Determinant

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Abstract

This is the note made by Len Fu during his learning progress in BIT. The main content is from *Linear Algebra Done Right* and *Linear Algebra Allenby*.

Contents

1 Permutation

1.1 N Permutation

An n-permutation is an arrangement of all the numbers $1, 2, ..., n$. The total number of n-permutations is $n!$.

1.2 Inversion and Inversion Number

Formally, for a sequence a_n with elements a_i and a_j with $i < j$, an inversion is present if $a_i > a_j$.

The inversion number of a permutation is the number of inversions in it.

A permutation with an odd inversion number is called an odd permutation. And a permutation with an even inversion number is called an even permutation.

1.3 Transposition

Formally, a transposition is a permutation that exchanges two elements and leaves all others unchanged. For example, in permutation $\tau = (1, 2, 3, 4, 5)$ the transposition $(1, 2)$ exchanges 1 and 2. Then the resulting permutation is $\tau = (2, 1, 3, 4, 5)$.

1.3.1 Theorem

A transposition changes the parity of a permutation. **Proof:**
Let $\tau = (i_1 i_2 ... i_j i_{j+1} ... i_n)$, and we exchange i_j and i_{j+1}, then the remain permutation $(i_1 i_2 ... i_j ... i_n)$ and $(i_1 i_2 ... i_{j+1} ... i_n)$ keep the same parity. But the paritr of $(i_j i_{j+1})$ changes, so the total parity of τ changes.

Now consider that if the transposition is between $(i_j i_k)$ like $(...j i_1 i_2 ... i_s k ...)$, then we first transpose s times to set j into i_s like $(...i_1 i_2 ... i_s j k ...)$. And we transpose j and k $(...i_1 i_2 ... i_s k j ...)$, then we transpose s times to set k into i_1 like $(...k i_1 i_2 ... i_s j ...)$. The total transposition is $2s + 1$. So the parity of the permutation changes.

Corollary 1.0.1. *In all n permutation, the number of even permutation is equal to the number of odd permutation, which is* $\frac{n!}{2}$.

Proof. Suppose there are s odd permutation, then there are t even permutation. Now transpose the first two elements of all even permutation, then we get s odd permutation. Then $s \leq t$, conversly, transpose the first two elements of all odd permutation, then we get t even permutation, and $t \leq s$. So $s = t = \frac{n!}{2}$. □

Corollary 1.0.2. *For*

$$a_{i_1 k_1} a_{i_2 k_2} \cdots a_{i_n k_n} \;=\; a_{1 j_1} a_{2 j_2} \cdots \; a_{n j_n}$$

the inversion number is

$$(-1)^{\tau(i_1 i_2 \cdots i_n) + \tau k_1 k_2 \cdots k_n} = (-1)^{\tau j_1 j_2 \cdots j_n}.$$

Proof. □

1.3.2 Theorem

Any n-permutation can be transposed from $(123...n)$ and the times of transposition equals to the inversion number of the permutation.

2 N-Order Determinant

2.1 Definition

2.1.1 n-order Determinant

The n-order determinant is a scalar value that can be computed from the elements of a square matrix of size $n \times n$.

Actually, it can be written abstract

$$\begin{aligned}
\det|A| &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
&= \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \\
&= \sum_{i_1 i_2 \cdots i_n} (-1)^{\tau(i_1 i_2 \cdots i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \\
&= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}
\end{aligned}$$

2.1.2 Minor

The minor of an element in a matrix is the determinant of the submatrix formed by deleting the row and column that contain the element. That is

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \cdots & & & \cdots & & & \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{in} \\ \cdots & & & \cdots & & & \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \cdots & & & \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

2.2 Expansion along a row or a column

Theorem 2.1. *The determinant of a n-order matrix equals to any row's or column's element multiplied by its algebraic cofactor. , like*

$$\det = \sum_{j=1}^{n} a_{ij} A_{ij} \textit{ for } i = 1, 2, \cdots, n = \sum_{i=1}^{n} a_{ij} A_{ij} \textit{ for } j = 1, 2, \cdots, n.$$

Proof. The factor is

$$(-1)^{i+j}(-1)^{\tau(j_1 j_2 \cdots j_{i-1} j_{i+1} \cdots j_n)} \; where \; (j_1 j_2 \cdots j_{i-1} j_{i+1} \cdots j_n) \; isapermutationof (12 \cdots j-1, j+1 \cdots n).$$

then

$$\begin{aligned} &(-1)^{i+j}(-1)^{\tau(j_1 j_2 \cdots j_{i-1} j_{i+1} \cdots j_n)} \\ &= (-1)^{(i-1)+(j-1)+\tau(j_1 j_2 \cdots j_{i-1} j_{i+1} \cdots j_n)} \\ &= (-1)^{\tau(i12 \cdots i-1, i+1 \cdots n)+(j-1)+\tau(j_1 j_2 \cdots j_{i-1} j_{i+1} \cdots j_n)} \end{aligned}$$

□

2.2.1 Algebraic Cofactor(Cofactor)

The algebraic cofactor of an element in a matrix is the product of the minor of the element and $(-1)^{i+j}$, where i is the row number and j is the column number of the element. We denote it by

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

2.2.2 k-minor k-cofactor and k-algebraic cofactor

A k-minor of a matrix is the determinant of a square submatrix obtained by **selecting** k rows and k columns from the original nn matrix, where n is the dimension of the matrix.

