Eigenvalue and Eigenvector

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Solution 0.1 (0.3.6)**.** For the first one, we suppose for all $x \in (A \cap B) \cup (A \cap C)$, then x is in A and in B or C. Now we consider two cases:

1. If x is in A and B, then $x \in A \cap B$ and $x \in A \cap C$, then $x \in (A \cap B) \cup (A \cap C)$.
2. If x is in A and C, then $x \in A \cap B$ and $x \in A \cap C$, then $x \in (A \cap B) \cup (A \cap C)$.

Thus we have $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Then we consider all $x \in A \cap (B \cup C)$, that x is in A or in B and C.

1. If x is in A, then $x \in A \cap B$ and $x \in A \cap C$, then $x \in (A \cap B) \cup (A \cap C)$.
2. If x is in B and C, then $x \in A \cap B$ and $x \in A \cap C$, then $x \in (A \cap B) \cup (A \cap C)$.

Thus we have $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

For the second one, we suppose for all $x \in (A \cup B) \cap (A \cup C)$, then x is in A and B or in A and C.

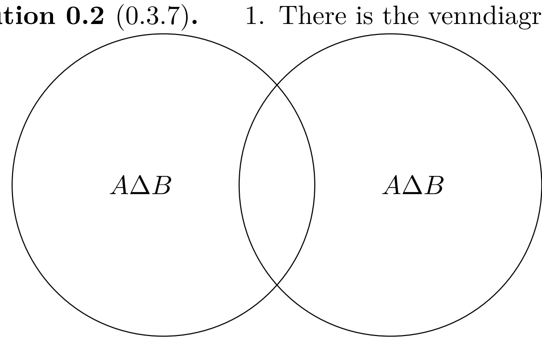
1. If x is in A and B, then $x \in A$, then $x \in A \cup (B \cap C)$.
2. If x is in A and C, then $x \in A$, then $x \in A \cup (B \cap C)$.

Thus we have $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Then we consider all $x \in A \cup (B \cap C)$, that x is in A or in B and C.

1. If x is in A, then $x \in A \cup B$ and $x \in A \cup C$, then $x \in (A \cup B) \cap (A \cup C)$.
2. If x is in B and C, then $x \in B$ and $x \in C$, then $x \in A \cup B$ and $x \in A \cup C$, then $x \in (A \cup B) \cap (A \cup C)$.

Thus we have $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution 0.2 (0.3.7)**.** 1. There is the venndiagram of the $A \Delta B$:



2. For all $x \in A \Delta B$, then x is in A or B but not in A and B.

   (a) If x is in A but not in B and A, then $x \in A \setminus B$ and $x \in (A \setminus B) \cup (B \setminus A)$.

   (b) If x is in B but not in B and A, then $x \in B \setminus A$ and $x \in (A \setminus B) \cup (B \setminus A)$.

   Thus we have $A \Delta B \subseteq (A \setminus B) \cup (B \setminus A)$. Inversely, for all $x \in (A \setminus B) \cup (B \setminus A)$, then :

   (a) If x is in $A \setminus B$, then $x \in A$ and $x \notin B$, then $x \in A \Delta B$.

   (b) If x is in $B \setminus A$, then $x \in B$ and $x \notin A$, then $x \in A \Delta B$.

Thus we have $(A \setminus B) \cup (B \setminus A) \subseteq A\Delta B$. Thus we have $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

3. For all $x \in A\Delta B$, then x is in A or B but not in A and B.

   (a) If x is in A but not in B and A, then $x \in A \cup B$ and $x \notin (A \cap B)$, then $x \in (A \cup B) \setminus (A \cap B)$.

   (b) If x is in B but not in B and A, then $x \in A \cup B$ and $x \notin (A \cap B)$, then $x \in (A \cup B) \setminus (A \cap B)$.

   Thus we have $A\Delta B \subseteq (A \cup B) \setminus (A \cap B)$. Inversely, for all $x \in (A \cup B) \setminus (A \cap B)$, then x is in A but not in B and A, or x is in B but not in B and A, then $x \in A\Delta B$. Thus we have $(A \cup B) \setminus (A \cap B) \subseteq A\Delta B$. Thus we have $A\Delta B = (A \cup B) \setminus (A \cap B)$.

