

MATH 3100 FALL 2020. LECTURE SUMMARIES

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1. 8/25

Section 1.1 in the textbook.

- (1) Sample space — long abstract definition which encompasses all possible mathematical models of randomness we are going to see in the course
- (2) Examples of sample spaces — coin tossing, dice rolling.
- (3) We are discussing finite sample spaces so far. Out of finite sample spaces, a special case is formed by *finite sample spaces with equally likely outcomes*. In them, we have $P(\omega) = \frac{1}{\#\Omega}$ for all $\omega \in \Omega$, and $P(A) = \frac{\#A}{\#\Omega}$ for all events A .
- (4) Repeated experiments, sample space $\Omega^n = \Omega \times \dots \times \Omega$ (Cartesian power), where

$$\Omega^n = \{(a_1, \dots, a_n) : a_i \in \Omega\}$$

is the space of ordered n -tuples of elements from Ω . The sample space Ω^n models the experiment corresponding to Ω , repeated (independently) n times.

- (5) Finer point. In uncountable sample spaces, usually it is not possible to define P consistently for all subsets. Therefore, we need to restrict the definition of event to “good” subsets of Ω .

2. 8/27

Section 1.2 in the textbook.

- (1) Random sampling. We stay in the scenario with finite sample spaces, equally likely outcomes.
- (2) We discuss three main sampling schemes of k objects out of n objects.
- (3) If we sample with replacement and order matters, then $\#\Omega = n^k$
- (4) If we sample without replacement and order matters, then

$$\#\Omega = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}.$$

If $k = n$, we talk about random permutations of n objects.

- (5) If we sample without replacement and order does not matter, then

$$\#\Omega = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

- (6) Hypergeometric distribution. Imagine we have n objects separated into a number of types $1, \dots, L$, and there are m_j objects of type j . So that $m_1 + \dots + m_L = n$. Sample k objects at random from n . The probability that in this sample there are p_j objects of type j is equal

to

$$\frac{\binom{m_1}{p_1} \cdots \binom{p_L}{m_L}}{\binom{n}{k}},$$

where $p_1 + \dots + p_L = k$.

3. 9/1

Sections 1.3 and 1.4 in the textbook.

- (1) Geometric distribution $P(k) = p^{k-1}(1-p)$, $k = 1, 2, \dots$. This is an example of an infinite Ω . Here Ω is countable. Countable and finite sample spaces have a special unifying name, “discrete sample spaces”.
- (2) Geometric series, its sum = $\frac{\text{first term}}{1 - \text{ratio}}$.
- (3) Continuous uniform distribution on $[0, 1]$ — another example of an infinite Ω . This Ω is uncountable.
- (4) $P(A)$ behaves like area of the event, both in continuous uniform case and in general (in some sense).
- (5) Operations on events and their probabilities: decomposition, complement, monotonicity, inclusion-exclusion.

4. 9/3

Sections 1.4 and 1.5 in the textbook.

- (1) Operations on events and their probabilities, and corresponding examples: decomposition, complements, monotonicity, inclusion-exclusion.
- (2) For monotonicity, a proof that we will see T with probability 1, after repeatedly tossing a fair coin.
- (3) For inclusion-exclusion, discussed a hard problem of computing the probability that no one has their own hat, if the hats are randomly permuted.
- (4) Random variable is a function on the sample space. This is the second fundamental definition of the course.
- (5) Discussed the definition, examples of random variables on discrete and continuous sample spaces.
- (6) Probability mass function (for discrete random variables). Probability distribution.

5. 9/8

Conditional probability and Bayes’ rule (Sections 2.1 and 2.2)

- (1) Definition of conditional probability $P(A | B)$
- (2) Multiplication rule $P(AB) = P(B)P(A | B)$.
- (3) Law of total probability $P(A) = \sum_{k=1}^N P(A | B_k)P(B_k)$, where $\Omega = \bigcup_{k=1}^N B_k$ is a partition of the sample space.
- (4) Bayes’s formula

$$P(B | A) = \frac{P(A | B)P(B)}{P(A)} = \frac{P(A | B)P(B)}{P(A | B)P(B) + P(A | B^c)P(B^c)}.$$

6. 9/10

Independence (section 2.3).

