Lectures on Random Matrices (Spring 2025)
Lecture 6: Double contour integral kernel. Steepest descent and local statistics

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Contents

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Notes for the lecturer

- GUE det structure
- Formulate Cauchy–Binet and Andreief
- Recall that $\rho_n = P_n$ and it is $(\det[\psi_i(x_j)]_{n\times n})^2$, then reproduce the proofs here.
- Recall the Christoffel–Darboux formula:

$$K_n(x,y) = \frac{e^{-\frac{x^2+y^2}{4}}}{\sqrt{2\pi}h_{n-1}} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y},$$

  here $h_{n-1} = \sqrt{2\pi}(n-1)!$.

1 Recap: Determinantal structure of the GUE

Last time, we proved the following result:

Theorem 1.1. *The GUE correlation functions are given by*

$$\rho_k(x_1,\dots,x_k) = \det\Big[K_n(x_i,x_j)\Big]_{i,j=1}^{k},$$

with the correlation kernel

$$K_n(x,y) = \sum_{j=0}^{n-1} \psi_j(x)\psi_j(y).$$

Here

$$\psi_j(x) = \frac{1}{\sqrt{h_j}} p_j(x)\, e^{-x^2/4},$$

where $p_j(x)$ are the monic Hermite polynomials, and h_j are the normalization constants so that $\psi_j(x)$ are orthonormal in $L^2(\mathbb{R})$.

For this theorem, we need Cauchy–Binet summation formula and Andreief identity (which is essentially the same as Cauchy–Binet, but when summation is replaced by integration). Having these, we can write

$$\begin{aligned}
\rho_k(x_1,\dots,x_k) &= \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} p(x_1,\dots,x_n)\,dx_{k+1}\cdots dx_n \\
&= \frac{1}{(n-k)!\,\widehat{Z}_{n,2}} \sum_{\substack{\sigma,\tau\in S_n \\ \sigma(k+1)=\tau(k+1),\dots,\sigma(n)=\tau(n)}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{i=1}^{k}\psi_{\sigma(i)-1}(x_i)\psi_{\tau(i)-1}(x_i) \\
&= \mathrm{const}_n \sum_{I\subseteq[n],\,|I|=k}\ \sum_{\sigma',\tau'\in S(I)} \operatorname{sgn}(\sigma')\operatorname{sgn}(\tau')\prod_{i=1}^{k}\psi_{\sigma'(i)-1}(x_i)\psi_{\tau'(i)-1}(x_i)
\end{aligned}$$

$$= \text{const}_n \sum_{I\subseteq[n],\,|I|=k} \det\left[\psi_{i_\alpha}(x_j)\right]_{\alpha,j=1}^{k} \det\left[\psi_{i_\alpha}(x_j)\right]_{\alpha,j=1}^{k},$$

where $I = \{i_1, \ldots, i_k\}$ is a subset of $[n]$ of size k, and $S(I)$ is the set of permutations of I. The last sum of products of two determinants is written by the Cauchy–Binet formula as

$$\text{const}_n \cdot \det\left[\sum_{j=0}^{n-1} \psi_j(x_\alpha)\psi_j(x_\beta)\right]_{\alpha,\beta=1}^{k},$$

and finally the constant is equal to 1 by Andreief identity.

2 Double Contour Integral Representation for the GUE Kernel

2.1 One contour integral representation for Hermite polynomials

Recall that the GUE kernel is defined by

$$K_N(x,y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y),$$

with the orthonormal functions

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\, p_n(x)\, e^{-x^2/4},$$

where the (monic, probabilists') Hermite polynomials are given by

$$p_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\, e^{-x^2/2}. \tag{2.1}$$

Note that the monic Hermite polynomials are uniquely defined by the orthogonality property. We are not proving (2.1) here, it is an exercise.

Lemma 2.1 (Generator function for Hermite polynomials)**.** *We have*

$$\exp\Big(xt - \frac{t^2}{2}\Big) = \sum_{n\geq 0} p_n(x) \frac{t^n}{n!}.$$

The series converges for all t since the left-hand side is an entire function of t.

