CS 106: Artificial Intelligence

Constraint Satisfaction Problems

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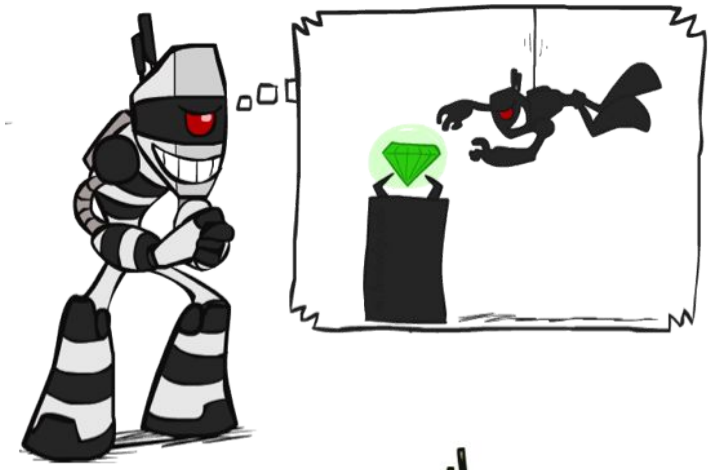
[These slides were adapted from the slides created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. http://ai.berkeley.edu.]

What is Search For?

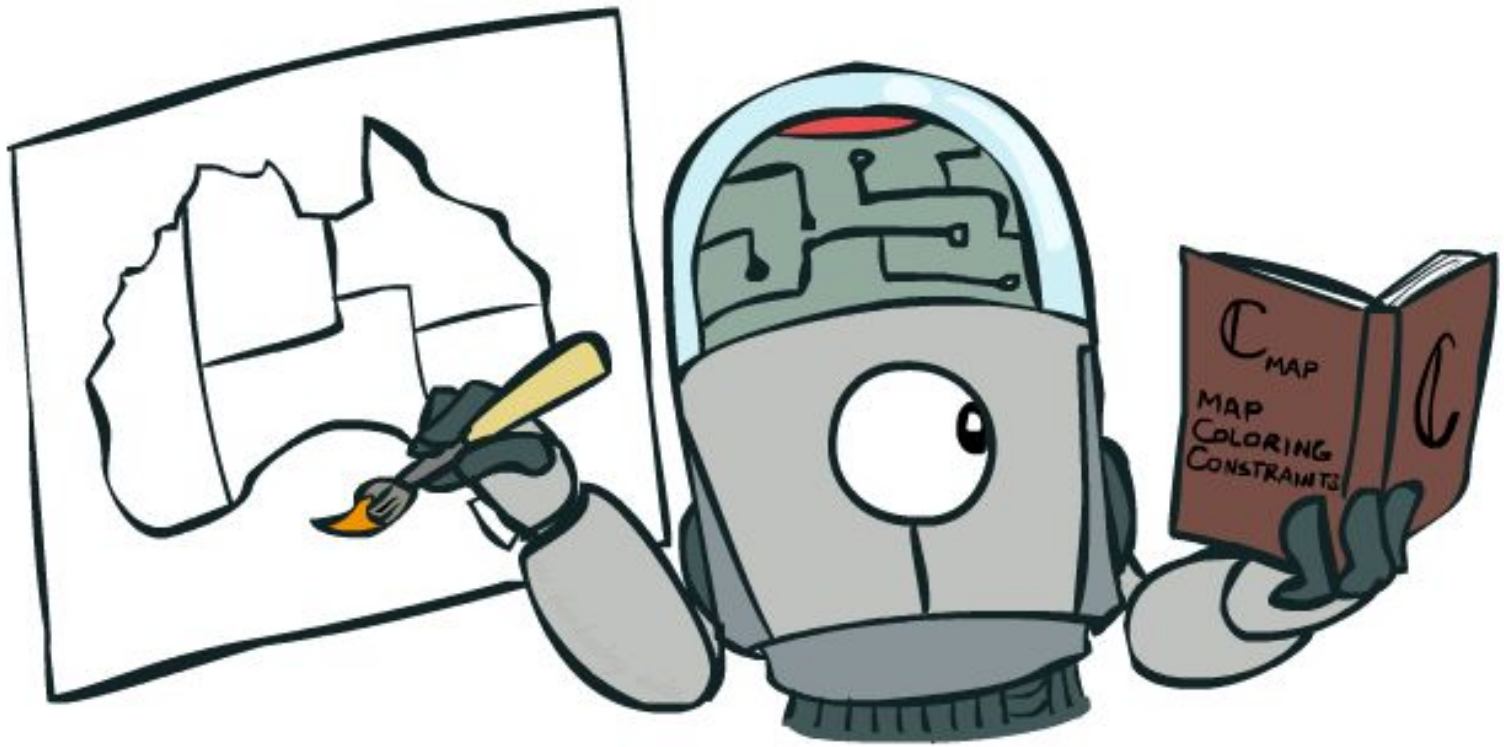
- Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space

- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems

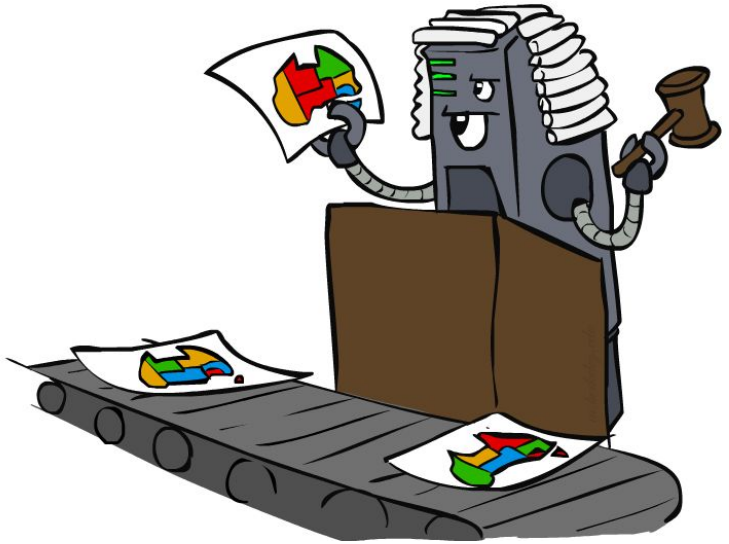


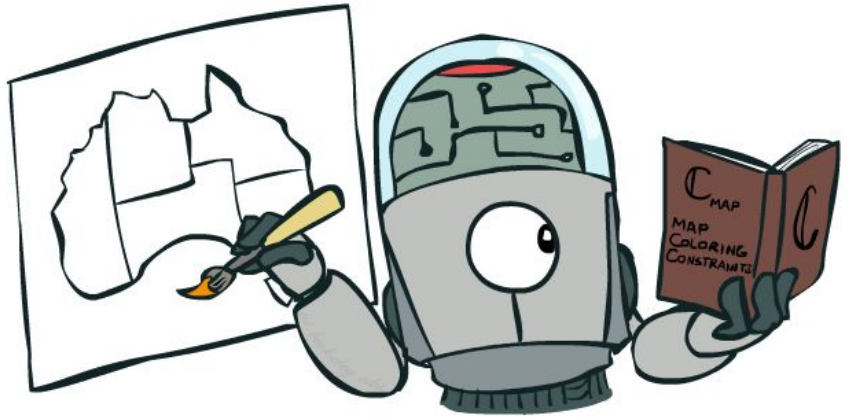
Constraint Satisfaction Problems



Constraint Satisfaction Problems

- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Allows useful general-purpose algorithms with more power than standard search algorithms



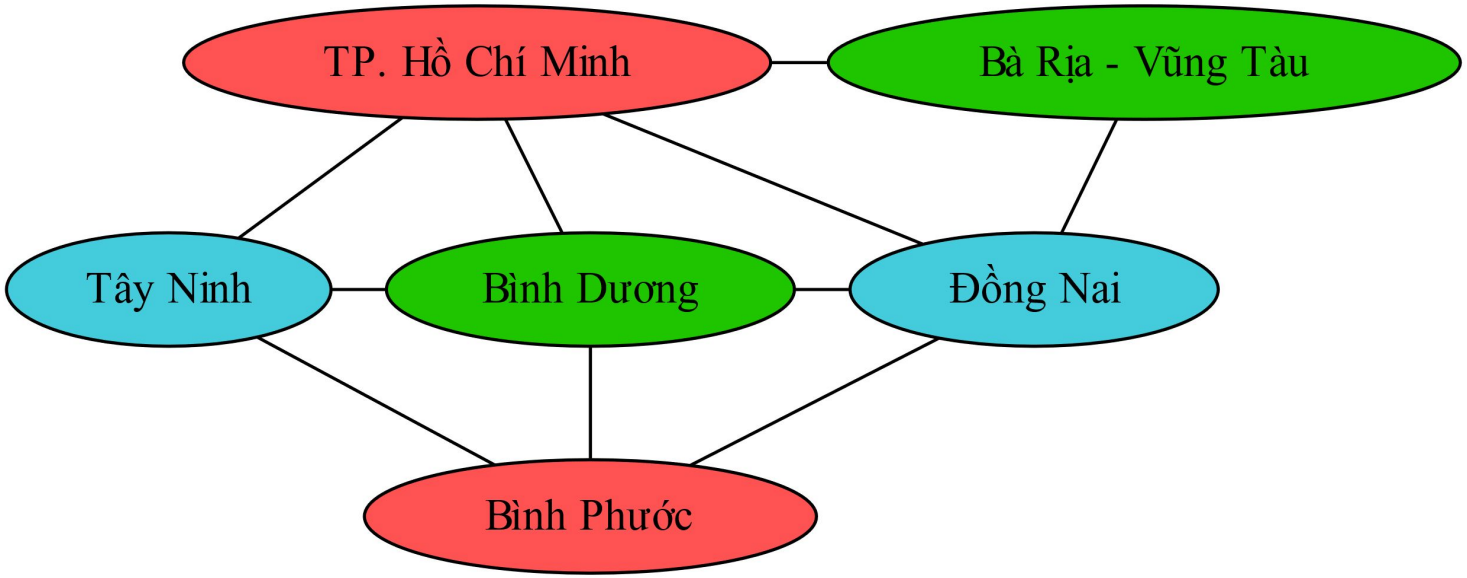


Constraint Satisfaction Problems

- A constraint satisfaction problem (CSP) is a tuple (X, D, C) where:
  - $X = \{x_1, x_2, \ldots, x_n\}$ is the set of variables.
  - $D = \{d_1, d_2, \ldots, d_n\}$ is the set of domains.
  - $C = \{c_1, c_2, \ldots, c_m\}$ is a set of constraints.
- For example, $x, y, z \in \{0,1\}, x + y = z$ is a CSP where:
  - Variables are: x, y, z
  - Domains are: $d_x = d_y = d_z = \{0,1\}$
  - There is a single constraint: $x + y = z$

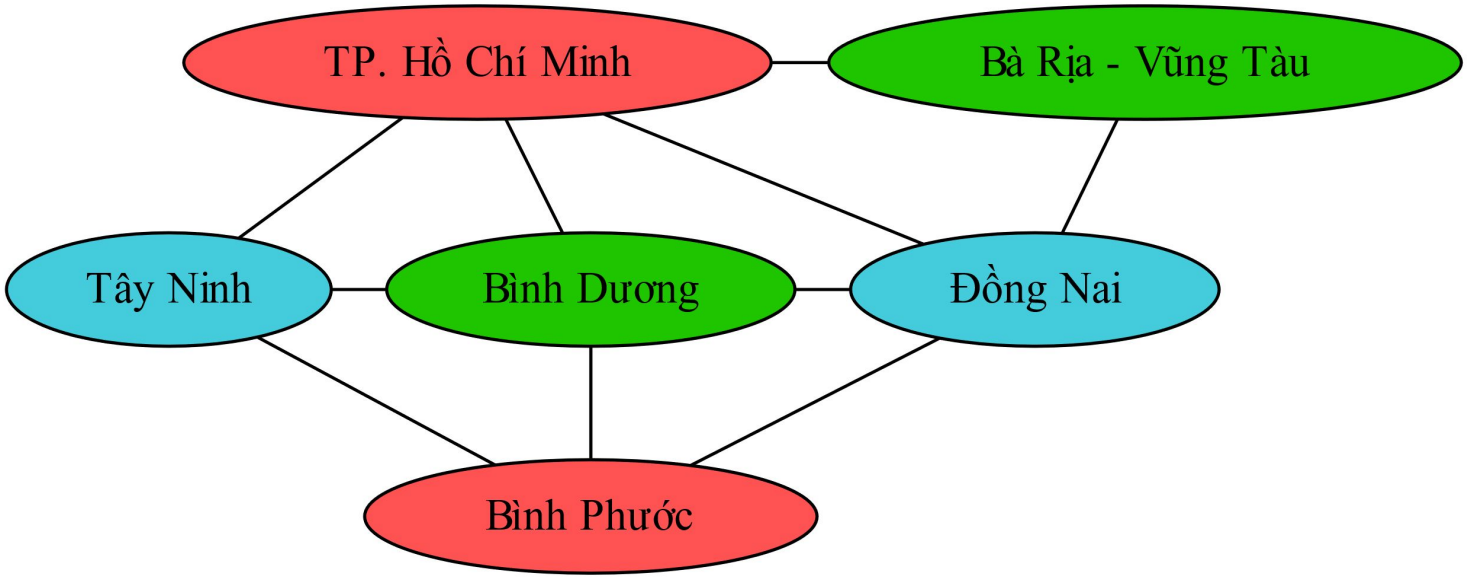
https://www.cs.upc.edu/~erodri/webpage/cps/cps.html

Constraint Graphs

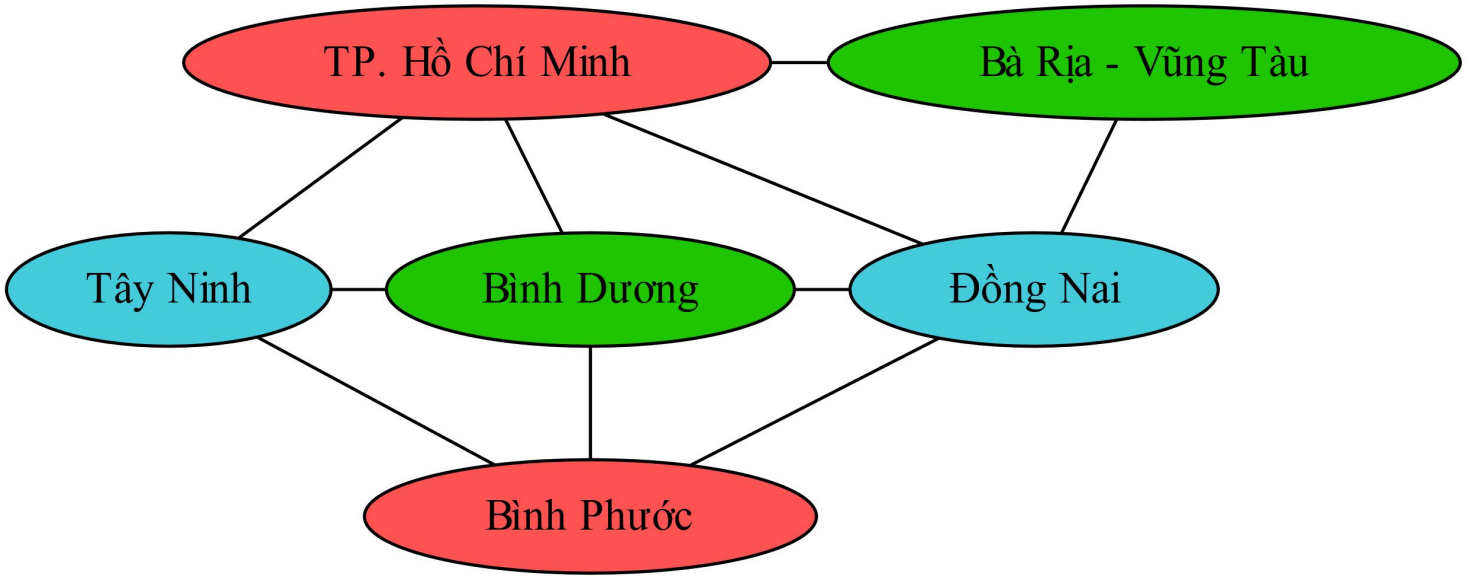


Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search.



