Lectures on Random Matrices (Spring 2025)
Lecture 15: Random Matrices and Topology

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Contents

*Course webpage • Live simulations • TeX Source • Updated at 09:18, Wednesday 23rd April, 2025

1 Introduction

In this wrap-up lecture, we go back to moments of random matrices, and outline their connection to topology (more precisely, to counting certain embedded graphs).

Remark 1.1. Throughout this lecture, to make an exact connection with the existing literature, the matrix size is denoted by N, and the small n is reserved to the order of the moment.

NOTE: course evaluations!

Todo on web: tables of contents in HTML + total PDF

2 Gluing polygons into surfaces

2.1 Gluing edges of a polygon

Consider a regular $2n$-gon with edges labeled by $1, \ldots, 2n$. We can glue the edges in pairs, so that the resulting surface is oriented.

Example 2.1. Consider a square. Recall that to obtain an orientable surface one must orient the square's boundary cyclically and then glue opposite sides with *opposite* orientations. There are three ways to glue the edges of a square. Note that in two cases, we get the sphere and in one case, the torus. The two spheres are obtained by gluing the edges in the same way, but this differs by a rotation — we consider these two cases as different.

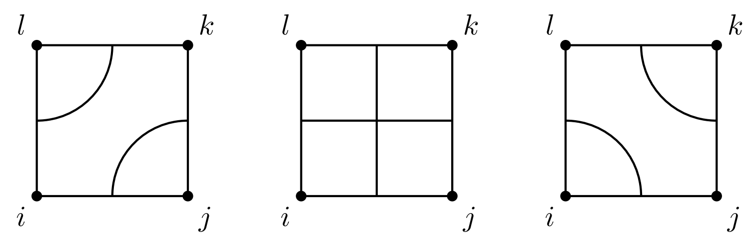

Figure 1: The three ways to glue edges of a square to make an orientable surface: two spheres (left and right) and one torus (center).

The boundary of the $2n$-gon becomes a graph embedded into the surface. It has exactly n edges and one face. It may have different number of vertices, and thus the number of vertices uniquely determines the genus of the surface:

$$V - E + F = 2 - 2g \implies g = \frac{n + 1 - V}{2}. \tag{2.1}$$

In the case of the square ($n = 2$), we have $V = 3$ and $g = 0$ for the sphere, and $V = 1$ and $g = 1$ for the torus.

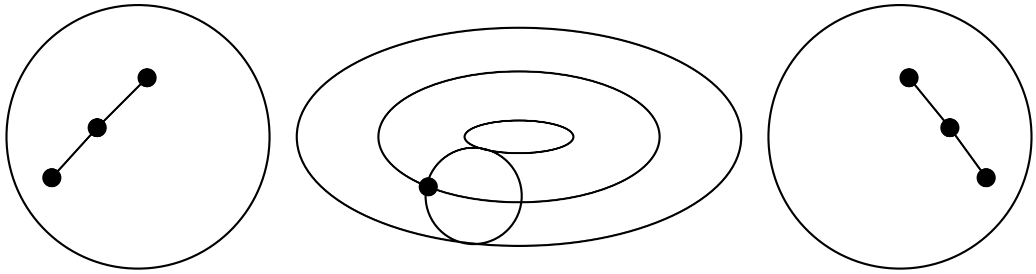

Figure 2: Surfaces corresponding to gluings: left and right show three-vertex trees (disk, sphere), center shows a one-vertex, one-face case (torus).

2.2 Starting to count

Proposition 2.2. *There is a total*

$$(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$$

ways to glue the edges of a $2n$*-gon into a surface.*

Proof. This is just the number of ways to pair $2n$ edges of the polygon. $\square$

Proposition 2.3. *The following are equivalent:*

1. *The surface is a sphere;*
2. *The graph on the surface is a tree;*
3. *The identification of the opposite edges of the polygon is a* noncrossing pairing *of the edges of the polygon.*

Proof. See Problem O.1. $\square$

There is $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ ways to get the sphere.

2.3 Dual picture

In the dual picture, we can consider a star with $2n$ half-edges. Then, we get a dual graph on the same surface. This graph has $V^* = 1$, $E^* = n$, but can have a variable number of faces (which corresponds to the genus):

$$F^* = n - 2g + 1.$$

When $n = 2$, for the sphere, we have $F^* = 3$, and for the torus, we have $F^* = 1$.

