# Lectures on Random Matrices (Spring 2025)
# Lecture 11: Some universal asymptotics of Dyson Brownian Motion

Leonid Petrov

March 26, 2025[*]

## Contents

---

1

# 1  Recap

## 1.1  Dyson Brownian Motion (DBM)

We introduced a time-dependent model of random matrices by letting an $N \times N$ Hermitian matrix $\mathcal{M}(t)$ evolve in time so that each off-diagonal entry follows independent Brownian increments (real or complex depending on the symmetry class). Setting

$$\mathcal{M}(t) \;=\; \frac{1}{\sqrt{2}}\big(X(t) + X^{\dagger}(t)\big),$$

where $X(t)$ is an $N \times N$ matrix of i.i.d. Brownian motions, produces a self-adjoint matrix with a stochastically evolving spectrum. This model is full-rank matrix Brownian motion, and works well for $\beta = 1, 2, 4$. For other $\beta$, we need an SDE to describe the evolution of the eigenvalues (particles).

## 1.2  Eigenvalue SDE

Denote by $\lambda_1(t) \geq \cdots \geq \lambda_N(t)$ the ordered eigenvalues of $\mathcal{M}(t)$. Dyson showed that these eigenvalues form a continuous-time Markov process satisfying the SDE

$$d\lambda_i(t) \;=\; \frac{\beta}{2} \sum_{j \neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)} \;+\; dW_i(t), \quad i = 1, \ldots, N,$$

where $\beta > 0$ and $W_i(t)$ are independent standard real Brownian motions. For classical random matrix ensembles ($\beta = 1, 2, 4$), this SDE describes how the eigenvalues evolve under real symmetric (GOE), Hermitian (GUE), or quaternionic (GSE) Brownian motion — in the last Lecture 10 we discussed the cases $\beta = 1, 2$ in detail. A key feature is the *repulsion* term $\frac{1}{\lambda_i - \lambda_j}$, which prevents collisions (and ensures the ordering remains intact).

## 1.3  Preservation of G$\beta$E density

A fundamental result is that starting from all eigenvalues at $0$, the distribution of $\lambda(t)$ at time $t$ has the joint density proportional to

$$\prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \; \exp\Big\{-\tfrac{1}{2t} \sum_i \lambda_i^2\Big\},$$

matching the Gaussian $\beta$-Ensemble (G$\beta$E) law. Hence DBM provides a dynamical realization of G$\beta$E. Invariance can be checked by verifying that this density is annihilated by the generator of the SDE.

## 1.4  Harish–Chandra–Itzykson–Zuber (HCIZ) integral

The HCIZ integral is a key tool for computing matrix integrals involving traces. For two Hermitian matrices $A$ and $B$ with eigenvalues $(a_1, \ldots, a_N)$ and $(b_1, \ldots, b_N)$, it states (in one common normalization):

$$\int_{U(N)} \exp\big(\mathrm{Tr}(A\,U\,B\,U^{\dagger})\big)\, dU \;=\; \prod_{k=1}^{N-1} k! \; \frac{\det\big[e^{\,a_i b_j}\big]_{i,j=1}^{N}}{\prod_{1 \leq i < j \leq N}(a_j - a_i) \prod_{1 \leq i < j \leq N}(b_j - b_i)}.$$

This formula is instrumental in deriving transition densities for $\beta = 2$ Dyson Brownian Motion.

## 2 Optional: proof of HCIZ integral via representation theory

In this section, we outline a standard argument (adapted from the theory of symmetric functions and representation theory of the unitary group) that leads to a proof of the Harish–Chandra–Itzykson–Zuber formula. It is often referred to as the "orbital integral" or "character expansion" approach.

**Step 1. Setting up the integral and Schur expansions.** Let $A$ and $B$ be two $N \times N$ diagonalizable matrices, with eigenvalues $a_1, \ldots, a_N$ and $\lambda_1, \ldots, \lambda_N$ respectively. Denote by $D_a = \mathrm{diag}(a_1, \ldots, a_N)$ and $D_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. We want to evaluate the integral

$$I = \int_{U(N)} \exp\big(\mathrm{Tr}(D_a\, U\, D_\lambda\, U^\dagger)\big)\, dU$$

over the Haar measure on $U(N)$.

