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A Control Theorist's Perspective on Dynamic Competitive Equilibria in Electricity Markets

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Abstract: We are moving towards a radical transformation of our energy systems. The success of the new paradigm created by the Smart Grid vision will require not only the creation and integration of new technologies into the grid, but also the redesign of the market structures coupled with it. In order to design the market structures for the grid of the future, economic models able to capture the new physical reality are the first requirement. In this paper, we present a general economic equilibrium model. The model is constructed using well-known control theory techniques, allowing a natural inclusion of dynamics, uncertainty in supply and demand, and other elements usually not considered in standard economic models.

Keywords: Economic equilibrium theory, electricity pricing, power transmission network.

1. INTRODUCTION

Power systems have always been a rich source of problems for control theorists. One of the most famous examples is the control of "singular perturbed dynamical systems", which was motivated in part by problems in power systems in the 1970s.

Today there is an urgent need for models and control techniques to address a massively complex and hitherto unseen dynamical system consisting of new energy sources, new information technologies and overlaid market mechanisms that are not completely understood (Massoud Amin and Wollenberg, 2005). We view control and system theory as one of the disciplines that will lead research to provide the insights needed to comprehend and control this complex network, and finally justifying the term "Smart Grid" (Santacana et al., 2010).

The focus of this paper is the market side of the Smart Grid. Our goal is to contribute to the understanding of the impact of dynamics, constraints, and uncertainty, that will become more acute with the increased deployment of renewable resources. Such sentiment is aptly conveyed by Smith et. al. in the recent article (Smith et al., 2010), where the authors write that, "*little consideration was given to market design and operation under conditions of high penetrations of remote, variable renewable generation, such as wind ... and solar energy, which had not yet appeared on the scene in any significant amounts.*" Our approach is the development of models able to characterize the competitive equilibria for a power network model that captures these complexities. This may be regarded as a stepping stone towards the creation of reliable markets for a smart grid.

In the spirit of the cold war Milton Friedman wrote, *"Fundamentally, there are only two ways of coordinating the economic activities of millions. One is central direction involving the use of coercion – the technique of the army and of the modern totalitarian state. The other is voluntary cooperation of individuals – the technique of the marketplace"* (Friedman, 1962). We hope that most economists today would see this as an extreme point of view, but we find that polarization inhibits discussion on market design even today. Of course there are more than two ways! What is missing is a firm science for making design choices in a dynamic market, taking into account the special features of that particular market.

What are the special features of an electricity market? The electricity market is a coupling of two constrained and highly complex dynamical systems; one physical and one economic. The physical system is a complex network consisting of power flowing through transmission lines, modulated by distributed generation units, Kirchhoff's laws, and operational and security constraints. Loads and generation are each subject to uncertainty. The economic system typically consists of coupled markets such as day-ahead and real-time auctions. These auctions are dynamic — prices vary by two orders of magnitude in many real-time markets today — and are subject to uncertainty because of the "rational agents" driving the economic side. The presence of such complicating factors in these coupled systems, along with the sometimes orthogonal nature of the physical objectives with respect to the economic goals of the market players, make electricity market design a challenging task.

Control Theorist's Perspective? In a typical control design, an engineer starts with a simple model of the physical system, then chooses a control approach (e.g., PID, H_∞, LQG, MPC, adaptive, or MDP), and creates a feedback control solution. The design is usually verified via simulation and/or experiments. Even then, it is likely

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that the design must be refined to accommodate particular features of the system. If all efforts fail, then the engineer might consider a redesign of the physical system. Typical market design halt at step one: Analysis is guided by idealized models of the behavior of players, and idealized models of the underlying physical reality. However, the underlying physical reality of electricity generation can impact deeply the outcome of a given market structure. This is why market behavior is frequently so different than what is predicted by economic models.

In this paper, we present a general economic equilibrium model that refines standard economic models (Aliprantis et al., 2002) by including dynamics, uncertainty in supply and demand, and operational constraints associated with generation and transmission. Using a Lagrangian decomposition that is standard in static economic analysis and certain dynamic economic analyses (Mas-Colell et al., 1995; Chow, 1997), we extend recent results in (Cho and Meyn, 2010; Meyn et al., 2010; Kizilkale and Mannor, 2010) and we provide conditions for the existence and optimality of a competitive equilibrium in this general case.