CSP Examples



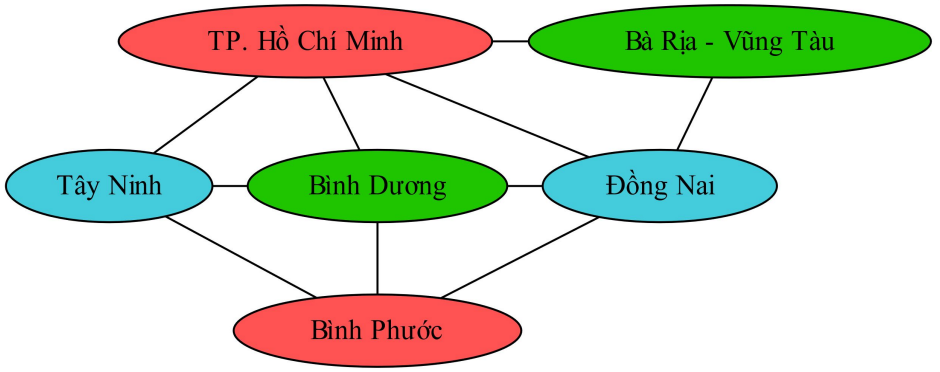
Example: Map Coloring

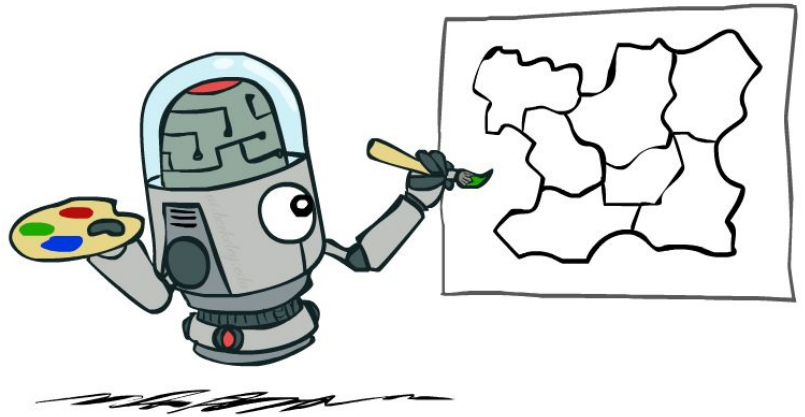
- Variables: $\{HCM, TN, BD, BP, ĐN, BR\}$
- Domains: $\{red, green, blue\}$
- Constraints: adjacent regions must have different colors

  Implicit: $HCM \neq TN$

  Explicit: $(HCM, TN) \in \{(red, green), (red, blue), \ldots\}$
- Solutions are assignments satisfying all constraints, e.g.:

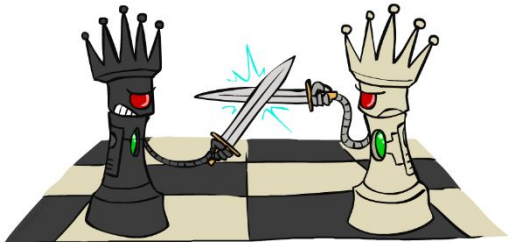
$$\{HCM = red, TN = blue, BD = green, \\ BP = red, ĐN = blue, BR = green\}$$

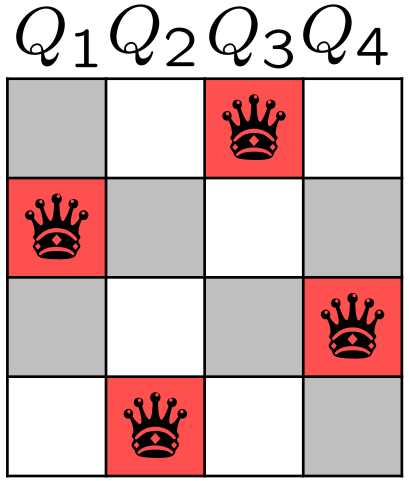




Example: N-Queens

- Variables: Q_k
- Domains: $\{1, 2, 3, \ldots N\}$
- Constraints:

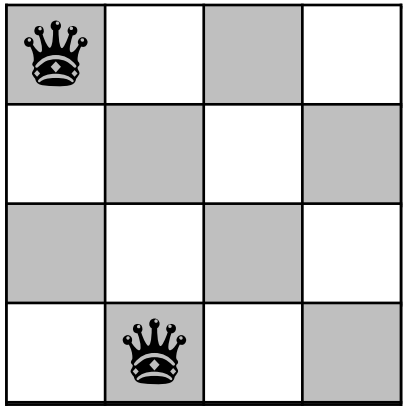




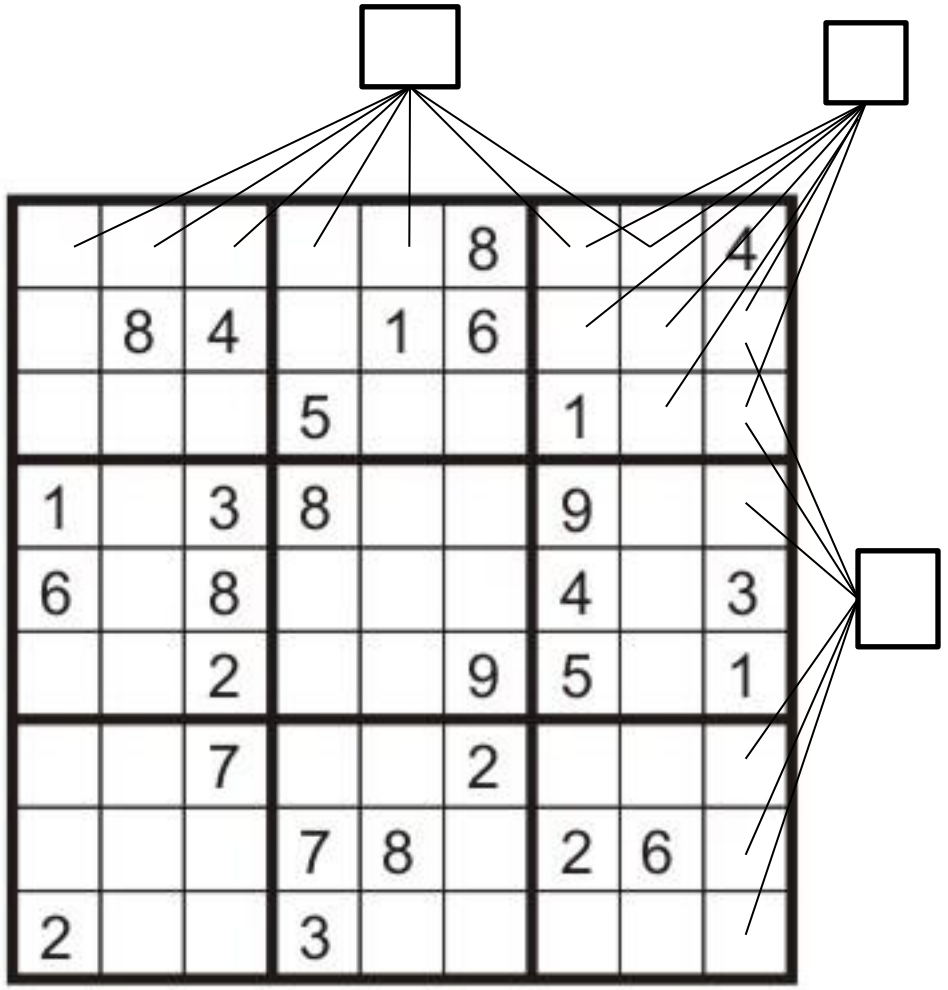
Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

Explicit: $(Q_1, Q_2) \in \{(1, 3), (1, 4), \ldots\}$

$\ldots$



Example: Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

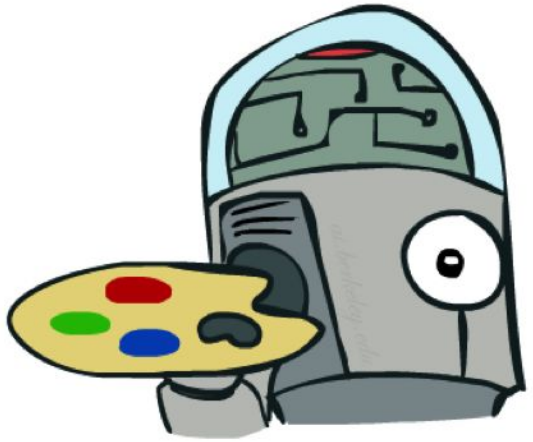
9-way alldiff for each row

9-way alldiff for each region

Varieties of CSPs and Constraints

Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job



- Continuous variables
  - E.g., start/end times for Hubble Telescope observations

Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains), e.g.:

$$TP.HCM \neq blue$$

  - Binary constraints involve pairs of variables, e.g.:

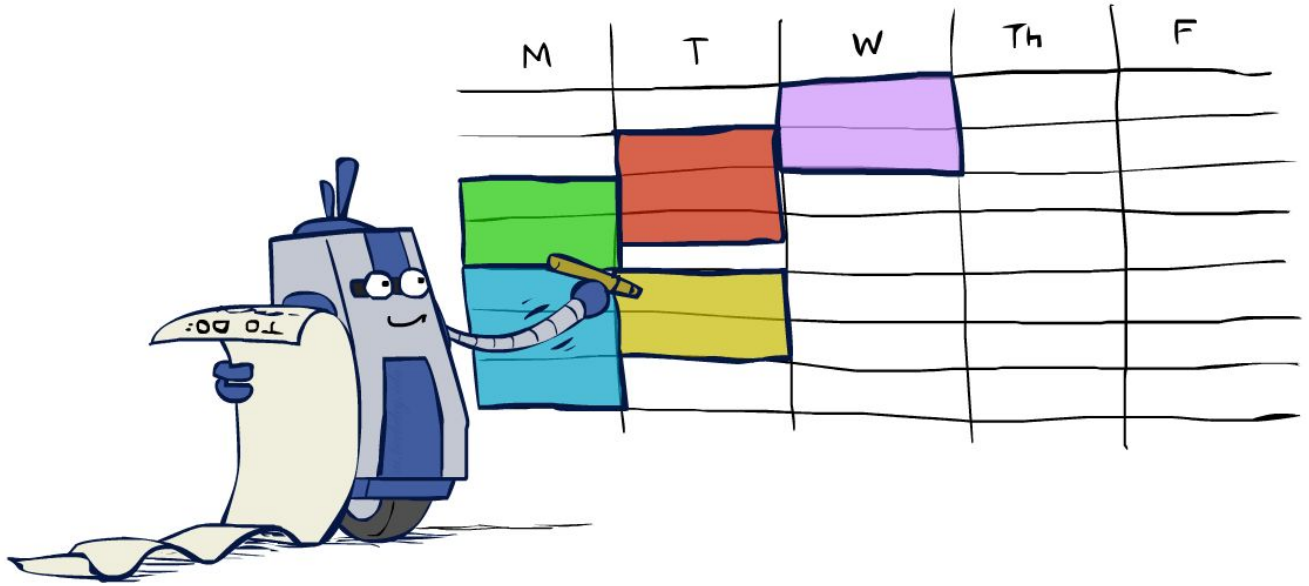
$$TP.HCM \neq Tay\ Ninh$$

  - Higher-order constraints involve 3 or more variables:
    e.g., Sudoku constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems

Real-World CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!

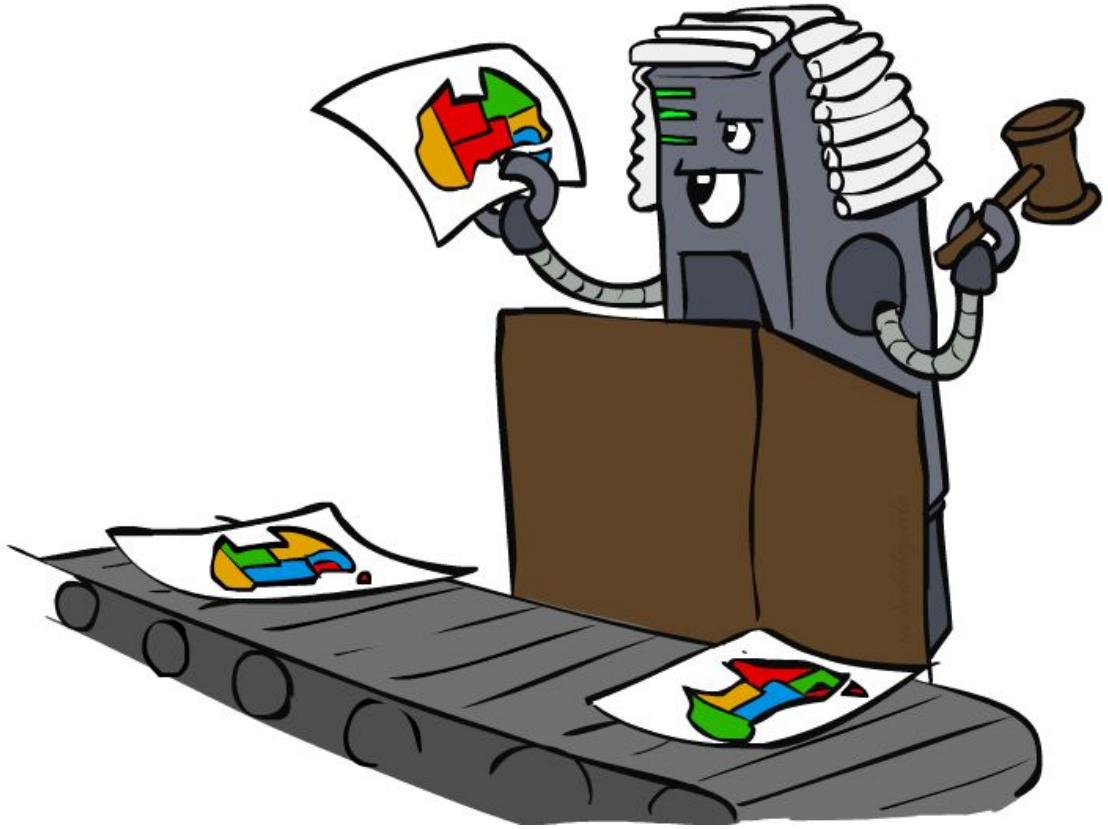


- Many real-world problems involve real-valued variables…

Solving CSPs

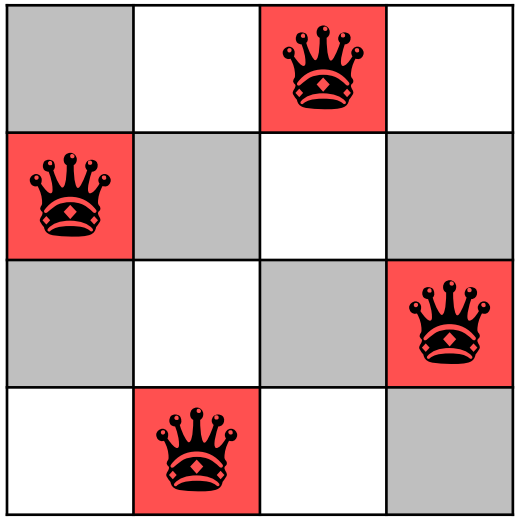
Standard Search Formulation

- Standard search formulation of CSPs
- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints
- We'll start with the straightforward, naïve approach, then improve it

