Determinant

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Abstract

This is the note made by Len Fu during his learning progress in BIT. The main content is from *Linear Algebra Done Right* and *Linear Algebra Allenby*.

Contents

1 Permutation

1.1 N Permutation

An n-permutation is an arrangement of all the numbers $1, 2, ..., n$. The total number of n-permutations is $n!$.

1.2 Inversion and Inversion Number

Formally, for a sequence a_n with elements a_i and a_j with $i < j$, an inversion is present if $a_i > a_j$.

The inversion number of a permutation is the number of inversions in it.

A permutation with an odd inversion number is called an odd permutation. And a permutation with an even inversion number is called an even permutation.

1.3 Transposition

Formally, a transposition is a permutation that exchanges two elements and leaves all others unchanged. For example, in permutation $\tau = (1, 2, 3, 4, 5)$ the transposition $(1, 2)$ exchanges 1 and 2. Then the resulting permutation is $\tau = (2, 1, 3, 4, 5)$.

1.3.1 Theorem

A transposition changes the parity of a permutation. **Proof:**
Let $\tau = (i_1 i_2 ... i_j i_{j+1} ... i_n)$, and we exchange i_j and i_{j+1}, then the remain permutation $(i_1 i_2 ... i_j ... i_n)$ and $(i_1 i_2 ... i_{j+1} ... i_n)$ keep the same parity. But the paritr of $(i_j i_{j+1})$ changes, so the total parity of τ changes.

Now consider that if the transposition is between $(i_j i_k)$ like $(...j i_1 i_2 ... i_s k...)$, then we first transpose s times to set j into i_s like $(...i_1 i_2 ... i_s j k...)$. And we transpose j and k $(...i_1 i_2 ... i_s k j...)$, then we transpose s times to set k into i_1 like $(...k i_1 i_2 ... i_s j...)$. The total transposition is $2s + 1$. So the parity of the permutation changes.

Corollary 1.0.1. *In all n permutation, the number of even permutation is equal to the number of odd permutation, which is* $\frac{n!}{2}$.

Proof. Suppose there are s odd permutation, then there are t even permutation. Now transpose the first two elements of all even permutation, then we get s odd permutation. Then $s \leq t$, conversly, transpose the first two elements of all odd permutation, then we get t even permutation, and $t \leq s$. So $s = t = \frac{n!}{2}$. □

Corollary 1.0.2. *For*

$$a_{i_1 k_1} a_{i_2 k_2} \cdots a_{i_n k_n} \ = \ a_{1 j_1} a_{2 j_2} \cdots \ a_{n j_n}$$

the inversion number is

$$(-1)^{\tau(i_1 i_2 \cdots i_n) + \tau k_1 k_2 \cdots k_n} = (-1)^{\tau j_1 j_2 \cdots j_n}.$$

Proof. When we move an element a_{ik}, the $\tau(i)$ and $\tau(k)$ change the parity simutaneously and theri sum's parity keeps unchanged. After finite transposition, we get

$$(-1)^{\tau(i_1 i_2 \cdots i_n)+\tau(k_1 k_2 \cdots k_n)} = (-1)^{\tau(12\cdots n)+\tau(j_1 j_2 \cdots j_n) = (-1)^{\tau(j_1 j_2 \cdots j_n)}}.$$

And it's obviously that we can have the same result if we have the expasion by column. □

Theorem 1.1. *Any n-permutation can be transposed from* $(123...n)$ *and the times of transposition equals to the inversion number of the permutation.*

2 N-Order Determinant

2.1 Definition

2.1.1 n-order Determinant

The n-order determinant is a scalar value that can be computed from the elements of a square matrix of size $n \times n$*.*

Actually, it can be written abstract

$$\begin{aligned}
\det|A| &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
&= \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \\
&= \sum_{i_1 i_2 \cdots i_n} (-1)^{\tau(i_1 i_2 \cdots i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \\
&= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}
\end{aligned}$$