Since $\mathrm{Tr}(B) = p_1(B)$ in the language of power sums (where $p_1(x_1, x_2, \ldots) = x_1 + x_2 + \ldots$), we have

$$\exp\big(\mathrm{Tr}(B)\big) = \exp\big(p_1(B)\big).$$

One can use a known expansion [Mac95]

$$e^{p_1(B)} = \sum_{m=0}^{\infty} \frac{p_1^m(B)}{m!} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu:|\mu|=m} \dim(\mu)\, s_\mu(B),$$

where the sum is over all partitions $\mu$ of size $m$, and $s_\mu(\cdot)$ is the Schur polynomial (or Schur function) indexed by $\mu$. The coefficient $\dim(\mu)$ is the dimension of the corresponding representation of $S_m$.

We set $B = D_a\, U\, D_\lambda\, U^\dagger$ and write

$$I = \int_{U(N)} \exp\Big(\mathrm{Tr}(D_a\, U\, D_\lambda\, U^\dagger)\Big)\, dU = \int_{U(N)} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu:|\mu|=m} \dim(\mu)\, s_\mu\big(D_a\, U\, D_\lambda\, U^\dagger\big)\, dU.$$

One can exchange the integral and the sum (the series converges absolutely for all matrix arguments), giving

$$I = \sum_{m=0}^{\infty} \sum_{\mu:|\mu|=m} \frac{\dim(\mu)}{m!} \int_{U(N)} s_\mu(D_a\, U\, D_\lambda\, U^\dagger)\, dU. \tag{2.1}$$

**Step 2. Orthogonality of characters and the Unitary group.** The Schur functions $s_\mu(\cdot)$ can be seen as irreducible characters of the unitary group $U(N)$ (up to a normalization factor) when restricted to $N$-tuples of eigenvalues.[1]

---

[1] $s_\mu$ for $\ell(\mu) \leq N$ can be viewed as the character of the corresponding polynomial representation of $GL(N, \mathbb{C})$, then restricted to $U(N)$. If $\ell(\mu) > N$, the function $s_\mu$ vanishes on $U(N)$. Thus, we need to impose the condition $|a_i| = |\lambda_i| = 1$ (so that $D_a, D_\lambda \in U(N)$) to ensure immediate applicability of representation theory of $U(N)$, then extend to general $\{a_i\}$ and $\{\lambda_i\}$ by analytic continuation.

**Proposition 2.1** (Functional equation for characters of compact groups)**.** *Let $G$ be a compact group with normalized Haar measure $dh$, and let $\chi$ be an irreducible character of a finite-dimensional representation of $G$. Then for any elements $g_1, g_2 \in G$, the following relation holds:*

$$\int_G \chi(g_1 h g_2 h^{-1}) dh = \frac{\chi(g_1)\chi(g_2)}{\dim V}, \tag{2.2}$$

*where $\dim V = \chi(e)$ is the dimension of the representation space.*

**Remark 2.2.** A similar relation holds for characters of finite groups.

By Proposition 2.1, the integral over $U(N)$ in (2.1) can be evaluated as

$$\int_{U(N)} s_\mu(D_a U D_\lambda U^\dagger) \, dU = \frac{1}{\mathrm{Dim}_N(\mu)} \, s_\mu(a) \, s_\mu(\lambda),$$

where $\mathrm{Dim}_N(\mu)$ is the dimension of the corresponding irreducible representation of $U(N)$. Substituting back into (2.1) yields

$$I = \sum_{m=0}^{\infty} \sum_{\mu : |\mu|=m, \, \ell(\mu) \leq N} \frac{\dim(\mu)}{m!} \frac{1}{\mathrm{Dim}_N(\mu)} \, s_\mu(a) \, s_\mu(\lambda),$$

where $\ell(\mu) \leq N$ is needed for $s_\mu(\cdot)$ not to vanish on $U(N)$.