Proof. Write the generating function as

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = \sum_{n\geq 0} \frac{(-1)^n t^n}{n!}\, e^{x^2/2} \frac{d^n}{dx^n}\, e^{-x^2/2}.$$

Since the factor $e^{x^2/2}$ does not depend on n, we can factor it out:

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = e^{x^2/2} \sum_{n\geq 0} \frac{(-t)^n}{n!} \frac{d^n}{dx^n}\, e^{-x^2/2}.$$

Now, recall Taylor's theorem: for any holomorphic function f we have

$$f(x-t)=\sum_{n\geq 0}\frac{(-t)^n}{n!}f^{(n)}(x).$$

Applying this with $f(x)=e^{-x^2/2}$, we deduce that

$$\sum_{n\geq 0}\frac{(-t)^n}{n!}\frac{d^n}{dx^n}\,e^{-x^2/2}=e^{-(x-t)^2/2}.$$

Thus, our generating function becomes

$$\sum_{n\geq 0}p_n(x)\frac{t^n}{n!}=e^{x^2/2}\,e^{-(x-t)^2/2},$$

as desired. □

By Cauchy's integral formula we can write using Lemma 2.1:

$$p_n(x)=\frac{n!}{2\pi i}\oint_C\frac{\exp\Big(xt-\frac{t^2}{2}\Big)}{t^{n+1}}\,dt, \tag{2.2}$$

where the contour C is a simple closed curve encircling the origin. Indeed, here we use the complex analysis property

$$\frac{1}{2\pi i}\oint_C\frac{1}{z^{k+1}}\,dz=\begin{cases}1, & \text{if } k=0,\\ 0, & \text{if } k\neq 0,\end{cases}$$

so (2.2) is simply a complex analysis version of the operation of extracting the coefficient of t^n in the Taylor expansion.

Therefore,

$$\psi_n(x)=\frac{1}{\sqrt{h_n}}\,p_n(x)\,e^{-x^2/4}=\frac{e^{-x^2/4}}{\sqrt{h_n}}\,\frac{n!}{2\pi i}\oint_C\frac{\exp\Big(xt-\frac{t^2}{2}\Big)}{t^{n+1}}\,dt.$$

2.2 Another contour integral representation for Hermite polynomials

We start with the Fourier transform identity

$$\int_{-\infty}^{\infty}\exp\Big(-\frac{t^2}{2}+i\,t\,x\Big)\,dt=\sqrt{2\pi}\,e^{-x^2/2}.$$

Differentiating both sides n times with respect to x yields

$$\frac{d^n}{dx^n}\Big(e^{-x^2/2}\Big)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(i\,t)^n\,e^{-t^2/2+i\,t\,x}\,dt.$$

Recalling the definition

$$p_n(x)=(-1)^n\,e^{x^2/2}\,\frac{d^n}{dx^n}\Big(e^{-x^2/2}\Big),$$

we obtain

$$p_n(x) = \frac{(-1)^n\, e^{x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i\,t)^n\, e^{-t^2/2+i\,t\,x}\, dt.$$

Next, perform the change of variable

$$s = i\,t, \quad \text{so that} \quad t = -i\,s, \quad dt = -i\,ds.$$

Under this substitution the factors transform as follows:

$$(i\,t)^n = s^n,$$

and the exponent becomes

$$-\frac{t^2}{2} + i\,t\,x = -\frac{(-i\,s)^2}{2} + i\,(-i\,s)\,x = \frac{s^2}{2} + s\,x.$$

Thus, the integral transforms into

$$\int_{-\infty}^{\infty} (i\,t)^n\, e^{-t^2/2+i\,t\,x}\, dt = -i \int_{-i\infty}^{i\infty} s^n\, e^{s^2/2+s\,x}\, ds.$$

Substituting back we have

$$p_n(x) = \frac{(-1)^n\, e^{x^2/2}}{\sqrt{2\pi}}\, (-i) \int_{-i\infty}^{i\infty} s^n\, e^{s^2/2+s\,x}\, ds.$$

That is,

$$p_n(x) = \frac{i\,(-1)^{n+1}\, e^{x^2/2}}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} s^n\, e^{s^2/2+s\,x}\, ds.$$

Finally, change the sign of s, and we get:

$$p_n(x) = \frac{i\, e^{x^2/2}}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} s^n\, e^{s^2/2-s\,x}\, ds.$$

Therefore,

$$\psi_n(x) = \frac{i\, e^{x^2/4}}{\sqrt{2\pi\, h_n}} \int_{-i\infty}^{i\infty} s^n\, e^{s^2/2-s\,x}\, ds.$$

