

MATH 3100 SPRING 2022. LECTURE SUMMARIES

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LECTURE 1. JANUARY 19

Introductory lecture, just saying hello to the students.

LECTURE 2. NO DEADLINE

Section 1.1

A recording of an in-class piece, explaining the definition of the probability space Ω , events \mathcal{F} , and probability measure $P(A)$.

LECTURE 3. JANUARY 21

Sections 1.1–1.2.

- Recalling (Ω, \mathcal{F}, P) .
- Finite sample spaces, when it is enough to specify $P(\{\omega\})$ for each singleton $\omega \in \Omega$. We only need to have $P(\{\omega\}) \geq 0$ and $\sum_{\omega \in \Omega} P(\{\omega\}) = 1$
- Biased coin, 2 flips — example
- Equally likely outcomes, $P(A) = \frac{\#A}{\#\Omega}$.
- Example with urns.

LECTURE 4. JANUARY 24

Section 1.2. Discussing 3 settings, when we sample k times from n objects. This is an instance of equally likely outcomes. Recall the factorial:

$$\begin{cases} n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n, & n \geq 1 \\ 0! = 1. \end{cases}$$

So, $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and so on.

- Sampling with replacement, order matters: $\#\Omega_k = n^k$.
- Sampling without replacement, order matters: $\#\Omega_k = n(n-1) \dots (n-k+1) = \frac{n!}{(n-k)!}$.
- Sampling without replacement, order does not matter:

$$\#\Omega_k = \frac{n(n-1) \dots (n-k+1)}{k!} = \frac{n!}{k! (n-k)!} = \binom{n}{k}.$$

The quantity $\binom{n}{k}$ has a special name, “binomial coefficient”.

LECTURE 5. JANUARY 26

Section 1.3.

- Probability space for rolling a die until you see a “6”
- How to sum the geometric progression
- Continuous sample spaces, $[0, 1]$ with length measure
- How to draw a random line in the plane (one example)

LECTURE 6. JANUARY 28

Section 1.4. Rules of probability (theory only).

- Venn diagrams. Probability “behaves like the area of a Venn diagram”
- Decomposing an event
- Complements, also $P(A) = P(AB) + P(AB^c)$
- Monotonicity of probability
- Inclusion-exclusion

LECTURE 7. JANUARY 31

Section 1.4. Examples on rules of probability — inclusion/exclusion principle; helpful passing to the complement event A^c .

A first look on random variables: if we flip a fair coin 5 times, and let Y be the number of Heads, what is the probability distribution of Y ? This is described in terms of the probability mass function, which in this case takes the form

$$P(Y = 0) = \frac{1}{32}, \quad P(Y = 1) = \frac{5}{32}, \quad P(Y = 2) = \frac{10}{32}, \quad P(Y = 3) = \frac{10}{32}, \quad P(Y = 4) = \frac{5}{32}, \quad P(Y = 5) = \frac{1}{32}.$$

LECTURE 8. FEB 2

Section 1.5. Random variables.

- Definition of a random variable
- Splitting of Ω into level sets
- Discrete, continuous, and other random variables
- Examples of discrete random variables
- Probability mass function (pmf)
- Distribution of a random variable, example of two random variables with the same distribution which are not the same.

LECTURE 9. FEBRUARY 4

Sections 2.1, 2.2. Conditional probabilities, Bayes rule.

- Conditional probability — motivating example
- Definition of $P(A | B)$
- Chain product of conditional probabilities
- Total probability formula (averaged conditional probabilities)
- Bayes rule
- An example, testing for rare diseases

LECTURE 10. FEBRUARY 7

Section 2.3. Independence.

- Recall Bayes rule. One more example.
- Independence of two events.
- Independence of A, B is equivalent to the independence of A^c, B and of A^c, B^c .
- Examples 2.18, 2.21
- Independence of multiple events means we need lots of product rules.
- Independence of random variables. Coin flips. Digits in uniformly random $\omega \in [0, 1]$.

LECTURE 11. FEBRUARY 11

Sections 3.1, 3.2.

- Discrete random variables. They are determined through pmf. Pmf properties (2) - non-negativity, sum to one. Probability through pmf as a sum.
- Continuous random variables, in the same fashion. Probability density function. Properties of pdf (2); Probability through integrals.
- Cumulative distribution function (cdf).
 - definition
 - how it looks for continuous r.v.; connection to pdf
 - how it looks for discrete r.v.

