NOTES ON RANDOM MATRICES

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(notes by Bryce Terwilliger; ...)

Abstract. These are notes for the MATH 8380 "Random Matrices" course at the University of Virginia in Spring 2016. The notes are constantly updated, and the latest version can be found at the git repository `https://github.com/lenis2000/RMT_Spring_2016`

(before TeXing, please familiarize yourself with style suggestions at `https://github.com/lenis2000/RMT_Spring_2016/blob/master/TeXing.md`)

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Lecture #1 on 1/20/2016

1. Introduction

1.1. Nonrandom and random matrices and their eigenvalues. The study of random matrices as a field is a patchwork of many fields. The main object we study is a probability distribution on a certain subset of the set of matrices $\mathrm{Mat}(N \times N,\ \mathbb{R} \text{ or } \mathbb{C})$, thus giving us a random matrix A.

Definition 1.1. An *eigenvalue* λ of the matrix A is a root of the polynomial $f(\lambda) = \det(A - \lambda I)$. Equivalently, λ is an eigenvalue if A if the matrix $A - \lambda I$ is not invertible. This second way of defining eigenvalues in fact works even when A is not a finite size matrix, but an operator in some infinite-dimensional space.

We will largely be only concerned with real eigenvalues. That is the eigenvalues of a real symmetric matrix over $\mathbb{R}$ or Hermitian over $\mathbb{C}$ that is where $A^* = A$.

Remark 1.2. The case when eigenvalues can be complex is also studied in the theory of random matrices, sometimes under the keyword *complex random matrices*. See, for example, [GT10] for a law of large numbers for complex eigenvalues.

Proposition 1.3. *Every eigenvalue of a Hermitian matrix is real.*

Proof. Let A be a Hermitian matrix so that $A^* = A$ (here and everywhere below $A*$ means $\overline{A^{\mathrm{T}}}$, i.e., transposition and complex conjugation). Let λ be an eigenvalue of A. Let v be a non-zero vector in the null space of $A - \lambda I$. Let $a = \overline{v^{\mathrm{T}}}v = |v|^2$, so that a is a positive real number. Let $b = \overline{v^{\mathrm{T}}}Av$. Then $\bar{b} = \overline{b^{\mathrm{T}}} = \overline{\overline{v^{\mathrm{T}}}Av}^{\mathrm{T}} = \overline{v^{\mathrm{T}}}\bar{A}^{\mathrm{T}}v = \overline{v^{\mathrm{T}}}Av = b$, so b is real. Since $b = \lambda a$, λ must be real. □

Date: January 23, 2016.

Let $\mathcal{H}_N$ be the set of $N \times N$ Hermitian matrices. For each N, let μ_N be a probability measure on $\mathcal{H}_N$ (it can be supported not by the whole $\mathcal{H}_N$, but by a subset of it, too). Then for each matrix $A \in \mathcal{H}_N$ we may order the real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_N$ of A.

A collection of probability measures μ_N on $\mathcal{H}_N$ for each $N \geq 1$ is said to be a *random matrix ensemble*. For such an ensemble, the eigenvalues $\lambda_1^{(N)} \geq \cdots \geq \lambda_N^{(N)}$ of matrices N form random point configurations on $\mathbb{R}$ with growing numbers of points. Our main goal is to study the asymptotic properties of these collections of points on $\mathbb{R}$, as $N \to \infty$.

References

e2010circular [GT10] F. Götze and A. Tikhomirov, *The circular law for random matrices*, Ann. Probab. **38** (2010), no. 4, 1444–1491, arXiv:0709.3995 [math.PR].