2.1.2 Minor

The minor of an element in a matrix is the determinant of the submatrix formed by deleting the row and column that contain the element. That is

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \cdots & & & \cdots & & & \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{in} \\ \cdots & & & \cdots & & & \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \cdots & & & \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

2.2 Expansion along a row or a column

Theorem 2.1 (Laplace Expansion Theorem)**.** *The determinant of a n-order matrix equals to any row's or column's element multiplied by its algebraic cofactor. , like*

$$\det = \sum_{j=1}^{n} a_{ij}A_{ij} \ for \ i = 1, 2, \cdots, n = \sum_{i=1}^{n} a_{ij}A_{ij} \ for \ j = 1, 2, \cdots, n.$$

Proof. The factor is

$$(-1)^{i+j}(-1)^{\tau(j_1j_2\cdots j_{i-1}j_{i+1}\cdots j_n)} \ where \ (j_1j_2\cdots j_{i-1}j_{i+1}\cdots j_n) \ is \ a \ permutation \ of \ (12\cdots j-1, j+1\cdots n).$$

then

$$
\begin{aligned}
&(-1)^{i+j}(-1)^{\tau(j_1j_2\cdots j_{i-1}j_{i+1}\cdots j_n)} \\
&= (-1)^{(i-1)+(j-1)+\tau(j_1j_2\cdots j_{i-1}j_{i+1}\cdots j_n)} \\
&= (-1)^{\tau(i12\cdots i-1,i+1\cdots n)+(j-1)+\tau(j_1j_2\cdots j_{i-1}j_{i+1}\cdots j_n)} \\
&= (-1)^{\tau(i12\cdots i-1,i+1\cdots n)+\tau(jj_1j_2\cdots j_{i-1}j_{i+1}\cdots j_n)} \qquad = (-1)^{\tau(12\cdots i-1,i,i+1\cdots n)+\tau(j_1j_2\cdots j_{i-1}j_ij_{i+1}\cdots j_n)} = (-1)^{\tau(j_1j_2\cdots j_{i-1}jj_{i+1}\cdots j_n)}
\end{aligned}
$$

Then the

$$a_{ij}A_{ij} = (-1)^{j_1j_2\cdots j_n}a_{1j_1}a_{2j_2}\cdots a_{nj_n}$$

is a part of the determinant. □

Theorem 2.2. *The sum of the product of any one row's (column's) elements and another row's (column's) elements' algebraic cofactor is 0. That is*

$$\sum_{k=1}^{n} a_{ik}A_{jk} = 0 \ for \ i \neq j$$

or

$$\sum_{k=1}^{n} a_{ki}A_{kj} = 0 \ for \ i \neq j.$$

Proof. Construct a matrix of its form and it's obvious that for two rows or two columns are the same, the determinant is zero. □

Then we can get an important formula

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \begin{cases} D \ j = i \\ 0 \ j \neq i \end{cases}$$

and

$$\sum_{k=1}^{n} a_{ki}A_{kj} = \begin{cases} D \ j = i \\ 0 \ j \neq i \end{cases} \quad .$$

2.2.1 Algebraic Cofactor(Cofactor)

The algebraic cofactor of an element in a matrix is the product of the minor of the element and $(-1)^{i+j}$*, where* i *is the row number and* j *is the column number of the element. We denote it by*

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

2.2.2 k-minor k-cofactor and k-algebraic cofactor

A k-minor of a matrix is the determinant of a square submatrix obtained by ***selecting*** k *rows and* k *columns from the original* nn *matrix, where* n *is the dimension of the matrix.*

A k-order principle minor is the determinant of a square submatrix obtained by ***selecting*** k *rows and* k *columns from the original* nn *matrix, where* n *is the dimension of the matrix and* $i_l = j_l \ for\ l = 1, 2, \cdots, k$.

A k-cofactor of a matrix is a determinant that the original determinant deleting the k-minor that remains as the follow order.

