Infinite Series and Infinite Products

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Abstract

This is the note of Infinite Series and Infinite Products, maded by Len Fu while his learning progress. The main content is from *Mathematical Analysis Tom A.Apostol*.

Contents

1 Convergence and Divergence

1.1 Basis

Definition 1.1 (Convergence)**.** A sequence of complex numbers $a_n \in C$ is called *convergent* if,

$$for\ every\ \epsilon > 0,\ there\ exists\ an\ N \in \mathbb{N}\ such\ that,\ |a_n - a| < \epsilon\ for\ all\ n \geq N.$$

If a_n converges to p, we write $\lim_{n \to \inf} a_n = p$ and call p the limit of the sequence. A sequence is called divergent if it is not convergent.

Definition 1.2 (Divergence)**.** A sequence of complex numbers $a_n \in C$ is called *divergent* if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that,

$$|a_n - a| \leq \epsilon\ for\ all\ n \geq N.$$

In this case we write $\lim_{n \to \infty} a_n = +\infty$.

1.2 Cauchy Sequence

Definition 1.3. A sequence in $\mathbb{C}$ is called a *Cauchy sequence* if it satisfies the *Cauchy condition*: for every $\epsilon > 0$ there is an integer N such that

$$|a_n - a_m| < \epsilon\ whenevern \geq N\ and\ m \geq N.$$

Obviously, being Cauchy Sequence means that the terms of the sequence are all arbitarily close to each other.

The Cauchy condition is particularly useful in establishing convergence when we do not know the actual value to which the sequence converges.

Proposition 1.1. *If a sequence is a Cauchy sequence, then it is bounded.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ is Cauchy.Pick an M such that for $n, k \geq M$, we have $|x_n - x_k| < 1$. In particular for all $n \geq M$,

$$|x_n - x_M| \geq 1.$$

Then we use the triangle inequality to obtain:

$$|x_n| - |x_M| \geq |x_n - x_M| < 1$$

then for all $n \geq M$,

$$|x_n| < 1 + |x_M|.$$

Now set

$$B := \max\{|x_1|, |x_2|, \cdots, |x_{M-1}|, 1 + |x_M|\}$$

Then B is an upper bound for the absolute sequence and it is bounded. □

Theorem 1.2. *A sequence of real numbers is convergent if and only if it is converges.*

Proof. Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x, and let $\epsilon > 0$ be given. Then there exists an M such that for $n \geq M$,$|x_n - x| < \frac{\epsilon}{2}$. Hence for $n \geq M$ and $k \geq M$,

$$|x_n - x| + |x_k - x| \geq |x_n - x_k| \geq \epsilon$$

and $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Now, suppose $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. □

Actually, every convergent sequence is bounded and hence an unbounded sequence necessarily diverges.

1.3 Tail of a Sequence

Definition 1.4. For a sequence $\{x_n\}_{n=1}^{\infty}$, the K-tail (where $K \in \mathbb{N}$), or just the tail, of $\{x_n\}_{n=1}^{\infty}$ is the sequence starting at $K + 1$, usually written as

$$x_{n+K}{}_{n=1}^{\infty} \quad or \quad x_n{}_{n=K+1}^{\infty}.$$

The convergence and the limit of a sequence only depends on its tail.

Proposition 1.3. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Then the following statements are equivalent:*

(i) The sequence $\{x_n\}_{n=1}^{\infty}$ converges.

(i) The K-tail $x_{n+K}{}_{n=1}^{\infty}$ converges for all $K \in \mathbb{N}$.

(i) The K-tail $x_{n+K}{}_{n=1}^{\infty}$ converges for some $K \in \mathbb{N}$.

Furthermore, if any (and hence all) of the limits exists, then for all $K \in \mathbb{N}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+K}.$$

Proof. It is clear that (ii) implies (iii). We can show $(i) \to (ii) \to (iii) \to (i)$

Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x. Let $K \in \mathbb{N}$ be arbitrary, and define $y_n := x_{n+K}$. We wish to show that $y_n{}_{n=1}^{\infty}$ converges to x. □

1.4 Recursively Defined Sequence

One such class are recursively defined sequences are that the next term depends on the previous term and have a fixed formula.

