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Module : CS1003 Assignment 3

Q1.

$$A = \begin{pmatrix} -5 & -2 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det A = -5(1) + 2(2) - 2(-1) = 1$$

$$\tilde{A} = \begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -5 & -2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} -5 & -2 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} -5 & -2 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} -5 & -2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & -2 \\ 2 & -4 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{1} \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & -2 \\ 2 & -4 & -1 \end{pmatrix}^T$$

$$\boxed{A^{-1} = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -3 & -4 \\ -1 & -2 & -1 \end{pmatrix}}$$

Q2. $A = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & 1 & 3 \end{pmatrix}$

$Av = \lambda v$

$(A - \lambda I)v = 0 \longrightarrow (A - \lambda I) = \begin{pmatrix} 4-\lambda & 2 & -1 \\ 2 & 4-\lambda & 1 \\ -1 & 1 & 3-\lambda \end{pmatrix}$

$$\det(A - \lambda I) = (4-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -1 & 3-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 4-\lambda \\ -1 & 1 \end{vmatrix}$$

$$= (4-\lambda)(11 + \lambda^2 - 7\lambda) - 2(7 - 2\lambda) - (6 - \lambda)$$

$$= 44 + 4\lambda^2 - 28\lambda - 11\lambda - \lambda^3 + 7\lambda^2 - 20 + 5\lambda$$

$$= -\lambda^3 + 11\lambda^2 - 34\lambda + 24$$

for $\det(A - \lambda I) = 0$:

$$\lambda^3 - 11\lambda^2 + 34\lambda - 24 = 0$$

when $\lambda = 1$, $1 - 11 + 34 - 24 = 0$

hence, $\lambda = 1$ is a solution to $\det(A - \lambda I) = 0$.

therefore $(\lambda - 1)$ is a factor of $\det(A - \lambda I)$.

$$\begin{array}{r|l} & \lambda^2 - 10\lambda + 24 \\ \hline \lambda - 1 & \lambda^3 - 11\lambda^2 + 34\lambda - 24 \\ & -(\lambda^3 - \lambda^2) \\ \hline & -10\lambda^2 + 34\lambda - 24 \\ & -(-10\lambda^2 + 10\lambda) \\ \hline & 24\lambda - 24 \\ & -(24\lambda - 24) \\ \hline & 0 \end{array}$$

hence $\det(A - \lambda I) = (\lambda - 1)(\lambda^2 - 10\lambda + 24) = 0$

$$(\lambda - 1)(\lambda - 4)(\lambda - 6) = 0$$

therefore, $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 6$ are eigenvalues of A.

Case 1: $\lambda = 1$

$$(A - \lambda I \mid 0) = \begin{array}{c} \left(\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 2 & 3 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right) \end{array} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

$$\xrightarrow[\substack{R2 \to R2 - 2R1 \\ R3 \to R3 - 3R1}]{\substack{R1 \leftrightarrow R3 \\ R1 \to R1 \times (-1)}} \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \end{array}\right)$$

$$\xrightarrow[R3 \to R3 - 5R2]{\substack{R2 \to R2 \times \frac{1}{5} \\ R1 \to R1 + R2}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

linear system: $x_1 - x_3 = 0$
$x_2 + x_3 = 0$

let $x_3 = t$, $t \in \mathbb{R}$

therefore, $x_1 = t$
$x_2 = -t$

thus, $\boxed{x = \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ for any } t \in \mathbb{R} \setminus \{0\} \text{ are eigenvectors of } A \text{ associated with the eigenvalue } \lambda = 1}$

Case 2: $\lambda = 4$

$$(A - \lambda I \mid 0) = \left(\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right) \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

$$\xrightarrow[R2 \to R2 - 2R1]{\substack{R1 \leftrightarrow R3 \\ R1 \to R1 \times (-1)}} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[R3 \to R3 - 2R2]{\substack{R2 \to R2 \times \frac{1}{2} \\ R1 \to R1 + R2}} \left(\begin{array}{cccc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

linear system: $x_1 + \frac{1}{2}x_3 = 0$

$x_2 - \frac{1}{2}x_3 = 0$

let $x_3 = t$, $t \in \mathbb{R}$

therefore. $x_1 = -\frac{t}{2}$

$x_2 = \frac{t}{2}$

thus,

$$\boxed{x = \begin{pmatrix} -t/2 \\ t/2 \\ t \end{pmatrix} = t\begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix} \text{ for any } t \in \mathbb{R}\setminus\{0\}}$$

are eigenvectors of A associated with the eigenvalue $\lambda = 4$

Case 3: $\lambda = 6$

$$(A - \lambda I \,\vdots\, 0) = \begin{array}{c} \left(\begin{array}{cccc} -2 & 2 & -1 & 0 \\ 2 & -2 & 1 & 0 \\ -1 & 1 & -3 & 0 \end{array}\right) \end{array} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

$$\xrightarrow[\substack{R2 \to R2 - 2R1 \\ R3 \to R3 + 2R1}]{\substack{R1 \leftrightarrow R3 \\ R1 \to R1 \times (-1)}} \left(\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right)$$

$$\xrightarrow[R1 \to R1 - 3R2]{\substack{R2 \to R2 \times (-\frac{1}{5}) \\ R3 \to R3 - 5R2}} \left(\begin{array}{cccc} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

linear system: $x_1 - x_2 = 0$

$x_3 = 0$

let $x_2 = t$, $t \in \mathbb{R}$

therefore $x_1 = t$

thus

$$\boxed{x = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for any } t \in \mathbb{R} \setminus \{0\}}$$

are eigenvectors of A associated with the eigenvalue $\lambda = 6$