2.4 Notation

Let us denote

$$\varepsilon_g(n) \coloneqq \text{number of ways to glue the edges of a } 2n\text{-gon into a surface of genus } g, \tag{2.2}$$

$$T_n(N) := \sum_{\text{gluings } \sigma} N^{V(\sigma)} = \sum_{g=0}^{\infty} \varepsilon_g(n) N^{n+1-2g}, \tag{2.3}$$

that is, this is the generating function of the gluings of the edges of a $2n$-gon, where N is the generating function variable.

Remark 2.4. The polynomial $T_n(N)$ has only powers of N of the same parity as n.

We have the first few polynomials (the case $n = 2$ corresponds to the square):

$$\begin{aligned}
T_1(N) &= N^2;\\
T_2(N) &= 2N^3 + N;\\
T_3(N) &= 5N^4 + 10N^2;\\
T_4(N) &= 14N^5 + 70N^3 + 21N;\\
T_5(N) &= 42N^6 + 420N^4 + 483N^2.
\end{aligned}$$

3 Harer–Zagier formula (statement)

Introduce the *exponential generating function* for the sequence $\{T_n(N)\}_{n\geq 0}$:

$$\begin{aligned}
T(N,s) &= 1 + 2Ns + 2s \sum_{n\geq 1} \frac{T_n(N)}{(2n-1)!!} s^n \\
&= 1 + 2Ns + 2N^2 s^2 + \frac{2}{3}(2N^3 + N)s^3 + \frac{2}{15}(5N^4 + 10N^2)s^4 + \dots
\end{aligned} \tag{3.1}$$

One of the goals of today's lecture is to prove the following:

Theorem 3.1 (Harer–Zagier formula [HZ86])**.** *For every $N \in \mathbb{Z}_{>0}$ one has the closed form*

$$T(N,s) = \left(\frac{1+s}{1-s}\right)^N. \tag{3.2}$$

Let us at least verify that the first few Taylor coefficients of (3.2) indeed coincide with those in (3.1). Write

$$\begin{aligned}
\left(\frac{1+s}{1-s}\right)^N &= (1+s)^N (1-s)^{-N} \\
&= \left(1 + Ns + \frac{N(N-1)}{2!}s^2 + \frac{N(N-1)(N-2)}{3!}s^3 + \dots\right) \\
&\quad \times \left(1 + Ns + \frac{N(N+1)}{2!}s^2 + \frac{N(N+1)(N+2)}{3!}s^3 + \dots\right).
\end{aligned}$$

Multiplying the two series and collecting terms up to s^3, we find

$$1 + 2Ns + 2N^2 s^2 + \frac{2}{3}(2N^3 + N)s^3 + \dots,$$

which matches the expansion (3.1) exactly.

Corollary 3.2. *For all $g \geq 0$ and $n \geq 0$, the numbers $\varepsilon_g(n)$ obey*

$$(n+2)\,\varepsilon_g(n+1) \;=\; (4n+2)\,\varepsilon_g(n) \;+\; (4n^3-n)\,\varepsilon_{g-1}(n-1), \tag{3.3}$$

with the initial condition

$$\varepsilon_g(0) \;=\; \begin{cases} 1, & g=0, \\ 0, & g \geq 1. \end{cases}$$

Proof. Follows from the identity

$$\left(\frac{1+s}{1-s}\right)^N = (1+s)(1+s+s^2+\ldots)\left(\frac{1+s}{1-s}\right)^{N-1}.$$

□

Corollary 3.3. *The number $\varepsilon_g(n)$ can be written as*

$$\varepsilon_g(n) \;=\; \frac{(2n)!}{(n+1)!\,(n-2g)!}\,\left[s^{2g}\right]\left(\frac{s/2}{\tanh(s/2)}\right)^{n+1},$$

where $\left[s^{2g}\right]f(s)$ denotes the coefficient of s^{2g} in the power-series expansion of $f(s)$.

One can define another family of coefficients:

$$C_g(n) \coloneqq \frac{2^g\varepsilon_g(n)}{\mathrm{Cat}_n}.$$

Then, (3.3) can be rewritten as

$$C_g(n+1) = C_g(n) + \binom{n+1}{2}C_{g-1}(n-1).$$

In particular, $C_g(n)$ is a positive integer, which is not straightforward from the definition of $\varepsilon_g(n)$.

