Infinite Series and Infinite Products

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Abstract

This is the note of Infinite Series and Infinite Products, maded by Len Fu while his learning progress. The main content is from *Mathematical Analysis Tom A.Apostol*.

Contents

1 Convergent and Divergent Sequences of Complex Numbers

1.1 Basis

Definition 1.1 (Convergence)**.** A sequence of complex numbers $a_n \in C$ is called *convergent* if,

$$for\ every\ \epsilon > 0,\ there\ exists\ an\ N \in \mathbb{N}\ such\ that,\ |a_n - a| < \epsilon\ for\ all\ n \geq N.$$

If a_n converges to p, we write $\lim_{n \to \inf} a_n = p$ and call p the limit of the sequence. A sequence is called divergent if it is not convergent.

Definition 1.2 (Divergence)**.** A sequence of complex numbers $a_n \in C$ is called *divergent* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that,

$$|a_n - a| \leq \epsilon\ for\ all\ n \geq N.$$

In this case we write $\lim_{n \to \infty} a_n = +\infty$.

1.2 Cauchy Sequence

Definition 1.3. A sequence in $\mathbb{C}$ is called a *Cauchy sequence* if it satisfies the *Cauchy condition*: for every $\epsilon > 0$ there is an integer N such that

$$|a_n - a_m| < \epsilon\ whenevern \geq N\ and\ m \geq N.$$

Obviously, being Cauchy Sequence means that the terms of the sequence are all arbitarily close to each other.

The Cauchy condition is particularly useful in establishing convergence when we do not know the actual value to which the sequence converges.

Proposition 1.1. *If a sequence is a Cauchy sequence, then it is bounded.*

Proof. Suppose $x_n{}_{n=1}^{\infty}$ is Cauchy.Pick an M such that for $n, k \geq M$, we have $|x_n - x_k| < 1$. In particular for all $n \geq M$,

$$|x_n - x_M| \geq 1.$$

Then we use the triangle inequality to obtain:

$$|x_n| - |x_M| \geq |x_n - x_M| < 1$$

then for all $n \geq M$,

$$|x_n| < 1 + |x_M|.$$

Now set

$$B := \max\{|x_1|, |x_2|, \cdots, |x_{M-1}|, 1 + |x_M|\}$$

Then B is an upper bound for the absolute sequence and it is bounded. □

Theorem 1.2. *A sequence of real numbers is convergent if and only if it is converges.*

Proof. Suppose $x_n{}_{n=1}^{\infty}$ converges to x, and let $\epsilon > 0$ be given. Then there exists an M such that for $n \geq M$,$|x_n - x| < \frac{\epsilon}{2}$. Hence for $n \geq M$ and $k \geq M$,

$$|x_n - x| + |x_k - x| \geq |x_n - x_k| \geq \epsilon$$

and $x_n{}_{n=1}^{\infty}$ is a Cauchy sequence.

Now, suppose $x_n{}_{n=1}^{\infty}$ is a Cauchy sequence.

□

1.3 Bounded and convergent

Every convergent sequence is bounded and hence an unbounded sequence necessarily diverges.

1.4 Subsequence

If a sequence a_n converges to p, then every subsequence a_{k_n} also converges to p.

If $\lim_{n\to\infty} -a_n = \infty$, we write $\lim_{n\to\infty} a_n = -\infty$ and say that a_n diverges to $-\infty$.

2 Limit Superior and Limit Inferior of a Real-Valued Sequence

2.1 Basis

Let a_n be a sequence of real numbers. Suppose there is a real number U satisfying the following two conditions:

1. For every $\epsilon > 0$, there exists an integer N such that $n > N$ implies

$$a_n < U + \epsilon.$$

2. Given $\epsilon > 0$ and given $m > 0$, there exists an integer $n > m$ such that

$$a_n > U - \epsilon.$$

Note. Statement (1) means that all terms of the sequence lie to the left of $U + \epsilon$. Statement (2) means that infinite terms of the sequence lie to the right of $U - \epsilon$.Every real sequence has a limit superior and a limit inferior in the extended real number $\mathbb{R}^*$.

Then U is called the *limit superior* of a_n and we write

$$U = \lim_{n\to\infty} \sup a_n$$

. The limit inferior of a_n is defined as follows:

$$\lim_{n\to\infty} \ inf\ a_n = -\lim_{n\to\infty}\ sup\ b_n,\ where\ b_n = -a_n\ for\ n = 1, 2, ..., n$$

.