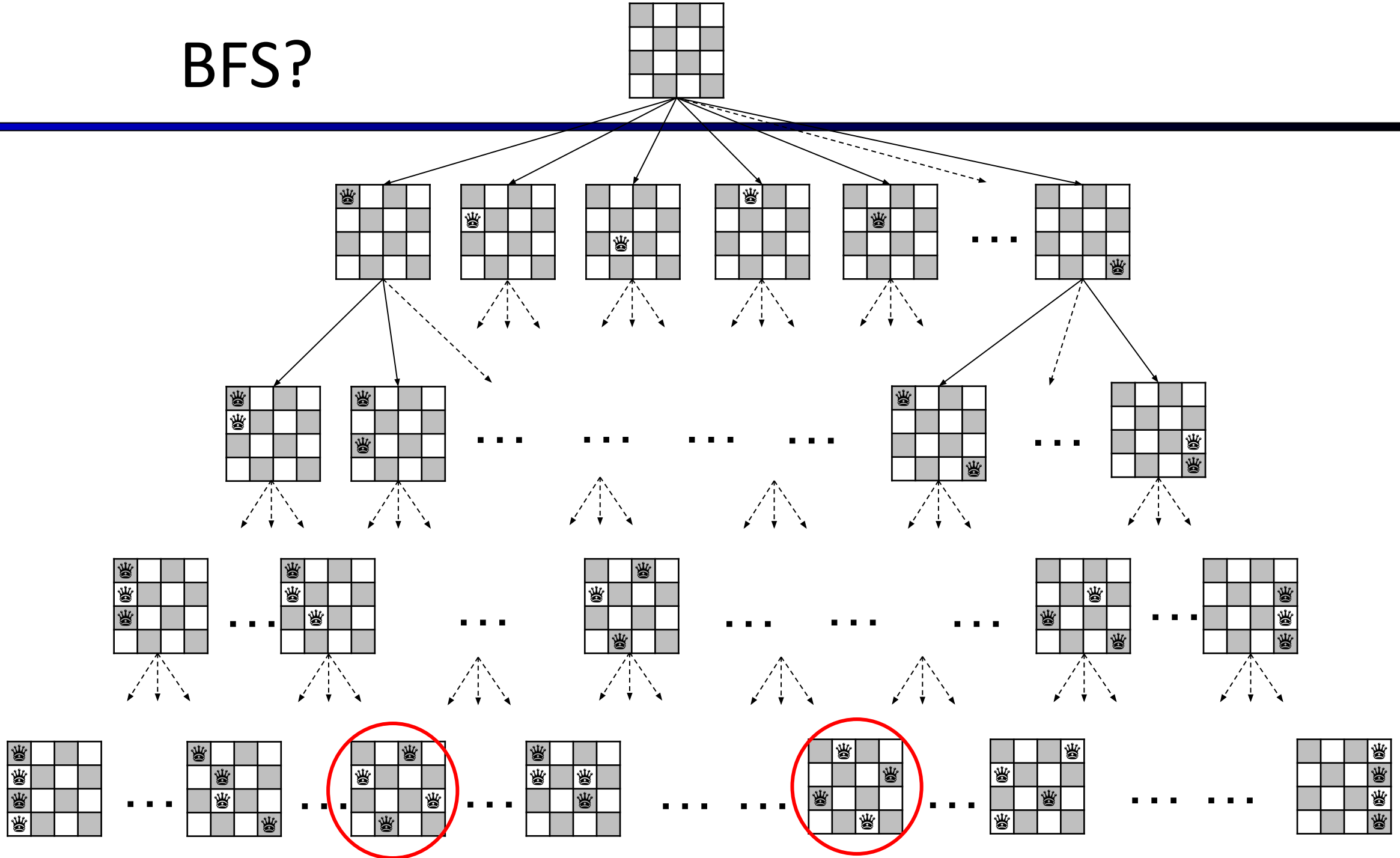


Search Methods

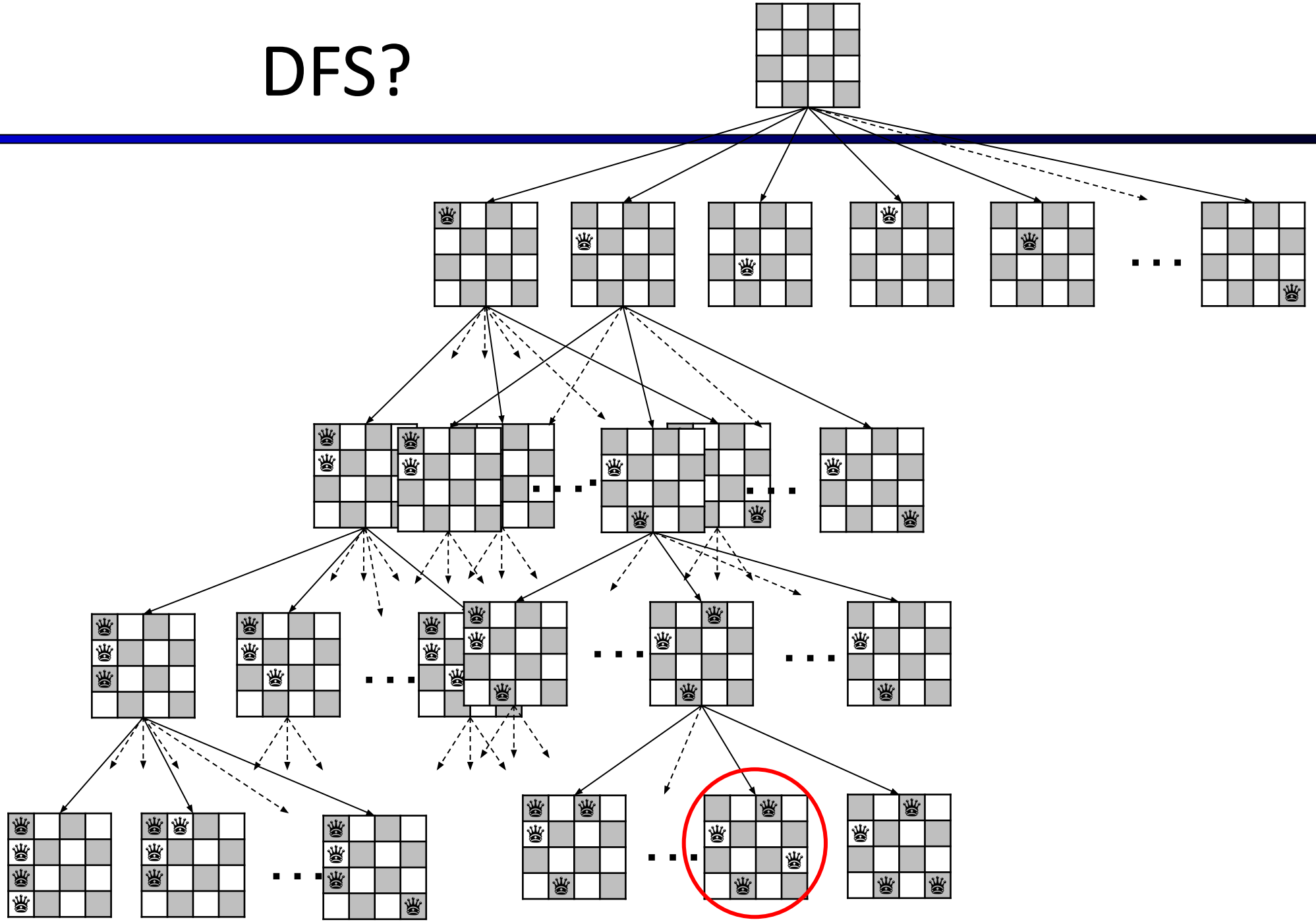
- What would BFS do?
- What would DFS do?
- What problems does naïve search have?



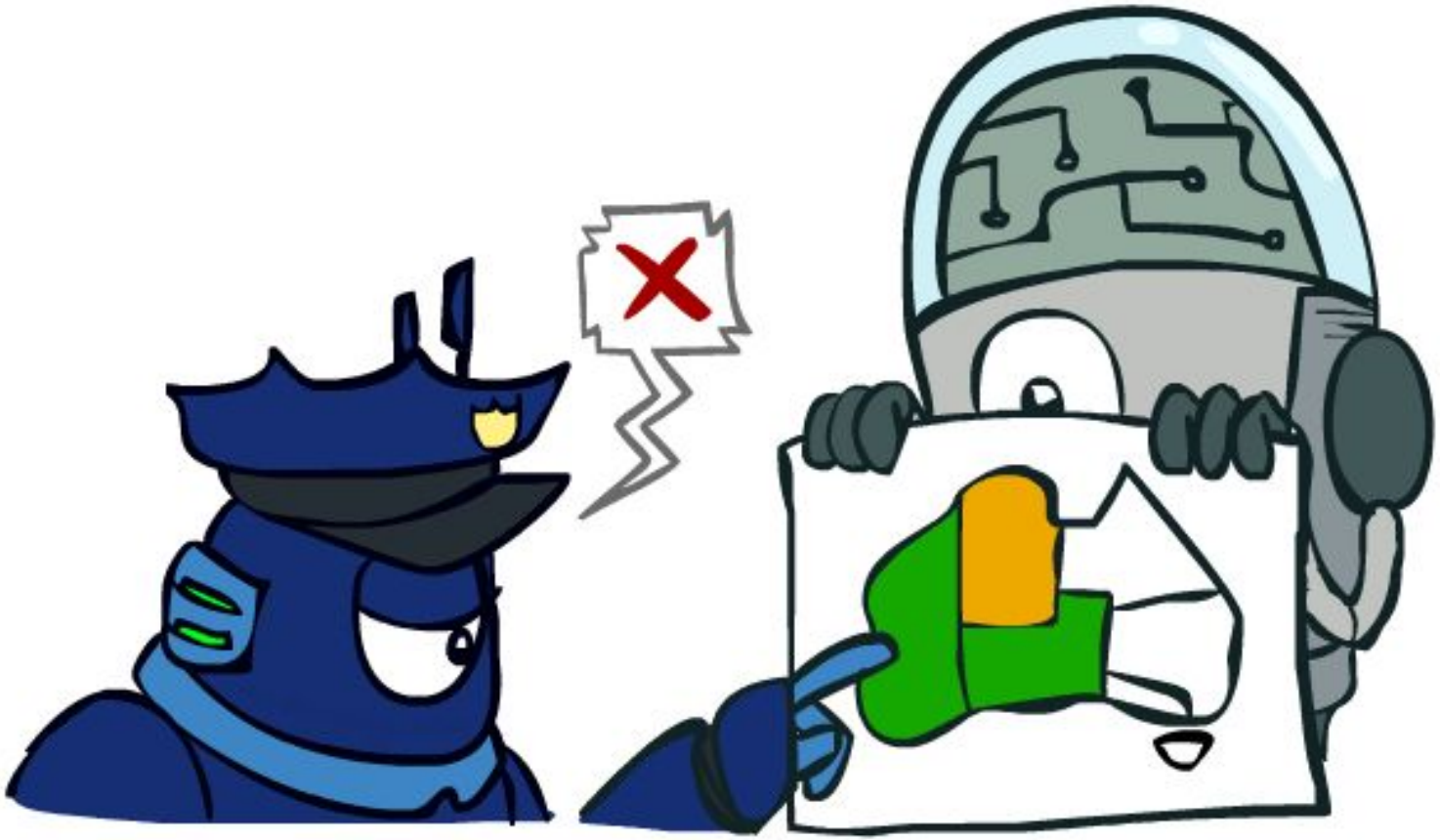
BFS?



DFS?

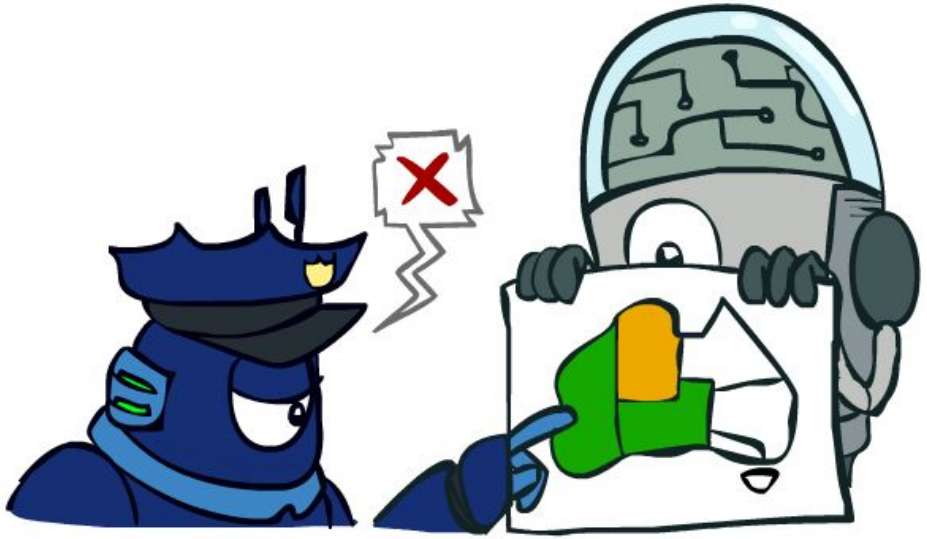


Backtracking Search

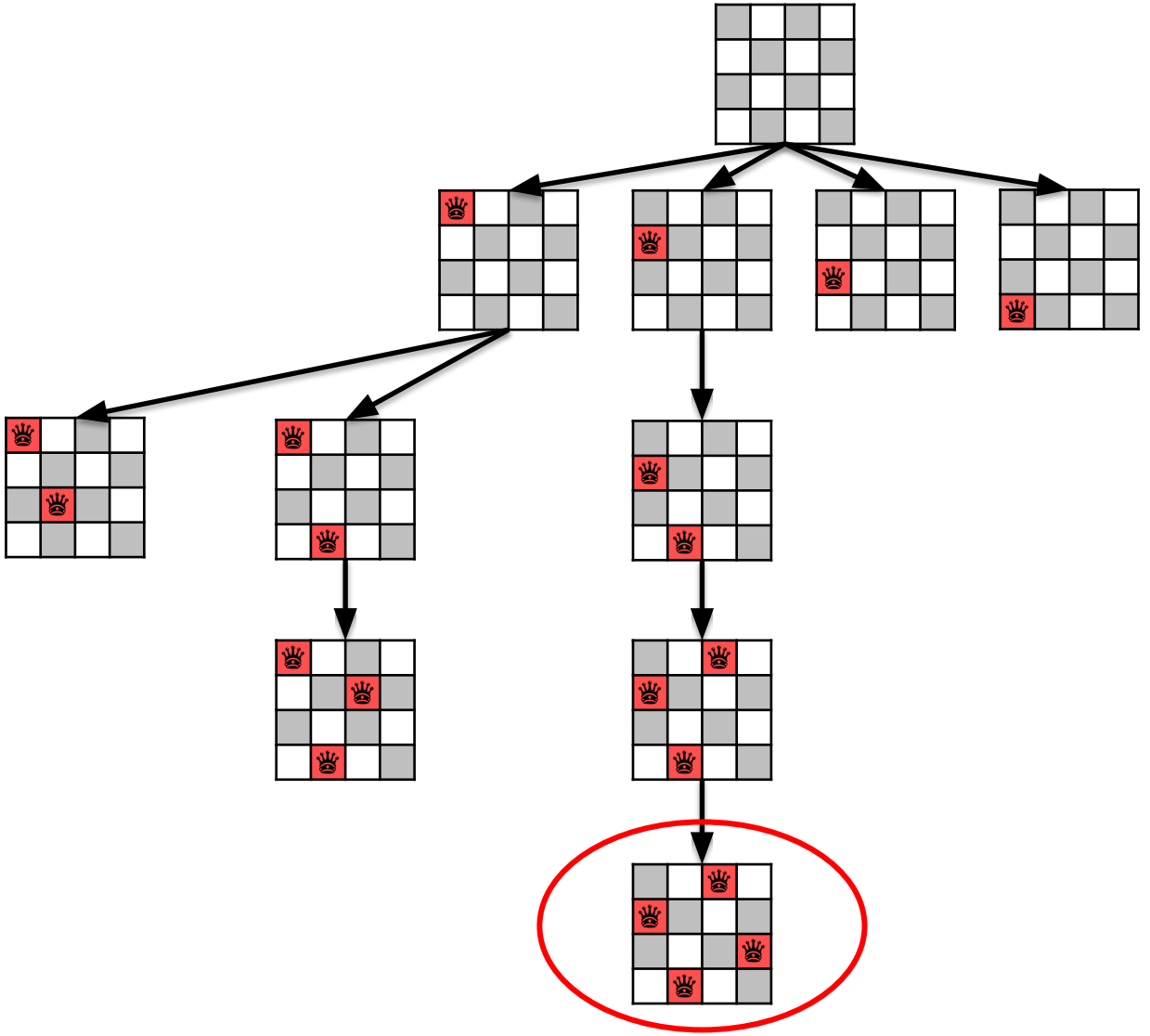


Backtracking Search

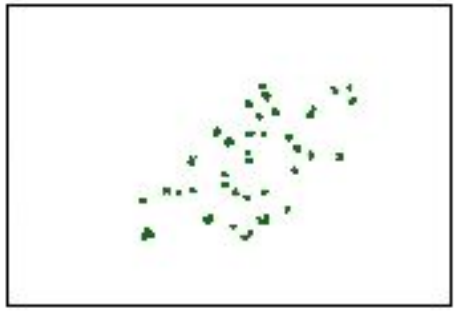
- Backtracking search is the basic uninformed algorithm for solving CSPs
- Idea 1: One variable at a time
  - Only need to consider assignments to a single variable at each step
- Idea 2: Check constraints as you go
  - i.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to check the constraints
  - "Incremental goal test"
- Depth-first search with these two improvements is called *backtracking search* (not the best name)
- Can solve n-queens for n ≈ 25



Backtracking Example



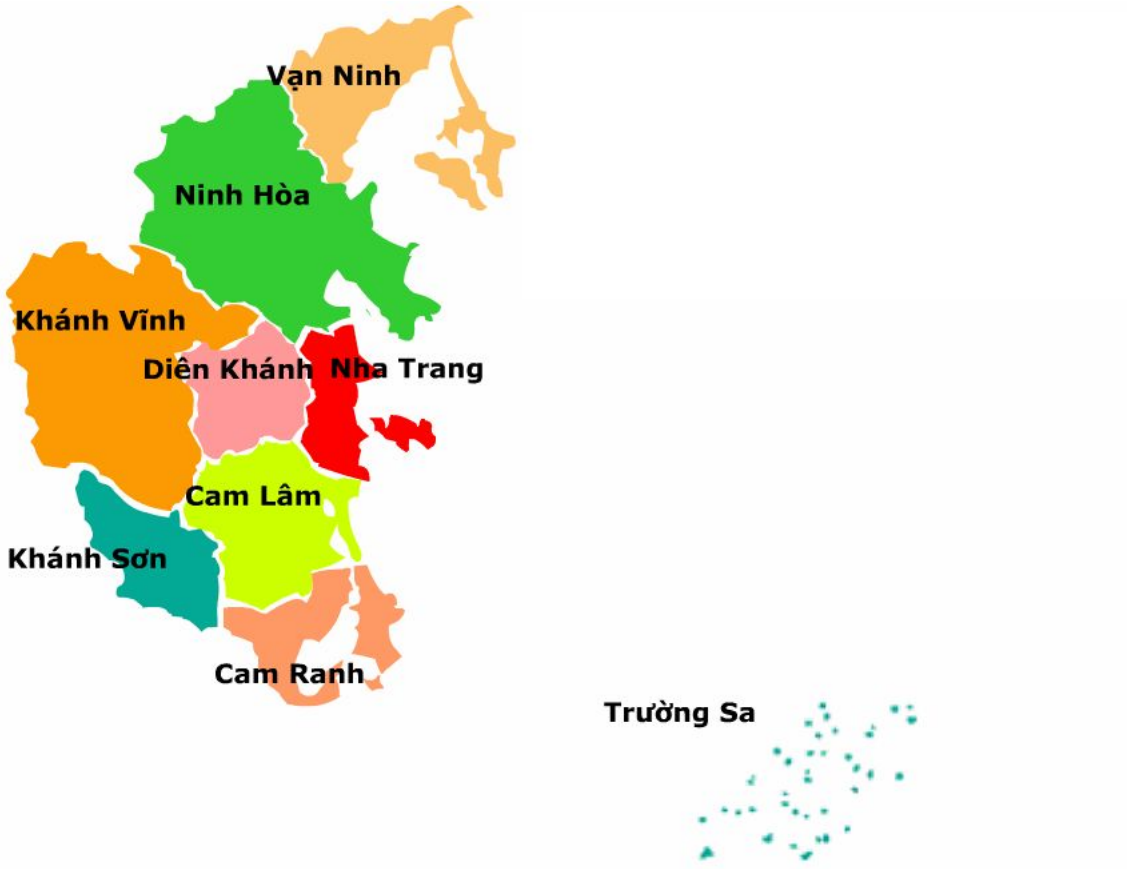
Map Coloring with Backtracking

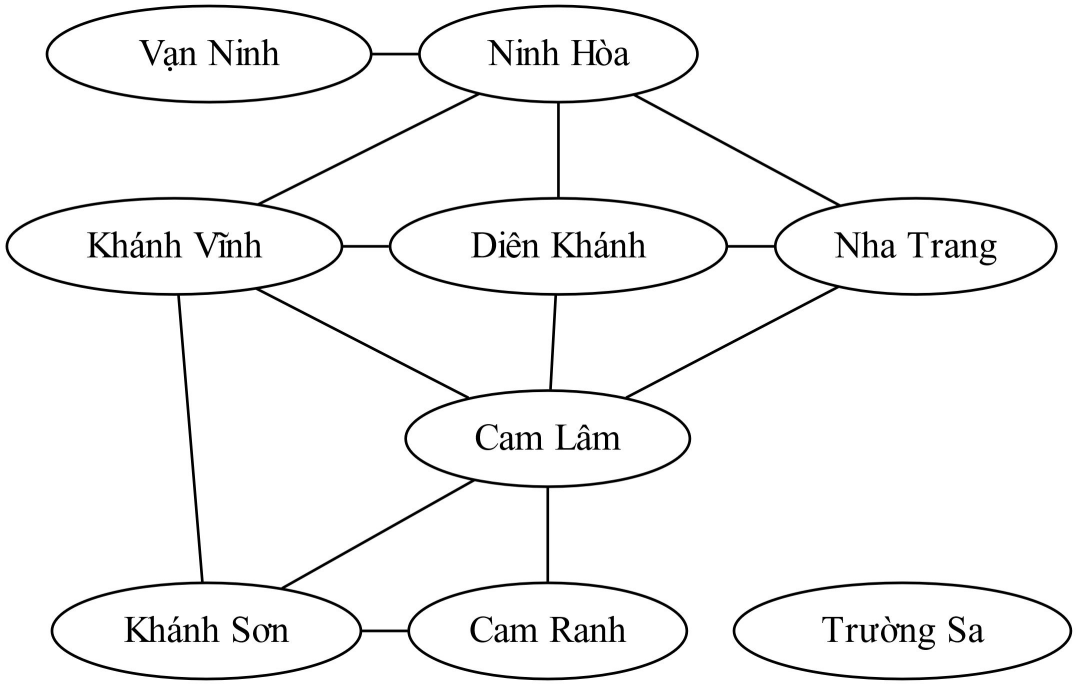


Where are we now?

