Chapter One

Uncertain Linear Optimization Problems and their Robust Counterparts

In this chapter, we introduce the concept of the uncertain Linear Optimization problem and its Robust Counterpart, and study the computational issues associated with the emerging optimization problems.

1.1 DATA UNCERTAINTY IN LINEAR OPTIMIZATION

Recall that the Linear Optimization (LO) *problem* is of the form

$$\min_x \left\{ c^T x + d : Ax \leq b \right\}, \tag{1.1.1}$$

where $x \in \mathbb{R}^n$ is the vector of *decision variables*, $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ form the *objective*, A is an $m \times n$ *constraint matrix*, and $b \in \mathbb{R}^m$ is the *right hand side vector*.

Clearly, the constant term d in the objective, while affecting the optimal value, does not affect the optimal solution, this is why it is traditionally skipped. As we shall see, when treating the LO problems with *uncertain data* there are good reasons not to neglect this constant term.

The *structure* of problem (1.1.1) is given by the number m of constraints and the number n of variables, while the *data* of the problem are the collection (c, d, A, b), which we will arrange into an $(m+1) \times (n+1)$ *data matrix*

$$D = \left[\begin{array}{c|c} c^T & d \\ \hline A & b \end{array} \right].$$

Usually not all constraints of an LO program, as it arises in applications, are of the form $a^T x \leq$ const; there can be linear "$\geq$" inequalities and linear equalities as well. Clearly, the constraints of the latter two types can be represented equivalently by linear "$\leq$" inequalities, and we will assume henceforth that these are the only constraints in the problem.

Typically, the data of real world LOs (Linear Optimization problems) is not known exactly. The most common reasons for data uncertainty are as follows:

- Some of data entries (future demands, returns, etc.) do not exist when the problem is solved and hence are replaced with their forecasts. These data entries are thus subject to *prediction errors*;

- Some of the data (parameters of technological devices/processes, contents associated with raw materials, etc.) cannot be measured exactly – in reality their values drift around the measured "nominal" values; these data are subject to *measurement errors*;

- Some of the decision variables (intensities with which we intend to use various technological processes, parameters of physical devices we are designing, etc.) cannot be implemented exactly as computed. The resulting *implementation errors* are equivalent to appropriate artificial data uncertainties.

  Indeed, the contribution of a particular decision variable x_j to the left hand side of constraint i is the product $a_{ij}x_j$. Hence the consequences of an additive implementation error $x_j \mapsto x_j + \epsilon$ are as if there were no implementation error at all, but the left hand side of the constraint got an extra additive term $a_{ij}\epsilon$, which, in turn, is equivalent to the perturbation $b_i \mapsto b_j - a_{ij}\epsilon$ in the right hand side of the constraint. The consequences of a more typical *multiplicative* implementation error $x_j \mapsto (1+\epsilon)x_j$ are as if there were no implementation error, but each of the data coefficients a_{ij} was subject to perturbation $a_{ij} \mapsto (1+\epsilon)a_{ij}$. Similarly, the influence of additive and multiplicative implementation error in x_j on the value of the objective can be mimicked by appropriate perturbations in d or c_j.

In the traditional LO methodology, a small data uncertainty (say, 1% or less) is just ignored; the problem is solved *as if* the given ("nominal") data were exact, and the resulting *nominal* optimal solution is what is recommended for use, in hope that small data uncertainties will not affect significantly the feasibility and optimality properties of this solution, or that small adjustments of the nominal solution will be sufficient to make it feasible. We are about to demonstrate that these hopes are not necessarily justified, and sometimes even small data uncertainty deserves significant attention.

1.1.1 Introductory Example

Consider the following very simple linear optimization problem:

Example 1.1.1. A company produces two kinds of drugs, DrugI and DrugII, containing a specific active agent A, which is extracted from raw materials purchased on the market. There are two kinds of raw materials, RawI and RawII, which can be used as sources of the active agent. The related production, cost, and resource data are given in table 1.1. The goal is to find the production plan that maximizes the profit of the company.

Parameter	DrugI	DrugII
Selling price, $ per 1000 packs	6,200	6,900
Content of agent A, g per 1000 packs	0.500	0.600
Manpower required, hours per 1000 packs	90.0	100.0
Equipment required, hours per 1000 packs	40.0	50.0
Operational costs, $ per 1000 packs	700	800

(a) Drug production data

Raw material	Purchasing price, $ per kg	Content of agent A, g per kg
RawI	100.00	0.01
RawII	199.90	0.02

(b) Contents of raw materials

Budget, $	Manpower, hours	Equipment, hours	Capacity of raw materials storage, kg
100,000	2,000	800	1,000

(c) Resources

Table 1.1 Data for Example 1.1.1.

The problem can be immediately posed as the following linear programming program:

(Drug):

$$\begin{array}{rcl}\text{Opt} & = & \min\Big\{ \overbrace{[100\cdot \mathit{RawI} + 199.90\cdot \mathit{RawII} + 700\cdot \mathit{DrugI} + 800\cdot \mathit{DrugII}]}^{\text{purchasing and operational costs}} \\ & & \quad - \underbrace{[6200\cdot \mathit{DrugI} + 6900\cdot \mathit{DrugII}]}_{\text{income from selling the drugs}}\Big\} \qquad \text{[minus total profit]}\end{array}$$

subject to

$$\begin{array}{lr}
0.01\cdot \mathit{RawI} + 0.02\cdot \mathit{RawII} - 0.500\cdot \mathit{DrugI} - 0.600\cdot \mathit{DrugII} \geq 0 & \text{[balance of active agent]}\\
\mathit{RawI} + \mathit{RawII} \leq 1000 & \text{[storage constraint]}\\
90.0\cdot \mathit{DrugI} + 100.0\cdot \mathit{DrugII} \leq 2000 & \text{[manpower constraint]}\\
40.0\cdot \mathit{DrugI} + 50.0\cdot \mathit{DrugII} \leq 800 & \text{[equipment constraint]}\\
100.0\cdot \mathit{RawI} + 199.90\cdot \mathit{RawII} + 700\cdot \mathit{DrugI} + 800\cdot \mathit{DrugII} \leq 100000 & \text{[budget constraint]}\\
\mathit{RawI}, \mathit{RawII}, \mathit{DrugI}, \mathit{DrugII} \geq 0 &
\end{array}$$

The problem has four variables — the amounts *RawI*, *RawII* (in kg) of raw materials to be purchased and the amounts *DrugI*, *DrugII* (in 1000 of packs) of drugs to be produced.

The optimal solution of our LO problem is

$$\text{Opt} = -8819.658;\ \mathit{RawI} = 0, \mathit{RawII} = 438.789, \mathit{DrugI} = 17.552, \mathit{DrugII} = 0.$$

Note that both the budget and the balance constraints are active (that is, the production process utilizes the entire 100,000 budget and the full amount of active agent contained in the raw materials). The solution promises the company a modest, but quite respectable profit of 8.8%.

1.1.2 Data Uncertainty and its Consequences

Clearly, even in our simple problem some of the data cannot be "absolutely reliable"; e.g., one can hardly believe that the contents of the active agent in the raw materials are exactly 0.01 g/kg for RawI and 0.02 g/kg for RawII. In reality, these contents vary around the indicated values. A natural assumption here is that the actual contents of active agent aI in RawI and aII in RawII are realizations of random variables somehow distributed around the "nominal contents" $anI = 0.01$ and $anII = 0.02$. To be more specific, assume that aI drifts in a 0.5% margin of anI, thus taking values in the segment $[0.00995, 0.01005]$. Similarly, assume that aII drifts in a 2% margin of $anII$, thus taking values in the segment $[0.0196, 0.0204]$. Moreover, assume that aI, aII take the two extreme values in the respective segments with probabilities 0.5 each. How do these perturbations of the contents of the active agent affect the production process? The optimal solution prescribes to purchase 438.8 kg of RawII and to produce 17.552K packs of DrugI (K stands for "thousand"). With the above random fluctuations in the content of the active agent in RawII, this production plan will be infeasible with probability 0.5, i.e., the actual content of the active agent in raw materials will be less than the one required to produce the planned amount of DrugI. This difficulty can be resolved in the simplest way: when the actual content of the active agent in raw materials is insufficient, the output of the drug is reduced accordingly. With this policy, the actual production of DrugI becomes a random variable that takes with equal probabilities the nominal value of 17.552K packs and the (2% less) value of 17.201K packs. These 2% fluctuations in the production affect the profit as well; it becomes a random variable taking, with probabilities 0.5, the nominal value 8,820 and the 21% (!) less value 6,929. The expected profit is 7,843, which is 11% less than the nominal profit 8,820 promised by the optimal solution of the nominal problem.

We see that in our simple example a pretty small (and unavoidable in reality) perturbation of the data may make the nominal optimal solution infeasible. Moreover, a straightforward adjustment of the nominally optimal solution to the actual data may heavily affect the quality of the solution.

