RM LECTURES 2/1 AND 2/3

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Lecture #1 on 2/1/2016

1. Elements of free probability

probability⟩?

Question: Suppose A_N and B_N are ensembles of (real symm) random matrices with $L_N(A_N B) \to \mu$ and $L_N(B_N) \to \nu$.[1]

$$L_N(A_N + B_N) \to ?$$

Example 1.1 (Real Wigner)**.** Suppose A_N and B_N are independent and real Wigner. Then $L_N(A_N) \to \mathsf{SC}$ and $L_N(B_N) \to \mathsf{SC}$. Then, $\dfrac{A_N + B_N}{\sqrt{2}}$ is real Wigner and so

$$\left(\frac{A_N + B_N}{\sqrt{2}}\right) \to \mathsf{SC}.$$

In general $Ł_N(A_N + B_N) \to \mu \boxplus \nu$, the **free convolution** of μ and ν. For A_N and B_N real Wigner,

$$\frac{\mathsf{SC} \boxplus \mathsf{SC}}{\sqrt{2}} = \mathsf{SC}.$$

In classical probability, if X and Y are iid random variables with $X + Y \overset{D}{=} cX$ for some constant c, X is called **stable**. An important such example is Gaussian random variables. Indeed, if X, Y are iid Gausian random variables, $N(0, 1)$. Then, $X + Y = \sqrt{2}X$. We will try to further draw out the analogy between SC and Gaussian distributions by looking at moments and cumulants.

1.1. **Classical Moments and Cumulants.** Given a random variable X, the n^{th} moment of X is given by

$$m_n(X) = \mathbb{E}(X^n) \text{ for } n \geq 0.$$

The **cumulants** (or half(semi)-invariants) of X form a sequence (c_n), where c_n is a homogeneous polynomial of moments m_k, $k \leq n$.

$$\begin{aligned}
c_0 &= 0 \\
c_1 &= m_1 = \mathbb{E}\, X \\
c_2 &= m_2 = \mathrm{Var}(X) = m_2 - m_1^2 \\
c_3 &= \text{skewness} = m_3 - 3m_2m_1 + 2m_1^3 \\
c_4 &= \text{kurtosis} = m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4 \\
&\vdots
\end{aligned}$$

[1] There are multiple scaling conventions: $L_N = \dfrac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i/\sqrt{N}}$ scales the spectrum. Other conventions scale entries.

Definition 1.2. The **Moment Generating Function** for a random variable X is given by

$$M(z) = \sum_{n=0}^{\infty} \frac{m_n(X) z^n}{n!} \tag{1.1}$$ Moment_Gen

The **Cumulant Generating Function** for X is

$$C(z) = \sum_{n=1}^{\infty} \frac{c_n(X) z^n}{n!} = \log(M(z)) \tag{1.2}$$ Cumulant_Gen

So, $M(z) = exp(C(z))$.

Now, if X and Y are independent random variables, then

$$c_2(X+Y) = \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = c_2(X) + c_2(Y).$$

In fact, this holds for all cumulants:

Proposition 1.3. *If X and Y are independent random variables, then for all $n \geq 0$,*

$$c_n(X+Y) = c_n(X) + c_n(Y). \tag{1.3}$$ lin_cumulant

Proof. Assume everything is nice for ind random variables X and Y. Then,

$$\begin{aligned} M_X(z) &= \mathbb{E}\, e^{zX} \\ &\Rightarrow \mathbb{E}(e^{zX+zY}) = \mathbb{E}\, e^{zX}\, \mathbb{E}\, e^{zY} \\ &\Rightarrow \log M_{X+Y}(z) = \log M_X(z) + \log M_Y(z) \end{aligned}$$

□

So, the cumuluants can be used to "linearize" moments for independent random variables. This isn't a true linearization because, for a random variable X and $\alpha \in \mathbb{R}$d,

$$c_n(\alpha X) = \alpha^n c_n(X). \tag{1.4}$$ lin_cumulant

To showcase the utility of the cumulants, we use them to prove the CLT:

