Lectures on Random Matrices (Spring 2025)
Lecture 11: Some universal asymptotics of Dyson Brownian Motion

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1 Recap

1.1 Dyson Brownian Motion (DBM)

We introduced a time-dependent model of random matrices by letting an $N \times N$ Hermitian matrix $\mathcal{M}(t)$ evolve in time so that each off-diagonal entry follows independent Brownian increments (real or complex depending on the symmetry class). Setting

$$\mathcal{M}(t) \;=\; \frac{1}{\sqrt{2}}\big(X(t) + X^{\dagger}(t)\big),$$

where $X(t)$ is an $N \times N$ matrix of i.i.d. Brownian motions, produces a self-adjoint matrix with a stochastically evolving spectrum. This model is full-rank matrix Brownian motion, and works well for $\beta = 1, 2, 4$. For other β, we need an SDE to describe the evolution of the eigenvalues (particles).

1.2 Eigenvalue SDE

Denote by $\lambda_1(t) \geq \cdots \geq \lambda_N(t)$ the ordered eigenvalues of $\mathcal{M}(t)$. Dyson showed that these eigenvalues form a continuous-time Markov process satisfying the SDE

$$d\lambda_i(t) \;=\; \frac{\beta}{2} \sum_{j \neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)} \;+\; dW_i(t), \quad i = 1, \ldots, N,$$

where $\beta > 0$ and $W_i(t)$ are independent standard real Brownian motions. For classical random matrix ensembles ($\beta = 1, 2, 4$), this SDE describes how the eigenvalues evolve under real symmetric (GOE), Hermitian (GUE), or quaternionic (GSE) Brownian motion — in the last Lecture 10 we discussed the cases $\beta = 1, 2$ in detail. A key feature is the *repulsion* term $\frac{1}{\lambda_i - \lambda_j}$, which prevents collisions (and ensures the ordering remains intact).

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1.3 Preservation of GβE density

A fundamental result is that starting from all eigenvalues at 0, the distribution of $\lambda(t)$ at time t has the joint density proportional to

$$\prod_{i<j} |\lambda_i - \lambda_j|^\beta \ \exp\Big\{-\tfrac{1}{2t}\sum_i \lambda_i^2\Big\},$$

matching the Gaussian β-Ensemble (GβE) law. Hence DBM provides a dynamical realization of GβE. Invariance can be checked by verifying that this density is annihilated by the generator of the SDE.

1.4 Harish–Chandra–Itzykson–Zuber (HCIZ) integral

The HCIZ integral is a key tool for computing matrix integrals involving traces. For two Hermitian matrices A and B with eigenvalues $(a_1, \ldots, a_N)$ and $(b_1, \ldots, b_N)$, it states (in one common normalization):

$$\int_{U(N)} \exp\big(\mathrm{Tr}(A\,U\,B\,U^\dagger)\big)\, dU \ = \ \prod_{k=1}^{N-1} k! \ \frac{\det\big[e^{\,a_i b_j}\big]_{i,j=1}^N}{\prod_{1\le i<j\le N}(a_j - a_i)\ \prod_{1\le i<j\le N}(b_j - b_i)}\,.$$

This formula is instrumental in deriving transition densities for $\beta = 2$ Dyson Brownian Motion.

2 Optional: proof of HCIZ integral via representation theory

3 Determinantal structure for $\beta = 2$

3.1 Transition density

Theorem 3.1 ($\beta = 2$ Dyson Brownian Motion Transition Probabilities)**.** *For $\beta = 2$, let $\lambda(t) = (\lambda_1(t) \ge \cdots \ge \lambda_N(t))$ follow Dyson Brownian Motion starting at $\lambda(0) = \mathbf{a} = (a_1 \ge \cdots \ge a_N)$. Then for each fixed time $t > 0$,*

$$P\big(\lambda(t) = \mathbf{x} \mid \lambda(0) = \mathbf{a}\big) \ = \ N!\,\big(\frac{1}{\sqrt{2\pi t}}\big)^N \prod_{1\le i<j\le N} \frac{x_i - x_j}{a_i - a_j} \ \det\Big[\exp\Big(-\frac{(x_i - a_j)^2}{2t}\Big)\Big]_{i,j=1}^N,$$

where $x_1 \ge \cdots \ge x_N$.

Proof. Consider an $N \times N$ Hermitian matrix process $X(t)$ whose entries perform independent complex Brownian motions (so that $X(t)$ is distributed as $A + \sqrt{t}\,\mathrm{GUE}$ at each fixed time, with $A = \mathrm{diag}(a_1, \ldots, a_N)$). Its eigenvalues $\lambda_1(t) \ge \cdots \ge \lambda_N(t)$ evolve exactly according to the $\beta = 2$ Dyson Brownian Motion.