https://vi.wikipedia.org/

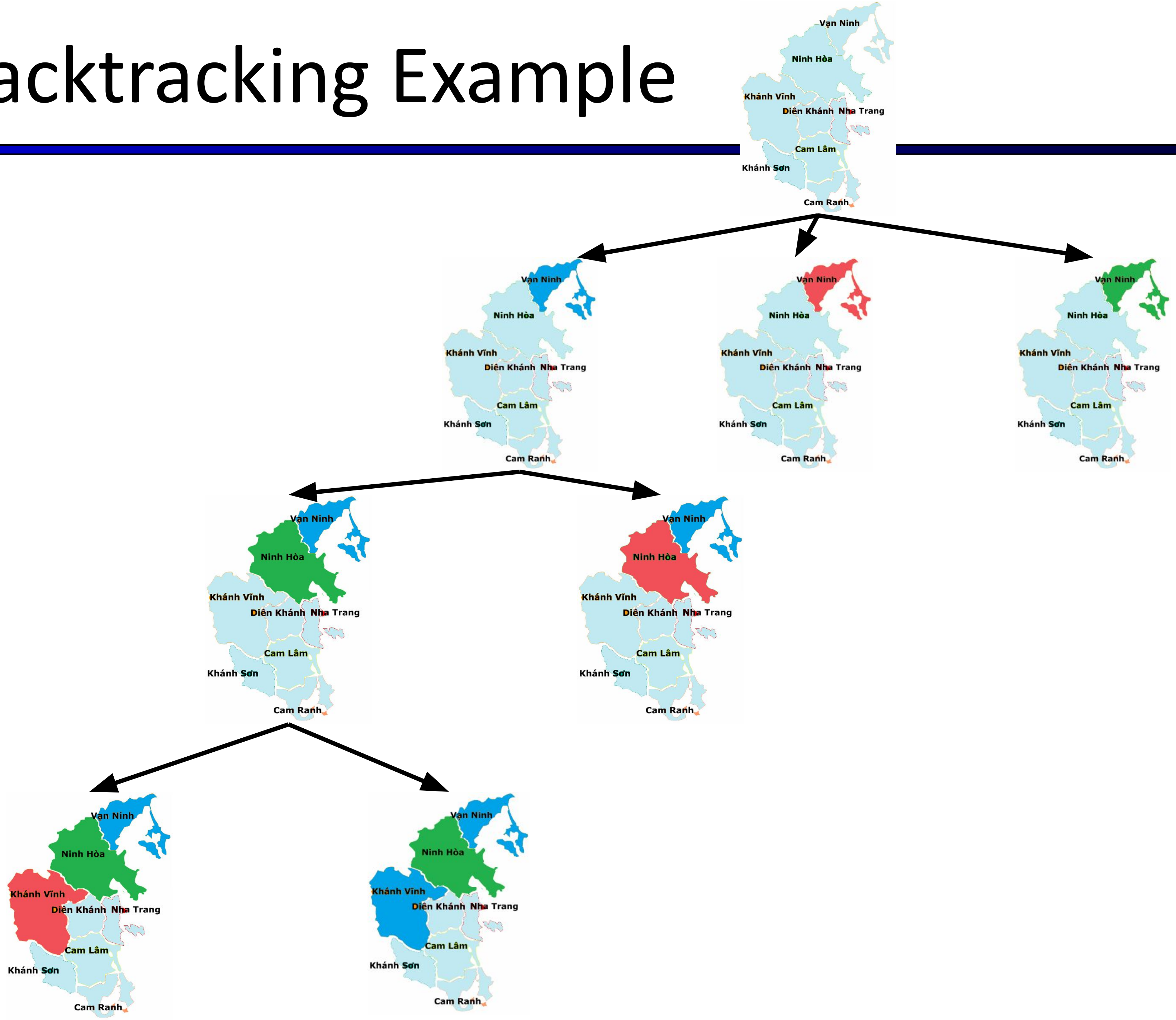
Map Coloring with Backtracking

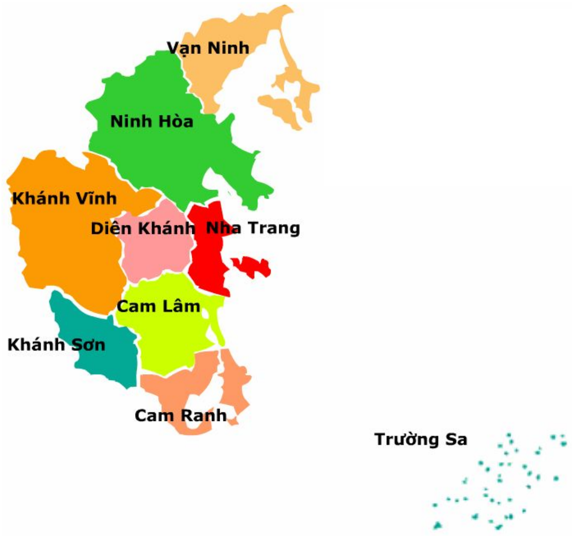




https://camranhcitygate.info/

Backtracking Example





Backtracking Search

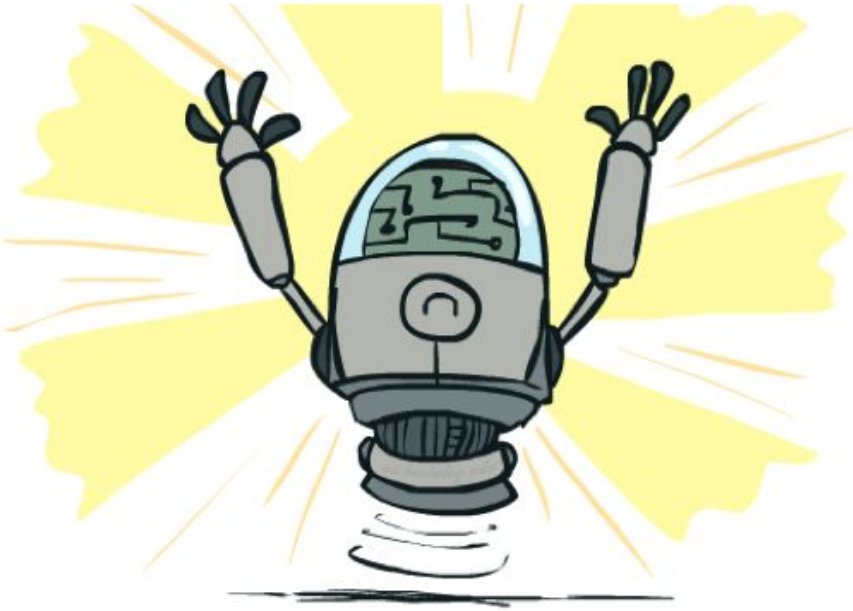
```
CSP-BACKTRACKING(PartialAssignment a)
    If a is complete then return a
    X <- select an unassigned variable
    D <- select an ordering for the domain of X
    For each value v in D do
        If v is consistent with a then
            Add (X = v) to a
            result <- CSP-BACKTRACKING(a)
            If result <> failure then return result
            Remove (X = v) from a
    Return failure
```

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the choice points?

https://www.dcc.fc.up.pt/~ines/aulas/1617/IA/cspNew.pdf

Improving Backtracking

- General-purpose ideas give huge gains in speed
- Filtering: Can we detect inevitable failure early?
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Structure: Can we exploit the problem structure?



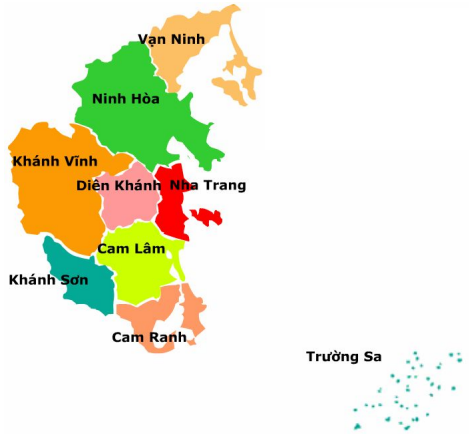
Filtering

Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: Cross off values that violate a constraint when added to the existing assignment

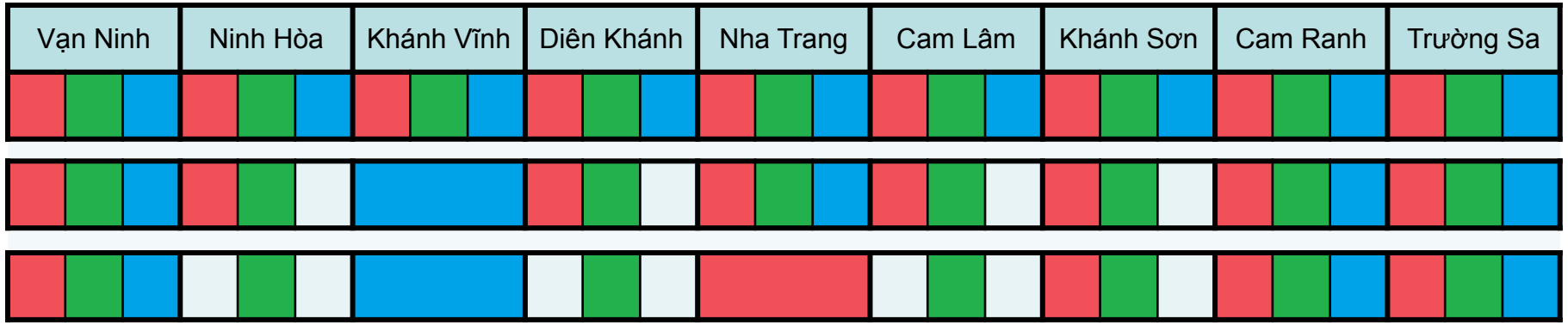
Filtering: Forward Checking

Vạn Ninh	Ninh Hòa	Khánh Vĩnh	Diên Khánh	Nha Trang	Cam Lâm	Khánh Sơn	Cam Ranh	Trường Sa



Filtering: Constraint Propagation

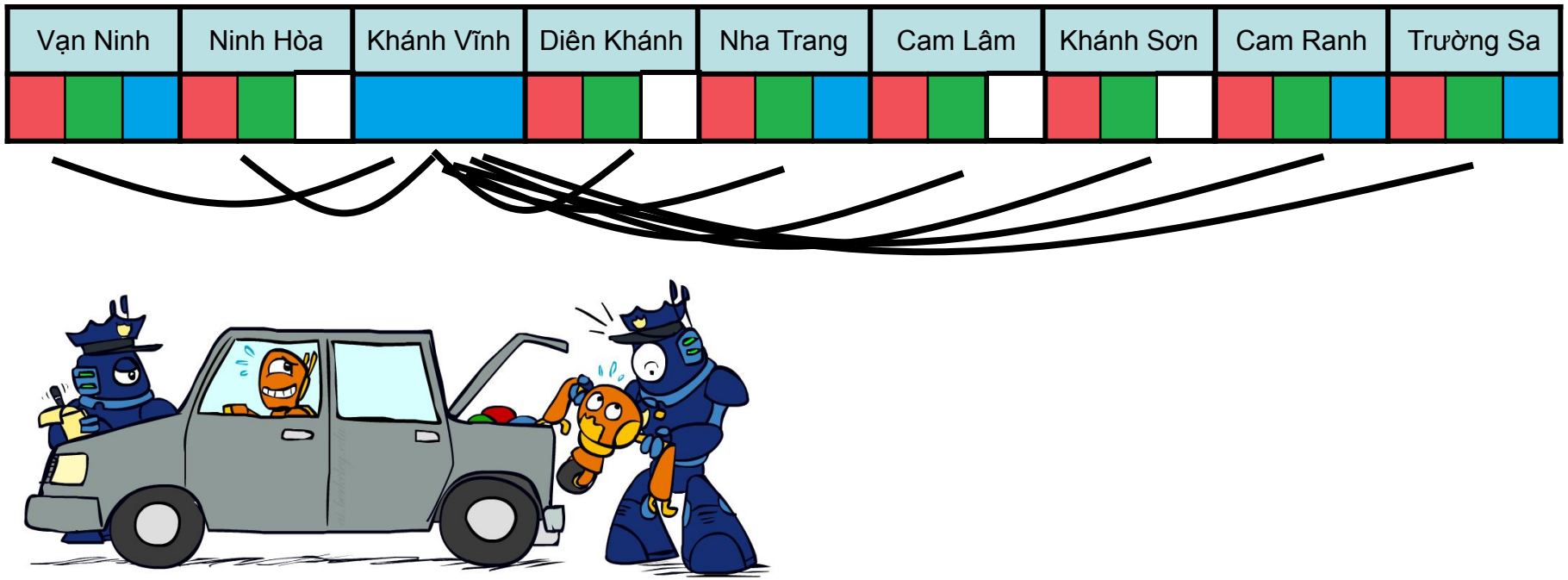
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



- Ninh Hòa and Diên Khánh cannot both be green!
- Why didn’t we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

Consistency of A Single Arc

- An arc X → Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint



Delete from the tail!

- Forward checking: Enforcing consistency of arcs pointing to each new assignment

Consistency of A Single Arc

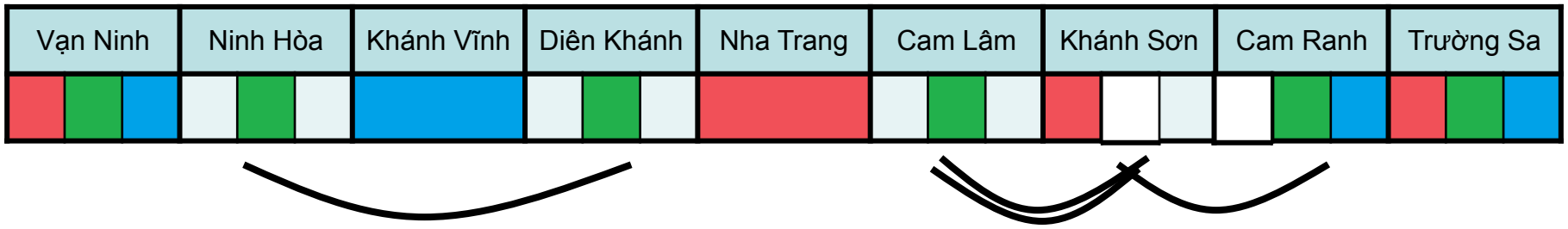
- Revise(i, j) removes values from d_i without support in d_j.

function Revise(i, j)
    change := false
    for each $a \in d_i$ **do**
        if $\forall_{b \in d_j} \neg c_{ij}(a, b)$ **then**
            change := true
            remove a **from** d_i
    return change

https://www.cs.upc.edu/~erodri/webpage/cps/cps.html

Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:



Remember: Delete from the tail!

- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

Enforcing Arc Consistency in a CSP

procedure AC-3(X, D, C)

$Q := \{(i, j), (j, i) \mid \ c_{ij} \in C\}$ // each pair is added twice

while $Q \neq \emptyset$ **do**

$(i, j) :=$ Fetch(Q) // selects and removes from Q

if Revise(i, j) **then**

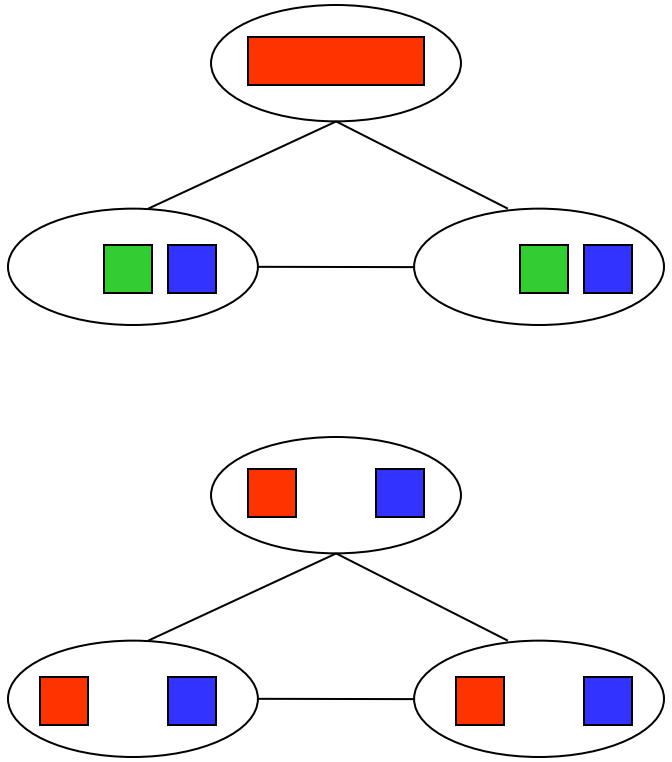
$Q := Q \cup \{(k, i) \mid \ c_{ki} \in C, k \neq j\}$

https://www.cs.upc.edu/~erodri/webpage/cps/cps.html

Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)

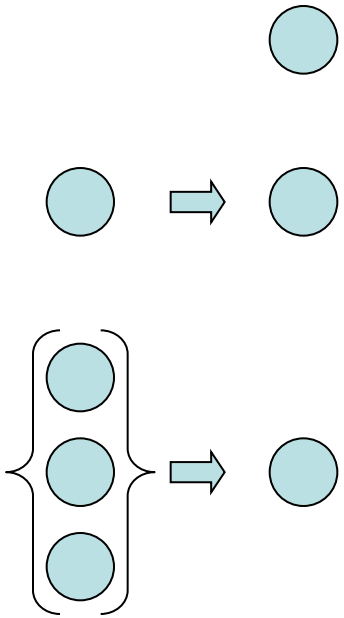
- Arc consistency still runs inside a backtracking search!



