Question 1

Say we roll a fair 6-sided die six times. Using the fact that each roll is an independent random event, what is the probability that we roll:

Question 1(a)

The sequence 1,1,2,2,3,3 ?

Solution -

$$2.14 \times 10^{-5}$$

Justification -

– The probability of rolling any specific side of a fair 6-faced die is $\frac{1}{6}$ on one roll.

– To obtain the specified sequence, the above event would happen 6 times, each of which are independent events, hence resulting in a probability of $(\frac{1}{6})^6 \approx 2.14 \times 10^{-5}$.

Question 1(b)

A three exactly 4 times?

Solution -

$$0.0080$$

Justification -

– With a fair 6-faced die, the probability of obtaining a 4 on one roll of the die is $\frac{1}{6}$ and the probability of getting a value other than 4 is $\frac{5}{6}$.

– The die is rolled 6 times, each of which are independent events. Out of these 6 times, exactly 4 of them would result in 4, hence for the other 2 rolls of the die, the results should be any value other than 4.

– Therefore, the probability would be $\binom{6}{4} \cdot (\frac{1}{6})^4 \cdot (\frac{5}{6})^2 \approx 0.0080$.

Question 1(c)

A single 1.

Solution -

$$0.402$$

Justification -

- Let E denote the event that the die only rolls a single 1 on 6 rolls.
- The probability of obtaining a 1 on a roll of a fair 6-faced die is $\frac{1}{6}$, while the probability of getting any value other than 1 is $\frac{5}{6}$.
- In event E, when the die is rolled 6 times, each of them being independent events, only one of these would result in a 1, and 5 of them would be some other value.
- Hence the probability here would be $P(E) = \binom{6}{1} \cdot (\frac{1}{6}) \cdot (\frac{5}{6})^5 \approx 0.402$.

Question 1(d)

One or more 1's.

Solution -

$$0.665$$

Justification -

- Let E denote the event that one or more 1's are obtained on a fair die, hence E^C would denote the event that there are no 1's obtained.
- In this scenario, the probability of rolling a 1 would be $\frac{1}{6}$, whilst the probability of rolling any value other than 1 is $\frac{5}{6}$ at each roll of the die.
- Rolling the die 6 times, each of them being independent events, would result in a probability of $P(E) = 1 - P(E^C) = 1 - \binom{6}{0} \cdot (\frac{1}{6})^0 \cdot (\frac{5}{6})^6 \approx 0.665$.

Question 2

Suppose one 6-sided and one 20-sided die are rolled. Let A be the event that the first die comes up 1 and B that the sum of the dice is 2. Are these events independent ? Explain using the formal definition of independence

Solution -

Events A and B are not independent.

Justification -

- There are two scenarios, one is that the first dice that comes with a 1 is the 6-sided die, the other is that the 1 comes up first on the 20-sided die.
- Suppose that the 6-sided die comes up with 1 first -
  - The probability that a fair 6-sided die would result in a 1 is $\frac{1}{6}$, hence $P(A) = \frac{1}{6}$.

- The minimum value on a die is 1, hence for the sum of the dice to be 2, the 20-sided die should result in a 1 to satisfy event B *i.e.* the only possible combination is $(die1, die2) = (1, 1)$ in a total of $6 \times 20 = 120$ permutations.
- Therefore, $P(B) = \frac{1}{6\times 20} \approx 0.0083$.
- According to the Law of total probability,

$$P(B) = P(B \cap A) + P(B \cap A^C)$$

$$P(B \cap A) = P(B) - P(B \cap A^C)$$
$$P(B \cap A) = P(B) - P(B \mid A^C)P(A^C)$$

Note that $P(A^C) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$ and $P(B \mid A^C) = 0$ this is because if the first die results in anything but 1, event B would never occur *e.g.* if the first die rolled a 2, the minimum sum of the two dice would be 3, exceeding 2. Therefore,

$$P(B \cap A) = \frac{1}{6 \times 20} - 0 \times \frac{1}{6} = \frac{1}{120} = P(B)$$
$$P(B \cap A) = P(A \cap B) = P(B)$$

- According to the definition of independence,

$$P(A)P(B) = \frac{1}{6} \cdot \frac{1}{120}, P(A \cap B) = \frac{1}{120}$$

$$P(A)P(B) = \frac{1}{720} \neq \frac{1}{120}$$
$$P(A)P(B) \neq P(A \cap B)$$

hence, events A and B are dependent.

- Suppose that the 20-sided die comes up with a 1 first -
  - Similar reasons to the above case, $P(A) = \frac{1}{20}$, $P(B) = \frac{1}{120}$.
  - We have proved above that $P(A \cap B) = P(B) = \frac{1}{120}$, hence, according to the definition of independence,

$$P(A)P(B) = \frac{1}{20} \cdot \frac{1}{120}, P(A \cap B) = \frac{1}{120}$$

$$P(A)P(B) = \frac{1}{2400} \neq \frac{1}{120}$$
$$P(A)P(B) \neq P(A \cap B)$$

hence events A and B are dependent.

- Both of the above cases yield the same result, therefore we conclude that events A and B are dependent.

