Mini Course: Dimers and Embeddings

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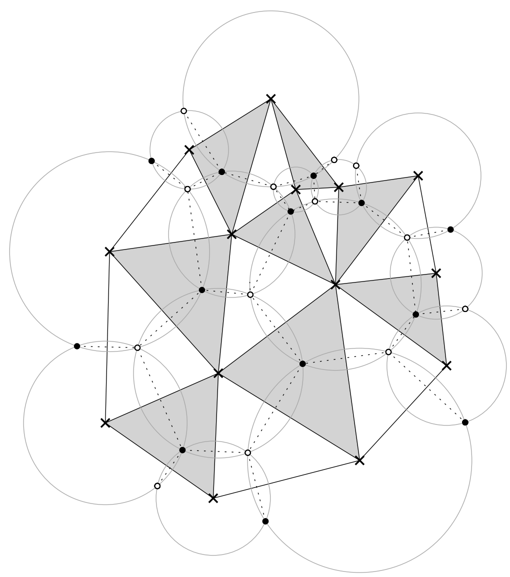
Exercise Session Three: t-embeddings and circle patterns

1. **Circle patterns and t-embeddings.** Show that *t-embeddings* are in bijection with *centers of circle patterns*, i.e. the **centers** of the circles are the vertices of the t-embedding.

   Remark. *Note that given a set of circle centers, there is actually a two parameter family of circle patterns with those circle centers.*



2. **Local moves.** Prove that t-embeddings are preserved under elementary transformations, see Figures 1–2.

   More precisely, show that

   a) replacement of a single edge with weight $v_1 + v_2$ by parallel edges with weights v_1, v_2 corresponds to adding a point dividing corresponding edges of the t-embedding in proportion $[v_1 : v_2]$; merging double edges corresponds to removing a degree 2 vertex of the t-embedding, which due to the angle condition has to lie on a line with its two adjacent vertices;

   b) contracting a degree 2 vertex corresponds to removing a degree two face of a t-embedding which can be seen as a diagonal of a face of the t-embedding; splitting a vertex of degree $d_1 + d_2$ to two vertices of degrees $d_1 + 1$ and $d_2 + 1$ and adding a degree two vertex between them corresponds to adding a diagonal to the corresponding face of the t-embedding with respect to the structure of the splitted graph;

   c) a spider move corresponds to a *central move* of points of t-embedding.

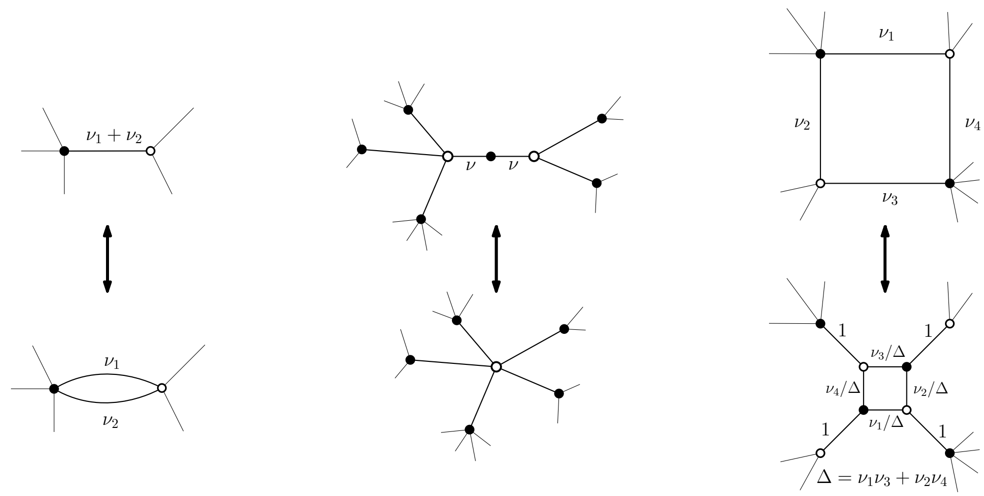


Figure 1: Elementary transformations of weighted bipartite graphs: (1) a single edge with weight $\nu_1+\nu_2$ can be replaced by parallel edges with weights ν_1,ν_2; double edges with weights ν_1,ν_2 can be merge to a single edge with weight $\nu_1+\nu_2$; (2) contracting a degree 2 vertex whose edges have equal weights; splitting a vertex of degree d_1+d_2 to two vertices of degrees d_1+1 and d_2+1 and adding a degree two vertex between them; (3) spider move, with the weight change as shown.