While this paper does not provide a direct solution to the market design problem, it provides a framework for constructing dynamic models for electricity markets, and methods for characterizing the resulting competitive equilibria. The dynamic model is constructed using techniques well known in the control community and effectively can handle the underlying physics of the power system while taking into account the economic aspects of electricity trading. In a forthcoming work (Wang et al., 2011), we explicitly tackle this problem and we quantify the impact of such physical constraints on the equilibrium prices.

The remainder of this paper contains three additional sections and is organized as follows. In Sec. 2 we present the economic and physical models of the electricity market. We devote Sec. 3 to characterize the competitive equilibrium in dynamic markets using a control-oriented scaffolding. The main results are the conditions for the existence of the competitive equilibrium in terms of duality concepts from nonlinear optimization theory. We provide concluding remarks and final thoughts in Sec. 4.

2. ELECTRICITY MARKET MODEL

Restructuring has provided increased opportunities for competition in the electricity industry. In the market environment, the market participants make decisions based on their own interests. These self-interested entities compete in the markets for the rights to serve the load or be served. The electricity prices and the quantities sold are determined by market rules, and the behavior of the participants.

Although the main reason for adopting energy markets has been to reduce electricity bills, reliability of service continues to remain an overriding concern of the system operators. As mentioned in the introduction, the reliability-driven operations – which are impacted by the physical constraints on the generation and the transmission – may conflict with the economic objectives of the market participants. Such conflicts are only intensified in a large power system consisting of multiple generators subject to minimum up/down-time constraints, ramping constraints and capacity constraints, connected to consumers via a complex, capacity-constrained transmission network. Since the aforementioned operational constraints on generation and transmission impact market decisions, we consider them explicitly in our analysis.

In what follows, we present a market model for energy in the perfect-competition setting of equilibrium economics. A critical assumption of this theory is that all players are "price takers". That is, no player can influence prices unilaterally. In every sense, this model is an appropriate representation of a perfect "free-market" as analyzed in typical economics text. Not surprisingly, we find that market outcomes reflect the standard economic conclusions for efficient markets in which prices equal marginal costs, but only *on average*. Because of dynamic constraints, the sample path property of prices in the equilibrium will show perverse volatility patterns that have negative impact on consumers, suppliers or both.

2.1 The players and the rules

For simplicity we restrict discussion to a market consisting of a single "consumer" and a single "supplier" that represent price taking consumers and suppliers distributed across the grid. It is assumed that N buses correspond to the nodes in a network, indexed by $1, 2, ..., N$. The L links in the network represent transmission lines, indexed by $\{1, 2, ..., L\}$. The network is assumed to be connected. We adopt a lossless DC model to represent the network (see (Wood and Wollenberg, 1996; Chen et al., 2006)).

For time $t \geq 0$, at each bus $n \in \{1, 2, ..., N\}$, there is an associated price $P_n(t)$ for energy traded at that bus. The *price-taking assumption* means that the price process $P_n(t)$ at a particular bus cannot be influenced by the actions of the consumers or suppliers.

Consumer We denote by $D_n(t)$ the demand at time t at bus n, and by $E_{\mathrm{D}n}(t)$ the energy withdrawn by the consumer at that bus. We assume that there is no free disposal for energy, which requires that $E_{\mathrm{D}n}(t) \leq D_n(t)$ for all t. If sufficient generation is available at bus n at time t, then $E_{\mathrm{D}n}(t) = D_n(t)$. In the event of insufficient generation, we have $E_{\mathrm{D}n}(t) < D_n(t)$, i.e., the consumer experiences a blackout.

The consumer obtains value on consuming energy and disutility for not meeting demand during a blackout. These are represented by possibly nonlinear functions,

$$\text{Utility of consumption:} \qquad v_n(E_{\mathrm{D}n}(t)) \tag{1a}$$

$$\text{Disutility of blackout:} \quad c_n^{\mathrm{bo}}(D_n(t) - E_{\mathrm{D}n}(t)) \tag{1b}$$

The consumer must pay for energy at price $P_n(t)$. We use $D(t)$, $E_{\mathrm{D}}(t)$, and $P(t)$ to denote the associated N-dimensional column vectors, and we use bold face font to denote the entire sample path. For instance, $\boldsymbol{P} := \{P(t) : t \geq 0\}$.

