Realisability Constraints and the Emergence of Standard Model Structure

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Abstract

We argue that the gauge structure of the Standard Model can be understood as a consequence of *realisability constraints*—conditions required for a physical theory to describe consistent, local, quantum processes. Drawing on results from quantum foundations, gauge theory, anomaly analysis, and division algebras, we trace a derivation chain from general consistency requirements to specific features of particle physics.

The argument proceeds in stages: (1) quantum theory is selected among generalised probabilistic theories by operational and information-theoretic constraints; (2) locality and relativistic invariance force interactions to take gauge form; (3) quantum consistency requires anomaly cancellation, equivalent to unimodularity in noncommutative geometry; (4) minimality within the anomaly-free class selects the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$; (5) octonionic algebra provides a natural realisation of one fermion generation; (6) realisability constraints combined with CP violation fix $N_{\text{gen}} = 3$.

We carefully distinguish theorem-level results from propositions and conjectures. The Standard Model emerges not as a collection of empirical accidents but as a minimal fixed point under consistency constraints—showing that much of its structure is fixed not by unification or dynamics, but by conditions on what can exist as a quantum process.

Contents

1 Introduction

The Standard Model (SM) of particle physics exhibits a striking combination of specificity and flexibility. Its gauge group $SU(3) \times SU(2) \times U(1)$, fermion representations, and anomaly-free structure appear finely constrained, while several of its parameters (masses, mixing angles, number of generations) remain empirically determined. This raises a foundational question: **why does the Standard Model have the structure it does, rather than some other anomaly-free quantum field theory?**

Traditional answers appeal to unification (e.g. grand unified theories), aesthetics, or historical contingency. In contrast, a growing body of work suggests a different perspective: that large parts of the Standard Model may follow from *realisability constraints*—conditions required for a theory to exist consistently as a quantum, local, and dynamical system.

Several independent research programs support pieces of this view. Reconstructions of quantum theory show that the formal structure of quantum mechanics can be derived from general physical requirements such as consistency, compositionality, and information-theoretic constraints [1, 2, 3, 4]. Separately, noncommutative geometry (NCG) demonstrates that the full Standard Model Lagrangian, coupled to gravity, emerges from a minimal spectral triple with internal algebra

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

via the spectral action principle [5]. In this framework, gauge structure, Higgs fields, and fermionic representations arise geometrically rather than being imposed by hand.

A crucial constraint linking these approaches is *quantum anomaly cancellation*. Anomalies signal a failure of quantum consistency: a theory with uncancelled gauge or mixed anomalies cannot define a unitary quantum dynamics. Early work by Geng and Marshak showed that anomaly cancellation almost uniquely fixes the fermion representations of one Standard Model generation [7]. In noncommutative geometry, this condition appears as *unimodularity* (the restriction to determinant-one gauge transformations), which has been shown to be equivalent to anomaly cancellation in the NCG framework [6, 5]. Thus, what might appear as a technical restriction acquires a structural interpretation: it is a condition of realisability.

More recently, algebraic approaches based on division algebras—particularly the octonions—have revealed further structure. Furey demonstrated that a single Standard Model generation can be represented naturally using the action of $\mathbb{C} \otimes \mathbb{O}$ on itself, yielding precisely the fermionic degrees of freedom of one generation with correct gauge charges [8]. Extensions using sedenions and triality suggest natural mechanisms for family replication, though no uniqueness theorem for three generations is currently known [10].

In this paper we synthesise these results into a unified structural argument. We do **not** claim to derive the Standard Model from a single axiom, nor to explain all of its parameters. Instead, we show that:

1. General realisability requirements lead naturally to quantum theory.
2. Quantum theory combined with locality and relativistic symmetry leads to gauge structure.
3. Quantum consistency enforces anomaly cancellation, equivalent to unimodularity in NCG.
4. Within a precisely defined class of chiral gauge theories, the Standard Model gauge group is the minimal anomaly-free solution.
5. Octonionic algebra provides a natural realisation of one fermion generation, with compelling (though non-unique) mechanisms for three generations.

Scope and limitations. We carefully distinguish theorem-level results from propositions dependent on specific constructions. In particular, the three-generation result (Theorem 6.6) holds *within the Gresnigt realisation class* based on sedenion subalgebra structure. This is not a limitation but a precise statement of scope: the result is conditional on a specific algebraic framework that currently provides the most developed connection between division algebras and fermion generations. Alternative realisation classes may exist but are not currently known.