⟨classic_clt⟩ **Theorem 1.4** (CLT)**.** *Let $X_1, X_2, ...$ be iid random variables with $\mathbb{E}\, X_i = 0$ and $\mathrm{Var}(X_i) = \mathrm{Var}(X_1) < \infty$ for all $i \geq 1$. Then,*

$$S_N = \frac{X_1 + ... + X_N}{\sqrt{N}} \xrightarrow{D} N(0, \mathrm{Var}(X_1)).$$

sketch. Note that, by (1.3) and (1.4), for each n and N,

$$c_n(S_N) = N^{-n/2} N c_n(X_1).$$

In particular, we have the following values: [2]

$$\begin{aligned} &n = 1 : c_1(S_N) = \mathbb{E}(X_1) = 0 \\ &n = 2 : c_2(S_N) = c_2(X_1) = \mathrm{Var}(X_1) \\ &n \geq 3 : c_n(S_N) \xrightarrow{N \to \infty} 0 \end{aligned}$$

So, the cumulant sequence of $\lim_N S_N$ is

$$(0, \mathrm{Var}(X_1), 0, 0, ...) \tag{1.5}$$ Cumulant_seq

[2]The argument for $c_n(S_N)$ vanishing in the limit is similar to the one from (**??**).

On the other hand, the moment generating function, (1.1), of $N(0,1)$ is

$$\begin{aligned} M(z) &= \int_{-\infty}^{\infty} e^{zX} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \\ &= e^{z^2/2} \int_{-\infty}^{\infty} \frac{e^{-(x-z)^2/2}}{\sqrt{2\pi}}\, dx \\ &= e^{z^2/2}. \end{aligned}$$

Hence, the cumulant generating fuction, (1.2), of $N(0,1)$ is

$$C(z) = z^2/2.$$

Whence, the cumulant sequence of $N(0,1)$ is

$$(0, 1, 0, 0, ...). \tag{1.6}$$ Cumulant_Seq

By comparing (1.5) and (1.6), we see that the cumulant sequences of S_N and $N(0,1)$ are the same up to a constant, which implies the CLT. □

As mentioned in the proof, the moment generating function of $N(0,1)$ is

$$M(z) = e^{z^2/2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{2^n n!} = \sum_{k=0}^{\infty} \frac{m_k z^k}{k!}.$$

Hence, $N(0,1)$ has moments

$$\begin{aligned} m_{2n+1} &= 0 \\ m_{2n} &= \frac{(2n)!}{2^n n!} = (2n-1)!! := (2n-1)(2n-3)\cdots 3\cdot 1 \end{aligned}$$

Furthermore, we know from (**??**) that the SC distribution has moments

$$\begin{aligned} m_{2n+1} &= 0 \\ m_{2n} &= \mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n} \end{aligned}$$

where Cat_n is the n^{th} Catalan number. Now, we have the following correspondence:

Classical:	moments	$\longleftrightarrow$	cumulants	$\longleftrightarrow$	$N(0,1)$ as simplest dist
RM:	moments	$\longleftrightarrow$	?	$\longleftrightarrow$	SC "simplest"

A natural approach would be to compute the cumulants of the SC distribution from its moments, but these aren't so nice. In fact,

$$\sum_{n=0}^{\infty} z^n \mathrm{Cat}_n = \frac{1-\sqrt{1-4z}}{2z},$$

and the moment generating function for SC is

$$M(z) = \sum_{n=0}^{\infty} \frac{\mathrm{Cat}_n z^n}{n!} = e^{2z}(I_0(2z) - I_1(2z))$$

where the I_i are modified Bessel functions:

$$I_\alpha(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\alpha+2n}}{n!\Gamma(n+1+\alpha)}.$$

We need a nicer analogue for cumulants.