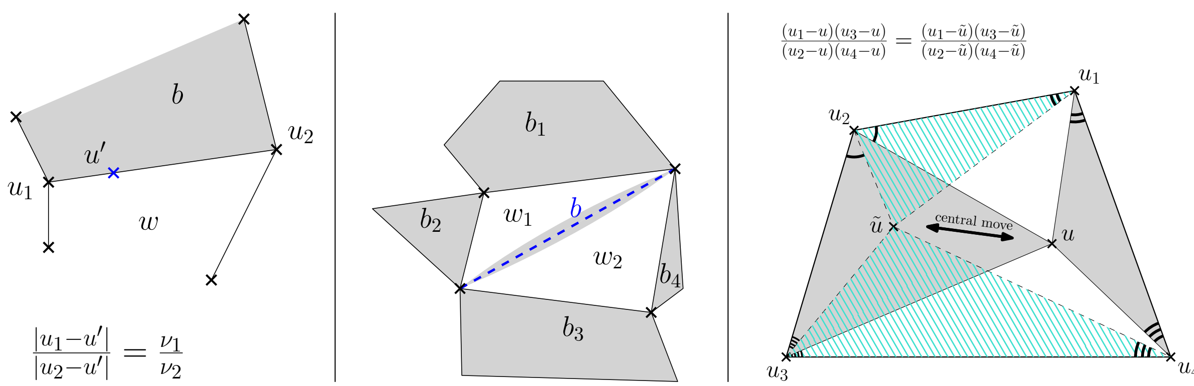


Figure 2: Transformations of t-embeddings: (1) adding / removing vertex of degree 2; (2) adding / removing diagonal of a face of a t-embedding; (3) central move (points obtained by a central move are isogonal conjugate).

3. Existence of t-embeddings. Show the existence of a t-embedding to a convex quadrilateral of the augmented dual graph of any weighted planar bipartite graph with outer face of degree four.

Hint: Use question 2. and Postnikov's theorem, which states the following: Any non degenerate bipartite graph with 4 marked boundary vertices $\mathrm{w}_1, \mathrm{b}_1, \mathrm{w}_2, \mathrm{b}_2$ *can be built up from the 4-cycle graph with vertices* $\mathrm{w}_1, \mathrm{b}_1, \mathrm{w}_2, \mathrm{b}_2$ *using a sequence of elementary transformations; moreover, the marked vertices remain in all intermediate graphs.*

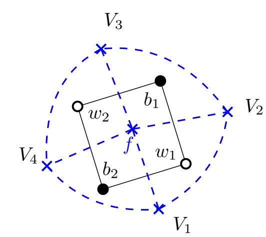
4. Origami map. Let $T_{(F,G)}$ be a t-embedding of a planar bipartite graph with edge Kasteleyn weights $K(b,w)$ given by a pair of Coulomb gauge functions (F,G), i.e. $dT(bw^*) = F(b)K(b,w)G(w)$, where bw^* is a dual edge corresponding to the primal edge bw, oriented with the white vertex on the left. Check that the discrete 1-form on edges of the dual graph given by

$$dO(bw^*) = F(b)K(b,w)\overline{G(w)}$$

defines an origami map.

5. Face weights are invariant under isometries in $\mathbb{R}^{2,2}$.

Consider a degree 4 face f with vertices w_1, b_1, w_2, b_2 as shown in the figure, and Kasteleyn weights $K(b_i, w_j)$. Define the *face weight*



$$X_f := -\frac{K(\mathrm{b}_1, \mathrm{w}_1)K(\mathrm{b}_2, \mathrm{w}_2)}{K(\mathrm{b}_1, \mathrm{w}_2)K(\mathrm{b}_1, \mathrm{w}_2)} > 0.$$

Show that

$$X_f = \frac{|\tilde{V}_2 - \tilde{V}_4|_{\mathbb{C}^{1,1}}}{|\tilde{V}_1 - \tilde{V}_3|_{\mathbb{C}^{1,1}}},$$

where $\tilde{V}_j = (T(V_j), O(V_j)) \in \mathbb{C}^{1,1}$ are lifts of dual vertices V_j to $\mathbb{C}^{1,1}$, using the t-embedding and origami map.

Recall that in $\mathbb{R}^{2,2}$ *one has*

$$\langle x, y \rangle = x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4.$$

Note that $\mathbb{R}^{2,2} \simeq \mathbb{C}^{1,1}$ *and for* $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{C}^{1,1}$ *one has* $\langle z, w \rangle = \mathrm{Re}(z_1\overline{w_1} - z_2\overline{w_2})$. *So in particular the norm above is given, for* $z_1, z_2 \in \mathbb{C}$, *by* $|(z_1, z_2)|_{\mathbb{C}^{1,1}} = |z_1|^2 - |z_2|^2$.