NOTES ON RANDOM MATRICES

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(notes by Bryce Terwilliger; . . .)

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1. Introduction

1.1. Nonrandom and random matrices and their eigenvalues. The study of random matrices as a field is a patchwork of many fields. The main object we study is a probability distribution on a certain subset of the set of matrices $\mathrm{Mat}(N \times N,\ \mathbb{R} \text{ or } \mathbb{C})$, thus giving us a random matrix A.

Definition 1.1. An *eigenvalue* λ of the matrix A is a root of the polynomial $f(\lambda) = \det(A - \lambda I)$. Equivalently, λ is an eigenvalue if A if the matrix $A - \lambda I$ is not invertible. This second way of defining eigenvalues in fact works even when A is not a finite size matrix, but an operator in some infinite-dimensional space.

We will largely be only concerned with real eigenvalues. That is the eigenvalues of a real symmetric matrix over $\mathbb{R}$ or Hermitian over $\mathbb{C}$ that is where $A^* = A$.

Remark 1.2. The case when eigenvalues can be complex is also studied in the theory of random matrices, sometimes under the keyword *complex random matrices*. See, for example, [GT10] for a law of large numbers for complex eigenvalues.

Proposition 1.3. *Every eigenvalue of a Hermitian matrix is real.*

Proof. Let A be a Hermitian matrix so that $A^* = A$ (here and everywhere below $A*$ means $\overline{A^{\mathrm{T}}}$, i.e., transposition and complex conjugation). Let λ be an eigenvalue of A. Let v be a non-zero vector in the null space of $A - \lambda I$. Let $a = \overline{v^{\mathrm{T}}}v = |v|^2$, so that a is a positive real number. Let $b = \overline{v^{\mathrm{T}}}Av$. Then $\bar{b} = \overline{b^{\mathrm{T}}} = \overline{\overline{v^{\mathrm{T}}}Av}^{\mathrm{T}} = \overline{v^{\mathrm{T}}}\bar{A}^{\mathrm{T}}v = \overline{v^{\mathrm{T}}}Av = b$, so b is real. Since $b = \lambda a$, λ must be real. $\square$

Let $\mathcal{H}_N$ be the set of $N \times N$ Hermitian matrices. For each N, let μ_N be a probability measure on $\mathcal{H}_N$ (it can be supported not by the whole $\mathcal{H}_N$, but by a subset of it, too). Then for each matrix $A \in \mathcal{H}_N$ we may order the real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_N$ of A.

A collection of probability measures μ_N on $\mathcal{H}_N$ for each $N \geq 1$ is said to be a *random matrix ensemble*. For such an ensemble, the eigenvalues $\lambda_1^{(N)} \geq \cdots \geq \lambda_N^{(N)}$ of matrices N form random point configurations on $\mathbb{R}$ with growing numbers of points. Our main goal is to study the asymptotic properties of these collections of points on $\mathbb{R}$, as $N \to \infty$.

Date: January 23, 2016.

References

e2010circular [GT10] F. Götze and A. Tikhomirov, *The circular law for random matrices*, Ann. Probab. **38** (2010), no. 4, 1444–1491, arXiv:0709.3995 [math.PR].