Determinant

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Abstract

This is the note maded by Len Fu while his learning progress. The main content is from *Linear Algebra Done Right* and *Linear Algebra Allenby*.

Contents

1 Permutation

1.1 N Permutation

An n-permutation is an arrangement of all the numbers $1, 2, ..., n$. The total number of n-permutations is $n!$.

1.2 Inversion and Inversion Number

Formally, for a sequence a_n with elements a_i and a_j with $i < j$, an inversion is present if $a_i > a_j$.

The inversion number of a permutation is the number of inversions in it.

A permutation with an odd inversion number is called an odd permutation. And a permutation with an even inversion number is called an even permutation.

1.3 Transposition

Formally, a transposition is a permutation that exchanges two elements and leaves all others unchanged. For example, in permutation $\tau = (1, 2, 3, 4, 5)$ the transposition $(1, 2)$ exchanges 1 and 2. Then the resulting permutation is $\tau = (2, 1, 3, 4, 5)$.

1.3.1 Theorem

A transposition changes the parity of a permutation. **Proof:**
Let $\tau = (i_1 i_2 ... i_j i_{j+1} ... i_n)$, and we exchange i_j and i_{j+1}, then the remain permutation $(i_1 i_2 ... i_j ... i_n)$ and $(i_1 i_2 ... i_{j+1} ... i_n)$ keep the same parity. But the paritr of $(i_j i_{j+1})$ changes, so the total parity of τ changes.

Now consider that if the transposition is between $(i_j i_k)$ like $(...j i_1 i_2 ... i_s k...)$, then we first transpose s times to set j into i_s like $(...i_1 i_2 ... i_s j k...)$. And we transpose j and k $(...i_1 i_2 ... i_s k j...)$, then we transpose s times to set k into i_1 like $(...k i_1 i_2 ... i_s j...)$. The total transposition is $2s + 1$. So the parity of the permutation changes.

Corollary 1 In all n permutation, the number of even permutation is equal to the number of odd permutation, which is $\frac{n!}{2}$.

Proof:

Suppose there are s odd permutation, then there are t even permutation. Now transpose the first two elements of all even permutation, then we get s odd permutation. Then $s \leq t$, conversly, transpose the first two elements of all odd permutation, then we get t even permutation, and $t \leq s$. So $s = t = \frac{n!}{2}$.

1.3.2 Theorem

Any n-permutation can be transposed from $(123...n)$ and the times of transposition equals to the inversion number of the permutation.

2 N-Order Determinant

2.1 Definition

The n-order determinant is a scalar value that can be computed from the elements of a square matrix of size $n \times n$.

Actually, it can be written abstract

$$\begin{aligned}\det|A| &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\ &= \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n} \\ &= \sum_{i_1 i_2 \cdots i_n} (-1)^{\tau(i_1 i_2 \cdots i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \\ &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}\end{aligned}$$

2.2 Properties

1. Transcope the matrix, the determinant does not change.

2.
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

3.
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 & b_2 & \cdots & b_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ c_1 & c_2 & \cdots & c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

4. If there are two rows or columns that are the same, the determinant is 0. is 0.

5. If there are two rows or columns are proportionable, the determinant is 0.

6. Add a row's or a column's k-times into another one, the determinant keeps the same.

7. Exchange two rows or columns, the determinant changes its sign.

Proof.

1. If we transcope the determinant,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \rightarrow \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & & & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

like this, then we expand the right one with respect to the rows like this

$$\sum_{i_1 i_2 \cdots i_n} (-1)^{\tau(i_1 i_2 \cdots i_n)} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n}$$

Actually it keeps from the left one.

2.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k\left(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}\right)$$

$$= k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

3.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (b_1 + c_1)A_{i1} + (b_2 + c_2)A_{i2} + \cdots + (b_n + c_n)A_{in}$$

$$= (b_1 A_{i1} + b_2 A_{i2} + \cdots + b_n A_{in}) + (c_1 A_{i1} + c_2 A_{i2} + \cdots + c_n A_{in})$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ b_1 & b_2 & \cdots & b_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & & & \cdots \\ c_1 & c_2 & \cdots & c_n \\ \cdots & & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

4.

$det = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau()}$ **5.**