Eigenvalue and Eigenvector

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Abstract

This is the note made by Len Fu during his learning progress in BIT. The main content is from *Linear Algebra Done Right* and *Linear Algebra Allenby*.

Contents

1 Similarity of the Matrix

1.1 Basis

Definition 1.1. Set $A, B \in C^{n\times n}$. If there exists an n-order invertible matrix P such that

$$P^{-1}AP = B$$

, we say that A and B are similar, denoted as $A \sim B$, and P is called the *similarity transformation* from A to B.

Properties 1.1 (Reflectivity)**.** $A \sim A$.

Properties 1.2 (Symmetry)**.** If $A \sim B$, then $B \sim A$.

Properties 1.3 (Transitivity)**.** If $A \sim B$ and $B \sim C$, then $A \sim C$.

1.2 Similar Diagonalization

Definition 1.2 (Digonalizable)**.** If there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where A is a square and D is a diagonal matrix. Then A is called *diagonalizable*.

2 Eigenvalue and Eigenvector of the Matrix

2.1 Basis

Definition 2.1. Set A as a $n \times n$ square, if there exists a number λ and $n - nonzero$ vector X, satisfying

$$AX = \lambda X \ or \ (\lambda I - A)X = 0$$

then we say that λ is an eigenvalue of A, and X is an eigenvector of A with eigenvalue λ.

Note.

1. Only squares have eigenvectors and eigenbvalues.
2. Eigenvector must be nonvector and eigenvalue can be zero.

Since $(\lambda I - A)X = 0$ and X is nonzero vector, then $\det(\lambda I - A)$ should be zero to ensure X is nonzero vector of the solution.

Consider the solution of $(\lambda I - A)X = 0$. The characteristic polynomial of A is

$$b_n\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0.$$

To solve the polynomial,

$$
\begin{vmatrix}
\lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\
-a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn}
\end{vmatrix}
$$

$$= b_n\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0$$

Consider the expasion of the determinant, except for

$$(\lambda - a_{11})(\lambda - a_{22})\cdots(\lambda - a_{nn})$$

other terms' highest order of λ is $n-2$. Then the coefficents

$$
\begin{cases}
b_n = 1 \\
b_{n-1} = -(a_{11} + a_{22} + \cdots + a_{nn}) = tr()
\end{cases}
$$