

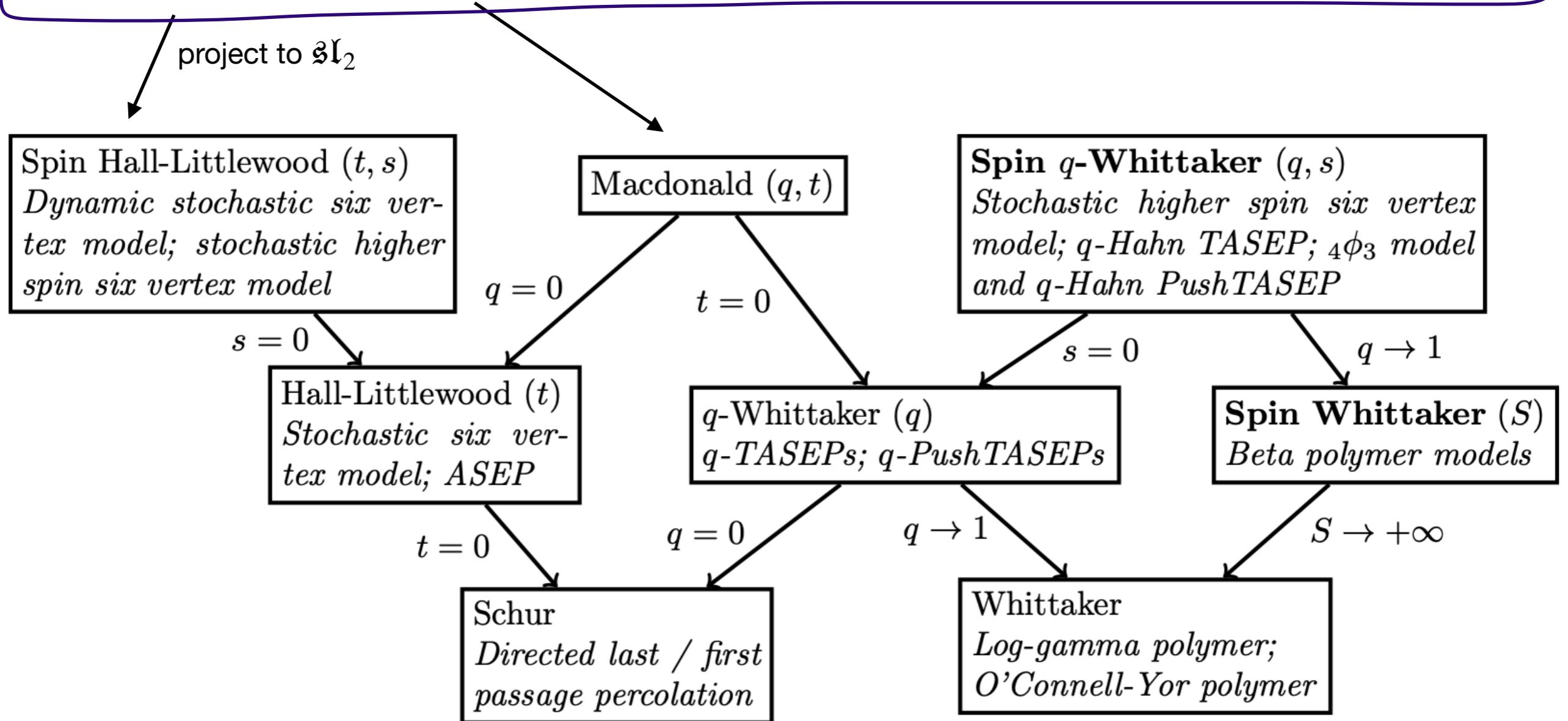
Symmetric functions from vertex models

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<https://arxiv.org/abs/2007.10886>
<https://arxiv.org/abs/2003.14260> (w. Matteo Mucciconi)

Nonsymmetric Macdonald and spin HL polynomials from coloured (\mathfrak{sl}_n) vertex models



1. A determinantal summation identity for spin Hall-Littlewood functions; application to ASEP
2. Spin q-Whittaker polynomials, spin Whittaker functions & connections to random polymers

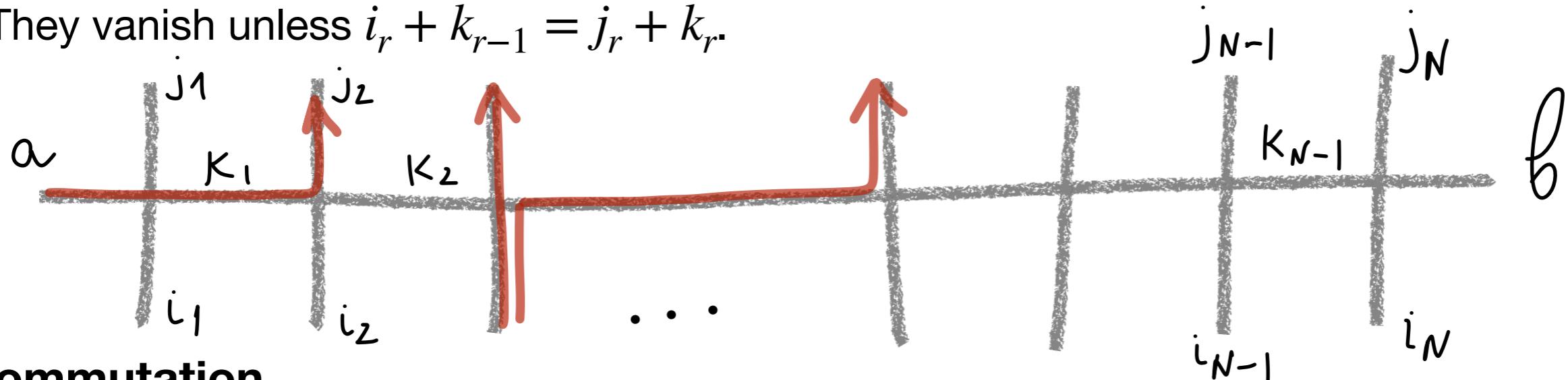
Row operators (transfer matrices) $T_x = T_x^{(a,b)}$

Let $V = \text{span}\{e_0, \dots, e_{r-1}\}$, and take $V^{\otimes N}$. A **transfer matrix** is an operator $V^{\otimes N} \rightarrow V^{\otimes N}$ defined as

$$T_x^{(a,b)}: e_{i_1} \otimes \dots \otimes e_{i_N} \mapsto \sum_{\vec{j}} \left(\prod_{r=1}^{N-1} w_x(i_r, k_{r-1}; j_r, k_r) \right) e_{j_1} \otimes \dots \otimes e_{j_N} \quad a, b, i_r, j_r, k_r \geq 0$$

Here x is a *spectral parameter*, and w_x are vertex weights.

They vanish unless $i_r + k_{r-1} = j_r + k_r$.



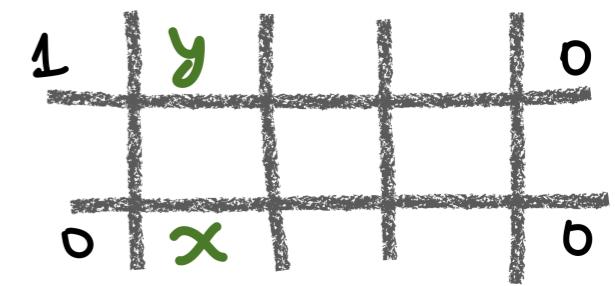
Commutation

In **integrable** models we have commuting transfer matrices $[T_x^{(0,0)}, T_y^{(0,0)}] = 0$.

