Exercise 6

Use Taylor series to derive the finite-difference approximations

$$\frac{y(x_i+h)-y(x_i-h)}{2h} \approx \frac{dy}{dx} \quad \text{and} \quad \frac{y(x_i+h)-2y(x_i)+y(x_i-h)}{h^2} \approx \frac{d^2y}{dx^2},$$

and obtain the order of the error incurred in the approximation.

a) $\frac{dy}{dx} = \frac{y(x_i+h)-y(x_i-h)}{2h}$ Using leap-frog method, we consider forward and backward approximation:

$$y(x_i+h) = y(x_i) + h\cdot\frac{dy(x_i)}{dx} + h^2\frac{d^2y(x_i)}{dx^2} + O(h^3) \quad ①$$

$$y(x_i-h) = y(x_i) - h\cdot\frac{dy(x_i)}{dx} + h^2\frac{d^2y(x_i)}{dx^2} + O(h^3) \quad ②$$

① − ②, then we get $\frac{dy(x_i)}{dx} = \frac{y(x_i+h)-y(x_i-h)}{2h} + O(h^2)$, the order of error is 2

b) $\frac{d^2y}{dx^2} = \frac{y(x_i+h)-2y(x_i)+y(x_i-h)}{h^2}$ consider forward and backward approximation again.

$$y(x_i+h) = y(x_i) + h\frac{dy(x_i)}{dx} + h^2\frac{d^2y(x_i)}{dx^2} + h^3\frac{d^3y(x_i)}{dx^3} + O(h^4) \quad ①$$

$$y(x_i-h) = y(x_i) - h\frac{dy(x_i)}{dx} + h^2\frac{d^2y(x_i)}{dx^2} - h^3\frac{d^3y(x_i)}{dx^3} + O(h^4) \quad ②$$

① + ②, then we get $\frac{d^2y(x_i)}{dx^2} = \frac{y(x_i+h)-2y(x_i)+y(x_i-h)}{h^2} + O(h^2)$, the order of the error is 2.