1.2. **Free Cumulants.** Recall that $m_2(\text{Gauss}) = (2n-1)!!$ and $m_2n(\mathsf{SC}) = \text{Cat}_n$; we need to define an analogue to cumulants so that we can still express

$$\begin{array}{ccc} M(z) & & = exp(C(z)) \\ \| & & \| \\ \displaystyle\sum_{n=0}^{\infty} \frac{m_n z^n}{n!} & & = \sum_{m=1}^{\infty} \frac{(C(z))^m}{m!} \end{array}$$

Our aim is to get z^n from the RHS. To that end, fix m. Then,

$$\frac{1}{m!}(C(z))^m = \frac{\left(\sum_{i=0}^{m} \frac{c_i z^i}{i!}\right)^m}{m!}.$$

So, from each bracket, take a partition $i_1, ..., i_m$ of n, i.e. $i_1 + ... + i_m = n$. Then, we can write

$$m_n = \sum_{\substack{m \\ i_1,\ldots,i_m}} \frac{c_{i_1} \cdots c_{i_m} n!}{m! i_1! \cdots i_m!}. \tag{1.7}$$ moment_multi

Notice that $\dfrac{n!}{i_1! \cdots i_m!}$ is a multinomial coefficient.

Recalling $M(z) = exp(C(z))$, we re-write the n^{th} moment as follows:

$$m_n = \sum_{\pi \in \mathcal{P}(n)} \prod_{B \in \pi} c_{|B|} \tag{1.8}$$ cumulants_to

where $\mathcal{P}(n)$ is the set of all partitions of $\{1, ..., n\}$ and $B \in \pi$ are the blocks of the partition π. Notice that in (1.7), m gives the number of blocks and $i_1, ..., i_m$ are the size of each block. With (1.8), we can give a (recursive) formula from moments to cumulants:

$$c_n = m_n - \sum_{\substack{\pi \in \mathcal{P}(n) \\ \#\text{ of blocks} \geq 2}} \prod_{B \in \pi} c_{|B|}$$

We can also interpret from $M(z) = exp(C(z))$ that, if m_n counts "objects", then c_n counts "connected objects" (because every object is the union of connected objects).

ting_graphs⟩? **Example 1.5.** If the objects in question are graphs, then

$$\begin{aligned} m_n &= \#\text{ of graphs } = 2^{\binom{n}{2}} \\ c_n &= \#\text{ of connected graphs} \end{aligned}$$

Applying (1.8) to the Gaussian, $N(0,1)$, we have

$$m_{2n} = (2n-1)!! = \sum_{\pi \in \mathcal{P}(2n)} \prod_{B \in \pi} c_{|B|}$$

However, since $c_n = 0$ for $n \neq 2$ and $c_2 = 1$, the sum is only over pair partitions (i.e. all blocks have size 2). Hence,

$$m_{2n} = (2n-1)!! = \#\text{of pair paritions}$$

Applying (1.8) to SC, we have

$$\mathrm{Cat}_{n/2} = m_n = \sum_{\pi \in NC(n)} \prod_{B \in \pi} K_{|B|} \tag{1.9}$$ free_cumulan

(1.10) {?}

Where $NC(n)$ is the number of non-crossing partitions of $\{1, ..., n\}$. Recall that Cat_n is the number of non-crossing pair partitions of $\{1, ..., n\}$. Hence, $k_{|B|} = 0$ unless $|B| = 2$.

In general, we define the **free cumulants** k_m of a rv by relating them to moments by

$$m_n = \sum_{\pi \in NC(n)} \prod_{B \in \pi} K_{|B|} \tag{1.11}$$?free_cumular

This yields a free cumulant sequence $(k_1, k_2, ...)$.

(Insert Figure Here)

In terms of counting graphs, k_n counts the "geometrically connected graphs" (i.e. a graph connected as a geometric object but not necessarily as a graph.)

(Insert Figure Here)

For SC, we see from (1.9) that the k_n sequence is $(0, 1, 0, 0, ...)$. Indeed, we know from (1.9) that $k_{|B|} = 0$ for $|B| \neq 2$. $k_2 = 1$ because

$$1 = \mathrm{Cat}_1 = m_2 = \sum_{\pi \in NC(2)} \prod_{B \in \pi} k_{|B|} = k_2.$$

Note that, because all partitions are non-crossing for $n \leq 3$, we have that

$$\begin{aligned} c_1 &= k_1 \\ c_2 &= k_2 \\ c_3 &= k_3. \end{aligned}$$

1.2.1. *Mixed Cumulants.* Recall that for for random variables, $X_1, ..., X_k$, we can write

$$m_n(X_1, ..., X_n) = \mathbb{E}[X_1, ..., X_n] = \sum_{\pi \in \mathcal{P}(n)} \prod_{B \in \pi} c_{|B|}(X_i; i \in B).$$

To adapt this to free cumulants, we replace $\mathcal{P}(n)$ with $NC(n)$ and $c_{|B|}(X_i : i \in B)$ with $K_{|B|}(X_i : i \in B)$.