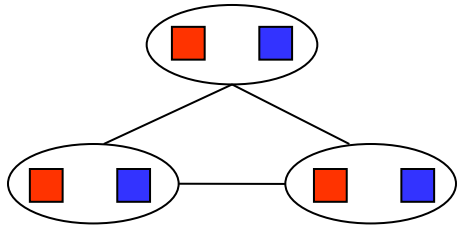
What went wrong here?

K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.

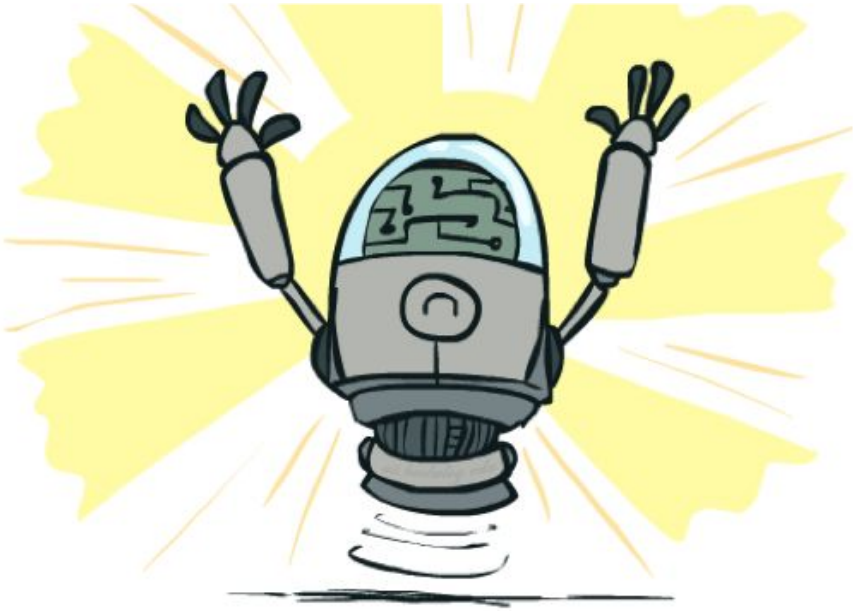


- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)



Improving Backtracking

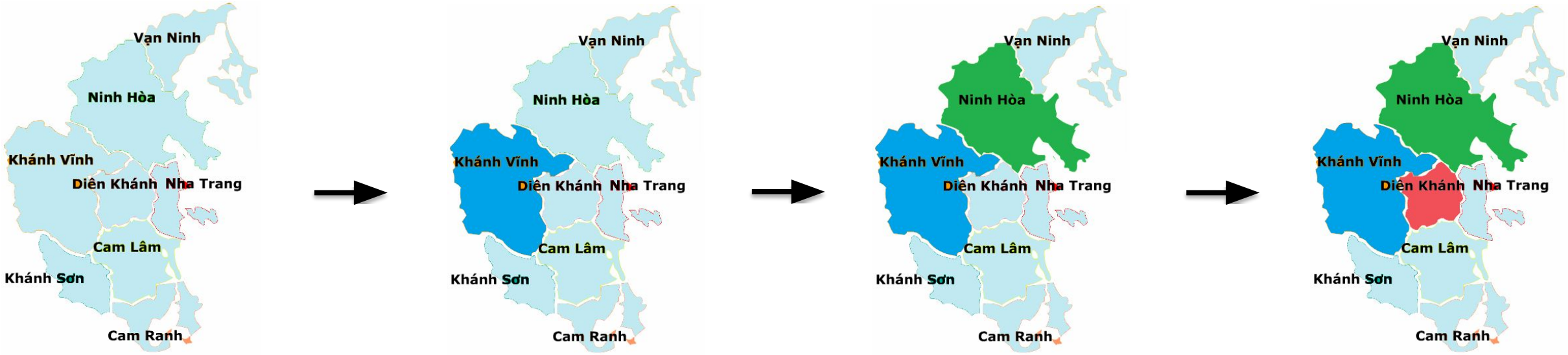
- General-purpose ideas give huge gains in speed
- *Filtering: Can we detect inevitable failure early?*
- **Ordering:**
  - **Which variable should be assigned next?**
  - **In what order should its values be tried?**
- Structure: Can we exploit the problem structure?



Ordering

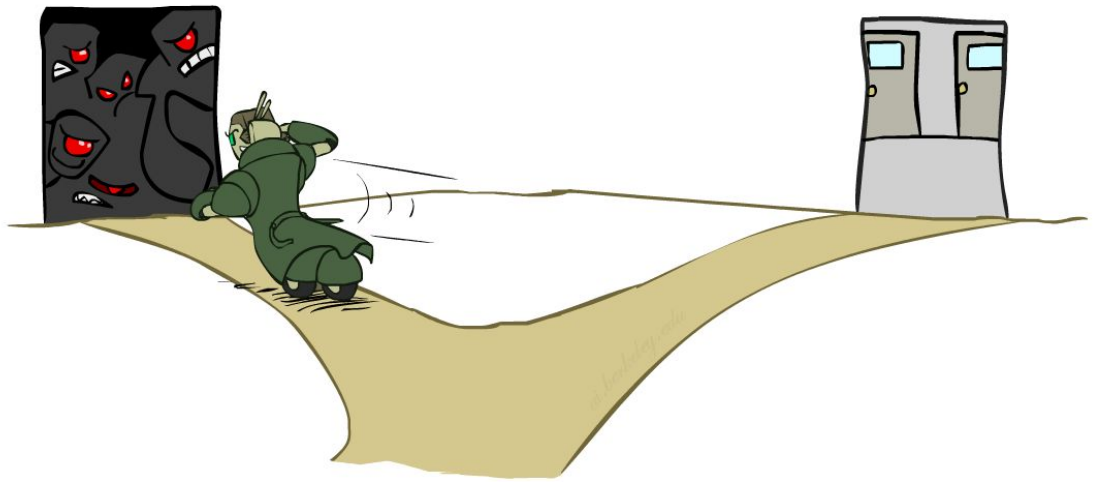
Ordering: Minimum Remaining Values

- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain



Ordering: Minimum Remaining Values

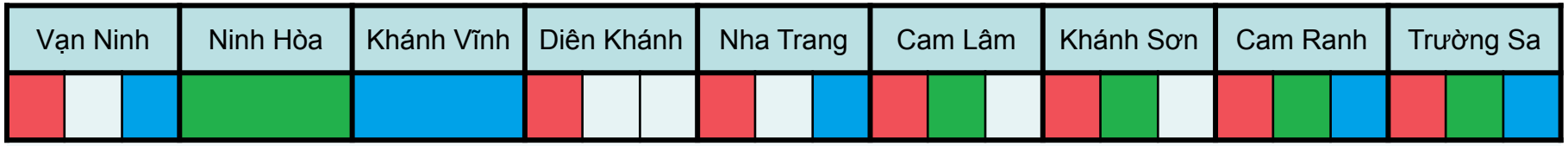
- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain



- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering

Ordering: Least Constraining Value

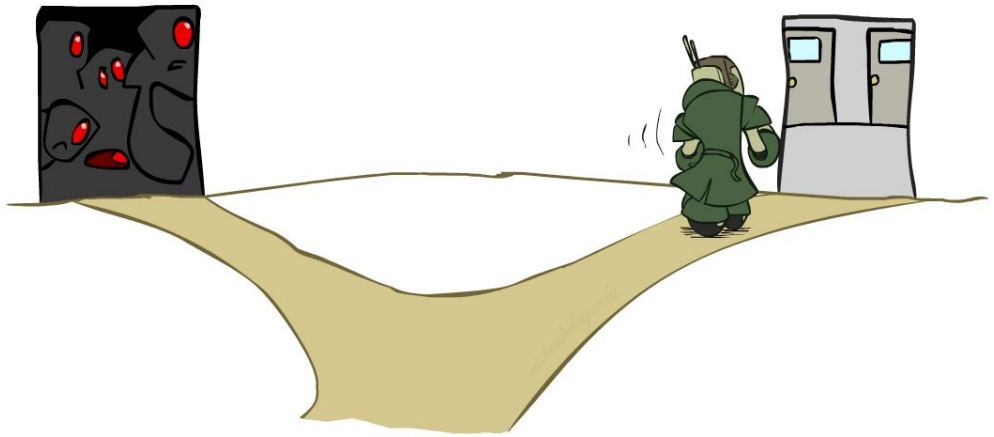
- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value* *(i.e.,* the one that rules out the fewest values in the remaining variables).



?

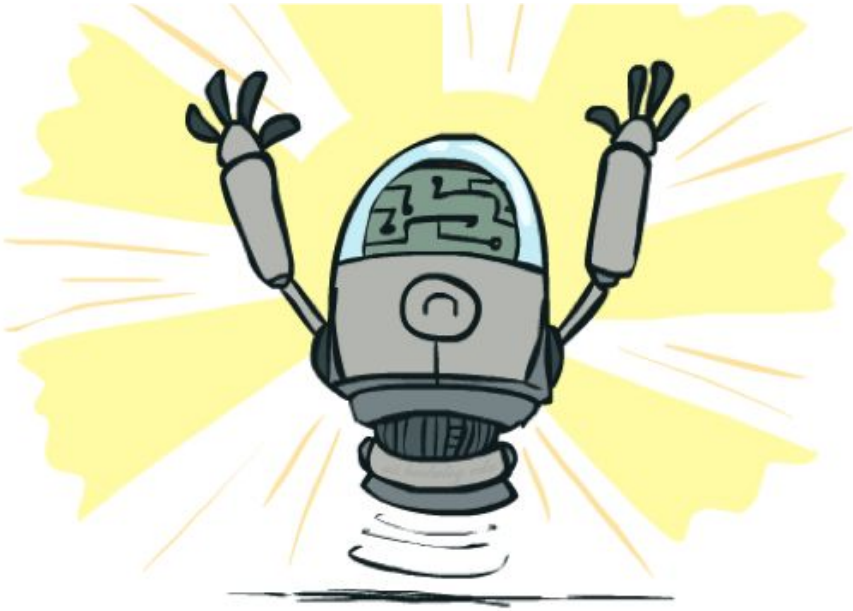
Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - I.e., the one that rules out the fewest values in the remaining variables
  - **Note that it may take some computation to determine this! (E.g., rerunning filtering)**
- Why least rather than most?
- Combining these ordering ideas makes 1000 queens feasible

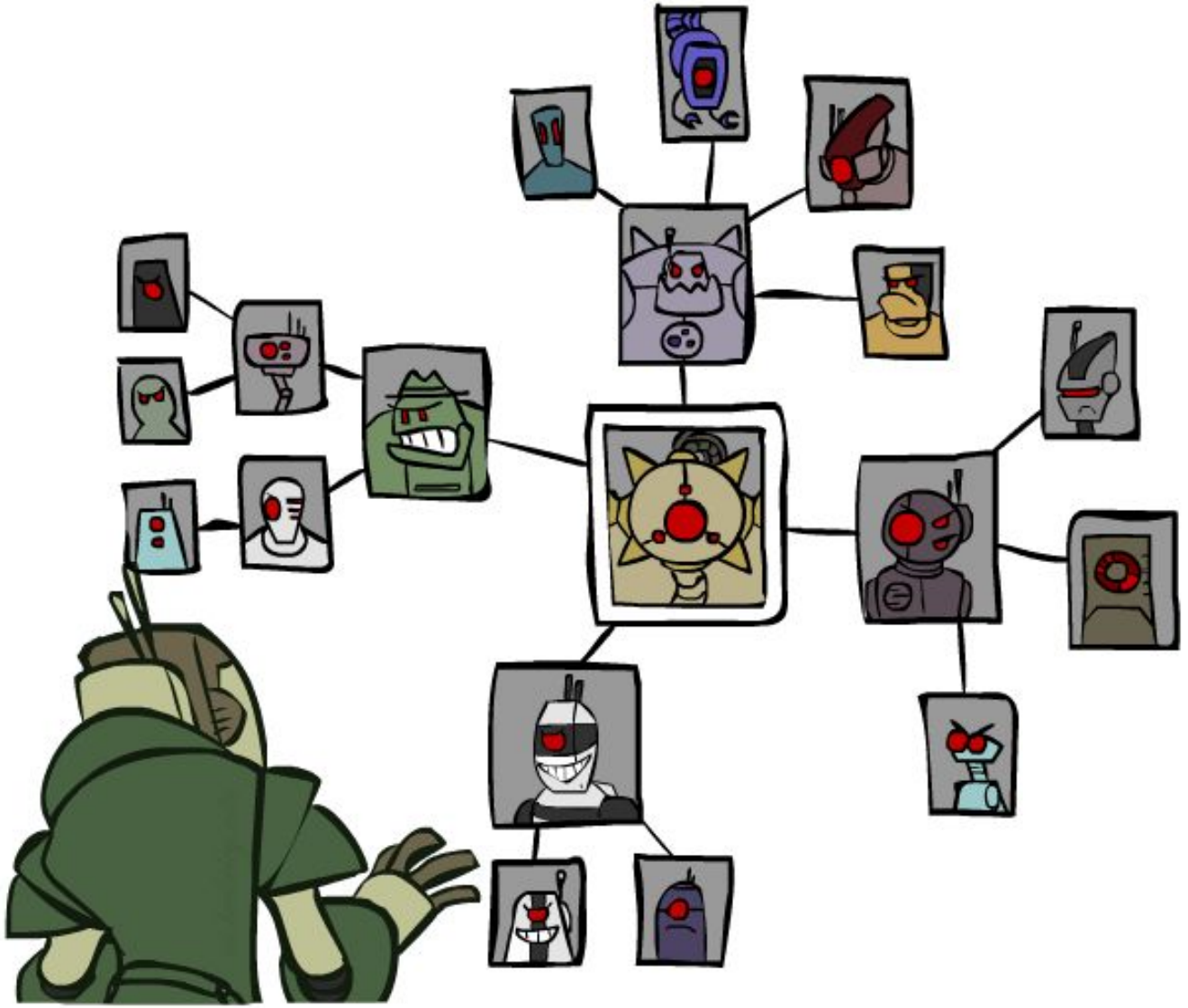


Improving Backtracking

- General-purpose ideas give huge gains in speed
- *Filtering: Can we detect inevitable failure early?*
- *Ordering:*
  - *Which variable should be assigned next?*
  - *In what order should its values be tried?*
- **Structure: Can we exploit the problem structure?**

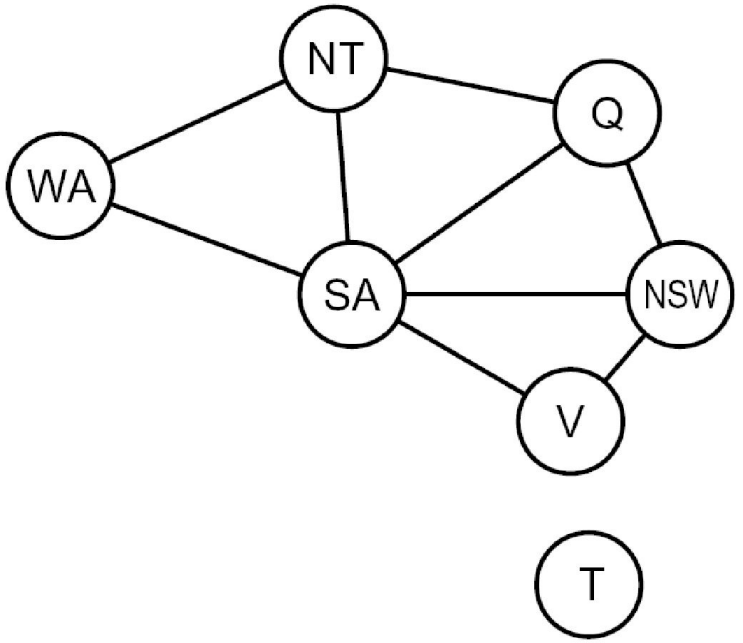


Structure

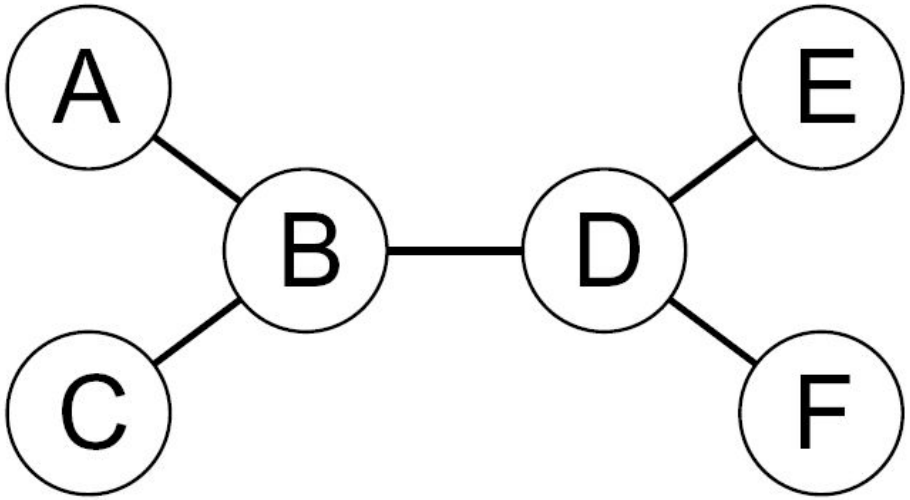


Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact
- Independent subproblems are identifiable as connected components of constraint graph
- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec

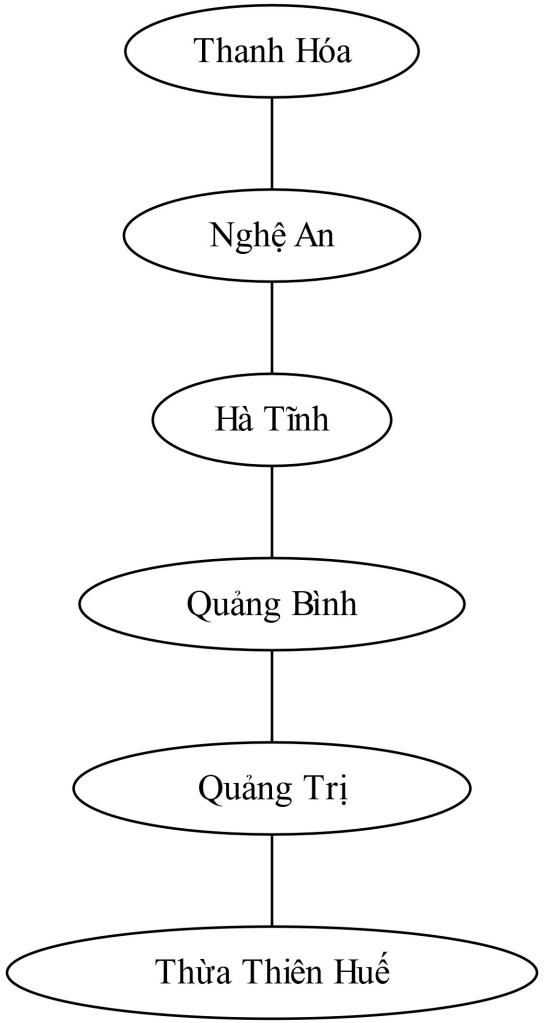


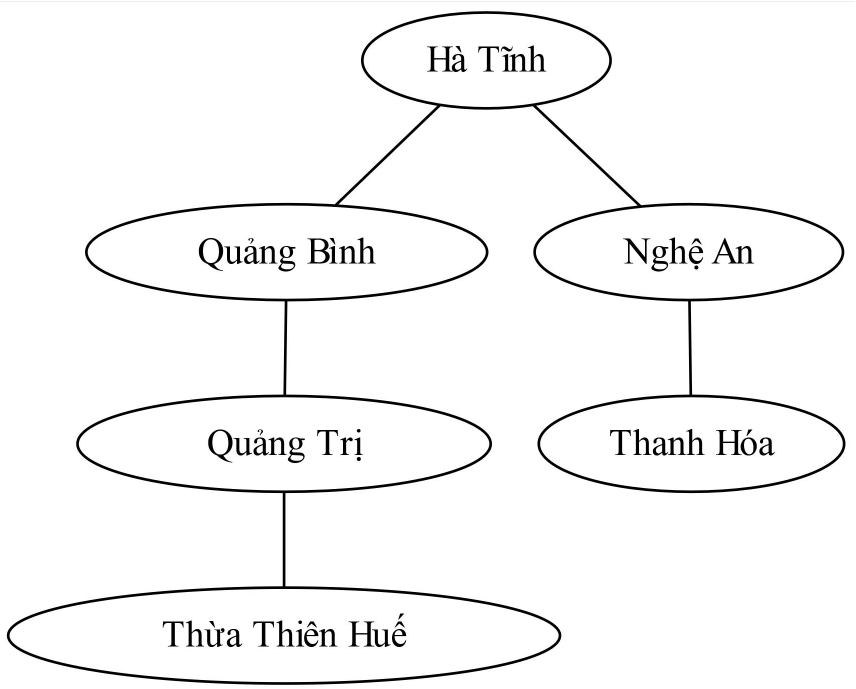
Tree-Structured CSPs



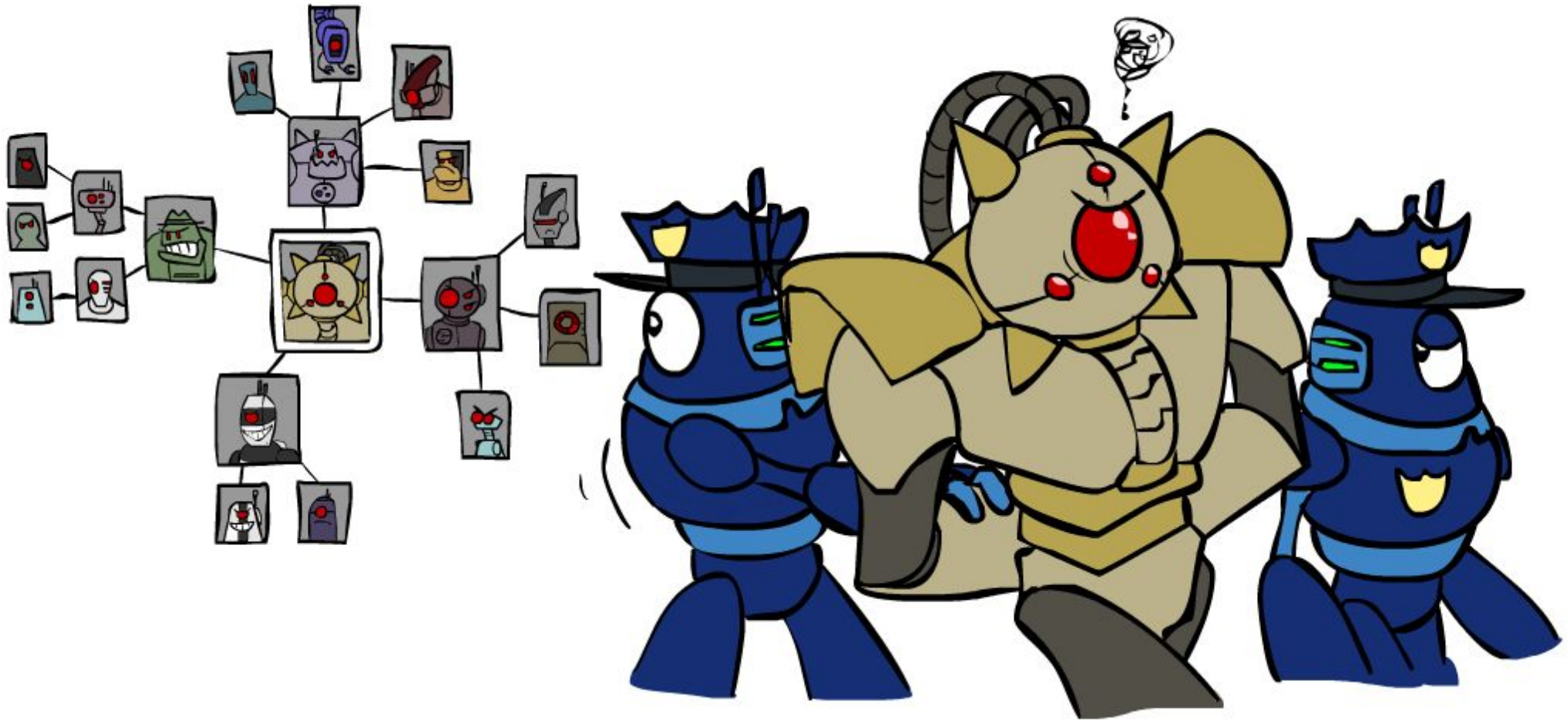
- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$

Tree-Structured CSPs

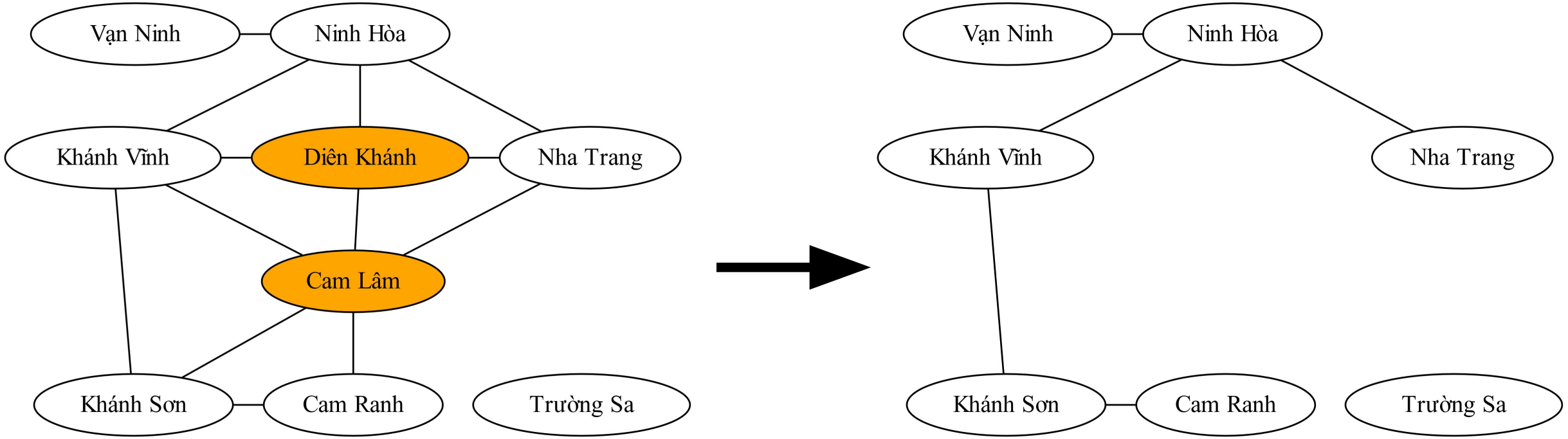




Improving Structure



Nearly Tree-Structured CSPs



- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

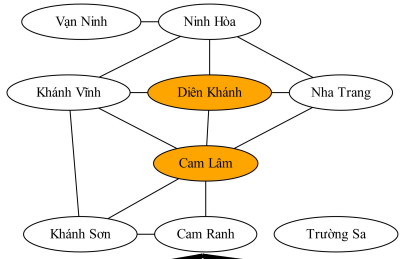
Cutset Conditioning

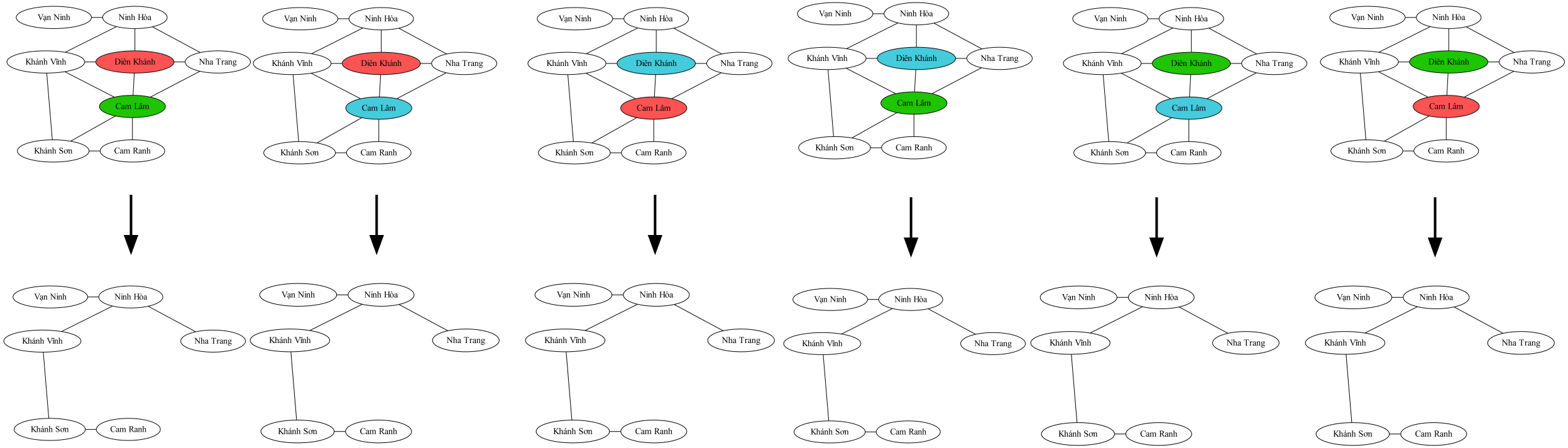
Choose a cutset

Instantiate the cutset (all possible ways)

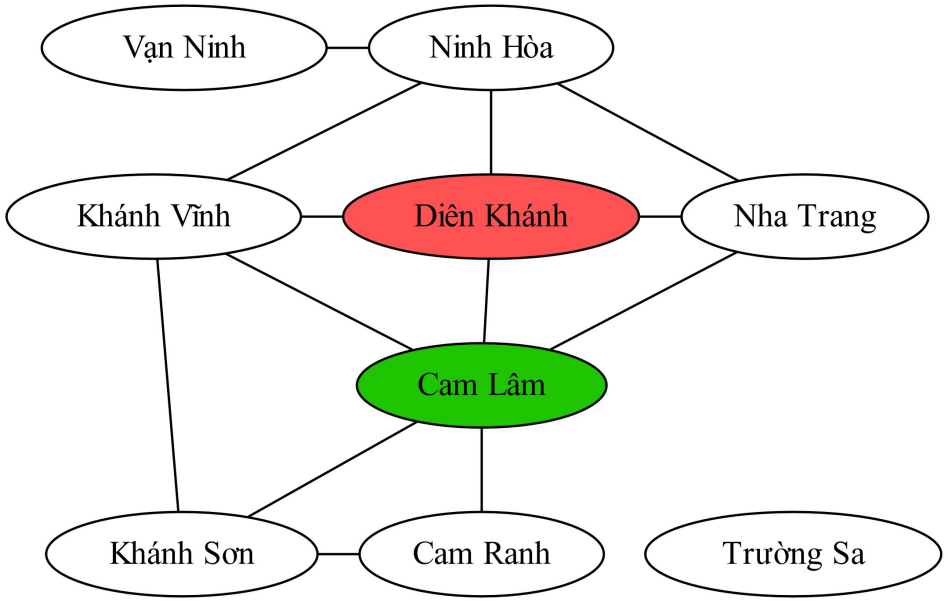
Compute residual CSP for each assignment

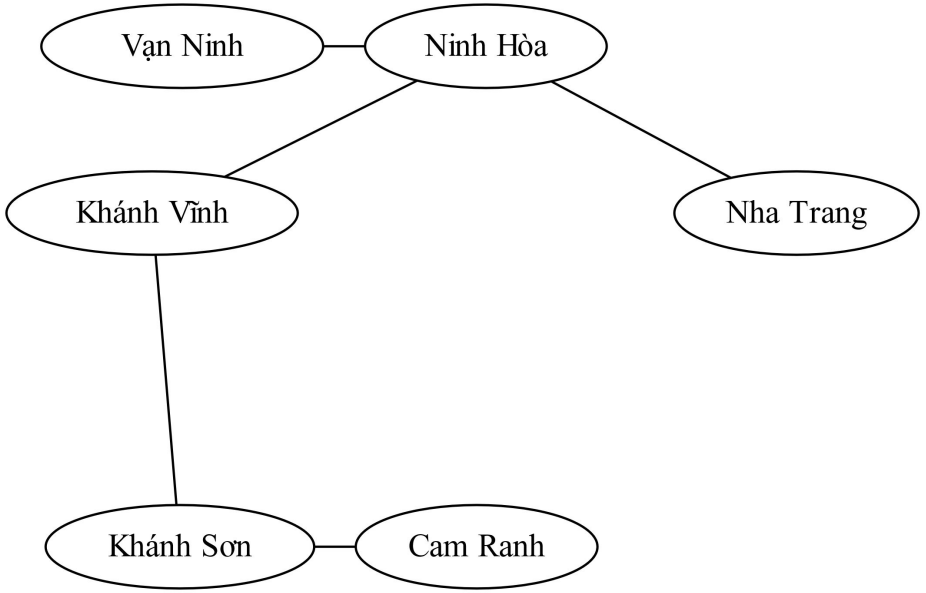
Solve the residual CSPs (tree structured)