4 Gaussian integrals and Wick formula

4.1 The standard one–dimensional Gaussian measure

Denote by

$$d\mu(x) \;=\; \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx, \qquad x \in \mathbb{R},$$

the *standard centred Gaussian measure.* We record the elementary facts that will be used repeatedly:

(i) **Normalization:** $\displaystyle\int_{\mathbb{R}} d\mu(x) = 1.$

(ii) **Odd moments vanish:** $\langle x^{2n+1}\rangle = 0.$

(iii) **Even moments:**

$$\langle x^{2n} \rangle \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-\frac{x^2}{2}}\, dx \;=\; (2n-1)!!, \qquad n \in \mathbb{N}.$$

(iv) **Characteristic (Fourier–Laplace) transform:**

$$\varphi(t) \;=\; \int_{\mathbb{R}} e^{itx}\, d\mu(x) \;=\; e^{-\frac{t^2}{2}}, \qquad t \in \mathbb{R}.$$

Here and below we use the convenient bracket notation $\langle f \rangle := \int_{\mathbb{R}} f(x)\, d\mu(x)$ for expectations.

Example 4.1. For $k = 1$ with variance 1 we have $\langle x^4 \rangle = 3\, \langle x^2 \rangle^2 = 3$. For degree 6 one finds $\langle x^6 \rangle = 15$. More generally, $\langle x^{2n} \rangle = (2n-1)!!$. This can be computed by a simple induction.

4.2 Gaussian measures on $\mathbb{R}^k$

Fix a positive–definite symmetric matrix $B \in \mathrm{Sym}_k^+(\mathbb{R})$ and set $C := B^{-1}$. The centred Gaussian measure with covariance C is

$$d\mu_B(x) \;=\; \underbrace{\Big[(2\pi)^{-k/2} (\det B)^{1/2}\Big]}_{=:\; Z_B^{-1}} \exp\!\Big(-\tfrac{1}{2}\langle Bx, x\rangle\Big)\, d^k x, \qquad x \in \mathbb{R}^k. \tag{4.1}$$

Orthogonal diagonalisation of B shows that the normalising prefactor indeed gives $\int_{\mathbb{R}^k} d\mu_B = 1$.

Basic facts.

$$\langle x_i \rangle = 0, \quad 1 \le i \le k; \tag{4.2}$$

$$\langle x_i x_j \rangle = C_{ij}, \quad 1 \le i, j \le k. \tag{4.3}$$

All higher moments are expressed in terms of the matrix C via Wick's formula in Section 4.3 below.

Remark 4.2. In this lecture, we consider only *centered* (mean zero) Gaussian measures.

4.3 Wick (Isserlis) formula

The essence of Wick's formula is that *every* moment of a centred Gaussian vector is a sum over pairwise contractions governed solely by the covariance matrix.

Theorem 4.3 (Wick's (or Isserlis') formula)**.** *Let $x = (x_1, \ldots, x_k)$ be distributed according to (4.1). For an integer $n \ge 1$ and indices $i_1, \ldots, i_{2n} \in \{1, \ldots, k\}$,*

$$\big\langle x_{i_1} \cdots x_{i_{2n}} \big\rangle \;=\; \sum_{p \in \mathrm{Pair}(2n)} \; \prod_{\{a,b\} \in p} C_{i_a i_b}, \tag{4.4}$$

where $\mathrm{Pair}(2n)$ *is the set of all* $(2n-1)!!$ *perfect pairings of* $\{1, \ldots, 2n\}$*. If the degree is* odd, *then the expectation vanishes.*

More generally, for any linear *functions (not necessarily distinct)* $f_1, \ldots, f_{2n}$ *of the variables* $x_1, \ldots, x_k$*, we have*

$$\left\langle f_1 \cdots f_{2n} \right\rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \ldots \langle f_{p_n} f_{q_n} \rangle, \tag{4.5}$$

where the sum is over all pairings of the indices $1, \ldots, 2n$*, and* $p_1 < p_2 < \ldots < p_n$, $q_1 < q_2 < \ldots < q_n$ *are the indices encoding the pairing.*

Sketch of proof. When $C = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_k^2)$, mixed covariances vanish and Wick's formula factorizes:

$$\left\langle x_1^{2n_1} \cdots x_k^{2n_k} \right\rangle \;=\; \prod_{i=1}^{k} (2n_i - 1)!!\; \sigma_i^{2n_i}, \qquad n_1, \ldots, n_k \in \mathbb{N}.$$

Indeed, pairings are allowed only between indices of the same variable, and then the number of pairings within one variable x_i is $(2n_i - 1)!!$.

The general case of Wick's formula follows from the diagonal case by making a linear change of variables which diagonalizes the covariance matrix, and using the linearity of (4.5). $\square$

Example 4.4. The one-dimensional integral $\langle x^4 \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}} dx$ can be computed using Wick's formula:

$$\langle f_1 f_2 f_3 f_4 \rangle = \langle f_1 f_2 \rangle \langle f_3 f_4 \rangle + \langle f_1 f_3 \rangle \langle f_2 f_4 \rangle + \langle f_1 f_4 \rangle \langle f_2 f_3 \rangle,$$

where $f_i(x) = x$ for $i = 1, 2, 3, 4$. We know this integral is equal to 3.