2.3 Normalization of Hermite polynomials

Lemma 2.2. *We have*

$$h_n = \int_{-\infty}^{\infty} p_n(x)^2\, e^{-x^2/2}\, dx = n!\sqrt{2\pi}.$$

Proof. Multiply the generating function

$$\exp\Big(xt - \frac{t^2}{2}\Big) = \sum_{n\geq 0} p_n(x) \frac{t^n}{n!}$$

with a second copy (with parameter s):

$$\exp\Big(xs - \frac{s^2}{2}\Big) = \sum_{m\geq 0} p_m(x)\frac{s^m}{m!}.$$

Then,

$$\exp\Big(xt - \frac{t^2}{2}\Big)\exp\Big(xs - \frac{s^2}{2}\Big) = \sum_{n,m\geq 0} p_n(x)p_m(x)\frac{t^n s^m}{n!m!}.$$

Integrate both sides against $e^{-x^2/2}\,dx$. Using the orthogonality

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)e^{-x^2/2}dx = h_n\delta_{nm},$$

the right-hand side becomes

$$\sum_{n\geq 0}\frac{h_n}{(n!)^2}(ts)^n.$$

On the left-hand side, we have

$$\int_{-\infty}^{\infty} e^{-x^2/2}\exp\Big(x(t+s) - \frac{t^2+s^2}{2}\Big)dx.$$

Completing the square in x or recalling the standard Gaussian integral yields

$$\sqrt{2\pi}\exp\Big(\frac{(t+s)^2-(t^2+s^2)}{2}\Big) = \sqrt{2\pi}\exp(ts).$$

Thus, we obtain

$$\sqrt{2\pi}\exp(ts) = \sum_{n\geq 0}\frac{h_n}{(n!)^2}(ts)^n.$$

Expanding the left side as

$$\sqrt{2\pi}\sum_{n\geq 0}\frac{(ts)^n}{n!},$$

and comparing coefficients, we conclude that

$$\frac{h_n}{(n!)^2} = \frac{\sqrt{2\pi}}{n!} \quad\Longrightarrow\quad h_n = n!\sqrt{2\pi}.$$

This completes the proof. $\square$

2.4 Double contour integral representation for the GUE kernel

We can sum up the kernel (essentially, this is another proof of the Christoffel–Darboux formula):

$$\begin{aligned} K_n(x,y) &= \sum_{k=0}^{n-1}\psi_k(x)\psi_k(y) \\ &= \frac{e^{\frac{x^2-y^2}{4}}}{(2\pi)^2}\oint_C dt\int_{-i\infty}^{i\infty} ds\,\exp\Big\{-\frac{t^2}{2}+xt+\frac{s^2}{2}-ys\Big\}\underbrace{\sum_{k=0}^{n-1}s^k t^{-k-1}}_{\frac{1-(s/t)^n}{t-s}}. \end{aligned} \tag{2.3}$$

Here we used the two contour integral representations for Hermite polynomials, and the explicit norm (Lemma 2.2). At this point, the t contour is a small circle around 0, and the s contour is a vertical line in the complex plane. Their mutual position can be arbitrary at this point — the s contour goes along the imaginary line. Indeed, the fraction $\frac{1-(s/t)^n}{t-s}$ does not have a singularity at $s = t$ due to the cancellation.

Let us now move the s contour to be to the left of the t contour, as in Figure 1. On the new contours, we have $|s| > |t|$. Now we can add the summands $s^k t^{-k-1}$ for all $k \le -1$ into the sum in (2.3). Indeed, for $|s| > |t|$, the series in k converges, while the summand $s^k t^{-k-1}$ has zero residue at 0 and thus adding the summands does not change the value of the integral.

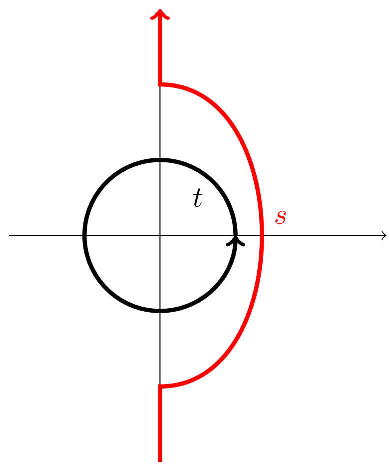

Figure 1: Integration contours for the GUE kernel (2.4).