Similar phenomenon can be met in many practical linear programs where at least part of the data are not known exactly and can vary around their nominal values. The consequences of data uncertainty can be much more severe than in our toy example. The analysis of linear optimization problems from the NETLIB collection[1] reported in [7] reveals that for 13 of 94 NETLIB problems, random 0.01% perturbations of the uncertain data can make the nominal optimal solution severely infeasible: with a non-negligible probability, it violates some of the constraints by

[1] A collection of LP programs, including those of real world origin, used as a standard benchmark for testing LP solvers.

50% and more. It should be added that in the general case (in contrast to our toy example) there is no evident way to adjust the optimal solution to the actual values of the data by a small modification, and there are cases when such an adjustment is in fact impossible; in order to become feasible for the perturbed data, the nominal optimal solution should be "completely reshaped."

The conclusion is as follows:

> *In applications of LO, there exists a real need of a technique capable of detecting cases when data uncertainty can heavily affect the quality of the nominal solution, and in these cases to generate a "reliable" solution, one that is immunized against uncertainty.*

We are about to introduce the *Robust Counterpart* approach to uncertain LO problems aimed at coping with data uncertainty.

1.2 UNCERTAIN LINEAR PROBLEMS AND THEIR ROBUST COUNTERPARTS

Definition 1.2.1. An uncertain Linear Optimization problem is a collection

$$\left\{\min_x \left\{c^T x + d : Ax \le b\right\}\right\}_{(c,d,A,b)\in\mathcal{U}} \tag{LO$_\mathcal{U}$}$$

of LO problems (instances) $\min\limits_x \left\{c^T x + d : Ax \le b\right\}$ of common structure (i.e., with common numbers m of constraints and n of variables) with the data varying in a given <u>uncertainty set</u> $\mathcal{U} \subset \mathbb{R}^{(m+1)\times(n+1)}$.

We always assume that the uncertainty set is parameterized, in an affine fashion, by *perturbation vector* ζ varying in a given *perturbation set* $\mathcal{Z}$:

$$\mathcal{U} = \left\{ \left[\begin{array}{c|c} c^T & d \\ \hline A & b \end{array}\right] = \underbrace{\left[\begin{array}{c|c} c_0^T & d_0 \\ \hline A_0 & b_0 \end{array}\right]}_{\substack{\text{nominal} \\ \text{data } D_0}} + \sum_{\ell=1}^{L} \zeta_\ell \underbrace{\left[\begin{array}{c|c} c_\ell^T & d_\ell \\ \hline A_\ell & b_\ell \end{array}\right]}_{\substack{\text{basic} \\ \text{shifts } D_\ell}} : \zeta \in \mathcal{Z} \subset \mathbb{R}^L \right\}. \tag{1.2.1}$$

For example, the story told in section 1.1.2 makes (Drug) an uncertain LO problem as follows:

- *Decision vector:* $x = [RawI; RawII; DrugI; DrugII]$;
- *Nominal data:* $D_0 = \left[\begin{array}{cccc|c} 100 & 199.9 & -5500 & -6100 & 0 \\ \hline -0.01 & -0.02 & 0.500 & 0.600 & 0 \\ 1 & 1 & 0 & 0 & 1000 \\ 0 & 0 & 90.0 & 100.0 & 2000 \\ 0 & 0 & 40.0 & 50.0 & 800 \\ 100.0 & 199.9 & 700 & 800 & 100000 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$

- *Two basic shifts:*

$$D_1 = 5.0\text{e-}5 \cdot \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad D_2 = 4.0\text{e-}4 \cdot \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- *Perturbation set:*

$$\mathcal{Z} = \left\{\zeta \in \mathbb{R}^2 : -1 \le \zeta_1, \zeta_2 \le 1\right\}.$$

This description says, in particular, that the only uncertain data in (Drug) are the coefficients *anI*, *anII* of the variables *RawI*, *RawII* in the balance inequality, (which is the first constraint in (Drug)), and that these coefficients vary in the respective segments $[0.01\cdot(1-0.005), 0.01\cdot(1+0.005)]$, $[0.02\cdot(1-0.02), 0.02\cdot(1+0.02)]$ around the nominal values 0.01, 0.02 of the coefficients, which is exactly what was stated in section 1.1.2.

Remark 1.2.2. If the perturbation set $\mathcal{Z}$ in (1.2.1) itself is represented as the image of another set $\widehat{\mathcal{Z}}$ under affine mapping $\xi \mapsto \zeta = p + P\xi$, then we can pass from perturbations ζ to perturbations ξ:

$$\begin{array}{rcl} \mathcal{U} & = & \left\{\left[\begin{array}{c|c} c^T & d \\ \hline A & b \end{array}\right] = D_0 + \sum\limits_{\ell=1}^{L} \zeta_\ell D_\ell : \zeta \in \mathcal{Z}\right\} \\ & = & \left\{\left[\begin{array}{c|c} c^T & d \\ \hline A & b \end{array}\right] = D_0 + \sum\limits_{\ell=1}^{L} [p_\ell + \sum\limits_{k=1}^{K} P_{\ell k}\xi_k] D_\ell : \xi \in \widehat{\mathcal{Z}}\right\} \\ & = & \left\{\left[\begin{array}{c|c} c^T & d \\ \hline A & b \end{array}\right] = \underbrace{\left[D_0 + \sum\limits_{\ell=1}^{L} p_\ell D_\ell\right]}_{\widehat{D}_0} + \sum\limits_{k=1}^{K} \xi_k \underbrace{\left[\sum\limits_{\ell=1}^{L} P_{\ell k} D_\ell\right]}_{\widehat{D}_k} : \xi \in \widehat{\mathcal{Z}}\right\}. \end{array}$$

It follows that when speaking about perturbation sets with simple geometry (parallelotopes, ellipsoids, etc.), we can normalize these sets to be "standard." For example, a parallelotope is by definition an affine image of a unit box $\{\xi \in \mathbb{R}^k : -1 \le \xi_j \le 1, j = 1, ..., k\}$, which gives us the possibility to work with the unit box instead of a general parallelotope. Similarly, an ellipsoid is by definition the image of a unit Euclidean ball $\{\xi \in \mathbb{R}^k : \|x\|_2^2 \equiv x^T x \le 1\}$ under affine mapping, so that we can work with the standard ball instead of the ellipsoid, etc. We will use this normalization whenever possible.

Note that a *family* of optimization problems like ($\mathrm{LO}_\mathcal{U}$), in contrast to a single optimization problem, is not associated by itself with the concepts of feasible/optimal solution and optimal value. How to define these concepts depends of course on the underlying "decision environment." Here we focus on an environment with the following characteristics:

A.1. All decision variables in ($\mathrm{LO}_{\mathcal{U}}$) represent "here and now" decisions; they should be assigned specific numerical values as a result of solving the problem *before* the actual data "reveals itself."

A.2. The decision maker is fully responsible for consequences of the decisions to be made when, and only when, the actual data is within the prespecified uncertainty set $\mathcal{U}$ given by (1.2.1).

A.3. The constraints in ($\mathrm{LO}_{\mathcal{U}}$) are "hard" — we cannot tolerate violations of constraints, even small ones, when the data is in $\mathcal{U}$.

The above assumptions determine, in a more or less unique fashion, what are the meaningful feasible solutions to the uncertain problem ($\mathrm{LO}_{\mathcal{U}}$). By A.1, these should be fixed vectors; by A.2 and A.3, they should be *robust feasible*, that is, they should satisfy all the constraints, whatever the realization of the data from the uncertainty set. We have arrived at the following definition.

Definition 1.2.3. A vector $x \in \mathbb{R}^n$ is a robust feasible solution to ($\mathrm{LO}_{\mathcal{U}}$), if it satisfies all realizations of the constraints from the uncertainty set, that is,

$$Ax \leq b \quad \forall (c, d, A, b) \in \mathcal{U}. \tag{1.2.2}$$

As for the objective value to be associated with a meaningful (i.e., robust feasible) solution, assumptions A.1 — A.3 do not prescribe it in a unique fashion. However, "the spirit" of these worst-case-oriented assumptions leads naturally to the following definition:

Definition 1.2.4. Given a candidate solution x, the robust value $\widehat{c}(x)$ of the objective in ($\mathrm{LO}_{\mathcal{U}}$) at x is the largest value of the "true" objective $c^T x + d$ over all realizations of the data from the uncertainty set:

$$\widehat{c}(x) = \sup_{(c,d,A,b)\in\mathcal{U}} [c^T x + d]. \tag{1.2.3}$$

After we agree what are meaningful candidate solutions to the uncertain problem ($\mathrm{LO}_{\mathcal{U}}$) and how to quantify their quality, we can seek the best robust value of the objective among all robust feasible solutions to the problem. This brings us to the central concept of this book, *Robust Counterpart* of an uncertain optimization problem, which is defined as follows:

Definition 1.2.5. The Robust Counterpart of the uncertain LO problem ($\mathrm{LO}_{\mathcal{U}}$) is the optimization problem

$$\min_x \left\{ \widehat{c}(x) = \sup_{(c,d,A,b)\in\mathcal{U}} [c^T x + d] : Ax \leq b \ \forall (c, d, A, b) \in \mathcal{U} \right\} \tag{1.2.4}$$

of minimizing the robust value of the objective over all robust feasible solutions to the uncertain problem.