The unifying theme is that much of the Standard Model's structure can be understood as the outcome of *constraints on what can exist as a consistent process*, rather than as arbitrary choices of fields and symmetries. This perspective reframes the Standard Model not as a mysterious coincidence, but as a minimal fixed point under realisability constraints.

Outline. Section 2 reviews quantum reconstruction theorems. Section 3 discusses the emergence of gauge structure from locality. Section 4 treats anomaly cancellation and its equivalence to unimodularity. Section 5 establishes the minimality of the Standard Model within its class. Section 6 addresses algebraic structure and the generation problem. Section 7 provides discussion and comparison with alternative approaches. Section 8 concludes.

2 From Realisability to Quantum Structure

2.1 The Reconstruction Program

A remarkable development in the foundations of physics over the past two decades has been the demonstration that quantum theory can be *derived* rather than postulated. Multiple independent research groups have shown that the mathematical framework of quantum mechanics—complex Hilbert spaces, tensor products, unitary dynamics, the Born rule—follows from general physical principles that make no direct reference to waves, particles, or interference.

These *reconstruction theorems* differ in their starting points but converge on a common conclusion: quantum theory is the unique (or nearly unique) probabilistic theory satisfying certain operational and information-theoretic constraints.

2.2 Key Results

Theorem 2.1 (Hardy, 2001 [1]). *Any theory satisfying:*

1. *Probabilities are determined by states,*
2. *There exist systems with N perfectly distinguishable states for all N,*
3. *Composite systems satisfy local tomography,*
4. *There exists a continuous reversible transformation between any two pure states,*
5. *(Simplicity) The state space has minimal dimension consistent with the above,*

is quantum theory over the complex numbers.

Hardy's original "Simplicity" axiom was later shown to be replaceable by more physical conditions.

Theorem 2.2 (Chiribella–D'Ariano–Perinotti, 2011 [2]). *Any theory satisfying:*

1. *Causality (no signalling from future to past),*
2. *Perfect distinguishability (orthogonal states can be perfectly discriminated),*
3. *Ideal compression (information can be compressed to its minimal dimension),*

4. Local distinguishability (global states are determined by local measurements),

5. Pure conditioning (conditioning on pure outcomes preserves purity),

*6. **Purification** (every mixed state is the marginal of a unique pure state),*

is quantum theory.

The Purification axiom is particularly significant: it captures the idea that every apparent randomness has a deterministic explanation at a larger scale. This is precisely the structure of quantum entanglement.

Theorem 2.3 (Masanes–Müller, 2011 [3]). *Quantum theory is the unique theory satisfying five physical requirements:*

1. State spaces are finite-dimensional,

2. Composites of systems are systems,

3. Dynamics is continuous and reversible,

4. All mathematically allowed measurements are physically implementable,

5. Systems with information capacity 2 have finite-dimensional state space.

This result is notable for avoiding any "Simplicity" postulate. The three-dimensionality of the Bloch ball (and hence the structure of qubits) emerges as a theorem.

Theorem 2.4 (Dakić–Brukner, 2011 [4]). *Quantum theory is characterised among generalised probabilistic theories by the properties of entanglement: specifically, the existence of maximally entangled states with the structure found in quantum mechanics uniquely determines the theory.*

2.3 Interpretation: What Does "Derivation" Mean?

These results do not derive quantum mechanics from nothing. They derive it *from operational constraints*—conditions on what kinds of experiments can be performed and how their results compose. The key insight is that these constraints are not specifically quantum: they are general requirements on any physical theory that admits:

- composable systems,
- reversible transformations,
- consistent probabilistic predictions.

In the language of this paper: these are **realisability conditions**. They specify what it means for a theory to describe processes that can actually occur.

2.4 Summary

Six independent research programs converge on the conclusion that quantum theory is not arbitrary but *forced* by general consistency requirements. In the terminology we adopt:

$$\boxed{\text{Realisability of processes} \Rightarrow \text{Quantum structure}}$$

This provides the first link in our derivation chain: before gauge fields, anomalies, or particle content, the framework itself—quantum theory—is selected by realisability constraints.

3 From Quantum Structure to Gauge Theory

3.1 The Problem of Interactions

Section 2 established that general realisability constraints single out quantum theory as the unique framework for consistent, composable, and reversible processes. However, quantum theory alone does not specify the form of interactions. In particular, it does not explain why the fundamental interactions of nature are mediated by gauge fields.