A k-order principle minor is the determinant of a square submatrix obtained by **selecting** k rows and k columns from the original nn matrix, where n is the dimension of the matrix and $i_l = j_l \; for \; l = 1, 2, \cdots, k$.

A k-cofactor of a matrix is a determinant that the original determinant deleting the k-minor that remains as the follow order.

A k-algebraic cofactor of a matrix is a product of $(-1)^{i_1, i_2, \cdots, i_k; j_1, j_2, \cdots, j_k}$ and the k-cofactor.

2.3 Properties

1. Transcope the matrix, the determinant does not change.

2.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

3.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1+c_1 & b_2+c_2 & \cdots & b_n+c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 & b_2 & \cdots & b_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ c_1 & c_2 & \cdots & c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

4. If there are two rows or columns that are the same, the determinant is 0. is 0.

5. If there are two rows or columns are proportionable, the determinant is 0.

6. Add a row's or a column's k-times into another one, the determinant keeps the same.

7. Exchange two rows or columns, the determinant changes its sign.

Proof.

1. If we transcope the determinant,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \rightarrow \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & & & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

like this, then we expand the right one with respect to the rows like this

$$\sum_{i_1 i_2 \cdots i_n} (-1)^{\tau(i_1 i_2 \cdots i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$$

Actually it keeps from the left one.

2.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k\left(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}\right)$$

$$= k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

3.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (b_1 + c_1)A_{i1} + (b_2 + c_2)A_{i2} + \cdots + (b_n + c_n)A_{in}$$

$$= (b_1A_{i1} + b_2A_{i2} + \cdots + b_nA_{in}) + (c_1A_{i1} + c_2A_{i2} + \cdots + c_nA_{in})$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 & b_2 & \cdots & b_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ c_1 & c_2 & \cdots & c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

4.

$det = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau()}$ **5.**

3 Cramer Rule

Theorem 3.1 (Cramer Rule)**.** *If the system of linear equations'*

$$Ax = b$$

coefficent matrix

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

its determinant is $d = |A| \neq 0$, then the system of linear equations has solution, and the solution is unique, and can be expressed as

$$x_1 = \frac{d_1}{d},\ x_2 = \frac{d_2}{d},\ \cdots,\ x_n = \frac{d_n}{d}.$$

where d_i is the determinant of the matrix whose the j-th column is replaced by the constant column vector b.

$$d_j = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & \cdots & a_{in} \\ \cdots & & \cdots & \cdots & \cdots & \\ a_{i1} & \cdots & a_{i,j-1} & b_i & \cdots & a_{in} \\ \cdots & & \cdots & \cdots & \cdots & \\ a_{n1} & \cdots & a_{n,j-1} & b_n & \cdots & a_{nn} \end{vmatrix},\ j = 1, 2, \cdots, n.$$

There are three results inside the theorem:

1. The system of linear equations has solution.
2. The solution is unique.
3. The solution is expressed as the formula.

Proof. □

Theorem 3.2. *If the homogenous system of linear equations' coefficent matrix's determinant $|A| \neq 0$, then it only has zero solution. On other word, if the homogenous system of linear equations has non-zero solution, then $|A| = 0$ is certainly.*

Proof. Using the Cramer Rule, we have $d_j = 0,\ j = 1, 2, \cdots, n$. That is to say,

$$(0, 0, \cdots, 0)$$

is its unique solution. □

4 Laplace Theorem

Lemma 4.0.1.

Theorem 4.1 (Laplace Theorem)**.** *If the system of linear equations*

Theorem 4.2 (Product Rule)**.** *Suppose there are two matrix A and B, and there determinant is $D_1 = |A|,\ D_2 = |B|$. Then the determinant of the product of A and B is*

$$C = D_1 D_2.$$

5 Adjoint Matrix

Definition 5.1. Set A as $n \times n$ matrix, and set A_{ij} as the determinant of the algebraic cofactor of a_{ij}. Then the

$$A^* = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

is called adjoint matrix of the matrix A.