The density of X at time t, viewed as a random matrix, is proportional to

$$\exp\Big(-\tfrac{1}{2t}\,\mathrm{Tr}\big(X - A\big)^2\Big).$$

If we replace A by $U\,A\,U^\dagger$ for any fixed unitary U, the law of X remains the same (this follows from the unitary invariance of the GUE). Thus the distribution of the eigenvalues of X is unchanged by such conjugation.

One writes

$$\int_{U(N)} \exp\Bigl(-\tfrac{1}{2t}\,\mathrm{Tr}\bigl(X - U\,A\,U^\dagger\bigr)^2\Bigr)\,dU \;=\; (\text{const.}) \times [\text{HCIZ integral in the variables } (X, A)],$$

which by the Harish–Chandra–Itzykson–Zuber formula leads to a product of determinants and a factor that is precisely

$$\exp\Bigl(-\tfrac{1}{2t}\sum_{i=1}^{N} x_i^2 - \tfrac{1}{2t}\sum_{i=1}^{N} a_i^2\Bigr)\,\frac{\det\Bigl[\exp\bigl(\frac{x_i\,a_j}{t}\bigr)\Bigr]}{\prod_{i<j}(x_i - x_j)\,(a_i - a_j)},$$

where $x_1, \ldots, x_N$ are the eigenvalues of X.

To convert this matrix distribution into a distribution on eigenvalues alone, we multiply by the usual Vandermonde Jacobian $\prod_{i<j}(x_i - x_j)^2$ (which comes from integrating out the unitary degrees of freedom). This produces exactly

$$N!\,\bigl(\tfrac{1}{\sqrt{2\pi t}}\bigr)^N \prod_{i<j}\frac{x_i - x_j}{a_i - a_j}\,\det\Bigl[\exp\Bigl(-\tfrac{(x_i - a_j)^2}{2t}\Bigr)\Bigr].$$

Hence we obtain the stated transition probability for the Dyson Brownian Motion at $\beta = 2$. $\square$

Remark 3.2. The factor $N!\,(\frac{1}{\sqrt{2\pi t}})^N$ arises naturally from normalizing the Gaussian increments and accounts for the ordering $\lambda_1 \geq \cdots \geq \lambda_N$. The determinant and product factors encode the eigenvalue "repulsion" characteristic of $\beta = 2$ random matrices.

3.2 Determinantal correlations

Theorem 3.3 (Determinantal structure for $\beta = 2$ DBM)**.** *Let $\{x_1(t), \ldots, x_n(t)\}$ be the eigenvalues at time $t > 0$ of the $\beta = 2$ Dyson Brownian Motion started at initial locations $(a_1, \ldots, a_n)$ at time 0. Equivalently, these $x_i(t)$ are the eigenvalues of*

$$A + \sqrt{t}\,G,$$

where $A = \mathrm{diag}(a_1, \ldots, a_n)$ and G is a random Hermitian matrix from the GUE. Then the (random) point configuration $\{x_i(t)\}$ forms a determinantal point process with correlation kernel

$$K_t(x, y) = \frac{1}{(2\pi i)^2\,t} \oint\oint \exp\Bigl(\frac{w^2 - 2\,y\,w}{2\,t}\Bigr) \Big/ \exp\Bigl(\frac{z^2 - 2\,x\,z}{2\,t}\Bigr) \prod_{i=1}^{n}\frac{w - a_i}{z - a_i}\,\frac{dw\,dz}{w - z}.$$

Here z goes around all the points $a_1, \ldots, a_n$, and the w contour passes from $-i\infty$ to $i\infty$, to the right of the z contour.

- If $a_1 = \cdots = a_n = 0$ and $t = 1$, this kernel reduces to the familiar correlation kernel of the GUE (see Lecture 6).
- One can use this formula to study the Baik–Ben Arous–Péché (BBP) [BBP05] phase transition for $\beta = 2$, which deals with finite rank perturbations of the GUE random matrix ensemble. Indeed, rank r perturbation corresponds to taking $a_1, \ldots, a_r \neq 0$, and $a_{r+1} = \cdots = a_n = 0$.

3.3 On the proof of determinantal structure

The idea of the proof of Theorem 3.3 is to represent the measure (the transition density) as a product of determinants. In general, if a measure is given as a product of determinants, there is a well-studied method (biorthogonal ensembles and, more generally, the Eynard–Mehta theorem) to compute the determinantal correlation kernel. We refer to [BR05], [Bor11] for a detailed exposition in the discrete case (which is arguably more transparent). The first step for the Dyson Brownian Motion is as follows.