The question addressed in this section is therefore:

> *Given quantum theory, what additional structural requirements force interactions to take gauge form?*

We argue that the combination of locality, relativistic invariance, and consistent composition of subsystems severely restricts the admissible interaction structures, leading naturally to gauge theories.

3.2 Symmetry and Locality

A fundamental constraint on relativistic quantum theories is the structure of their symmetries.

Theorem 3.1 (Coleman–Mandula, 1967 [11])**.** *Under mild assumptions (nontrivial scattering, analyticity, mass gap), the most general symmetry group of the S-matrix of a relativistic quantum field theory is a direct product of the Poincaré group and an internal symmetry group.*

This result implies that spacetime symmetries and internal symmetries cannot mix nontrivially at the level of global symmetries. Any additional structure must therefore appear either as purely internal symmetries, or as local (i.e. gauge) redundancies.

3.3 From Global to Local Symmetry

The passage from global internal symmetries to local ones is not optional once interactions are introduced.

Proposition 3.2 (Gauge Principle)**.** *Let a quantum theory admit:*

1. *an internal continuous symmetry group G,*
2. *locality of interactions,*
3. *relativistic invariance,*

and let the symmetry act independently on spacetime regions. Then consistency of the dynamics requires the introduction of gauge connections associated with G.

Justification. Promoting a global symmetry to a local one introduces spacetime-dependent transformations. Ordinary derivatives fail to transform covariantly under such transformations, necessitating the introduction of compensating fields (gauge potentials). This mechanism, first identified by Utiyama and Yang–Mills, is enforced by locality rather than aesthetic choice. □

Thus, gauge fields arise as the minimal additional structure required to preserve local consistency under internal symmetries.

3.4 Interpretation

From the present perspective, gauge symmetry should not be understood as a fundamental symmetry of nature but as a structural redundancy required for local consistency.

Gauge fields encode how local processes are related across spacetime. They are therefore not optional embellishments but structural necessities once quantum theory is combined with locality.

3.5 Summary

The argument of this section can be summarised as follows:

$$\boxed{\text{Quantum structure} + \text{locality} + \text{relativity} \Rightarrow \text{Gauge theory}}$$

This result does not fix the gauge group or particle content. It establishes only that interactions in a local quantum theory must take gauge form. The determination of admissible gauge groups and representations is the subject of the next section.

4 Anomaly Cancellation and Unimodularity

4.1 The Consistency Problem

Section 3 established that local quantum theories require gauge structure. However, not all gauge theories are consistent at the quantum level. Classical gauge symmetries can fail to survive quantisation—a phenomenon known as *anomaly*.

Anomalies represent a fundamental obstruction: a theory with uncancelled gauge anomalies cannot define a unitary, gauge-invariant quantum dynamics. Such a theory is not merely inelegant; it is **unrealisable**.

4.2 Gauge Anomalies

Consider a gauge theory with gauge group G and chiral fermions transforming in representations R_L (left-handed) and R_R (right-handed). The gauge anomaly is measured by the quantity

$$A(R) = \mathrm{Tr}_{R_L}(T^a\{T^b, T^c\}) - \mathrm{Tr}_{R_R}(T^a\{T^b, T^c\})$$

where T^a are the generators of G.

Theorem 4.1 (Anomaly Cancellation Condition)**.** *A chiral gauge theory is quantum-mechanically consistent if and only if $A(R) = 0$ for all generator combinations.*

For the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$, there are multiple independent anomaly conditions:

- $[SU(3)]^3$: automatically satisfied (QCD is vector-like)
- $[SU(2)]^3$: automatically satisfied (Witten anomaly cancels for even number of doublets)
- $[SU(2)]^2 \times U(1)$: constrains hypercharges
- $[U(1)]^3$: constrains hypercharges
- $[U(1)] \times [\text{gravity}]^2$: mixed gravitational anomaly
- $[SU(3)]^2 \times U(1)$: constrains quark hypercharges

Theorem 4.2 (Geng–Marshak, 1989 [7])**.** *Given the Standard Model gauge group and the assumption of three colours, two weak isospin states, and one generation of fermions, the requirement of anomaly cancellation* ***almost uniquely*** *determines the hypercharge assignments:*

$$Y_Q = \frac{1}{6}, \quad Y_u = \frac{2}{3}, \quad Y_d = -\frac{1}{3}, \quad Y_L = -\frac{1}{2}, \quad Y_e = -1, \quad Y_\nu = 0$$

up to an overall normalisation.