With this extension of the sum, formula (2.3) becomes

$$K_n(x,y) = \frac{e^{(y^2-x^2)/4}}{(2\pi)^2} \oint_C dt \int_{-i\infty}^{i\infty} ds \, \frac{\exp\left\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\right\}}{s-t} \left(\frac{s}{t}\right)^n . \tag{2.4}$$

Remark 2.3. The s contour passes to the right of the t contour, but it might as well pass to the left of it. Indeed, one can deform the s contour to the left while picking the residue at $s = t$:

$$\operatorname{Res}_{s=t} \frac{\exp\left\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\right\}}{s-t} \left(\frac{s}{t}\right)^n = -e^{t(x-y)}.$$

This function is entire in t, and its integral over the t contour is zero. Therefore, there is no difference where the s contour passes with respect to the t contour.

2.5 Conjugation of the kernel

The kernel $K_n(x,y)$ contains a factor $e^{\frac{y^2-x^2}{4}} = g(x)/g(y)$, where $g(\cdot)$ is a nonvanishing function. This factor can be safely removed, since in all determinants $\det[K_n(x_i,x_j)]_{i,j=1}^k$ representing the correlation functions, the conjugation factors $g(x_i)/g(x_j)$ do not affect the value of the determinant. Thus, we can and will deal with the correlation kernel

$$K_n(x,y) = \frac{1}{(2\pi)^2} \oint_C dt \int_{-i\infty}^{i\infty} ds \, \frac{\exp\left\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\right\}}{s-t} \left(\frac{s}{t}\right)^n, \tag{2.5}$$

and will use the same notation for it. Throughout the asymptotic analysis in Section 4 below, other conjugation factors may appear, but we can similarly remove them.

2.6 Extensions

Many other versions of the GUE / unitary invariant ensembles admit determinantal structure:

1. The GUE corners process [JN06]
2. The Dyson Brownian motion (e.g., add a GUE to a diagonal matrix) [NF98]
3. GUE corners plus a fixed matrix [FF14]
4. Corners invariant ensembles with fixed eigenvalues $UDU^{\dagger}$, where D is a fixed diagonal matrix and U is Haar distributed on the unitary group [Met13]

We will discuss the corners process structure in the next Lecture 7.

3 Steepest descent — generalities for single integrals

3.1 Setup

In many problems arising in random matrix theory—as well as in asymptotic analysis more generally—it is necessary to evaluate integrals of the form

$$I(\Lambda) = \int_\gamma e^{\Lambda f(z)} \phi(z) \, dz, \tag{3.1}$$

where

- $\Lambda > 0$ is a large parameter,
- $f(z)$ and $\phi(z)$ are holomorphic functions in a neighborhood of the contour $\gamma \subset \mathbb{C}$,
- and the contour γ is chosen in such a way that the integral converges.

The *method of steepest descent* (also known as the *saddle point method*) provides a systematic procedure for obtaining the asymptotic behavior of $I(\Lambda)$ as $\Lambda \to +\infty$.

The key observation is that for large Λ, the exponential term $e^{\Lambda f(z)}$ is highly oscillatory or decaying, so that the main contributions to the integral come from small neighborhoods of points where the real part of $f(z)$ is maximal. Moreover, since we can deform the integration contour γ to pick points where $\operatorname{Re} f(z)$ is even bigger, it makes sense to find points *not only on the original* *contour* where $\operatorname{Re} f(z)$ is maximal. Such *critical* (or *saddle*) points are found from the equation with the complex derivative:

$$f'(z) = 0$$

Indeed, since $\operatorname{Re} f(z)$ is harmonic and $f(z)$ satisfies the Cauchy–Riemann equations, the condition $f'(z) = 0$ is equivalent to the condition that $\operatorname{Re} f(z)$ has zero gradient. Moreover, by harmonicity, all critical points of $\operatorname{Re} f(z)$ are saddle-like.

Once the saddle points are identified, one deforms the contour γ to Γ so that Γ passes through the saddle point(s) with the maximal value of $\operatorname{Re} f(z)$, and, moreover, such that on the rest of the new contour Γ the real part of $f(z)$ is strictly less than the value(s) at the saddle point(s). The decrease of $\operatorname{Re} f(z)$ along Γ may be ensured if one picks Γ to be *steepest descent* for $\operatorname{Re} f(z)$. By holomorphicity of $f(z)$, the steepest descent of Re is equivalent to the condition that the imaginary part of $f(z)$ is constant along Γ.

Remark 3.1. In practical applications, one does not need Γ to be fully steepest descent (it is usually hard to control). One can either choose Γ to be steepest descent in a neighborhood of the critical point and estimate the real part outside, or simply estimate the change of $\operatorname{Re} f(z)$ directly along a given contour.