An optimal solution to the Robust Counterpart is called a robust optimal solution to ($\mathrm{LO}_{\mathcal{U}}$), and the optimal value of the Robust Counterpart is called the robust optimal value of ($\mathrm{LO}_{\mathcal{U}}$).

In a nutshell, the robust optimal solution is simply "the best uncertainty-immunized" solution we can associate with our uncertain problem.

Example 1.1.1 continued. Let us find the robust optimal solution to the uncertain problem (Drug). There is exactly one uncertainty-affected "block" in the data, namely, the coefficients of *RawI*, *RawII* in the balance constraint. A candidate solution is thus robust feasible if and only if it satisfies all constraints of (Drug), except for the balance constraint, *and* it satisfies the "worst" realization of the balance constraint. Since *RawI*, *RawII* are nonnegative, the worst realization of the balance constraint is the one where the uncertain coefficients *anI*, *anII* are set to their minimal values in the uncertainty set (these values are 0.00995 and 0.0196, respectively). Since the objective is not affected by the uncertainty, the robust objective values are the same as the original ones. Thus, the RC (Robust Counterpart) of our uncertain problem is the LO problem

$$\begin{array}{l}
\text{RC(Drug):}\\
\text{RobOpt} = \min\{-100\cdot \mathit{RawI} - 199.9\cdot \mathit{RawII} + 5500\cdot \mathit{DrugI} + 6100\cdot \mathit{DrugII}\}\\
\ \text{subject to}\\
\begin{array}{rcl}
0.00995\cdot \mathit{RawI} + 0.0196\cdot \mathit{RawII} - 0.500\cdot \mathit{DrugI} - 0.600\cdot \mathit{DrugII} & \geq & 0\\
\mathit{RawI} + \mathit{RawII} & \leq & 1000\\
90.0\cdot \mathit{DrugI} + 100.0\cdot \mathit{DrugII} & \leq & 2000\\
40.0\cdot \mathit{DrugI} + 50.0\cdot \mathit{DrugII} & \leq & 800\\
100.0\cdot \mathit{RawI} + 199.90\cdot \mathit{RawII} + 700\cdot \mathit{DrugI} + 800\cdot \mathit{DrugII} & \leq & 100000\\
\mathit{RawI}, \mathit{RawII}, \mathit{DrugI}, \mathit{DrugII} & \geq & 0
\end{array}
\end{array}$$

Solving this problem, we get

$$\text{RobOpt} = -8294.567; \mathit{RawI} = 877.732, \mathit{RawII} = 0, \mathit{DrugI} = 17.467, \mathit{DrugII} = 0.$$

The "price" of robustness is the reduction in the promised profit from its nominal optimal value 8819.658 to its robust optimal value 8294.567, that is, by 5.954%. This is much less than the 21% reduction of the actual profit to 6,929 which we may suffer when sticking to the nominal optimal solution when the "true" data are "against" it. Note also that the structure of the robust optimal solution is quite different from the one of the nominal optimal solution: with the robust solution, we shall buy only raw materials RawI, while with the nominal one, only raw materials RawII. The explanation is clear: with the nominal data, RawII as compared to RawI results in a bit smaller per unit price of the active agent (9,995 \$/g vs. 10,000 \$/g). This is why it does not make sense to use RawI with the nominal data. The robust optimal solution takes into account that the uncertainty in *anI* (i.e., the variability of contents of active agent in RawI) is 4 times smaller than that of *anII* (0.5% vs. 2%), which ultimately makes it better to use RawI.

1.2.1 More on Robust Counterparts

We start with several useful observations.

A. The Robust Counterpart (1.2.4) of $\mathrm{LO}_{\mathcal{U}}$ can be rewritten equivalently as the problem

$$\min_{x,t}\left\{t : \left.\begin{array}{rcl} c^T x - t & \leq & -d \\ Ax & \leq & b \end{array}\right\} \forall (c,d,A,b) \in \mathcal{U}\right\}. \tag{1.2.5}$$

Note that we can arrive at this problem in another fashion: we first introduce the extra variable t and rewrite instances of our uncertain problem $(\mathrm{LO}_{\mathcal{U}})$ equivalently as

$$\min_{x,t}\left\{t:\ \begin{array}{rcl} c^Tx - t & \leq & -d \\ Ax & \leq & b \end{array}\right\},$$

thus arriving at an equivalent to $(\mathrm{LO}_{\mathcal{U}})$ uncertain problem in variables x, t with the objective t that is not affected by uncertainty at all. The RC of the reformulated problem is exactly (1.2.5). We see that

> *An uncertain LO problem can always be reformulated as an uncertain LO problem with certain objective. The Robust Counterpart of the reformulated problem has the same objective as this problem and is equivalent to the RC of the original uncertain problem.*

As a consequence, *we lose nothing when restricting ourselves with uncertain LO programs with certain objectives* and we shall frequently use this option in the future.

> We see now why the constant term d in the objective of (1.1.1) should not be neglected, or, more exactly, should not be neglected if it is uncertain. When d is certain, we can account for it by the shift $t \mapsto t - d$ in the slack variable t which affects only the optimal value, but not the optimal solution to the Robust Counterpart (1.2.5). When d is uncertain, there is no "universal" way to eliminate d without affecting the optimal solution to the Robust Counterpart (where d plays the same role as the right hand sides of the original constraints).

B. Assuming that $(\mathrm{LO}_{\mathcal{U}})$ is with certain objective, the Robust Counterpart of the problem is

$$\min_x \left\{c^Tx + d : Ax \leq b,\ \forall (A, b) \in \mathcal{U}\right\} \tag{1.2.6}$$

(note that the uncertainty set is now a set in the space of the constraint data $[A, b]$). We see that

> *The Robust Counterpart of an uncertain LO problem with a certain objective is a purely "constraint-wise" construction: to get RC, we act as follows:*
>
> - *preserve the original certain objective as it is, and*
> - *replace every one of the original constraints*
>
> $$(Ax)_i \leq b_i \Leftrightarrow a_i^Tx \leq b_i \tag{C$_i$}$$
>
> *(a_i^T is i-th row in A) with its Robust Counterpart*
>
> $$a_i^Tx \leq b_i\ \forall [a_i; b_i] \in \mathcal{U}_i, \tag{RC(C$_i$)}$$
>
> *where $\mathcal{U}_i$ is the projection of $\mathcal{U}$ on the space of data of i-th constraint:*
>
> $$\mathcal{U}_i = \{[a_i; b_i] : [A, b] \in \mathcal{U}\}.$$

In particular,

> The RC of an uncertain LO problem with a certain objective remains intact when the original uncertainty set $\mathcal{U}$ is extended to the direct product
> $$\widehat{\mathcal{U}} = \mathcal{U}_1 \times ... \times \mathcal{U}_m$$
> of its projections onto the spaces of data of respective constraints.

Example 1.2.6. The RC of the system of uncertain constraints

$$\{x_1 \geq \zeta_1,\ x_2 \geq \zeta_2\} \tag{1.2.7}$$

with $\zeta \in \mathcal{U} := \{\zeta_1 + \zeta_2 \leq 1, \zeta_1, \zeta_2 \geq 0\}$ is the infinite system of constraints

$$x_1 \geq \zeta_1,\ x_1 \geq \zeta_2\ \ \forall \zeta \in \mathcal{U};$$

on variables x_1, x_2. The latter system is clearly equivalent to the pair of constraints

$$x_1 \geq \max_{\zeta \in \mathcal{U}} \zeta_1 = 1,\ x_2 \geq \max_{\zeta \in \mathcal{U}} \zeta_2 = 1. \tag{1.2.8}$$

The projections of $\mathcal{U}$ to the spaces of data of the two uncertain constraints (1.2.7) are the segments $\mathcal{U}_1 = \{\zeta_1 : 0 \leq \zeta_1 \leq 1\}$, $\mathcal{U}_2 = \{\zeta_2 : 0 \leq \zeta_2 \leq 1\}$, and the RC of (1.2.7) w.r.t.[2] the uncertainty set $\widehat{\mathcal{U}} = \mathcal{U}_1 \times \mathcal{U}_2 = \{\zeta \in \mathbb{R}^2 : 0 \leq \zeta_1, \zeta_2 \leq 1\}$ clearly is (1.2.8).

> The conclusion we have arrived at seems to be counter-intuitive: it says that it is immaterial whether the perturbations of data in different constraints are or are not linked to each other, while intuition says that such a link should be important. We shall see later (chapter 14) that this intuition is valid when a more advanced concept of *Adjustable* Robust Counterpart is considered.

C. If x is a robust feasible solution of (C_i), then x remains robust feasible when we extend the uncertainty set $\mathcal{U}_i$ to its convex hull $\mathrm{Conv}(\mathcal{U}_i)$. Indeed, if $[\bar{a}_i; \bar{b}_i] \in \mathrm{Conv}(\mathcal{U}_i)$, then

$$[\bar{a}_i; \bar{b}_i] = \sum_{j=1}^{J} \lambda_j [a_i^j; b_i^j],$$

with appropriately chosen $[a_i^j; b_i^j] \in \mathcal{U}_i$, $\lambda_j \geq 0$ such that $\sum_j \lambda_j = 1$. We now have

$$\bar{a}_i^T x = \sum_{j=1}^{J} \lambda_j [a_i^j]^T x \leq \sum_j \lambda_j b_i^j = \bar{b}_i,$$

where the inequality is given by the fact that x is feasible for $\mathrm{RC}(\mathrm{C}_i)$ and $[a_i^j; b_i^j] \in \mathcal{U}_i$. We see that $\bar{a}_i^T x \leq \bar{b}_i$ for all $[\bar{a}_i; \bar{b}_i] \in \mathrm{Conv}(\mathcal{U}_i)$, QED.