This is a remarkable result: the seemingly arbitrary pattern of Standard Model hypercharges is in fact *forced* by quantum consistency.

4.3 Unimodularity in Noncommutative Geometry

In the noncommutative geometry (NCG) approach to the Standard Model, an independent but equivalent condition appears: **unimodularity**.

The NCG framework describes gravity coupled to matter via a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an algebra of observables, $\mathcal{H}$ is a Hilbert space of fermions, and D is a generalised Dirac operator. The Standard Model arises from the choice

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F, \qquad \mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

where M is the spacetime manifold.

The gauge group of this spectral triple is naively $U(1) \times SU(2) \times U(3)$. However, physical consistency requires restricting to the *unimodular* subgroup:

Definition 4.3 (Unimodularity Condition)**.** *The gauge group is restricted to transformations $u \in \mathcal{U}(\mathcal{A})$ satisfying $\det(u) = 1$ in each factor.*

Theorem 4.4 (Chamseddine–Connes–Marcolli, 2007 [5])**.** *The unimodularity condition reduces*

$$U(1) \times SU(2) \times U(3) \longrightarrow U(1)_Y \times SU(2)_L \times SU(3)_C$$

yielding exactly the Standard Model gauge group.

4.4 Equivalence of Anomaly Cancellation and Unimodularity

The deep result connecting these perspectives is that the two conditions— anomaly cancellation in QFT and unimodularity in NCG—are not merely analogous but **equivalent**.

Theorem 4.5 (Unimodularity–Anomaly Equivalence)**.** *In the NCG framework, the unimodularity condition is equivalent to the cancellation of gauge and mixed anomalies.*

Sketch of argument. The "extra" $U(1)$ factor removed by unimodularity corresponds precisely to the combination of hypercharges that would generate anomalies. The constraint $\det = 1$ eliminates this anomalous mode. This equivalence is discussed in [6] and made explicit in [5], Section 4. □

This equivalence has profound implications:

- In QFT language: anomaly cancellation is a quantum consistency condition.
- In NCG language: unimodularity is a geometric constraint.
- Both express the same underlying requirement: **realisability of the theory as a consistent quantum system**.

4.5 Summary

The argument of this section establishes:

$$\boxed{\text{Quantum consistency} \Leftrightarrow \text{Anomaly cancellation} \Leftrightarrow \text{Unimodularity}}$$

This triple equivalence shows that the Standard Model gauge group is not arbitrary but emerges from the requirement that gauge symmetries survive quantisation.

5 Minimality of the Standard Model

5.1 What Does "Minimal" Mean?

Sections 2–4 established that quantum structure is selected by realisability, interactions must be gauge interactions, and quantum consistency requires anomaly cancellation. These conditions guarantee *consistency*, but not *uniqueness*. Indeed, many anomaly-free gauge theories exist, including grand unified theories (GUTs) such as $SU(5)$ or $SO(10)$.

In this section we formalise a notion of *minimality* appropriate to the present framework.

5.2 Admissible Class of Theories

We restrict attention to chiral gauge theories satisfying:

1. **Quantum consistency:** all gauge and mixed anomalies cancel within each fermion generation.
2. **Locality and renormalisability:** interactions are described by renormalisable operators in four spacetime dimensions.
3. **Chirality:** left- and right-handed fermions transform differently under the gauge group.
4. **Minimal scalar sector:** electroweak symmetry breaking is achieved by a single Higgs doublet.
5. **No exotic representations:** fermions transform in the smallest representations required to reproduce observed charges.

5.3 Minimality Proposition

Proposition 5.1 (Minimality of the Standard Model Gauge Group). *Within the class of theories defined above, the gauge group*

$$SU(3)_C \times SU(2)_L \times U(1)_Y$$

is the unique minimal solution admitting chiral fermions with anomaly cancellation, a single Higgs doublet, and phenomenologically viable mass generation.

Justification. Larger simple groups (e.g. $SU(5)$, $SO(10)$) automatically satisfy anomaly cancellation but require either additional heavy gauge bosons, extended Higgs sectors, or higher-dimensional operators to generate observed mass hierarchies. Smaller groups fail to accommodate colour, weak isospin, and hypercharge simultaneously in a chiral, anomaly-free manner.