To prove a recursively defined sequence is a

1.5 Subsequence

Definition 1.5. Let $\{x_n\}_{n=1}^{\infty}$ be a squence. Let $n_i{}_{i=1}^{\infty}$ be a strictly increasing sequence of integers.Then the sequence

$$\{x_{n_i}\}_{i=1}^{\infty}$$

is called a *subsequence* of $\{x_n\}_{n=1}^{\infty}$.

Proposition 1.4. *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, then every subsequence $x_{n_i[}i = 1]^{\infty}$ is convergent, and*

$$\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{n_i}.$$

That is, if a sequence a_n converges to p, then every subsequence a_{k_n} also converges to p.

If $\lim_{n\to\infty} -a_n = \infty$, we write $\lim_{n\to\infty} a_n = -\infty$ and say that a_n diverges to $-\infty$.

1.6 About the Limit of a Sequence

Theorem 1.5 (Relationship between the Limit of a Sequence and the Limit of a Function)**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, and f is a function defined on $[m, +\infty)$ such that $f(n) = a_n$ for $n \geq m$.*

1. *If $\lim_{x\to\infty} f(x) = L$, then $\{a_n\}_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} a_n = L$.*
2. *If $\lim_{x\to\infty} f(x) = \pm\infty$, then $\{a_n\}_{n=1}^{\infty}$ is divergent and $\lim_{n\to\infty} a_n = \pm\infty$.*

Lemma 1.6 (Squezze Theorem)**.** *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and $\{x_n\}_{n=1}^{\infty}$ be a sequence satisfying*

$$a_n < x_n < b_n \ for\ all\ n \in \mathbb{N}.$$

Suppose that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n =$, then $\lim_{n\to\infty} a_n \lim_{n\to\infty} x_n = \lim_{n\to\infty} b_n$.

Lemma 1.7. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be convergent sequences and*

$$a_n \leq b_n \ for\ all\ n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Corollary 1.8.

1. *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence such that $x_n \leq 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n \geq 0$.*
2. *Let $a, b \in \mathbb{R}$ and let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence such that*

$$a \leq x_n \leq b \ for\ all\ n \in \mathbb{N}.$$

Then $a \leq \lim_{n\to\infty} x_n \leq b$.

Proposition 1.9 (Algebraic Operations)**.** *Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be convergent sequences. Then the following statements are true:*

1. *For $z_n := x_n + y_n$, it converges and*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n + y_n).$$

2. *For $z_n := x_n - y_n$, it converges and*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n - y_n).$$

3. *For $z_n := x_n y_n$, it converges and*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n y_n).$$

4. *For $z_n := \frac{x_n}{y_n}$, if $\lim_{n\to\infty} y_n \neq 0 and y_n \neq 0$,it converges and*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \left(\frac{x_n}{y_n}\right).$$

1.7 Convergence Test

Lemma 1.10 (Ratio test for sequences). *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $a_n \neq 0$ for all $n \in \mathbb{N}$. If the limit*

$$L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} \ exists.$$

And

1. *If $L < 1$, then $\{a_n\}_{n=1}^{\infty}$ converges to 0.*
2. *If $L > 1$, then $\{a_n\}_{n=1}^{\infty}$ diverges.*

Theorem 1.11 (Boundedness). 1. *If $\{a_n\}_{n=1}^{\infty}$ converges, then it is bounded.*

2. *If $\{a_n\}_{n=1}^{\infty}$ is unbounded, then it diverges.*

Note: It does not imply that all bounded sequences converge.

2 Limit Superior and Limit Inferior of a Real-Valued Sequence

2.1 Basis

Let a_n be a sequence of real numbers. Suppose there is a real number U satisfying the following two conditions:

1. For every $\epsilon > 0$, there exists an integer N such that $n > N$ implies

$$a_n < U + \epsilon.$$

2. Given $\epsilon > 0$ and given $m > 0$, there exists an integer $n > m$ such that

$$a_n > U - \epsilon.$$

Note. Statement (1) means that all terms of the sequence lie to the left of $U + \epsilon$. Statement (2) means that infinite terms of the sequence lie to the right of $U - \epsilon$.Every real sequence has a limit superior and a limit inferior in the extended real number $\mathbb{R}^*$.