By similar reasons, the set of robust feasible solutions to (C_i) remains intact when we extend $\mathcal{U}_i$ to the closure of this set. Combining these observations with **B.**, we arrive at the following conclusion:

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[2] abbr. for "with respect to"

The Robust Counterpart of an uncertain LO problem with a certain objective remains intact when we extend the sets $\mathcal{U}_i$ of uncertain data of respective constraints to their closed convex hulls, and extend $\mathcal{U}$ to the direct product of the resulting sets.

In other words, we lose nothing when assuming from the very beginning that the sets $\mathcal{U}_i$ of uncertain data of the constraints are closed and convex, and $\mathcal{U}$ is the direct product of these sets.

In terms of the parameterization (1.2.1) of the uncertainty sets, the latter conclusion means that

When speaking about the Robust Counterpart of an uncertain LO problem with a certain objective, we lose nothing when assuming that the set $\mathcal{U}_i$ of uncertain data of i-th constraint is given as

$$\mathcal{U}_i = \left\{ [a_i; b_i] = [a_i^0; b_i^0] + \sum_{\ell=1}^{L_i} \zeta_\ell [a_i^\ell; b_i^\ell] : \zeta \in \mathcal{Z}_i \right\}, \tag{1.2.9}$$

with a closed and convex perturbation set $\mathcal{Z}_i$.

D. An important modeling issue. In the usual — with certain data — Linear Optimization, constraints can be modeled in various equivalent forms. For example, we can write:

$$\begin{array}{ll} (a) & a_1x_1 + a_2x_2 \le a_3 \\ (b) & a_4x_1 + a_5x_2 = a_6 \\ (c) & x_1 \ge 0, x_2 \ge 0 \end{array} \tag{1.2.10}$$

or, equivalently,

$$\begin{array}{ll} (a) & a_1x_1 + a_2x_2 \le a_3 \\ (b.1) & a_4x_1 + a_5x_2 \le a_6 \\ (b.2) & -a_5x_1 - a_5x_2 \le -a_6 \\ (c) & x_1 \ge 0, x_2 \ge 0. \end{array} \tag{1.2.11}$$

Or, equivalently, by adding a slack variable s,

$$\begin{array}{ll} (a) & a_1x_1 + a_2x_2 + s = a_3 \\ (b) & a_4x_1 + a_5x_2 = a_6 \\ (c) & x_1 \ge 0, x_2 \ge 0, s \ge 0. \end{array} \tag{1.2.12}$$

However, when (part of) the data $a_1, ..., a_6$ become *uncertain*, not all of these equivalences remain valid: the RCs of our now uncertainty-affected systems of constraints are not equivalent to each other. Indeed, denoting the uncertainty set by $\mathcal{U}$, the RCs read, respectively,

$$\left.\begin{array}{ll} (a) & a_1x_1 + a_2x_2 \le a_3 \\ (b) & a_4x_1 + a_5x_2 = a_6 \\ (c) & x_1 \ge 0, x_2 \ge 0 \end{array}\right\} \forall a = [a_1; ...; a_6] \in \mathcal{U}. \tag{1.2.13}$$

$$\left.\begin{array}{ll} (a) & a_1x_1 + a_2x_2 \le a_3 \\ (b.1) & a_4x_1 + a_5x_2 \le a_6 \\ (b.2) & -a_5x_1 - a_5x_2 \le -a_6 \\ (c) & x_1 \ge 0, x_2 \ge 0 \end{array}\right\} \forall a = [a_1; ...; a_6] \in \mathcal{U}. \tag{1.2.14}$$

$$\left.\begin{array}{ll} (a) & a_1x_1 + a_2x_2 + s = a_3 \\ (b) & a_4x_1 + a_5x_2 = a_6 \\ (c) & x_1 \ge 0, x_2 \ge 0, s \ge 0 \end{array}\right\} \forall a = [a_1; ...; a_6] \in \mathcal{U}. \tag{1.2.15}$$

It is immediately seen that while the first and the second RCs are equivalent to each other,[3] they are *not* equivalent to the third RC. The latter RC is more conservative than the first two, meaning that whenever (x_1, x_2) can be extended, by a properly chosen s, to a feasible solution of (1.2.15), (x_1, x_2) is feasible for (1.2.13)≡(1.2.14) (this is evident), but not necessarily vice versa. In fact, the gap between (1.2.15) and (1.2.13)≡(1.2.14) can be quite large. To illustrate the latter claim, consider the case where the uncertainty set is

$$\mathcal{U} = \{a = a_\zeta := [1+\zeta; 2+\zeta; 4-\zeta; 4+\zeta; 5-\zeta; 9] : -\rho \le \zeta \le \rho\},$$

where ζ is the data perturbation. In this situation, $x_1 = 1$, $x_2 = 1$ is a feasible solution to (1.2.13)≡(1.2.14), provided that the uncertainty level ρ is $\le 1/3$:

$$(1+\zeta)\cdot 1 + (2+\zeta)\cdot 1 \le 4-\zeta\,\forall(\zeta : |\zeta| \le \rho \le 1/3)\ \&(4+\zeta)\cdot 1 + (5-\zeta)\cdot 1 = 9\ \forall\zeta.$$

At the same time, when $\rho > 0$, our solution $(x_1 = 1, x_2 = 1)$ cannot be extended to a feasible solution of (1.2.15), since the latter system of constraints is infeasible and remains so even after eliminating the equality (1.2.15.b).

> Indeed, in order for x_1, x_2, s to satisfy (1.2.15.a) for all $a \in \mathcal{U}$, we should have
>
> $$x_1 + 2x_2 + s + \zeta[x_1 + x_2] = 4 - \zeta\ \forall(\zeta : |\zeta| \le \rho);$$
>
> when $\rho > 0$, we therefore should have $x_1 + x_2 = -1$, which contradicts (1.2.15.c)

The origin of the outlined phenomenon is clear. Evidently the inequality $a_1x_1 + a_2x_2 \le a_3$, where all a_i and x_i are fixed reals, holds true if and only if we can "certify" the inequality by pointing out a real $s \ge 0$ such that $a_1x_1 + a_2x_2 + s = a_3$. When the data a_1, a_2, a_3 become uncertain, the restriction on (x_1, x_2) to be robust feasible for the uncertain inequality $a_1x_1 + a_2x_2 \le a_3$ for all $a \in \mathcal{U}$ reads, "in terms of certificate," as

$$\forall a \in \mathcal{U}\, \exists s \ge 0 : a_1x_1 + a_2x_2 + s = a_3,$$

that is, the certificate s should be allowed to depend on the true data. In contrast to this, in (1.2.15) we require from both the decision variables x *and the slack variable ("the certificate")* s to be independent of the true data, which is by far too conservative.

What can be learned from the above examples is that when modeling an uncertain LO problem one should avoid whenever possible converting inequality

[3]Clearly, this always is the case when an equality constraint, certain or uncertain alike, is replaced with a pair of opposite inequalities.

constraints into equality ones, unless all the data in the constraints in question are certain. Aside from avoiding slack variables,[4] this means that restrictions like "total expenditure cannot exceed the budget," or "supply should be at least the demand," which in LO problems with certain data can harmlessly be modeled by equalities, in the case of uncertain data should be modeled by inequalities. This is in full accordance with common sense saying, e.g., that when the demand is uncertain and its satisfaction is a must, it would be unwise to forbid surplus in supply. Sometimes a good for the RO methodology modeling requires eliminating "state variables" — those which are readily given by variables representing actual decisions — via the corresponding "state equations." For example, time dynamics of an inventory is given in the simplest case by the state equations

$$\begin{aligned} x_0 &= c \\ x_{t+1} &= x_t + q_t - d_t,\ t = 0, 1, ..., T, \end{aligned}$$

where x_t is the inventory level at time t, d_t is the (uncertain) demand in period $[t, t+1)$, and variables q_t represent actual decisions – replenishment orders at instants $t = 0, 1, ..., T$. A wise approach to the RO processing of such an inventory problem would be to eliminate the state variables x_t by setting

$$x_t = c + \sum_{\tau=1}^{t-1} q_\tau,\ t = 0, 1, 2, ..., T + 1,$$

and to get rid of the state equations. As a result, typical restrictions on state variables (like "x_t should stay within given bounds" or "total holding cost should not exceed a given bound") will become uncertainty-affected inequality constraints on the actual decisions q_t, and we can process the resulting inequality-constrained uncertain LO problem via its RC.[5]

1.2.2 What is Ahead

After introducing the concept of the Robust Counterpart of an uncertain LO problem, we confront two major questions:

i) What is the "computational status" of the RC? When is it possible to process the RC efficiently?

ii) How to come-up with meaningful uncertainty sets?

