The Fibonacci Identity Paradox

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April 30, 2025

Abstract

This note presents a mathematical puzzle involving an apparent contradiction in Fibonacci number identities. We demonstrate a sequence of seemingly correct algebraic steps that lead to a nonsensical conclusion, then reveal the subtle error that creates this mathematical illusion.

1 Introduction

The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by the recurrence relation:

$$F_0 = 0 \tag{1}$$
$$F_1 = 1 \tag{2}$$
$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \tag{3}$$

This sequence (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...) has been studied extensively and appears in numerous mathematical contexts and natural phenomena.

2 The Paradox

Paradox 1. *Consider the following algebraic manipulation involving Fibonacci numbers:*

$$\begin{aligned}
F_{n+1} \cdot F_{n-1} - F_n^2 &= F_{n+1} \cdot F_{n-1} - F_n \cdot F_n && (4)\\
&= F_{n+1} \cdot F_{n-1} - F_n(F_{n-1} + F_{n-2}) && (5)\\
&= F_{n+1} \cdot F_{n-1} - F_n \cdot F_{n-1} - F_n \cdot F_{n-2} && (6)\\
&= F_{n-1}(F_{n+1} - F_n) - F_n \cdot F_{n-2} && (7)\\
&= F_{n-1} \cdot F_n - F_n \cdot F_{n-2} && (8)\\
&= F_n(F_{n-1} - F_{n-2}) && (9)\\
&= F_n \cdot F_{n-3} && (10)
\end{aligned}$$

Therefore, we have derived:

$$F_{n+1} \cdot F_{n-1} - F_n^2 = F_n \cdot F_{n-3} \tag{11}$$

But it is well-known that for the Fibonacci sequence:

$$F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n \tag{12}$$

This gives us:

$$(-1)^n = F_n \cdot F_{n-3} \tag{13}$$

Which is clearly false for most values of n!

3 The Investigation

Dialogue 1. ***Undergraduate***: *Wait, I derived* $F_{n+1} \cdot F_{n-1} - F_n^2 = F_n \cdot F_{n-3}$*, but I know it should equal* $(-1)^n$*. Did I make a mistake?*

Algebraist: *Let me check your work... hmm, each step seems correct. You used the recurrence relation properly.*

Number Theorist: *But it can't be right. Let's try* $n = 5$*. We have* $F_6 \cdot F_4 - F_5^2 = 8 \cdot 3 - 5^2 = 24 - 25 = -1 = (-1)^5$*. But your formula gives* $F_5 \cdot F_2 = 5 \cdot 1 = 5$*. That's not equal to* -1*!*

Logician: *So we have a contradiction. Either the well-known identity is wrong, or there's an error in the derivation.*

Computer Scientist: *Let me write a program to check...*

Combinatorialist: *Wait! I think I see the issue...*

4 The Resolution

Resolution 1. *The error occurs in the step:*

$$F_{n-1}(F_{n+1} - F_n) - F_n \cdot F_{n-2} = F_{n-1} \cdot F_n - F_n \cdot F_{n-2} \tag{14}$$

Here, we claimed that $F_{n+1} - F_n = F_n$*, which is incorrect. The correct relation is:*

$$\begin{aligned} F_{n+1} - F_n &= (F_n + F_{n-1}) - F_n \qquad (15) \\ &= F_{n-1} \qquad (16) \end{aligned}$$

With this correction, the derivation becomes:

$$\begin{aligned} F_{n-1}(F_{n+1} - F_n) - F_n \cdot F_{n-2} &= F_{n-1} \cdot F_{n-1} - F_n \cdot F_{n-2} \qquad (17) \\ &= F_{n-1}^2 - F_n \cdot F_{n-2} \qquad (18) \end{aligned}$$

And this does not lead to the claimed result. In fact, continuing correctly:

$$\begin{aligned} F_{n+1} \cdot F_{n-1} - F_n^2 &= F_{n-1}^2 - F_n \cdot F_{n-2} \qquad (19) \\ & \qquad (20) \end{aligned}$$

We can use a similar approach to show that $F_{n-1}^2 - F_n \cdot F_{n-2} = (-1)^n$*, which is consistent with the known identity.*

5 A Humorous Dialogue on the Resolution

Dialogue 2. ***Undergraduate***: *So I made a simple arithmetic error? That's embarrassing.*

Algebraist: *Don't feel bad. Fiddling with recurrence relations can be tricky.*

Number Theorist: *In mathematics, we often learn more from our mistakes than our successes.*

Combinatorialist: *Besides, Fibonacci himself probably made this exact error at some point.*

Computer Scientist: *[looking up from laptop] My program confirms that* $F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n$ *for all values of n from 2 to 1000.*

Logician: *That's not a proof.*

Computer Scientist: *But it's pretty convincing empirical evidence!*

Undergraduate: *Wait, I have another idea...*

Everyone else: *[groans]*

6 Bonus: The Golden Connection

This identity has a fascinating connection to the golden ratio. Let $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\psi = \frac{1-\sqrt{5}}{2} \approx -0.618$. The closed-form expression for the Fibonacci numbers is:

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}} \tag{21}$$

Using this formula, we can prove directly that:

$$F_{n+1} \cdot F_{n-1} - F_n^2 = \frac{(\phi^{n+1} - \psi^{n+1})(\phi^{n-1} - \psi^{n-1})}{5} - \frac{(\phi^n - \psi^n)^2}{5} \tag{22}$$

$$= \frac{1}{5}[(\phi^{n+1} - \psi^{n+1})(\phi^{n-1} - \psi^{n-1}) - (\phi^n - \psi^n)^2] \tag{23}$$

After expanding and using the facts that $\phi \cdot \psi = -1$ and $\phi + \psi = 1$, this simplifies to $(-1)^n$.

7 Conclusion

This paradox demonstrates how carefully we must handle recurrence relations and algebraic manipulations. A single misstep can lead to a plausible-looking but entirely incorrect result. As the mathematician John von Neumann once said: "In mathematics, you don't understand things. You just get used to them." This paradox helps us understand why that might be true!

References

[1] Fibonacci, L. (1202). "Liber Abaci."

[2] Livio, M. (2002). "The Golden Ratio: The Story of Phi, the World's Most Astonishing Number."

[3] Gardner, M. (1956). "Mathematics, Magic and Mystery."