Lectures on Random Matrices (Spring 2025)
Lecture 15: Random Matrices and Topology

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Contents

1 Introduction

In this wrap-up lecture, we go back to moments of random matrices, and outline their connection to topology (more precisely, to counting certain embedded graphs).

Remark 1.1. Throughout this lecture, to make an exact connection with the existing literature, the matrix size is denoted by N, and the small n is reserved to the order of the moment.

*Course webpage • Live simulations • TeX Source • Updated at 14:26, Tuesday 22nd April, 2025

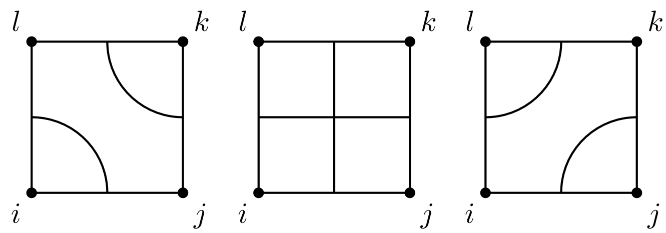
NOTE: course evaluations!
Todo: tables of contents in HTML + total PDF

2 Gluing polygons into surfaces

2.1 Gluing edges of a polygon

Consider a regular $2n$-gon with edges labeled by $1, \dots, 2n$. We can glue the edges in pairs, so that the resulting surface is oriented.

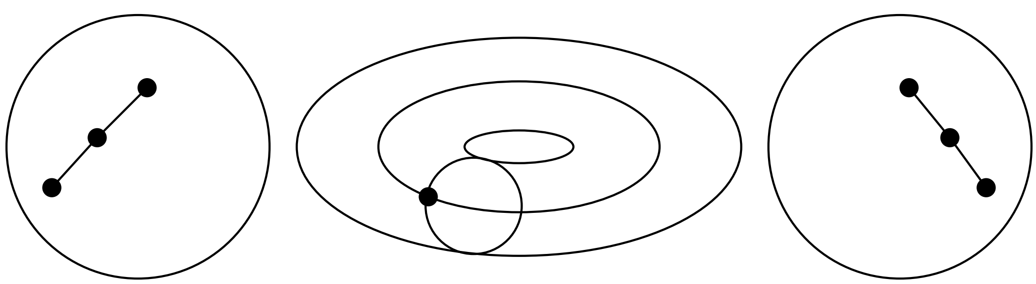
Example 2.1. Consider a square. Recall that to obtain an orientable surface one must orient the square's boundary cyclically and then glue opposite sides with *opposite* orientations. There are three ways to glue the edges of a square. Note that in two cases, we get the sphere and in one case, the torus. The two spheres are obtained by gluing the edges in the same way, but this differs by a rotation — we consider these two cases as different.



The boundary of the $2n$-gon becomes a graph embedded into the surface. It has exactly n edges and one face. It may have different number of vertices, and thus the number of vertices uniquely determines the genus of the surface:

$$V - E + F = 2 - 2g \implies g = \frac{n+1-V}{2}.$$

In the case of the square ($n = 2$), we have $V = 3$ and $g = 0$ for the sphere, and $V = 1$ and $g = 1$ for the torus.



2.2 Starting to count

Proposition 2.2. *There is a total*

$$(2n-1)!! = (2n-1)(2n-3)\cdots 3 \cdot 1$$

ways to glue the edges of a $2n$-gon into a surface.

Proof. This is just the number of ways to pair $2n$ edges of the polygon. $\square$

Proposition 2.3. *The following are equivalent:*

1. *The surface is a sphere;*
2. *The graph on the surface is a tree;*
3. *The identification of the opposite edges of the polygon is a* noncrossing pairing *of the edges of the polygon.*

Proof. See Problem O.1. $\square$

There is $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ ways to get the sphere.

2.3 Dual picture

In the dual picture, we can consider a star with $2n$ half-edges. Then, we get a dual graph on the same surface. This graph has $V^* = 1$, $E^* = n$, but can have a variable number of faces (which corresponds to the genus):

$$F^* = n - 2g + 1.$$

When $n = 2$, for the sphere, we have $F^* = 3$, and for the torus, we have $F^* = 1$.

2.4 Notation

Let us denote

$$\begin{aligned} \varepsilon_g(n) &\coloneqq \text{number of ways to glue the edges of a } 2n\text{-gon into a surface of genus } g, \\ T_n(N) &\coloneqq \sum_{\text{gluings } \sigma} N^{V(\sigma)} = \sum_{g=0}^{\infty} \varepsilon_g(n) N^{n+1-2g}, \end{aligned}$$

that is, this is the generating function of the gluings of the edges of a $2n$-gon, where N is the generating function variable.

