

Lectures on Random Matrices (Spring 2025)

Lecture 2: Wigner semicircle law

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Wednesday, January 15, 2025*

Notes for the lecturer

PREP:

1. Start: Catalan number formula
2. Moments of SC need to be computed
3. SC is uniquely determined by its moments; need Carleman criterion to show that the moments determine the distribution.
4. from expected moments to the full convergence, some analysis needed

1 Recap

We are working on the Wigner semicircle law.

1. Wigner matrices W : real symmetric random matrices with iid entries X_{ij} , $i > j$ (mean 0, variance σ^2); and iid diagonal entries X_{ii} (mean 0, some other variance and distribution).
2. Empirical spectral distribution (ESD)

$$\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i/\sqrt{n}},$$

which is a random probability measure on \mathbb{R} .

3. Semicircle distribution μ_{sc} :

$$\mu_{\text{sc}}(dx) = \frac{1}{2\pi} \sqrt{4 - x^2} dx, \quad x \in [-2, 2].$$

4. Computation of expected traces of powers of W . We showed that

$$\int_{\mathbb{R}} x^k \nu_n(dx) \rightarrow \# \{\text{rooted planar trees with } k/2 \text{ edges}\}.$$

*[Course webpage](#) • [TeX Source](#) • Updated at 12:01, Tuesday 14th January, 2025

2 Two computations

First, we finish the combinatorial part, and match the limiting expected traces of powers of W to moments of the semicircle law.

2.1 Moments of the semicircle law

We also need to match the Catalan numbers to the moments of the semicircle law. Let $k = 2m$, and we need to compute the integral

$$\int_{-2}^2 x^{2m} \frac{1}{2\pi} \sqrt{4 - x^2} dx.$$

By symmetry, we write:

$$\int_{-2}^2 x^{2m} \rho(x) dx = \frac{2}{\pi} \int_0^2 x^{2m} \sqrt{4 - x^2} dx.$$

Using the substitution $x = 2 \sin \theta$, we have $dx = 2 \cos \theta d\theta$. The integral becomes:

$$\frac{2}{\pi} \int_0^{\pi/2} (2 \sin \theta)^{2m} (2 \cos \theta) (2 \cos \theta d\theta) = \frac{2^{2m+2}}{\pi} \int_0^{\pi/2} \sin^{2m} \theta \cos^2 \theta d\theta.$$

Using $\cos^2 \theta = 1 - \sin^2 \theta$, we split the integral:

$$\frac{2^{2m+2}}{\pi} \left(\int_0^{\pi/2} \sin^{2m} \theta d\theta - \int_0^{\pi/2} \sin^{2m+2} \theta d\theta \right).$$

Using the standard formula

$$\int_0^{\pi/2} \sin^{2n} \theta d\theta = \frac{\pi}{2} \frac{(2n)!}{2^{2n} (n!)^2}, \quad (2.1)$$

we compute each term:

$$\frac{2^{2m+2}}{\pi} \left(\frac{\pi}{2} \frac{(2m)!}{2^{2m} (m!)^2} - \frac{\pi}{2} \frac{(2m+2)!}{2^{2m+2} ((m+1)!)^2} \right).$$

After simplification, this becomes C_m , the m -th Catalan number.

2.2 Counting trees and Catalan numbers

Throughout this section, for a random matrix trace moment of order k , we use $m = k/2$ as our main parameter. Note that m can be arbitrary (not necessarily even).

Definition 2.1 (Dyck Path). A *Dyck path* of semilength m is a sequence of $2m$ steps in the plane, each step being either $(1, 1)$ (up step) or $(1, -1)$ (down step), starting at $(0, 0)$ and ending at $(2m, 0)$, such that the path never goes below the x -axis. We denote an up step by U and a down step by D .

Definition 2.2 (Rooted Plane Tree). A *rooted plane tree* is a tree with a designated root vertex where the children of each vertex have a fixed left-to-right ordering. The size of such a tree is measured by its number of edges, which we denote by m .

Definition 2.3 (Catalan Numbers). The sequence of *Catalan numbers* $\{C_m\}_{m \geq 0}$ is defined recursively by:

$$C_0 = 1, \quad C_{m+1} = \sum_{j=0}^m C_j C_{m-j} \quad \text{for } m \geq 0. \quad (2.2)$$

Alternatively, they have the closed form:

$$C_m = \frac{1}{m+1} \binom{2m}{m} = \binom{2m}{m} - \binom{2m}{m+1}. \quad (2.3)$$

These numbers appear naturally in the moments of random matrices, where $m = k/2$ for trace moments of order k .

Lemma 2.4. *Formulas (2.2) and (2.3) are equivalent.*

Proof. One can check that the closed form satisfies the recurrence relation by direct substitution. The other direction involves generating functions. Namely, (2.2) can be rewritten for the generating function

$$C(z) = \sum_{m=0}^{\infty} C_m z^m$$

as

$$C(z) = 1 + zC(z)^2.$$

Solving for $C(z)$, we get

$$C(z) = \frac{1 \pm \sqrt{1-4z}}{2z}. \quad (2.4)$$

We need to pick the solution which is nonsingular at $z = 0$, and it corresponds to the minus sign. Taylor expansion of the right-hand side of (2.4) at $z = 0$ gives the closed form. \square

Remark 2.5. Catalan numbers enumerate many (too many!) combinatorial objects. For a comprehensive treatment, see [Sta15].

Proposition 2.6 (Dyck Path–Rooted Tree Correspondence). *For any m , there exists a bijection between the set of Dyck paths of semilength m and the set of rooted plane trees with m edges.*

Proof. Given a Dyck path of semilength m , we build the corresponding rooted plane tree as follows (see Figure 1 for an illustration):

1. Start with a single root vertex
2. Read the Dyck path from left to right:
 - For each up step (U), add a new child to the current vertex
 - For each down step (D), move back to the parent of the current vertex



This is clearly a bijection, and we are done. \square

Proposition 2.7. *The number of Dyck paths of semilength m satisfies the Catalan recurrence (2.2).*

4

3 Analysis steps in the proof

We are done with combinatorics, and it remains to justify that the computations lead to the desired semicircle law from [Lecture 1](#).

B Problems (due 2025-02-15)

B.1 Standard formula

Prove formula [\(2.1\)](#):

$$\int_0^{\pi/2} \sin^{2n} \theta \, d\theta = \frac{\pi}{2} \frac{(2n)!}{2^{2n} (n!)^2}.$$

References

[Sta15] R. Stanley, *Catalan numbers*, Cambridge University Press, 2015. [↑3](#)

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