- (1) Some hints on the most challenging problems from Problem Set 3.
- (2) Independence. Independence algebraically means product rule.
- (3) Independence of two events.
- (4) Mutual independence and pairwise independence of several events.
- (5) Electric circuits example.
- (6) Model of arbitrary many independent events with $P(A_j) = \frac{1}{2}$ on $\Omega = [0, 1]$.
- (7) Independence of random variables.

7. 9/15

Section 2.4 and some parts from 2.5.

- (1) Recall independence.
- (2) Independent events from independent events, for example, AB^c and C^c are independent if A, B, C are mutually independent.
- (3) Independent trials. Sample space.
- (4) Proof that all the probabilities sum to one.
- (5) Bernoulli, binomial, geometric distributions.
- (6) Conditional independence (brief discussion).
- (7) Hypergeometric distribution.

8. 9/17

Section 3.1.

- (1) Probability mass function, pmf (for discrete distributions). Examples, properties.
- (2) Probability density function, pdf (for continuous distributions). Examples, properties.
- (3) (Continuous) uniform distribution.
- (4) Pdf as a derivative / infinitesimal description.
- (5) Example with a uniform point in a disc and the pdf for R , the distance from the point to 0.

9. 9/24

Sections 3.2 and 3.3.

- (1) Cumulative distribution function (cdf)
 - Motivation for cdf
 - Definition of cdf
 - Relation between cdf and pmf for discrete random variables
 - Relation between cdf and pdf for continuous random variables
 - Properties of the cdf
- (2) Expectation.
 - Expectation of a discrete random variable
 - Expectation of a continuous random variable
 - Expectation of geometric, Bernoulli, binomial random variables. Method of derivatives.
 - Formula for the expectation of a function of a random variable.
 - Nonexistence of expectation.

10. 9/29

Sections 3.3 and 3.4 (further discussion of expectation, and variance).

- (1) Properties of random variables (table from the textbook)
- (2) Expectation which is infinite. Expectation of the hypergeometric distribution.
- (3) Indicator random variables.
- (4) Expectation of a function of a random variable (further discussion).
- (5) (begin part 2 of the video) Variance - definition and formula $E(X^2) - (EX)^2$.
- (6) Variance of Bernoulli and binomial random variables.
- (7) Hypergeometric variance (no computation, just showing you the formula which is quite complex).
- (8) Expectation and variance of $aX + b$.
- (9) $Var(X) = 0$ if and only if $P(X = a) = 1$ for some a .
- (10) Variance of geometric and uniform distributions.

11. 10/1

Gaussian (normal) distribution. Sections 3.5.

- (1) Gaussian distribution — standard $\mathcal{N}(0, 1)$
- (2) Getting the probability density normalizing constant $\sqrt{2\pi}$
- (3) Examples with the table
- (4) Expectation and variance of the standard normal random variable
- (5) Generic normal random variable $\mathcal{N}(\mu, \sigma^2)$

12. 10/6

Central limit theorem and law of large numbers (all for the binomial distribution).

- (1) Graphs of binomial pmf for large n
- (2) CLT: formulation. Limit, and normal approximation with $\Phi(x)$.
- (3) CLT: examples
- (4) Continuity correction
- (5) CLT: idea of proof
- (6) Law of large numbers for the binomial distribution

13. 10/8

Applications of the Central Limit Theorem (section 4.3).

- (1) Confidence intervals for the unknown p of the binomial distribution,

$$P(|p - \hat{p}| > \varepsilon) \geq 2\Phi(2\varepsilon\sqrt{n}) - 1$$

- (2) Various examples with confidence intervals
- (3) Maximum likelihood estimate
- (4) One more example of the use of Central Limit Theorem (airplane overbooking problem)

14. 10/13

Poisson distribution, Poisson process, exponential distribution. Sections 4.4 – 4.6.