Solution 0.3 (0.3.8)**.** (a) $\{6k | k \in \mathbb{N}\}$.

(b) $\{k | k \in \mathbb{N} \; but \; k \neq 1\}$

(c) $\{0\}$

Solution 0.4 (0.3.14)**.** Using principle of induction, we have:
Basis statement: For n=1, left side is 1, right side is 1, thus the statement is true.
Induction step: Suppose the statement is true for $n = k$, then we have:

$$1^3 + 2^3 + \cdots + k^3 = (\frac{k(k+1)}{2})^2.$$

Now consider $n = k + 1$, since

$$\begin{aligned}
(\frac{(k+1)(k+1+1)}{2})^2 - (\frac{k(k+1)}{2})^2 = \frac{k^4 + 6k^3 + 13k^2 + 2k + 4 - (k^4 + 2k^3 + k^2)}{4} &= k^3 + 3k^2 + 3K + 1 \\
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (\frac{k(k+1)}{2})^2 + (\frac{(k+1)(k+1+1)}{2})^2 - (\frac{k(k+1)}{2})^2 \\
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (\frac{(k+1)(k+2)}{2})^2
\end{aligned}$$

Thus we have the statement is true for $n = k + 1$, thus the statement is true for all $n \in \mathbb{N}$.

Solution 0.5 (0.3.15)**.** Using principle of induction, we have:
Basis statement: For n=1, $n^3 + 5n \equiv 0 (mod \; 6)$.
Induction step: Suppose the statement is true for $n = k$, then we have:

$$k^3 + 5k \equiv 0 (mod \; 6).$$

Now consider $n = k + 1$, since

$$\begin{aligned} & (k+1)^3 + 5(k+1) \\ =& k^3 + 5k + 6 + 3k(k+1) \\ & k^3 + 5k + 6 + 3k(k+1)(mod\ 6) \\ \equiv& (k^3 + 5k)(mod\ 6) + 0 + 3k(k+1)(mod\ 6) \\ \equiv& 0 + 3k(k+1)(mod\ 6) \\ \equiv& k(k+1)(mod\ 2) \end{aligned}$$

Actually $k(k+1) \equiv 0(mod\ 2)$ for all $k \in \mathbb{N}$, then we have

$$(k+1)^3 + 5(k+1) \equiv 0(mod\ 6).$$

Thus we have the statement is true for $n = k + 1$, thus the statement is true for all $n \in \mathbb{N}$.

Solution 0.6 (0.3.16)**.** Define $f(n) = n^3 - 2(n+5)^2,\ n \in \mathbb{N}$, and suppose n_0 is the smallest integer such that $f(n) > 0$.Using principle of induction, we have:

Basis statement: For $n = n_0$, $f(n) > 0$, the statement is ture.

Induction step: Suppose the statement is true for $n = k$, then we have:

$$f(k) = k^3 - 2k^2 - 20k - 50 > 0.$$

Now consider $n = k + 1$, since

$$\begin{aligned} f(k+1) =& (k+1)^3 - 2(k+1)^2 - 20(k+1) - 50 \\ =& k^3 + 3k^2 + 3k + 1 - 2k^2 - 4k - 2 - 20k - 20 - 50 \\ =& k^3 - 2k^2 - 20k - 50 + 3k^2 - k - 21 \end{aligned}$$

Obviously, $3 < n_0$ and $3 < k$, and $3k^2 - k - 21 > 3 * 3^2 - 3 - 21 = 3 > 0$, thus

$$f(k+1) > 0 + 0 = 0.$$

Thus we have the statement is true for $n = k + 1$, thus the statement is true for all $n \geq n_0$.

Solution 0.7 (0.3.17)**.**

$$\{n | n \geq 2\ and\ n \in \mathbb{N}\}$$

Solution 0.8 (0.3.18)**.** Using principle of strong induction, we consider a subset S_n of $\mathbb{N}$ with n elements, now:"

Basis statement: For $n = 1$, the element is the smallest element, then the statement is true for n=1.

Strong induction step: Suppose the statement is true for $n = k$, then there exists a smallest element x_k in S_k, and now we insert an element a_{k+1} into the set S_k as S_{k+1}. And x_k or a_{k+1} is the smallest element, then there exists a smallest element in S_{k+1},then the statement is true for $n = k + 1$.