Continuous Functions

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Abstract

This is the note of Continous Functions, maded by Len Fu while his learning progress. The main content is from *Mathematical Analysis Tom A.Apostol* , *Basic Analysis I Jiří Lebl* the course of *Mathematical Analysis Z.H.Zhao BIT 2024 Fall*. Due to urgent the urgent time, the most proof of the propositions and theorems are quoted from *Basic Analysis I Jiří Lebl*.

In this chapter, you need to learn:

1. Limit
2. Continuity

Contents

1 Limit

1.1 Cluster Points

Definition 1.1. Let $S \subset \mathbb{R}$ be a set. A number $x \in \mathbb{R}$ is called a cluster point of S if for every $\epsilon > 0$ the set $(x - \epsilon, x + \epsilon) \cap S \setminus \{x\}$ is non-empty.

Proposition 1.1. *Let $S \subset \mathbb{R}$ be a set. Then x is a cluster point of S if and only if there exists a convergent sequence of numbers $\{x_n\}_{n=1}^{\infty}$ such that $x_n \neq x$ and $x_n \in S$ for all n, and $\lim_[n \to \infty] x_n = x$.*

Proof. To prove the sequence converges to x, we can choose $\epsilon = \frac{1}{n}$. To prove a sequece can reveal that x is the cluster point of S, emmmm, they are obviously equal. $\square$

1.2 Limit of Functions

Definition 1.2. Let $f : S \to \mathbb{R}$ be a function and c is a cluster point of $S \subset \mathbb{R}$. Suppose there exists an $L \in \mathbb{R}$ and for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in S$ such that $|x - c| < \delta$, we have

$$|f(x) - L| < \epsilon.$$

We then say $f(x)$ converges to L as x goes to c, and we write

$$f(x) \to L \; as \; x \to c.$$

We say L is a limit of $f(x)$ as x goes to c, and if L is unique(it is), we write

$$\lim_{x \to c} f(x) := L.$$

If no such L exists, then we say that the limit does not exist or that f diverges at c.

Proposition 1.2. *Let c be a cluster point of $S \subset \mathbb{R}$ and let $f : S \to \mathbb{R}$ be a functino such that $f(x)$ converges as x goes to c. Then the limit of $f(x)$ as x goes to c is unique.*

Proof. Let L_1 and L_2 be two numbers that both satisfy the definition. Take an $\epsilon > 0$ and find a $\delta_1 > 0$ such that $|f(x) - L_1| < \epsilon/2$ for all $x \in S \setminus \{c\}$ with $|x - c| < \delta_1$. Also find $\delta_2 > 0$ such that $|f(x) - L_2| < \epsilon/2$ for all $x \in S \setminus \{c\}$ with $|x - c| < \delta_2$. Then we have

$$|f(x) - L_1| < \epsilon/2 \; and \; |f(x) - L_2| < \epsilon/2,$$

put $\delta = min(\delta_1, \delta_2)$, we have

$$|f(x) - L_1| < \epsilon/2 \; and \; |f(x) - L_2| < \epsilon/2, \; for \; all \; x \in S \setminus \{c\} \; with \; |x - c| < \delta$$

. Then we have

$$|L_1 - L_2| < |f(x) - L_1| + |f(x) - L_2| < \epsilon, \; and \; L_1 = L_2$$

$\square$

Definition 1.3 (Infinite Limit)**.** Let $f : S \to \mathbb{R}$ be a function and suppose S has ∞ as a cluster point. We say $f(x)$ diverges to infinity as x goes to ∞ if for every $N \in \mathbb{R}$ there exists an $M \in \mathbb{R}$ such that

$$f(x) > N$$

whenever $x \in S$ and $x \geq M$. We write

$$\lim_{x \to \infty} f(x) := \infty,$$

or we say that $f(x) \to \infty$ as $x \to \infty$.

Proposition 1.3 (Compositions)**.** *Suppose $f : A \to B, g : B \to \mathbb{R}, A, B \subset \mathbb{R}, a \in \mathbb{R} \cup \{-\infty, \infty\}$ is a cluster point of A, and $b \in \mathbb{R} \cup \{-\infty, \infty\}$ is a cluster point of B. Suppose*

$$\lim_{x \to a} f(x) = b \quad \textit{and} \quad \lim_{y \to b} g(y) = c$$

for some $c \in \mathbb{R} \cup \{-\infty, \infty\}$. If $b \in B$, then suppose $g(b) = c$. Then

$$\lim_{x \to a} g(f(x)) = c.$$

1.3 Sequential Limits

Lemma 1.4. *Let $S \subset \mathbb{R}$, let c be a cluster point of S, let $f : S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$. Then $f(x) \to L$ as $x \to c$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S \setminus \{c\}$ for all n, and such that $\lim_{n \to \infty} x_n = c$, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.*

Corollary 1.5. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are functions such that the limits of $f(x)$ and $g(x)$ as x goes to c both exist, and*

$$f(x) \leq g(x) \quad \textit{for all}\, x \in S \setminus \{c\}.$$

Then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Corollary 1.6. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as x goes to c exists. Suppose there are two real numbers a and b such that*

$$a \leq f(x) \leq b \quad \textit{for all}\, x \in S \setminus \{c\}.$$

Then

$$a \leq \lim_{x \to c} f(x) \leq b.$$

Corollary 1.7. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$, and $h : S \to \mathbb{R}$ are functions such that*

$$f(x) \leq g(x) \leq h(x) \quad \textit{for all}\, x \in S \setminus \{c\}.$$

Suppose the limits of $f(x)$ and $h(x)$ as x goes to c both exist, and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x).$$

Then the limit of $g(x)$ as x goes to c exists and

$$\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x).$$

Corollary 1.8. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are functions such that the limits of $f(x)$ and $g(x)$ as x goes to c both exist. Then*

(i) $\lim_{x\to c}(f(x) + g(x)) = (\lim_{x\to c} f(x)) + (\lim_{x\to c} g(x))$.

