Linear Mapping and Linear Transformation

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Abstract

This is the note of Linear Mapping and Linear Transformation, maded by Len Fu while his learning progress. The main content is from *Linear Algebra Done Right* , . and *Linear Algebra Allenby*. It's also the notes from the classes of BIT.

Contents

1 Linear Mapping

To prove a mapping is not linear mapping, you just need to find a counterexample.

1.1 Mapping

Definition 1.1 (Mapping)**.** Suppose X and Y are two non-empty sets. A mapping σ from X to Y, denoted as $\sigma : X \to Y$, is a rule that assigns to each element $x \in X$ exactly one element y in the set Y. The assignment $y = \sigma(x)$ is called the image of a under the mapping σ, and x is called the preimage.

Properties 1.1 (Domain)**.** Every element x in the set X must be mapped to some element in Y.

Properties 1.2 (Uniqueness)**.** For each x in X, there is a unique y in Y such that $\sigma(x) = y$.

The set X is called the domain of the mapping σ, and the set Y is called the codomain. The image of a set, which is the set of all elements in Y that are mapped to by elements in X, denoted by as $Im(\sigma) or \sigma(X)$. , or

$$\sigma(X) = y \in Y | \exists x \in X \ such \ that \ \sigma(x) = y.$$

Definition 1.2 (Injective Mapping)**.** A mapping σ is called an *injective mapping* or an *onto mapping* if for each y in Y, there is a unique x in X such that $\sigma(x) = y$. Formally, for all $x_1, x_2 \in X$, if $\sigma(x_1) = \sigma(x_2)$, then $x_1 = x_2$.

Definition 1.3 (Surjective Mapping)**.** A mapping σ is called a *surjective mapping* or a *onto mapping* if for every y in Y, there exists an x in X such that $\sigma(x) = y$.

Definition 1.4 (Bijective Mapping)**.** A mapping σ is called a *bijective mapping* or a *onto mapping* if it is both injective and surjective.

Definition 1.5 (Product of mappings)**.** Set σ as a mapping from X to Y, and τ as a mapping from Y to Z, then we can define a new mapping $\tau \circ \sigma$ from X to Z by

$$\tau \circ \sigma(x) = \tau(\sigma(x)), \ for \ all \ x \in X.$$

1.2 Linear Mapping

Definition 1.6 (Linear Mapping)**.** Set the V_1 and V_2 as vector spaces on the field F. If a mapping τ from V_1 to V_2 keeps the adding property and the scalar multiplication property, then we say that τ is a *linear mapping* or a *linear transformation.*

$$\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta), \ \sigma(k\alpha) = k\sigma(\alpha), \ for \ any \ \alpha \ and \ \beta \in V_1, k \in \ F.$$

The necessary and sufficient condition for a Linear Mapping is

$$\sigma(k\alpha + l\beta) = k\sigma(\alpha) + l\sigma(\beta).$$

1.3 Unitary Mapping

Set V as a vector space on the field F, a mapping

$$\epsilon : V \to V$$

is defined as $\epsilon(\alpha) = \alpha,\ for\ all\ \alpha \in V$.

1.4 Zero Mapping

Set the V_1 and V_2 as vector spaces on the field F. A mapping

$$\tau : V_1 \to V_2$$

is defined as $\tau(0) = 0,\ for\ all\ \alpha \in V_1$.

1.5 Properties

If τ is a linear mapping, then it has follow properties:

Properties 1.3. $\tau(\theta) = \theta,\ \tau(-\alpha) = -\tau(\alpha)$

Properties 1.4. Linear Mappings keep the linear combination and linear coefficents unchanged.

Properties 1.5. Linear Mappings transform the linear relative vector group into another linear relative groups.

1.6 Matrix Reprentation of the Linear Mapping

Definition 1.7. Set σ as a linear mapping from V_1 to V_2, choose a basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ in the V_1 and choose a basis $\beta_1, \beta_2, \cdots, \beta_m$ in the V_2. If the image of the basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ is

$$\begin{cases} \sigma(\alpha_1) = a_{11}\beta_1 + a_{21}\beta_2 + \cdots + a_{m1}\beta_m \\ \sigma(\alpha_2) = a_{12}\beta_1 + a_{22}\beta_2 + \cdots + a_{m2}\beta_m \\ \cdots \\ \sigma(\alpha_n) = a_{1n}\beta_1 + a_{2n}\beta_2 + \cdots + a_{mn}\beta_m \end{cases}$$

and can be expressed as

$$[\sigma(\alpha_1), \sigma(\alpha_2), \cdots, \sigma(\alpha_n)] = [\beta_1, \beta_2, \cdots, \beta_m]A.$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called the linear mapping matrix of σ under the basis α and β.