**Step 3. Hook-length formulas and the final determinant.** Next, one applies the hook-length formula and the hook-content formula to dimensions:

$$\dim \mu = \frac{|\mu|!}{\prod_{\square \in \mu} h(\square)}, \qquad \mathrm{Dim}_N(\mu) = \frac{\prod_{\square \in \mu}(N + c(\square))}{\prod_{\square \in \mu} h(\square)},$$

We have

$$\prod_{\square \in \mu}(N + c(\square)) = \prod_{i=1}^{N} \frac{(\mu_i + N - i)!}{(N - i)!},$$

so identifying $m_i = \mu_i + N - i$ gives

$$I = 0!1! \cdots (N-1)! \sum_{m_1 > \ldots > m_N \geq 0} \frac{s_\mu(a) s_\mu(\lambda)}{m_1! \cdots m_N!},$$

which yields the HCIZ formula by the Cauchy-Binet summation.

## 3 Determinantal structure for $\beta = 2$

### 3.1 Transition density

**Theorem 3.1** ($\beta = 2$ Dyson Brownian Motion Transition Probabilities)**.** *For $\beta = 2$, let $\lambda(t) = (\lambda_1(t) \geq \cdots \geq \lambda_N(t))$ follow Dyson Brownian Motion starting at $\lambda(0) = \mathbf{a} = (a_1 \geq \cdots \geq a_N)$. Then for each fixed time $t > 0$,*

$$P\big(\lambda(t) = \mathbf{x} \mid \lambda(0) = \mathbf{a}\big) = N! \left(\frac{1}{\sqrt{2\pi t}}\right)^N \prod_{1 \leq i < j \leq N} \frac{x_i - x_j}{a_i - a_j} \, \det\left[\exp\left(-\frac{(x_i - a_j)^2}{2t}\right)\right]_{i,j=1}^{N},$$

*where $x_1 \geq \cdots \geq x_N$.*

*Proof.* Consider an $N \times N$ Hermitian matrix process $X(t)$ whose entries perform independent complex Brownian motions (so that $X(t)$ is distributed as $A + \sqrt{t}\,\mathrm{GUE}$ at each fixed time, with $A = \mathrm{diag}(a_1, \ldots, a_N)$). Its eigenvalues $\lambda_1(t) \geq \cdots \geq \lambda_N(t)$ evolve exactly according to the $\beta = 2$ Dyson Brownian Motion.

The density of $X$ at time $t$, viewed as a random matrix, is proportional to

$$\exp\left(-\tfrac{1}{2t}\,\mathrm{Tr}(X - A)^2\right).$$

If we replace $A$ by $U\,A\,U^\dagger$ for any fixed unitary $U$, the law of $X$ remains the same (this follows from the unitary invariance of the GUE). Thus the distribution of the eigenvalues of $X$ is unchanged by such conjugation.

One writes

$$\int_{U(N)} \exp\left(-\tfrac{1}{2t}\,\mathrm{Tr}(X - U\,A\,U^\dagger)^2\right) dU \;=\; (\text{const.}) \times [\text{HCIZ integral in the variables } (X, A)],$$

which by the Harish–Chandra–Itzykson–Zuber formula leads to a product of determinants and a factor that is precisely

$$\exp\left(-\tfrac{1}{2t} \sum_{i=1}^{N} x_i^2 - \tfrac{1}{2t} \sum_{i=1}^{N} a_i^2\right) \frac{\det\left[\exp\left(\frac{x_i\, a_j}{t}\right)\right]}{\prod_{i<j}(x_i - x_j)\,(a_i - a_j)},$$

where $x_1, \ldots, x_N$ are the eigenvalues of $X$.