The first of these questions, to be addressed in depth in section 1.3, is a "structural" one: what should be the structure of the uncertainty set in order to make the RC computationally tractable? Note that the RC as given by (1.2.5) or (1.2.6) is a *semi-infinite* LO program, that is, an optimization program with simple linear

[4]Note that slack variables do not represent actual decisions; thus, their presence in an LO model contradicts assumption A.1, and thus can lead to too conservative, or even infeasible, RCs.
[5]For more advanced robust modeling of uncertainty-affected multi-stage inventory, see chapter 14.

objective and *infinitely many* linear constraints. In principle, such a problem can be "computationally intractable" — NP-hard.

Example 1.2.7. Consider an uncertain "essentially linear" constraint

$$\{\|Px - p\|_1 \leq 1\}_{[P;p]\in\mathcal{U}}\,, \tag{1.2.16}$$

where $\|z\|_1 = \sum\limits_j |z_j|$, and assume that the matrix P is certain, while the vector p is uncertain and is parameterized by perturbations from the unit box:

$$p \in \{p = B\zeta : \|\zeta\|_\infty \leq 1\}\,,$$

where $\|\zeta\|_\infty = \max\limits_\ell |\zeta_\ell|$ and B is a given positive semidefinite matrix. To check whether $x = 0$ is robust feasible is exactly the same as to verify whether $\|B\zeta\|_1 \leq 1$ whenever $\|\zeta\|_\infty \leq 1$; or, due to the evident relation $\|u\|_1 = \max\limits_{\|\eta\|_\infty\leq 1} \eta^T u$, the same as to check whether $\max\limits_{\eta,\zeta}\left\{\eta^T B\zeta : \|\eta\|_\infty \leq 1, \|\zeta\|_\infty \leq 1\right\} \leq 1$. The maximum of the bilinear form $\eta^T B\zeta$ with positive semidefinite B over η, ζ varying in a convex symmetric neighborhood of the origin is always achieved when $\eta = \zeta$ (you may check this by using the polarization identity $\eta^T B\zeta = \frac{1}{4}(\eta+\zeta)^T B(\eta+\zeta) - \frac{1}{4}(\eta-\zeta)^T B(\eta-\zeta)$). Thus, to check whether $x = 0$ is robust feasible for (1.2.16) is the same as to check whether the maximum of a given nonnegative quadratic form $\zeta^T B\zeta$ over the unit box is ≤ 1. The latter problem is known to be NP-hard,[6] and therefore so is the problem of checking robust feasibility for (1.2.16).

The second of the above is a modeling question, and as such, goes beyond the scope of purely theoretical considerations. However, theory, as we shall see in section 2.1, contributes significantly to this modeling issue.

1.3 TRACTABILITY OF ROBUST COUNTERPARTS

In this section, we investigate the "computational status" of the RC of uncertain LO problem. The situation here turns out to be as good as it could be: we shall see, essentially, that *the RC of the uncertain LO problem with uncertainty set $\mathcal{U}$ is computationally tractable whenever the convex uncertainty set $\mathcal{U}$ itself is computationally tractable.* The latter means that we know in advance the affine hull of $\mathcal{U}$, a point from the relative interior of $\mathcal{U}$, and we have access to an efficient *membership oracle* that, given on input a point u, reports whether $u \in \mathcal{U}$. This can be reformulated as a precise mathematical statement; however, we will prove a slightly restricted version of this statement that does not require long excursions into complexity theory.

1.3.1 The Strategy

Our strategy will be as follows. First, we restrict ourselves to uncertain LO problems with a certain objective — we remember from item **A** in Section 1.2.1 that we lose

[6] In fact, it is NP-hard to compute the maximum of a nonnegative quadratic form over the unit box with inaccuracy less than 4% [61].

nothing by this restriction. Second, all we need is a "computationally tractable" representation of the RC of a *single* uncertain linear constraint, that is, an equivalent representation of the RC by an explicit (and "short") system of efficiently verifiable convex inequalities. Given such representations for the RCs of every one of the constraints of our uncertain problem and putting them together (cf. item **B** in Section 1.2.1), we reformulate the RC of the problem as the problem of minimizing the original linear objective under a finite (and short) system of explicit convex constraints, and thus — as a computationally tractable problem.

To proceed, we should explain first what does it mean to represent a constraint by a system of convex inequalities. Everyone understands that the system of 4 constraints on 2 variables,

$$x_1 + x_2 \leq 1, x_1 - x_2 \leq 1, -x_1 + x_2 \leq 1, -x_1 - x_2 \leq 1, \tag{1.3.1}$$

represents the nonlinear inequality

$$|x_1| + |x_2| \leq 1 \tag{1.3.2}$$

in the sense that both (1.3.2) and (1.3.1) define the same feasible set. Well, what about the claim that the system of 5 linear inequalities

$$-u_1 \leq x_1 \leq u_1, -u_2 \leq x_2 \leq u_2, u_1 + u_2 \leq 1 \tag{1.3.3}$$

represents the same set as (1.3.2)? Here again everyone will agree with the claim, although we cannot justify the claim in the former fashion, since the feasible sets of (1.3.2) and (1.3.3) live in different spaces and therefore cannot be equal to each other!

What actually is meant when speaking about "equivalent representations of problems/constraints" in Optimization can be formalized as follows:

Definition 1.3.1. A set $X^+ \subset \mathbb{R}^n_x \times \mathbb{R}^k_u$ is said to represent a set $X \subset \mathbb{R}^n_x$, if the projection of X^+ onto the space of x-variables is exactly X, i.e., $x \in X$ if and only if there exists $u \in \mathbb{R}^k_u$ such that $(x, u) \in X^+$:

$$X = \left\{x : \exists u : (x, u) \in X^+\right\}.$$

A system of constraints $\mathcal{S}^+$ in variables $x \in \mathbb{R}^n_x$, $u \in \mathbb{R}^k_u$ is said to represent a system of constraints $\mathcal{S}$ in variables $x \in \mathbb{R}^n_x$, if the feasible set of the former system represents the feasible set of the latter one.

With this definition, it is clear that the system (1.3.3) indeed represents the constraint (1.3.2), and, more generally, that the system of $2n+1$ linear inequalities

$$-u_j \leq x_j \leq u_j,\ j = 1, ..., n, \sum_j u_j \leq 1$$

in variables x, u represents the constraint

$$\sum_j |x_j| \leq 1.$$

To understand how powerful this representation is, note that to represent the same constraint in the style of (1.3.1), that is, without extra variables, it would take as much as 2^n linear inequalities.

Coming back to the general case, assume that we are given an optimization problem

$$\min_x \{f(x) \text{ s.t. } x \text{ satisfies } \mathcal{S}_i,\ i = 1, ..., m\}, \tag{P}$$

where $\mathcal{S}_i$ are systems of constraints in variables x, and that we have in our disposal systems $\mathcal{S}_i^+$ of constraints in variables x, v^i which represent the systems $\mathcal{S}_i$. Clearly, the problem

$$\min_{x,v^1,...,v^m} \left\{f(x) \text{ s.t. } (x, v^i) \text{ satisfies } \mathcal{S}_i^+,\ i = 1, ..., m\right\} \tag{P$^+$}$$

is equivalent to (P): the x component of every feasible solution to (P$^+$) is feasible for (P) with the same value of the objective, and the optimal values in the problems are equal to each other, so that the x component of an ϵ-optimal (in terms of the objective) feasible solution to (P$^+$) is an ϵ-optimal feasible solution to (P). We shall say that (P$^+$) represents equivalently the original problem (P). What is important here, is that a representation can possess desired properties that are absent in the original problem. For example, an appropriate representation can convert the problem of the form $\min\limits_x\{\|Px-p\|_1 : Ax \leq b\}$ with n variables, m linear constraints, and k-dimensional vector p, into an LO problem with $n+k$ variables and $m+2k+1$ linear inequality constraints, etc. Our goal now is to build a representation capable of expressing equivalently a semi-infinite linear constraint (specifically, the robust counterpart of an uncertain linear inequality) as a finite system of explicit convex constraints, with the ultimate goal to use these representations in order to convert the RC of an uncertain LO problem into an explicit (and as such, computationally tractable) convex program.

The outlined strategy allows us to focus on a *single* uncertainty-affected linear inequality — a family

$$\left\{a^T x \leq b\right\}_{[a;b]\in\mathcal{U}}, \tag{1.3.4}$$

of linear inequalities with the data varying in the uncertainty set

$$\mathcal{U} = \left\{[a; b] = [a^0; b^0] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell; b^\ell] : \zeta \in \mathcal{Z}\right\} \tag{1.3.5}$$

— and on "tractable representation" of the RC

$$a^T x \leq b \quad \forall \left([a; b] = [a^0; b^0] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell; b^\ell] : \zeta \in \mathcal{Z}\right) \tag{1.3.6}$$

of this uncertain inequality.

By reasons indicated in item **C** of Section 1.2.1, we assume from now on that the associated perturbation set $\mathcal{Z}$ is convex.

1.3.2 Tractable Representation of (1.3.6): Simple Cases

We start with the cases where the desired representation can be found by "bare hands," specifically, the cases of *interval* and *simple ellipsoidal* uncertainty.