- (1) Poisson distribution
- (2) Poisson distribution: mean and variance
- (3) Poisson approximation to the binomial distribution
- (4) Poisson approximation vs normal approximation

- (5) Poisson process — idea from bus arrivals
- (6) Exponential distribution, derivation from the bus arrival process
- (7) Exponential distribution: mean and variance

15. 10/15

Exponential distribution, Poisson process, gamma distribution (sections 4.4–4.6).

- (1) Exponential distribution. Definition and examples
- (2) Memorylessness of the exponential distribution
- (3) Exponential as a limit of geometric distribution (connection to the coin-flipping setup)
- (4) Poisson process on the line, rigorous definition
- (5) Poisson process, example
- (6) Poisson process in space (illustration)
- (7) Gamma distribution from Poisson process
- (8) Binomial distribution from Poisson process

16. 10/20

Finishing up chapter 4, and surveying the material of chapter 5 (which we mostly skip). Preparing for chapter 6.

- (1) Expectation and variance of the Gamma distribution
- (2) Section 5.1 — moment generating function and its usefulness in computing all moments $\mathbb{E}(X^n)$ of the random variable.
- (3) Section 5.2 — Distribution of a function of a random variable. How the pdf is transformed.
- (4) Overview of the topics of chapter 6. Beginning of discrete joint distributions.

17. 10/27

Joint distributions. Sections 6.1–6.2.

- (1) Joint pmf (discrete random variables)
- (2) Multinomial distribution
- (3) Examples on the multinomial distribution
- (4) Joint pdf (continuous random variables)
- (5) Nonexistence of some joint pdfs (example with $X = Y$)
- (6) Example. Crash course on double integrals \iint
- (7) Marginal pdfs and one more example

18. 10/29

Joint distributions. Sections 6.2–6.3.

- (1) Joint uniform distribution in a region D of \mathbb{R}^2 or higher dimensions.
- (2) Examples. Marginal distributions.
- (3) Joint pdf/pmf and independence.
- (4) Functions of independent collections. Minimum and maximum.
- (5) Minimum and maximum of uniform and exponential random variables.
- (6) Product rule for expectations.

19. 11/3

Convolution of random variables (distribution of $X + Y$, where X, Y are independent). Many examples. (Section 7.1). Poisson process via convolution (section 7.3).

20. 11/5

Expectation, variance, and covariance for multiple random variables. Sections 8.1, 8.2

- (1) Linearity of expectation, for any random variables (section 8.1). Application to binomial and hypergeometric distributions' expectations.
- (2) Product rule for expectations for independent random variables (section 8.2). Summation rule for variances. Application to binomial distribution's variance. Sample mean.
- (3) Covariance, some properties (continued next time).

21. 11/10

Covariance and correlation coefficient. Bivariate normal random variables. Section 8.4, and Pitman's textbook (which is optional but has a somewhat shorter exposition).

- (1) Covariance and its properties
- (2) Covariance for multinomial random variables
- (3) Correlation coefficient
- (4) Bivariate normal: independent standard
- (5) Bivariate normal: dependent standard
- (6) Bivariate normal: general
- (7) Example problem (heights of parents and children)

22. 11/12

Inequalities, law of large numbers. Sections 9.1, 9.2

- (1) Markov inequality, proof
- (2) Chebyshev inequality, proof
- (3) Law of large numbers, proof

23. 11/17

Central limit theorem and examples (Section 9.3)

- (1) Central limit theorem (CLT) — binomial and general
- (2) Many examples of uses of CLT
- (3) Normal approximations of some known distributions (Poisson, gamma)
- (4) New topic, conditioning. First example.

24. 11/19

Conditioning (10.1 – 10.2).

- (1) Recall the conditioning.
- (2) Condition on a random variable, discrete case.
- (3) Examples of conditioning in discrete case.
- (4) From conditional to joint pdfs.
- (5) Application of conditioning to Poisson process.
- (6) Definitions and some examples of conditioning in continuous case.

25. 11/24

Conditioning. Conditional expectation. Sections 10.2 – 10.3.

- (1) Continuous examples and conditioning.
- (2) Conditional expectation with respect to a random variable — motivating example.
- (3) Conditional expectation $E(X | Y)$ with respect to a random variable — definition.
- (4) Examples.