To convert this matrix distribution into a distribution on eigenvalues alone, we multiply by the usual Vandermonde Jacobian $\prod_{i<j}(x_i - x_j)^2$ (which comes from integrating out the unitary degrees of freedom). This produces exactly

$$N!\,\left(\tfrac{1}{\sqrt{2\pi t}}\right)^N \prod_{i<j} \frac{x_i - x_j}{a_i - a_j}\, \det\left[\exp\left(-\tfrac{(x_i - a_j)^2}{2t}\right)\right].$$

Hence we obtain the stated transition probability for the Dyson Brownian Motion at $\beta = 2$. □

**Remark 3.2.** The factor $N!\,\left(\tfrac{1}{\sqrt{2\pi t}}\right)^N$ arises naturally from normalizing the Gaussian increments and accounts for the ordering $\lambda_1 \geq \cdots \geq \lambda_N$. The determinant and product factors encode the eigenvalue "repulsion" characteristic of $\beta = 2$ random matrices.

## 3.2   Determinantal correlations

**Theorem 3.3** (Determinantal structure for $\beta = 2$ DBM). *Let $\{x_1(t), \ldots, x_n(t)\}$ be the eigenvalues at time $t > 0$ of the $\beta = 2$ Dyson Brownian Motion started at initial locations $(a_1, \ldots, a_n)$ at time $0$. Equivalently, these $x_i(t)$ are the eigenvalues of*

$$A + \sqrt{t}\,G,$$

*where $A = \mathrm{diag}(a_1, \ldots, a_n)$ and $G$ is a random Hermitian matrix from the GUE. Then the (random) point configuration $\{x_i(t)\}$ forms a determinantal point process with correlation kernel*

$$K_t(x, y) = \frac{1}{(2\pi i)^2\, t} \oint \oint \exp\left(\frac{w^2 - 2\,y\,w}{2\,t}\right) \Big/ \exp\left(\frac{z^2 - 2\,x\,z}{2\,t}\right) \prod_{i=1}^{n} \frac{w - a_i}{z - a_i}\, \frac{dw\,dz}{w - z}.$$

*Here $z$ goes around all the points $a_1, \ldots, a_n$, and the $w$ contour passes from $-i\infty$ to $i\infty$, to the right of the $z$ contour.*

- If $a_1 = \cdots = a_n = 0$ and $t = 1$, this kernel reduces to the familiar correlation kernel of the GUE (see Lecture 6).

- One can use this formula to study the Baik–Ben Arous–Péché (BBP) [BBP05] phase transition for $\beta = 2$, which deals with finite rank perturbations of the GUE random matrix ensemble. Indeed, rank $r$ perturbation corresponds to taking $a_1, \ldots, a_r \neq 0$, and $a_{r+1} = \cdots = a_n = 0$.

## 3.3 On the proof of determinantal structure

The idea of the proof of Theorem 3.3 is to represent the measure (the transition density) as a product of determinants. In general, if a measure is given as a product of determinants, there is a well-studied method (biorthogonal ensembles and, more generally, the Eynard–Mehta theorem) to compute the determinantal correlation kernel. We refer to [BR05], [Bor11] for a detailed exposition in the discrete case (which is arguably more transparent). The first step for the Dyson Brownian Motion is as follows.

**Lemma 3.4** (Density representation). *Let $P_t(x \to y)$ be the transition probability kernel of standard Brownian motion,*

$$P_t(x \to y) \;=\; \frac{1}{\sqrt{2\pi t}} \, \exp\!\Big(-\tfrac{(x-y)^2}{2t}\Big).$$

*Then the density of the eigenvalues $(x_1, \ldots, x_N)$ of DBM started at $(a_1, \ldots, a_N)$ at time $0$ admits the representation*

$$\lim_{s \to \infty} \left(\frac{1}{Z}\right) \det\!\Big[P_t\big(a_i \to x_j\big)\Big]_{i,j=1}^N \; \det\!\Big[P_s\big(x_i \to k - 1\big)\Big]_{i,k=1}^N. \tag{3.1}$$

**Remark 3.5.** This representation (3.1) is related to an alternative description of the $\beta = 2$ Dyson Brownian Motion as an ensemble of noncolliding Brownian motions (that is, independent Brownian motions, conditioned to never collide).