Some examples.

LECTURE 11.1. FEBRUARY 11

This is an example of finding cdfs of random variables (X, Y) which are coordinates of a uniformly random point thrown into some figure in the plane.

LECTURE 12. FEBRUARY 14

Section 3.3. Expectation.

- Expectation - discrete rv
- Expectation - continuous rv
- Example - Bernoulli; binomial without proof
- Example - geometric
- Derivative technique
- Use derivatives to get expectation of geometric
- Application to binomial distribution.
- One more example of the computation of expectation for a continuous random variable.

LECTURE 13. FEBRUARY 21

Sections 3.1–3.4. Expectation, variance, derivative method.

- Expectation of a function of a random variable. Discussion and definitions for discrete and continuous cases.
- Variance. Definition, discussion.
- Two formulas for the variance. If $E(X) = \mu$, then

$$\text{Var}(X) = E((X - \mu)^2) = E(X^2) - \mu^2.$$

The first formula explains why variance is nonnegative, and the second formula is more practical for computation of the variance.

- Discussion of the derivative method. Application to compute

$$E(X), \quad E\left(\frac{1}{1+X}\right)$$

for the Poisson random variable, where $p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \dots$

LECTURE 14. FEBRUARY 25

Sections 3.5, 4.1. Normal (= Gaussian) distribution, normal approximation.

- (neat thing) Expectation through cdf $F_X(x)$
- Linearity of E , transformations of Var
- Normal distribution (standard)
- Normal table
- Normalization, general normal distribution
- Approximation of the binomial distribution — Central Limit Theorem (CLT)
- On the proof of CLT

LECTURE 15 PART 1. MARCH 2

Sections 4.1, 4.2.

- Recall Central Limit Theorem.
- Law of Large Numbers. Discussion: frequency interpretation of probability.
- Law of Large Numbers. Proof from Central Limit Theorem.

LECTURE 15 PART 2. MARCH 2

Section 4.4 (beginning). Poisson approximation of the binomial distribution — computation of the limit of $P(S_n = k)$ as $n \rightarrow +\infty$, $p \rightarrow 0$, $np \rightarrow \lambda$ (where $\lambda > 0$ is a fixed real number), and k is fixed.

LECTURE 16. MARCH 4

Section 4.4 and 4.5. Poisson random variable, exponential random variable. Their pmf (Poisson) and pdf (exponential); expectation, and variance.

If $X \sim \text{Poiss}(\lambda)$ and $T \sim \text{Exp}(\lambda)$, then

$$E(X) = Var(X) = \lambda, \quad E(T) = \frac{1}{\lambda}, \quad Var(T) = \frac{1}{\lambda^2}.$$

LECTURE 17. MARCH 14

Sections 4.4–4.6. Poisson process from coin flipping.

- Two descriptions of the coin-flipping process: through number of heads having binomial distribution; and through inter-heads intervals which have geometric distribution.
- Limit as $n \rightarrow \infty$ and $p = \lambda/n$. Geometric distribution becomes exponential, and binomial distribution becomes Poisson.
- The whole coin-flipping process in the scaling limit turns into a new device, the Poisson process. Poisson process is a random collection of points on the nonnegative real line, with Poisson and exponential distributions built into its definition.

LECTURE 18. MARCH 16

Sections 4.5–4.6 and 7.3. Poisson processes I.

- Poisson process — definition through point-count random variables N_A .
- Extracting exponential distributions from Poisson process.

LECTURE 19. MARCH 18

Sections 4.5–4.6 and 7.3. Poisson process II.

- From Poisson process to gamma distributions.
- Emergence of uniform and binomial distributions from Poisson process.
- Remark about Poisson process in space.

LECTURE 20. MARCH 21

Section 5.2 — distribution of a function of a random variable.

LECTURE 20. MARCH 21. PART 2

Queuing systems — stochastic modeling via Poisson processes.

LECTURE 21. MARCH 23

Joint distributions of random variables. Beginning chapter 6 in the textbook.

- Joint pdf of two random variables
- Independence in terms of joint pdf
- $P(a < X \leq b, c < Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y) dx dy$
- Application to a joint distribution of (T, W) in a Poisson process (where T is the time between bus 1 and bus 3, and W is the time between bus 3 and bus 4)
- Discrete pmf
- Marginal pmfs
- Independence in terms of pmfs

LECTURE 22. MARCH 25

Joint distributions, discrete and continuous. Chapter 6 of the textbook.

