Lectures on Random Matrices (Spring 2025)
Lecture 2: Wigner semicircle law

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Notes for the lecturer

PREP:

1. Start: Catalan number formula
2. Moments of SC need to be computed
3. SC is uniquely determined by its moments; need Carleman criterion to show that the moments determine the distribution.
4. from expected moments to the full convergence, some analysis needed

0.1 Moments of the semicircle law

We also need to match the Catalan numbers to the moments of the semicircle law. Let $k = 2m$, and we need to compute the integral

$$\int_{-2}^{2} x^{2m} \frac{1}{2\pi} \sqrt{4 - x^2}\, dx.$$

By symmetry, we write:

$$\int_{-2}^{2} x^{2m} \rho(x)\, dx = \frac{2}{\pi} \int_{0}^{2} x^{2m} \sqrt{4 - x^2}\, dx.$$

Using the substitution $x = 2 \sin\theta$, we have $dx = 2\cos\theta\, d\theta$. The integral becomes:

$$\frac{2}{\pi} \int_0^{\pi/2} (2\sin\theta)^{2m} (2\cos\theta)(2\cos\theta\, d\theta) = \frac{2^{2m+2}}{\pi} \int_0^{\pi/2} \sin^{2m}\theta \cos^2\theta\, d\theta.$$

Using $\cos^2\theta = 1 - \sin^2\theta$, we split the integral:

$$\frac{2^{2m+2}}{\pi} \left(\int_0^{\pi/2} \sin^{2m}\theta\, d\theta - \int_0^{\pi/2} \sin^{2m+2}\theta\, d\theta \right).$$

*Course webpage • TeX Source • Updated at 02:03, Tuesday 14th January, 2025

Using the standard formula

$$\int_0^{\pi/2} \sin^{2n}\theta \, d\theta = \frac{\pi}{2} \frac{(2n)!}{2^{2n}(n!)^2}, \tag{0.1}$$

we compute each term:

$$\frac{2^{2m+2}}{\pi} \left(\frac{\pi}{2} \frac{(2m)!}{2^{2m}(m!)^2} - \frac{\pi}{2} \frac{(2m+2)!}{2^{2m+2}((m+1)!)^2} \right).$$

After simplification, this becomes C_m, the m-th Catalan number.

A Problems (due 2025-02-15)

A.1 Standard formula

Prove formula (0.1):

$$\int_0^{\pi/2} \sin^{2n}\theta \, d\theta = \frac{\pi}{2} \frac{(2n)!}{2^{2n}(n!)^2}.$$

References

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