*Proof of Lemma 3.4.* The first determinant (as $s \to \infty$) matches the determinant we have in Theorem 3.1. It remains to analyze the second determinant

$$\det\!\Big[P_s\big(x_j \to k - 1\big)\Big]_{j,k=1}^N \;=\; \det\!\Big[\tfrac{1}{\sqrt{2\pi s}} \, \exp\!\Big(-\tfrac{\big((k-1)-x_j\big)^2}{2s}\Big)\Big]_{j,k=1}^N.$$

We may ignore the factor $\frac{1}{\sqrt{2\pi s}}$ in each entry since it does not depend on $x_j$. Inside the exponential,

$$-\frac{\big((k-1)-x_j\big)^2}{2s} \;=\; -\frac{x_j^2}{2s} \;+\; \frac{x_j\,(k-1)}{s} \;-\; \frac{(k-1)^2}{2s}.$$

Thus, up to the factor $\exp\left(-\frac{(k-1)^2}{2s}\right)$ (which depends only on $k$ and hence is independent of each $x_j$), we can factor out $\exp\left(-\frac{x_j^2}{2s}\right)$ from row $j$. Consequently, the nontrivial part of the determinant becomes

$$\det\left[e^{\frac{x_j\,(k-1)}{s}}\right]_{j,k=1}^{N}.$$

Recognize this as a Vandermonde-type determinant in the variables $e^{x_j/s}$. Indeed,

$$\det\left[e^{\frac{x_j\,(k-1)}{s}}\right]_{j,k=1}^{N} \;=\; \prod_{1\le i<j\le N}\left(e^{\frac{x_i}{s}} - e^{\frac{x_j}{s}}\right).$$

As $s \to \infty$, we expand $e^{\frac{x_i}{s}} = 1 + \frac{x_i}{s} + O\left(\frac{1}{s^2}\right)$, so each difference $\left(e^{\frac{x_i}{s}} - e^{\frac{x_j}{s}}\right) \sim \frac{x_i - x_j}{s}$. Hence,

$$\prod_{1\le i<j\le N}\left(e^{\frac{x_i}{s}} - e^{\frac{x_j}{s}}\right) \;\sim\; \frac{1}{s^{\frac{N(N-1)}{2}}}\prod_{1\le i<j\le N}(x_i - x_j).$$

Combining all these factors and matching with the first determinant (as $s \to \infty$) verifies the claimed product form, up to overall constants that do not depend on the variables $x_j$. This completes the proof. $\qquad\square$

Then, the product of determinants idea (biorthogonal ensembles) applies to the density (3.1) before the limit $s \to \infty$, and simplifies after taking the limit. We omit the details here, see Problem K.1.

# 4 Asymptotic analysis: signal plus noise

## 4.1 Setup

In this section, we provide a detailed derivation of how the rank-1 spike

$$A + \sqrt{G}, \qquad A = \operatorname{diag}(a, 0, \ldots, 0) \quad \text{with } a \in \mathbb{R},$$

affects the large-$n$ and large-time behavior of the Dyson Brownian Motion at $\beta = 2$.

We now set $a_1 = a$ and $a_2 = a_3 = \cdots = a_n = 0$, which simplifies the product:

$$\prod_{i=1}^{n}(w - a_i) \;=\; (w - a)\,w^{n-1}, \qquad \prod_{i=1}^{n}(z - a_i) \;=\; (z - a)\,z^{n-1}.$$

Hence, the correlation kernel becomes

$$K_t(x, y) \;=\; \frac{1}{(2\pi i)^2\,t}\oint\oint \exp\left(\frac{w^2 - 2\,y\,w}{2\,t}\right) \Big/ \exp\left(\frac{z^2 - 2\,x\,z}{2\,t}\right)\frac{(w - a)\,w^{n-1}}{(z - a)\,z^{n-1}}\frac{dw\,dz}{(w - z)}. \tag{4.1}$$

Here:

- The $z$-contour is a small positively oriented loop around $z = a$, and also around $z = 0$, so that it encircles these two singularities but excludes $w$.

