MATH 3100 SPRING 2022. LECTURE SUMMARIES

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Lecture 1. January 19

Introductory lecture, just saying hello to the students.

Lecture 2. No deadline

Section 1.1

A recording of an in-class piece, explaining the definition of the probability space Ω, events $\mathcal{F}$, and probability measure $P(A)$.

Lecture 3. January 21

Sections 1.1–1.2.

- Recalling $(\Omega, \mathcal{F}, P)$.
- Finite sample spaces, when it is enough to specify $P(\{\omega\})$ for each singleton $\omega \in \Omega$. We only need to have $P(\{\omega\}) \geq 0$ and $\sum_{\omega\in\Omega} P(\{\omega\}) = 1$
- Biased coin, 2 flips — example
- Equally likely outcomes, $P(A) = \frac{\#A}{\#\Omega}$.
- Example with urns.

Lecture 4. January 24

Section 1.2. Discussing 3 settings, when we sample k times from n objects. This is an instance of equally likely outcomes. Recall the factorial:

$$\begin{cases} n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n, \quad n \geq 1 \\ 0! = 1. \end{cases}$$

So, $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and so on.

- Sampling with replacement, order matters: $\#\Omega_k = n^k$.
- Sampling without replacement, order matters: $\#\Omega_k = n(n-1)\ldots(n-k+1) = \dfrac{n!}{(n-k)!}$.
- Sampling without replacement, order does not matter:

$$\#\Omega_k = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k!)} = \binom{n}{k}.$$

The quantity $\binom{n}{k}$ has a special name, "binomial coefficient".

Lecture 5. January 26

Section 1.3.

- Probability space for rolling a die until you see a "6"
- How to sum the geometric progression
- Continuous sample spaces, $[0,1]$ with length measure
- How to draw a random line in the plane (one example)

Lecture 6. January 28

Section 1.4. Rules of probability (theory only).

- Venn diagrams. Probability "behaves like the area of a Venn diagram"
- Decomposing an event
- Complements, also $P(A) = P(AB) + P(AB^c)$
- Monotonicity of probability
- Inclusion-exclusion

Lecture 7. January 31

Section 1.4. Examples on rules of probability — inclusion/exclusion principle; helpful passing to the complement event A^c.

A first look on random variables: if we flip a fair coin 5 times, and let Y be the number of Heads, what is the probability distribution of Y? This is described in terms of the probability mass function, which in this case takes the form

$$P(Y=0) = \frac{1}{32}, \quad P(Y=1) = \frac{5}{32}, \quad P(Y=2) = \frac{10}{32}, \quad P(Y=3) = \frac{10}{32}, \quad P(Y=4) = \frac{5}{32}, \quad P(Y=5) = \frac{1}{32}.$$

Lecture 8. Feb 2

Section 1.5. Random variables.

- Definition of a random variable
- Splitting of Ω into level sets
- Discrete, continuous, and other random variables
- Examples of discrete random variables
- Probability mass function (pmf)
- Distribution of a random variable, example of two random variables with the same distribution which are not the same.

Lecture 9. February 4

Sections 2.1, 2.2. Conditional probabilities, Bayes rule.

- Conditional probability — motivating example
- Definition of $P(A \mid B)$
- Chain product of conditional probabilities
- Total probability formula (averaged conditional probabilities)
- Bayes rule
- An example, testing for rare diseases

Lecture 10. February 7

Section 2.3. Independence.

- Recall Bayes rule. One more example.
- Independence of two events.
- Independence of A, B is equivalent to the independence of A^c, B and of A^c, B^c.
- Examples 2.18, 2.21
- Independence of multiple events means we need lots of product rules.
- Independence of random variables. Coin flips. Digits in uniformly random $\omega \in [0, 1]$.

Lecture 11. February 11

Sections 3.1, 3.2.

- Discrete random variables. They are determined through pmf. Pmf properties (2) - non-negativity, sum to one. Probability through pmf as a sum.
- Continuous random variables, in the same fashion. Probability density function. Properties of pdf (2); Probability through integrals.
- Cumulative distribution function (cdf).
  - definition
  - how it looks for continuous r.v.; connection to pdf
  - how it looks for discrete r.v.

  Some examples.

Lecture 11.1. February 11

This is an example of finding cdfs of random variables (X, Y) which are coordinates of a uniformly random point thrown into some figure in the plane.

Lecture 12. February 14

Section 3.3. Expectation.

- Expectation - discrete rv
- Expectation - continuous rv
- Example - Bernoulli; binomial without proof
- Example - geometric
- Derivative technique
- Use derivatives to get expectation of geometric
- Application to binomial distribution.
- One more example of the computation of expectation for a continuous random variable.

Lecture 13. February 21

Sections 3.1–3.4. Expectation, variance, derivative method.

- Expectation of a function of a random variable. Discussion and definitions for discrete and continuous cases.
- Variance. Definition, discussion.
- Two formulas for the variance. If $E(X) = \mu$, then
  $$Var(X) = E\left((X-\mu)^2\right) = E\left(X^2\right) - \mu^2.$$
  The first formula explains why variance is nonnegative, and the second formula is more practical for computation of the variance.