Properties 5.1. If A is a square matrix,

$$AA^* = \det AI.$$

Proof.

$$A^*A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1i} & A_{2i} & \cdots & A_{ni} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nj} & a_{nn} \end{bmatrix}$$

$$\begin{aligned} [A^*A]_{ij} &= a_{1j}A_{1i} + a_{2j}A_{2i} + \cdots + a_{nj}A_{nj} \\ &= \sum_{k=1}^{n} a_{kj}A_{ki} \\ &= \begin{cases} \det A & j = i \\ 0 & j \neq i \end{cases} \end{aligned}$$

$$A^*A = \begin{bmatrix} \det A & 0 & 0 & \cdots & 0 \\ 0 & \det A & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \det A \end{bmatrix} = \det AI$$

□

Theorem 5.1. *A square matrix A is invertible if and only if its determinant is not equal to zero. When it is* *invertible,*

$$A^{-1} = \frac{1}{\det A} A^*.$$

Proof. **Sufficiency:** If $\det A \neq 0$, then

$$(\frac{1}{\det A} A^*)A = I$$

and $A^{-1} = \frac{1}{\det A} A^*$.

Neccessity: If A is invertible, then $AA^{-1} = I$.

$$\det AA^{-1} = \det A \det A^{-1} = 1$$

thus $\det A \neq 0$. □

6 Determinant and the rank of a Matrix

Theorem 6.1. *Set A as a n-square, then A is full rank if and only if*

$$\det A \neq 0.$$

Note. Three conditions following are equivalent.

1. A is full rank.
2. A is invertible.
3. $\det A \neq 0$.

Theorem 6.2. *Set A as a $m \times n$ matrix, then $r(A) = r$ if and only if there exists a r-rank minor that is not zero, and all r+1-rank minor is zero.*

Note. The rank of the matrix equals to the highest rank of the non-zero minor.

Proof. **Neccessity:** If $r(A) = r$, then there the first r rows of the matrix is linear irrelative and the rank of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rn} \end{bmatrix}$$

is r. Then the rank of its transcope is also m. Then its m-rank minor, which is $r \times r$ square is not zero.

Select any $r + 1$ rows of the matrix, then the rank of it is $r + 1$, and its $r + 1 \times r + 1$ square minor's rank is r. Since $r < r + 1$, the determinant of the $r + 1 \times n$ submatrix is zero.

Then for all rows larger than r+1, the determinant of the submatrix is zero and any k-rank minor is zero($k > r$).

Sufficiency: If there exists a r-rank minor that is not zero, and all r+1-rank minor is zero. Assume that the rank of the matrix is k, since r+1-rank minor is zero, from the **Neccessity** we know that $k < r + 1$. Suppose $k < r$, then all the r-minor is zero, it contradicts to the **Neccessity**. Thus, $k \leq r$, so $k = r$. $\square$

7 Exercise

Exercise 7.1 (Vandermonde Determinant)**.**

$$d = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & a_3^n & \cdots & a_n^n \end{vmatrix}$$

is the n-order determinant of the Vandermonde matrix. Now we prove that for any $n \geq 2$, d equals to the product of these n numbers all possible differences.

Solution 7.1.1. Use the method of induction, for $k = 2$, we have

$$\begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = (a_2 - a_1).$$

Assume that for $k = n - 1$ the result keeps, then we consider the $k = n$ case.

$$\begin{aligned}
d_n &= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \\ 0 & a_2(a_2 - a_1) & a_3(a_3 - a_1) & \cdots & a_n(a_n - a_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_2^{n-2}(a_2 - a_1) & a_3^{n-2}(a_3 - a_1) & \cdots & a_n^{n-2}(a_n - a_1) \end{vmatrix} \\
&= \begin{vmatrix} a_2 - a_1 & a_3 - a_1 & \cdots & a_n - a_1 \\ a_2(a_2 - a_1) & a_3(a_3 - a_1) & \cdots & a_n(a_n - a_1) \\ \vdots & \vdots & \ddots & \vdots \\ a_2^{n-2}(a_2 - a_1) & a_3^{n-2}(a_3 - a_1) & \cdots & a_n^{n-2}(a_n - a_1) \end{vmatrix} \\
&= (a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2} \end{vmatrix} \\
&= (a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1) \times d_{n-1}
\end{aligned}$$

Then the result holds for $k = n$. For simplicity, we write

$$d_n = \prod_{1 \leq i < j \leq n} (a_j - a_i).$$

Exercise 7.2. Assume that A is a invertible 4-square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Find the solution of the

$$\begin{cases} a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = b_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = b_4 \end{cases}$$

Solution 7.2.1. Actually, since

$$\sum_{j=1}^{4} a_{ij}A_{ij} = \begin{cases} \det A, \ i = j \\ 0, \ i \neq j \end{cases}$$

we know that $(A_{11}, A_{12}, A_{13}, A_{14})^T$ is a solution. Since A is a invertible 4-square matrix, the rank of A is 4 and the coefficent matrix's rank is 3. Thus there is only one fundmental solution vector.

$$k(A_{11}, A_{12}, A_{13}, A_{14})^T \ for \ k \in F$$