Real Numbers

Len Fu

11.29.2024

Abstract

Contents

1 Basic Properties

2 The Set of Real Numbers

2.1

2.2 Archimedean Property

Theorem 2.1. *1. (Archimedean Property) If $x, y \in \mathbb{R}$ and $x > 0$, then there exists an $n \in \mathbb{N}$ such that*

$$nx > y.$$

2. ($\mathbb{Q}$ is dense in $\mathbb{R}$) If $x, y \in \mathbb{R}$ and $x < y$, then there exists an $r \in \mathbb{Q}$ such that

$$x < r < y.$$

Proof. Consider (i), for every real number $t := \frac{y}{x}$

Consider (ii), first assume $x \leq 0$, and $y - x > 0$, then there exists an $n \in \mathbb{N}$ such that $n(y-x) > 1$, and $y - x > \frac{1}{n}$. And there has a least integer $m > nx$, divide through by n we get $x < \frac{m}{n}$.

If $m > 1$, then $m - 1 \in \mathbb{N}$ and $m - 1 \leq nx$. If $m = 1$, $m - 1 = 0 \leq nx$. That is to say $nx \geq m - 1$.

Then $y > x + \frac{1}{n} \geq \frac{m}{n} > x$, that is $\mathbb{Q}$ is dense in $\mathbb{R}$. □

2.3 Inf and Sup

Proposition 2.1. *Let $A, B \subset \mathbb{R}$ be nonempty sets such that $x \geq y$ whenever $x \in A$ and $y \in B$. Then A is bounded above, B is bounded below, and $\sup A \geq \inf B$.*

Proof. □

Proposition 2.2.

Definition 2.1. Let $A \subset \mathbb{R}$ be a set.

1. If A is empty, then $\sup A := -\infty$.
2. If A is empty, then $\inf A := \infty$.
3. If A is not bounded above, then $\sup A := \infty$.
4. If A is not bounded below, then $\inf A := -\infty$.

And $\mathbb{R}^* = \mathbb{R} \bigcup \infty, -\infty$ is defined as **the set of Extended Real Numbers**

But we must leave $\infty - \infty, 0 \cdot \pm\infty,\ and\ \frac{\pm\infty}{\pm\infty}$ as undefined.

2.4 Absolute Value and Bounded Functions

Proposition 2.3 (Triangle Inquality)**.** *Let $x, y \in \mathbb{R}$ and $x > 0$, then $|x + y| \leq |x| + |y|$.*

Corollary 2.1.1. *Let $x, y \in \mathbb{R}$. (i) (reverse triangle inequality) $||x| - |y|| \leq |x - y|$. (ii) $|x - y| \leq |x| + |y|$.*

Definition 2.2 (Bounded Functions)**.** Suppose $f : D \to \mathbb{R}$ is a function. We say f is **bounded** if there exists a constant $M \in \mathbb{R}$ such that $|f(x)| \leq M$ whenever $x \in D$.

Proposition 2.4. *If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions and $f(x) \leq g(x)$ for all $x \in D$ then*

$$\sup_{x \in D} f(x) \leq \sup_{x \in D} g(x) \ and \ \inf_{x \in D} f(x) \leq \sup_{x \in D} g(x).$$

2.5 Intervals and the size of $\mathbb{R}$

Proposition 2.5. *A set $I \subset \mathbb{R}$ is an interval if and only if I contains at least 2 points and for all $a, c \in I$ and $a < b < c$, we have $b \in I$.*

2.6 Decimal Representation of the Reals

We represent rational numbers with positive integer M, K and digts $d_K d_{K-1} \cdots d_1 d_0 d_{-1} \cdots d_{-M+1} d_M$ such that

$$x = d_K 10^K + d_{K-1} 10^{K-1} + \cdots + d_1 10^1 + d_0 10^0 + d_{-1} 10^{-1} + \cdots + d_{-M+1} 10^{-M+1} + d_M 10^{-M}$$

and call D_n the **truncation of x to n decimal digits**.

However for irrarional numbers, we can not represent them in this way. And for some infinite curcilation, we can not represent them in this way either.

For every real number $x \in (0, 1]$, we define

$$x = \sup_{n \in \mathbb{N}} D_n = \sup_{n \in \mathbb{N}} \left(\frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} \right).$$

Proposition 2.6. *(i) Every infinite sequece of digts $0.d_1 d_2 \cdots$ represents a unique real number $x \in (0, 1]$, and*

$$D_n \leq x \leq D_n + \frac{1}{10^n} \; for \; all \; n \in \mathbb{N}.$$

(ii) For every real number $x \in (0, 1]$, there exists an infinite sequence of digts $0.d_1 d_2 \cdots$ that represents x. There exists a unique representation such that

$$D_n < x \leq D_n + \frac{1}{10^n} \; for \; all \; n \in \mathbb{N}.$$

Proposition 2.7. *If $x \in (0, 1]$ is a rational number and $x = 0.d_1 d_2 \cdots$, then the decimal digits eventually start repeating. That is, there are positive integers N and P, such that for all $n \geq N$, $d_n = d_{n+P}$.*