Random Fibonacci Words

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Abstract

Fibonacci words are words of 1's and 2's, graded by the total sum of the digits. They form a differential poset ($\mathbb{YF}$) which is an estranged cousin of the Young lattice powering irreducible representations of the symmetric group. We introduce families of "coherent" measures on $\mathbb{YF}$ depending on many parameters, which come from the theory of clone Schur functions [Oka94]. We characterize parameter sequences ensuring positivity of the measures, and we describe the large-scale behavior of some ensembles of random Fibonacci words. The subject has connections to total positivity of tridiagonal matrices, Stieltjes moment sequences, orthogonal polynomials from the (q-)Askey scheme, and residual allocation (stick-breaking) models.

What is this text

These are notes for a chalk talk, prepared based on the paper [PS24], in the "extended lecture notes" style, similar to my random matrix course. Along the notes, there are numerous skipped details, which are left as exercises for the reader.

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1 Motivation 1. De Finetti's theorem and Pascal triangle

1.1

Definition 1.1. A sequence $X_1, X_2, \ldots$ of binary random variables (taking values in $\{0, 1\}$) is called *exchangeable* if for any n and any permutation σ of $\{1, 2, \ldots, n\}$ the joint distribution of $X_1, X_2, \ldots, X_n$ is the same as the joint distribution of $X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}$.

Exchangeable sequences are more than just Bernoulli iid sequences with some parameter $p \in [0,1]$. Consider the Polya urn scheme.

Start with an urn containing b black and w white balls. At each step, draw a ball uniformly at random from the urn and put it back along with another ball of the same color.

The sequence of ball colors drawn from the urn is exchangeable.

At time n, there are n new balls in the urn, and the distribution of the number of, say, black balls,

$$\mathbb{P}\,(\text{black} = k) = M_n(k), \qquad k = 0, 1, \dots, n,$$

is called the (n-th) *coherent measure*. We can talk about S_n, the random variable which is the number of black balls drawn by time n. The coherent measures M_n for various n satisfy certain linear recurrence relations.

One can convince oneself that the space of coherent measures is the same as the space of exchangeable random sequences of 0's and 1's. This space is a convex set, moreover, it is a simplex.

Definition 1.2. A point A in a convex linear set is called *extremal* if it cannot be written as a convex combination of other points in the set. A simplex is a convex set in which every point is a unique convex combination of extremal points.

Examples: triangle vs square vs disc.

Extreme points of the simplex corresponding to the Pascal triangle are given by iid sequences, that is, Bernoulli product measures on $\{0,1\}^\infty$. This is de Finetti's theorem.

1.2

Coherent measures on Pascal triangle are related to exchangeable sequences of 0's and 1's. The *boundary* of the Pascal triangle encodes all possible coherent measures via the law of large numbers,

$$\frac{S_n}{n} \to \mu \quad \text{on} \quad [0,1].$$

Extreme measures correspond to delta point masses. For example, the Polya urn for $a = b = 1$ corresponds to μ being the uniform measure on $[0,1]$.

1.3 Lonely paths

There are two distinguished paths in the Pascal triangle, the *lonely paths* $0 \to 00 \to 000 \to \dots$ and $1 \to 11 \to 111 \to \dots$, which are characterized by the property that [GK00b]

> All but finitely many vertices in the path have a single immediate predecessor.

These paths correspond to the extreme measures with $\mu = \delta_0$ and $\mu = \delta_1$, respectively.

It turns out that all other extreme measures on the Pascal triangle are obtained by a *"convex interpolation" of these two lonely path measures*. Note that this interpolation is *not* the same as the convex combination of coherent measures, so the points $p \in (0,1)$ are still extremal for the space of coherent measures. However, the boundary of the Pascal triangle clearly contains the linear piece between δ_0 and δ_1.

2 Motivation 2. Young lattice

The Young lattice $\mathbb{Y}$ of integer partitions ordered by the relation "adding a box" encodes another meaningful structure — irreducible representations of the symmetric groups. The boundary encodes the irreducible representations of the infinite symmetric group $S(\infty)$.

2.1

The Young lattice is a *differential poset* [Sta88], [Fom94], in the sense that

> for each λ, there is one more element in the set $\{\nu\colon \nu = \lambda + \square\}$ than in the set $\{\mu\colon \mu = \lambda - \square\}$.

Differential poset property implies that for f^λ the number of paths from $\varnothing$ to λ, we have

$$\sum_{|\lambda|=n} (f^\lambda)^2 = n!, \qquad \text{define} \qquad M_n(\lambda) := \frac{(f^\lambda)^2}{n!}.$$

The measure M_n is called *Plancherel*, it is coherent and extremal. It corresponds to the regular representation of $S(\infty)$, which is irreducible.

2.2

There are two lonely paths here, as well — corresponding to growing one-row and one-column partitions.