Remark 4.5. Note that in the second part of Theorem 4.3, the linear functions f_j must be not *affine*, but truly *linear*, that is, $f_j(0, \ldots, 0) = 0$. See Problem O.2.

5 GUE integrals and gluing polygons

We will now apply Wick's formula to compute the moments of traces of GUE matrices. Recall that in Lecture 1 and Lecture 2 we worked with general Wigner matrices (real symmetric or Hermitian), and now we will deal with the special case of GUE, Gaussian Hermitian matrices. Here, the Gaussian distribution will allow us to connect the moments of traces of GUE matrices to the topology of surfaces.

5.1 Traces of powers, again

Let $\mathcal{H}_N$ be the space of $N \times N$ Hermitian matrices, and μ on $\mathcal{H}_N$ be the GUE measure, with complex variances 1 for the diagonal and off-diagonal entries. Let us begin by an example with $n = 2$.

Consider the integral

$$\int_{\mathcal{H}_N} \operatorname{tr}(H^4)\, d\mu(H).$$

Here the integrand is a sum of monomials,

$$\operatorname{tr}(H^4) \;=\; \sum_{i,j,k,l=1}^{N} h_{ij}\, h_{jk}\, h_{kl}\, h_{li}.$$

Since each entry h_{pq} is a linear function of the real and imaginary parts of H, we may apply Wick's formula:

$$\big\langle h_{ij}h_{jk}h_{kl}h_{li}\big\rangle = \langle h_{ij}h_{jk}\rangle\,\langle h_{kl}h_{li}\rangle + \langle h_{ij}h_{kl}\rangle\,\langle h_{jk}h_{li}\rangle + \langle h_{ij}h_{li}\rangle\,\langle h_{jk}h_{kl}\rangle. \tag{5.1}$$

Lemma 5.1. *We have $\langle h_{ij}h_{ji}\rangle = 1$, and all other second moments are zero.*

Proof. This is straightforward from the independence of real and imaginary parts of the entries of H. □

Let us inspect each term in (5.1) separately:

- In the first product $\langle h_{ij}h_{jk}\rangle$ is nonzero only when $i = k$, and then equals 1. Likewise $\langle h_{kl}h_{li}\rangle = 1$ only when $k = i$. Summing over all i, j, k, l with $i = k$ gives N^3.
- In the second product $\langle h_{ij}h_{kl}\rangle\,\langle h_{jk}h_{li}\rangle$ is nonzero only if $i = j = k = l$, and then each factor equals 1. Hence this term contributes N.
- The third product is identical in structure to the first and therefore contributes another N^3.

There is a one-to-one correspondence between these three terms in (5.1) and the three pairings of the edges of a square (see Figure 1). Each pairing contributes $N^{V(\sigma)}$, where $V(\sigma)$ is the number of vertices in the glued graph.

Putting everything together, we get

$$\int_{\mathcal{H}_N} \operatorname{tr}(H^4)\,d\mu(H) \;=\; 2\,N^3 + N \;=\; T_2(N),$$

where $T_2(N)$ is defined by (2.3).

In a similar manner, we obtain the following:

Proposition 5.2. *For any $n \geq 1$, we have*

$$\int_{\mathcal{H}_N} \operatorname{tr}(H^{2n})\,d\mu(H) \;=\; T_n(N).$$

Odd moments (expectations of $\operatorname{tr}(H^{2n+1})$*) vanish.*

Proof sketch. The idea why we get the genus will be evident from a larger example. Let $n = 4$, so we are dealing with a sum of N^8 monomials of the form

$$h_{i_1i_2}h_{i_2i_3}h_{i_3i_4}h_{i_4i_5}h_{i_5i_6}h_{i_6i_7}h_{i_7i_8}h_{i_8i_1}.$$

Choose an arbitrary Wick pairing (there are $7!! = 105$ of them). For instance, pair

$$h_{i_1i_2} \text{ with } h_{i_4i_5}, \quad h_{i_2i_3} \text{ with } h_{i_5i_6}, \quad h_{i_3i_4} \text{ with } h_{i_8i_1}, \quad h_{i_6i_7} \text{ with } h_{i_7i_8}.$$