Or we can use another definition:

Definition 2.1.

Corollary 2.1. *Let a_n be a sequence of real numbers. Then we have:*

1. $\lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \inf b_n$.

2. *The sequence converges if, and only if,* $\limsup_{n\to\infty} a_n$ *and* $\liminf_{n\to\infty} a_n$ *are both finite and equal, in which case* $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

3. *The sequence diverges to* $+\infty$ *if, and only if,* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = +\infty$.

4. *The sequence diverges to* $-\infty$ *if, and only if,* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -\infty$.

Note. A sequence for which $\limsup_{n\to\infty} a_n \neq \liminf_{n\to\infty} a_n$ is said to oscillate.

Proof. **1.** From definition, denote $U = \limsup_{n\to\infty} a_n$ and $L = \liminf_{n\to\infty} a_n$. For every $\epsilon_1 > 0$, $b_n < -L + \epsilon_1$, where $b_n = -a_n$. And for every $\epsilon_2 > 0$, $a_n < U + \epsilon_2$.

$$
\begin{aligned}
-a_n &< -L + \epsilon_1 \\
a_n &> L - \epsilon_1 \\
a_n &< U + \epsilon_2 \\
L - \epsilon_1 < a_n &< U + \epsilon_2 \\
L &< U + \epsilon_1 + \epsilon_2
\end{aligned}
$$

Since ϵ_1 and ϵ_2 is arbitary positive, we have $L \leq U$, that is $\lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \inf b_n$.
2.

$\square$

2.2 Monotune Sequence

Theorem 2.2. *A monotonic sequence converges if, and only if, it is bounded.*

Proof. If $a_n \uparrow$, $\lim_{n\to\infty} = \sup a_n : n = 1, 2, \cdots$. If $a_n \downarrow$, $\lim_{n\to\infty} = \inf a_n : n = 1, 2, \cdots$. $\square$

3 Infinite Series

Let a_n be a sequence of real or comcplex numbers, and form a new sequence s_n as follows:

$$s_n = a_1 + a_2 + \cdots + a_n \ (n = 1, 2, \cdots)$$

3.1 Basis

Definition 3.1 (Series)**.** The orderd pair of sequences a_n, s_n is called an infinite series. The number s_n is called the *nth partial sum* of the series. The series is said to *converge* or to *diverge* according as s_n is convergent or divergent. The following symbols are used to denote series:

$$a_1 + a_2 + \cdots + a_n + \cdots\ ,\ , a_1 + a_2 + a_3 + \cdots\ ,\ , \sum_{k=1}^{\infty} a_k.$$

Definition 3.2 (Patial Sum)**.** A series converges if the sequence $s_k{}_{k=1}^{\infty}$ defined by

$$s_k := \sum_{n=1}^{k} x_n = x_1 + x_2 + x_3 + \cdots + x_k + \cdots$$

converges. The numbers s_k are called the *partial sums.*

Note. The letter k used in $\sum_{k=1}^{\infty} a_k$ is a **"dummy variable"** and may be replaced by any other letter.

If the sequence s_n defined as previous converges to s, the number s is called the *sum* of the series, and we write

$$s = \sum_{k=1}^{\infty} a_k$$

.

Corollary 3.1. *Let $a = \sum a_n$ and $b = \sum b_n$ be convergent series. Then, for every α and β, the series $\sum(\alpha a_n + \beta b_n)$ converges to the sum $(\alpha a + \beta b)$.*

Proof. $\sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$. □

Corollary 3.2. *Assume that $a_n \neq 0$ for each $n = 1, 2, \cdots$ Then $\sum a_n$ converges if, and only if, the sequence of patial sums is bounded above.*

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Then we can apply 2.2 □

Theorem 3.3 (Telescoping series)**.** *Let a_n and b_n be two sequences such that $a_n = b_{n+1} - b_n$ for $n = 1, 2, \cdots$. Then $\sum a_n$ converges if, and only if, $\lim_{n\to\infty} \sum b_n$ exists, in which case we have $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} b_n - b_1$.*

Theorem 3.4 (Cauchy condition for series)**.** *The series $\sum a_n$ converges if, and only if, for every $\epsilon > 0$, there exists an integer N such that $n > N$ implies*