1.3. **Free Independence.** Recall that in the classical case, for independent random variables X and Y and each $n \geq 1$,

$$c_n(X + Y) = c_n(X) + c_n(Y).$$

So, we'd like to define free independence such that the same holds for free cumulans, i.e. if X and Y are free independent random variables, then for all $n \geq 1$

$$k_n(X + Y) = k_n(X) + k_n(Y).$$

Note that this will not be the same as regular independence. Indeed, SC is not stable, but if you have two SC free independent random variables, then

$$X + Y \stackrel{D}{=} SC.$$

Exercise 1.6. X and Y are (classicaly) independent) random variables iff $c_n(X, X, ...X, Y, ..., Y) = 0$ and all joint mixed cumulants are zero.

This leads us to a natural definition of free independence.

Definition 1.7. X and Y are **free independent** if $k_n(X, X, ..., X, Y, ..., Y) = 0$ and all joint mixed free cumulants are zero.

However, free independence of classical random variables is problematic. To see this, consider free independent classical random variables X and Y. Then,

$$m_4(X, X, Y, Y) = m_4(X, Y, X, Y)$$

where

$$m_4(X, X, Y, Y) = k_2(X, X)k_2(Y, Y) + k_2(X, X) + k_1(Y)^2 + k_2(Y, Y)k_1(X)^2 + k_1(X)^2k_1(Y)^2$$

and

$$m_4(X, Y, X, Y) = k_2(X, X)k_1(Y)^2 + k_1(X)^2k_2(Y, Y) + k_1(X)^2k_2(Y)^2.$$

Hence $k_2(X, X)k_2(Y, Y) = 0$, which means that, for classical random variables, free independence requires one to be constant. The solution is non-commutivity, whence

$$m_4(X, X, Y, Y) \neq m_4(X, Y, X, Y).$$

1.4. **Free Random Variables.** The idea for free random variables is due to Voiculescu (1982) [VDN92]. The space $L^\infty(\Omega, \mathcal{F}, P)$ is a commutative von Neumann algebra. Hence, a natural definition for a non-commutative (or free) random variable is an element of a non-commutative von Neumann Algebra.

To define a von Neuman algebra, first, let H be a Hilbert space (complete inner product space) and $B(H)$ the space of all bounded linear operators on H (bounded with respect to the norm induced by the inner product $(\cdot|\cdot)$ on H). $B(H)$ has a natural involution given by

$$(Tx|y) = (x|T^*y)$$

for any $T \in B(H)$ and $x, y \in H$. In addition to the topology on $B(H)$ induced by the norm, there is a **weak operator topology**. We say that for $T_n, T \in B(H)$, $T_n \to T$ in the weak operator topology if for any $x, y \in H$

$$(T_nx|y) \to (Tx|y).$$

Now, we may define a von Neumann Algebra

Definition 1.8. A **von Neumann algebra** is a unital $*$-closed subalgebra of $B(H)$, which is closed with respect to the weak operator topology.

So, we'd like to replace $L^\infty(\Omega, \mathcal{F}, P)$ with a von Neumann algebra A.

1.5. **The Trace as Expectation.** In lieu of classical expectation, we have a trace on our von Neumann algebra.

Definition 1.9. A **trace** on a von Neumann algebra A is a functional $\tau : A \to \mathbb{C}$ such that

(1) $\tau(1) = 1$ (where $1 \in A$ is the identity element)
(2) $\tau(ab) = \tau(ba)$ for every $a, b \in A_+$
(3) $\tau(a^*a) \geq 0$ for every $a \in A$.

The trace is **faithful** if $\tau(a) = 0$ iff $a = 0$.

So, we have the following correspondences:

Classical	Free
$L^\infty(\Omega, \mathcal{F}, P)$	a von Neumann algebra A
$\mathbb{E}[\cdot]$	$\tau[\cdot]$
$m_n(X)$	$m_n(a) = \tau(a^n), a \in A_+$

Now, we define (free) independence for free random variables.