A k-algebraic cofactor of a matrix is a product of $(-1)^{i_1, i_2, \cdots, i_k; j_1, j_2, \cdots, j_k}$ *and the k-cofactor.*

2.3 Properties

1. *Transcope the matrix, the determinant does not change.*

2. $$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

3. $$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 & b_2 & \cdots & b_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ c_1 & c_2 & \cdots & c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

4. *If there are two rows or columns that are the same, the determinant is 0.*

5. *If there are two rows or columns are proportionable, the determinant is 0.*

6. *Add a row's or a column's k-times into another one, the determinant keeps the same.*

7. *Exchange two rows or columns, the determinant changes its sign.*

Proof.

1. *If we transcope the determinant,*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \rightarrow \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & & & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

like this, then we expand the right one with respect to the rows like this

$$\sum_{i_1 i_2 \cdots i_n} (-1)^{\tau(i_1 i_2 \cdots i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$$

Actually it keeps from the left one.

2.

$$\begin{aligned} &\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\ &= k\left(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}\right) \\ &= k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{aligned}$$

3.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (b_1 + c_1)A_{i1} + (b_2 + c_2)A_{i2} + \cdots + (b_n + c_n)A_{in}$$

$$= (b_1 A_{i1} + b_2 A_{i2} + \cdots + b_n A_{in}) + (c_1 A_{i1} + c_2 A_{i2} + \cdots + c_n A_{in})$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 & b_2 & \cdots & b_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ c_1 & c_2 & \cdots & c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

4. *Assume that the l-th and k-th rows are the same,* $det = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} a_{1 j_1 a_{2 j_2} \cdots a_{n j_n}}$ *Since* $a_{li} = a_{ki}$ *for* $i = 1, 2, \cdots, n$*, then* $a_{l j_l} a_{k j_k} = a_{l j_k} a_{k j_l}$ *and the parity changes. The sum is zero.*

5. *Use **2** and **4***

6. *Use **3** and **5***

7. *Obviously.*

Properties 2.1. For n-square matrix A,

1. $\det(kA) = k^n \det A$

2. $\det A^T = \det A$

3. $\det A^{-1} = (\det A)^{-1}$

4. $\det \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} = \det A \det D = \det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$

5. If A is orthogonal matrix, then $\det A = \pm 1$

6. If $A \xrightarrow{+cR_i} B$, then $\det B = c \det A$

7. If $A \xrightarrow{R_j + kR_i} B$, then $\det B = \det A$

8. If $A \xrightarrow{R_{ij}} B$, then $\det B = -\det A$

3 Application

3.1 Cramer Rule

Theorem 3.1 (Cramer Rule)**.** *If the system of linear equations'*

$$Ax = b$$

coefficent matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

its determinant is $d = |A| \neq 0$, then the system of linear equations has solution, and the solution is unique, and can be expressed as

$$x_1 = \frac{d_1}{d},\ x_2 = \frac{d_2}{d},\ \cdots,\ x_n = \frac{d_n}{d}.$$

where d_i is the determinant of the matrix whose the j-th column is replaced by the constant column vector b.

$$d_j = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & \cdots & a_{in} \\ \cdots & & \cdots & \cdots & \cdots & \\ a_{i1} & \cdots & a_{i,j-1} & b_i & \cdots & a_{in} \\ \cdots & & \cdots & \cdots & \cdots & \\ a_{n1} & \cdots & a_{n,j-1} & b_n & \cdots & a_{nn} \end{vmatrix},\ j = 1, 2, \cdots, n.$$

There are three results inside the theorem:

1. *The system of linear equations has solution.*
2. *The solution is unique.*
3. *The solution is expressed as the formula.*

Proof. □

Theorem 3.2. *If the homogenous system of linear equations' coefficent matrix's determinant $|A| \neq 0$, then it only has zero solution. On other word, if the homogenous system of linear equations has non-zero solution, then $|A| = 0$ is certainly.*