For $a \neq b$ there might not be commutation, but rather a more complicated instance of the **Yang-Baxter equation**, like

$$T_x^{(0,0)} T_y^{(1,0)} = \frac{x-y}{x-qy} T_y^{(1,0)} T_x^{(0,0)} + \frac{(1-q)x}{x-qy} T_y^{(0,0)} T_x^{(1,0)}.$$

However, passing to $N \rightarrow \infty$ might simplify this to a commutation relation, which leads to **partition functions symmetric in spectral parameters x_i** .



Part 1

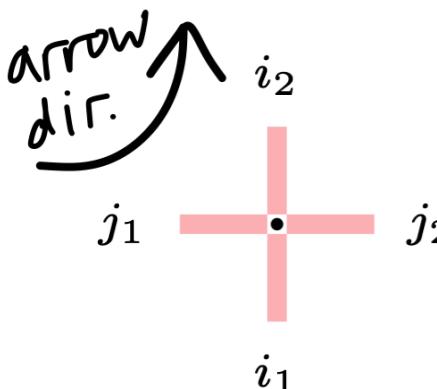
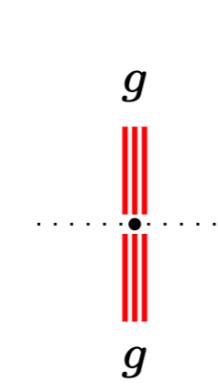
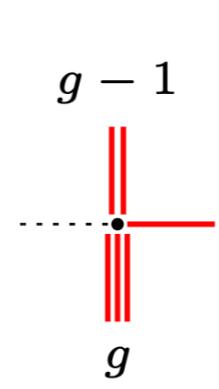
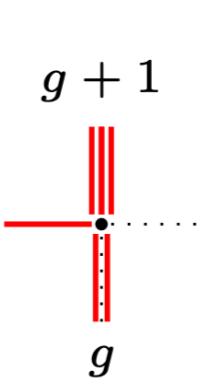
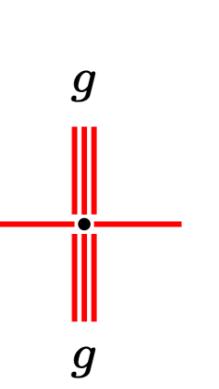
Determinantal summation identities for spin Hall-Littlewood functions

First, we focus on the **spin Hall-Littlewood functions** - a simpler example.

[Borodin 2014], [Borodin-P. 2016]

As an application we will get a new determinantal summation formula. [P. 2020]

Spin Hall-Littlewood vertex weights

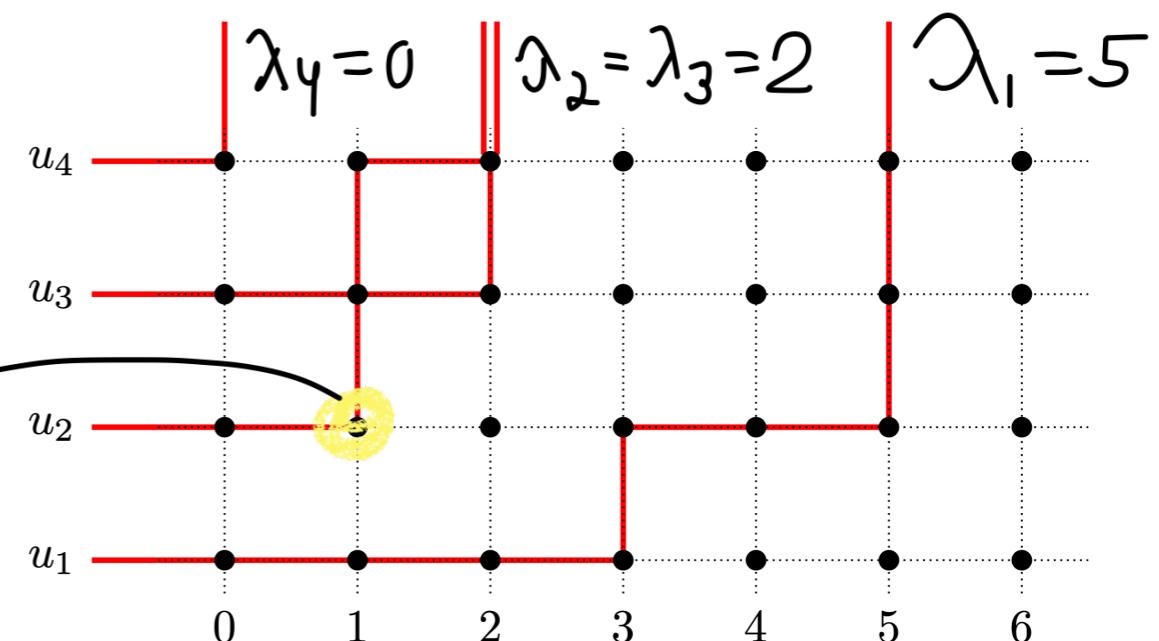
				
$w_{u,s}(i_1, j_1; i_2, j_2)$	$\frac{1 - suq^g}{1 - su}$	$\frac{u(1 - s^2 q^{g-1})}{1 - su}$	$\frac{1 - q^{g+1}}{1 - su}$	$\frac{u - sq^g}{1 - su}$