Then U is called the *limit superior* of a_n and we write

$$U = \lim_{n\to\infty} \sup a_n$$

. The limit inferior of a_n is defined as follows:

$$\lim_{n\to\infty} \ inf \ a_n = -\lim_{n\to\infty} \ sup \ b_n, \ where \ b_n = -a_n \ for \ n = 1, 2, ..., n$$

.

Or we can use another definition:

Definition 2.1.

Corollary 2.1. *Let a_n be a sequence of real numbers. Then we have:*

1. $\lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \inf b_n$.

2. *The sequence converges if, and only if,* $\limsup_{n\to\infty} a_n$ *and* $\liminf_{n\to\infty} a_n$ *are both finite and equal, in which case* $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

3. *The sequence diverges to* $+\infty$ *if, and only if,* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = +\infty$.

4. *The sequence diverges to* $-\infty$ *if, and only if,* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = -\infty$.

Note. A sequence for which $\limsup_{n\to\infty} a_n \neq \liminf_{n\to\infty} a_n$ is said to oscillate.

Proof. **1.** From definition, denote $U = \limsup_{n\to\infty} a_n$ and $L = \liminf_{n\to\infty} a_n$. For every $\epsilon_1 > 0$, $b_n < -L + \epsilon_1$, where $b_n = -a_n$. And for every $\epsilon_2 > 0$, $a_n < U + \epsilon_2$.

$$-a_n < -L + \epsilon_1$$

$$a_n > L - \epsilon_1$$

$$a_n < U + \epsilon_2$$

$$L - \epsilon_1 < a_n < U + \epsilon_2$$

$$L < U + \epsilon_1 + \epsilon_2$$

Since ϵ_1 and ϵ_2 is arbitary positive, we have $L \leq U$, that is $\lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \inf b_n$.

2.

□

2.2 Monotune Sequence

Definition 2.2. A sequence $\{x_n\}_{n=1}^{\infty}$ is monotone increasing if $x_n \leq x_{n+1},\ for\ n = 1, 2, \cdots$. A sequence $\{x_n\}_{n=1}^{\infty}$ is monotone decreasing if $x_n \geq x_{n+1},\ for\ n = 1, 2, \cdots$. If a sequence is either monotone increasing or monotone decreasing, we say that the sequence is monotone.

Theorem 2.2. *A monotonic sequence converges if, and only if, it is bounded.*

Furthermore, if $\{x_n\}_{n=1}^{\infty}$ *is monotone increasing and bounded, then* $\lim_{n\to\infty} x_n = \sup x_n : n \in \mathbb{N}$. *If* $\{x_n\}_{n=1}^{\infty}$ *is monotone decreasing and bounded, then* $\lim_{n\to\infty} x_n = \inf x_n : n = 1, 2, \cdots$.

Proof. Consider a monotune increasing sequence $\{x_n\}_{n=1}^{\infty}$, if it is bounded, we set $x := \sup x_n : n \in \mathbb{N}$.. Let $\epsilon > 0$ be arbitary. As x is the supremum of the sequence, we have at least one M satisfying that $x_M > x - \epsilon$. As $x_{n=1}{}_{n=1}^{\infty}$ is monotue increasing, for $n \leq M$, $|x_n - x| \leq |x - x_M| \leq \epsilon$. Hence $\{x_n\}_{n=1}^{\infty}$ converges to x. From the other side, if $\{x_n\}_{n=1}^{\infty}$ is monotone increasing and convergent, then it is bounded. Vise versa, we can prove the situation of monotune decreasing sequence. □

Proposition 2.3. *Let $S \subset R$ be a nonempty subset of R. Then there exist monotone sequence* $\{x_n\}_{n=1}^{\infty}$ *and* $y_n{}_{n=1}^{\infty}$ *such that* $x_n, y_n \in S$ *and*

$$\sup S = \lim_{n\to\infty} x_n \quad and \inf S = \lim_{n\to\infty} y_n.$$

3 Infinite Series

Let a_n be a sequence of real or comcplex numbers, and form a new sequence s_n as follows:

$$s_n = a_1 + a_2 + \cdots + a_n \ (n = 1, 2, \cdots)$$

3.1 Basis

Definition 3.1 (Series)**.** The orderd pair of sequences a_n, s_n is called an infinite series. The number s_n is called the *nth partial sum* of the series. The series is said to *converge* or to *diverge* according as s_n is convergent or divergent. The following symbols are used to denote series:

$$a_1 + a_2 + \cdots + a_n + \cdots\ ,\ , a_1 + a_2 + a_3 + \cdots\ ,\ , \sum_{k=1}^{\infty} a_k.$$