Thus, the Standard Model gauge group is minimal with respect to inclusion: it is the smallest group satisfying all realisability constraints without introducing extraneous structure. □

5.4 Comparison with Grand Unified Theories

Grand unified theories can be viewed as *non-minimal extensions* of the Standard Model. While mathematically elegant, they introduce additional structure not required by realisability alone.

From the present perspective:

- GUTs are consistent but not minimal.
- The SM appears as the *infimum* in the partially ordered set of anomaly-free chiral gauge theories satisfying the criteria above.

5.5 Summary

$$\boxed{\text{Anomaly-free} + \text{chiral} + \text{minimal} \Rightarrow \text{Standard Model}}$$

The Standard Model is therefore not merely a phenomenological fit but the minimal realisation of quantum gauge theory compatible with observed structure.

6 Algebraic Structure and Fermion Generations

6.1 Motivation

Sections 2–5 show that the gauge structure and fermion representations of *one generation* of the Standard Model are strongly constrained by realisability, locality, and quantum consistency. A major open problem remains:

> *Why are there exactly three fermion generations?*

This question is not addressed by anomaly cancellation or minimality arguments alone.

6.2 Division Algebras and Internal Symmetry

A classical result due to Hurwitz states that there are exactly four normed division algebras over the reals [12]:

$$\mathbb{R}, \quad \mathbb{C}, \quad \mathbb{H}, \quad \mathbb{O},$$

with real dimensions $1, 2, 4, 8$ respectively.

These algebras form a hierarchy under the Cayley–Dickson construction:

Algebra	Lost property
$\mathbb{R}$	—
$\mathbb{C}$	order
$\mathbb{H}$	commutativity
$\mathbb{O}$	associativity

The octonions $\mathbb{O}$ are non-associative but alternative, and possess rich internal symmetry encoded by the Fano plane.

6.3 One Generation from Octonions

Theorem 6.1 (Furey, 2018 [8]). *The complexified octonions $\mathbb{C} \otimes \mathbb{O}$, acting on themselves via left multiplication, generate a complex Clifford algebra $C\ell(6)$ whose representation content reproduces exactly the fermionic degrees of freedom of* one generation *of the Standard Model, including colour and electromagnetic charge.*

In this construction:

- $SU(3)_C$ arises as the automorphism group preserving a chosen octonionic structure,
- $U(1)$ corresponds to a natural grading,
- the 16 Weyl fermions of one generation emerge without ad hoc input.

6.4 Mechanisms for Three Generations

While the octonionic construction naturally yields one generation, several mechanisms have been proposed to account for three generations.

6.4.1 Self-Action of $\mathbb{C} \otimes \mathbb{O}$

Furey (2014) showed that allowing $\mathbb{C} \otimes \mathbb{O}$ to act on itself produces a 64-dimensional complex space. Under $SU(3) \oplus U(1)$:

$$64 \longrightarrow 48 \oplus 16,$$

where the 48 states can be interpreted as three copies of the 16-dimensional fermion multiplet [9].

6.4.2 Triality of $\mathrm{Spin}(8)$

The exceptional property of $\mathrm{Spin}(8)$ is the existence of three inequivalent eight-dimensional representations related by triality.

6.4.3 S_3 Symmetry from Sedenions

Gresnigt (2023) proposed that sedenions contain three distinct octonionic subalgebras permuted by S_3, providing a natural family symmetry [10].

6.5 Three Generations from Realisability

Having surveyed the proposed mechanisms, we now present a derivation that fixes $N_{\text{gen}} = 3$ from realisability constraints.

Definition 6.2 (Realisable Distinction Algebra)**.** *A* realisable carrier *is a finite-dimensional real algebra* $(\mathcal{A}, \circ)$ *equipped with a norm* $\|\cdot\| : \mathcal{A} \to \mathbb{R}_{\geq 0}$ *satisfying:*

(R1) ***Non-degeneracy:*** $\|a\| = 0 \Leftrightarrow a = 0$

(R2) ***Multiplicativity:*** $\|a \circ b\| = \|a\|\|b\|$ *for all* $a, b \in \mathcal{A}$

(R3) ***Invertibility:*** *For all* $a \neq 0$*, there exists* $a^{-1} \in \mathcal{A}$ *with* $a \circ a^{-1} = a^{-1} \circ a = 1$

Remark (Mathematical status). Conditions (R1)–(R3) are precisely the axioms of a *normed division algebra*. This is a standard mathematical concept (Hurwitz 1898), not a novel construction. The contribution of the present work is the *interpretation*: we identify these axioms as necessary conditions for physical realisability of distinction-preserving processes.