Definition 1.10. Elements a and b in a von Neumann algebra A are **freely independent** if for all polynomials $f_1, g_1, ..., f_k, g_k$ such that $\tau(f_i(a)) = \tau(g_j(b)) = 0$, we have that

$$\tau(f_1(a)g_1(b)f_2(a)g_2(b)\cdots g_k(b)) = 0.$$

Note that this is essentially requiring that joint mixed cumulants are zero.

Definition 1.11. Elements a and b in a von Neumann algebra A are **classicaly independent** if $ab = ba$ and for all polynomials f and g such that $\tau(f(a)) = \tau(g(b)) = 0$, we have that

$$\tau(f(a)(b)) = 0.$$

Definition 1.12. For a sequence (A_n, τ_N) of von Neumann algebras with traces and sequences (a_N) and (b_N) with $a_N, b_N \in A_N$ for each N, we say that (a_N) and (b_N) are **asymptotically free** if for all polynomials $f_1, g_1, ..., f_k, g_k$ such that $\tau(f_i(a_N)) \to 0$ and $\tau(g_j(b_N)) \to 0$, we have that

$$\tau(f_1(a_N)g_1(b_N)f_2(a_N)g_2(b_N)\cdots g_k(b_N)) \to 0.$$

ree_ensemble⟩ **Example 1.13.** Take the von Neumann algebras A_N to be collection of NxN Hermitian random matrices (i.e. the entries a_{ij} are bounded random variables) with traces

$$\tau_N = \frac{1}{N}\,\mathbb{E}\,\mathrm{tr}(A),$$

and consider sequences $(X_N), (Y_N)$ of NxN (Hermitian) random matrices with

$$L_N(X_N) \to \mu L_N(Y_N) \to \nu$$

where the limiting spectral densities have bounded support. For each N, let U_N be a Haar-distributed unitary matrix that is independent of X_N and Y_N. Then, X_N and $U_N Y_N U_N^*$ are asymptotically free.

1.6. **Free CLT and the SC Law (for Gaussian Matrices).**

?⟨free_clt⟩? **Theorem 1.14** (Free CLT)**.** *Let $X_1, ...$ be a free family of id random variables in (A, τ).*

$$\frac{X_1 + ... + X_N}{\sqrt{N}} \to \mathsf{SC}.$$

That is,

$$\tau\left[\frac{X_1 + ... + X_n}{\sqrt{N}}\right] \to Cat_N.$$

Proof. Idea Use free cumulants as in 1.4. □

With this, we have a proof of the SC Law for Gaussian matrices.

Idea. Let $G_N^{(1)}, ..., G_N^{(N)}$ be independent copies of the Gaussian Hermitian random matrices. Then for $U_N^{(k)}, 1 \le k \le N$, Haar-distributed unitarie, $U_N^{(k)} G_N^{(k)} (U_N^{(k)})^* \sim G_N$, and further,

$$G_N \overset{D}{=} \frac{U_N^{(1)} G_N^{(1)} (U_N^{(1)})^* + ... + U_N^{(N)} G_N^{(N)} (U_N^{(N)})^*}{\sqrt{N}}.$$

Hence, we can take the G_N as a.s. free; so by the free CLT, G_N converge to an element with spectral density SC. □

1.7. **Free Convolution and Voiculescu's Algorithm.**

Definition 1.15. Let X_N and Y_N be random matrix ensembles with $L_N(X_N) \to \mu$ and $L_N(Y_N) \to \nu$ as in 1.13 where μ and ν have compact support. Then we define the **free convolution** of μ and ν by

$$\mu \boxplus \nu = \lim L_N(X_N + U_N Y_N U_N^*). \tag{1.12}$$?free_convolu

Note that $k_n(\mu \boxplus \nu) = k_n(\mu) + k_n(\nu)$ for each n.

