MATH 3100 FALL 2020. LECTURE SUMMARIES

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1. 8/25

Section 1.1 in the textbook.

(1) Sample space — long abstract definition which encompasses all possible mathematical models of randomness we are going to see in the course
(2) Examples of sample spaces — coin tossing, dice rolling.
(3) We are discussing finite sample spaces so far. Out of finite sample spaces, a special case is formed by *finite sample spaces with equally likely outcomes*. In them, we have $\mathsf{P}(\omega) = \dfrac{1}{\#\Omega}$ for all $\omega \in \Omega$, and $\mathsf{P}(A) = \dfrac{\#A}{\#\Omega}$ for all events A.
(4) Repeated experiments, sample space $\Omega^n = \Omega \times \ldots \times \Omega$ (Cartesian power), where
$$\Omega^n = \{(a_1, \ldots, a_n) \colon a_i \in \Omega\}$$
is the space of ordered n-tuples of elements from Ω. The sample space Ω^n models the experiment corresponding to Ω, repeated (independently) n times.
(5) Finer point. In uncountable sample spaces, usually it is not possible to define P consistently for all subsets. Therefore, we need to restrict the definition of event to "good" subsets of Ω.

2. 8/27

Section 1.2 in the textbook.

(1) Random sampling. We stay in the scenario with finite sample spaces, equally likely outcomes.
(2) We discuss three main sampling schemes of k objects out of n objects.
(3) If we sample with replacement and order matters, then $\#\Omega = n^k$
(4) If we sample without replacement and order matters, then
$$\#\Omega = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}.$$
If $k = n$, we talk about random permutations of n objects.
(5) If we sample without replacement and order does not matter, then
$$\#\Omega = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}.$$
(6) Hypergeometric distribution. Imagine we have n objects separated into a number of types $1, \ldots, L$, and there are m_j objects of type j. So that $m_1 + \ldots + m_L = n$. Sample k objects at random from n. The probability that in this sample there are p_j objects of type j is equal

to

$$\frac{\binom{m_1}{p_1} \dots \binom{p_L}{m_L}}{\binom{n}{k}},$$

where $p_1 + \ldots + p_L = k$.

3. 9/1

Sections 1.3 and 1.4 in the textbook.

(1) Geometric distribution $\mathsf{P}(k) = p^{k-1}(1-p)$, $k = 1, 2, \ldots$. This is an example of an infinite Ω. Here Ω is countable. Countable and finite sample spaces have a special unifying name, "discrete sample spaces".
(2) Geometric series, its sum $= \dfrac{\text{first term}}{1 - \text{ratio}}$.
(3) Continuous uniform distribution on $[0, 1]$ — another example of an infinite Ω. This Ω is uncountable.
(4) $\mathsf{P}(A)$ behaves like area of the event, both in continuous uniform case and in general (in some sense).
(5) Operations on events and their probabilities: decomposition, complement, monotonicity, inclusion-exclusion.

4. 9/3

Sections 1.4 and 1.5 in the textbook.

(1) Operations on events and their probabilities, and corresponding examples: decomposition, complements, monotonicity, inclusion-exclusion.
(2) For monotonicity, a proof that we will see T with probability 1, after repeatedly tossing a fair coin.
(3) For inclusion-exclusion, discussed a hard problem of computing the probability that no one has their own hat, if the hats are randomly permuted.
(4) Random variable is a function on the sample space. This is the second fundamental definition of the course.
(5) Discussed the definition, examples of random variables on discrete and continuous sample spaces.
(6) Probability mass function (for discrete random variables). Probability distribution.

5. 9/8

Conditional probability and Bayes' rule (Sections 2.1 and 2.2)

(1) Definition of conditional probability $\mathsf{P}(A \mid B)$
(2) Multiplication rule $\mathsf{P}(AB) = \mathsf{P}(B)\,\mathsf{P}(A \mid B)$.
(3) Law of total probability $\mathsf{P}(A) = \sum_{k=1}^{N} \mathsf{P}(A \mid B_k)\,\mathsf{P}(B_k)$, where $\Omega = \bigcup_{k=1}^{N} B_k$ is a partition of the sample space.
(4) Bayes's formula

$$\mathsf{P}(B \mid A) = \frac{\mathsf{P}(A \mid B)\mathsf{P}(B)}{\mathsf{P}(A)} = \frac{\mathsf{P}(A \mid B)\mathsf{P}(B)}{\mathsf{P}(A \mid B)\mathsf{P}(B) + \mathsf{P}(A \mid B^c)\mathsf{P}(B^c)}.$$

6. 9/10

Independence (section 2.3).

(1) Some hints on the most challenging problems from Problem Set 3.
(2) Independence. Independence algebraically means product rule.
(3) Independence of two events.
(4) Mutual independence and pairwise independence of several events.
(5) Electric circuits example.
(6) Model of arbitrary many independent events with $\mathsf{P}(A_j) = \frac{1}{2}$ on $\Omega = [0, 1]$.
(7) Independence of random variables.

7. 9/15

Section 2.4 and some parts from 2.5.