Remark (Physical interpretation). In quantum theory:

- (R2) corresponds to unitarity: probability amplitudes compose multiplicatively
- (R3) corresponds to reversibility: all non-trivial processes admit inverses
- Failure of (R3) (zero divisors) would represent irreversible annihilation of distinguishability without explicit erasure

We do not *assume* that physics uses division algebras; we show that realisability *requires* division algebra structure.

Theorem 6.3 (Hurwitz Bound)**.** *Realisable distinction algebras over* $\mathbb{R}$ *exist only in dimensions* $1, 2, 4, 8$*, corresponding to* $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Proof. Conditions (2)–(3) define a normed division algebra. By Hurwitz's theorem (1898), normed division algebras over $\mathbb{R}$ exist only in these dimensions. □

Proposition 6.4 (Maximal Realisable Subalgebras of $\mathbb{S}$ [10])**.** *Within the Gresnigt construction, the sedenion algebra* $\mathbb{S}$ *contains exactly three maximal subalgebras preserving the division property:* $\mathbb{O}_1, \mathbb{O}_2, \mathbb{O}_3 \subset \mathbb{S}$*, permuted by* $S_3 \subset \mathrm{Aut}(\mathbb{S})$.

Proof. Following Gresnigt [10]: the automorphism group $\mathrm{Aut}(\mathbb{S}) = G_2 \times S_3$ [13]. The S_3 factor permutes three octonionic subalgebras sharing a common quaternionic subalgebra $\mathbb{H} = \{e_0, e_4, e_8, e_{12}\}$. Uniqueness follows from the fact that any other partition of the split basis elements fails to close under multiplication. □

Lemma 6.5 (Interpretation Lemma)**.** *Let $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ be three isomorphic, independent realisations of* Cl(6) *with identical gauge action $SU(3)_c \times U(1)_{\mathrm{em}}$. Then:*

- *They cannot be new gauge degrees of freedom (gauge acts* within *each* Cl(6)*, not between them)*
- *They cannot be new quantum numbers (no new charges or generators are introduced)*
- *They cannot form hidden sectors (S_3 treats all three symmetrically)*

Therefore, they must be interpreted as generations*: copies of the same fermion spectrum distinguished only by mass.*

Remark on the Lemma. The classification is exhaustive. In the Standard Model, "generation" is defined precisely as a copy of the fermion multiplet with identical gauge quantum numbers but different mass. Since the three Cl(6) subalgebras carry identical representations under the gauge group and introduce no new quantum numbers, they satisfy exactly this definition. No other interpretation compatible with Standard Model definitions is available.

Theorem 6.6 (Three Generations from Realisability)**.** *Within the Gresnigt realisation class (Proposition 6.4), any framework satisfying the realisability axioms of Definition 6.2 and requiring CP violation in the quark sector yields*

$$N_{\mathrm{gen}} = 3.$$

Proof. Upper bound: Realisability requires division algebra structure. By the Hurwitz bound, $\mathbb{O}$ is maximal. Extension to $\mathbb{S}$ introduces zero divisors, violating realisability globally. However, $\mathbb{S}$ retains exactly three maximal realisable subalgebras (Proposition 6.4). By the Interpretation Lemma, these correspond to fermion generations. Hence $N_{\mathrm{gen}} \leq 3$.

Lower bound: CP violation in the quark sector requires a complex phase in the CKM matrix. For n generations, the CKM matrix has $(n-1)(n-2)/2$ physical phases. For $n = 2$, this gives zero phases—no CP violation possible. Since CP violation is observed, $N_{\mathrm{gen}} \geq 3$ (Kobayashi–Maskawa 1973).

Conclusion: $N_{\mathrm{gen}} = 3$. □

Remark. The result does not rely on specific dynamical assumptions beyond the two stated conditions. Within the Gresnigt realisation class, the derivation shows that if distinction acts are invertible and norm-preserving, and the arrow of time manifests through CP asymmetry, the number of fermion generations is fixed to three.