Now, we'd like to have an algorithm for determining the free convolution of two spectram measures. To do this, we first take the Cauchy Transform each measure:

$$\begin{aligned} G_\mu(z) &= \int_{\mathbb{R}} \frac{d\mu(x)}{z-x} \\ &= \int_{\mathbb{R}} \frac{z^{-1} d\mu(x)}{1 - x/z} \\ &= \int_{\mathbb{R}} \sum_{n=0}^{\infty} z^{-1} \frac{x^n}{z^n} d\mu(x) \\ &= \sum_{n=0}^{\infty} z^{-1} \frac{x^n}{z^n} d\mu(x) \\ &= \sum_{n=0}^{\infty} \frac{m_n(\mu)}{z^{n+1}} \end{aligned}$$

Now, we take the **free cumulant generating function**

$$K(z) = 1 + \sum_{n=1}^{\infty} k_n z^n,$$

and define the transform

$$V(z) = \frac{1}{z} K_n(z).$$

Now, by the relations between m_n and k_n given in (1.9), we can see that $V(G(z)) = z$, i.e. V and G are inverses.

⟨G_SC⟩ **Example 1.16** (Semi-Circle)**.** We compute G_{SC} using the Cauchy Transform:

$$\begin{aligned} G_{SC}(z) &= \int_{-2}^{2} \frac{\frac{1}{2\pi}\sqrt{1-4x^2}}{z-x} dx \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{Cat}_n}{z^{2n+1}} \\ &= \frac{1}{z} \left(\frac{1 - \sqrt{1 - 4/z^2}}{2} \right) \\ &= \frac{1}{2} z - \frac{1}{2} \sqrt{z^2 - 4} \end{aligned}$$

Since we know the free cumulant sequence for SC is $(0, 1, 0, 0, ...)$, we can compute

$$V_{SC}(z) = V(z) = \frac{1}{z} + z$$

Hence, we can regain G_{SC}

$$\frac{1}{G} + G = z \Rightarrow G^2 - zG + 1 = 0$$
$$\Rightarrow G = \frac{1}{2}(z - \sqrt{z^2 - 4}).$$

Theorem 1.17 (Voiculescu's Algorithm for Free Convolution)**.** *Let X_N and Y_N be random matrix ensembles as in 1.13, where μ and ν have compact support.*

(1) Compute $G_X(z) = \mathbb{E}(z - X)^{-1}$ and $G_Y(z) = \mathbb{E}(z - Y)^{-1}$.
(2) Solve $G_X(V_X(z)) = z$ and $G_Y(V_Y(z)) = z$ subject to $V(z) \sim 1/z$ at 0.
(3) Compute ***Voiculescu's R-transforms***

$$R_X(z) = V_X(z) - 1/zR_Y(z) = V_Y(z) - 1/z$$

Then, $V_{X+Y} = 1/z + R_X + R_Y$.
(4) Finally, solve $V_{X+Y}(G_{X+Y}(z)) = z$ subject to $G(z) \sim 1/z$ at ∞.
(5) Recover the distribution using the Inversion Formula, given below (1.13)*.*

How do we recover the distribution $\mu \boxplus \nu$ once we know $G_{\mu\boxplus\nu}$?
We can use the following:

$$G_{\mu\boxplus\nu} = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}} \to m_n \to \mathbb{E}\, e^{zX}$$

and then recover $\mu \boxplus \nu$ with the inverse Fourier Transform.

However, if we assume $\mu \boxplus \nu$ is "nice" (i.e. has continuous density), we can use a more direct approach. First, let $\rho(t)$ be the density for $\mu \boxplus \nu$ with compact support on the interval I. Then, using the substitution $z = x + iy$, we get

$$G(z) = \int_I \frac{\rho(t)\ dt}{z - t}$$
$$= \int_I \rho(t) \left[\frac{x - t}{(x - t)^2 + y^2} - i\frac{y}{(x - t)^2 + y^2}\right] dt.$$

Now, we find $\lim_{y\to 0^+} \int_a^b Im(G(x + iy))\ dx$.

$$\int_a^b Im(G(x + iy))\ dx = \left[\int_I \rho(t)\ dt\right] \left[\int_a^b \frac{-y}{y^2 + (x - t)^2}\right] dx$$
$$= \int_I \rho(t)\ dt \left[\arctan\left(\frac{x - t}{y}\right)\right]_a^b$$
$$= \int_I \rho(t)\ dt \left[\arctan\left(\frac{b - t}{y}\right) - \arctan\left(\frac{a - t}{y}\right)\right]$$