Theorem 1.1. *If σ is a linear mapping from V_1 to V_2, take a basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ in the V_1, and take a basis $\beta_1, \beta_2, \cdots, \beta_m$ in the V_2, then the linear mapping matrix of σ under the basis α and β is A.*

For every $\alpha \in V$, if the coordinate of α under the basis α is $(x_1, x_2, \cdots, x_n)^T$, then the coordinate of $\sigma(\alpha)$ under the basis β is $(y_1, y_2, \cdots, y_n)^T$. Then

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Proof. Since

$$[\sigma(\alpha_1), \sigma(\alpha_2), \cdots, \sigma(\alpha_n)] = [\beta_1, \beta_2, \cdots, \beta_m]A,$$

$$\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \sigma(\alpha) = [\sigma(\beta_1), \sigma(\beta_2), \cdots, \sigma(\beta_n)] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

and

$$\sigma\alpha = \sigma([\alpha_1, \alpha_2, \cdots, \alpha_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix})$$

$$= [\sigma(\alpha_1, \alpha_2, \cdots, \alpha_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad = [\beta_1, \beta_2, \cdots, \beta_n]A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then we hava

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

□

2 Exercise

Exercise 2.1. Set $D : R[x]_{n+1} \to R[x]_n$ as *Derivative Map*, you should find the matrix reprentation of D under the basis $1, x, x^2, \cdots, x_n$ and $1, x, x^2, \cdots, x_{n-1}$.

Solution 2.1.1. Set $f_1 = 1, f_2 = x, \cdots, f_{n+1} = x^n$, then

$$D(f_1) = 0, D(f_2) = 1, D(f_3) = 2x, \cdots, D(f_{n+1}) = nx^{n-1}.$$

$$\begin{cases} D(f_1) = 0f_1 + 0f_2 + 0f_3 + \cdots + 0f_{n-1} \\ D(f_2) = 1f_1 + 0f_2 + 0f_3 + \cdots + 0f_{n-1} \\ D(f_3) = 0f_1 + 2f_2 + 0f_3 + \cdots + 0f_{n-1} \\ \vdots \\ D(f_{n+1}) = 0f_1 + 0f_2 + \cdots + nf_{n-1} \end{cases}$$

$$\begin{aligned} &[D(f_1), D(f_2), D(f_3), \cdots, D(f_{n+1})] \\ &= [0, 1, 2x, \cdots, nx^{n-1}] \\ &= [1, x, \cdots, x^{n-1}] \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix} \end{aligned}$$

Thus, the matrix representation of D under the basis $1, x, x^2, \cdots, x^n$ and basis $1, x, x^2, \cdots, x^{n-1}$ is

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

Exercise 2.2. In R^3, we form a mapping $\sigma : R^3 \to R^3$ by $\sigma[(x_1, x_2, x_3)] = (x_3, 0, x_2 - 2x_1),\ (x_1, x_2, x_3) \in R$.

1. Prove σ is a linear mapping.

2. Find the matrix representation of σ under the basis $(1, 0, 0), (1, 1, 0), (1, 1, 1)$.

Solution 2.2.1. Choose any (x_1, x_2, x_3) and $(y_1, y_2, y_3) \in R^3$, $k \in R$, Since

$$\begin{aligned} \sigma[(x_1, x_2, x_3) + (y_1, y_2, y_3)] &= \sigma[(x_1 + y_1, x_2 + y_2, x_3 + y_3)] \\ &= (x_3 + y_3, 0, x_2 + y_2 - 2(x_1 + y_1)) \\ &= (x_3, 0, x_2 - 2x_1) + (y_3, 0, y_2 - 2y_1) \\ &= \sigma[(x_1, x_2, x_3)] + \sigma[(y_1, y_2, y_3)] \\ \sigma[k(x_1, x_2, x_3)] &= \sigma[(kx_1, kx_2, kx_3)] \\ &= (kx_3, 0, kx_2 - 2kx_1) \\ &= k(x_3, 0, x_2 - 2x_1) \\ &= k\sigma[(x_1, x_2, x_3)] \end{aligned}$$

Solution 2.2.2. Choose the natrual basis of R^3, $(1, 0, 0), (0, 1, 0), (0, 0, 1)$.

$$\begin{aligned} \sigma[(1, 0, 0)] &= (0, 0, -2) \\ \sigma[(0, 1, 0)] &= (0, 0, 1) \\ \sigma[(0, 0, 1)] &= (1, 0, 0) \end{aligned}$$

and

$$\begin{cases} (0,0,-2) & = a_{11}(1,0,0) + a_{12}(1,1,0) + a_{13}(1,1,1) \\ (0,0,1) & = a_{21}(1,0,0) + a_{22}(1,1,0) + a_{23}(1,1,1) \\ (1,0,0) & = a_{31}(1,0,0) + a_{32}(1,1,0) + a_{33}(1,1,1) \end{cases}$$

$$\begin{cases} a_{11} = 0,\ a_{12} = 2,\ a_{13} = -2 \\ a_{21} = 0,\ a_{22} = -1,\ a_{23} = 1 \\ a_{31} = 1,\ a_{32} = 0,\ a_{33} = 0 \end{cases}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

That is the answer.