In other words, consider the product

$$\big\langle h_{i_1i_2}h_{i_4i_5}\big\rangle\big\langle h_{i_2i_3}h_{i_5i_6}\big\rangle\big\langle h_{i_3i_4}h_{i_8i_1}\big\rangle\big\langle h_{i_6i_7}h_{i_7i_8}\big\rangle. \tag{5.2}$$

Each factor in (5.2) is usually 0; if any of them vanishes, so does the whole product. For the product to be non–zero, *every* factor must equal 1, which imposes the constraints

$$\langle h_{i_1 i_2} h_{i_4 i_5} \rangle = 1 \iff i_1 = i_5,\ i_2 = i_4; \qquad \langle h_{i_2 i_3} h_{i_5 i_6} \rangle = 1 \iff i_2 = i_6,\ i_3 = i_5;$$

$$\langle h_{i_3 i_4} h_{i_8 i_1} \rangle = 1 \iff i_3 = i_1,\ i_4 = i_8; \qquad \langle h_{i_6 i_7} h_{i_7 i_8} \rangle = 1 \iff i_6 = i_8,\ i_7 = i_7.$$

Altogether we obtain the *chain of equalities*

$$i_1 = i_5 = i_3 = i_1, \qquad i_2 = i_4 = i_8 = i_6 = i_2, \qquad i_7 = i_7,$$

which leaves i_1, i_2, i_7 free and therefore yields N^3 admissible index choices. So, the contribution of the pairing (5.2) equals N^3.

Now, consider an octagon ($2n = 8$), and glue its sides in pairs as illustrated in Figure 3. Since the edges are identified, we have also identification of the vertices:

$$i_1 = i_5,\ i_2 = i_4, \qquad i_2 = i_6,\ i_3 = i_5, \qquad i_3 = i_1,\ i_4 = i_8, \qquad i_6 = i_8.$$

We thus see that the eight initial vertices collapse into

$$i_1 = i_5 = i_3, \quad i_2 = i_4 = i_6 = i_8, \quad i_7 = i_7,$$

producing $V(\sigma) = 3$ vertices in the resulting map; hence the gluing σ shown in Figure 3 contributes $N^{V(\sigma)} = N^3$. By (2.1), we get a torus. □

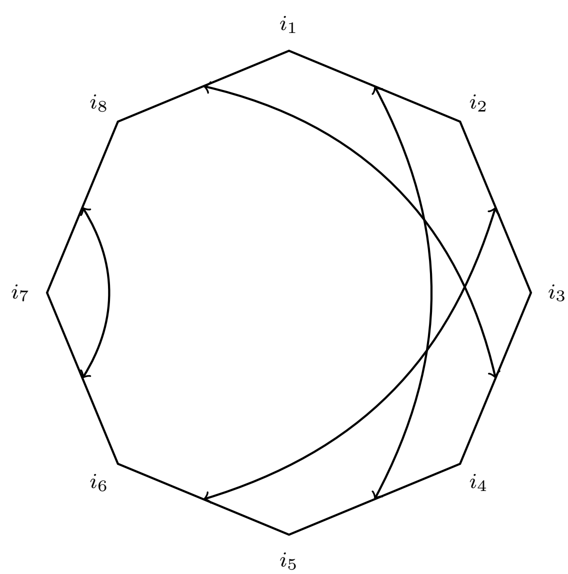

Figure 3: An 8–gon with pairwise–identified sides corresponding to the Wick pairing considered in the proof of Proposition 5.2.

5.2 Computing traces of powers

We now invoke the powerful technique of dealing with integrals over GUE through their spectrum. Recall that we have the following change of measure formula. For any function f on $\mathcal{H}_N$ which depends only on the eigenvalues $\lambda_1, \ldots, \lambda_N$ of H, we have

$$\int_{\mathcal{H}_N} f(H)\, d\mu(H) = c_N \int_{\mathbb{R}^N} f(\lambda_1, \ldots, \lambda_N) \prod_{1 \le i < j \le N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^{N} \mu(d\lambda_i),$$

where $\mu(H)$ is the GUE distribution on $\mathcal{H}_N$, and $\mu(dx)$ is the one-dimensional Gaussian measure on $\mathbb{R}$. The constant c_N depends only on N (we computed it in, e.g., Lecture 5 using determinantal structure of the GUE eigenvalues).

Lemma 5.3. *The function*

$$t(N, n) := \frac{T_n(N)}{(2n-1)!!}$$

is a polynomial in the variable n, of degree $N-1$.

In particular, $t(1, n) = 1$.