Question 3

Say a hacker has a list of n distinct password candidates, only one of which will successfully log her into a secure system.

Question 3(a)

If she tries passwords from the list uniformly at random, deleting those passwords that do not work, what is the probability that her first successful login will be (exactly) on her $k-th$ try?

Solution -

$$\frac{1}{n}$$

Justification -

- If the hacker tries k times and successfully logs in on the $k-th$ try -

$$k = 1, P = \frac{1}{n}$$

$$k = 2, P = \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}$$

  where $(n-1)$ number of passwords left after the failure at attempt 1. Similarly, if she succeeds on the third try,

$$k = 3, P = \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{1}{n-2} = \frac{1}{n}$$

  where $(n-1)$ is the number of passwords left after the failure at attempt 1, and $(n-2)$ is the number of passwords left after the consecutive failure at attempt 2.

- Hence, the probability of succeeding on the $k-th$ try is

$$\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot ... \cdot \frac{n-k+1}{n-k+2} \cdot \frac{1}{n-k+1} = \frac{1}{n}$$

Question 3(b)

When $n = 6$ and $k = 3$ what is the value of this probability ?

Solution -

$$0.167$$

Justification -

- According to part (a), $P = \frac{1}{n} = \frac{1}{6} \approx 0.167$.

Question 3(c)

Now say the hacker tries passwords from the list at random, but does not delete previously tried passwords from the list. She stops after her first successful login attempt. What is the probability that her first successful login will be (exactly) on her $k-th$ try?

Solution -

$$(\frac{n-1}{n})^{k-1} \cdot \frac{1}{n}$$

Justification -

- In this case, the probability of success, at the consecutive events are,

$$k = 1, P = \frac{1}{n}$$
$$k = 2, P = \frac{n-1}{n} \cdot \frac{1}{n}$$

  where $(n-1)$ is the number of incorrect passwords in the possible candidates. Similarly, if attempt 3 is a success, then,

$$k = 3, P = (\frac{n-1}{n})^2 \cdot \frac{1}{n}$$

  because the hacker does not delete the previous tried incorrect passwords from the list, the probability of failure for any try would be $\frac{n-1}{n}$.

- In conclusion, the probability that her first successful login will be exactly on her $k-th$ try is

$$(\frac{n-1}{n})^{k-1} \cdot \frac{1}{n}$$

Question 3(d)

When $n = 6$ and $k = 3$ what is the value of this probability ? Hint: use the fact that the outcome of each try is an independent random event (since passwords are selected uniformly at random at each attempt)

Solution -

$$0.116$$

Justification -

- According to the part (b),

$$P = (\frac{n-1}{n})^{k-1} \cdot \frac{1}{n}$$
$$P = (\frac{6-1}{6})^{3-1} \cdot \frac{1}{6}$$
$$P = (\frac{5}{6})^2 \cdot \frac{1}{6} \approx 0.116$$

Question 4

A website wants to detect if a visitor is a robot. They decide to deploy three CAPTCHA tests that are hard for robots and if the visitor fails in one of the tests, they are flagged as a possible robot. The probability that a human succeeds at a single test is 0.95, while a robot only succeeds with probability 0.3. Assume all tests are independent.

Question 4(a)

If a visitor is actually a robot, what is the probability they get flagged?

Solution -

$$0.973$$

Justification -

- Let E denote the event that the robot succeeds all test, hence E^C denotes that the robot is flagged.
- The probability of success of a robot is 0.3, suppose the robot is not flagged and passes all 3 of the CAPTCHA tests, the probability of this occurring is $P(E) = 0.3^3$
- Therefore, $P(E^C) = 1 - P(E) = 1 - 0.3^3 \approx 0.973$

Question 4(b)

If a visitor is human, what is the probability they get flagged?

Solution -

$$0.143$$

Justification -

- Let F denote the event that the human succeeds all test, hence F^C denotes that the human is flagged.
- The probability of success of a human is 0.95, suppose the human passes all tests and is not flagged, the probability of this happening is $P(F) = 0.95^3$
- Therefore, $P(F^C) = 1 - P(F) = 1 - 0.95^3 \approx 0.143$

Question 4(c)

The fraction of visitors on the site that are robots is 1/10. Suppose a visitor gets flagged. What is the probability that visitor is a robot? Hint: use Bayes Rule.

Solution -

$$0.431$$

Justification -

- Let A denote the event that the visitor is a robot, and A^C denote the event that the visitor is human. Therefore $P(A) = 0.1, P(A^C) = 0.9$.
- Let B denote the event that the visitor is flagged, and let B^C denote the event that is the visitor is not flagged.
- Therefore, according to parts (a) and (b) $P(B \mid A) = 0.973, P(B \mid A^C) = 0.143$
- According to Bayes rule,

  $$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

  According to the Law of total probability,

  $$P(flagged) = P(flagged \cap robot) + P(flagged \cap human)$$

  $$P(B) = P(B \mid A) \cdot P(A) + P(B \mid A^C) \cdot P(A^C)$$

  Therefore, substituting this into the previous,

  $$P(A \mid B) = \frac{0.973 \times 0.1}{0.973 \times 0.1 + 0.143 \times 0.9} \approx 0.431$$