Proof. Using the Cramer Rule, we have $d_j = 0,\ j = 1, 2, \cdots, n.$ That is to say,

$$(0, 0, \cdots, 0)$$

is its unique solution. □

3.2 Product Rule

3.2.1 Elementary Matrix

Theorem 3.3. *The determinant of the elementary matrix is not zero and*

$$\det E_{ij} = -1 \ \ \det E_i(c) = c \neq 0 \ \ \det E_{ij}(k) = 1.$$

Theorem 3.4. *If P is an elementary matrix then*

$$\det PA = \det P \det A.$$

3.2.2 Product Rule

Theorem 3.5 (Product Rule)**.** *Suppose there are two matrix A and B, and there determinant is $D_1 = |A|,\ D_2 = |B|$. Then the determinant of the product of A and B is*

$$C = D_1 D_2.$$

Proof. If A is an invertible n-square, then A can be expressed as the multiplication of elementary metrices

$$A = P_s P_{s-1} \cdots P_1,$$

thus

$$\det A = \det P_s \det P_{s-1} \cdots \det P_1$$

and

$$\det AB = \det P_s \det P_{s-1} \cdots \det P_1 \det B = \det A \det B = D_1 D_2.$$

If A is not an invertible matrix, then the $r(A) < n$ and there exists a couples of P, Q satisfying that

$$P_l P_{l-1} \cdots P_1 A Q_1 Q_2 \cdots Q_t = diag(1, \cdots, 1, 0, \cdots, 0) := \Lambda,$$

where the rank of Λ is $r(A)$, then

$$A = P_1^{-1} P_2^{-1} \cdots P_l^{-1} \Lambda Q_t^{-1} Q_{t-1}^{-1} \cdots Q_1^{-1},$$

then

$$\det \Lambda \cdots = 0$$

$$\det A = 0$$

$$\det AB = 0.$$

$\square$

Corollary 3.5.1. *If $A_i (i = 1, 2, \cdots, s)$ is n-square, then*

$$\det(A_1 A_2 \cdots A_s) = \det A_1 A_2 \cdots A_s.$$

4 Adjoint Matrix

Definition 4.1. Set A as $n \times n$ matrix, and set A_{ij} as the determinant of the algebraic cofactor of a_{ij}. Then the

$$A^* = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

is called adjoint matrix of the matrix A.

Properties 4.1. If A is a square matrix,

$$AA^* = \det AI.$$

Proof.

$$A^*A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1i} & A_{2i} & \cdots & A_{ni} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nj} & a_{nn} \end{bmatrix}$$

$$\begin{aligned} [A^*A]_{ij} &= a_{1j}A_{1i} + a_{2j}A_{2i} + \cdots + a_{nj}A_{nj} \\ &= \sum_{k=1}^{n} a_{kj}A_{ki} \\ &= \begin{cases} \det A & j = i \\ 0 & j \neq i \end{cases} \end{aligned}$$

$$A^*A = \begin{bmatrix} \det A & 0 & 0 & \cdots & 0 \\ 0 & \det A & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \det A \end{bmatrix} = \det AI$$

$\square$

Theorem 4.1. *A square matrix A is invertible if and only if its determinant is not equal to zero. When it is invertible,*

$$A^{-1} = \frac{1}{\det A} A^*.$$

Proof. **Sufficiency:** If $\det A \neq 0$, then

$$(\frac{1}{\det A} A^*)A = I$$

and $A^{-1} = \frac{1}{\det A} A^*$.

Neccessity: If A is invertible, then $AA^{-1} = I$.

$$\det AA^{-1} = \det A \det A^{-1} = 1$$

thus $\det A \neq 0$. □

5 Determinant and the rank of a Matrix

Theorem 5.1. *Set A as a n-square, then A is full rank if and only if*

$$\det A \neq 0.$$

Note. *Three conditions following are equivalent.*

1. *A is full rank.*
2. *A is invertible.*
3. $\det A \neq 0$.