Proof of Lemma 5.3. Consider $f(H) = \operatorname{tr}(H^{2n})$, so $f(\lambda_1, \ldots, \lambda_N) = \lambda_1^{2n} + \ldots + \lambda_N^{2n}$. Because the integral over the GUE spectrum is symmetric in the λ_i, we may replace the trace by $N\lambda_1^{2n}$.

Express the squared Vandermonde determinant as a polynomial in λ_1 and integrate in the remaining variables $\lambda_2, \ldots, \lambda_N$. This reduces the multiple integral to a one–dimensional integral whose integrand is a polynomial in λ_1 of degree $2n + 2N - 2$. For fixed N the coefficients of this polynomial are constants; its leading coefficient equals Nc_N.

When we integrate each monomial λ_1^{2n+2k} and divide the result by $(2n-1)!!$, we obtain

$$\frac{(2n+2k-1)!!}{(2n-1)!!},$$

which is a polynomial in n of degree k. Hence $t(N, n)$ is a polynomial in n of degree $N - 1$, as desired. □

5.3 Proof of Harer–Zagier formula

Assume now that the vertices of the $2n$-gon are colored in (at most) N colors. A gluing is said to be *compatible with the coloring* if only vertices of the *same* color may (but are not required to) be glued to one another.

Lemma 5.4. *The number $T_n(N)$ is precisely the number of gluings of a $2n$-gon that are compatible with some coloring of its vertices in (at most) N colors.*

Proof. After gluing, the boundary of the polygon becomes an embedded graph with V vertices. color each of those V vertices with one of the N colors. Any such coloring induces a coloring of the original polygon, and the given gluing is compatible with it. There are exactly N^V such colorings of the graph (note that adjacent vertices are *not* required to have different colors). □

Let $\tilde{T}_n(N)$ denote the number of gluings of the $2n$-gon that are compatible with colorings in *exactly* N colors. Choosing which L colors are actually used and then coloring the vertices gives

$$T_n(N) \;=\; \sum_{L=1}^{N} \binom{N}{L} \tilde{T}_n(L).$$

Remark 5.5. This combinatorial technique is extremely standard, and it is useful here.

Clearly, $\tilde{T}_0(N) = \tilde{T}_1(N) = \tilde{T}_{N-2}(N) = 0$, because the graph on the surface has at most $n+1$ vertices (and that is possible only when the graph is a tree). Hence no coloring with more than $n+1$ different colors can be compatible with any gluing.

Define

$$\tilde{t}(N,n) \;=\; \frac{\tilde{T}_n(N)}{(2n-1)!!}.$$

The function $\tilde{t}(N,n)$ is a polynomial in n of degree $N-1$ by Lemma 5.3. We just found the roots of this polynomial: its $N-1$ roots are $0,1,2,\ldots,N-2$. Therefore, there exists a constant A_N such that

$$\tilde{t}(N,n) = A_N\, n(n-1)(n-2)\ldots(n-N+2) = A_N\,(N-1)!\binom{n}{N-1}.$$

Substituting this into the expression for $T_n(N)$, we obtain

$$T_n(N) = (2n-1)!!\sum_{L=1}^{N} A_L \binom{n}{L-1}\binom{N}{L}(L-1)!!.$$

Now, consider $T_n(N)$ as a polynomial in N. Its leading coefficient (the coefficient of N^{n+1}) equals

$$(2n-1)!!\,\frac{A_{n+1}}{(n+1)!}\,n!.$$

On the other hand, this coefficient is known to be the n-th Catalan number (since the surface is a sphere, and we are enumerating trees):

$$(2n-1)!!\,\frac{A_{n+1}}{(n+1)!}\,n! = \mathrm{Cat}_n = \frac{(2n)!}{n!\,(n+1)!}.$$

Hence $A_{n+1} = 2^n/n!$, and therefore, we have found

$$T_n(N) = (2n-1)!!\sum_{L=1}^{N} 2^{L-1}\binom{n}{L-1}\binom{N}{L}(L-1)!!.$$

Because $\binom{n}{L-1} = 0$ when $n < L-1$, the number of non–zero summands is $\min\{N, n+1\}$.

As the last step in the proof of Theorem 3.1, we note that the last formula is exactly the series expansion of

$$\left(\frac{1+s}{1-s}\right)^N$$

in powers of s. Indeed,

$$1 + 2Ns + 2s\sum_{n=1}^{\infty} \frac{T_n(N)}{(2n-1)!!}\, s^n \;=\; 1 + \sum_{L=1}^{N} 2^L \binom{N}{L} \sum_{n=L-1}^{\infty} \binom{n}{L-1}\, s^{n+1}$$

$$= \sum_{L=0}^{N} \binom{N}{L} \left(\frac{2s}{1-s}\right)^L = \left(\frac{1+s}{1-s}\right)^N .$$

This completes the proof of Theorem 3.1.