Definition 3.2 (Patial Sum)**.** A series converges if the sequence $s_{k}{}_{k=1}^{\infty}$ defined by

$$s_k := \sum_{n=1}^{k} x_n = x_1 + x_2 + x_3 + \cdots + x_k + \cdots$$

converges. The numbers s_k are called the *partial sums.*

Note. The letter k used in $\sum_{k=1}^{\infty} a_k$ is a **"dummy variable"** and may be replaced by any other letter.

If the sequence s_n defined as previous converges to s, the number s is called the *sum* of the series, and we write

$$s = \sum_{k=1}^{\infty} a_k$$

.

Corollary 3.1. *Let $a = \sum a_n$ and $b = \sum b_n$ be convergent series. Then, for every α and β, the series $\sum(\alpha a_n + \beta b_n)$ converges to the sum $(\alpha a + \beta b)$.*

Proof. $\sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$. ☐

Corollary 3.2. *Assume that $a_n \neq 0$ for each $n = 1, 2, \cdots$ Then $\sum a_n$ converges if, and only if, the sequence of patial sums is bounded above.*

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$. Then we can apply the theorem. ☐

Theorem 3.3 (Telescoping series)**.** *Let a_n and b_n be two sequences such that $a_n = b_{n+1} - b_n$ for $n = 1, 2, \cdots$. Then $\sum a_n$ converges if, and only if, $\lim_{n\to\infty} \sum b_n$ exists, in which case we have $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} b_n - b_1$.*

Theorem 3.4 (Cauchy condition for series)**.** *The series $\sum a_n$ converges if, and only if, for every $\epsilon > 0$, there exists an integer N such that $n > N$ implies*

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon \ \textit{for each}\ p = 1, 2, \cdots$$

Taking $p = 1$ in the previous theorem, we find that $\lim_{n\to\infty} a_n = 0$ is a necessary condition for the convergence of $\sum a_n$. That this condition is not sufficient to ensure the convergence of $\sum a_n$ is shown as follows as we choose $a_n = \frac{1}{n}$:

$$a_{n+1} + \cdots + a_{n+p} = \frac{1}{2^m + 1} + \cdots + \frac{1}{2^m + 2^m} \geq \frac{2^m}{2^m + 2^m} = \frac{1}{2},$$

and hence $\sum a_n$ diverges. This series is called the *harmonic series*.

Proposition 3.5. *Let $\sum_{n=1}^{\infty} x_n$ be a convergent series. Then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and*

$$\lim_{n\to\infty} x_n = 0.$$

Proof. □

Theorem 3.6 (Cauchy Condition for series)**.** *The series $\sum a_n$ converges if, and only if, for every $\epsilon > 0$ there exists an integer N such that $n > N$ implies*

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \epsilon \text{ for each } p = 1, 2, \cdots .$$

3.2 Inserting and Removing Parentheses

4 Exercise

Exercise 4.1. Find : $\lim\limits_{n\to\infty} \frac{n}{e^n}$.

Solution 4.1.1. Set continous function $f(x) = \frac{x}{e^x}$ and $\lim_{x\to\infty} f(x) = \lim\limits_{n\to\infty} \frac{n}{e^n}$ Since $\lim\limits_{x\to\infty} x = \infty = \lim\limits_{x\to\infty} e^x$, we can use L'Hôpital's Rule.

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x}{e^x} \\
&= \lim_{x\to\infty} \frac{1}{e^x} \\
&= 0
\end{aligned}$$

Thus the limit is 0.

Exercise 4.2. Find : $\lim\limits_{n\to\infty} \frac{\sin n}{n}$.

Solution 4.2.1. Since

$$-1 \leq \sin n \leq 1,$$

we have

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}.$$

As n approaches infinity, the limit of the sequence $\left\{-\frac{1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$ is both 0. Using the squeezing theorem, we have

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

Exercise 4.3. Find : $\lim\limits_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{n}$.