The welfare of the consumer at time t is the signed sum of his benefits and costs:

$$\mathcal{W}_{\rm D}(t) := \sum_n \bigl[v_n(E_{{\rm D}n}(t)) \; - \; c_n^{\rm bo}(D_n(t) - E_{{\rm D}n}(t)) \\ - \; P_n(t) E_{{\rm D}n}(t) \bigr] \, . \tag{2}$$

Observe that prices are determined by location. In the language of today's markets they are *locational prices.*

Supplier We denote by $E_{{\rm S}n}(t)$ and $R_{{\rm S}n}(t)$ the energy and reserve produced by the supplier at bus n. The operational and physical constraints on the production of energy and reserve are expressed abstractly as

$$(\boldsymbol{E}_{\rm S}, \boldsymbol{R}_{\rm S}) \in \mathsf{X}_{\rm S} \, . \tag{3}$$

These constraints include ramping constraints on generation imposed by the physics of both generators and the grid. However, at this level of generality the specific details of the constraints are unimportant.

The production cost at time t for energy injected at bus n is denoted by $c_n^{\rm E}(E_{{\rm S}n}(t))$, and for the reserve provided at that bus is $c_n^{\rm R}(R_{{\rm S}n}(t))$. The supplier receives the revenue $P_n E_{{\rm S}n}(t)$ for producing energy. The welfare of the supplier at time t is the difference between his revenue and the costs,

$$\mathcal{W}_{\rm S}(t) := \sum_n \bigl[P_n E_{{\rm S}n}(t) - c_n^{\rm E}(E_{{\rm S}n}(t)) - c_n^{\rm R}(R_{{\rm S}n}(t)) \bigr] \, . \tag{4}$$

Network To capture the impact of network constraints and exploit network structure we introduce a third player – the network. This is motivated in part by current practice: The transmission grid is operated by a third entity (neither the consumers nor the suppliers) in every electricity market operating in the world today.

The first constraint faced by the network is based on the assumption that it is lossless, so it neither generates nor consumes energy. Consequently, the network is subject to the supply-demand balance constraint,

$$1^T E_{\rm S}(t) = 1^T E_{\rm D}(t) \, , \quad t \ge 0 \, . \tag{5}$$

The next set of constraints are due to the limitations of transmission. Suppose bus 1 is selected as the reference bus, based on which the *injection shift factor matrix* $H \in [-1,1]^{N \times L}$ is defined, where H_{nl} denote the power distributed on line l when 1 MW is injected into bus n and withdrawn at the reference bus (Wood and Wollenberg, 1996; Chen et al., 2006). Let $f_l^{\max}$ denote the capacity of transmission line l. On letting $H_l \in \mathbb{R}^N$ denote the l-th column of H, the capacity constraint for line l is expressed as,

$$-f_l^{\max} \le (E_{\rm S} - E_{\rm D})^T H_l \le f_l^{\max} \, . \tag{6}$$

In the market analysis that follows we find it convenient to introduce a "network welfare function" to define a competitive equilibrium for the power grid market model. The welfare of the network at time t represents the "toll charges" for the transmission of energy. At time t, this is defined by,

$$\mathcal{W}_{\rm T}(t) := \sum_n \bigl[P_n(E_{{\rm D}n}(t) - E_{{\rm S}n}(t)) \bigr] \, . \tag{7}$$

The introduction of $\mathcal{W}_{\rm T}$ is purely for the sake of analysis. In Sec. 3 we assume that the "network" wishes to maximize its welfare, subject to the constraints (5) and (6), which are collectively summarized by the notation,

$$(\boldsymbol{E}_{\rm S}, \boldsymbol{E}_{\rm D}) \in \mathsf{X}_{\rm T} \, . \tag{8}$$

2.2 Information and Uncertainty

In addition to the physical constraints captured by $\mathsf{X}_{\rm S}$ and $\mathsf{X}_{\rm T}$, in every market there are informational constraints. In this paper, we adopt the highly idealized assumption that both sides of the market share a common information set. To model this, and also the impact of uncertainty and volatility, we opt for a stochastic model. Hence the processes described in the previous pages are all stochastic, and assumed to be adapted to a filtration $\{\mathcal{H}_t : t \ge 0\}$.

The consumer and supplier's objective function is the long-run discounted expected profit with discount rate γ, represented by

$$K_{\rm D} := \mathsf{E}\Bigl[\int e^{-\gamma t} \mathcal{W}_{\rm D}(t) \, dt \Bigr] \, ,$$

The long-run discounted welfare of the supplier $K_{\rm S}$, and the network $K_{\rm T}$, are defined similarly. The consumer, supplier, and network each aim to optimize their respective mean discounted mean welfare $K_{\rm D}$, $K_{\rm S}$, and $K_{\rm T}$. These quantities will in general depend on the initial condition of the system. We suppress this dependency whenever possible.