Symmetric functions

$$\lambda = (\lambda_1 \geq \dots \geq \lambda_N \geq 0)$$

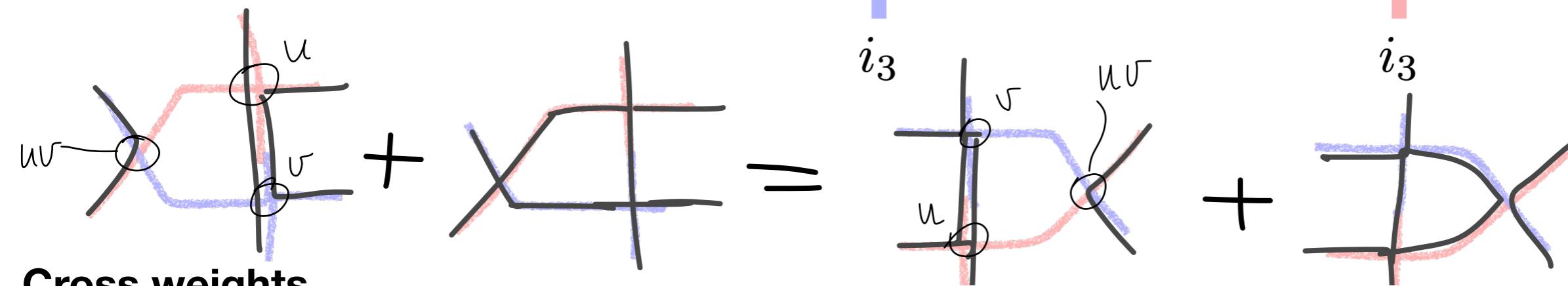
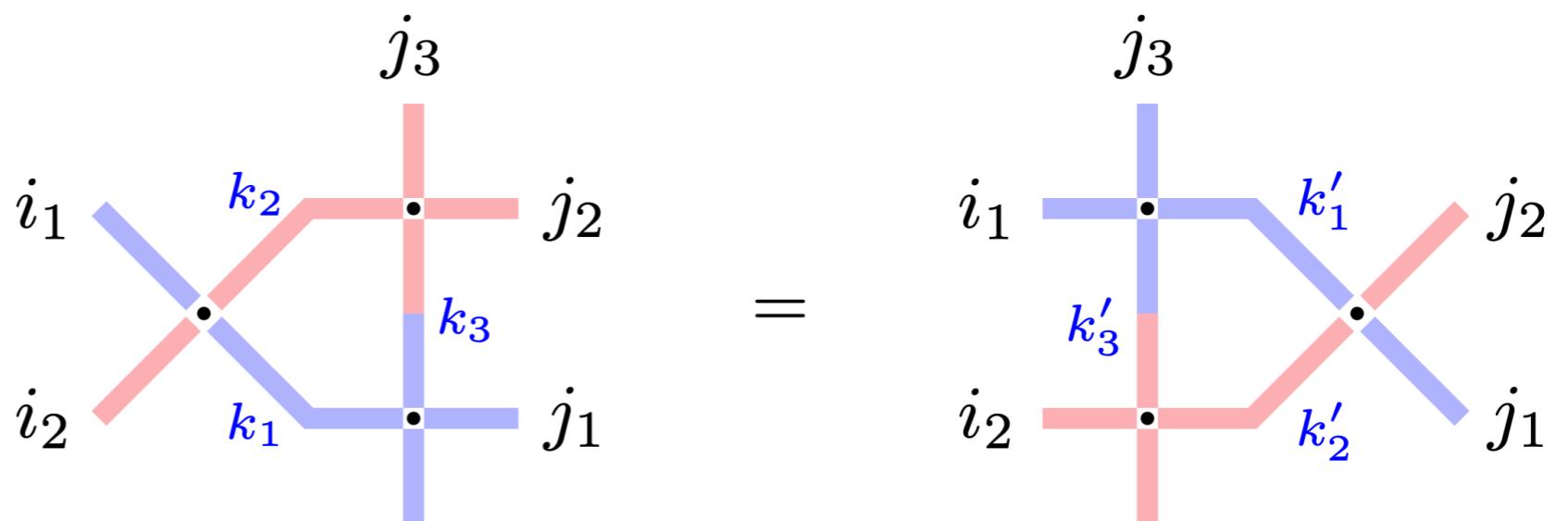
$$F_\lambda(u_1, \dots, u_N)$$

Weight
 $w_{u_2, \lambda_1, \lambda_1}$

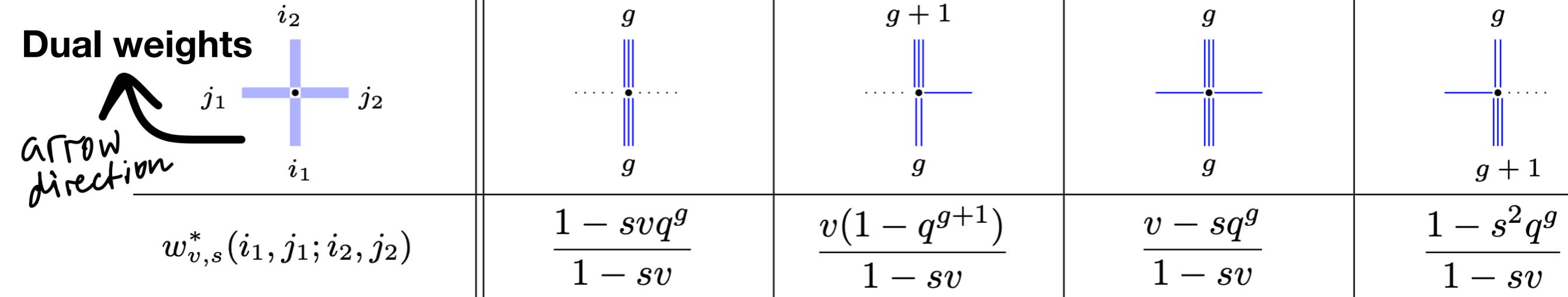
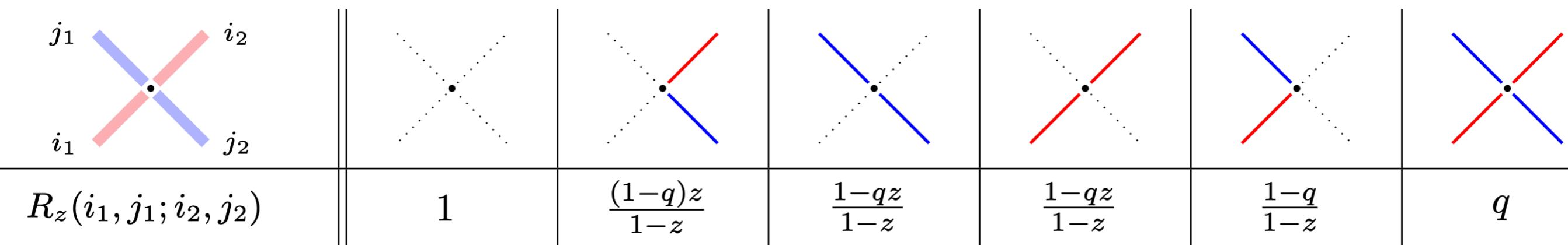


Yang-Baxter equation

Example:



Cross weights

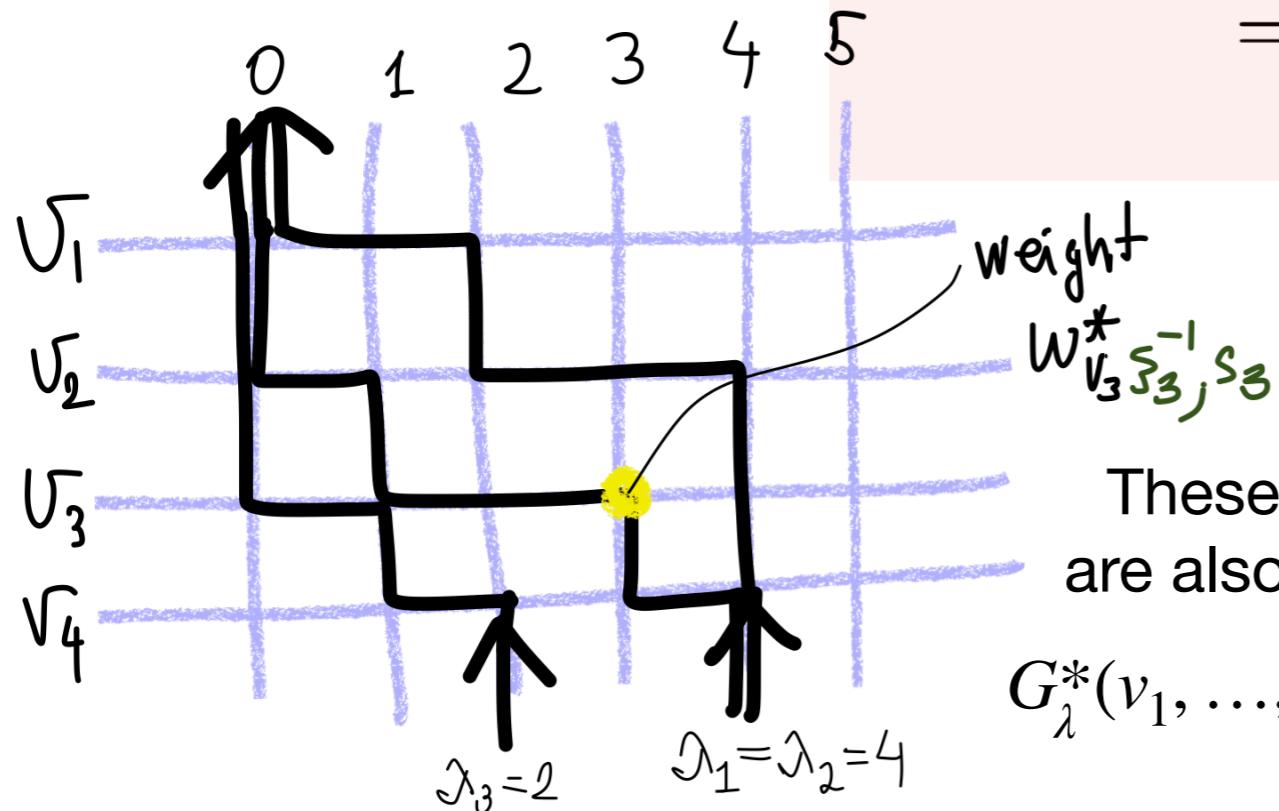


Cauchy identity

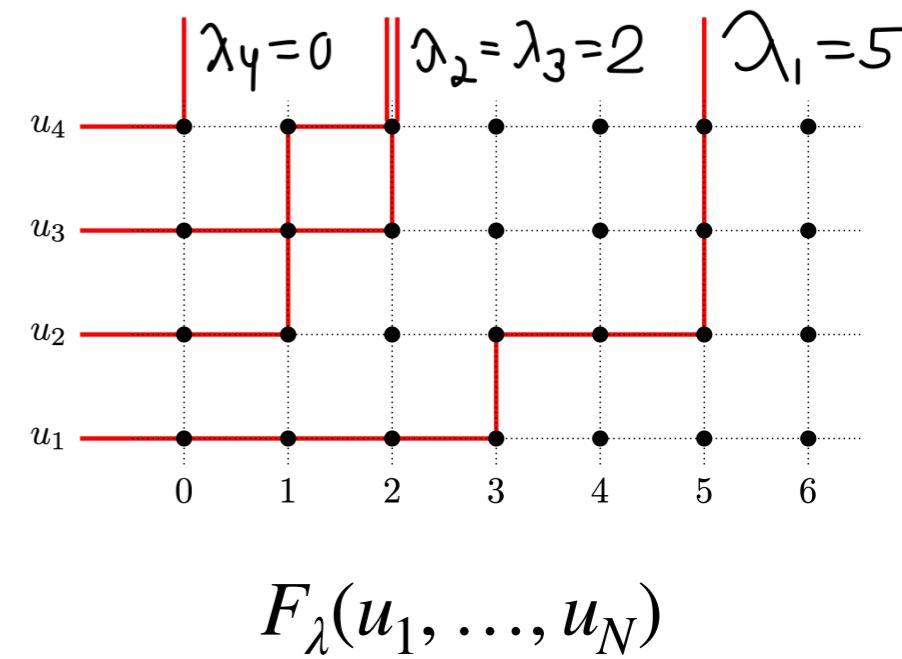
[Borodin-P. 2016]

$$\sum_{\lambda} F_{\lambda}(u_1, \dots, u_N) G_{\lambda}^*(v_1, \dots, v_M) \underbrace{\quad}_{\sum |u_i v_j| < c}$$

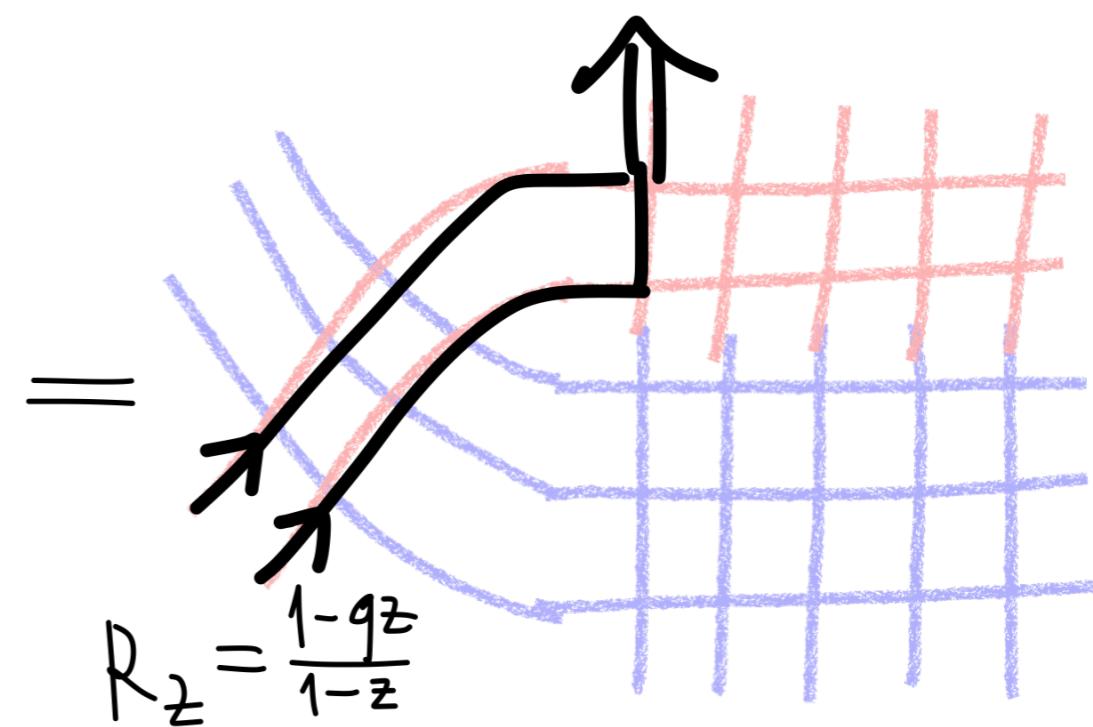
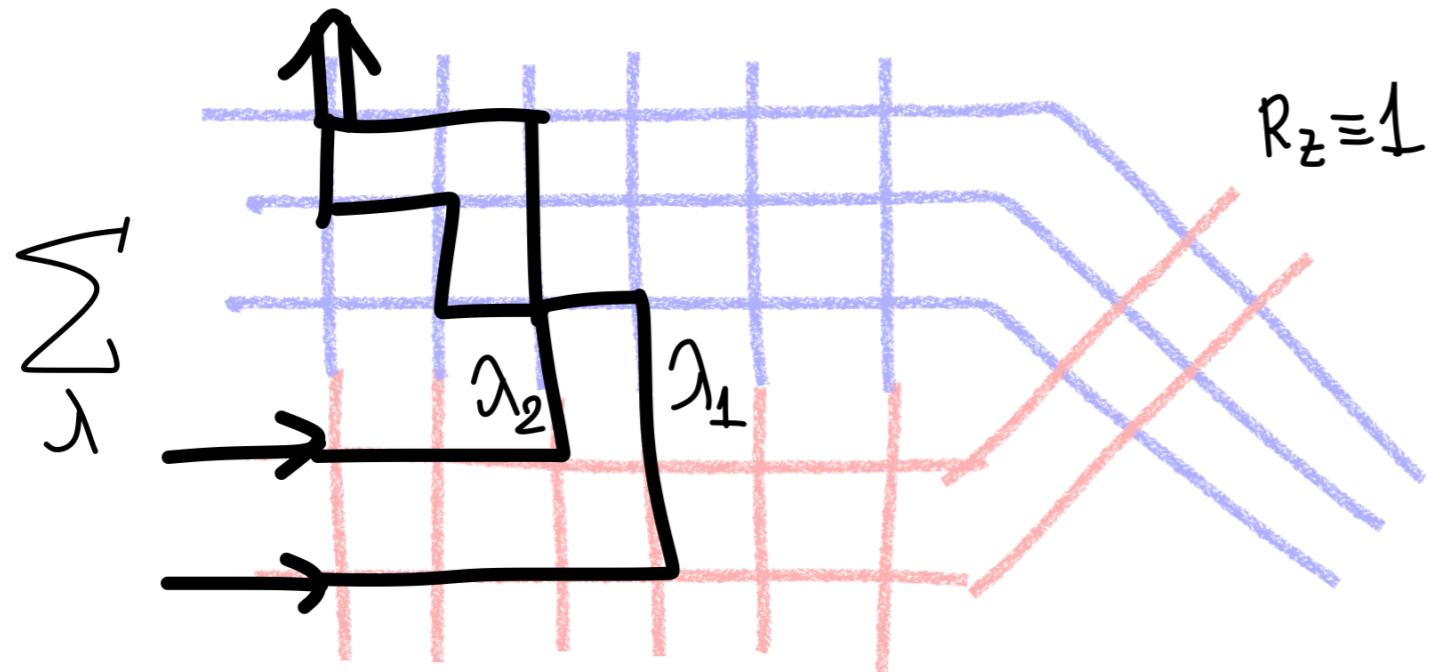
$$= \frac{(q; q)_N}{\prod_{i=1}^N (1 - s_0 \xi_0 u_i)} \prod_{i=1}^N \prod_{j=1}^M \frac{1 - qu_i v_j}{1 - u_i v_j}$$



