3). Using proposition(ii):

$$\begin{aligned} x + y > 0 + 0 = 0 \ y - x >& 0 \\ (y - x)(y + x) >& 0 \\ y^2 - x^2 > 0 \\ y^2 >& x^2 \end{aligned}$$

Solution 3.0.3. 1.1.4
A is an ordered subset of ordered subset B, since all infs and sups exist, from the definition we know that:

$$there\ exists\ an\ \sup A\ \in B,\ for\ all\ x \in A,\ x \leq \sup A.$$

And

$$there\ exists\ an\ \sup B\ \in S,\ for\ all\ x \in B,\ x \leq \sup B.$$

Since $\sup A$ is in B, then $\sup A \leq \sup B$. Vise versa, $\inf B \leq \inf A$.

And for a nonempty set with inf and sup, it obeys that inf $\leq$ sup, thus $\inf A \leq \sup A$.

Above all, we have proved that

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

Solution 3.0.4 (1.1.5). We assume the supremum exists and denote the supremum of A as $\sup A$. From the defition of supremum, since $b \in A$, we get that $b \leq \sup A$. From another side, we know that b is an upper bound of A, thus $b \geq \sup A$. Obviously $b = \sup A$.

Solution 3.0.5. 1.2.3
To prove (iii), we suppose that b is an upper bound of A, that is, $y \leq b$ for all $y \in A$. For $x > 0$ we have $xy \leq xb$ for all $y \in A$, and so xb is an upper bound of xA. In particular, b is sup of A. We have $\sup xA \leq x \sup A$.

To prove the inverse inequality, suppose c is a upper bound of xA, thus $xy \leq c$ for all $y \in A$, and we have $y \leq \frac{c}{x}$ which reveals that $\frac{c}{x}$ is an upper bound of A. In particular, c is the sup of xA, we have $\sup A \leq \frac{\sup xA}{x}$.And we have $\sup xA = x \sup A$. Vise versa, it remains for inf as (iv).

To prove (v), we suppose that b is an lower bound of A, that is, $y \geq b$ for all $y \in A$. For $x < 0$ we have $xy \leq bx$ for all$y \in A$, and bx is an upper bound of xA. In particular, b is inf of A.We have $\sup xA \leq x \inf A$.

To prove the inverse inequality, suppose c is a upper bound of xA, thus $xy \leq c$ for all $y \in A$, and we have $y \geq \frac{c}{x}$ which reveals that $\frac{c}{x}$ is an lower bound of A. In particular, c is the sup of xA. We have $\sup xA \geq x \inf A$. And we have $\sup xA = x \inf A$ Vise versa, it remains for sup as (vi).

Solution 3.0.6 (1.2.5). Now we assume that $\sqrt{3}$ is rational and denote it as $\frac{p}{q}$ where p, q are irreducible. Then we have $p^2 = 3q^2$, we can see that $p = 3k$ for some $k \in N^*$, then $q^2 = 3k^2$. We conclude that both p and q are multiple of 3, contradicting to the assumption. So the assumption fails, $\sqrt{3}$ is irrarional.

Solution 3.0.7 (1.2.8). For every pair of $x, y \in \mathbb{R}$, we have that $\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}} \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we have that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \in \mathbb{R}$ for some $r \in \mathbb{Q}$. Then we have that $x < \sqrt{2}r < y$, which implies that there exists an irrarional number r^* such that $x < r^* < y$.

Solution 3.0.8. 1.2.9
We set p and q is an upper bound of A and B correspondingly, for all $a \in A, b \in B$. Now we set $c = a + b \in C$, and we have

$$c = a + b \leq p + b \leq p + q$$

we can see that C is upper bounded. Then $p + q$ is an upper bound of C and in particular, p, q are sup of A and B respectively. Then we have $\sup C \leq \sup A + \sup B$.

To prove the inverse inequality, as we have known that C is upper bounded, we set c as an upper bound of C. and for all $a \in A, b \in B$, we have $a + b \leq c$, then $a \leq c - b$ for all $a \in A$ showing that $c - b$ is an upper bound of A and in particular, c is sup of C. Then $\sup C - b$ is an upper bound of A and we have $\sup A \leq \sup C - b$, or equally, $b \leq \sup C - \sup A$. Follow the same procedure, we have $\sup B \leq \sup C - \sup A$ that is $\sup A + \sup B \leq \sup C$. And we see that $\sup A + \sup B = \sup C$. Vise versa, it remains the same as it changes from sup to inf.

Solution 3.0.9 (1.2.10)**.** Emmmmm, I don't think it differs in a large extent from the thinking chain of exercise[1.2.3] and exercise[1.2.9]. So let me skip this exercise.

Solution 3.0.10. 1.2.11
To prove the statement, we first take the set $A = \{a \in \mathbb{R} | a^n < x\}$. We need to show that A is bounded above and has a supremum, which can be proved that it is the unique $r = x^{\frac{1}{n}}$ we want.
Step1(Ensure the exsistence of supA): For $x > 1$, if $a > x$, we have $a^n > x^n$ contradicting to the assumption, thus $a < x$ which reveals that A is upper bounded. For $x < 1$, then a should be less than 1, which reveals that A is upper bounded. And whether x is larger than 1 or not, $\frac{x}{2}^n < x^n$, thus A is not empty. Thus there must exist the supremum.
Step2(Show $r = x^{\frac{1}{n}}$): Suppose the sup of A is r.