To emphasize the similarity between static and dynamic equilibrium theory we adopt the following Hilbert-space notation: For two stochastic processes $\boldsymbol{F}$ and $\boldsymbol{G}$, each adapted to $\mathcal{H}_t$, we denote

$$\langle \boldsymbol{F}, \boldsymbol{G} \rangle := \mathsf{E}\Bigl[\int e^{-\gamma t} F(t) G(t) \, dt \Bigr]. \tag{9}$$

For example, using this notation we have $K_{\rm S} = \langle \boldsymbol{\mathcal{W}}_{\rm S}, \mathbf{1} \rangle$, where $\mathbf{1}$ denotes the process that is identically unity.

3. EQUILIBRIA AND EFFICIENCY

The competitive equilibrium of economics is used as a vehicle to study the outcomes of a market under a set of idealized assumptions (Aliprantis et al., 2002). It is a widely accepted benchmark for evaluating real markets outcomes. If the behavior of a market nearly matches the behavior predicted by the competitive equilibrium, then the market is deemed to be functioning well. The existence and optimality of a competitive equilibrium have been studied extensively, though typically under ideal assumptions that include continuity and convexity of cost and utility functions, and models typically disregard dynamics.

In the electricity industry, physical and operational limits of the facilities and network impose stringent constraints on the behavior of the market participants. This can present challenges when applying the usual market analysis tools to evaluate the market. A complete understanding of the impact of these physical characteristics on market outcomes remains an open question. We believe that a better science for dynamic, networked markets is a key requirement for the electricity market designs of the future.

The usual definition of a competitive equilibrium with two players is based on the respective optimization problems of the supplier and consumer,

$$\boldsymbol{E}_{\mathrm{D}} \in \arg\max_{\boldsymbol{E}_{\mathrm{D}}} \langle \boldsymbol{\mathcal{W}}_{\mathrm{D}}, \mathbf{1} \rangle \,, \tag{10}$$

$$(\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}) \in \arg\max_{\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}} \langle \boldsymbol{\mathcal{W}}_{\mathrm{S}}, \mathbf{1} \rangle, \tag{11}$$

where the welfare functions are given in (2) and (4). We adopt the same conventions in our dynamic analysis. However, in the equilibrium definition that follows we introduce the third player – the network – to account for network constraints (and also rule out "arbitrage" – an issue to be illustrated with examples in a lengthier version of this paper under preparation).

Definition 1. A *competitive equilibrium* is a quadruple of process vectors: consumed energy, supplied energy, supplied reserve, and energy price, denoted as $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$, which satisfies the following conditions:

(i) $(\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}})$ solves (11), subject to the operational/physical constraint (3).
(ii) $\boldsymbol{E}_{\mathrm{D}}$ solves (10).
(iii) The pair $(\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}})$ optimizes the welfare function of the network,
$$(\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}) \in \arg\max_{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}} \langle \boldsymbol{\mathcal{W}}_{\mathrm{T}}, \mathbf{1} \rangle, \tag{12}$$
subject to the supply-demand balance constraints, and transmission constraints (8).

The supplier, consumer and network are also subject to the measurability constraints outlined in Sec. 2.2 in their respective optimization problems. □

Note that the set of feasible strategies for the consumer are *not* subject to the constraints (3) or (8). The rationale is at the heart of competitive equilibrium theory: It is believed that in the long-run, the system will evolve to satisfy the preferences of consumers.

To evaluate the welfare performance of the market, we introduce a *social planner* who aims to maximize the economic well-being of everyone in the system. We stress that there is no actual planner — this is another analytical device.

The social planner optimizes based on the total welfare, denoted

$$\mathcal{W}_{\mathrm{tot}}(t) := \mathcal{W}_{\mathrm{S}}(t) + \mathcal{W}_{\mathrm{D}}(t) + \mathcal{W}_{\mathrm{T}}(t) \,. \tag{13}$$

The total welfare is independent of prices, as $\mathcal{W}_{\mathrm{tot}}(t)$ is equal to the sum of $\{v_n(E_{\mathrm{D}n}(t)) - c_n^{\mathrm{bo}}(D_n(t) - E_{\mathrm{D}n}(t)) - c_n^{\mathrm{E}}(E_{\mathrm{S}n}(t)) - c_n^{\mathrm{R}}(R_{\mathrm{S}n}(t))\}$ over all nodes n.