Theorem 5.2. *Set A as a $m \times n$ matrix, then $r(A) = r$ if and only if there exists a r-rank minor that is not zero, and all r+1-rank minor is zero.*

Note. *The rank of the matrix equals to the highest rank of the non-zero minor.*

Proof. **Neccessity:** If $r(A) = r$, then there the first r rows of the matrix is linear irrelative and the rank of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{bmatrix}$$

is r. Then the rank of its transcope is also m. Then its m-rank minor, which is $r \times r$ square is not zero.

Select any $r+1$ rows of the matrix, then the rank of it is $r+1$, and its $r+1 \times r+1$ square minor's rank is r. Since $r < r+1$, the determinant of the $r+1 \times n$ submatrix is zero.

Then for all rows larger than r+1, the determinant of the submatrix is zero and any k-rank minor is zero($k > r$).

Sufficiency: If there exists a r-rank minor that is not zero, and all r+1-rank minor is zero. Assume that the rank of the matrix is k, since r+1-rank minor is zero, from the **Neccessity** we know that $k < r + 1$. Suppose $k < r$, then all the r-minor is zero, it contradicts to the **Neccessity**. Thus, $k \leq r$, so $k = r$. $\square$

6 Exercise

Exercise 6.1 (Vandermonde Determinant)**.**

$$d = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & a_3^n & \cdots & a_n^n \end{vmatrix}$$

is the n-order determinant of the Vandermonde matrix. Now we prove that for any $n \geq 2$, d equals to the product of these n numbers all possible differences.

Solution 6.1.1. Use the method of induction, for $k = 2$, we have

$$\begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = (a_2 - a_1).$$

Assume that for $k = n - 1$ the result keeps, then we consider the $k = n$ case.

$$\begin{aligned}
d_n &= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \\ 0 & a_2(a_2 - a_1) & a_3(a_3 - a_1) & \cdots & a_n(a_n - a_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_2^{n-2}(a_2 - a_1) & a_3^{n-2}(a_3 - a_1) & \cdots & a_n^{n-2}(a_n - a_1) \end{vmatrix} \\
&= \begin{vmatrix} a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \\ a_2(a_2 - a_1) & a_3(a_3 - a_1) & \cdots & a_n(a_n - a_1) \\ \vdots & \vdots & \ddots & \vdots \\ a_2^{n-2}(a_2 - a_1) & a_3^{n-2}(a_3 - a_1) & \cdots & a_n^{n-2}(a_n - a_1) \end{vmatrix} \\
&= (a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2} \end{vmatrix} \\
&= (a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1) \times d_{n-1}
\end{aligned}$$

Then the result holds for $k = n$. For simplicity, we write

$$d_n = \prod_{1 \le i < j \le n} (a_j - a_i).$$

Exercise 6.2. Assume that A is a invertible 4-square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Find the solution of the

$$\begin{cases} a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \end{cases}$$

Solution 6.2.1. Actually, since

$$\sum_{j=1}^{4} a_{ij}A_{ij} = \begin{cases} \det A, \ i = j \\ 0, \ i \neq j \end{cases}$$

we know that $(A_{11}, A_{12}, A_{13}, A_{14})^T$ is a solution. Since A is a invertible 4-square matrix, the rank of A is 4 and the coefficent matrix's rank is 3. Thus there is only one fundmental solution vector.

$$k(A_{11}, A_{12}, A_{13}, A_{14})^T \ for \ k \in F$$

Exercise 6.3. Calculate the determinant

$$D_n \begin{vmatrix} a_1 + b_1 & a_2 & \cdots & a_n \\ a_1 & a_2 + b_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n + b_n \end{vmatrix}, \ b_1 b_2 \cdots b_n \neq 0.$$

Solution 6.3.1.

$$
\begin{aligned}
D_n &= \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & a_1+b_1 & a_2 & \cdots & a_n \\ 0 & a_1 & a_2+b_2 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_1 & a_2 & \cdots & a_n+b_n \end{vmatrix} \\
&= \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & b_1 & 0 & \cdots & 0 \\ -1 & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & b_n \end{vmatrix}
\end{aligned}
$$