These functions
are also symmetric



Proof of the identity



(only one nontrivial configuration)

Particular cases of spin Hall-Littlewood functions F_λ

- $s_k \equiv 0$ - Hall-Littlewood symmetric polynomials (well, q should be denoted by t then)

- $s_k \equiv 0, q = 0$ - Schur symmetric polynomials. Cauchy identity is well-known:

$$\sum_{\lambda} s_{\lambda}(x_1, \dots, x_N) s_{\lambda}(y_1, \dots, y_M) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

- (remark) New observation **[P. 2020]** - spin Hall-Littlewood functions reduce as $s_k = 0, \xi_k = 1$ ($k \geq 1$), and $s_0 \xi_0 \rightarrow 0, s_0 / \xi_0 \rightarrow q^{1-N}$ to the Hall-Littlewood limit of **interpolation Macdonald polynomials**, namely, $I_{\lambda}(u_1, \dots, u_N; 0, 1/q)$. This also extends to coloured setting (with nonsymmetric interpolation HL polynomials).

- (remark) $q = 0$, Grothendieck-like polynomials

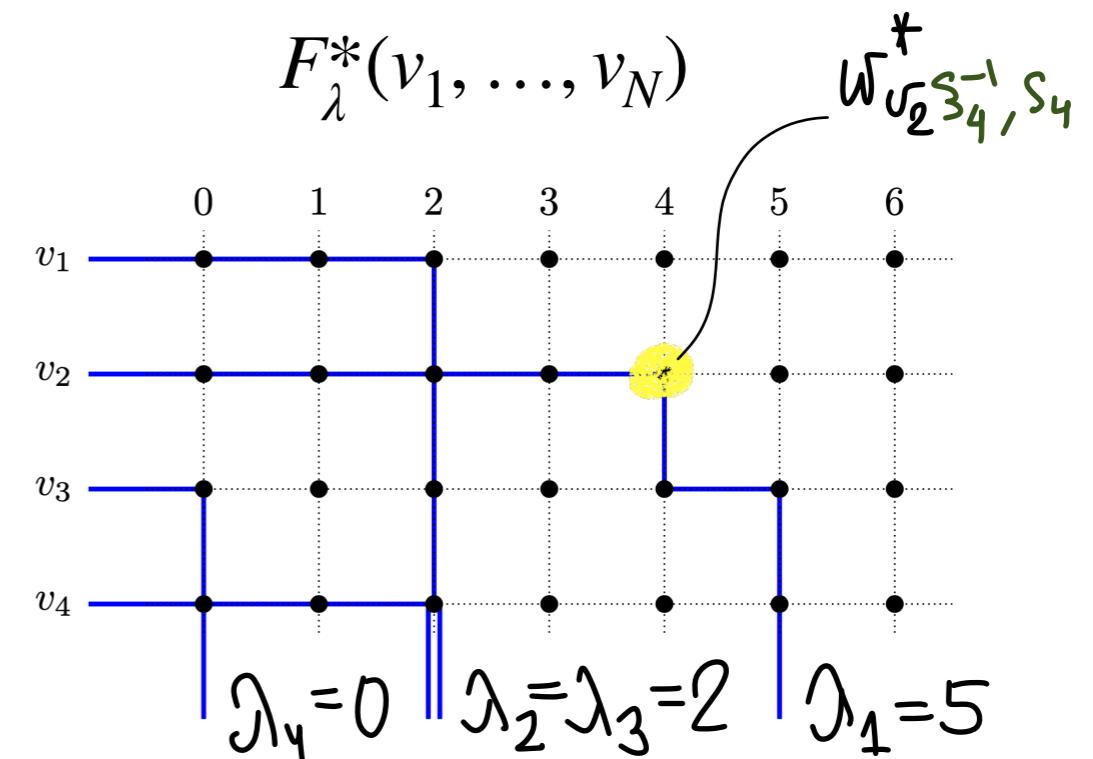
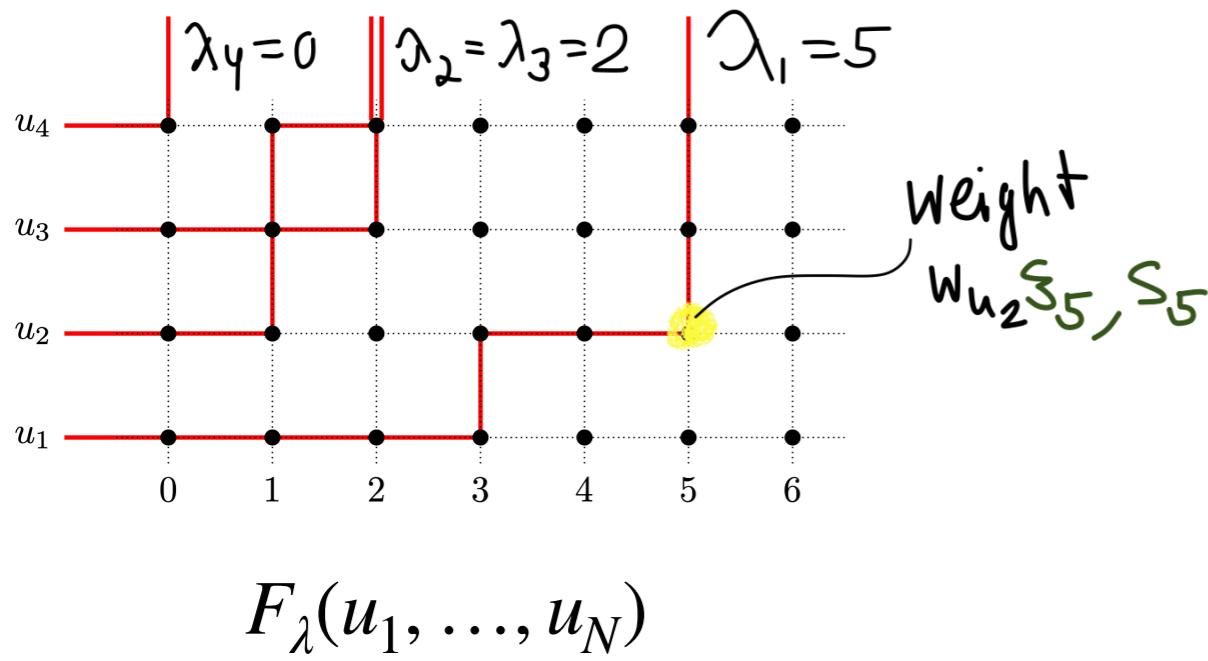
- (main point) $s_k \equiv 1/\sqrt{q}, \xi_k \equiv 1$ - eigenfunctions of the ASEP (**Asymmetric simple exclusion process**). More on ASEP - a little later.