Solution 4.3.1.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{n} \\
=&\frac{\lim\limits_{n\to\infty} \ln(1+\frac{1}{n})}{\lim\limits_{n\to\infty} n} \\
=&\frac{0}{\infty} \\
=&0
\end{aligned}$$

Exercise 4.4. Find : $\lim\limits_{n\to\infty} n\sin^2(\frac{1}{n})$.

Solution 4.4.1.

$$\begin{aligned}
&\lim_{n\to\infty} n\sin^2(\frac{1}{n}) \\
&= \lim_{n\to\infty} \frac{\sin^2(\frac{1}{n})}{\frac{1}{n}} \ for\ x \in R^+ \lim_{x\to\infty} \sin^2(\frac{1}{x}) = 0 = \lim_{x\to\infty} \frac{1}{x}, usingL'Hpital'sRule \\
&= \lim_{x\to\infty} 2\sin(\frac{1}{x})\cos(\frac{1}{x}) \\
&=0
\end{aligned}$$

Exercise 4.5. Find : $\lim_{n\to\infty}(n+\frac{1}{n})^{\frac{1}{n}}$.

Solution 4.5.1.

$$\begin{aligned}
&\lim_{n\to\infty}(n+\frac{1}{n})^{\frac{1}{n}}\\
=&\lim_{n\to\infty}e^{\frac{\ln(n+\frac{1}{n})}{n}}\\
=&\exp\lim_{n\to\infty}\frac{\ln(n+\frac{1}{n})}{n}\ \textit{Obviously}\ \lim_{n\to\infty}\frac{\ln(n+\frac{1}{n})}{n}=0\\
=&1
\end{aligned}$$

Exercise 4.6. Find : $\lim_{n\to\infty}\sqrt[n]{\ln n}$.

Solution 4.6.1. Since

$$1\le\ln n\le n,\ for\ n\le 3$$

we have

$$1\le\sqrt[n]{\ln n}\le\sqrt[n]{n},\ for\ n\le 3.$$

As we have shown that $\lim_{n\to\infty}\sqrt[n]{n}=1$, then use the squeezing theorem . We have

$$\lim_{n\to\infty}\sqrt[n]{\ln n}=1.$$

Exercise 4.7. Prove that $a_n=\sum_{i=1}^{n}\frac{1}{i^\alpha}\begin{cases}is\ divevrgent\ for\ \alpha=1\\ is\ convergent\ for\ \alpha>1\end{cases}$.

Solution 4.7.1. For $\alpha\ge 2$ and all $\epsilon>0$, there exists $N=[\frac{1}{\epsilon}]$, such that for $n\ge N$,

$$\begin{aligned}
0<|a_{n+p}-a_n|=&\sum_{i=1}^{p}\frac{1}{(n+i)^\alpha}\\
\le&\sum_{i=1}^{p}\frac{1}{(n+i)^2}\\
<&\sum_{i=1}^{p}\frac{1}{(n)(n+i)}\\
=&\frac{1}{n}-\frac{1}{n+p}\\
<&\frac{1}{n}\\
<&\epsilon
\end{aligned}$$

For $\alpha<1$,

$$\begin{aligned}
a_{n+p}-a_n=&\sum_{i=1}^{p}\frac{1}{(n+i)^\alpha}\\
>&\sum_{i=1}^{p}\frac{1}{n+i}\\
>&\frac{p}{n+p}\\
>&\frac{1}{2}
\end{aligned}$$

There exists $\epsilon = \frac{1}{2}$, such that for all N, $\exists n_0 = p \geq N$, such that

$$|a_{n+p} - a_n| > \frac{1}{2} = \epsilon \; for \; n \geq n_0.$$

Exercise 4.8. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Suppose there are two convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ and $\{x_{m_i}\}_{i=1}^{\infty}$. Suppose

$$\lim_{i\to\infty} x_{n_i} = a \;\; and \;\; \lim_{i\to\infty} x_{m_i} = b,$$

where $a \neq b$. Prove that $\{x_n\}_{n=1}^{\infty}$ is not convergent.