Now we assume that $r^n < x$, and we first choose a number $0 < h < 1$. We can have

$$\begin{aligned}
&(r+h)^n - r^n \\
=&h * Poly(r, h) \ (where \ Poly(r, h) = \sum a_i r^i h^{n-i-1} \ a_i > 1) \\
<&h * Poly(r, 1)
\end{aligned}$$

Then we set $h < \frac{x - r^n}{Poly(r,1)}$, we have

$$(r + h)^n < x.$$

That is to say, there exists a number $h > 0$ such that $(r + h)^n < x$. And we know that $r + h \in A$ and $(r + h) > r$ contradicting to $r = \sup A$, thus $r^n \geq x$.

And now we assume that $r^n > x$, then we set $0 < h < 1$, and we have

$$\begin{aligned}
&r^n - (r-h)^n \\
=&h * Poly(r, -h) \ (where \ Poly(r, -h) = \sum a_i r^i (-h)^{n-i-1} \ a_i > 1) \\
<&h * Poly(r, 1)
\end{aligned}$$

Then we set $h < \frac{r^n - x}{Poly(r,1)}$, we have

$$(r - h)^n > x.$$

That is to say there exists a number $h > 0$ such that $(r - h)^n > x$. And we know that $r - h \notin A$ and there doesn't exisit an $x \in [r - h, r]$ satisfying $x \in A$ contradicting to $r = \sup A$(proposition 1.2.8 basic property of sup), thus $r^n \leq x$.

Ok then we have $r^n = x$. To prove its uniqueness, suppose that there are two numbers r_1, r_2 satisfying, and we assume that $r_1 < r_2$, and we can get $x < x$ as a consequence. Obviously it's wrong, thus $r_1 = r_2$. And we ensure the uniqueness of r by contradiction.

Solution 3.0.11. 1.2.13
Using principle of induction, we have: **Base Case:** If $n = 1$, the inequality is trivially satisfied. **Induction Step:** Assume the inequality holds for $n = k$, then we can write

$$(1 + x)^k - (1 + kx) \geq 0.$$

Now we consider the inequality for $n = k + 1$:

$$\begin{aligned}
&(1+x)^{k+1} - (1+(k+1)x) \\
=&(1+x)^k(1+x) - 1 - kx - x \\
\geq&(1+kx)(1+x) - 1 - kx - x \\
=&kx^2 \\
\geq& 0
\end{aligned}$$

Obviously, the inequality holds for $n = k + 1$ as well. **Conclusion:** By the principle of induction, we have shown that for any $n \in \mathbb{N}$, the inequality is satisfied.

Solution 3.0.12. 1.2.15

(a)) We set $A = \{x \in \mathbb{Q} | x < y\}$. First y is an upper bound of A, we need to prove that y is the sup of A. beginsinglespace A is upper bounded and nonempty, the sup is exsisting and we denote it as r. Assume that $r \neq y$, that is equally $r < y$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exsists an rational number x such that $r < x < y$. Then we know that $x \in A$, then $x < r$, which contradicts to $r \neq y$. Thus $y = r$ is the sup of A.

(b)) We set $\inf A$ as y, from the definition of Dedekind cuts, we know that there is no largest element in A, that is for any $a \in A$, it must be $a < y$. Thus $A \subset \{x \in \mathbb{Q} | x < y\}$.

Now we choose $b \in \{x \in \mathbb{Q} | x < y\}$, since y is the sup of A, then for any $\epsilon > 0$, there exists $a \in A$, satisfying $y - \epsilon < a < y$. We choose $y - x$ as ϵ, then $x < a$ and we know that $x \in A$, that is $\{x \in \mathbb{Q} | x < y\} \subset A$.

And we have

$$A = \{x \in \mathbb{Q} | x < y\}, \ where\ y = \sup A.$$

(c)) $f : \mathbb{R} \to Dedekind\ Cuts,\ f(r) = \{x \in \mathbb{Q} | x < r\}\ \ r \in R.$

Solution 3.0.13 (1.3.3)**.** Skip.

Solution 3.0.14 (1.3.4)**.** If a is a lower bound of $f(D)$, then $a \leq f(x) \leq g(x)$, thus a is also a lower bound of $g(D)$ and we choose the inf of $f(D)$.

$$\inf_{y \in D} f(y) \leq g(x),\ for\ all\ x \in D$$

and $\inf_{y \in D} f_y$ is a lower bound of $g(D)$ and less than the inf of $g(D)$:

$$\inf_{x \in D} f(x) \leq \inf_{x \in D} g(x).$$

Solution 3.0.15. 1.3.5

(a) Since $f(x) \leq g(y)$ for all $x \in D$ and $y \in D$, then $g(y)$ is an upper bound of $f(D)$, $\sup_{x \in D} \leq g(y)$ for all $y \in D$. Then $\sup_{x \in D}$ is a lower bound of $g(D)$, and we get $\inf_{x \in D} \sup_{x \in D} \leq g(y)$.

(b) $D = [0, 1], f(x) = x, g(x) = x + 0.5$.

Solution 3.0.16 (1.3.6)**.** Now we rewrite the proposition's condition: If $f : D \to \mathbb{R}^*$ and $g : D \to \mathbb{R}^*$...

Now the inf and sup is well defined even f and g are not bounded functions. And the proving procedure remains unchanged.

Solution 3.0.17. 1.3.7

(a) For all $x \in D$, we have $f(x) \leq \sup_{x \in D} f(x)$ and $g(x) \leq \sup_{x \in D} g(x)$. Thus $f(x) + g(x) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x)$ for all $x \in D$, and $\sup_{x \in D} f(x) + \sup_{x \in D} g(x)$ is an upper bound of $f(x) + g(x)$. And

$$\sup_{x \in D} (f(x) + g(x)) \leq \sup_{x \in D} f(x) + \sup_{x \in D} g(x).$$

Vise versa, it remains for the inf.

(b) $\sin x$ and $\cos x$