In ASEP specialization, at most one vertical path is allowed per edge. So Cauchy identity **does not** degenerate to ASEP because G_{λ}^* does not make sense.

Using Yang-Baxter equation, we obtain a Cauchy-like identity which makes sense in the ASEP specialization.

Refined Cauchy identity

Hall-Littlewood case: [Wheeler - Zinn Justin 2015]



[P. 2020] For any $\gamma \neq 0$ we have

$$\sum_{\lambda=(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)} \frac{(\gamma q; q)_{m_0(\lambda)} (\gamma^{-1} s_0^2; q)_{m_0(\lambda)}}{(q; q)_{m_0(\lambda)} (s_0^2; q)_{m_0(\lambda)}} F_\lambda(u_1, \dots, u_N) F_\lambda^*(v_1, \dots, v_N)$$

$$= \prod_{j=1}^N \frac{1}{(1 - s_0 \xi_0 u_j)(1 - \xi_0^{-1} s_0 v_j)} \frac{\prod_{i,j=1}^N (1 - qu_i v_j)}{\prod_{1 \leq i < j \leq N} (u_i - u_j)(v_i - v_j)}$$

$$\times \det \left[\frac{(1 - \gamma)(q - \gamma^{-1} s_0^2)(1 - u_i v_j) + (1 - q)(1 - \xi_0 s_0 u_i)(1 - \xi_0^{-1} s_0 v_j)}{(1 - u_i v_j)(1 - qu_i v_j)} \right]_{i,j=1}^N.$$

$$(a; q)_k = (1-a)(1-aq)\dots(1-aq^{k-1})$$

$$m_0(\lambda) = N - \ell(\lambda) = \#\text{ of zeroes } |u_i v_j| < c$$

$$\begin{aligned}
& \sum_{\lambda=(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)} \frac{(\gamma q; q)_{m_0(\lambda)} (\gamma^{-1} s_0^2; q)_{m_0(\lambda)}}{(q; q)_{m_0(\lambda)} (s_0^2; q)_{m_0(\lambda)}} F_\lambda(u_1, \dots, u_N) F_\lambda^*(v_1, \dots, v_N) \\
&= \prod_{j=1}^N \frac{1}{(1 - s_0 \xi_0 u_j)(1 - \xi_0^{-1} s_0 v_j)} \frac{\prod_{i,j=1}^N (1 - qu_i v_j)}{\prod_{1 \leq i < j \leq N} (u_i - u_j)(v_i - v_j)} \\
&\quad \times \det \left[\frac{(1 - \gamma)(q - \gamma^{-1} s_0^2)(1 - u_i v_j) + (1 - q)(1 - \xi_0 s_0 u_i)(1 - \xi_0^{-1} s_0 v_j)}{(1 - u_i v_j)(1 - qu_i v_j)} \right]_{i,j=1}^N.
\end{aligned}$$

In particular, for $\gamma = 1$:

$$\sum_{\lambda} F_\lambda(u_1, \dots, u_N) F_\lambda^*(v_1, \dots, v_N) = \frac{(1 - q)^N \prod_{i,j=1}^N (1 - qu_i v_j)}{\prod_{1 \leq i < j \leq N} (u_i - u_j)(v_i - v_j)} \det \left[\frac{1}{(1 - u_i v_j)(1 - qu_i v_j)} \right]_{i,j=1}^N$$