2.3

All extreme coherent measures on the Young lattice are given by specializations of Schur symmetric functions, and have the form

$$M_n(\lambda) = s_\lambda(\vec{\alpha}; \vec{\beta}; \gamma) \cdot f^\lambda.$$

The problem of describing the boundary of $\mathbb{Y}$ is equivalent to the problem of finding parameters $\vec{\alpha}, \vec{\beta}, \gamma$ such that the Schur functions $s_\lambda(\vec{\alpha}; \vec{\beta}; \gamma)$ are nonnegative for all λ.

Schur functions are (essentially) determinants, and for the Young lattice, we have a great match between these multiparameter functions and extreme coherent measures. The algebraic combinatorial property of the Schur polynomials which connects them to the Young lattice is the Pieri rule:

$$p_1 s_\lambda = \sum_{\nu\colon \nu = \lambda + \square} s_\nu.$$

Remark 2.1. The parameters $\vec{\alpha}; \vec{\beta}; \gamma$ encode the law of large numbers for the growing random Young diagram. The parameters α_i and β_i are the lengths of the i-th row and column scaled by n^{-1}, and γ is the scaled excess $1 - \sum(\alpha_i + \beta_i)$. For the Plancherel measure, rows and columns grow as $\sqrt{n}$, so $\alpha_i = \beta_i = 0$ and $\gamma = 1$.

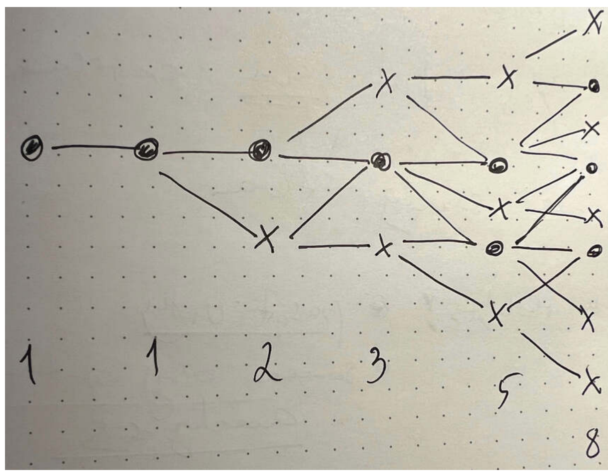
3 Another differential poset — the Young–Fibonacci lattice

3.1

A natural question arises: do other differential posets exist? Indeed, there exists another fundamental example, denoted $\mathbb{YF}$, which, upon first examination, might seem contrived and unnatural. (While there also exists a family of posets interpolating between $\mathbb{YF}$ and $\mathbb{Y}$, we shall not explore that here.)

3.2

The Young–Fibonacci lattice $\mathbb{YF}$ can be formed starting from the single edge $\varnothing \to 1$, by successive reflection. We then encode the new vertices as starting from 1 (followed by the old vertex index from the level $n-1$), and the reflected vertices as starting from 2 (followed by the old vertex index from the level $n-2$).



3.3

$\mathbb{YF}$ is a graded poset formed by Fibonacci words (binary words whose digits lie in $\{1, 2\}$), graded by the sum of their digits.

We denote the set of all Fibonacci words of weight n by $\mathbb{YF}_n$. Clearly, the total number of such words is the nth Fibonacci number (with $F_0 = F_1 = 1$). The poset $\mathbb{YF}$ is then the disjoint union of all $\mathbb{YF}_n$ for $n = 0, 1, 2, \ldots$, with rank function given by the weight $|w| = n$. We always identify the empty word $\varnothing$ with $\mathbb{YF}_0$.

Definition 3.1 (Young–Fibonacci Partial Order)**.** We say a Fibonacci word w *covers* another Fibonacci word v if $|v| = |w| - 1$ and one can transform w to v by one of the following rules:

1. If $w = 1\,v$, then we delete the leftmost 1 to obtain v.
2. If $w = 2\,u$ for some u, then we obtain v by turning the leftmost 2 into a 1 or by removing the leftmost inserted 1 after a 2.

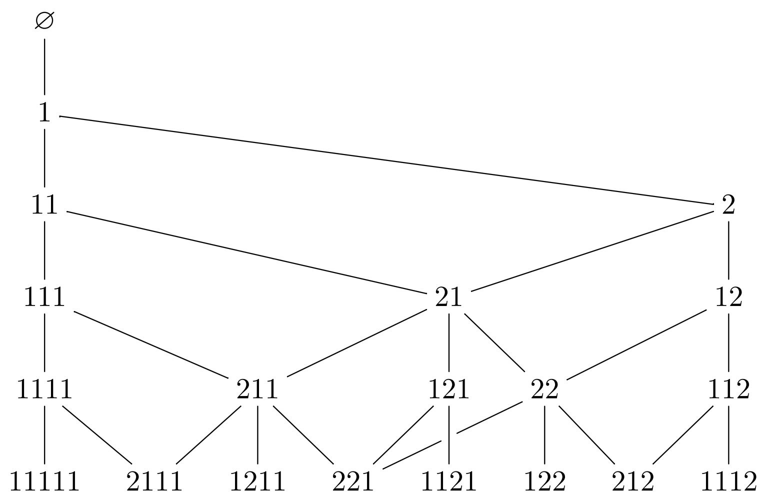


Figure 1: The Young–Fibonacci lattice up to level $n = 5$.