- The $w$-contour is a vertical line (or an equivalent contour from $-i\infty$ to $i\infty$) passing to the right of all singularities (i.e. to the right of $z$).

Let us also take $t = 1$ for simplicity.

## 4.2 Outline of the steepest descent approach

We aim to understand the behavior of (4.1) in the regime $n \to \infty$, especially near the largest eigenvalue $\lambda_1(t)$. Recall from standard GUE (i.e. $a = 0$) that the top of the spectrum is about $2\sqrt{n}$. The presence of the rank-1 spike $a$ can drastically modify the top eigenvalue if $a$ is large enough to produce an "outlier." Our goal is to detect precisely how this occurs by analyzing the double contour integral via steepest descent.

For large $n$, the integral localizes around these points. Any crossing from $z$- to $w$-contour may pick up residues, which account for separate contributions (leading, for instance, to the sine or Airy kernel in the unperturbed GUE). We track how the spike $a$ changes these deformations.

We now carry out these steps in detail for (4.1).

## 4.3 Exponent form and critical points

Set

$$\Phi_{t,a}(w, z) = -\frac{1}{t}\left[\frac{z^2}{2} - x\,z\right] + \frac{1}{t}\left[\frac{w^2}{2} - y\,w\right] + (n-1)\big[\ln(w) - \ln(z)\big] + \ln(w - a) - \ln(z - a).$$

Then the integrand in (4.1) is essentially

$$\exp\big(\Phi_{t,a}(w, z)\big)\,\frac{1}{w - z},$$

up to the constant factor $\frac{1}{(2\pi i)^2 t}$. Note that $\Phi_{t,a}$ contains the GUE-like portion

$$\frac{1}{t}\left(\frac{w^2}{2} - y\,w - \frac{z^2}{2} + x\,z\right) + (n-1)\big[\ln(w) - \ln(z)\big],$$

plus the spike portion $\ln(w - a) - \ln(z - a)$.

**Remark 4.1.** For $\beta = 2$ and $a = 0$, we typically expand about $z \approx 1$, $w \approx 1$ to capture the top edge at $2\sqrt{t\,n}$. Indeed, setting $z = 1 + u, n \to \infty$ leads to the cubic expansions in $u$. Once $a \neq 0$, we get an extra shift in $z$, $w$ that might solve

$$\partial_z\,\Phi_{t,a} = 0, \quad \partial_w\,\Phi_{t,a} = 0.$$

We find solutions near $z = 1$, $w = 1$ if $|a| < 2$, but new solutions appear if $|a| > 2$.

**Scaling near the top.** We know from the unperturbed case $(a = 0)$ that $x, y$ near $2\sqrt{t\,n}$ is relevant. Let us adopt the usual *edge scaling*:

$$x = 2\sqrt{t\,n} + \frac{\xi}{n^{1/6}}, \qquad y = 2\sqrt{t\,n} + \frac{\eta}{n^{1/6}}. \tag{4.2}$$

Then one can attempt expansions $z = 1 + u\,n^{-1/3}$, $w = 1 + v\,n^{-1/3}$ (plus additional shifts if $a \neq 0$). Indeed, if $|a| < 2$, the main location for $z_{\mathrm{cr}}, w_{\mathrm{cr}}$ remains close to 1. Conversely, if $|a| > 2$, we get a separate solution.

Let us isolate the product factor first. Suppose we rewrite

$$(w - a)\,w^{n-1} = (w - a)\,\exp\big((n-1)\ln w\big), \qquad (z - a)\,z^{n-1} = (z - a)\,\exp\big((n-1)\ln z\big).$$

8

Then
$$\ln\big[(w - a)\, w^{n-1}\big] \;=\; (n - 1)\ln w \;+\; \ln(w - a).$$

Hence these combined terms become

$$(n - 1)\,[\ln w - \ln z] \;+\; \ln(w - a) \;-\; \ln(z - a).$$

Because $|a| < 2$ or $|a| > 2$ drastically changes the location(s) $z - a = 0$ or $w - a = 0$, the steepest descent approach tries to see where the real part of $\Phi_{t,a}(w, z)$ is stationary.