(ii) $\lim_{x\to c}(f(x) - g(x)) = (\lim_{x\to c} f(x)) - (\lim_{x\to c} g(x))$.

(iii) $\lim_{x\to c}(f(x)g(x)) = (\lim_{x\to c} f(x)) (\lim_{x\to c} g(x))$.

(iv) If $\lim_{x\to c} g(x) \neq 0$ and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Corollary 1.9. *Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ is a function such that the limit of $f(x)$ as x goes to c exists. Then*

$$\lim_{x\to c} |f(x)| = \left|\lim_{x\to c} f(x)\right|.$$

1.4 Limits of restrictions and one-sided limits

Definition 1.4. Let $f : S \to \mathbb{R}$ be a function and $A \subset S$. Define the function $f|_A : A \to \mathbb{R}$ by

$$f|_A(x) := f(x) \quad \text{for } x \in A.$$

We call $f|_A$ the restriction of f to A.

Proposition 1.10. *Let $S \subset \mathbb{R}$, $c \in \mathbb{R}$, and let $f : S \to \mathbb{R}$ be a function. Suppose $A \subset S$ is such that there is some $\alpha > 0$ such that*

$$(A \setminus \{c\}) \cap (c - \alpha, c + \alpha) = (S \setminus \{c\}) \cap (c - \alpha, c + \alpha).$$

(i) The point c is a cluster point of A if and only if c is a cluster point of S.

(ii) Supposing c is a cluster point of S, then $f(x) \to L$ as $x \to c$ if and only if $f|_A(x) \to L$ as $x \to c$.

Proof. To prove (i), let c be a cluster point of A. Since $A \subset S$, then if

$$(A \setminus \{c\}) \cap (c - \epsilon, c + \epsilon)$$

is non-empty for all $\epsilon > 0$, then $(S \setminus \{c\}) \cap (c - \epsilon, c + \epsilon)$ is non-empty. Thus c is a cluster point of S. If c is not a cluster point of S, for $\epsilon < \alpha$, we get that $(A \setminus \{c\}) \cap (c - \epsilon, c + \epsilon) = (S \setminus \{c\}) \cap (c - \epsilon, c + \epsilon)$ This is true for all $\varepsilon < \alpha$ and hence $(A \setminus \{c\}) \cap (c - \varepsilon, c + \varepsilon)$ must be nonempty for all $\varepsilon > 0$. Thus c is a cluster point of A.

Now suppose c is a cluster point of S and $f(x) \to L$ as $x \to c$. That is, for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$. Because $A \subset S$, if $x \in A \setminus \{c\}$, then $x \in S \setminus \{c\}$, and hence $f|_A(x) \to L$ as $x \to c$.

Finally, suppose $f|_A(x) \to L$ as $x \to c$ and let $\epsilon > 0$ be given. There is a $\delta' > 0$ such that if $x \in A \setminus \{c\}$ and $|x - c| < \delta'$, then $|f|_A(x) - L| < \varepsilon$. Take $\delta := \min\{\delta', \alpha\}$. Now suppose $x \in S \setminus \{c\}$ and $|x - c| < \delta$. As $|x - c| < \alpha$, we find $x \in A \setminus \{c\}$, and as $|x - c| < \delta'$, we get

$$|f(x) - L| = |f|_A(x) - L| < \varepsilon.$$

The hypothesis on A in the proposition is necessary. For an arbitrary restriction we generally get an implication in only one direction, see Exercise 3.1.6. The usual notation for the limit is

$$\lim_{\substack{x \to c \\ x \in A}} f(x) := \lim_{x \to c} f|_A(x).$$

A common use of restriction with respect to limits, which does not satisfy the hypothesis in the proposition, is the so-called one-sided limit*. $\square$

Definition 1.5. Let $f : S \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$. If c is a cluster point of $S \cap (c, \infty)$ and the limit of the restriction of f to $S \cap (c, \infty)$ as $x \to c$ exists, define

$$\lim_{x \to c^+} f(x) := \lim_{x \to c} f|_{S \cap (c,\infty)}(x).$$

Similarly, if c is a cluster point of $S \cap (-\infty, c)$ and the limit of the restriction as $x \to c$ exists, define

$$\lim_{x \to c^-} f(x) := \lim_{x \to c} f|_{S \cap (-\infty,c)}(x).$$

Proposition 1.11. *Let $S \subset \mathbb{R}$ be such that c is a cluster point of both $S \cap (-\infty, c)$ and $S \cap (c, \infty)$, let $f : S \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$. Then c is a cluster point of S and*

$$\lim_{x \to c} f(x) = L \quad \text{if and only if} \quad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

That is, a limit at c exists if and only if both one-sided limits exist and are equal. The proof is a straightforward application of the definition of limit and is left as an exercise. The key point is that

$$(S \cap (-\infty, c)) \cup (S \cap (c, \infty)) = S \setminus \{c\}.$$

1.5 Limits at Infinity

Definition 1.6. We say ∞ is a cluster point of $S \subset \mathbb{R}$ if for every $M \in \mathbb{R}$, there exists an $x \in S$ such that $x \geq M$. Similarly, $-\infty$ is a cluster point of $S \subset \mathbb{R}$ if for every $M \in \mathbb{R}$, there exists an $x \in S$ such that $x \leq M$.

Let $f : S \to \mathbb{R}$ be a function, where ∞ is a cluster point of S. If there exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that

$$|f(x) - L| < \varepsilon$$

whenever $x \in S$ and $x \geq M$, then we say $f(x)$ converges to L as x goes to ∞. We call L the limit and write

$$\lim_{x \to \infty} f(x) := L.$$

Alternatively we write $f(x) \to L$ as $x \to \infty$.