3.4

The Young–Fibonacci lattice is a differential poset. Hence, we have

$$\sum_{|w|=n} (\dim w)^2 = n!,$$

and we can define the *Plancherel measure*

$$M_n(w) \coloneqq \frac{(\dim w)^2}{n!}.$$

Note that the $\mathbb{YF}$-dimension is very different from the Young lattice one. For $w \in \mathbb{YF}_n$ of the form $w = a_1\, a_2 \cdots a_\ell$, we have

$$\dim(w) \;=\; \prod_{\substack{1 \le j \le \ell \\ a_j = 2}} \bigl(|u_j| + 1\bigr),$$

where u_j is the subword to the right of the j-th digit.

The Plancherel measure is extremal.

3.5 Boundary problem

We would like to understand the boundary of $\mathbb{YF}$. As in the Young and Pascal cases, the boundary should capture the law of large numbers for the growing Fibonacci words.

3.6 Lonely paths

In contrast with the Young lattice and the Pascal triangle, the Young–Fibonacci lattice has many lonely paths. Namely, there is a lonely path from each Fibonacci word w:

$$1w, \quad 11w, \quad 111w, \quad \dots$$

We denote it by $1^\infty w$. Lonely paths correspond to extreme measures, so the boundary has a "discrete component" $1^\infty\mathbb{YF}$.

The full boundary looks as the Plancherel point, connected to all points $1^\infty w$, $w \in \mathbb{YF}$, by linear segments (via the "convex interpolation" as in the Pascal case — recall that these segments are still extremal for coherent measures). Graphically, the boundary is a "star" with the Plancherel point in the center.

3.7 Boundary description — references

The boundary of the Young–Fibonacci lattice was established in the following works:

- [GK00b] described the Martin boundary, which is the set of all coherent measures obtained by finite rank approximation. It remained an open problem to show that this list is of extreme measures.
- [GK00a], shown that the Plancherel measure is extremal (ergodic), by considering the scaling limit of Plancherel random Fibonacci words. They essentially show that this limit is incompatible with any other possible point from the Martin boundary, thus leading to the extremality.
- Preprints [BE20], [Evt20] established the full boundary description by showing the extremality (ergodicity) of all coherent measures.

3.8 How about Schur functions?

While we now understand the boundary's structure, a natural question arises: are there elegant functions, analogous to determinantal Schur functions, that capture the combinatorial properties of this lattice? Indeed, such functions exist - the *clone Schur functions* introduced by Okada [Oka94]. These functions were specifically developed to provide an algebraic framework for the Young–Fibonacci lattice, paralleling how classical Schur functions encode the structure of the Young lattice.

The clone Schur functions $s_w(\vec{x} \mid \vec{y})$ (definition later) satisfy a Pieri rule:

$$x_{|w|+1} s_w(\vec{x} \mid \vec{y}) = \sum_{v\colon v \nearrow w} s_v(\vec{x} \mid \vec{y}).$$

There are *clone coherent measures* defined from clone Schur functions,

$$M_n(w) = s_w(\vec{\alpha}; \vec{\beta}; \gamma) \cdot \dim w,$$

but **they are not extremal** (except for the Plancherel case).

3.9 Now, briefly, what we do with this

We get the following main results:

1. Complete classification of clone coherent measures which are positive. This is related to total positivity of tridiagonal matrices and Stieltjes moment problems. In fact, we obtain a new, narrower notion of tridiagonal positivity called *Fibonacci positivity*.

2. We describe a number of examples of Fibonacci positive specializations.

3. For several Fibonacci positive specializations, we consider the large-scale behavior of random Fibonacci words.

4 Clone Schur functions and positivity

4.1 Definition

Let $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ be two families of indeterminates. Define two sequences of tridiagonal determinants as follows:

$$A_\ell(\vec{x} \mid \vec{y}) := \det \underbrace{\begin{pmatrix} x_1 & y_1 & 0 & \cdots \\ 1 & x_2 & y_2 & \\ 0 & 1 & x_3 & \\ \vdots & & & \ddots \end{pmatrix}}_{\ell \times \ell \text{ tridiagonal matrix}}, \qquad B_{\ell-1}(\vec{x} \mid \vec{y}) := \det \underbrace{\begin{pmatrix} y_1 & x_1 y_2 & 0 & \cdots \\ 1 & x_3 & y_3 & \\ 0 & 1 & x_4 & \\ \vdots & & & \ddots \end{pmatrix}}_{\ell \times \ell \text{ tridiagonal matrix}}. \tag{4.1}$$

Here $\ell \geq 0$. For a sequence $\vec{u} = (u_1, u_2, \ldots)$, denote its *shift* by $\vec{u} + \ell = (u_{1+\ell}, u_{2+\ell}, \ldots)$, where $\ell \in \mathbb{Z}_{\geq 0}$.