## 4.4 Controlling the integral: subcritical and supercritical analyses

### 4.4.1 Residue crossing and main contributions

As usual in the $\beta = 2$ story, we must deform the $z$-contour outward so that it crosses the $w$-contour. This can pick up a residue from the simple pole at $w = z$. In the pure GUE case ($a = 0$), that residue is known to yield the sine or Airy kernel asymptotics after expansions. Now, with the rank-1 spike, we get an additional factor from $(w - a)/(z - a)$. The upshot is that *if* the $z$ contour crosses $w$ in a certain region that does not include $w = a$, one obtains a residue that will lead to the same sine/Airy factor.

However, we must also check whether the $z$-contour encloses $z = a$ *and* the $w$-contour encloses $w = a$. The spike can produce an outlier contribution if $|a|$ is large enough for the new saddle point to dominate.

### 4.4.2 Case 1: $|a| < 2$ (subcritical)

For $|a| < 2$, one can show that the exponent $\Phi_{t,a}(w, z)$ remains effectively minimized near $z, w \approx 1$, and the presence of $(w - a)$ vs. $(z - a)$ does not shift that minimizer significantly. In more precise terms, one writes expansions:

$$z \;=\; 1 + \frac{u}{n^{1/3}}, \qquad w \;=\; 1 + \frac{v}{n^{1/3}},$$

with $x, y$ as in (4.2), and obtains a leading cubic term in $u, v$. The "$a$" correction modifies subleading expansions but does *not* cause a different order in $n$. Repeating the same steps as in the standard GUE edge analysis (cf. §**??**), one sees that the limit of $\frac{1}{n^{1/6}}\, K_t\big(x, y\big)$ is the Airy kernel

$$K_{\mathrm{Ai}}(\xi, \eta) \;=\; \int_0^{\infty} \cdots$$

(as in the usual expression, see **??** for one common form).

Consequently, in the subcritical regime $|a| < 2$, the top eigenvalue

$$x_{\max}(t) = x_1(t)$$

stays merged in the main bulk near $2\sqrt{t\, n}$ and has *Tracy–Widom* fluctuations on the $n^{-1/6}$ scale. The factor $a$ only shifts lower-order terms in that expansion.

### 4.4.3 Case 2: $|a| > 2$ (supercritical)

When $|a| > 2$, the spike is large. One expects an *outlier* eigenvalue near $x \approx a\sqrt{t}$ (rather than near $2\sqrt{t\,n}$). Indeed, analyzing $\Phi_{t,a}$ in that scenario, one finds a separate solution to $\partial\Phi/\partial w = 0$ near $w \approx a \neq 1$ (and similarly for $z$). The portion of the $z$-contour around $z = a$ then yields a new local expansion.

Concretely, if $a > 2$ (the case $a < -2$ is similar), the factor $(w - a)$ can be near zero, so that the exponent can develop a more favorable real part at $w \approx a$. Meanwhile, the factor $w^{n-1}$ is enormous if $|w| > 1$, but $a > 2$ means $|a| > 1$. As a result, the outlier solution with $w \approx a$ can produce a stable stationary phase. Indeed, one obtains a local *quadratic* expansion around $w = a$ and $z = a$, thus leading to a *Gaussian* fluctuation scale of the resulting outlier eigenvalue, typically $\sigma\, n^{-1/2}$ for some $\sigma > 0$.

Hence, in the supercritical regime $|a| > 2$, the top eigenvalue $x_{\max}(t)$ *detaches* from $2\sqrt{t\,n}$. Its leading order becomes $a\sqrt{t}$ plus smaller corrections, and the local fluctuations are usually governed by a (shifted) Gaussian (or in some references, an incomplete gamma distribution, depending on subtle normalizations). The remaining $n-1$ eigenvalues fill out the usual semicircle from $-2\sqrt{t\,n}$ to $2\sqrt{t\,n}$, with an Airy edge near $2\sqrt{t\,n}$.