Definition 2. The social planner's problem (SPP) is

$$\max_{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}} \langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1} \rangle, \tag{14}$$

subject to all constraints, (3) and (8).

Its solution is called an *efficient allocation.* □

We assume throughout the paper that the SPP (14) has a solution, denoted by $(\boldsymbol{E}_{\mathrm{D}}^*, \boldsymbol{E}_{\mathrm{S}}^*, \boldsymbol{R}_{\mathrm{S}}^*)$.

Let $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$ be a competitive equilibrium. If $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ is an efficient allocation, then we say that the equilibrium is efficient. If every competitive equilibrium is efficient, then we say that the *first welfare theorem* holds. On the other hand, let $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ be an efficient allocation. If we can construct a price process $\boldsymbol{P}$ such that $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$ becomes a competitive equilibrium, we say the allocation is supported by the price $\boldsymbol{P}$. If every efficient allocation can be supported, then we say that the *second welfare theorem* holds.

The first and second fundamental theorems of welfare economics are each implied by Theorem 8 that follows:

Theorem 3. (First Fundamental Theorem). Any competitive equilibrium, if it exists, is efficient. □

The proof is given at the end of this section. The second welfare follows similarly:

Theorem 4. (Second Fundamental Theorem). If the market admits a competitive equilibrium, then for any efficient allocation $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$, there exists a supporting price process $\boldsymbol{P}$ such that $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$ constitutes a competitive equilibrium. □

Before proceeding to the analysis, we first explain how this model is related to the single bus / single consumer model of (Cho and Meyn, 2010). In this prior work, the decision variables were taken to be generation capacity $\boldsymbol{G}$ and reserve, where $\boldsymbol{G}$ coincides with $\boldsymbol{E}+\boldsymbol{R}$ in the notation of this paper. The generation was assumed normalized (the deviation from the day ahead market), so that negative values for $G(t)$ were allowed. The set $\mathbf{X}_{\mathrm{S}}^{\circ}$ in (Cho and Meyn, 2010) would be defined by ramp constraints on $\boldsymbol{G}$: For all $t_1 > t_0 \geq 0$,

$$\zeta^- \leq \frac{E_{\mathrm{S}}(t_1) - E_{\mathrm{S}}(t_0)}{t_1 - t_0} + \frac{R_{\mathrm{S}}(t_1) - R_{\mathrm{S}}(t_0)}{t_1 - t_0} \leq \zeta^+ \,. \tag{15}$$

In this prior work the utility of consumption was assumed to be of the form $v \min(G, D)$ for a constant $v > 0$. The disutility of blackout was taken to be piecewise linear: zero for $R > 0$, and proportional to $D - G$ otherwise. The analogous functions for the model introduced here are the linear and piecewise linear functions,

$$\begin{aligned} &\text{Utility of consumption: } vE \,, \\ &\text{Disutility of blackout: } c^{\mathrm{bo}} \max(D - E, 0) \,, \end{aligned} \tag{16}$$

where v and c^{bo} are constants. Based on these modeling assumptions, and specific statistical assumptions on demand, the first and second welfare theorems were established in this prior work.

The market analysis that follows is based on Lagrangian relaxations.

Definition 5. The Lagrangian of the SPP is

$$\begin{aligned} \mathcal{L} = &- \langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1} \rangle + \langle \boldsymbol{\lambda}, (1 \cdot \boldsymbol{E}_{\mathrm{D}} - 1 \cdot \boldsymbol{E}_{\mathrm{S}}) \rangle \\ &+ \sum_l \langle \boldsymbol{\mu}_l^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle \\ &+ \sum_l \langle \boldsymbol{\mu}_l^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle \,, \end{aligned}$$

where $\mu_l^+(t) \geq 0$ and $\mu_l^-(t) \geq 0$ for all t and l. □

A key step is to define the candidate price process $\boldsymbol{P}$ as

$$P_n(t) := \lambda(t) + \sum_l (\mu_l^-(t) - \mu_l^+(t)) H_{ln}, \quad t \geq 0, \ n \geq 1. \tag{17}$$