Example 1.3.2. Consider the case of *interval uncertainty*, where $\mathcal{Z}$ in (1.3.6) is a box. W.l.o.g.[7] we can normalize the situation by assuming that

$$\mathcal{Z} = \mathrm{Box}_1 \equiv \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \le 1\}.$$

In this case, (1.3.6) reads

$$\begin{array}{lll} & [a^0]^T x + \sum\limits_{\ell=1}^L \zeta_\ell [a^\ell]^T x \le b^0 + \sum\limits_{\ell=1}^L \zeta_\ell b^\ell & \forall (\zeta : \|\zeta\|_\infty \le 1) \\ \Leftrightarrow & \sum\limits_{\ell=1}^L \zeta_\ell [[a^\ell]^T x - b^\ell] \le b^0 - [a^0]^T x & \forall (\zeta : |\zeta_\ell| \le 1, \ \ell = 1, ..., L) \\ \Leftrightarrow & \max\limits_{-1 \le \zeta_\ell \le 1} \left[\sum\limits_{\ell=1}^L \zeta_\ell [[a^\ell]^T x - b^\ell] \right] \le b^0 - [a^0]^T x & \end{array}$$

The concluding maximum in the chain is clearly $\sum\limits_{\ell=1}^L |[a^\ell]^T x - b^\ell|$, and we arrive at the representation of (1.3.6) by the explicit convex constraint

$$[a^0]^T x + \sum_{\ell=1}^L |[a^\ell]^T x - b^\ell| \le b^0, \tag{1.3.7}$$

which in turn admits a representation by a system of linear inequalities:

$$\left\{ \begin{array}{l} -u_\ell \le [a^\ell]^T x - b^\ell \le u_\ell, \ \ell = 1, ..., L, \\ [a^0]^T x + \sum\limits_{\ell=1}^L u_\ell \le b^0. \end{array} \right. \tag{1.3.8}$$

Example 1.3.3. Consider the case of *ellipsoidal uncertainty* where $\mathcal{Z}$ in (1.3.6) is an ellipsoid. W.l.o.g. we can normalize the situation by assuming that $\mathcal{Z}$ is merely the ball of radius Ω centered at the origin:

$$\mathcal{Z} = \mathrm{Ball}_\Omega = \{\zeta \in \mathbb{R}^L : \|\zeta\|_2 \le \Omega\}.$$

In this case, (1.3.6) reads

$$\begin{array}{lll} & [a^0]^T x + \sum\limits_{\ell=1}^L \zeta_\ell [a^\ell]^T x \le b^0 + \sum\limits_{\ell=1}^L \zeta_\ell b^\ell & \forall (\zeta : \|\zeta\|_2 \le \Omega) \\ \Leftrightarrow & \max\limits_{\|\zeta\|_2 \le \Omega} \left[\sum\limits_{\ell=1}^L \zeta_\ell [[a^\ell]^T x - b^\ell] \right] \le b^0 - [a^0]^T x & \\ \Leftrightarrow & \Omega \sqrt{\sum\limits_{\ell=1}^L ([a^\ell]^T x - b^\ell)^2} \le b^0 - [a^0]^T x, & \end{array}$$

and we arrive at the representation of (1.3.6) by the explicit convex constraint ("conic quadratic inequality")

$$[a^0]^T x + \Omega \sqrt{\sum_{\ell=1}^L ([a^\ell]^T x - b^\ell)^2} \le b^0. \tag{1.3.9}$$

---

[7] abbr. for "without loss of generality."

1.3.3 Tractable Representation of (1.3.6): General Case

Now consider a rather general case when the perturbation set $\mathcal{Z}$ in (1.3.6) is given by a *conic representation* (cf. section A.2.4 in Appendix):

$$\mathcal{Z} = \left\{\zeta \in \mathbb{R}^L : \exists u \in \mathbb{R}^K : P\zeta + Qu + p \in \mathbf{K}\right\}, \tag{1.3.10}$$

where $\mathbf{K}$ is a closed convex pointed cone in $\mathbb{R}^N$ with a nonempty interior, P, Q are given matrices and p is a given vector. In the case when $\mathbf{K}$ is *not* a polyhedral cone, assume that this representation is strictly feasible:

$$\exists(\bar{\zeta}, \bar{u}) : P\bar{\zeta} + Q\bar{u} + p \in \mathrm{int}K. \tag{1.3.11}$$

Theorem 1.3.4. Let the perturbation set $\mathcal{Z}$ be given by (1.3.10), and in the case of non-polyhedral $\mathbf{K}$, let also (1.3.11) take place. Then the semi-infinite constraint (1.3.6) can be represented by the following system of conic inequalities in variables $x \in \mathbb{R}^n, y \in \mathbb{R}^N$:

$$\begin{array}{l} p^T y + [a^0]^T x \le b^0, \\ Q^T y = 0, \\ (P^T y)_\ell + [a^\ell]^T x = b^\ell, \ell = 1, ..., L, \\ y \in \mathbf{K}_*, \end{array} \tag{1.3.12}$$

where $\mathbf{K}_* = \{y : y^T z \ge 0 \, \forall z \in \mathbf{K}\}$ is the cone dual to $\mathbf{K}$.

Proof. We have

$$\begin{array}{cl} & x \text{ is feasible for (1.3.6)} \\ \Leftrightarrow & \sup\limits_{\zeta\in\mathcal{Z}} \Big\{ \underbrace{[a^0]^T x - b^0}_{d[x]} + \sum\limits_{\ell=1}^{L} \zeta_\ell \underbrace{\left[[a^\ell]^T x - b^\ell\right]}_{c_\ell[x]} \Big\} \le 0 \\ \Leftrightarrow & \sup\limits_{\zeta\in\mathcal{Z}} \left\{c^T[x]\zeta + d[x]\right\} \le 0 \\ \Leftrightarrow & \sup\limits_{\zeta\in\mathcal{Z}} c^T[x]\zeta \le -d[x] \\ \Leftrightarrow & \max\limits_{\zeta, v} \left\{c^T[x]\zeta : P\zeta + Qv + p \in \mathbf{K}\right\} \le -d[x]. \end{array}$$

The concluding relation says that x is feasible for (1.3.6) if and only if the optimal value in the conic program

$$\max_{\zeta, v} \left\{c^T[x]\zeta : P\zeta + Qv + p \in \mathbf{K}\right\} \tag{CP}$$

is $\le -d[x]$. Assume, first, that (1.3.11) takes place. Then (CP) is strictly feasible, and therefore, applying the Conic Duality Theorem (Theorem A.2.1), the optimal value in (CP) is $\le -d[x]$ if and only if the optimal value in the conic dual to the (CP) problem

$$\min_y \left\{p^T y : Q^T y = 0, P^T y = -c[x], y \in \mathbf{K}_*\right\}, \tag{CD}$$

is attained and is $\le -d[x]$. Now assume that $\mathbf{K}$ is a polyhedral cone. In this case the usual LO Duality Theorem, (which does not require the validity of (1.3.11)), yields exactly the same conclusion: the optimal value in (CP) is $\le -d[x]$ if and only if the optimal value in (CD) is achieved and is $\le -d[x]$. In other words, under the premise of the Theorem, x is feasible for (1.3.6) if and only if (CD) has a feasible solution y with $p^T y \leq -d[x]$. □

Observing that nonnegative orthants, Lorentz and Semidefinite cones are self-dual, we derive from Theorem 1.3.4 the following corollary:

Corollary 1.3.5. Let the nonempty perturbation set in (1.3.6) be:

(i) polyhedral, i.e., given by (1.3.10) with a nonnegative orthant $\mathbb{R}^N_+$ in the role of $\mathbf{K}$, or

(ii) conic quadratic representable, i.e., given by (1.3.10) with a direct product $\mathbf{L}^{k_1} \times ... \times \mathbf{L}^{k_m}$ of Lorentz cones $\mathbf{L}^k = \{x \in \mathbb{R}^k : x_k \geq \sqrt{x_1^2 + ... + x_{k-1}{}^2}\}$ in the role of $\mathbf{K}$, or

(iii) semidefinite representable, i.e., given by (1.3.10) with the positive semidefinite cone $\mathbf{S}^k_+$ in the role of $\mathbf{K}$.

In the cases of (ii), (iii) assume in addition that (1.3.11) holds true. Then the Robust Counterpart (1.3.6) of the uncertain linear inequality (1.3.4) — (1.3.5) with the perturbation set $\mathcal{Z}$ admits equivalent reformulation as an explicit system of

— linear inequalities, in the case of (i),

— conic quadratic inequalities, in the case of (ii),

— linear matrix inequalities, in the case of (iii).

In all cases, the size of the reformulation is polynomial in the number of variables in (1.3.6) and the size of the conic description of $\mathcal{Z}$, while the data of the reformulation is readily given by the data describing, via (1.3.10), the perturbation set $\mathcal{Z}$.