- Main formulas about discrete and continuous joint distributions:
 - (1) Integrate to 1
 - (2) Marginal distributions
 - (3) Probabilities of events
 - (4) Expectation of a function of a random variable
- Double integral via iterated integrals, general algorithm.
- Double integral via iterated integrals — example of a triangle.

LECTURE 23. APRIL 1

Joint distributions and independence. Section 7.1

- Independent random variables, how does pmf and pdf look like
- Example
- pmf of $X+Y$ - convolution of discrete distributions
- example with Poisson (which also follows from Poisson process)
- pdf of $X+Y$ - convolution of continuous distributions

- example with sum of uniform, and sum of exponentials

LECTURE 24. APRIL 4

Convolution of normals. Indicators.

- Sum of independent normal random variables is always normal. Proof for the case of equal variances
- Indicator random variables. Definition.
- Indicator random variables. Main property: $E(I_A + I_B) = E(I_A) + E(I_B)$
- An example: solving a problem using indicators.

LECTURE 26. APRIL 6

Linearity of expectation. Section 8.1

- Proof of linearity of expectation for joint pdfs
- Proof of linearity of expectation for joint pmf (both random variables discrete), short discussion
- Proof of linearity of expectation using indicators and simple random variables, using approximation.

LECTURE 26. APRIL 8

Expectation, covariance, variance, and independence

- Product rule for expectations
- Counterexample for product rule for expectations
- Covariance (definition)
- Variance of the sum of independent random variables
- Application to binomial distribution.

LECTURE 27. APRIL 11

Covariance and correlation (Section 8.4). Markov inequality (Section 9.1).

- Example, covariance of I_A, I_B .
- Bilinearity of covariance.
- Cauchy–Schwarz inequality

$$|Cov(X, Y)| \leq \sqrt{Var(X) \cdot Var(Y)}$$

- Correlation coefficient
- Markov inequality: For $X \geq 0$ and $a > 0$, we have

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

LECTURE 28. APRIL 13

Sections 9.1–9.3.

- Markov inequality
- Chebyshev inequality
- Law of Large Numbers for arbitrary random variables (with proof)
- Central Limit Theorem for arbitrary random variables (without proof)

LECTURE 28_I. APRIL 15

Examples on the Central Limit Theorem

- Normal approximation to Poisson distribution for large λ
- Discussion of the continuity correction in the general Central Limit Theorem
- Normal approximation to $\text{Gamma}(n, \lambda)$ for fixed λ and large n
- Illustrations of approximations

LECTURE 29. APRIL 20

Conditioning of discrete random variables. Section 10.1

- Reading the formula for averaged conditional probabilities as an expectation.
- Events and random variables are almost the same concepts
- Conditioning a random variable on an event.
- $E(X) = E(E(X | B_Y)) = \sum_{j=1}^n E(X | B_j)P(B_j)$, formula for averaged conditional expectation.
- Example with computing the expectation and variance of the geometric random variable using conditioning (instead of infinite series!)
- Conditioning on a random variable Y .
- Main formula to rule them all: $E(X) = E(E(X | Y))$.

LECTURE 30. APRIL 22

Conditional distributions, discrete. Section 10.1

- Constructing joint pmf from marginal and conditional pmf.
- Example of conditioning Y on X “in the other direction” in the problem from April 20’s class.
- Example 10.12 in the textbook, discussed.

LECTURE 31. APRIL 25

Continuous conditional distributions. Section 10.2

- conditional density; definition
- check it’s a density, too!
- conditional density; example with triangle
- conditional probabilities and expectations in continuum (definitions)
- example 10.20

LECTURE 32. APRIL 27-29

The last lecture. Conditional distributions, section 10.

- Conditional expectations $E(g(X) | Y = y)$ in discrete and continuous setting — reminder
- Conditional expectation as a random variable
- Double averaging formula — proof in the continuous case
- Further properties of conditioning — linearity, independence, and $E(g(X)h(Y) | Y) = h(Y)E(g(X) | Y)$
- $E(X | X + Y)$ in the case of independent uniform
- $E(X_1 | S_n)$ in the general iid case
- $P(X > Y)$ for independent exponentials, using conditioning

- $E(X \mid \min(X, Y))$ for independent exponentials (I am not sure about this computation — please check my work)
- Flipping coins with unknown (uniform) probability of success