From the definitions, we conclude that the Lagrangian can be expressed as,

$$
\begin{aligned}
\mathcal{L} = &- \sum_n \{ \langle v_n(\boldsymbol{E}_{\mathrm{D}n}) - c_n^{\mathrm{bo}}(\boldsymbol{D}_n - \boldsymbol{E}_{\mathrm{D}n}), \mathbf{1} \rangle - \langle \boldsymbol{P}_n, \boldsymbol{E}_{\mathrm{D}n} \rangle \} \\
&- \sum_n \{ \langle \boldsymbol{P}_n, \boldsymbol{E}_{\mathrm{S}n} \rangle - \langle c_n^{\mathrm{E}}(\boldsymbol{E}_{\mathrm{S}n}) + c_n^{\mathrm{R}}(\boldsymbol{R}_{\mathrm{S}n}), \mathbf{1} \rangle \} \qquad (18) \\
&- \sum_l \langle \boldsymbol{\mu}_l^+ + \boldsymbol{\mu}_l^-, f_l^{\max} \rangle \,.
\end{aligned}
$$

Therefore, $\mathcal{L}$ is a constant minus the sum of supplier and consumer welfare functions, with $\mathcal{W}_{\mathrm{D}}$ and $\mathcal{W}_{\mathrm{D}}$ defined using this price $\boldsymbol{P}$.

Definition 6. The dual functional for the SPP is

$$
h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) = \min_{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}} \mathcal{L} \,. \qquad (19)
$$

The following weak duality bound follows since the minimization in (19) amounts to a relaxation of the SPP (14):

Lemma 7. (Weak Duality). For any allocation $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ and Lagrangian multiplier $(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$ with $\boldsymbol{\mu}^+, \boldsymbol{\mu}^- \geq 0$, we have

$$
-\langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1} \rangle \geq h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) \,. \qquad (20)
$$

An equality in (20) implies that strong duality holds. The main result of this section characterizes the existence of a competitive equilibrium in terms of strong duality:

Theorem 8. (Existence of Competitive Equilibrium). The market admits a competitive equilibrium if and only if the SPP satisfies strong duality.

Proof. We first prove the *sufficient condition*: strong duality implies existence of competitive equilibrium. Since strong duality holds, we have

$$
-\langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1} \rangle = h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) \,. \qquad (21)
$$

Suppose that the allocation $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ is feasible for the SPP. We then construct a competitive equilibrium with price as given in (17).

The feasibility of the triple $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ for SPP implies $1^T E_{\mathrm{S}}(t) = 1^T E_{\mathrm{D}}(t)$ for all t, and hence

$$
\begin{aligned}
\mathcal{L} = -\langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1} \rangle &+ \sum_l \langle \boldsymbol{\mu}_l^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle \\
&+ \sum_l \langle \boldsymbol{\mu}_l^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle \,.
\end{aligned}
$$

Feasibility also implies $-f_l^{\max} \leq (E_{\mathrm{S}}(t) - E_{\mathrm{D}}(t))^T H_l \leq f_l^{\max}$, and given the non-negativity of $\boldsymbol{\mu}^+, \boldsymbol{\mu}^-$, we have

$$
\begin{aligned}
&\langle \boldsymbol{\mu}^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle \leq 0 \,, \\
\text{and } &\langle \boldsymbol{\mu}^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle \leq 0 \,.
\end{aligned}
$$

This together with (21) gives $\mathcal{L} \leq h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$ But, by the definition in (19), we have $\mathcal{L} \geq h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$, so that we obtain the identity,

$$
h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) = \mathcal{L} \,. \qquad (22)
$$

This identity implies that $\boldsymbol{E}_{\mathrm{D}}$ maximizes the consumer's welfare, $\{\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ maximizes the supplier's welfare, and

$$
\begin{aligned}
&\langle \boldsymbol{\mu}^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle = 0 \,, \\
\text{and } &\langle \boldsymbol{\mu}^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle = 0 \,.
\end{aligned}
$$

Using the prices $\{\boldsymbol{P}_n = \boldsymbol{\lambda} + \sum_l (\boldsymbol{\mu}_l^- - \boldsymbol{\mu}_l^+) H_{ln}\}$ defined in (17), we substitute the above two equations into the network welfare expression to obtain,

$$
\begin{aligned}
&\Big\langle \sum_n [\boldsymbol{P}_n(\boldsymbol{E}_{\mathrm{D}n} - \boldsymbol{E}_{\mathrm{S}n})], \mathbf{1} \Big\rangle \\
&= \Big\langle \sum_n \Big[\sum_l (\boldsymbol{\mu}_l^- - \boldsymbol{\mu}_l^+) H_{ln} (\boldsymbol{E}_{\mathrm{D}n} - \boldsymbol{E}_{\mathrm{S}n}) \Big], \mathbf{1} \Big\rangle \\
&= \Big\langle \sum_l (\boldsymbol{\mu}_l^+ + \boldsymbol{\mu}_l^-) \cdot f_l^{\max}, \mathbf{1} \Big\rangle \,.
\end{aligned}
$$