Since $\arctan\left(\frac{b-t}{y}\right) - \arctan\left(\frac{a-t}{y}\right) = 0$ for $t \notin [a, b]$, we have

$$\arctan\left(\frac{b - t}{y}\right) - \arctan\left(\frac{a - t}{y}\right) \to_{y\to 0^+} \frac{\pi}{2} - \frac{-\pi}{2}$$

Hence,

$$\int_a^b Im(G(x+iy))\ dx \to_{y\to 0^+} \int_I \pi\rho(t)\ dt.$$

This gives us an **Inversion Formula**

$$\rho(x) = \lim_{y\to 0^+} \frac{1}{\pi} Im(G(x+iy)). \tag{1.13}$$

Inversion_For

1.7.1. *Examples.*

Example 1.18. From the computations in 1.16, we can show that

$$\frac{\mathsf{SC} \boxplus \mathsf{SC}}{\sqrt{2}} = \mathsf{SC}.$$

Example 1.19 (Bernoulli)**.** Suppose $\mu = \nu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. Note that classical convolution yields

$$\mu * \nu = \frac{1}{4}\delta_{-2} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_2.$$

To find $\mu \boxplus \nu$, we follow Voiculescu's Algorithm:

(1) $G(z) = \mathbb{E}\,\frac{1}{z-x} = \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right) = \frac{z}{z^2-1}$

(2) Solving $z = G(V(z)) = \dfrac{V}{V^2-1}$ yields $zV^2 - V - z = 0$. So,

$$V(z) = \frac{1 \pm \sqrt{1+4z^2}}{2z}.$$

Since $V(z) = \frac{K(z)}{z} \sim \frac{1}{z}$ at $z = 0$, we choose

$$V(z) = \frac{1 + \sqrt{1+4z^2}}{2z}.$$

(3) $V_{\mu+\nu}(z) = V_\mu(z) + V_\nu(z) - \dfrac{1}{z} = \dfrac{\sqrt{1+4z^2}}{z}.$

(4) Solving $z = V(G(z)) = \dfrac{\sqrt{1+4G^2}}{G}$ yields

$$G(z) = \pm(z^2-4)^{-1/2}.$$

Since $zG(z) \to 1$ at $z = 0$, we choose

$$G(z) = \frac{1}{\sqrt{z^2-4}}.$$

(5) From (1.13), we see that the distribution $\rho(x)$ for $\mu \boxplus \nu$ is

$$\begin{aligned}\rho(x) &= \lim_{y\to 0^+} Im\left(\frac{1}{\sqrt{(x+iy)^2-4}}\right)\\ &= \begin{cases} 0 & ,|x| > 2 \\ \frac{1}{\pi\sqrt{4-x^2}} & ,|x| \le 2\end{cases}\end{aligned}$$

This is the arcsine distribution, which is quite different from the classical convolution.

Example 1.20 (Free Poisson Theorem)**.** Let $\alpha > 0, \lambda > 1$, and

$$\mu_N = (1 - \lambda/N)\delta_0 + \lambda/N\delta_\alpha.$$

Classically,

$$\mu_N^{*N} \to_{N\to\infty} \text{Poisson} \sim \lambda \text{ on } \{0, \alpha, 2\alpha, ...\}.$$

In the free case, if we compute the R transform, multiply by N, and then invert, we can show that $\mu_N^{\boxplus N}$ converges to density $\frac{1}{2\pi\alpha t}\sqrt{4\lambda\alpha^2 - (t - \alpha(1-\lambda))^2}$, which is a Marchenko-Pastur distribution.

Example 1.21 (Cauchy Density)**.** For the Cauchy density $C = \dfrac{1}{\pi(1+x^2)}$ on $\mathbb{R}$,

$$\mu * C = \mu \boxplus C.$$

:notes_Free⟩? 1.8. **Notes and references.** Free probability was invented by Voiculescu in the 1980s (e.g., [VDN92] gives an early introduction). In our discussions we mainly follow Chapter 5 of [AGZ10], and also lecture notes [NL12].

Lecture #2 on 2/3/2016

References

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