UVA Summer School Invariant Measures PS 4

July 18, 2024

The number of stars next to an exercise indicates the expected time needed to fully solve the exercise.

Exercise 1 [*]

Prove Lemma 6.1, which we state in the following. Let $N \in \mathbb{N}_+$, and let $(\eta_t)_{t\geq 0}$ be the open TASEP with respect to boundary parameters $\alpha, \beta > 0$. There exists a coupling between $(\eta_t)_{t\geq 0}$ and the environment $(\omega_v)_{v\in\mathcal{S}_N}$ such that the respective growth interface $(G_t)_{t\geq 0}$ of $(\eta_t)_{t\geq 0}$ satisfies almost surely for all $t \geq 0$ and $i \in [N]$

$$\{\eta_t(i) = 0\} = \{g_t^i - g_t^{i-1} = \mathrm{e}_1\}. \tag{1}$$

Moreover, when starting the open TASEP from the empty initial condition $\mathbf{0}$, we have that for all $t \geq 0$ and $n \in \mathbb{N}_+$

$$\{\mathcal{J}_t > n\} = \{T((0,0),(n,n)) \leq t\}. \tag{2}$$

Exercise 2 [**]

Prove Lemma 6.6, which we state in the following. For all $\eta \in \Omega_N$, we recall the function

$$f_N(\eta) := \sum_{\zeta\in\Omega_N} A^{(h_\eta(N)-h_\zeta(N))}(AC)^{-\min_{0\leq j\leq N}(h_\eta(j)-h_\zeta(j))} \tag{3}$$

Let $\alpha, \beta \in (0,1)$, and let B be a basic weight function for the open TASEP with respect to boundary parameters α and β. Then

$$B(\eta) = f_N(\eta) \tag{4}$$

for all $N \in \mathbb{N}_+$, and all $\eta \in \Omega_N$.

Exercise 3 [**]

The goal of this exercise is to show Lemma 6.2, which we state in the following. Consider the disagreement process $(\xi_t)_{t\geq 0}$ on the segment of size N with a single second class particle starting at x, let γ_0 be the initial growth interface on the strip of width $N+1$ corresponding to the configuration where we replace the second class particle in ξ_0 by a $(0,1)$ pair, and take $m = x$ in

$$g_0^m - g_0^{m-1} = \mathrm{e}_1 \quad \text{and} \quad g_0^m - g_0^{m+1} = \mathrm{e}_2.$$

There exists a coupling such that

$$X_t - X_0 = (\phi_n^1 - \phi_1^1) - (\phi_n^2 - \phi_1^2) \tag{5}$$

holds almost surely for all $t \in [T(\gamma_0, \phi_n), T(\gamma_0, \phi_{n+1}))$ with $t < \tau_{\mathrm{ex}}$, and $n \in \mathbb{N}_+$. Moreover, if

$$\{(k+\ell, k-\ell) : k \in \{0,\dots,N+1\}\} \subseteq \Gamma^+ \text{ or } \{(k+\ell, k-\ell) : k \in \{0,\dots,N+1\}\} \subseteq \Gamma^- \tag{6}$$

holds for some $k > N$, then the second class particle in the disagreement process leaves the segment until time $T(\gamma_0, \{(k+\ell, k-\ell) : k \in \{0,\dots,N+1\}\})$.

Exercise 4 [***]

Consider a sequence of open TASEPs on $\{1, \ldots, N\}$ with jump rates α_N and β_N, for $N = 1, 2, \ldots$. Consider the height function

$$H_N(i) = \tau_1 + \cdots + \tau_i \quad \text{for } 1 \le i \le N.$$

under the stationary measure μ_N.

For fixed $u, v \in \mathbb{R}$, assume $\alpha_N, \beta_N \to 1/2$ with

$$C_N = \frac{1 - \alpha_N}{\alpha_N} = e^{-u/\sqrt{N}}, \quad A_N = \frac{1 - \beta_N}{\beta_N} = e^{-v/\sqrt{N}}.$$

Show that

$$\left\{ \frac{1}{\sqrt{N}} \left(2H_N(\lfloor xN \rfloor) - \lfloor xN \rfloor \right) \right\}_{x \in [0,1]} \Rightarrow \{B_x + X_x\}_{x \in [0,1]} \qquad \text{as } N \to \infty,$$

where the convergence is in Skorokhod's space $D[0,1]$ of càdàg functions, processes B, X are independent processes on $[0,1]$ with continuous trajectories, B is a Brownian motion of variance $1/2$, and the law of X is given by the Radon-Nikodym derivative

$$\frac{d\mathbb{P}_X}{d\mathbb{P}_B}(\omega) = \frac{1}{\mathfrak{K}(u,v)} \exp\left((u+v) \min_{0 \le x \le 1} \omega(x) - v\omega(1) \right), \quad \omega \in C[0,1], \tag{7}$$

where $\mathfrak{K}(u,v)$ is a normalization constant.
Hint: Use the two-layer expression of the open TASEP stationary measure given by Theorem 6.5.

We mention that the process $(B + X)$ defined above is the stationary measure of the (conjectural) KPZ fixed point on the interval $[0,1]$ with boundary parameters $u, v \in \mathbb{R}$. It is expected to arise as the scaling limits of stationary measures of all models in the KPZ class on an interval.