Remark 1.3.6. **A.** Usually, the cone $\mathbf{K}$ participating in (1.3.10) is the direct product of simpler cones $\mathbf{K}^1, ..., \mathbf{K}^S$, so that representation (1.3.10) takes the form

$$\mathcal{Z} = \{\zeta : \exists u^1, ..., u^S : P_s\zeta + Q_s u^s + p_s \in \mathbf{K}^s,\ s = 1, ..., S\}. \tag{1.3.13}$$

In this case, (1.3.12) becomes the system of conic constraints in variables $x, y^1, ..., y^S$ as follows:

$$\begin{array}{l}
\sum\limits_{s=1}^{S} p_s^T y^s + [a^0]^T x \leq b^0, \\
Q_s^T y^s = 0,\ s = 1, ..., S, \\
\sum\limits_{s=1}^{S} (P_s^T y^s)_\ell + [a^\ell]^T x = b^\ell, \ell = 1, ..., L, \\
y^s \in \mathbf{K}^s_*, s = 1, ..., S,
\end{array} \tag{1.3.14}$$

where K^s_* is the cone dual to K^s.

B. Uncertainty sets given by LMIs seem "exotic"; however, they can arise under quite realistic circumstances, see section 1.4.

1.3.3.1 Examples

We are about to apply Theorem 1.3.4 to build tractable reformulations of the semi-infinite inequality (1.3.6) in two particular cases. While at a first glance no natural "uncertainty models" lead to the "strange" perturbation sets we are about to consider, it will become clear later that these sets are of significant importance — they allow one to model *random* uncertainty.

Example 1.3.7. *$\mathcal{Z}$ is the intersection of concentric co-axial box and ellipsoid, specifically,*

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : -1 \le \zeta_\ell \le 1, \ell \le L, \sqrt{\sum_{\ell=1}^{L} \zeta_\ell^2/\sigma_\ell^2} \le \Omega\}, \tag{1.3.15}$$

where $\sigma_\ell > 0$ and $\Omega > 0$ are given parameters.

Here representation (1.3.13) becomes

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : P_1\zeta + p_1 \in \mathbf{K}^1, P_2\zeta + p_2 \in \mathbf{K}^2\},$$

where

- $P_1\zeta \equiv [\zeta; 0]$, $p_1 = [0_{L\times 1}; 1]$ and $\mathbf{K}^1 = \{(z,t) \in \mathbb{R}^L \times \mathbb{R} : t \ge \|z\|_\infty\}$, whence $\mathbf{K}^1_* = \{(z,t) \in \mathbb{R}^L \times \mathbb{R} : t \ge \|z\|_1\}$;
- $P_2\zeta = [\Sigma^{-1}\zeta; 0]$ with $\Sigma = \mathrm{Diag}\{\sigma_1, ..., \sigma_L\}$, $p_2 = [0_{L\times 1}; \Omega]$ and $\mathbf{K}^2$ is the Lorentz cone of the dimension $L+1$ (whence $\mathbf{K}^2_* = \mathbf{K}^2$)

Setting $y^1 = [\eta_1; \tau_1]$, $y^2 = [\eta_2; \tau_2]$ with one-dimensional τ_1, τ_2 and L-dimensional η_1, η_2, (1.3.14) becomes the following system of constraints in variables τ, η, x:

$$\begin{array}{lrcl}
(a) & \tau_1 + \Omega\tau_2 + [a^0]^T x & \le & b^0, \\
(b) & (\eta_1 + \Sigma^{-1}\eta_2)_\ell & = & b^\ell - [a^\ell]^T x, \ \ell = 1, ..., L, \\
(c) & \|\eta_1\|_1 & \le & \tau_1 \quad [\Leftrightarrow [\eta_1; \tau_1] \in \mathbf{K}^1_*], \\
(d) & \|\eta_2\|_2 & \le & \tau_2 \quad [\Leftrightarrow [\eta_2; \tau_2] \in \mathbf{K}^2_*].
\end{array}$$

We can eliminate from this system the variables τ_1, τ_2 — for every feasible solution to the system, we have $\tau_1 \ge \bar{\tau}_1 \equiv \|\eta_1\|_1$, $\tau_2 \ge \bar{\tau}_2 \equiv \|\eta_2\|_2$, and the solution obtained when replacing τ_1, τ_2 with $\bar{\tau}_1$, $\bar{\tau}_2$ still is feasible. The reduced system in variables x, $z = \eta_1$, $w = \Sigma^{-1}\eta_2$ reads

$$\begin{array}{rcl}
\sum\limits_{\ell=1}^{L} |z_\ell| + \Omega\sqrt{\sum\limits_\ell \sigma_\ell^2 w_\ell^2} + [a^0]^T x & \le & b^0, \\
z_\ell + w_\ell & = & b^\ell - [a^\ell]^T x, \ \ell = 1, ..., L,
\end{array} \tag{1.3.16}$$

which is also a representation of (1.3.6), (1.3.15).

Example 1.3.8. [*"budgeted uncertainty"*] Consider the case where $\mathcal{Z}$ is the intersection of $\|\cdot\|_\infty$- and $\|\cdot\|_1$-balls, specifically,

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \le 1, \|\zeta\|_1 \le \gamma\}, \tag{1.3.17}$$

where γ, $1 \le \gamma \le L$, is a given "uncertainty budget."

Here representation (1.3.13) becomes

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^L : P_1\zeta + p_1 \in \mathbf{K}^1, P_2\zeta + p_2 \in \mathbf{K}^2\},$$

where

- $P_1\zeta \equiv [\zeta; 0]$, $p_1 = [0_{L\times 1}; 1]$ and $\mathbf{K}^1 = \{[z;t] \in \mathbb{R}^L \times \mathbb{R} : t \geq \|z\|_\infty\}$, whence $\mathbf{K}^1_* = \{[z;t] \in \mathbb{R}^L \times \mathbb{R} : t \geq \|z\|_1\}$;
- $P_2\zeta = [\zeta; 0]$, $p_2 = [0_{L\times 1}; \gamma]$ and $\mathbf{K}^2 = \mathbf{K}^1_* = \{[z;t] \in \mathbb{R}^L \times \mathbb{R} : t \geq \|z\|_1\}$, whence $\mathbf{K}^2_* = \mathbf{K}^1$.

Setting $y^1 = [z; \tau_1]$, $y^2 = [w; \tau_2]$ with one-dimensional τ and L-dimensional z, w, system (1.3.14) becomes the following system of constraints in variables τ_1, τ_2, z, w, x:

$$\begin{array}{llcl}
(a) & \tau_1 + \gamma\tau_2 + [a^0]^T x & \leq & b^0, \\
(b) & (z+w)_\ell & = & b^\ell - [a^\ell]^T x,\ \ell = 1, ..., L, \\
(c) & \|z\|_1 & \leq & \tau_1 \quad [\Leftrightarrow [\eta_1; \tau_1] \in \mathbf{K}^1_*], \\
(d) & \|w\|_\infty & \leq & \tau_2 \quad [\Leftrightarrow [\eta_2; \tau_2] \in \mathbf{K}^2_*].
\end{array}$$

Same as in Example 1.3.7, we can eliminate the τ-variables, arriving at a representation of (1.3.6), (1.3.17) by the following system of constraints in variables x, z, w:

$$\begin{array}{rcl}
\sum\limits_{\ell=1}^{L} |z_\ell| + \gamma \max\limits_\ell |w_\ell| + [a^0]^T x & \leq & b^0, \\
z_\ell + w_\ell & = & b^\ell - [a^\ell]^T x,\ \ell = 1, ..., L,
\end{array} \tag{1.3.18}$$

which can be further converted into the system of linear inequalities in z, w and additional variables.

1.4 NON-AFFINE PERTURBATIONS

In the first reading this section can be skipped.

So far we have assumed that the uncertain data of an uncertain LO problem are *affinely* parameterized by a perturbation vector ζ varying in a closed convex set $\mathcal{Z}$. We have seen that this assumption, combined with the assumption that $\mathcal{Z}$ is computationally tractable, implies tractability of the RC. What happens when the perturbations enter the uncertain data in a nonlinear fashion? Assume w.l.o.g. that every entry a in the uncertain data is of the form

$$a = \sum_{k=1}^{K} c_k^a f_k(\zeta),$$

where c_k^a are given coefficients (depending on the data entry in question) and $f_1(\zeta), ..., f_K(\zeta)$ are certain basic functions, perhaps non-affine, defined on the perturbation set $\mathcal{Z}$. Assuming w.l.o.g. that the objective is certain, we still can define the RC of our uncertain problem as the problem of minimizing the original objective over the set of robust feasible solutions, those which remain feasible for all values of the data coming from $\zeta \in \mathcal{Z}$, but what about the tractability of this RC? An immediate observation is that the case of nonlinearly perturbed data can be immediately reduced to the one where the data are affinely perturbed. To this end, it suffices to pass from the original perturbation vector ζ to the new vector

$$\widehat{\zeta}[\zeta] = [\zeta_1; ...; \zeta_L; f_1(\zeta); ...; f_K(\zeta)].$$

As a result, the uncertain data become *affine* functions of the new perturbation vector $\widehat{\zeta}$ which now runs through the image $\widetilde{\mathcal{Z}} = \widehat{\zeta}[\mathcal{Z}]$ of the original uncertainty set $\mathcal{Z}$ under the mapping $\zeta \mapsto \widehat{\zeta}[\zeta]$. As we know, in the case of affine data perturbations the RC remains intact when replacing a given perturbation set with its closed convex hull. Thus, we can think about our uncertain LO problem as an affinely perturbed problem where the perturbation vector is $\widehat{\zeta}$, and this vector runs through the closed convex set $\widehat{\mathcal{Z}} = \mathrm{cl}\,\mathrm{Conv}(\widehat{\zeta}[\mathcal{Z}])$. We see that formally speaking, the case of general-type perturbations can be reduced to the one of affine perturbations. This, unfortunately, does not mean that non-affine perturbations do not cause difficulties. Indeed, in order to end up with a computationally tractable RC, we need more than affinity of perturbations and convexity of the perturbation set — we need this set to be computationally tractable. And the set $\widehat{\mathcal{Z}} = \mathrm{cl}\,\mathrm{Conv}(\widehat{\zeta}[\mathcal{Z}])$ may fail to satisfy this requirement even when both $\mathcal{Z}$ and the *nonlinear* mapping $\zeta \mapsto \widehat{\zeta}[\zeta]$ are simple, e.g., when $\mathcal{Z}$ is a box and $\widehat{\zeta} = [\zeta; \{\zeta_\ell \zeta_r\}_{\ell,r=1}^L]$, (i.e., when the uncertain data are quadratically perturbed by the original perturbations ζ).