Lemma 3.4 (Density representation)**.** *Let $P_t(x \to y)$ be the transition probability kernel of standard Brownian motion,*

$$P_t(x \to y) \;=\; \frac{1}{\sqrt{2\pi\, t}}\,\exp\Bigl(-\tfrac{(x-y)^2}{2\,t}\Bigr).$$

Then the density of the eigenvalues $(x_1, \dots, x_N)$ of DBM started at $(a_1, \dots, a_N)$ at time 0 admits the representation

$$\lim_{s\to\infty}\left(\frac{1}{Z}\right)\,\det\Bigl[P_t\bigl(a_i \to x_j\bigr)\Bigr]_{i,j=1}^{N}\,\det\Bigl[P_s\bigl(x_i \to k-1\bigr)\Bigr]_{i,k=1}^{N}. \tag{3.1}$$

Remark 3.5. This representation (3.1) is related to an alternative description of the $\beta = 2$ Dyson Brownian Motion as an ensemble of noncolliding Brownian motions (that is, independent Brownian motions, conditioned to never collide).

Proof of Lemma 3.4. The first determinant (as $s \to \infty$) matches the determinant we have in Theorem 3.1. It remains to analyze the second determinant

$$\det\Bigl[P_s\bigl(x_j \to k-1\bigr)\Bigr]_{j,k=1}^{N} \;=\; \det\Bigl[\tfrac{1}{\sqrt{2\pi\, s}}\,\exp\Bigl(-\tfrac{\bigl((k-1)-x_j\bigr)^2}{2\,s}\Bigr)\Bigr]_{j,k=1}^{N}.$$

We may ignore the factor $\frac{1}{\sqrt{2\pi\, s}}$ in each entry since it does not depend on x_j. Inside the exponential,

$$-\,\frac{\bigl((k-1)-x_j\bigr)^2}{2\,s} \;=\; -\,\frac{x_j^2}{2\,s} \;+\; \frac{x_j\,(k-1)}{s} \;-\; \frac{(k-1)^2}{2\,s}.$$

Thus, up to the factor $\exp\bigl(-\frac{(k-1)^2}{2\,s}\bigr)$ (which depends only on k and hence is independent of each x_j), we can factor out $\exp\bigl(-\frac{x_j^2}{2\,s}\bigr)$ from row j. Consequently, the nontrivial part of the determinant becomes

$$\det\Bigl[e^{\frac{x_j\,(k-1)}{s}}\Bigr]_{j,k=1}^{N}.$$

Recognize this as a Vandermonde-type determinant in the variables $e^{x_j/s}$. Indeed,

$$\det\Bigl[e^{\frac{x_j\,(k-1)}{s}}\Bigr]_{j,k=1}^{N} \;=\; \prod_{1\le i<j\le N}\Bigl(e^{\frac{x_i}{s}} - e^{\frac{x_j}{s}}\Bigr).$$

As $s \to \infty$, we expand $e^{\frac{x_i}{s}} = 1 + \frac{x_i}{s} + O\bigl(\frac{1}{s^2}\bigr)$, so each difference $\Bigl(e^{\frac{x_i}{s}} - e^{\frac{x_j}{s}}\Bigr) \sim \frac{x_i - x_j}{s}$. Hence,

$$\prod_{1\le i<j\le N}\Bigl(e^{\frac{x_i}{s}} - e^{\frac{x_j}{s}}\Bigr) \;\sim\; \frac{1}{s^{\frac{N(N-1)}{2}}}\prod_{1\le i<j\le N}(x_i - x_j).$$

Combining all these factors and matching with the first determinant (as $s \to \infty$) verifies the claimed product form, up to overall constants that do not depend on the variables x_j. This completes the proof. $\square$

Then, the product of determinants idea (biorthogonal ensembles) applies to the density (3.1) before the limit $s \to \infty$, and simplifies after taking the limit. We omit the details here, see Problem K.1.

K Problems (due 2025-04-29)

K.1 Biorthogonal ensembles

Derive Theorem 3.3 from Lemma 3.4 using the orthogonalization process similar to Lecture 5, and then taking the limit as $s \to \infty$.

K.2 Scaling of the kernel

Let $a_i = 0$ in Theorem 3.3. Find α such that $t^\alpha K_t(x/\sqrt{t}, y/\sqrt{t})$ is independent of t. Can you explain this value of α?

References

[BBP05] J. Baik, G. Ben Arous, and S. Péché, *Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices*, Ann. Probab. **33** (2005), no. 5, 1643–1697. arXiv:math/0403022 [math.PR]. ↑3

[Bor11] A. Borodin, *Determinantal point processes*, Oxford handbook of random matrix theory, 2011. arXiv:0911.1153 [math.PR]. ↑4

[BR05] A. Borodin and E.M. Rains, *Eynard–Mehta theorem, Schur process, and their Pfaffian analogs*, J. Stat. Phys **121** (2005), no. 3, 291–317. arXiv:math-ph/0409059. ↑4

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