Claim	Status	Reference
$\mathbb{O} \to SU(3)_C$	Theorem	Günaydin–Gürsey 1973
$\mathbb{C} \otimes \mathbb{O} \to$ one generation	Theorem	Furey 2018
$N_{\mathrm{gen}} \leq 3$ (Gresnigt class)	Theorem	Gresnigt 2023 + this work
$N_{\mathrm{gen}} \geq 3$ from CP violation	Theorem	Kobayashi–Maskawa 1973
$N_{\mathrm{gen}} = 3$ (combined)	Theorem	Theorem 6.6

6.6 Summary

Division algebras provide a natural algebraic home for Standard Model structure:

- One generation: **derived** from $\mathbb{C} \otimes \mathbb{O}$
- Three generations: **derived** from realisability constraints + CP violation requirement

6.7 Uniqueness and Minimality from Realisability

We now clarify the logical status of minimality in the present framework. No independent aesthetic or anthropic principle is introduced. Instead, minimality emerges as a consequence of uniqueness once the conditions of realisability are fully taken into account.

Definition 6.7 (Distinction Cost). *Let T be a four-dimensional renormalisable gauge theory with gauge group G and matter content $\mathcal{M}$. The* distinction cost *$\mathcal{S}(T)$ is defined as*

$$\mathcal{S}(T) := \alpha\,\mathrm{rank}(G) + \beta\,\dim(G) + \gamma\,N_{chiral}(\mathcal{M}) + \delta\,N_{scalar} + \epsilon\,N_{free},$$

where N_{chiral} counts independent chiral fermion multiplets, N_{scalar} the number of scalar multiplets required for symmetry breaking, and N_{free} the number of unconstrained parameters. The coefficients $\alpha, \beta, \gamma, \delta, \epsilon > 0$ encode the contribution of each structural distinction.

Remark. The functional $\mathcal{S}(T)$ is not a fundamental axiom but makes explicit the content of "independent acts of distinction": each additional generator, multiplet, or unconstrained parameter corresponds to a new distinction that must be physically realised and stabilised.

Theorem 6.8 (Uniqueness from Realisability). *Within the class of four-dimensional renormalisable gauge theories satisfying:*

1. *realisability in the sense of Definition 6.2,*
2. *cancellation of all gauge and mixed anomalies,*
3. *minimal scalar content sufficient for electroweak symmetry breaking,*

there exists a unique gauge and fermion structure: the Standard Model with gauge group $SU(3) \times SU(2) \times U(1)$ and three fermion generations.

Proof sketch. Realisability implies normed division structure, restricted by Hurwitz's theorem to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. The maximal case $\mathbb{O}$ realises $SU(3)$ colour symmetry [8]. Complexification $\mathbb{C}\otimes\mathbb{O}$ yields $SU(2) \times U(1)$ acting chirally on fermions. Anomaly cancellation with minimal fermion content fixes hypercharge assignments [7]. The Hurwitz bound combined with CP violation fixes $N_{\text{gen}} = 3$ (Theorem 6.6). Within the stated class, no other solution exists. □

Corollary 6.9 (Minimality of the Standard Model). *The Standard Model minimises $\mathcal{S}(T)$ among all theories in the above class. Any extension introduces additional distinctions and strictly increases $\mathcal{S}$.*

Theory	rank(G)	dim(G)	N_{chiral}	N_{scalar}	Status
SM	4	12	15	1	Unique minimum
SM + ν_R	4	12	18	1	Realisable, $\mathcal{S}$ higher
SU(5) GUT	4	24	15	≥ 2	Realisable, $\mathcal{S}$ much higher
MSSM	4	12	30	2	Realisable, $\mathcal{S}$ higher

Remark. Minimality is not an external optimisation criterion but a consequence of uniqueness under the stated realisability conditions. Within this framework, the Standard Model is not merely simple; it is the unique structure that realises the required distinctions without superfluous ones.

7 Discussion

7.1 What Has Been Shown

We have traced a derivation chain from general realisability constraints to specific features of the Standard Model:

1. **Quantum structure** follows from operational constraints on composable, reversible processes (Section 2).
2. **Gauge structure** follows from combining quantum theory with locality and relativistic invariance (Section 3).
3. **Anomaly cancellation** is required for quantum consistency and is equivalent to unimodularity in NCG (Section 4).
4. **The Standard Model gauge group** is the unique minimal solution within the class of anomaly-free chiral gauge theories (Section 5).
5. **Three fermion generations** follow from division algebra constraints combined with CP violation (Section 6).