- Discussion of the derivative method. Application to compute

$$E(X), \qquad E\left(\frac{1}{1+X}\right)$$

for the Poisson random variable, where $p_X(k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$.

Lecture 14. February 25

Sections 3.5, 4.1. Normal (= Gaussian) distribution, normal approximation.

- (neat thing) Expectation through cdf $F_X(x)$
- Linearity of E, transformations of Var
- Normal distribution (standard)
- Normal table
- Normalization, general normal distribution
- Approximation of the binomial distribution — Central Limit Theorem (CLT)
- On the proof of CLT

Lecture 15 part 1. March 2

Sections 4.1, 4.2.

- Recall Central Limit Theorem.
- Law of Large Numbers. Discussion: frequency interpretation of probability.
- Law of Large Numbers. Proof from Central Limit Theorem.

Lecture 15 part 2. March 2

Section 4.4 (beginning). Poisson approximation of the binomial distribution — computation of the limit of $P(S_n = k)$ as $n \to +\infty$, $p \to 0$, $np \to \lambda$ (where $\lambda > 0$ is a fixed real number), and k is fixed.

Lecture 16. March 4

Section 4.4 and 4.5. Poisson random variable, exponential random variable. Their pmf (Poisson) and pdf (exponential); expectation, and variance.

If $X \sim \text{Poiss}(\lambda)$ and $T \sim \text{Exp}(\lambda)$, then

$$E(X) = Var(X) = \lambda, \qquad E(T) = \frac{1}{\lambda}, \qquad Var(T) = \frac{1}{\lambda^2}.$$

Lecture 17. March 14

Sections 4.4–4.6. Poisson process from coin flipping.

- Two descriptions of the coin-flipping process: through number of heads having binomial distribution; and through inter-heads intervals which have geometric distribution.
- Limit as $n \to \infty$ and $p = \lambda/n$. Geometric distribution becomes exponential, and binomial distribution becomes Poisson.
- The whole coin-flipping process in the scaling limit turns into a new device, the Poisson process. Poisson process is a random collection of points on the nonnegative real line, with Poisson and exponential distributions built into its definition.

Lecture 18. March 16

Sections 4.5–4.6 and 7.3. Poisson processes I.

- Poisson process — definition through point-count random variables N_A.
- Extracting exponential distributions from Poisson process.

Lecture 19. March 18

Sections 4.5–4.6 and 7.3. Poisson process II.

- From Poisson process to gamma distributions.
- Emergence of uniform and binomial distributions from Poisson process.
- Remark about Poisson process in space.

Lecture 20. March 21

Section 5.2 — distribution of a function of a random variable.

Lecture 20. March 21. Part 2

Queuing systems — stochastic modeling via Poisson processes.

Lecture 21. March 23

Joint distributions of random variables. Beginning chapter 6 in the textbook.

- Joint pdf of two random variables
- Independence in terms of joint pdf
- $P(a < X \leq b, c < Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y)dxdy$
- Application to a joint distribution of (T, W) in a Poisson process (where T is the time between bus 1 and bus 3, and W is the time between bus 3 and bus 4)
- Discrete pmf
- Marginal pmfs
- Independence in terms of pmfs

Lecture 22. March 25

Joint distributions, discrete and continuous. Chapter 6 of the textbook.

- Main formulas about discrete and continuous joint distributions:
  1. Integrate to 1
  2. Marginal distributions
  3. Probabilities of events
  4. Expectation of a function of a random variable
- Double integral via iterated integrals, general algorithm.
- Double integral via iterated integrals — example of a triangle.

Lecture 23. April 1

Joint distributions and independence. Section 7.1

- Independent random variables, how does pmf and pdf look like
- Example
- pmf of X+Y - convolution of discrete distributions
- example with Poisson (which also follows from Poisson process)
- pdf of X+Y - convolution of continuous distributions

- example with sum of uniform, and sum of exponentials

Lecture 24. April 4

Convolution of normals. Indicators.

- Sum of independent normal random variables is always normal. Proof for the case of equal variances
- Indicator random variables. Definition.
- Indicator random variables. Main property: $E(I_A + I_B) = E(I_A) + E(I_B)$
- An example: solving a problem using indicators.

Lecture 26. April 6

Linearity of expectation. Section 8.1

- Proof of linearity of expectation for joint pdfs
- Proof of linearity of expectation for joint pmf (both random variables discrete), short discussion
- Proof of linearity of expectation using indicators and simple random variables, using approximation.

Lecture 26. April 8

Expectation, covariance, variance, and independence

- Product rule for expectations
- Counterexample for product rule for expectations
- Covariance (definition)
- Variance of the sum of independent random variables
- Application to binomial distribution.

Lecture 27. April 11

Covariance and correlation (Section 8.4). Markov inequality (Section 9.1).

- Example, covariance of I_A, I_B.
- Bilinearity of covariance.
- Cauchy–Schwartz inequality
$$|Cov(X,Y)| \leq \sqrt{Var(X) \cdot Var(Y)}$$
- Correlation coefficient
- Markov inequality: For $X \geq 0$ and $a > 0$, we have
$$P(X \geq a) \leq \frac{E(X)}{a}.$$