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon\ for\ each\ p = 1, 2, \cdots$$

Taking $p = 1$ in the previous theorem, we find that $\lim_{n\to\infty} a_n = 0$ is a necessary condition for the convergence of $\sum a_n$. That this condition is not sufficient to ensure the convergence of $\sum a_n$ is shown as follows as we choose $a_n = \frac{1}{n}$:

$$a_{n+1} + \cdots + a_{n+p} = \frac{1}{2^m + 1} + \cdots + \frac{1}{2^m + 2^m} \geq \frac{2^m}{2^m + 2^m} = \frac{1}{2},$$

and hence $\sum a_n$ diverges. This series is called the *harmonic series.*

Proposition 3.5. *Let $\sum_{n=1}^{\infty} x_n$ be a convergent series. Then the sequence $x_n{}_{n=1}^{\infty}$ is convergent and*

$$\lim_{n\to\infty} x_n = 0.$$

Proof. □

3.2 Inserting and Removing Parentheses

4 Exercise

Exercise 4.1. Find : $\lim\limits_{n\to\infty} \frac{n}{e^n}$.

Solution 4.1.1. Set continous function $f(x) = \frac{x}{e^x}$ and $\lim_{x\to\infty} f(x) = \lim\limits_{n\to\infty} \frac{n}{e^n}$ Since $\lim\limits_{x\to\infty} x = \infty = \lim\limits_{x\to\infty} e^x$, we can use L'Hôpital's Rule.

$$\begin{aligned} &\lim_{x\to\infty} \frac{x}{e^x} \\ &= \lim_{x\to\infty} \frac{1}{e^x} \\ &= 0 \end{aligned}$$

Thus the limit is 0.

Exercise 4.2. Find : $\lim\limits_{n\to\infty} \frac{\sin n}{n}$.

Solution 4.2.1. Since

$$-1 \leq \sin n \leq 1,$$

we have

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

As n approaches infinity, the limit of the sequence $\left\{-\frac{1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$ is both 0. Using the squeezing theorem, we have

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Exercise 4.3. Find : $\lim\limits_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{n}$.

Solution 4.3.1.

$$\begin{aligned} &\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{n} \\ =&\frac{\lim\limits_{n\to\infty} \ln(1+\frac{1}{n})}{\lim\limits_{n\to\infty} n} \\ =&\frac{0}{\infty} \\ =&0 \end{aligned}$$

Exercise 4.4. Find : $\lim\limits_{n\to\infty} n\sin^2(\frac{1}{n})$.

Solution 4.4.1.

$$\begin{aligned} &\lim_{n\to\infty} n\sin^2(\frac{1}{n}) \\ &= \lim_{n\to\infty} \frac{\sin^2(\frac{1}{n})}{\frac{1}{n}} \ for\ x \in R^+ \lim_{x\to\infty} \sin^2(\frac{1}{x}) = 0 = \lim_{x\to\infty} \frac{1}{x}, usingL'Hpital'sRule \\ &= \lim_{x\to\infty} 2\sin(\frac{1}{x})\cos(\frac{1}{x}) \\ &=0 \end{aligned}$$

Exercise 4.5. Find : $\lim_{n\to\infty}(n+\frac{1}{n})^{\frac{1}{n}}$.

Solution 4.5.1.

$$\begin{aligned}
&\lim_{n\to\infty}(n+\frac{1}{n})^{\frac{1}{n}}\\
=&\lim_{n\to\infty}e^{\frac{\ln(n+\frac{1}{n})}{n}}\\
=&\exp\lim_{n\to\infty}\frac{\ln(n+\frac{1}{n})}{n}\ \textit{Obviously}\ \lim_{n\to\infty}\frac{\ln(n+\frac{1}{n})}{n}=0\\
=&1
\end{aligned}$$

Exercise 4.6. Find : $\lim_{n\to\infty}\sqrt[n]{\ln n}$.

Solution 4.6.1. Since

$$1\le\ln n\le n,\ \ for\ n\le 3$$

we have

$$1\le\ \sqrt[n]{\ln n}\le\ \sqrt[n]{n},\ \ for\ n\le 3.$$

As we have shown that $\lim_{n\to\infty}\sqrt[n]{n}=1$, then use the squeezing theorem . We have

$$\lim_{n\to\infty}\sqrt[n]{\ln n}=1.$$