(1) Recall independence.
(2) Independent events from independent events, for example, AB^c and C^c are independent if A, B, C are mutually independent.
(3) Independent trials. Sample space.
(4) Proof that all the probabilities sum to one.
(5) Bernoulli, binomial, geometric distributions.
(6) Conditional independence (brief discussion).
(7) Hypergeometric distribution.

8. 9/17

Section 3.1.

(1) Probability mass function, pmf (for discrete distributions). Examples, properties.
(2) Probability density function, pdf (for continuous distributions). Examples, properties.
(3) (Continuous) uniform distribution.
(4) Pdf as a derivative / infinitesimal description.
(5) Example with a uniform point in a disc and the pdf for R, the distance from the point to 0.

9. 9/24

Sections 3.2 and 3.3.

(1) Cumulative distribution function (cdf)
    - Motivation for cdf
    - Definition of cdf
    - Relation between cdf and pmf for discrete random variables
    - Relation between cdf and pdf for continuous random variables
    - Properties of the cdf
(2) Expectation.
    - Expectation of a discrete random variable
    - Expectation of a continuous random variable
    - Expectation of geometric, Bernoulli, binomial random variables. Method of derivatives.
    - Formula for the expectation of a function of a random variable.
    - Nonexistence of expectation.

10. 9/29

Sections 3.3 and 3.4 (further discussion of expectation, and variance).

(1) Properties of random variables (table from the textbook)
(2) Expectation which is infinite. Expectation of the hypergeometric distribution.
(3) Indicator random variables.
(4) Expectation of a function of a random variable (further discussion).
(5) (begin part 2 of the video) Variance - definition and formula $E(X^2) - (EX)^2$.
(6) Variance of Bernoulli and binomial random variables.
(7) Hypergeometric variance (no computation, just showing you the formula which is quite complex).
(8) Expectation and variance of $aX + b$.
(9) $Var(X) = 0$ if and only if $P(X = a) = 1$ for some a.
(10) Variance of geometric and uniform distributions.

11. 10/1

Gaussian (normal) distribution. Sections 3.5.

(1) Gaussian distribution — standard $\mathcal{N}(0, 1)$
(2) Getting the probability density normalizing constant $\sqrt{2\pi}$
(3) Examples with the table
(4) Expectation and variance of the standard normal random variable
(5) Generic normal random variable $\mathcal{N}(\mu, \sigma^2)$

12. 10/6

Central limit theorem and law of large numbers (all for the binomial distribution).

(1) Graphs of binomial pmf for large n
(2) CLT: formulation. Limit, and normal approximation with $\Phi(x)$.
(3) CLT: examples
(4) Continuity correction
(5) CLT: idea of proof
(6) Law of large numbers for the binomial distribution

13. 10/8

Applications of the Central Limit Theorem (section 4.3).

(1) Confidence intervals for the unknown p of the binomial distribution,
$$P(|p - \hat{p}| > \varepsilon) \geq 2\Phi(2\varepsilon\sqrt{n}) - 1$$
(2) Various examples with confidence intervals
(3) Maximum likelihood estimate
(4) One more example of the use of Central Limit Theorem (airplane overbooking problem)

14. 10/13

Poisson distribution, Poisson process, exponential distribution. Sections 4.4 – 4.6.

(1) Poisson distribution
(2) Poisson distribution: mean and variance
(3) Poisson approximation to the binomial distribution
(4) Poisson approximation vs normal approximation

(5) Poisson process — idea from bus arrivals
(6) Exponential distribution, derivation from the bus arrival process
(7) Exponential distribution: mean and variance

15. 10/15

Exponential distribution, Poisson process, gamma distribution (sections 4.4–4.6).

(1) Exponential distribution. Definition and examples
(2) Memorylessness of the exponential distribution
(3) Exponential as a limit of geometric distribution (connection to the coin-flipping setup)
(4) Poisson process on the line, rigorous definition
(5) Poisson process, example
(6) Poisson process in space (illustration)
(7) Gamma distribution from Poisson process
(8) Binomial distribution from Poisson process

16. 10/20

Finishing up chapter 4, and surveying the material of chapter 5 (which we mostly skip). Preparing for chapter 6.

(1) Expectation and variance of the Gamma distribution
(2) Section 5.1 — moment generating function and its usefulness in computing all moments $\mathbb{E}(X^n)$ of the random variable.
(3) Section 5.2 — Distribution of a function of a random variable. How the pdf is transformed.
(4) Overview of the topics of chapter 6. Beginning of discrete joint distributions.

17. 10/27

Joint distributions. Sections 6.1–6.2.

(1) Joint pmf (discrete random variables)
(2) Multinomial distribution
(3) Examples on the multinomial distribution
(4) Joint pdf (continuous random variables)
(5) Nonexistence of some joint pdfs (example with $X = Y$)
(6) Example. Crash course on double integrals $\iint$
(7) Marginal pdfs and one more example

18. 10/29

Joint distributions. Sections 6.2–6.3.

(1) Joint uniform distribution in a region D of $\mathbb{R}^2$ or higher dimensions.
(2) Examples. Marginal distributions.
(3) Joint pdf/pmf and independence.
(4) Functions of independent collections. Minimum and maximum.
(5) Minimum and maximum of uniform and exponential random variables.
(6) Product rule for expectations.