Similarly, if $-\infty$ is a cluster point of S and there exists an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is an $M \in \mathbb{R}$ such that

$$|f(x) - L| < \varepsilon$$

whenever $x \in S$ and $x \leq M$, then we say $f(x)$ converges to L as x goes to $-\infty$. Alternatively, we write $f(x) \to L$ as $x \to -\infty$. We call L a limit and, if unique, write

$$\lim_{x \to -\infty} f(x) := L.$$

Proposition 1.12. *The limit at ∞ and $-\infty$ aas defined above is unique if it exists.*

Lemma 1.13. *Suppose $f : S \to \mathbb{R}$ is a function, ∞ is a cluster point of $S \subset \mathbb{R}$, and $L \in \mathbb{R}$. Then*

$$\lim_{x \to \infty} f(x) = L \quad \textit{if and only if} \quad \lim_{n \to \infty} f(x_n) = L$$

for all sequences $\{x_n\}_{n=1}^{\infty}$ in S such that $\lim_{n \to \infty} x_n = \infty$.

Proof. First suppose $f(x) \to L$ as $x \to \infty$. Given an $\varepsilon > 0$, there exists an M such that for all $x \geq M$, we have $|f(x) - L| < \varepsilon$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in S such that $\lim_{n \to \infty} x_n = \infty$. Then there exists an N such that for all $n \geq N$, we have $x_n \geq M$. And thus $|f(x_n) - L| < \varepsilon$.

We prove the converse by contrapositive. Suppose $f(x)$ does not go to L as $x \to \infty$. This means that there exists an $\varepsilon > 0$, such that for every $n \in \mathbb{N}$, there exists an $x \in S$, $x \geq n$, let us call it x_n, such that $|f(x_n) - L| \geq \varepsilon$. Consider the sequence $\{x_n\}_{n=1}^{\infty}$. Clearly $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L. It remains to note that $\lim_{n \to \infty} x_n = \infty$, because $x_n \geq n$ for all n. $\square$

2 Continuity

2.1 Basis

Definition 2.1. Suppose $S \subset \mathbb{R}$ and $c \in S$. We say $f : S \to \mathbb{R}$ is *continuous at c* if for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in S$ and $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$. When $f : S \to \mathbb{R}$ is continuous at all $c \in S$, then we simply say f is a *continuous function*.

Proposition 2.1. *Consider a function $f : S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}$ and let $c \in S$. Then:*

1. *If c is not a cluster point of S, then f is continuous at c.*
2. *If c is a cluster point of S, then f is continuous at c if and only if the limit of $f(x)$ as $x \to c$ exists and*

$$\lim_{x \to c} f(x) = f(c).$$

3. *The function f is continuous at c if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n \in S$ and $\lim_{n \to \infty} x_n = c$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(c)$.*

Proof. We start with (i). Suppose c is not a cluster point of S. Then there exists a $\delta > 0$ such that $S \cap (c - \delta, c + \delta) = \{c\}$. For any $\epsilon > 0$, simply pick this given δ. The only $x \in S$ such that $|x - c| < \delta$ is $x = c$. Then $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$.

Let us move to (ii). Suppose c is a cluster point of S. Let us first suppose that $\lim_{x \to c} f(x) = f(c)$. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that if $x \in S \setminus \{c\}$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Also $|f(c) - f(c)| = 0 < \epsilon$, so the definition of continuity at c is satisfied. On the other hand, suppose f is continuous at c. For every $\epsilon > 0$, there exists a $\delta > 0$ such that for $x \in S$ where $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$. Then the statement is, of course, still true if $x \in S \setminus \{c\} \subset S$. Therefore, $\lim_{x \to c} f(x) = f(c)$.

For (iii), first suppose f is continuous at c. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \in S$ and $\lim_{n \to \infty} x_n = c$. Let $\epsilon > 0$ be given. Find a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in S$ where $|x - c| < \delta$. Find an $M \in \mathbb{N}$ such that for $n \geq M$, we have $|x_n - c| < \delta$. Then for $n \geq M$, we have that $|f(x_n) - f(c)| < \epsilon$, so $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(c)$.

We prove the other direction of (iii) by contrapositive. Suppose f is not continuous at c. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$, there exists an $x \in S$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ as follows. Let $x_n \in S$ be such that $|x_n - c| < 1/n$ and $|f(x_n) - f(c)| \geq \epsilon$. Now $\{x_n\}_{n=1}^{\infty}$ is a sequence in S such that $\lim_{n \to \infty} x_n = c$ and such that $|f(x_n) - f(c)| \geq \epsilon$ for all $n \in \mathbb{N}$. Thus $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(c)$. It may or may not converge, but it definitely does not converge to $f(c)$. $\square$

Proposition 2.2. *Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be functions continuous at $c \in S$.*

(i) The function $h : S \to \mathbb{R}$ defined by $h(x) := f(x) + g(x)$ is continuous at c.

(ii) The function $h : S \to \mathbb{R}$ defined by $h(x) := f(x) - g(x)$ is continuous at c.

(iii) The function $h : S \to \mathbb{R}$ defined by $h(x) := f(x)g(x)$ is continuous at c.

(iv) If $g(x) \neq 0$ for all $x \in S$, the function $h : S \to \mathbb{R}$ given by $h(x) := \frac{f(x)}{g(x)}$ is continuous at c.

Proposition 2.3. *Let $A, B \subset \mathbb{R}$ and $f : B \to \mathbb{R}$ and $g : A \to B$ be functions. If g is continuous at $c \in A$ and f is continuous at $g(c)$, then $f \circ g : A \to \mathbb{R}$ is continuous at c.*

2.2 Discontinuous Functions

Proposition 2.4. *Let $f : S \to \mathbb{R}$ be a function and $c \in S$. Suppose there exists a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in S$ for all n, and $\lim_{n \to \infty} x_n = c$ such that $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(c)$. Then f is discontinuous at c.*

2.3 Uniform Continuity

Definition 2.2. Let $S \subset \mathbb{R}$, and let $f : S \to \mathbb{R}$ be a function. Suppose for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x, c \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Then we say f is *uniformly continuous*.