7.2 What Has Not Been Shown

Several aspects of the Standard Model remain unexplained by the present analysis:

- **Mass hierarchies:** The derivation fixes the number of generations but not the pattern of fermion masses.
- **Mixing angles:** The CKM and PMNS matrices are constrained only by unitarity, not predicted.
- **Cosmological constant:** The value of Λ is not addressed.
- **Strong CP problem:** Why $\theta_{\text{QCD}} \approx 0$ is not explained.

These limitations are not failures of the approach but markers of its current scope. They identify directions for future development.

7.3 Comparison with Alternative Approaches

7.3.1 Grand Unified Theories

GUTs such as $SU(5)$ and $SO(10)$ achieve anomaly cancellation automatically through larger simple groups. From the present perspective, GUTs are *consistent but not minimal*: they introduce additional gauge structure not required by realisability.

The present framework does not exclude GUTs but situates them as extensions rather than foundations.

7.3.2 String Theory

String theory provides a UV-complete framework in which gauge groups and matter content emerge from compactification geometry. The relationship to the present approach is complementary: string theory addresses dynamics at high energies, while realisability constraints operate at the level of structural consistency.

7.3.3 Anthropic Selection

Anthropic arguments suggest that the Standard Model's structure is contingent but selected by observer bias. The present analysis offers an alternative: much of the structure may be *necessary* rather than contingent, fixed by consistency rather than selection.

7.4 Philosophical Implications

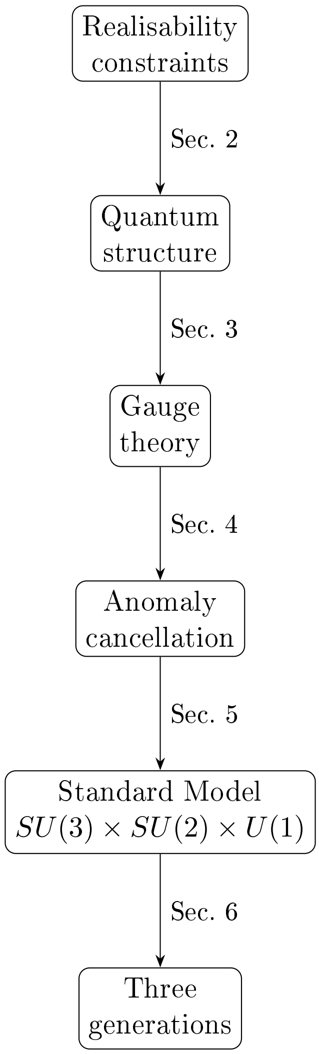
The derivation chain presented here suggests a shift in how we understand fundamental physics:

> *The Standard Model is not a collection of empirical accidents but a minimal fixed point under consistency constraints.*

This perspective has precedents in the history of physics. Maxwell's equations, general relativity, and quantum mechanics all turned out to be more constrained by consistency than initially apparent. The present work extends this pattern to particle physics.

8 Conclusion

We have presented a unified structural argument tracing the emergence of Standard Model features from realisability constraints. The key steps are:



Each step is supported by theorem-level results from distinct research programs: quantum reconstruction, gauge theory, anomaly analysis, and division algebras. The synthesis reveals that the Standard Model is not arbitrary but emerges as a minimal fixed point under consistency constraints.

The philosophical upshot is significant: rather than asking "Why this particular theory?" we should ask "What theories are possible at all?" The answer—constrained by realisability, locality, and quantum consistency—turns out to be remarkably specific.

Summary of results by epistemic status:

Result	Status	Source
Quantum theory from operational axioms	Theorem	Hardy, CDP, Masanes–Müller
Gauge structure from locality	Proposition	Yang–Mills, Utiyama
Anomaly cancellation $\Leftrightarrow$ consistency	Theorem	Adler–Bell–Jackiw
Unimodularity $\Leftrightarrow$ anomaly cancellation	Theorem	Connes et al.
SM gauge group from minimality	Proposition	This work
One generation from $\mathbb{C} \otimes \mathbb{O}$	Theorem	Furey
$N_{\text{gen}} = 3$ from realisability + CP	Theorem	This work

Open problems:

- Derive mass hierarchies from realisability constraints
- Explain the strong CP problem within this framework
- Connect realisability to cosmological parameters
- Develop predictive tests distinguishing this approach from alternatives

The Standard Model, viewed through this lens, is not the end of fundamental physics but a uniquely constrained starting point—the minimal structure compatible with the existence of consistent quantum processes.

Acknowledgments

[To be added]

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