Remark 3.2. The function $\phi(z)$ might not be holomorphic, and might have poles. The deformation of the contour from γ to Γ might pick residues at these poles. These residues can be harmless (easy to account for) or not (hard to account for; or affect the asymptotics of the integral), and one has to be careful with the contour deformation.

Despite the caveats in Remarks 3.1 and 3.2, in what follows in this section we will discuss the easiest case of steepest descent analysis. We also assume that there is only one saddle point z_0 to take care of.

3.2 Saddle points and steepest descent paths

Definition 3.3 (Saddle point)**.** A point $z_0 \in \mathbb{C}$ is called a *saddle point* of $f(z)$ if

$$f'(z_0) = 0.$$

We shall assume in what follows that at every saddle point under consideration the second derivative satisfies

$$f''(z_0) \neq 0.$$

Definition 3.4 (Steepest descent path)**.** Let z_0 be a saddle point of $f(z)$. A curve $\Gamma \subset \mathbb{C}$ passing through z_0 is called a *steepest descent path* for $f(z)$ if along Γ the imaginary part of $f(z)$ is constant (i.e., $\operatorname{Im}\big(f(z)\big) = \operatorname{Im}\big(f(z_0)\big)$ for all $z \in \Gamma$), which implies that the real part $\operatorname{Re}\big(f(z)\big)$ decreases away from z_0.

In a neighborhood of a saddle point z_0,

$$z = z_0 + w, \qquad f(z) = f(z_0) + \frac{1}{2} f''(z_0) w^2 + O(w^3).$$

If we denote

$$f''(z_0) = |f''(z_0)| e^{i\theta_0},$$

then writing $w = r\, e^{i\varphi}$, we obtain

$$f(z) = f(z_0) + \frac{1}{2}|f''(z_0)|r^2 e^{i(2\varphi+\theta_0)} + O(r^3).$$

For the imaginary part to remain constant in a neighborhood of z_0, and, moreover, for the phase of the quadratic term to be π modulo 2π, one must choose φ so that

$$2\varphi + \theta_0 = \pi \pmod{2\pi}. \tag{3.2}$$

We need the phase π so that the exponent is negative, for the integral to converge.

There are two directions satisfying (3.2) through z_0, and we use both of them for our contour Γ. Along these directions, one finds that

$$\mathrm{Re}\big(f(z)\big) = \mathrm{Re}\big(f(z_0)\big) - \frac{1}{2}|f''(z_0)|r^2 + O(r^3),$$

so that $\mathrm{Re}(f(z))$ is maximal at $z = z_0$ and decays quadratically as one moves away from z_0 along the steepest descent paths.

3.3 Local asymptotic evaluation near a saddle point

Assume now that the contour γ in (3.1) has been deformed so that it passes through a saddle point z_0 along a steepest descent path. In a small neighborhood of z_0, we write

$$z = z_0 + w,$$

so the local contribution of a neighborhood of z_0 to the integral is

$$I_{z_0}(\Lambda) = e^{\Lambda f(z_0)}\phi(z_0)\int_{-\infty}^{\infty} e^{\Lambda \frac{1}{2} f''(z_0) w^2}\, dw\Big(1 + O\Big(\frac{1}{\Lambda}\Big)\Big). \tag{3.3}$$

Here the integration is taken along the steepest descent direction, so that the quadratic term in the exponent is real and negative. (That is, by the choice (3.2), we have $\mathrm{Re}\big(f''(z_0)w^2\big) = -|f''(z_0)|r^2$.) Then the Gaussian integral evaluates to

$$\int_{-\infty}^{\infty} e^{-\Lambda \frac{|f''(z_0)|}{2} r^2}\, dr = \sqrt{\frac{2\pi}{\Lambda|f''(z_0)|}}.$$

Hence, we arrive at the following fundamental result.

Theorem 3.5 (Local asymptotics via steepest descent)**.** *Let z_0 be a saddle point of $f(z)$ with $f'(z_0) = 0$ and $f''(z_0) \neq 0$, and assume that $\phi(z)$ is holomorphic in a neighborhood of z_0. Then, as $\Lambda \to +\infty$, the contribution of a small neighborhood of z_0 to the integral (3.1) is given by*

$$I_{z_0}(\Lambda) \sim e^{\Lambda f(z_0)}\,\phi(z_0)\sqrt{\frac{2\pi}{\Lambda\,|f''(z_0)|}}, \qquad \Lambda \to +\infty. \tag{3.4}$$

Moreover, the behavior (3.4) captures the full asymptotic behavior of the integral (3.1) as long as on the new contour Γ, the real part of $f(z)$ is maximized at z_0 and is separated from $\mathrm{Re}\, f(z_0)$ everywhere else on Γ outside of a small neighborhood of z_0.