Theorem 2.5. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous.*

Proof. We prove the statement by contrapositive. Suppose f is not uniformly continuous. We will prove that there is some $c \in [a,b]$ where f is not continuous. Let us negate the definition of uniformly continuous. There exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exist points x, y in $[a,b]$ with $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

So for the $\varepsilon > 0$ above, we find sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $|x_n - y_n| < 1/n$ and such that $|f(x_n) - f(y_n)| \geq \varepsilon$. By Bolzano-Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Let $c := \lim_{k\to\infty} x_{n_k}$. As $a \leq x_{n_k} \leq b$ for all k, we have $a \leq c \leq b$. Estimate

$$|y_{n_k} - c| = |y_{n_k} - x_{n_k} + x_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| < 1/n_k + |x_{n_k} - c|.$$

As $1/n_k$ and $|x_{n_k} - c|$ both go to zero when k goes to infinity, $\{y_{n_k}\}_{k=1}^{\infty}$ converges and the limit is c. We now show that f is not continuous at c. Estimate

$$\begin{aligned} |f(x_{n_k}) - f(c)| &= |f(x_{n_k}) - f(y_{n_k}) + f(y_{n_k}) - f(c)| \\ &\geq |f(x_{n_k}) - f(y_{n_k})| - |f(y_{n_k}) - f(c)| \\ &\geq \varepsilon - |f(y_{n_k}) - f(c)|. \end{aligned}$$

Or in other words,

$$|f(x_{n_k}) - f(c)| + |f(y_{n_k}) - f(c)| \geq \varepsilon.$$

At least one of the sequences $\{f(x_{n_k})\}_{k=1}^{\infty}$ or $\{f(y_{n_k})\}_{k=1}^{\infty}$ cannot converge to $f(c)$, otherwise the left-hand side of the inequality would go to zero while the right-hand side is positive. Thus f cannot be continuous at c. □

2.3.1 Continuous Extension

Lemma 2.6. *Let $S \subset \mathbb{R}$ and let $f : S \to \mathbb{R}$ be a uniformly continuous function. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in S. Then $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy.*

Proof. Let $\varepsilon > 0$ be given. There is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in S$ and $|x - y| < \delta$. Find an $M \in \mathbb{N}$ such that for all $n, k \geq M$, we have $|x_n - x_k| < \delta$. Then for all $n, k \geq M$, we have $|f(x_n) - f(x_k)| < \varepsilon$. □

Proposition 2.7. *function $f : (a,b) \to \mathbb{R}$ is uniformly continuous if and only if the limits*

$$L_a := \lim_{x\to a} f(x) \quad \textit{and} \quad L_b := \lim_{x\to b} f(x)$$

exist and the function $\widetilde{f} : [a,b] \to \mathbb{R}$ defined by

$$\widetilde{f}(x) := \begin{cases} f(x) & \textit{if } x \in (a,b), \\ L_a & \textit{if } x = a, \\ L_b & \textit{if } x = b \end{cases}$$

is continuous.

Proof. First, if L_a exists, then $\lim_{x\to a} \widetilde{f}(x)$ exists and equal to L_a. Similarly, if L_b exists, then $\lim_{x\to b} \widetilde{f}(x)$ exists and equal to L_b. Then

Now, we prove the other direction. If $\widetilde{f}(x)$ is defined as above and , then

$\square$

2.3.2 Lipschitz Continuity

Definition 2.3. A function $f : S \to \mathbb{R}$ is Lipschitz continuous*, if there exists a $K \in \mathbb{R}$, such that

$$|f(x) - f(y)| \leq K|x - y| \quad \text{for all } x \text{ and } y \text{ in } S.$$

Proposition 2.8. *A Lipschitz continuous function is uniformly continuous.*

Proof. Let $f : S \to \mathbb{R}$ be a function and let K be a constant such that $|f(x) - f(y)| \leq K|x - y|$ for all x, y in S. Let $\varepsilon > 0$ be given. Take $\delta := \varepsilon/K$. For all x and y in S such that $|x - y| < \delta$,

$$|f(x) - f(y)| \leq K|x - y| < K\delta = K\frac{\varepsilon}{K} = \varepsilon.$$

Therefore, f is uniformly continuous. $\square$

3 Extreme and Intermediate Value Theorems

3.1 Min-Max or Extreme Value Theorem

Lemma 3.1. *A continuous function $f : [a, b] \to \mathbb{R}$ is bounded.*

Proof. We prove the claim by contrapositive. Suppose f is not bounded. Then for each $n \in \mathbb{N}$, there is an $x_n \in [a, b]$, such that

$$|f(x_n)| \geq n.$$

The sequence $\{x_n\}_{n=1}^{\infty}$ is bounded as $a \leq x_n \leq b$. By the Bolzano-Weierstrass theorem, there is a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$. Let $x := \lim_{i\to\infty} x_{n_i}$. Since $a \leq x_{n_i} \leq b$ for all i, then $a \leq x \leq b$. The sequence $\{f(x_{n_i})\}_{i=1}^{\infty}$ is not bounded as $|f(x_{n_i})| \geq n_i \geq i$. Thus f is not continuous at x as

$$f(x) = f\left(\lim_{i\to\infty} x_{n_i}\right), \quad \text{but} \quad \lim_{i\to\infty} f(x_{n_i}) \text{ does not exist.}$$

$\square$

Theorem 3.2. *A continuous function $f : [a, b] \to \mathbb{R}$ achieves both an absolute minimum and an absolute maximum on $[a, b]$.*

Proof. The lemma says that f is bounded, so the set $f([a, b]) = \{f(x) : x \in [a, b]\}$ has a supremum and an infimum. There exist sequences in the set $f([a, b])$ that approach its supremum and its infimum. That is, there are sequences $\{f(x_n)\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$, where x_n and y_n are in $[a, b]$, such that

$$\lim_{n\to\infty} f(x_n) = \inf f([a, b]) \quad \text{and} \quad \lim_{n\to\infty} f(y_n) = \sup f([a, b]).$$