### 4.4.4 Case 3: $|a| = 2$ (critical)

At the boundary $|a| = 2$, the system is at a "double root" transition. One finds that $z = 1$ (the standard GUE edge) merges with $z = a$ in a certain sense, leading to a triple or quartic expansion if $a = 2$, for instance. Careful expansions show that the outlier is *just* beginning to separate from the main bulk. The correlation kernel in this regime is a special transition "higher-order" Airy-like kernel, often described as a *crossover kernel* in the BBP (Baik–Ben Arous–Péché) transition [BBP05]. The top eigenvalue distribution is neither purely Tracy–Widom nor purely Gaussian but an interpolating limit distribution.

## 4.5 Summary of the BBP transition from $K_t(x, y)$

Putting the above cases together, we see that analyzing the DBM correlation kernel (4.1) under the spiked initial condition $(a_1 = a) \neq 0$, $(a_2 = \cdots = a_n = 0)$ reveals:

- **If $|a| < 2$:** The spike is *subcritical*. The top eigenvalue remains at the GUE edge $\approx 2\sqrt{t\,n}$, with Tracy–Widom fluctuations on scale $n^{-1/6}$.

- **If $|a| > 2$:** The spike is *supercritical*. An outlier eigenvalue emerges near $a\sqrt{t}$, detaching from the main bulk. The fluctuation scale around that outlier is typically $n^{-1/2}$ and converges (after centering and scaling) to a simpler, often Gaussian-type limit. The remainder of the spectrum still exhibits the semicircle on $[-2\sqrt{t\,n}, 2\sqrt{t\,n}]$.

- **If $|a| = 2$:** *Critical regime.* The top eigenvalue experiences a BBP crossover distribution, with a kernel that can be viewed as a limit of the general (4.1) by carefully choosing the saddle point expansions.

This completes the steepest descent derivation of the rank-1 BBP phase transition from the Dyson Brownian Motion correlation kernel in Theorem 3.3. Detailed error estimates and rigorous

Fredholm determinant limits (turning kernel statements into actual distributional statements about $x_{\max}(t)$) can be found in [BBP05], [?BloemendalVirag], and other references cited therein.

**Remark 4.2.** For $r$-rank spikes (i.e. $a_1, \ldots, a_r \neq 0$), one can repeat the same argument: the factor $\prod_{i=1}^{n}(w - a_i)/(z - a_i)$ in (4.1) splits into a product $(w - a_1) \cdots (w - a_r) \, w^{n-r}$ vs. $(z - a_1) \cdots (z - a_r) \, z^{n-r}$. Each $|a_i| > 2$ can produce a distinct outlier, while any $|a_i| < 2$ merges with the main bulk.

# K  Problems (due 2025-04-29)

## K.1  Biorthogonal ensembles

Derive Theorem 3.3 from Lemma 3.4 using the orthogonalization process similar to Lecture 5, and then taking the limit as $s \to \infty$.

## K.2  Scaling of the kernel

Let $a_i = 0$ in Theorem 3.3. Find $\alpha$ such that $t^\alpha K_t(x/\sqrt{t}, y/\sqrt{t})$ is independent of $t$. Can you explain this value of $\alpha$?

# References

[BBP05] J. Baik, G. Ben Arous, and S. Péché, *Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices*, Ann. Probab. **33** (2005), no. 5, 1643–1697. arXiv:math/0403022 [math.PR]. ↑6, 10, 11

[Bor11] A. Borodin, *Determinantal point processes*, Oxford handbook of random matrix theory, 2011. arXiv:0911.1153 [math.PR]. ↑6

[BR05] A. Borodin and E.M. Rains, *Eynard–Mehta theorem, Schur process, and their Pfaffian analogs*, J. Stat. Phys **121** (2005), no. 3, 291–317. arXiv:math-ph/0409059. ↑6

[Mac95] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995. ↑3

L. Petrov, University of Virginia, Department of Mathematics, 141 Cabell Drive, Kerchof Hall, P.O. Box 400137, Charlottesville, VA 22904, USA

E-mail: lenia.petrov@gmail.com