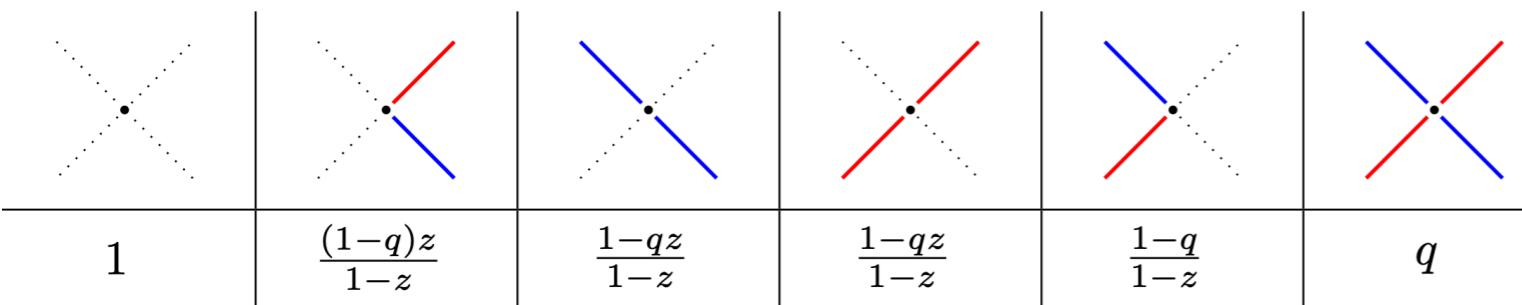
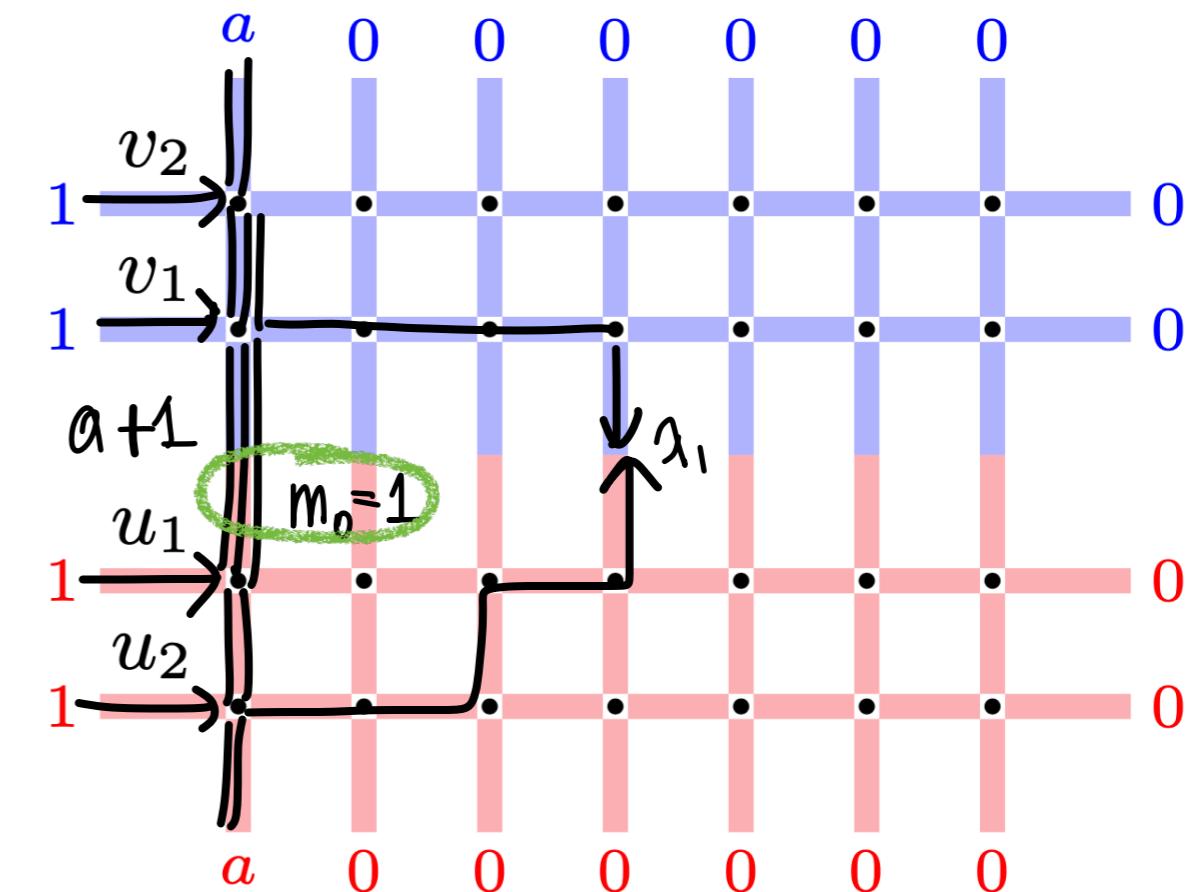
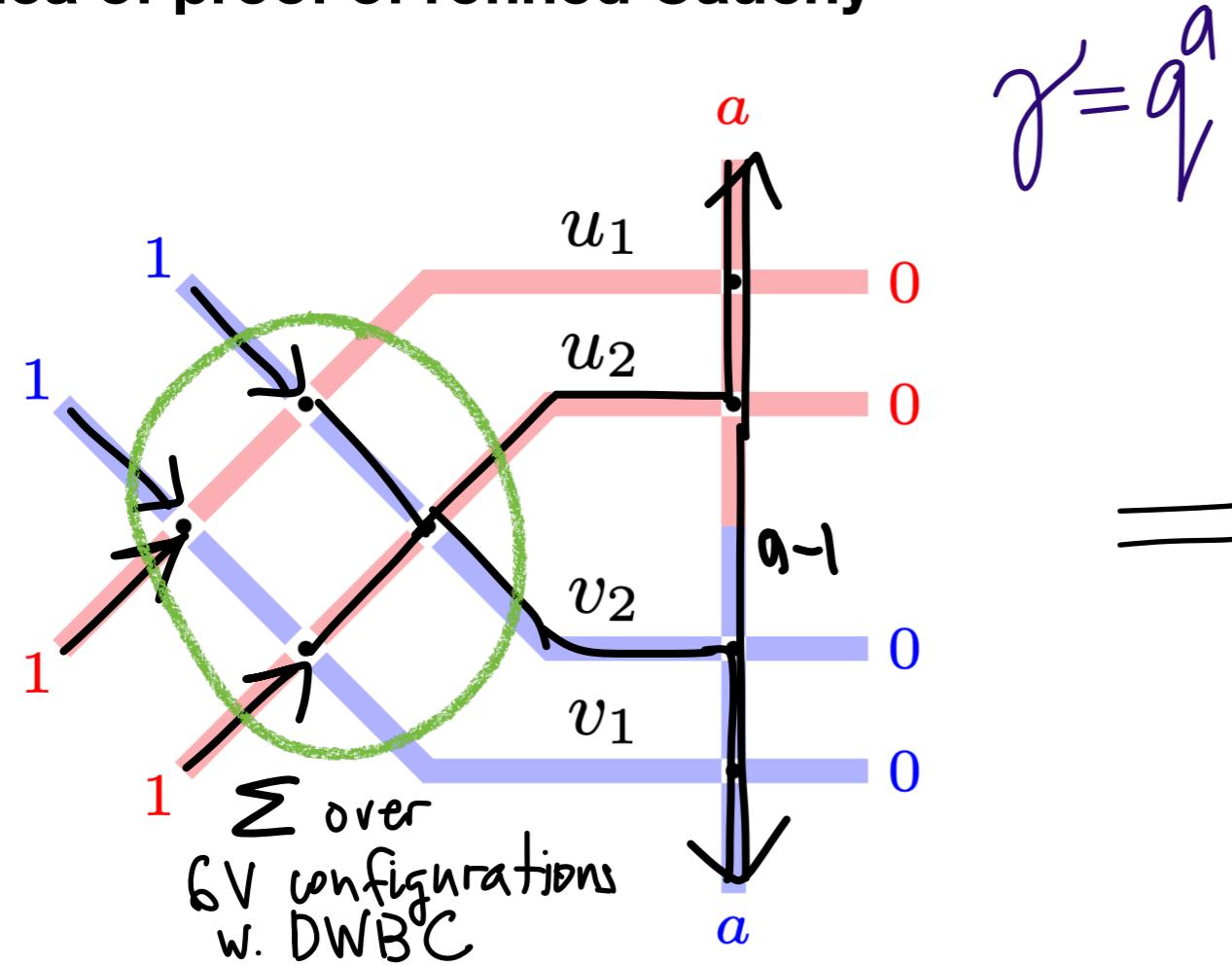
Izergin-Korepin det

[P. 2020] Another form of the same determinant:

$$\begin{aligned}
& \frac{\prod_{i,j=1}^N (1 - qu_i v_j)}{\prod_{1 \leq i < j \leq N} (v_i - v_j)} \det \left[\frac{(1 - \gamma)(q - \gamma^{-1} s_0^2)(1 - u_i v_j) + (1 - q)(1 - \xi_0 s_0 u_i)(1 - \xi_0^{-1} s_0 v_j)}{(1 - u_i v_j)(1 - qu_i v_j)} \right]_{i,j=1}^N \\
&= \det \left[u_j^{N-i-1} \left\{ (1 - s_0 \xi_0 u_j) \left(u_j - \frac{s_0}{\xi_0} \right) \prod_{l=1}^N \frac{1 - qu_j v_l}{1 - u_j v_l} \right. \right. \\
&\quad \left. \left. - \gamma^{-1} q^{N-i} (\gamma - s_0 \xi_0 u_j) \left(\gamma qu_j - \frac{s_0}{\xi_0} \right) \right\} \right]_{i,j=1}^N.
\end{aligned}$$