We are not done yet; we need to find where the minima and the maxima are. The problem is that the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ need not converge. We know $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are bounded (their elements belong to a bounded interval $[a,b]$). Apply the Bolzano-Weierstrass theorem, to find convergent subsequences $\{x_{n_i}\}_{i=1}^{\infty}$ and $\{y_{m_i}\}_{i=1}^{\infty}$. Let

$$x := \lim_{i\to\infty} x_{n_i} \quad \text{and} \quad y := \lim_{i\to\infty} y_{m_i}.$$

As $a \leq x_{n_i} \leq b$ for all i, we have $a \leq x \leq b$. Similarly, $a \leq y \leq b$. So x and y are in $[a,b]$. A limit of a subsequence is the same as the limit of the sequence, and we can take a limit past the continuous function f:

$$\inf f([a,b]) = \lim_{n\to\infty} f(x_n) = \lim_{i\to\infty} f(x_{n_i}) = f\left(\lim_{i\to\infty} x_{n_i}\right) = f(x).$$

Similarly,

$$\sup f([a,b]) = \lim_{n\to\infty} f(y_n) = \lim_{i\to\infty} f(y_{m_i}) = f\left(\lim_{i\to\infty} y_{m_i}\right) = f(y).$$

Hence, f achieves an absolute minimum at x and an absolute maximum at y. □

3.2 Bolzano's Intermediate Value Theorem

Lemma 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) < 0$ and $f(b) > 0$. Then there exists a number $c \in (a,b)$ such that $f(c) = 0$.*

Theorem 3.4 (Bolzano's intermediate value theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Suppose $y \in \mathbb{R}$ is such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$. Then there exists a $c \in (a,b)$ such that $f(c) = y$.*

Corollary 3.5. *If $f : [a,b] \to \mathbb{R}$ is continuous, then the direct image $f([a,b])$ is a closed and bounded interval or a single number.*

Proposition 3.6. *Let $f(x)$ be a polynomial of odd degree. Then f has a real root.*

4 Exercise

Exercise 4.1 (Dirichlet function)**.**

$$f(x) = \begin{cases} 1, \ x \ is \ rational \\ 0, \ x \ isirrational \end{cases}$$

The function is discontinuous at all $c \in \mathbb{R}$.

Solution 4.1.1. If c is rational, take a sequence $\{x_n\}_{n=1}^{\infty}$ of rational numbers such that $\lim_{n\to\infty} x_n = c$. Then $f(x_n) = 0$ and $\lim_{n\to\infty} f(x_n) = 0$, but $f(c) = 1$. And vise versa.

Exercise 4.2 (3.1.7)**.** Find an example of a function $f : [-1, 1] \to \mathbb{R}$, where for $A := [0, 1]$, we have $f|_A(x) \to 0$ as $x \to 0$, but the limit of $f(x)$ as $x \to 0$ does not exist. Note why you cannot apply Proposition 3.1.15.

Solution 4.2.1 (3.1.7)**.**

$$f(x) = \begin{cases} x, \ x \in [0, 1] \\ \sin \frac{1}{x}, \ x \in [-1, 0) \end{cases}$$

Exercise 4.3 (3.1.8)**.** Find example functions f and g such that the limit of neither $f(x)$ nor $g(x)$ exists as $x \to 0$, but such that the limit of $f(x) + g(x)$ exists as $x \to 0$.

Solution 4.3.1 (3.1.8)**.**

$$f(x) = \begin{cases} x, \ x \in [0, 1] \\ \sin \frac{1}{x}, \ x \in [-1, 0) \end{cases} \qquad g(x) = \begin{cases} x, \ x \in [0, 1] \\ -\sin \frac{1}{x}, \ x \in [-1, 0) \end{cases}$$

Exercise 4.4 (3.1.9)**.** Let c_1 be a cluster point of $A \subset \mathbb{R}$ and c_2 be a cluster point of $B \subset \mathbb{R}$. Suppose $f : A \to B$ and $g : B \to \mathbb{R}$ are functions such that $f(x) \to c_2$ as $x \to c_1$ and $g(y) \to L$ as $y \to c_2$. If $c_2 \in B$, also suppose that $g(c_2) = L$. Let $h(x) := g(f(x))$ and show $h(x) \to L$ as $x \to c_1$. Hint: Note that $f(x)$ could equal c_2 for many $x \in A$, see also Exercise 3.1.14.

Solution 4.4.1 (3.1.9)**.** Set $\epsilon > 0$, there exists a $\delta' > 0$ such that for all $y \in B$ such that $|y - c_2| < \delta'$, we have $|g(y) - L| < \epsilon$. And we can have that for $\delta' > 0$, there exists $\delta > 0$ such that for all $x \in A$ such that $|x - c_1| < \delta$, we have $|f(x) - c_2| < \delta'$. Then we have for $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ such that $|x - c_1| < \delta$, we have $|f(x) - c_2| < \delta'$ and $|g(f(x)) - L| < \epsilon$.

Exercise 4.5 (3.1.10)**.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ be a function such that for every sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. Prove that f is constant, that is, $f(x) = f(y)$ for all $x, y \in \mathbb{R}$.

Solution 4.5.1 (3.1.10)**.** Now we construct a divergent sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_{2n-1} = a, x_{2n} = b, a \neq b \ a, b \in \mathbb{R}$, then $\{f(x_{2n-1})\}_{n=1}^{\infty}$ converges to $f(a)$ and $\{f(x_{2n})\}_{n=1}^{\infty}$ converges to $f(b)$. If $f(a) \neq f(b)$, $\{f(x_n)\}_{n=1}^{\infty}$ is divergent, contradictint with the assumption. Then we know that $a = b$. Then for $a, b \in \mathbb{R}$, we have that $f(x) = f(a) = f(b)$, so f is constant.