This is independent of $\{\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$, which implies that the welfare of the network is maximized under the prices $\{\boldsymbol{P}_n\}$. Thus, we conclude that $\boldsymbol{P}$ as defined in (17) is the equilibrium price as claimed.

Next we establish *necessity*: existence of a competitive equilibrium implies strong duality. Suppose that $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$ is a competitive equilibrium. Then we know that $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}\}$ maximizes the network welfare when the price is $\boldsymbol{P}$. The Lagrangian associated with the maximization of network welfare is expressed as follows: For any $\boldsymbol{\mu}^+, \boldsymbol{\mu}^- \geq 0$,

$$
\begin{aligned}
\mathcal{L}_{\mathrm{T}} = &- \sum_n \langle \boldsymbol{P}_n, (\boldsymbol{E}_{\mathrm{D}n} - \boldsymbol{E}_{\mathrm{S}n}) \rangle + \langle \boldsymbol{\lambda}, (1^T \boldsymbol{E}_{\mathrm{D}} - 1^T \boldsymbol{E}_{\mathrm{S}}) \rangle \\
&+ \sum_l \langle \boldsymbol{\mu}_l^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle \qquad (23) \\
&+ \sum_l \langle \boldsymbol{\mu}_l^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle \,.
\end{aligned}
$$

The maximization of network welfare is a linear program, and hence the optimum satisfies the KKT conditions. As a consequence, associated with the constraints $\dfrac{\partial \mathcal{L}_{\mathrm{T}}}{\partial E_{\mathrm{D}n}} = \dfrac{\partial \mathcal{L}_{\mathrm{T}}}{\partial E_{\mathrm{S}n}} = 0$, there exist $\{\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-\}$ such that (17) holds:

$$
\boldsymbol{P}_n = \boldsymbol{\lambda} + \sum_l (\boldsymbol{\mu}_l^- - \boldsymbol{\mu}_l^+) H_{ln} \,.
$$

Moreover, by complementary-slackness, we have

$$
\begin{aligned}
\langle \boldsymbol{\mu}^+, (\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle = 0 \,; \\
\langle \boldsymbol{\mu}^-, -(\boldsymbol{E}_{\mathrm{S}} - \boldsymbol{E}_{\mathrm{D}})^T H_l - f_l^{\max} \rangle = 0 \,.
\end{aligned} \qquad (24)
$$

Next, we investigate $h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$ for the SPP, using the multipliers from the maximization of network welfare. Since $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}, \boldsymbol{P}\}$ is a competitive equilibrium, $\boldsymbol{E}_{\mathrm{D}}$ maximizes the consumer's welfare, and $\{\boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ maximizes the supplier's welfare. Based on the form (17) for $\boldsymbol{P}$ we conclude that

$$
\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\} \in \operatorname*{arg\,min}_{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}} \mathcal{L} \,.
$$

Substituting $\{\boldsymbol{E}_{\mathrm{D}}, \boldsymbol{E}_{\mathrm{S}}, \boldsymbol{R}_{\mathrm{S}}\}$ into the definition of $\mathcal{L}$, and applying the complementary slackness equation (24),

$$
-\langle \mathcal{W}_{\mathrm{tot}}, \mathbf{1} \rangle = h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) \,.
$$

That is, strong duality holds. □

We stress that the theorem characterizes prices in any competitive equilibrium:

Corollary 9. The only candidates for prices in a competitive equilibrium are given by (17), based on the optimal Lagrangian multipliers.

Proof. This is because only the optimal multipliers could possibly lead to strong duality. □

Theorem 8 tells us that computation of prices and quantities can be decoupled: The quantities $\{\boldsymbol{E}_{\rm D}, \boldsymbol{E}_{\rm S}, \boldsymbol{R}_{\rm S}\}$ are obtained through the solution of the SPP, and the price process $\boldsymbol{P}^e$ is obtained as a solution to its dual. The following corollary underlines this point. If $\boldsymbol{P}^e$ supports one competitive equilibrium, then it supports any other competitive equilibrium.

