Lectures on Random Matrices (Spring 2025)
Lecture 2: Wigner semicircle law

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Notes for the lecturer

PREP:

1. Start: Catalan number formula
2. Moments of SC need to be computed
3. SC is uniquely determined by its moments; need Carleman criterion to show that the moments determine the distribution.
4. from expected moments to the full convergence, some analysis needed

1 Recap

We are working on the Wigner semicircle law.

1. Wigner matrices W: real symmetric random matrices with iid entries X_{ij}, $i > j$ (mean 0, variance σ^2); and iid diagonal entries X_{ii} (mean 0, some other variance and distribution).
2. Empirical spectral distribution (ESD)
$$\nu_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i/\sqrt{n}},$$
which is a random probability measure on $\mathbb{R}$.
3. Semicircle distribution μ_{sc}:
$$\mu_{\mathrm{sc}}(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,dx, \qquad x \in [-2,2].$$
4. Computation of expected traces of powers of W. We showed that
$$\int_{\mathbb{R}} x^k \nu_n(dx) \to \#\left\{\text{rooted planar trees with } k/2 \text{ edges}\right\}.$$

*Course webpage • TeX Source • Updated at 02:41, Tuesday 14th January, 2025

2 Two computations

First, we finish the combinatorial part, and match the limiting expected traces of powers of W to moments of the semicircle law.

2.1 Moments of the semicircle law

We also need to match the Catalan numbers to the moments of the semicircle law. Let $k = 2m$, and we need to compute the integral

$$\int_{-2}^{2} x^{2m} \frac{1}{2\pi} \sqrt{4 - x^2}\, dx.$$

By symmetry, we write:

$$\int_{-2}^{2} x^{2m} \rho(x)\, dx = \frac{2}{\pi} \int_{0}^{2} x^{2m} \sqrt{4 - x^2}\, dx.$$

Using the substitution $x = 2\sin\theta$, we have $dx = 2\cos\theta\, d\theta$. The integral becomes:

$$\frac{2}{\pi} \int_{0}^{\pi/2} (2\sin\theta)^{2m} (2\cos\theta)(2\cos\theta\, d\theta) = \frac{2^{2m+2}}{\pi} \int_{0}^{\pi/2} \sin^{2m}\theta \cos^2\theta\, d\theta.$$

Using $\cos^2\theta = 1 - \sin^2\theta$, we split the integral:

$$\frac{2^{2m+2}}{\pi} \left(\int_{0}^{\pi/2} \sin^{2m}\theta\, d\theta - \int_{0}^{\pi/2} \sin^{2m+2}\theta\, d\theta \right).$$

Using the standard formula

$$\int_{0}^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2} \frac{(2n)!}{2^{2n}(n!)^2}, \tag{2.1}$$

we compute each term:

$$\frac{2^{2m+2}}{\pi} \left(\frac{\pi}{2} \frac{(2m)!}{2^{2m}(m!)^2} - \frac{\pi}{2} \frac{(2m+2)!}{2^{2m+2}((m+1)!)^2} \right).$$

After simplification, this becomes C_m, the m-th Catalan number.

2.2 Counting trees and Catalan numbers

3 Analysis steps in the proof

We are done with combinatorics, and it remains to justify that the computations lead to the desired semicircle law from Lecture 1.

A Problems (due 2025-02-15)

A.1 Standard formula

Prove formula (2.1):

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2}\frac{(2n)!}{2^{2n}(n!)^2}.$$

References

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