Exercise 4.6 (3.2.3)**.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational,} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

Using the definition of continuity, directly prove that f is continuous at 1 and discontinuous at 2.

Solution 4.6.1 (3.2.3)**.** For f at 1, for $\epsilon > 0$, there exists $\delta = \epsilon > 0$ for rational x, such that $|x-1| < \delta$ and $|f(x)-1| = |x-1| < \epsilon$. For $\delta' > 0$, for irrational x, we have $|x-1| < \delta'$, so $|f(x)-1| = |x^2-1| = |(x+1)(x-1)| < |\delta'(\delta'+2)| < |\delta'$. If $\epsilon > 2$, we set $\delta' = 1/2$, then we have $|f(x)-1| < 1.5 < \epsilon$. If $\epsilon \leq 2$, we set $\delta' = \epsilon/3 < 2/3 < 1$, then we have $|f(x)-1| < \epsilon/3 * (2+1) < \epsilon$. So f is continuous at 1.

For f at 2, for $\epsilon > 0$, there exists $\delta = \epsilon > 0$ for rational x, such that $|x-2| < \delta$ and $|f(x)-2| = |x-2| < \epsilon$. For $\delta' > 0$, for irrational x, we can have an x such that $2 < x < 2+\delta'$, then $|f(x)-f(2)| = |x^2-2| > 2$, so f is discontinuous at 2.

Exercise 4.7 (3.2.4)**.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

Solution 4.7.1 (3.2.4)**.**

Exercise 4.8. 3.2.4: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

Solution 4.8.1 (3.2.4)**.** Consider the sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n = \frac{1}{(2n+1/2)\pi}$ for $n = 1, 2, 3, \ldots$. Then $f(x_n) = \sin(1/x_n) = 1$, don't converge to $f(0) = 0$. So f is not continuous at 0.

Exercise 4.9. 3.2.5: Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Is f continuous? Prove your assertion.

Solution 4.9.1 (3.2.5)**.**

$$\begin{aligned} -1 \leq \sin(1/x) \leq 1, &\quad x \neq 0 \\ -x \leq x\sin(1/x) \leq x, &\quad x =\neq 0 \\ \lim_{x\to 0}(-x) \leq \lim_{x\to 0} f(x) \leq \lim_{x\to}(x) & \\ 0 \leq \lim_{x\to 0} f(x) \leq 0 & \end{aligned}$$

That is $\lim_{x\to 0} f(x) = f(0)$. So f is continuous at 0.

Exercise 4.10. 3.2.6: Prove Proposition 3.2.5.

Solution 4.10.1 (3.2.6)**.** You just need to use the properties of limits and you can prove it easily.

But here to prove it, we prove it from the definition.

(i) $|h(x) - h(c)| = |f(x) - f(c) + g(x) - g(c)| \leq |f(x) - f(c)| + |g(x) - g(c)|$. Then you can follow the normal process to prove it.

(ii) same as (i)

(iii) $|h(x) - h(c)| = |f(x)g(x) - f(c)g(c)| = |f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)| \leq |f(x)(g(x) - g(c)) + (f(x) - f(c))g(c)| \leq |f(x)||(g(x) - g(c))| + |(f(x) - f(c))||g(c)|$. As we know that $f(x)$ is continuous at c, thus it is bounded around $f(c)$, like this, for $\epsilon_1 > 0$, we can find a $delta_1 > 0$, for all $|x - c| < \delta_1$, we have $|f(x)| < max|f(c) - \epsilon_1|, |f(c) + \epsilon_1|$, we denote the last term as M. Then we can have that

$$|h(x) - h(c)| < M||(g(x) - g(c))| + |(f(x) - f(c))||g(c)| < M|(g(x) - g(c))| + \epsilon_1|g(c)|.$$

For $\epsilon_2 > 0$, we can find a $delta_2 > 0$, for all $|x - c| < \delta_2$, we have $|g(x) - g(c)| < \epsilon_2$, then we have that for $|x - c| < min(\delta_1, \delta_2)$, we have that

$$|h(x) - h(c)| < M\epsilon_2 + \epsilon_1|g(c)| < |f(c)|\epsilon_2 + \epsilon_1|g(c)| + \epsilon_1\epsilon_2.$$

Hence for all $\epsilon > 0$, we can find a pair of $\epsilon_1, \epsilon_2 > 0$ such that

$$|f(c)|\epsilon_2 + \epsilon_1|g(c)| + \epsilon_1\epsilon_2 < \epsilon,$$

and for each ϵ_1, ϵ_2, we can find $\delta_1, \delta_2 > 0$, and then $\delta = min(\delta_1, \delta_2)$ for all $|x - c| < \delta$, we have

$$|h(x) - h(c)| < \epsilon.$$

Then $h(x)$ is continuous at c.

(iv) Simiar as (iii), $|h(x) - h(c)| < |\frac{(f(x)-f(c))g(c)+f(c)(g(x)-g(c))}{g(c)g(x)}| < |\frac{(f(x)-f(c))g(c)}{g(c)g(x)}| + \frac{f(c)(g(x)-g(c))}{g(c)g(x)}$, then the left process is nothing defferent with (iii).

Exercise 4.11. 3.2.9: Give an example of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that the function h, defined by $h(x) := f(x) + g(x)$, is continuous, but f and g are not continuous. Can you find f and g that are nowhere continuous, but h is a continuous function?

Solution 4.11.1 (3.2.9)**.**

$$f(x) = \begin{cases} 1, \ x \ is \ rational \\ 0, \ x \ is \ irrational \end{cases}$$

$$g(x) = \begin{cases} 0, \ x \ is \ rational \\ 1, \ x \ is \ irrational \end{cases}$$

Exercise 4.12. 3.2.10: Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Solution 4.12.1 (3.2.10)**.** Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then we for all irrational number x, we can find a rationl sequence $\{x_n\}_[n = 1]^\infty$ converges to x, that is

$$\lim_{n\to\infty} x_n = x,$$

then we have that $\lim_{n\to\infty} f(x_n) = f(x)$ since $f(x)$ is a continuous function. It is the same for the $g(x)$. Then we have that $\lim_{n\to\infty} g(x_n) = g(x)$. As we know that $f(y) = g(y)$ for y is rational, then we have that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n)$. Obviously, $f(x) = g(x)$ for all irrational x. Combine two conditions, we know that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Exercise 4.13. 3.2.11: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $f(c) > 0$. Show that there exists an $\alpha > 0$ such that for all $x \in (c - \alpha, c + \alpha)$, we have $f(x) > 0$.