Definition 4.1. For any Fibonacci word w, define the (*biserial*) *clone Schur function* $s_w(\vec{x} \mid \vec{y})$ through the following recurrence:

$$s_w(\vec{x} \mid \vec{y}) := \begin{cases} A_k(\vec{x} \mid \vec{y}), & \text{if } w = 1^k \text{ for some } k \geq 0, \\ B_k\big(\vec{x} + |u| \mid \vec{y} + |u|\big) \cdot s_u(\vec{x} \mid \vec{y}), & \text{if } w = 1^k 2u \text{ for some } k \geq 0. \end{cases} \tag{4.2}$$

Note that these functions are not symmetric in the variables, and the order in the sequences $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ is important.

4.2 Positivity problem: reduction to tridiagonal matrices

For the positivity of the functions $s_w(\vec{x} \mid \vec{y})$, it is necessary that the infinite tridiagonal matrix

$$\begin{pmatrix} x_1 & y_1 & 0 & \cdots \\ 1 & x_2 & y_2 & \\ 0 & 1 & x_3 & \\ \vdots & & & \ddots \end{pmatrix} \tag{4.3}$$

is totally positive (that is, all its minors that are not identically zero must be positive).

Total positivity of tridiagonal matrices is a well-known phenomenon [FZ99]. We have a stronger requirement than just the total positivity of (4.3) — we need the total positivity of another family of matrices,

$$\mathcal{B}_r(\vec{x} \mid \vec{y}) := \begin{pmatrix} y_{r+1} & x_{r+1} y_{r+2} & 0 & \cdots \\ 1 & x_{r+3} & y_{r+3} & \\ 0 & 1 & x_{r+4} & \\ \vdots & & & \ddots \end{pmatrix},$$

for all r. The tridiagonal matrix (4.3) is a good starting point, though: it allows us to reparameterize

$$x_k = 1 + t_{k-1}, \qquad y_k = t_k, \qquad t_0 = 0, \qquad t_j > 0, \quad j \geq 1.$$

(There are some obvious renormalizations of the parameters $\vec{x}, \vec{y}$ which we ignore, and focus only on the primary case.)

4.3 Fibonacci positivity: result

There are two classes of $\vec{t}$-sequences for which the specializations of clone Schur functions are positive.

Theorem 4.2. *All Fibonacci positive sequences $(\vec{x}, \vec{y})$ have the form*

$$x_k = c_k(1 + t_{k-1}), \qquad y_k = c_k c_{k+1} t_k, \qquad k \geq 1,$$

where $\vec{c}$ is an arbitrary positive sequence, and $\vec{t} = (t_1, t_2, \ldots)$ (with $t_0 = 0$, for convenience) is a positive real sequence of one of the two types:

- *(divergent type) The infinite series*

$$1 + t_1 + t_1 t_2 + t_1 t_2 t_3 + \ldots \tag{4.4}$$

  diverges, and $t_{m+1} \geq 1 + t_m$ for all $m \geq 1$;

- *(convergent type) The series (4.4) converges, and*

$$1 + t_{m+3} + t_{m+3} t_{m+4} + t_{m+3} t_{m+4} t_{m+5} + \ldots \geq \frac{t_{m+1}}{t_{m+2}(1 + t_m - t_{m+1})}, \qquad \text{for all } m \geq 0.$$

The sequences $\vec{c}$ and $\vec{t}$ are determined by $(\vec{x}, \vec{y})$ uniquely.

A divergent type sequence can be written as

$$t_k = k + \varepsilon_1 + \ldots + \varepsilon_k,$$

where $\varepsilon_j \geq 0$. Then the matrices (4.3) and $\mathcal{B}_r$ have all minors either identically zero, or element of $\mathbb{Z}[\varepsilon_1, \varepsilon_2, \ldots]$ with *positive coefficients.*

4.4 Examples for which we do scaling limits

- Plancherel: $x_k = y_k = k$, so $t_k = k$;
- A two-parameter deformation: $x_k = k + \rho + \sigma - 2$, $y_k = (k + \sigma - 1)\rho$, where $\sigma \geq 1$ and $0 < \rho \leq 1$.

Other examples come from orthogonal polynomials in the (q-)Askey scheme. We describe the framework next.

4.5 Stieltjes moment problem

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