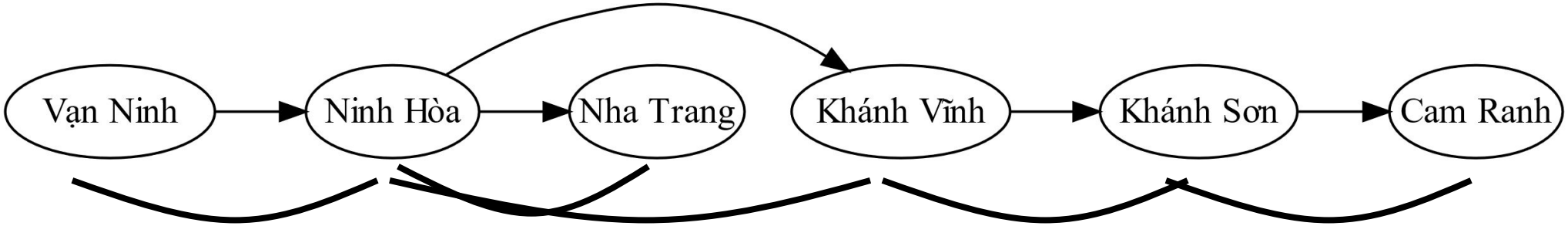
Cutset Conditioning

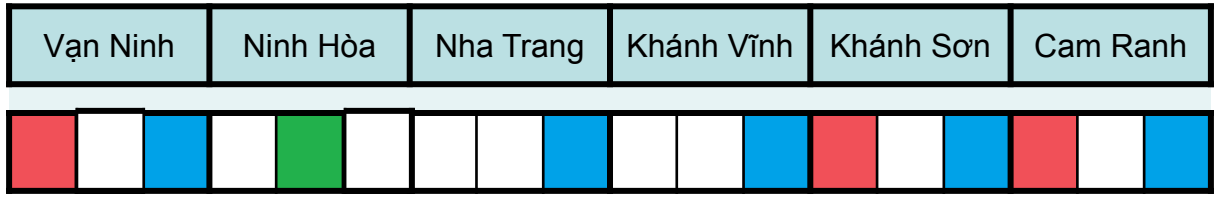




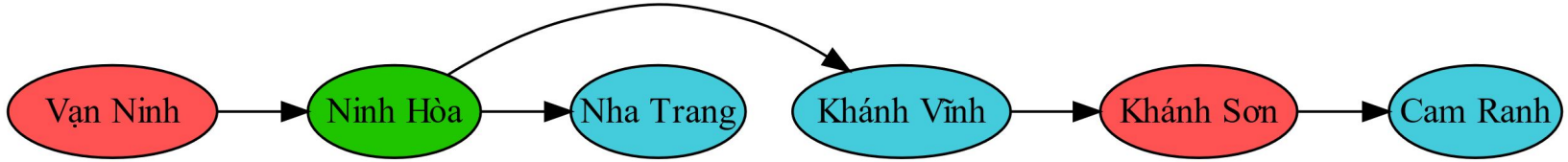
Tree-Structured CSP

- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children

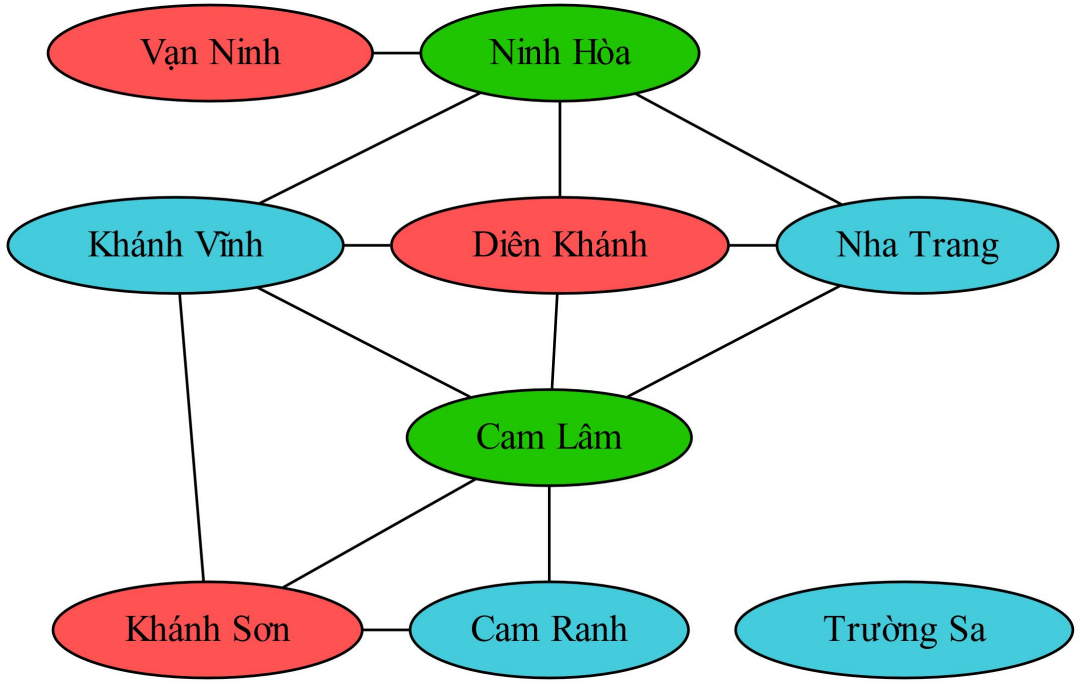


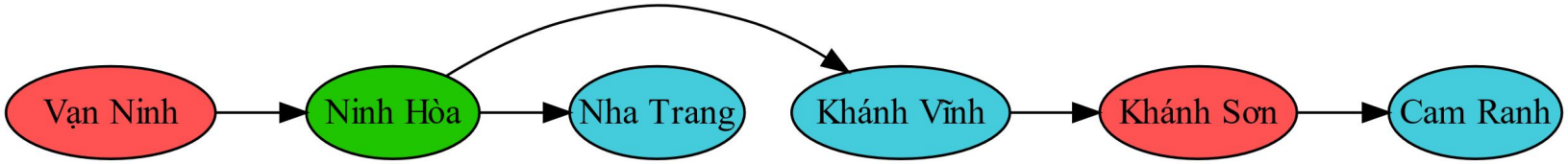


  - Remove backward: For i = n : 2, apply Remove-Inconsistency(Parent(X_i),X_i)
  - Assign forward: For i = 1 : n, assign X_i consistently with Parent(X_i)

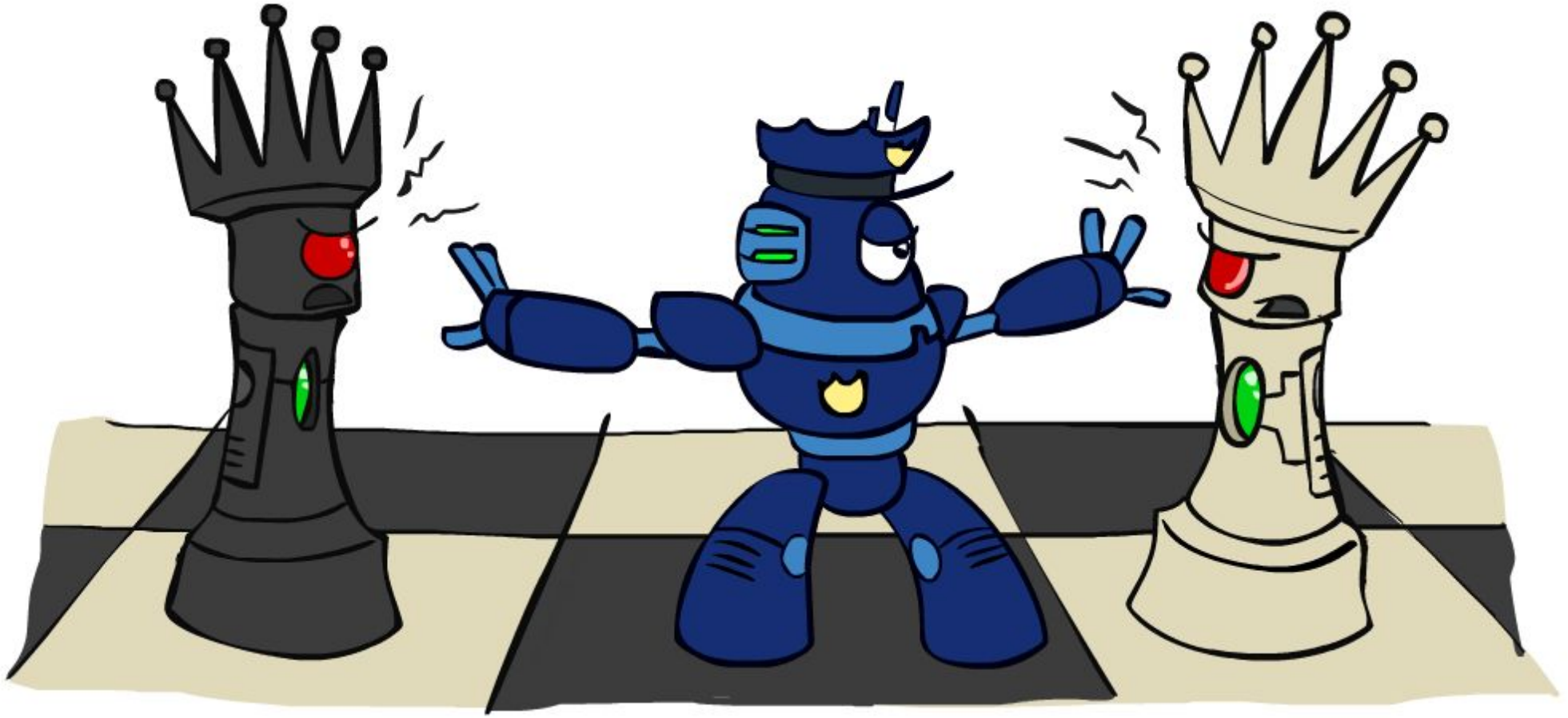


Tree-Structured CSP





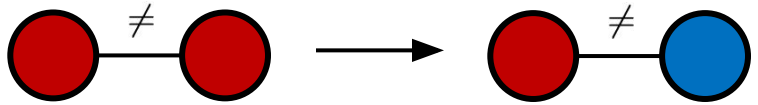
Iterative Improvement



Iterative Algorithms for CSPs

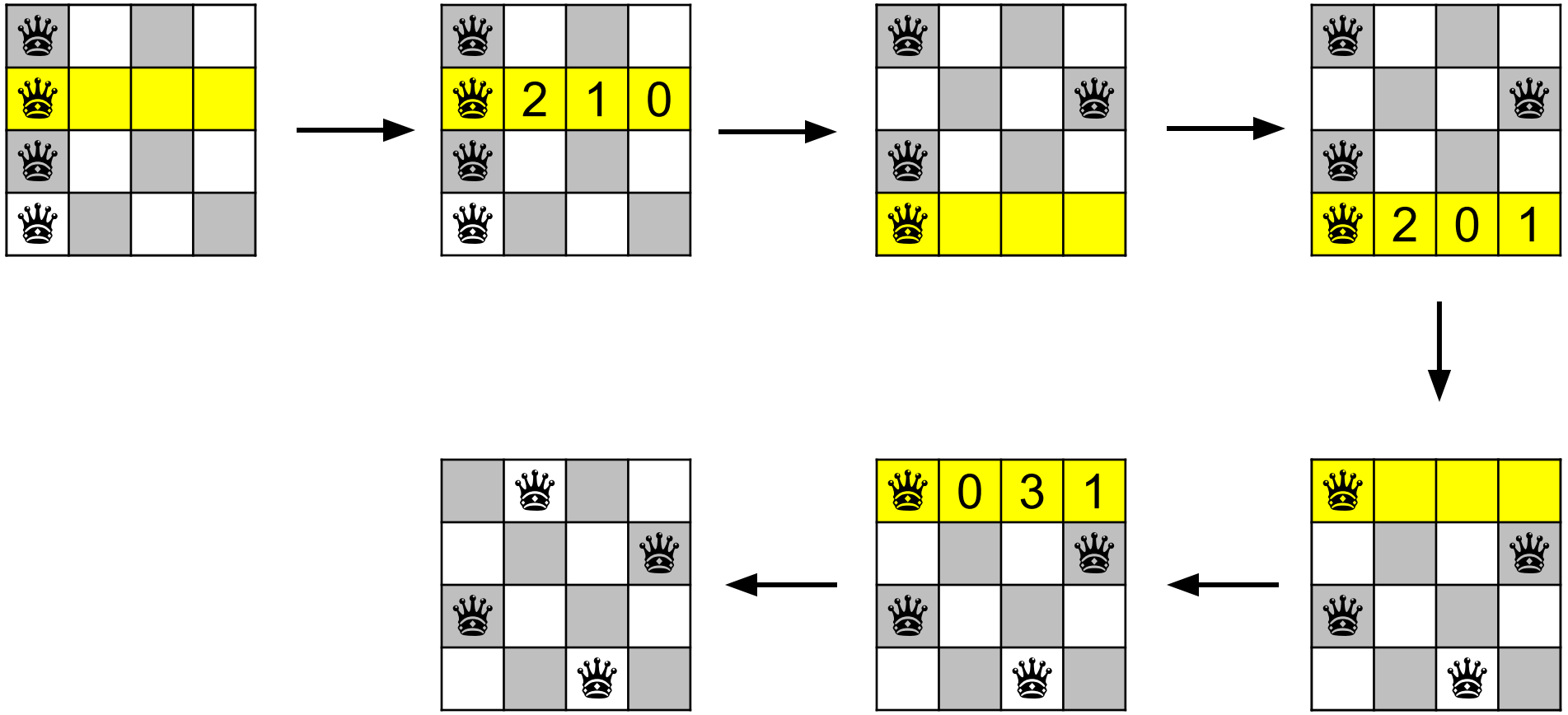
- Local search methods typically work with “complete” states, i.e., all variables assigned

- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.



- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with h(n) = total number of violated constraints

Example: 4-Queens

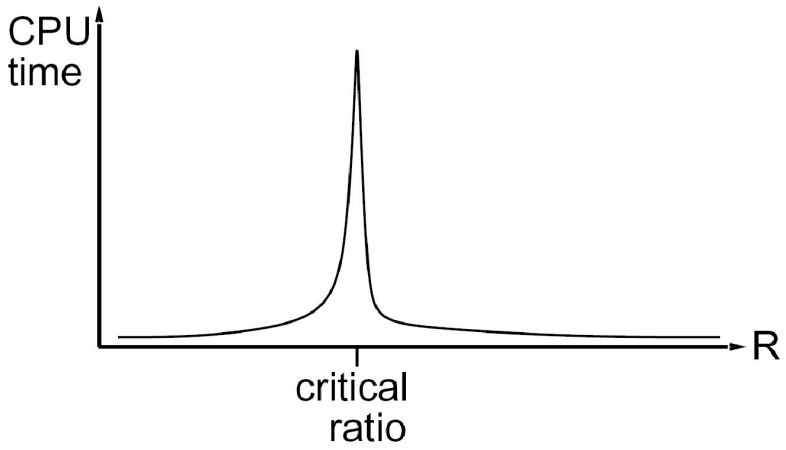


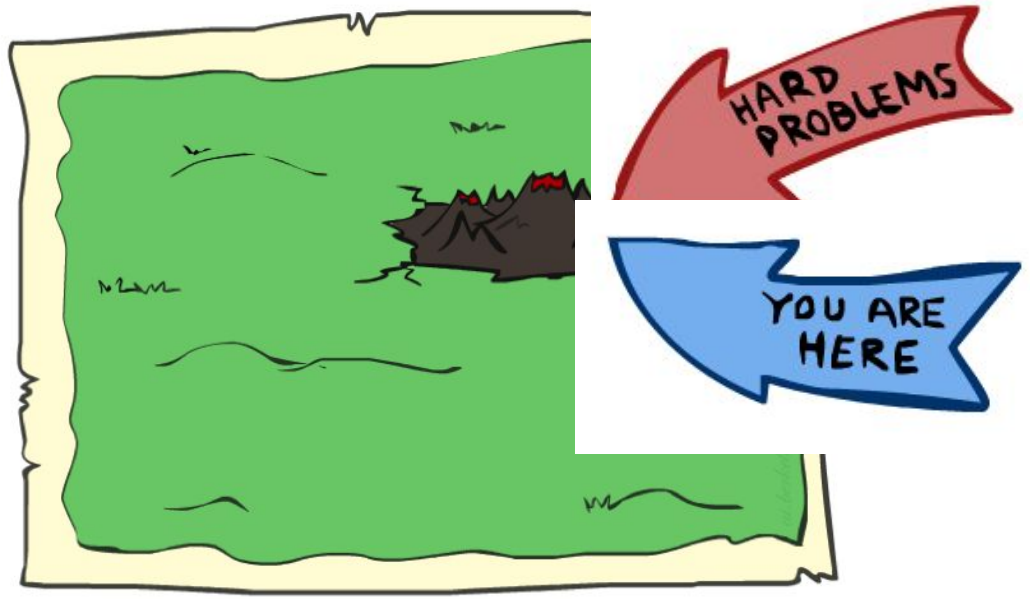
Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!

- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

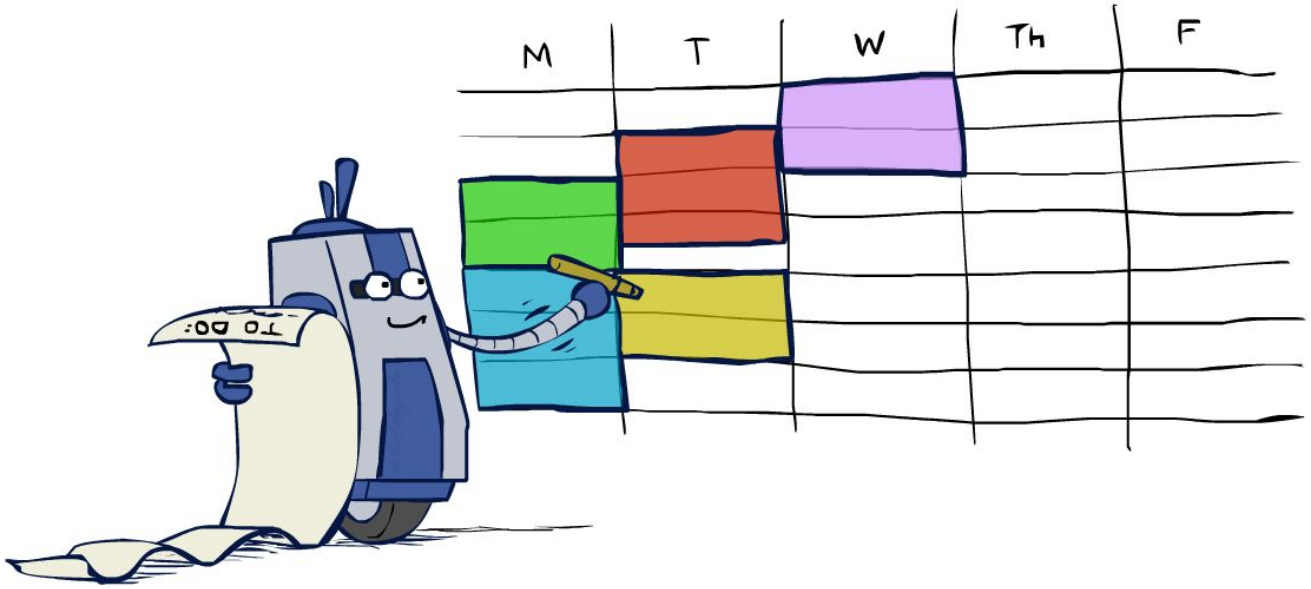
$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$





Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - Structure
- Iterative min-conflicts is often effective in practice