Solution 4.13.1 (3.2.11)**.** Since $f(x)$ is continuous, for $x = c$, we have that for every $\epsilon > 0$, we have a $\delta > 0$, for all $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$. Then $f(c) - \epsilon < f(x) < f(c) + \epsilon$, we choose $\epsilon < f(c)$ then $f(x) > 0$. That is there exists $\alpha > 0$ satisfying the condition.

Exercise 4.14. 3.2.12: Let $f : \mathbb{Z} \to \mathbb{R}$ be a function. Show that f is continuous.

Solution 4.14.1 (3.2.12)**.** Actually, for every $\epsilon > 0$, there exists $\delta = 1$, for all $|x - c| < \delta(\mathbb{Z}$ is continuous), we have $|f(x) - f(c)| < \epsilon$. Then $f(x)$ is continuous.

Exercise 4.15. 3.2.13: Let $f : S \to \mathbb{R}$ be a function and $c \in S$, such that for every sequence $\{x_n\}_{n=1}^\infty$ in S with $\lim_{n\to\infty} x_n = c$, the sequence $\{f(x_n)\}_{n=1}^\infty$ converges. Show that f is continuous at c.

Solution 4.15.1 (3.2.13)**.** It's a strange problem, can I prove that the sequence must converge to $f(c)$?

Exercise 4.16. 3.2.14: Suppose $f : [-1, 0] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are continuous and $f(0) = g(0)$. Define $h : [-1, 1] \to \mathbb{R}$ by $h(x) := f(x)$ if $x \le 0$ and $h(x) := g(x)$ if $x > 0$. Show that h is continuous.

Solution 4.16.1 (3.2.14)**.** For $x = 0$, $h(x) = 0$, $\lim_{x\to 0^-} h(x) = 0$, $\lim_{x\to 0^+} h(x) = 0$, then h is continuous at 0. For $x \in (0, 1]$, $h(x) = g(x)$, $g(x)$ is continuous in $(0, 1]$, then h is continuous in $(0, 1]$. For $x \in [-1, 0)$, $h(x) = f(x)$, $f(x)$ is continuous in $[-1, 0)$, then h is continuous in $[-1, 0)$.

Exercise 4.17. 3.2.15: Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$, and suppose $f : \mathbb{R} \to \mathbb{R}$ is such that $|f(x) - f(y)| \le g(x - y)$ for all x and y. Show that f is continuous.

Solution 4.17.1 (3.2.15)**.** As $x \to y$, we have $0 \le \lim_{x\to y} |f(x) - f(y)| \le \lim_{x\to y} g(x - y) = 0$. Then $\lim_{x\to y} f(x) = f(y)$ for all x, y, then f is continuous.

Exercise 4.18 (3.3.4)**.** Let

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f has the intermediate value property. That is, whenever $a < b$, if there exists a y such that $f(a) < y < f(b)$ or $f(a) > y > f(b)$, then there exists a $c \in (a, b)$ such that $f(c) = y$.

Solution 4.18.1 (3.3.4)**.** For $ab > 0$, $f(x)$ is continuous in $[a, b]$ and it is obvious.

For $ab < 0$, why not suppose that $a < 0$ and $b > 0$, then we consider the interval $(0, b)$. Actually, near the 0, $f(x)$ oscillates between $[-1, 1]$ infinity. No matter how we choose b, the range of $f((0, b))$ is always $[-1, 1]$. Thus there at least exists a c such that $f(c) = y$.

Exercise 4.19 (3.3.12)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $x \leq f(x) \leq x + 1$ for all $x \in \mathbb{R}$. Find $f(\mathbb{R})$.

Solution 4.19.1 (3.3.12)**.** $\mathbb{R}$

Exercise 4.20 (3.3.13)**.** *True/False, prove or find a counterexample.* If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f|_{\mathbb{Z}}$ is bounded, then f is bounded.

Solution 4.20.1 (3.3.13)**.** False. For example, $f(x) = \sin(\pi x) \ln(x^2 + 1)$ is continuous, but f is not bounded.

Exercise 4.21 (3.3.14)**.** Suppose $f : [0, 1] \to (0, 1)$ is a bijection. Prove that f is not continuous

Solution 4.21.1 (3.3.14)**.** If $f(x)$ is continuos and a bijection, then $f(x)$ should hava a minimum and maximum. But $(0, 1)$ doesn't have a maximum and minimum. Then f is not continuous.

Exercise 4.22 (3.3.15)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous.

a) Prove that if there is a c such that $f(c)f(-c) < 0$, then there is a $d \in \mathbb{R}$ such that $f(d) = 0$.

b) Find a continuous function f such that $f(\mathbb{R}) = \mathbb{R}$, but $f(x)f(-x) \geq 0$ for all $x \in \mathbb{R}$.

Solution 4.22.1 (3.3.15)**.** a is easy to prove. b is $f(x) = x^3$.

Exercise 4.23 (3.3.16)**.** Suppose $g(x)$ is a monic polynomial of even degree d, that is,

$$g(x) = x^d + b_{d-1}x^{d-1} + \cdots + b_1 x + b_0,$$

for some real numbers $b_0, b_1, \ldots, b_{d-1}$. Show that g achieves an absolute minimum on $\mathbb{R}$.