Corollary 10. If $\{\boldsymbol{E}^1_{\rm D}, \boldsymbol{E}^1_{\rm S}, \boldsymbol{R}^1_{\rm S}, \boldsymbol{P}^1\}$ and $\{\boldsymbol{E}^2_{\rm D}, \boldsymbol{E}^2_{\rm S}, \boldsymbol{R}^2_{\rm S}, \boldsymbol{P}^2\}$ are two competitive equilibria, then $\{\boldsymbol{E}^2_{\rm D}, \boldsymbol{E}^2_{\rm S}, \boldsymbol{R}^2_{\rm S}, \boldsymbol{P}^1\}$ is also a competitive equilibrium.

Proof. Due to the necessary condition of Theorem 8 there exists Lagrange multipliers $(\boldsymbol{\lambda}_1, \boldsymbol{\mu}^+_1, \boldsymbol{\mu}^-_1)$ and $(\boldsymbol{\lambda}_2, \boldsymbol{\mu}^+_2, \boldsymbol{\mu}^-_2)$ corresponding to the two equilibria, such that

$$h(\boldsymbol{\lambda}_1, \boldsymbol{\mu}^+_1, \boldsymbol{\mu}^-_1) = -\langle \boldsymbol{\mathcal{W}}_{\rm tot1}, \mathbf{1}\rangle = -\langle \boldsymbol{\mathcal{W}}_{\rm tot2}, \mathbf{1}\rangle \,.$$

By the sufficient condition of Theorem 8 any of these price and quantity pair satisfying strong duality will constitute a competitive equilibrium. □

Proof of Theorem 3. By the proof of necessary condition, for any competitive equilibrium $\{\boldsymbol{E}_{\rm D}, \boldsymbol{E}_{\rm S}, \boldsymbol{R}_{\rm S}, \boldsymbol{P}\}$, there exists some $(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$, such that

$$h(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) = -\langle \mathcal{W}_{\rm tot}, \mathbf{1}\rangle \,.$$

By weak duality (20), $\{\boldsymbol{E}_{\rm D}, \boldsymbol{E}_{\rm S}, \boldsymbol{R}_{\rm S}\}$ is an efficient allocation. □

4. CONCLUSIONS

We introduced in this paper a dynamic competitive equilibria model for power systems based on a theoretical scaffolding familiar to the decision and control systems community. The model is flexible enough to include the impact of a wide range of constraints. In (Wang et al., 2011) we show how the results here can be used to explain extreme price volatility seen today in electricity markets around the world.

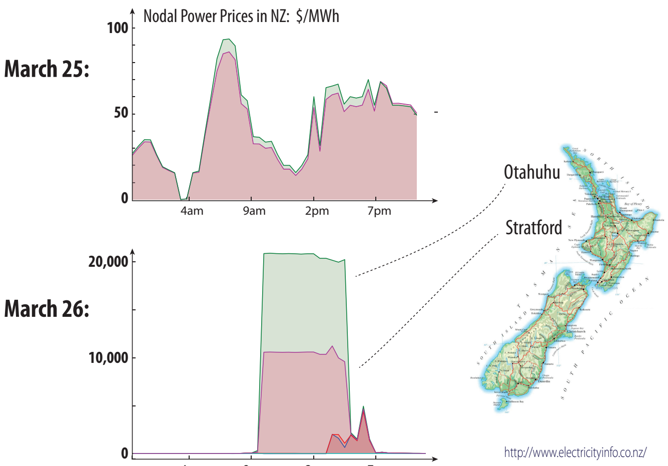

Fig. 1. Prices over 100 times the average meant that generators made 25 million dollars in six hours.

However, we must stress that the results of this paper do not take us out of the narrow focus of *equilibrium* analysis. Consider the recent data shown in Fig. 1, which is causing panic today in New Zealand. *Is this a competitive equilibrium*? The results of this paper do not rule out these dramatic price dynamics, so we cannot answer.

Moreover, we do not know if the equilibrium introduced in this paper is stable in any meaningful sense, and we have not addressed the impact of strategic behavior. To address these critical issues, we believe it is necessary to put more structure on the problem. These will take the form of constraints on the behaviors of the participants in the market so the overall system will be more predictable — the same way that we obtain reliability in our transportation systems and communication systems. We will then have a system that is analyzable, so that market architectures can be constructed, and the impact of strategic behavior can be evaluated.

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