We are about to present two generic cases where the difficulty just outlined does not occur (for justification and more examples, see section 14.3.2).

Ellipsoidal perturbation set $\mathcal{Z}$, quadratic perturbations. Here $\mathcal{Z}$ is an ellipsoid, and the basic functions f_k are the constant, the coordinates of ζ and the pairwise products of these coordinates. This means that the uncertain data entries are quadratic functions of the perturbations. W.l.o.g. we can assume that the ellipsoid $\mathcal{Z}$ is centered at the origin: $\mathcal{Z} = \{\zeta : \|Q\zeta\|_2 \leq 1\}$, where $\mathrm{Ker}Q = \{0\}$. In this case, representing $\widehat{\zeta}[\zeta]$ as the matrix $\left[\begin{array}{c|c} & \zeta^T \\ \hline \zeta & \zeta\zeta^T \end{array}\right]$, we have the following semidefinite representation of $\widehat{\mathcal{Z}} = \mathrm{cl}\,\mathrm{Conv}(\widehat{\zeta}[\mathcal{Z}])$:

$$\widehat{\mathcal{Z}} = \left\{ \left[\begin{array}{c|c} & w^T \\ \hline w & W \end{array}\right] : \left[\begin{array}{c|c} 1 & w^T \\ \hline w & W \end{array}\right] \succeq 0, \mathrm{Tr}(QWQ^T) \leq 1 \right\}$$

(for proof, see Lemma 14.3.7).

Separable polynomial perturbations. Here the structure of perturbations is as follows: ζ runs through the box $\mathcal{Z} = \{\zeta \in \mathbb{R}^L : \|\zeta\|_\infty \leq 1\}$, and the uncertain data entries are of the form

$$a = p_1^a(\zeta_1) + ... + p_L^a(\zeta_L),$$

where $p_\ell^a(s)$ are given algebraic polynomials of degrees not exceeding d; in other words, the basic functions can be split into L groups, the functions of ℓ-th group being $1 = \zeta_\ell^0, \zeta_\ell, \zeta_\ell^2, ..., \zeta_\ell^d$. Consequently, the function $\widehat{\zeta}[\zeta]$ is given by

$$\widehat{\zeta}[\zeta] = [[1; \zeta_1; \zeta_1^2; ...; \zeta_1^d]; ...; [1; \zeta_L; \zeta_L^2; ...; \zeta_L^d]].$$

Setting $P = \{\widehat{s} = [1; s; s^2; ...; s^d] : -1 \leq s \leq 1\}$, we conclude that $\widetilde{\mathcal{Z}} = \widehat{\zeta}[\mathcal{Z}]$ can be identified with the set $P^L = \underbrace{P \times ... \times P}_{L}$, so that $\widehat{\mathcal{Z}}$ is nothing but the set

$\underbrace{\mathcal{P} \times \ldots \times \mathcal{P}}_{L}$, where $\mathcal{P} = \mathrm{Conv}(P)$. It remains to note that the set $\mathcal{P}$ admits an explicit semidefinite representation, see Lemma 14.3.4.

1.5 EXERCISES

Exercise 1.1. Consider an uncertain LO problem with instances

$$\min_x \left\{ c^T x : Ax \le b \right\} \qquad [A : m \times n]$$

and with simple interval uncertainty:

$$\mathcal{U} = \{(c, A, b) : |c_j - c_j^{\mathrm{n}}| \le \sigma_j, |A_{ij} - A_{ij}^{\mathrm{n}}| \le \alpha_{ij}, |b_i - b_i^{\mathrm{n}}| \le \beta_i \forall i, j\}$$

($^{\mathrm{n}}$ marks the nominal data). Reduce the RC of the problem to an LO problem with m constraints (not counting the sign constraints on the variables) and $2n$ nonnegative variables.

Exercise 1.2. Represent the RCs of every one of the uncertain linear constraints given below:

$$\begin{array}{ll} a^T x \le b, [a; b] \in \mathcal{U} = \{[a; b] = [a^{\mathrm{n}}; b^{\mathrm{n}}] + P\zeta : \|\zeta\|_p \le \rho\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad [p \in [1, \infty]] & (a) \\ a^T x \le b, [a; b] \in \mathcal{U} = \{[a; b] = [a^{\mathrm{n}}; b^{\mathrm{n}}] + P\zeta : \|\zeta\|_p \le \rho, \zeta \ge 0\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad [p \in [1, \infty]] & (b) \\ a^T x \le b, [a; b] \in \mathcal{U} = \{[a; b] = [a^{\mathrm{n}}; b^{\mathrm{n}}] + P\zeta : \|\zeta\|_p \le \rho\} & \\ \qquad\qquad\qquad\qquad\qquad\qquad [p \in (0, 1)] & (c) \end{array}$$

as explicit convex constraints.

Exercise 1.3. Represent in tractable form the RC of uncertain linear constraint

$$a^T x \le b$$

with $\cap$-ellipsoidal uncertainty set

$$\mathcal{U} = \{[a, b] = [a^{\mathrm{n}}; b^{\mathrm{n}}] + P\zeta : \zeta^T Q_j \zeta \le \rho^2, 1 \le j \le J\},$$

where $Q_j \succeq 0$ and $\sum_j Q_j \succ 0$.

1.6 NOTES AND REMARKS

NR 1.1. The paradigm of Robust Linear Optimization in the form considered here goes back to A.L. Soyster [109], 1973. To the best of our knowledge, in two subsequent decades there were only two publications on the subject [52, 106]. The activity in the area was revived circa 1997, independently and essentially simultaneously, in the frameworks of both Integer Programming (Kouvelis and Yu [70]) and Convex Programming (Ben-Tal and Nemirovski [3, 4], El Ghaoui et al. [49, 50]). Since 2000, the RO area is witnessing a burst of research activity in both theory and applications, with numerous researchers involved worldwide. The magnitude and diversity of the related contributions make it beyond our abilities to discuss them here. The reader can get some impression of this activity from [9, 16, 110, 89] and references therein.

NR 1.2. By itself, the RO methodology can be applied to every optimization problem where one can separate numerical data (that can be partly uncertain) from a problem's structure (that is known in advance and common for all instances of the uncertain problem). In particular, the methodology is fully applicable to uncertain *mixed integer* LO problems, where part of the decision variables are restricted to be integer. Note, however, that tractability issues, (which are our main focus in this book), in Uncertain LO with real variables and Uncertain Mixed-Integer LO need quite different treatment. While Theorem 1.3.4 is fully applicable to the mixed integer case and implies, in particular, that the RC of an uncertain mixed-integer LO problem $\mathcal{P}$ with a polyhedral uncertainty set is an explicit mixed-integer LO program with exactly the same integer variables as those of the instances of $\mathcal{P}$, the "tractability consequences" of this fact are completely different from those we made in the main body of this chapter. With no integer variables, the fact that the RC is an LO program straightforwardly implies tractability of the RC, while in the presence of integer variables no such conclusion can be made. Indeed, in the mixed integer case already the instances of the uncertain problem $\mathcal{P}$ typically are intractable, which, of course, implies intractability of the RC. In the case when the instances of $\mathcal{P}$ are tractable, the "fine structure" of the instances responsible for this rare phenomenon usually is destroyed when passing to the mixed-integer reformulation of the RC. There are some remarkable exceptions to this rule (see, e.g., [25]); however, in general the Uncertain Mixed-Integer LO is incomparably more complex computationally than the Uncertain LO with real variables. As it was already stated, our book is primarily focused on tractability issues of RO, and in order to get positive results in this direction, we restrict ourselves to uncertain problems with well-structured convex (and thus tractable) instances.

NR 1.3. Tractability of the RC of an uncertain LO problem with a tractable uncertainty set was established in the very first papers on convex RO. Theorem 1.3.4 and Corollary 1.3.5 are taken from [5].