Remark 2.4. The polynomial $T_n(N)$ has only powers of N of the same parity as n.

We have the first few polynomials (the case $n = 2$ corresponds to the square):

$$\begin{aligned} T_1(N) &= N^2; \\ T_2(N) &= 2N^3 + N; \\ T_3(N) &= 5N^4 + 10N^2; \\ T_4(N) &= 14N^5 + 70N^3 + 21N; \\ T_5(N) &= 42N^6 + 420N^4 + 483N^2. \end{aligned}$$

3 Harer–Zagier formula (statement)

Introduce the *exponential generating function* for the sequence $\{T_n(N)\}_{n\geq 0}$:

$$\begin{aligned} T(N,s) \ &= 1 + 2Ns + 2s \sum_{n\geq 1} \frac{T_n(N)}{(2n-1)!!}\, s^n \\ &= 1 + 2Ns + 2N^2 s^2 + \frac{2}{3}\big(2N^3+N\big)s^3 + \frac{2}{15}\big(5N^4+10N^2\big)s^4 + \dots \end{aligned} \tag{3.1}$$

One of the goals of today's lecture is to prove the following:

Theorem 3.1 (Harer–Zagier formula [HZ86])**.** *For every $N \in \mathbb{Z}_{>0}$ one has the closed form*

$$T(N,s) \ = \ \left(\frac{1+s}{1-s}\right)^N. \tag{3.2}$$

Let us at least verify that the first few Taylor coefficients of (3.2) indeed coincide with those in (3.1). Write

$$\begin{aligned} \left(\frac{1+s}{1-s}\right)^N &= (1+s)^N(1-s)^{-N} \\ &= \left(1 + Ns + \frac{N(N-1)}{2!}s^2 + \frac{N(N-1)(N-2)}{3!}s^3 + \dots\right) \\ &\quad \times \left(1 + Ns + \frac{N(N+1)}{2!}s^2 + \frac{N(N+1)(N+2)}{3!}s^3 + \dots\right). \end{aligned}$$

Multiplying the two series and collecting terms up to s^3, we find

$$1 + 2Ns + 2N^2s^2 + \frac{2}{3}(2N^3+N)s^3 + \dots,$$

which matches the expansion (3.1) exactly.

Corollary 3.2. *For all $g \geq 0$ and $n \geq 0$, the numbers $\varepsilon_g(n)$ obey*

$$(n+2)\,\varepsilon_g(n+1) \ = \ (4n+2)\,\varepsilon_g(n) \ + \ (4n^3-n)\,\varepsilon_{g-1}(n-1), \tag{3.3}$$

with the initial condition

$$\varepsilon_g(0) \ = \ \begin{cases} 1, & g=0, \\ 0, & g\geq 1. \end{cases}$$

Proof. Follows from the identity

$$\left(\frac{1+s}{1-s}\right)^N = (1+s)(1+s+s^2+\dots)\left(\frac{1+s}{1-s}\right)^{N-1}.$$

□

Corollary 3.3. *The number $\varepsilon_g(n)$ can be written as*

$$\varepsilon_g(n) \;=\; \frac{(2n)!}{(n+1)!\,(n-2g)!}\left[s^{2g}\right]\left(\frac{s/2}{\tanh(s/2)}\right)^{n+1},$$

where $\left[s^{2g}\right]f(s)$ denotes the coefficient of s^{2g} in the power-series expansion of $f(s)$.

One can define another family of coefficients:

$$C_g(n) \coloneqq \frac{2^g\varepsilon_g(n)}{\mathrm{Cat}_n}.$$

Then, (3.3) can be rewritten as

$$C_g(n+1) = C_g(n) + \binom{n+1}{2}C_{g-1}(n-1).$$

In particular, $C_g(n)$ is a positive integer, which is not straightforward from the definition of $\varepsilon_g(n)$.

4 Gaussian integrals and Wick formula

5 GUE integrals and gluing polygons

6 Multi-matrix models

7 Two-matrix models and the Ising model

O Problems (due 2025-04-29)

O.1 Gluing a Sphere

Show that for a connected, orientable surface formed by gluing the edges of a $2n$-gon in pairs, the following are equivalent:

1. The resulting surface is a sphere.
2. The embedded graph formed by the identification is a tree.
3. The pairing of edges corresponds to a *noncrossing pairing* (i.e., when the edges are arranged around the polygon in order, the identifications can be drawn inside the disk without crossings).

(This is the proof of Proposition 2.3.)

References

[HZ86] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), 457–485. ↑4

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