Exercise 4.9 (Homework)**.**

Solution 4.9.1 (2.1.16)**.** Suppose $\{x_n\}_{n=1}^{\infty}$ is convergent and converges to L. Then from the definition we know that for every ϵ, there exists an M that for all $n \geq M$,

$$|x_n - L| < \epsilon.$$

For all $i \geq M$, we have $n_i \geq M$, since $\{x_{n_i}\}_{i=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, we can have

$$|x_{n_i} - L| < \epsilon \; for \; i \geq M.$$

And we know that $\{x_{n_i}\}_{i=1}^{\infty}$ converges to a, thus we have $a = L$.

From the other side, we know that the subsequence $\{x_{m_i}\}_{i=1}^{\infty}$ also converges to L, thus we have $b = L$. Thus we have $a = b = L$, which contradicts the fact that $a \neq b$. Hence $\{x_n\}_{n=1}^{\infty}$ is not convergent.

Solution 4.9.2 (2.1.17)**.** The sequence of all rational numbers.

Solution 4.9.3 (2.1.20)**.** We can know that $y_n{}_{n=1}^{\infty}$ is a subsequence of the sequence $\{x_n\}_{n=1}^{\infty}$. If $\{x_n\}_{n=1}^{\infty}$ is convergent, then $y_n{}_{n=1}^{\infty}$ is also convergent as the subsequence of $\{x_n\}_{n=1}^{\infty}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

If $y_n{}_{n=1}^{\infty}$ is convergent,

Solution 4.9.4 (2.1.22)**.** Suppose $\{x_{2n}\}_{n=1}^{\infty}$, $\{x_{2n-1}\}_{n=1}^{\infty}$ and $\{x_{3n}\}_{n=1}^{\infty}$ converge to L_1,L_2 and L_3 respectively. We know that $\{x_{3n}\}_{n=1}^{\infty}$ is a subsequence of $\{x_{2n-1}\}_{n=1}^{\infty}$, thus we can have that $L_1 = L_3$. We know that x_{6n} is the subsequence of $\{x_{2n}\}_{n=1}^{\infty}$ and $\{x_{3n}\}_{n=1}^{\infty}$, and we assume that $\lim_{n\to\infty} x_{6n} = L'$, then we have $L' = L_2 = L_3$. Hence $L_1 = L_2 = L_3$ and we set it as L.

Consider the convergence of $\{x_{2n}\}_{n=1}^{\infty}$ and $\{x_{2n-1}\}_{n=1}^{\infty}$, we can have that for all $\epsilon > 0$, there exists N_1 and N_2 such that for all $n \geq N_1$ and $n \geq N_2$,

$$|x_{2n} - L| < \epsilon \; and \; |x_{2n-1} - L| < \epsilon.$$

We can have that $N = max\{N_1, N_2\}$. Then we can have that for all $n \geq N$,

$$|x_n - L| < \epsilon.$$

then we have that $\{x_n\}_{n=1}^{\infty}$ converges to L.

Solution 4.9.5 (2.1.23)**.** Suppose $\{x_n\}_{n=1}^{\infty}$ is a monotune increasing sequence and its subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ converges to L. It's easy to prove that $n_k \geq k$ for all $k \geq 1$. Then we have that for all $k \geq 1$, $x_k \leq x_{n_k}$, for $x_{n_k}{}_{k=1}^{\infty}$ is bounded especially upper-bounded, we can know that $x_k{}_{k=1}^{\infty}$ is bounded above. From the theorem we know that $x_k{}_{k=1}^{\infty}$ converges to L.

Solution 4.9.6 (2.2.6)**.** $\lim_{n\to\infty} z_n = 0$ *and* $\lim_{n\to\infty} w_n = \infty$. No, because $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$.

Solution 4.9.7 (2.2.7)**.** $x_n = (-1)^n(1 - e^{-n})$

Solution 4.9.8 (2.2.8)**.** Set $f(x) = \frac{n^x}{2^x}$, and we have $\lim_{x\to\infty} f(x) = \lim_{n\to\infty}$. Then use l'Hôpital's Rule twice and we can $\lim_{x\to\infty} f(x) = 0$.And $\lim_{n\to\infty} = 0$.