6 Going further: Multi-matrix models

6.1 Maps with several faces and Feynman diagrams

Fix a composition $\mathbf{k} = (k_1, \ldots, k_\ell)$ with $k_1 + \cdots + k_\ell = n$. For a GUE matrix H let

$$M_{\mathbf{k}}(N) \;:=\; \Big\langle \mathrm{tr}(H^{2k_1})\ \mathrm{tr}(H^{2k_2}) \cdots \mathrm{tr}(H^{2k_\ell}) \Big\rangle.$$

Write each trace as a $2k_i$-*valent star*: a cyclicly ordered vertex with $2k_i$ labelled half-edges. Wick's formula pairs the $2n$ half-edges in every possible way; a pairing σ produces an oriented ribbon graph $\mathcal{G}(\sigma)$ (dual to the collection of stars), with

- $F = \ell$ faces (the original traces),
- $E = n$ edges (Wick pairings),
- a vertex count $V(\sigma)$.

On the other hand, we may count the star picture, then $V(\sigma)$ becomes $F^*(\sigma)$, the number of faces in the star picture.

Each pairing contributes $N^{V(\sigma)}$, so

$$M_{\mathbf{k}}(N) \;= \sum_{\sigma} N^{V(\sigma)} = \sum_{\sigma} N^{F^*(\sigma)}.$$

The sum (the matrix integral) enumerates maps by genus. One can also write the matrix integral $\langle \mathrm{tr}(H)^{\alpha_1}\, \mathrm{tr}(H^2)^{\alpha_2} \ldots \rangle$ as a sum over all possible embedded graphs (into surfaces of various genera), with α_1 vertices of degree 1, α_2 vertices of degree 2, etc. The sum needs to be normalized by the number of automorphisms of the graph, more precisely, the matrix integral is equal to

$$\underbrace{\alpha_1! \ldots \alpha_k! \cdot 1^{\alpha_1} 2^{\alpha_2} \ldots k^{\alpha_k}}_{c_\alpha} \sum_{\Gamma} \frac{N^{F(\Gamma)}}{|\,\mathrm{Aug}(\Gamma)|}.$$

Example 6.1. There are two embedded graphs with one vertex and two cycles. For the sphere, the automorphisms of the graph are 2 (the two cycles can be interchanged), and for the torus, the automorphisms are 1 (the two cycles cannot be interchanged). The contribution of the sphere is $2N^2$, and the contribution of the torus is N^2. We have $\alpha_2 = 1$ (and all other α_i are zero), so $c_\alpha = 4$, and we get

$$4\left(\frac{N^3}{2} + \frac{N}{4}\right) = 2N^3 + N,$$

which is the moment $\langle \mathrm{tr}\left(H^4\right)\rangle$, as it should be.

Remark 6.2. Even the case $N = 1$ is of interest:

$$\sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} = \frac{1}{\alpha_1! \ldots \alpha_k! \cdot 1^{\alpha_1} 2^{\alpha_2} \ldots k^{\alpha_k}} \langle x^{\sum i\alpha_i} \rangle = \frac{(-1 + 2\sum i\alpha_i)!!}{\alpha_1! \ldots \alpha_k! \cdot 1^{\alpha_1} 2^{\alpha_2} \ldots k^{\alpha_k}}.$$

6.2 Two–matrix model and the Ising interaction

Take two independent GUE matrices $H, G \in \mathcal{H}_N$ and fix a real coupling c with $|c| < 1$. Define

$$Z_N(c) \;=\; \int_{\mathcal{H}_N \times \mathcal{H}_N} \exp\Big\{-\tfrac{N}{2}\,\mathrm{tr}(H^2 + G^2) + cN\ \mathrm{tr}(HG)\Big\}\, d\mu(H)\, d\mu(G).$$

This is again a Gaussian integral, but it is over correlated indices. Namely, we have

$$\langle h_{ij} h_{ji} \rangle = \frac{1}{1 - c^2}, \qquad \langle g_{ij} g_{ji} \rangle = \frac{1}{1 - c^2}, \qquad \langle h_{ij} g_{ji} \rangle = \frac{c}{1 - c^2},$$

and all other covariances vanish.