Under appropriate assumptions (typically, if f and ϕ are holomorphic on a neighborhood that can be reached by the deformed contour and if the contributions away from the saddle points are exponentially small), one may show that the error in approximating the full integral by the sum of the local contributions is itself exponentially small relative to the leading order terms. In many cases, the next-order corrections can be computed by carrying the expansion in (3.3) to higher order in w. (See, e.g., [Olv74] for a systematic treatment.)

4 Steepest descent for the GUE kernel

4.1 Scaling

Let us now consider the GUE kernel (2.5),

$$K_n(x,y) = \frac{1}{(2\pi)^2} \oint_C dt \int_{-i\infty}^{i\infty} ds \, \frac{\exp\left\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\right\}}{s-t} \left(\frac{s}{t}\right)^n,$$

where the integration contours are as in Figure 1.

We know from the Wigner semicircle law (established for real symmetric matrices with general iid entries in in Lecture 2, and for the GUE in Lecture 4) that the eigenvalues live on the scare $\sqrt{n}$. This means that to capture the local asymptotics, we need to scale

$$x = X\sqrt{n} + \Delta x, \qquad y = Y\sqrt{n} + \Delta y, \qquad \Delta x, \Delta y \in \mathbb{R}. \tag{4.1}$$

Moreover, if $X \neq Y$ (i.e., different global positions), one can check that the kernel vanishes. In other words, the local behaviors at different global positions are independent. See Problem F.1. In what follows, we take $Y = X$.

Let us also make a change of the integration variables:

$$t = z\sqrt{n}, \qquad s = w\sqrt{n}.$$

The integration contours for z and w look the same as in Figure 1, up to a rescaling. However, as 0 and $t = s$ are the only singularities in the integrand, we can deform the z, w contours as we wish, while keeping $|z| < |w|$ and the general shape as in Figure 1.

We thus have:

$$\begin{aligned} &K_n(X\sqrt{n} + \Delta x, X\sqrt{n} + \Delta y) \\ &\quad = \frac{\sqrt{n}}{(2\pi)^2} \oint_C dz \int_{-i\infty}^{i\infty} dw \, \frac{\exp\left\{n\left(\log w - \log z + \frac{w^2}{2} - \frac{z^2}{2} + X(z-w) + \frac{z\Delta x - w\Delta y}{\sqrt{n}}\right)\right\}}{w-z}. \end{aligned} \tag{4.2}$$

Remark 4.1. The logarithms in the exponent are harmless, since for the estimates we only need the real parts of the logarithms, and for the main contributions, we will have $z \approx w$, so any phases of the logarithms would cancel.

The asymptotic analysis of double contour integrals like (4.2) in the context of determinantal point processes was pioneered in [Oko02, Section 3].

4.2 Critical points

Let us define

$$S(z) \coloneqq \frac{z^2}{2} + \log z - Xz.$$

Then the exponent contains $n\,(S(w) - S(z))$. According to the steepest descent ideology, we should deform the integration contours to pass through the critical point(s) z_{cr} of $S(z)$. Moreover, the new w contour should maximize the real part of $S(z)$ at z_{cr}, and the new z contour should minimize it. If $S''(z_{cr}) \neq 0$, it is possible to locally choose such contours, they will be perpendicular to each other at z_{cr}.

Thus, we need to find the critical points of $S(z)$. They are found from the quadratic equation:

$$S'(z) = z + \frac{1}{z} - X = 0, \qquad z_{cr} = \frac{X \pm \sqrt{X^2 - 4}}{2}.$$

Depending on whether $|X| < 2$, there are three cases. We will consider them in Sections 4.3 to 4.5 below.

4.3 Imaginary critical points: $|X| < 2$

4.4 Real critical points: $|X| > 2$

4.5 Double critical points: $|X| = 2$

F Problems (due 2025-03-12)

F.1 Different global positions

Show that if in (4.1) we take $X \neq Y$, then $K_n(x, y)$ vanishes as $n \to +\infty$. Moreover, establish that the decay is exponential in n.

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