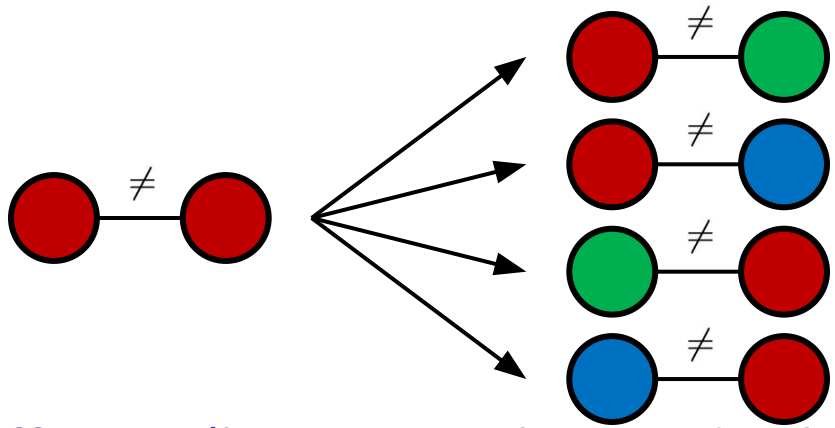


Local Search

Local Search

- Tree search keeps unexplored alternatives on the fringe (ensures completeness)

- Local search: improve a single option until you can't make it better
- New successor function: local changes



- Generally much faster and more memory efficient (but incomplete and suboptimal)
- Pretty much unavoidable when the state is "yourself"

Hill Climbing

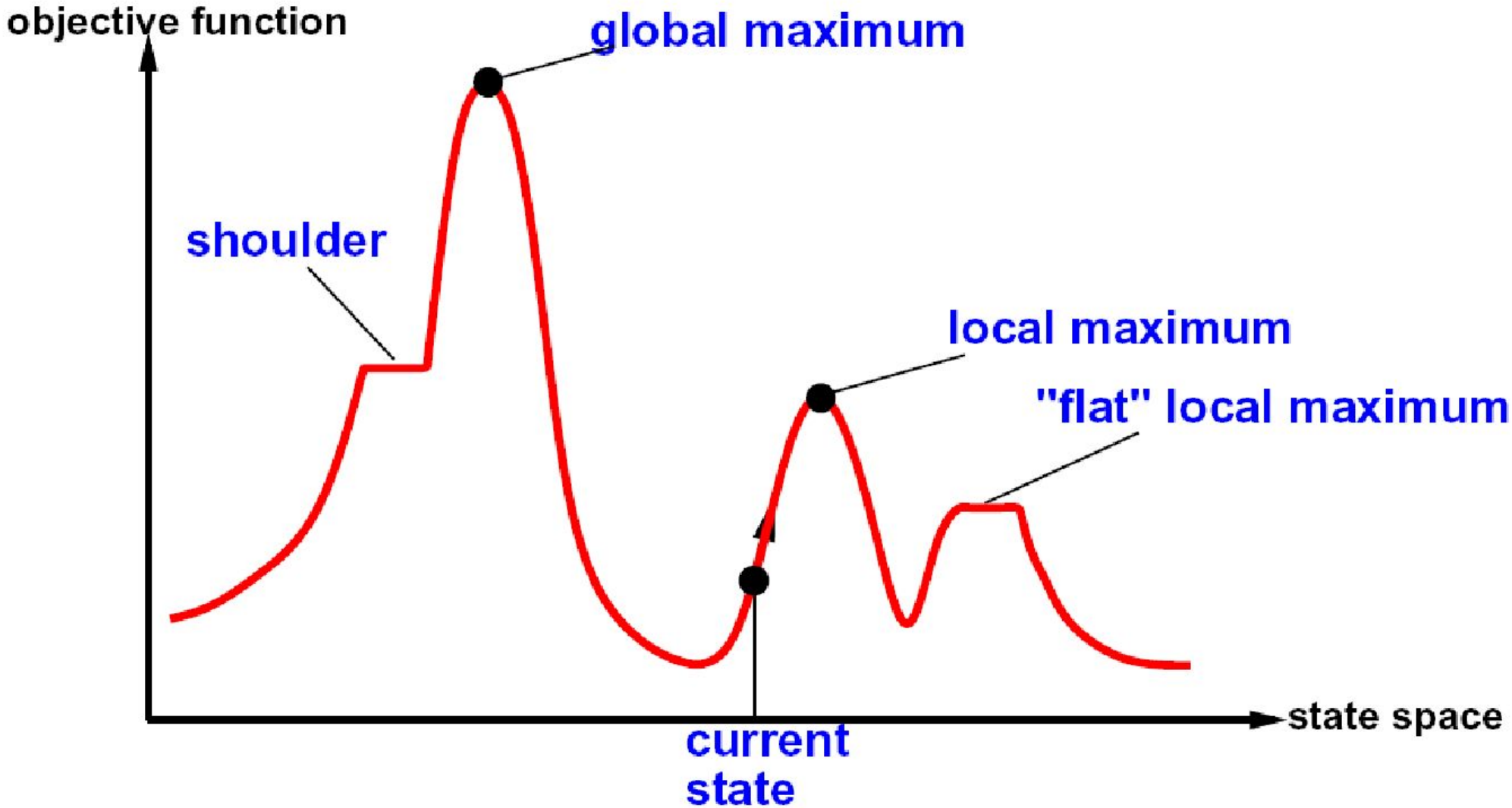
- Simple, general idea:
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit

Hill Climbing

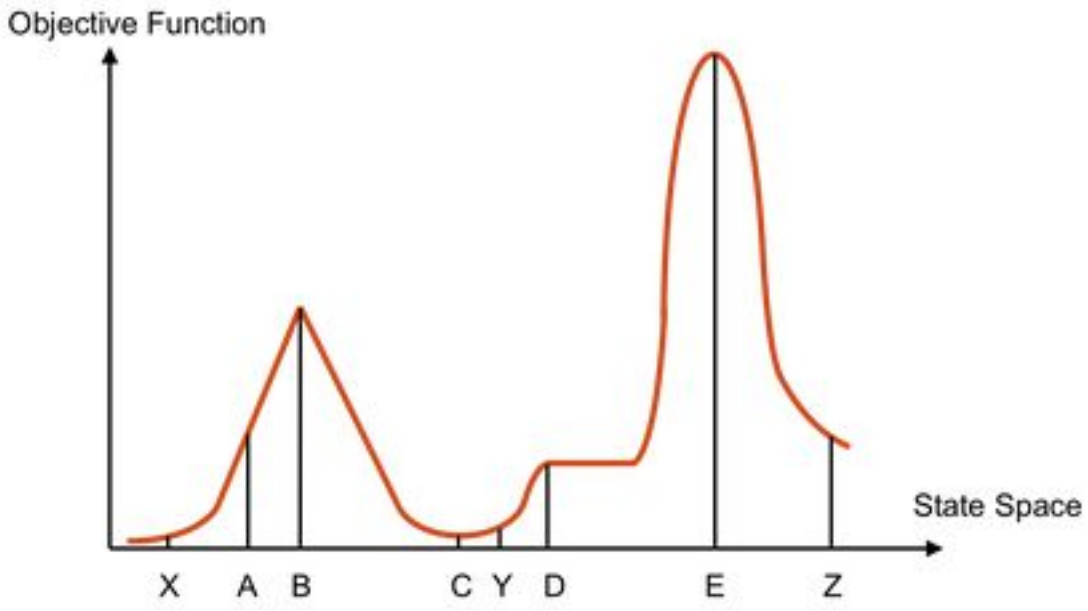
```
function HILL-CLIMBING( problem) returns a state that is a local maximum
    inputs: problem, a problem
    local variables: current, a node
                     neighbor, a node

    current ← MAKE-NODE(INITIAL-STATE[problem])
    loop do
        neighbor ← a highest-valued successor of current
        if VALUE[neighbor] ≤ VALUE[current] then return STATE[current]
        current ← neighbor
    end
```

Hill Climbing Diagram



Hill Climbing Quiz



Starting from X, where do you end up ?

Starting from Y, where do you end up ?

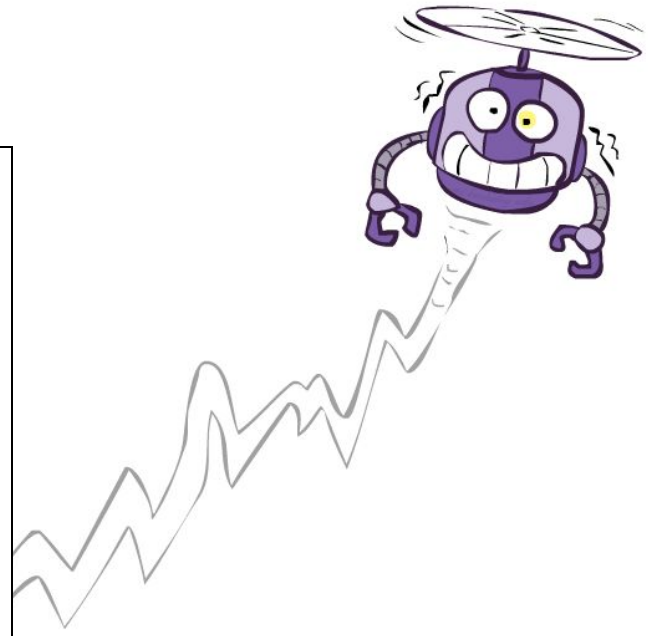
Starting from Z, where do you end up ?

Simulated Annealing

- Idea: Escape local maxima by allowing downhill moves
  - But make them rarer as time goes on

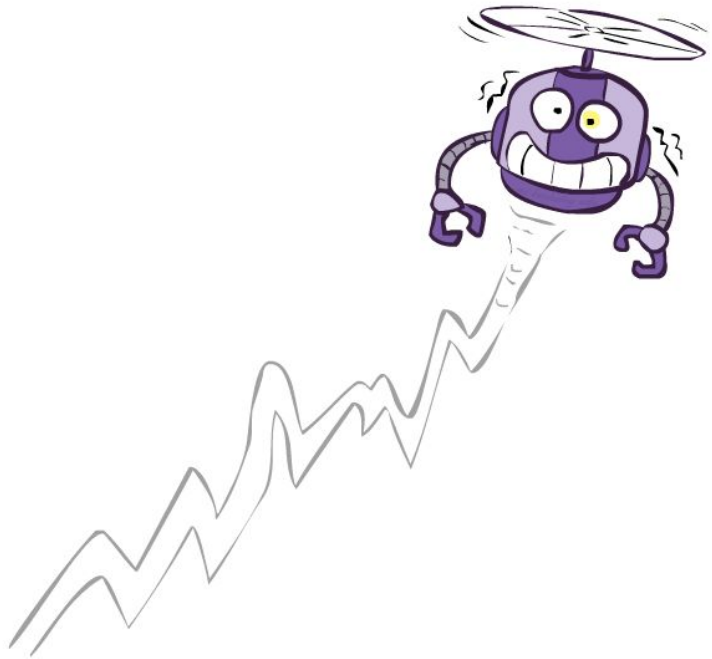
```
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
  inputs: problem, a problem
          schedule, a mapping from time to "temperature"
  local variables: current, a node
                   next, a node
                   T, a "temperature" controlling prob. of downward steps

  current ← MAKE-NODE(INITIAL-STATE[problem])
  for t ← 1 to ∞ do
      T ← schedule[t]
      if T = 0 then return current
      next ← a randomly selected successor of current
      ΔE ← VALUE[next] – VALUE[current]
      if ΔE > 0 then current ← next
      else current ← next only with probability e^(Δ E/T)
```



Simulated Annealing

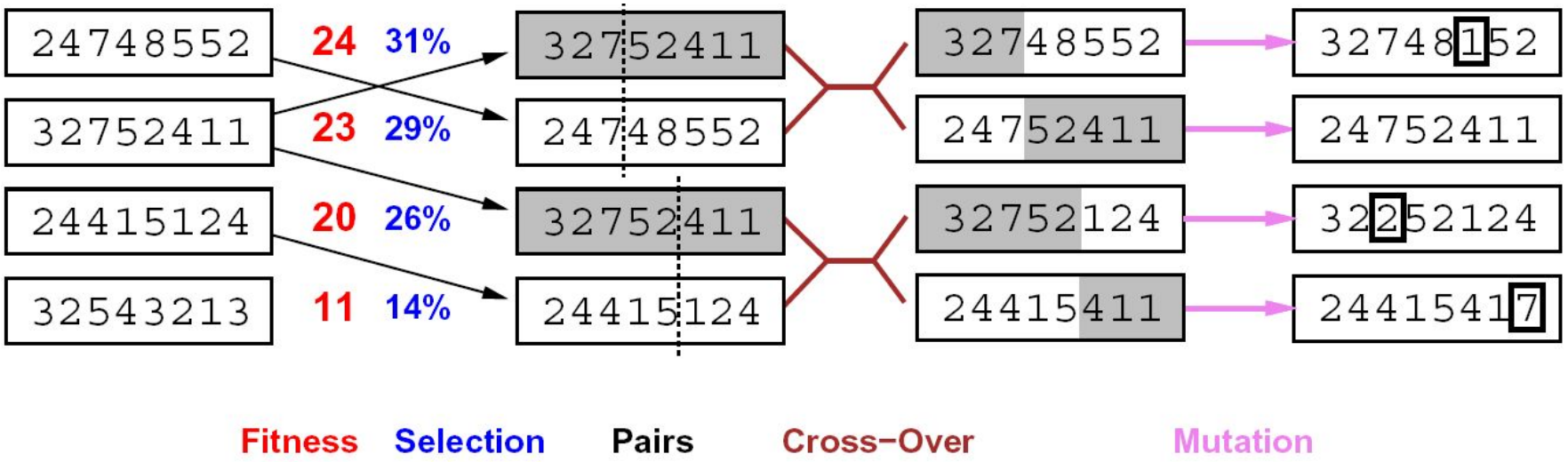
- Theoretical guarantee:
  - Stationary distribution (Boltzmann): $p(x) \propto e^{\frac{E(x)}{kT}}$
  - If T decreased slowly enough,
    will converge to optimal state!

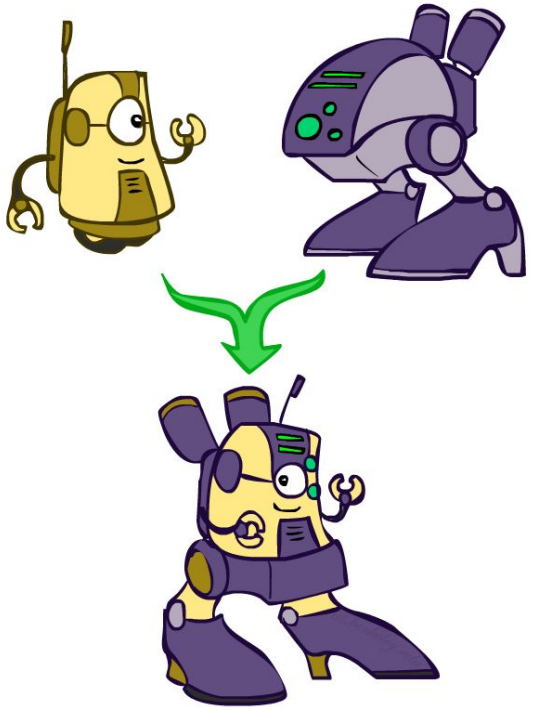


- Is this an interesting guarantee?

- Sounds like magic, but reality is reality:
  - The more downhill steps you need to escape a local optimum, the less likely you are to ever make them all in a row
  - "Slowly enough" may mean exponentially slowly
  - Random restart hillclimbing also converges to optimal state…

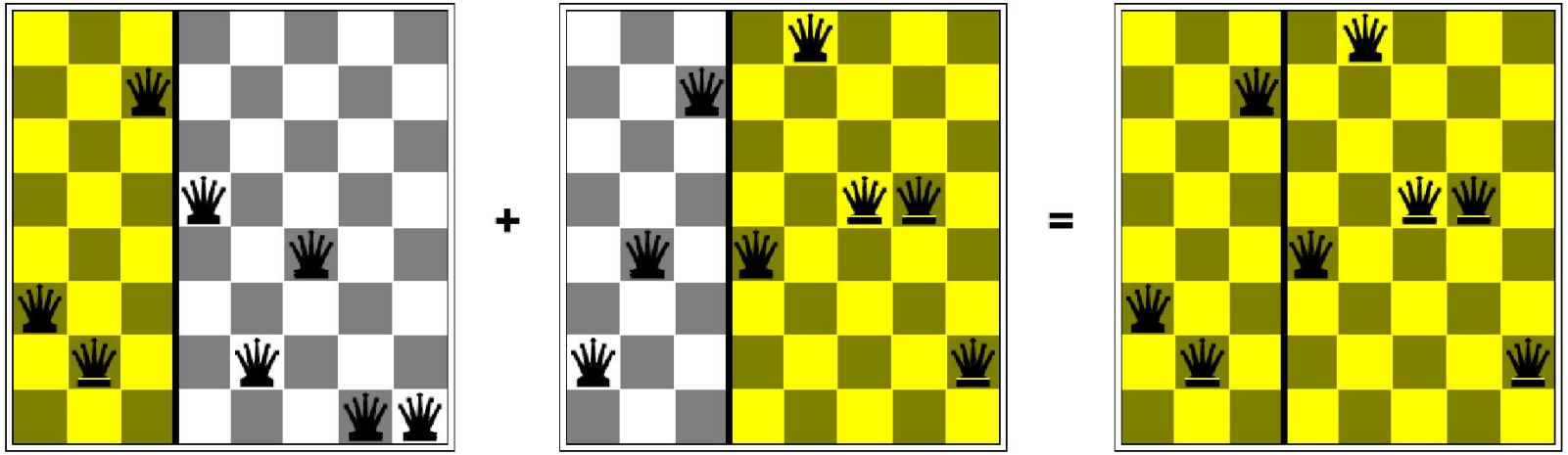
Genetic Algorithms





- Genetic algorithms use a natural selection metaphor
  - Keep best N hypotheses at each step (selection) based on a fitness function
  - Also have pairwise crossover operators, with optional mutation to give variety
- Possibly the most misunderstood, misapplied (and even maligned) technique around

Example: N-Queens



- Why does crossover make sense here?
- When wouldn't it make sense?
- What would mutation be?
- What would a good fitness function be?