Now, we may consider matrix integral, for example,

$$\int e^{-t\,\mathrm{tr}\left(H^4 + G^4\right)} d\mu(H) d\mu(G).$$

This integral is a generating function of four-valent maps (with an arbitrary number of vertices), but this time assign one of two “states” to each vertex of the map. We label these states H and G; this labeling simply means that the vertex represents either $\mathrm{tr}\, H^4$ or $\mathrm{tr}\, G^4$, respectively. Now, if two vertices are connected by an edge and they are in the same state, the contribution of the edge to the sum (i.e., in the ”perturbation theory series”) is equal to $1/(1 - c^2)$, and if the vertices are in different states, the contribution of this edge is equal to $c/(1-c^2)$. The summation is carried out over all maps and all possible combinations of vertex states.

The model we obtained is very reminiscent of the classical Ising model in statistical physics, only in this case the model is considered not on a regular lattice, but on the set of maps (and the summation is carried out not only over the states of the system, but also over all maps). Physicists call this model the “Ising model on a dynamical lattice” [IZ80]. This model shares many common phenomena with the regular Ising model, such as the presence of phase transitions.

6.3 Outlook: meanders and higher matrix species

Adding further matrix species (and therefore edge colors) opens the door to many classical counting problems. A three-matrix integral, laid out in the cited PDF under the title “Enumeration of meanders”, produces the generating series for plane meanders—closed curves crossing a line through $2n$ bridges. The table included near the end of the PDF lists the first values

$M_1 = 1,\ M_2 = 2,\ M_3 = 8, \dots$ and shows that M_n grows asymptotically like $C\,\rho^{2n} n^{-3/2}$ with $\rho \approx 3.20$. A detailed derivation is postponed to the probabilistic part of the course; the essential moral here is that the Feynman-diagram machinery scales seamlessly to these richer combinatorial classes.

O Problems (due 2025-04-29)

O.1 Gluing a Sphere

Show that for a connected, orientable surface formed by gluing the edges of a $2n$-gon in pairs, the following are equivalent:

1. The resulting surface is a sphere.
2. The embedded graph formed by the identification is a tree.
3. The pairing of edges corresponds to a *noncrossing pairing* (i.e., when the edges are arranged around the polygon in order, the identifications can be drawn inside the disk without crossings).

(This is the proof of Proposition 2.3.)

O.2 Wick's formula for affine functions

Consider the integrals of the form

$$I(a_1, \dots, a_k) := \int_{-\infty}^{\infty} \prod_{i=1}^{k} (x - a_i)\ \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx,$$

where $a_1, \dots, a_k \in \mathbb{R}$ are fixed parameters.

Compute $I(a_1, a_2)$ and $I(a_1, a_2, a_3, a_4)$ explicitly as polynomials in $a_1, \dots, a_4$, and compare $I(a_1, a_2, a_3, a_4)$ with the Wick-like expansion.

O.3 GOE and non-orientable surfaces

Let $S \in \mathrm{Sym}_N(\mathbb{R})$ be drawn from the *Gaussian Orthogonal Ensemble* (GOE), so that the entries are centred Gaussians with covariances

$$\langle s_{ij} s_{kl} \rangle \ = \ \delta_{ik}\,\delta_{jl} \ + \ \delta_{il}\,\delta_{jk}, \qquad 1 \le i, j, k, l \le N.$$

Because of the second term, Wick's pairings now allow *reversals* of indices, and the polygon-gluing picture changes accordingly.

(a) Show that each Wick contraction contributing to the moment $\displaystyle\int_{\mathrm{Sym}_N(\mathbb{R})} \mathrm{tr}(S^{2n})\, d\mu(S)$ corresponds to a pairing of the $2n$ edges of a $2n$-gon in which *half* of the identifications are orientation-preserving and the other half are orientation-reversing. Conclude that the resulting surface is, in general, **non-orientable**. (A convenient measure of non-orientability is the *cross-cap number* γ, so that the Euler characteristic is $\chi = 2 - \gamma$.)

(b) Let $\tilde{\varepsilon}_\gamma(n)$ be the number of pairings producing a surface with γ cross-caps. Prove that

$$2^n \int_{\mathrm{Sym}_N(\mathbb{R})} \operatorname{tr}(S^{2n})\, d\mu(S) = \sum_\sigma N^{V(\sigma)}.$$

(c) Compute $\tilde{\varepsilon}_\gamma(n)$ explicitly for $n = 1, 2, 3$ and identify the corresponding additional non-orientable surfaces (real projective plane, Klein bottle, ...).

(d) Derive a recurrence relation for the numbers $\tilde{\varepsilon}_\gamma(n)$ analogous to the Harer–Zagier recurrence. (*Hint:* keep track of how many of the $2n$ edges are glued with or without a twist.)

References

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