Solution 4.23.1 (3.3.16)**.** Since $g(x)$ tends to $+\infty$ as $x \to -\infty$ and $x \to \infty$. Consider $g(0)$, there exists an $N > 0$, for all $x \geq N$, $g(x) > g(0)$, and there exists an $M < 0$, for all $x \leq M$, $g(x) > g(0)$. And in the interval $[M, N]$, there exists a minima $g(x_0)$. For $x \in [M, N]$, $g(x) \geq g(x_0)$. For $x \in (-\infty, M)$, $g(x) > g(0) \geq g(x_0)$. For $x \in (N, \infty)$, $g(x) > g(0) \geq g(x_0)$. Then $g(x_0)$ is the absolute minimum.

Exercise 4.24 (3.4.7)**.** Let $f : (0, 1) \to \mathbb{R}$ be a bounded continuous function. Show that the function $g(x) := x(1 - x)f(x)$ is uniformly continuous.

Solution 4.24.1 (3.4.7)**.** Since $f(x)$ is bounded, then $\lim_{x \to 0} f(x)$ or $\lim_{x \to 1} f(x)$ don't go to infinity. Then we denote them as L_0 and L_1, then we extend $f(x)$ into $[0, 1]$, it is uniformly continuous. And for $x(1 - x)$ is uniformly continuous, then $g(x)$ is uniformly continuous.

Exercise 4.25 (3.4.8)**.** Show that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) := \sin(1/x)$ is not uniformly continuous.

Solution 4.25.1 (3.4.8)**.** For $\epsilon = 1/2$, there exists $\delta > 0$, we choose $x = \frac{1}{2n\pi}, y = \frac{1}{(2n+1/2)\pi}$, it's obvious that

$$|x - y| = \frac{1/2}{2n(2n + 1/2)\pi}$$

if there exists a δ, then no matter how small it is, we can always find that there exists a N such that for $n \geq N$,$|x - y| < \delta$, satisfying the condition. Then we know that

$$|f(x) - f(y)| = 1 > \epsilon$$

So f is not uniformly continuous.

Exercise 4.26 (3.4.10)**.** a) Find a continuous $f : (0, 1) \to \mathbb{R}$ and a sequence $\{x_n\}_{n=1}^{\infty}$ in $(0, 1)$ that is Cauchy, but such that $\{f(x_n)\}_{n=1}^{\infty}$ is not Cauchy.

b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $\{x_n\}_{n=1}^{\infty}$ is Cauchy, then $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy.

Solution 4.26.1 (3.4.10)**.** a. $f(x) = 1/x$, $x_n = \frac{1}{n}$, then $f(x_n) = n$, but $f(x_n)$ is not Cauchy.

b. Since $\{x_n\}_{n=1}^{\infty}$ is Cauchy, it is bounded and we denote them as L,U. Then we can extend the domain of $f(x)$ into a closed interval and $f(x)$ is uniformly continuous. Then $f(x_n)$ is Cauchy.

Exercise 4.27 (3.4.11)**.** a) If $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are uniformly continuous, then $h : S \to \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is uniformly continuous.

b) If $f : S \to \mathbb{R}$ is uniformly continuous and $a \in \mathbb{R}$, then $h : S \to \mathbb{R}$ given by $h(x) := af(x)$ is uniformly continuous.

Solution 4.27.1 (3.4.11)**.** There is nothing defferent with the proof of normal continuity

Exercise 4.28 (3.4.12)**.** Prove:

a) If $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are Lipschitz, then $h : S \to \mathbb{R}$ given by $h(x) := f(x) + g(x)$ is Lipschitz.

b) If $f : S \to \mathbb{R}$ is Lipschitz and $a \in \mathbb{R}$, then $h : S \to \mathbb{R}$ given by $h(x) := af(x)$ is Lipschitz.

Solution 4.28.1 (3.4.12)**.** Emmmmmm, it's nothing defferent with the proof of normal continuity.

Exercise 4.29 (3.5.2)**.** Let $f : [1, \infty) \to \mathbb{R}$ be a function. Define $g : (0, 1] \to \mathbb{R}$ via $g(x) := f(1/x)$. Using the definitions of limits directly, show that $\lim_{x \to 0^+} g(x)$ exists if and only if $\lim_{x \to \infty} f(x)$ exists, in which case they are equal.

Solution 4.29.1 (3.5.2)**.** Suppose that $\lim_{x \to 0^+} g(x)$ exists and equals to L, then for $\epsilon > 0$, there exists an $\delta > 0$, for all $x < \delta$, $|g(x) - L| < \epsilon$ that is $|f(1/x) - L| < \epsilon$. Now set $y = 1/x$, for all $y > 1/\delta$, $|f(y) - L| < \epsilon$, then $\lim_{x \to \infty} f(x) = L$. Vise versa, you can prove that $\lim_{x \to 0^+} g(x) = \lim_{x \to \infty} f(x)$ if they exist.

Exercise 4.30 (3.5.7)**.** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Consider $S := \mathbb{N} \subset \mathbb{R}$, and $f : S \to \mathbb{R}$ defined by $f(n) := x_n$. Show that the two notions of limit,

$$\lim_{n \to \infty} x_n \quad \text{and} \quad \lim_{x \to \infty} f(x)$$

are equivalent. That is, show that if one exists so does the other one, and in this case they are equal.

Solution 4.30.1 (3.5.7)**.** If $\lim_{x\to\infty} f(x) = L$ exists, then for all $\epsilon > 0$, there exists $N > 0$, for all $x \geq N$, $|f(x) - L| < \epsilon$. Since $f(n) = x_n$, then for all $n \geq N$, $|x_n - L| < \epsilon$. And we have $\lim_{n\to\infty} x_n = L$.

If $\lim_{n\to\infty} x_n = L$ exists, then for all $\epsilon > 0$, there exists $N > 0$, for all $n \geq N$, $|x_n - L| < \epsilon$. Since, $f(n) = x_n$, then for all $n \geq N$, $|f(n) - L| < \epsilon$, that is $\lim_{x\to\infty} f(x) = L$.