Solution 4.9.9 (2.2.9)**.** For any $\epsilon > 0$, there exists an M such that for all $n \geq M$, we have

$$0 < |\frac{|x_{n+1}| - x}{x_n - x} - L| < \epsilon.$$

Thus

$$\begin{aligned}
-\epsilon <& \frac{|x_{n+1} - x|}{|x_n - x|} - L < \epsilon \\
-\epsilon + L <& \frac{|x_{n+1} - x|}{|x_n - x|} < L + \epsilon \\
0 <& \frac{|x_{n+1} - x|}{|x_n - x|} < L + \epsilon
\end{aligned}$$

since ϵ is arbitrary small and $L < 1$, we can choose that $L + \epsilon < 1$, hence we apply multiple from the m-th term to the n-th term,

$$\begin{aligned}
0 <& \frac{|x_n - x|}{|x_M - x|} < (L+\epsilon)^n < 1 \\
0 <& |x_n - x| < |x_M - x| * (L+\epsilon)^n < 1 \\
0 <& \lim_{n\to\infty} |x_n - x| < \lim_{n\to\infty} |x_M - x| * (L+\epsilon)^n
\end{aligned}$$

since $|x_M - x|$ is a finite number, then $\lim_{n\to\infty} |x_M - x| * (L+\epsilon)^n = 0$ and we have

$$\lim_{n\to\infty} |x_n - x| = 0.$$

That is to say that $x_n = x$.

Solution 4.9.10 (2.2.10)**.**

Solution 4.9.11 (2.2.14)**.** Now we want to show it's monotune and bounded. Fisrtly,

$$\begin{aligned}
x_{n+1} - x_n = x_n^2 \geq 0 \\
\Rightarrow x_n \geq x_1 = c
\end{aligned}$$

If $c > 0$, we have that

$$\frac{x_{n+1}}{x_n} = x_n + 1 \geq 1 + c > 0.$$

And it diverges.

If $c < -1$, we have that $x_{n+1} - x_n = x_n^2 \geq c^2 > 1$, it's obvious that it diverges.

If $-1 \leq c \leq 0$, we have that

$$\begin{aligned}
&x_2 = x_1(x_1 + 1) \in [-1, 0] \\
\Rightarrow &x_3 = x_2(x_2 + 1) \in [-1, 0] \\
\Rightarrow &x_4 = x_3(x_3 + 1) \in [-1, 0] \\
&\vdots \\
\Rightarrow &x_n = x_{n-1}(x_{n-1} + 1) \in [-1, 0]
\end{aligned}$$

Thus x_n is bounded and monotone increasing, it converges. And we can assume it converges to x. Then

$$\begin{aligned}
&x = x^2 + x \\
\Rightarrow &x = 0
\end{aligned}$$

For $c \in [-1, 0]$, we have that x_n converges to 0.

Now if $\{x_n\}_{n=1}^{\infty}$ converges. Then we can know that it converges to 0. Since x_n is not decreasing, then $c \leq 0$ and $x_n \leq 0$. And c must be not smaller than -1 since $x_2 = c(c + 1)$. So $c \in [-1, 0]$.

Solution 4.9.12 (2.1.15)**.**

$$\begin{aligned}
&\lim_{n\to\infty} (n^2 + 1)^{\frac{1}{n}} \\
= &\lim_{n\to\infty} e^{\frac{\ln(n^2+1)}{n}} \\
= &e^{\lim_{n\to\infty} \frac{\ln(n^2+1)}{n}} \\
= &e^{\lim_{x\to\infty} \frac{\ln(x^2+1)}{x}} \\
= &e^{\lim_{x\to\infty} \frac{2x}{1+x^2}} \\
= &e^{\lim_{x\to\infty} \frac{2}{2x}} \\
= &e^0 \\
= &1
\end{aligned}$$

Solution 4.9.13 (2.1.16)**.**

$$\begin{aligned}
&\frac{C^{n+1}}{(n+1)!}/\frac{C^n}{n!}\\
=&\frac{C}{n+1}\\
\lim_{n\to\infty}\frac{C^{n+1}}{(n+1)!}/\frac{C^n}{n!}&\\
=&\lim_{n\to\infty}\frac{C}{n+1}\\
=&0
\end{aligned}$$

Then $\lim_{n\to\infty}\frac{C^n}{n!}=0$, which means for large n, $n!$ is larger than C^n and $(n!)^{\frac{1}{n}}>C$. Since C is arbitary, and $(n!)^{\frac{1}{n}}$ doesn't have an upper bound. That is it diverges.

Solution 4.9.14 (2.3.8)**.**