Eigenvalue and Eigenvector

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Abstract

This is the note made by Len Fu during his learning progress in BIT. The main content is from *Linear Algebra Done Right* and *Linear Algebra Allenby*.

Contents

1 Similarity of the Matrix

1.1 Basis

Definition 1.1. Set $A, B \in C^{n\times n}$. If there exists an n-order invertible matrix P such that

$$P^{-1}AP = B$$

, we say that A and B are similar, denoted as $A \sim B$, and P is called the *similarity transformation* from A to B.

Properties 1.1 (Reflectivity)**.** $A \sim A$.

Properties 1.2 (Symmetry)**.** If $A \sim B$, then $B \sim A$.

Properties 1.3 (Transitivity)**.** If $A \sim B$ and $B \sim C$, then $A \sim C$.

Properties 1.4.

1. $P^{-1}(A_1 + A_2 + \cdots + A_n)P = P^{-1}A_1P + P^{-1}A_2P + \cdots + P^{-1}A_nP = P^{-1}\sum_{i=1}^{n} P^{-1}A_iP$.
2. $P^{-1}(kA)P = kP^{-1}AP$.

1.2 Conditions of Similar Digonalizablity

Definition 1.2 (Digonalizable)**.** If there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where A is a square and D is a diagonal matrix. Then A is called *diagonalizable*.

Theorem 1.1. *A n-square A is similar diagonalizable if and only if A has n linear irrelative eigenvectors.*

Proof.

$$P^{-1}AP = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$
$$AP = Pdiag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

Now set $P = [X_1, X_2, \cdots, X_n]$, and

$$[AX_1, AX_2, \cdots, AX_n] = [\lambda_1 X_1, \lambda_2 X_2, \cdots, \lambda_n X_n]$$

then we have

$$(A - \lambda_i)X_i = 0, \ for \ i = 1, 2, \cdots, n.$$

Since P is invertible, we can find n linear irrelative vectors $X_1, X_2, \cdots, X_n$. And $X_1, X_2, \cdots, X_n$ are n linear irrelative eigenvectors of A and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A.

Inversely, if A has n linear irrelative eigenvectors $X_1, X_2, \cdots, X_n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, satisfying that

$$AX_i = \lambda_i X_i, \ for \ i = 1, 2, \cdots, n.$$

Set $P = [X_1, X_2, \cdots, X_n]$ and obviously P is invertible, and

$$P^{-1}AP = diag\lambda_1, \lambda_2, \cdots, \lambda_n$$

which reveals that A is similar diagonalizable. □

2 Eigenvalue and Eigenvector of the Matrix

2.1 Basis

Definition 2.1. Set A as a $n \times n$ square, if there exists a number λ and $n - nonzero$ vector X, satisfying

$$AX = \lambda X \ or \ (\lambda I - A)X = 0$$

then we say that λ is an eigenvalue of A, and X is an eigenvector of A with eigenvalue λ.

Note.

1. Only squares have eigenvectors and eigenvalues.

2. Eigenvector must be nonvector and eigenvalue can be zero.

Definition 2.2. $(\lambda I - A)$ is the eigrnmatrix of A. $|\lambda I - A|$ is the eigenpolynomial of A. $|\lambda I - A| = 0$ is the eigenequation of the matrix A.

Then the eigenvector of A with eigenvalue λ is the combination of the solution vectors of $(\lambda I - A)X = 0$.

Since $(\lambda I - A)X = 0$ and X is nonzero vector, then $\det(\lambda I - A)$ should be zero to ensure X is nonzero vector of the solution.

Consider the solution of $(\lambda I - A)X = 0$. The characteristic polynomial of A is

$$b_n\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0.$$

To solve the polynomial,

$$\begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

$$= b_n\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0$$

Consider the expasion of the determinant, except for

$$(\lambda - a_{11})(\lambda - a_{22})\cdots(\lambda - a_{nn})$$

other terms' highest order of λ is $n-2$. Then the coefficents

$$\begin{cases} b_n = 1 \\ b_{n-1} = -(a_{11} + a_{22} + \cdots + a_{nn}) = tr() \end{cases}$$

And we divide the determinant into two parts and one is

$$\begin{vmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

the other one doesn't contribute to the b_0, thus

$$b_0 = (-1)^n |A|.$$

From the polynomial theorem,

$$f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = 0.$$

Then

$$b_0 = (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n \ b_{n-1} = -(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$$

And there are some properties of the **Eigenvalue and Eigenvector**.

Properties 2.1.
1. $|A| = \lambda_1 \lambda_2 \cdots \lambda_n$.
2. $trA = \sum_{i=1}^{n} \lambda_i$
3. If $X_1, X_2, \cdots, X_s$ are eigenvectors of A that belong to eigenvalue λ_0, then the linear combination of $X_1, X_2, \cdots, X_s$ is also an eigenvector of A that belongs to eigenvalue λ_0. And all eigenvectors plus zero vector forms an **eigenspace** of A with eigenvalue λ_0, denoted as V_{λ_0} and it's a solution space of $(\lambda_0 I - A)X = 0$.

Properties 2.2. If λ is an eigenvalue of A with eigenvector X, then we have

1. $k\lambda$ is the eigenvalue of kA.
2. λ^m is the eigenvalue of $A^m (m \in N^*)$.
3. $f(\lambda)$ is the eigenvalue of $f(A)$ if f is a polynomial transformation.

4. When A is invertible, λ^{-1} is the eigenvalue of A^{-1}

And X is the eigenvector of matrices above with corresponding eigenvalue.

Properties 2.3. The matrix A and A^T have the same **spectrum**.

3 Exercise

Exercise 3.1. The eigenvalues of A are $1, 2, 3$, find the eigenvalues of $A^2 - 2I$.

Solution 3.1.1. We know that $A \sim diag(1, 2, 3)$ and then $A^2 \sim diag(1, 4, 9)$. Then $(A^2 - 2I) \sim (diag(1, 4, 9) - 2I) = diag(-1, 2, 7)$, then the eigenvalues of $A^2 - 2I$ are $-1, 2, 7$.

Exercise 3.2. Sovle the maxima of the

$$f(x_1, x_2) = 2x_1^2 + 2x_1x_2 + 2x_2^2$$

under the constraint $x_1^2 + x_2^2 = 1$ using **Lagrange Multipliers**.

Solution 3.2.1. Using Lagrange Multipliers,

$$L(x_1, x_2) = 2x_1^2 + 2x_1x_2 + 2x_2^2 - \lambda(x_1^2 + x_2^2 - 1)$$

and