Inspired by [Cuenca 2017]

Idea of proof of refined Cauchy



Left-hand side is a determinant:
identified through Lagrange interpolation

(method goes back to 1980s work of Izergin and Korepin on six-vertex model
with domain wall boundary conditions)

Case $s_0 = 0$ and connection to Macdonald averages

$$\begin{aligned}
 & \prod_{i,j=1}^N \frac{1}{1 - qu_i v_j} \sum_{\lambda \in \text{Sign}_N} (\gamma q; q)_{m_0(\lambda)} \tilde{F}_\lambda(u_1, \dots, u_N) \tilde{F}_\lambda^*(v_1, \dots, v_N) \\
 &= \sum_{\lambda \in \text{Sign}_N} \prod_{j=1}^N (1 - (\gamma q) q^{\lambda_j + N - j}) s_\lambda(u_1, \dots, u_N) s_\lambda(v_1, \dots, v_N) \\
 &= \det [\dots]
 \end{aligned}$$

functions at $s_0=0$

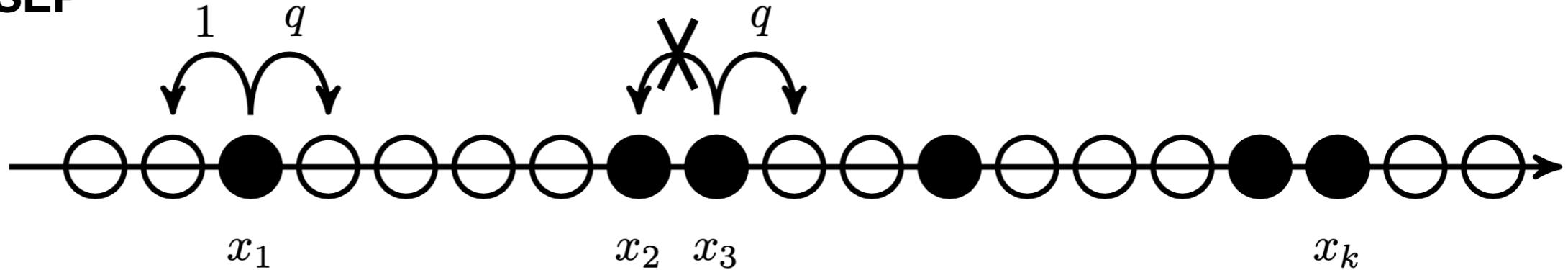
$$\mathbb{E}_{\text{sHL}(q)}(-\zeta; q)_{m_0(\lambda)} = \mathbb{E}_{\text{MM}(q, q)}^{\text{Schur measure}} \prod_{j=1}^N (1 + \zeta q^{\lambda_j + N - j})$$

\parallel

Also related to the distribution of the height function of the **stochastic six-vertex model**

$$= \mathbb{E}_{\text{MM}(q, t)}^{\text{Macdonald measure}} \prod_{j=1}^N (1 + \zeta q^{\lambda_j} t^{N-j})$$

Reduction to ASEP



The eigenfunctions of the Markov generator of ASEP are

$$\Psi_{\vec{x}}^r(\vec{z}) = \sum_{\sigma \in S_N} \prod_{1 \leq i < j \leq N} \frac{z_{\sigma(i)} - q z_{\sigma(j)}}{z_{\sigma(i)} - z_{\sigma(j)}} \prod_{i=1}^N \left(\frac{1 - z_{\sigma(i)}}{1 - z_{\sigma(i)}/q} \right)^{-x_i}, \quad \mathcal{A} \Psi_{\vec{x}}^r(\vec{z}) = \text{ev}(\vec{z}) \Psi_{\vec{x}}^r(\vec{z});$$

(Coordinate Bethe Ansatz)

$$\text{ev}(\vec{z}) = - \sum_{j=1}^N \frac{(1-q)^2}{(1-z_j)(1-q/z_j)}.$$

[Tracy-Widom 2007] Transition function of ASEP is (all contours are around 1)

$$P_t(\vec{x} \rightarrow \vec{y}) = \frac{1}{N!(2\pi i)^N} \oint dz_1 \dots \oint dz_N \frac{\prod_{i < j} (z_i - z_j)^2}{\prod_{i \neq j} (z_i - qz_j)} \\ \times \prod_{j=1}^N \frac{1 - 1/q}{(1 - z_j)(1 - z_j/q)} \exp\{t \cdot \text{ev}(\vec{z})\} \Psi_{\vec{x}}^r(\vec{z}) \Psi_{\vec{y}}^\ell(\vec{z}),$$

Summation identities for ASEP eigenfunctions

[Tracy-Widom 2007]

$$\sum_{0 \leq x_1 < x_2 < \dots < x_N} \Psi_{\vec{x}}^{\ell}(\vec{z}) = \frac{(-q)^{\frac{N(N-1)}{2}} (1 - z_1/q) \dots (1 - z_N/q)}{(1 - 1/q)^N z_1 \dots z_N}$$

(one of the “magic identities”)

[Corwin-Liu 2019, unpublished], [P. 2020]

$$\begin{aligned} & \sum_{0 \leq x_1 < x_2 < \dots < x_N} \Psi_{\vec{x}}^r(\vec{z}) \Psi_{\vec{x}}^{\ell}(\vec{w}) \\ &= \prod_{j=1}^N (1 - z_j)(1 - w_j/q) \frac{(1/q - 1)^{-N} \prod_{i,j=1}^N (z_i - q w_j)}{\prod_{1 \leq i < j \leq N} (z_i - z_j)(w_j - w_i)} \det \left[\frac{1}{(z_i - w_j)(z_i - q w_j)} \right]_{i,j=1}^N \end{aligned}$$

An example two-time quantity in ASEP:

$$\text{Prob}(x_1(t_1) \geq k_1, x_1(t_2) \geq k_2) = \sum_{x'_1 \geq k_1, x''_1 \geq k_2} P_{t_1}(\vec{x} \rightarrow \vec{x}') P_{t_2 - t_1}(\vec{x}' \rightarrow \vec{x}'')$$

[P. 2020]

$$\begin{aligned} \text{Prob}(x_1(t_1) \geq k_1, x_1(t_2) \geq k_2) &= \frac{(-1)^N q^{\frac{N(N-1)}{2}}}{(N!)^2 (2\pi i)^{2N}} \oint \frac{dz_1}{1-z_1} \cdots \oint \frac{dz_N}{1-z_N} \oint \frac{dw_1}{w_1} \cdots \oint \frac{dw_N}{w_N} \\ &\times \frac{\prod_{i < j} (z_i - z_j)(w_i - w_j) \prod_{i,j=1}^N (w_i - qz_j)}{\prod_{i \neq j} (z_i - qz_j)(w_i - qw_j)} \det \left[\frac{1}{(w_i - z_j)(w_i - qz_j)} \right]_{i,j=1}^N \\ &\times \exp\{t_1 \mathbf{ev}(\vec{z}) + (t_2 - t_1) \mathbf{ev}(\vec{w})\} \prod_{j=1}^N \left(\frac{1 - z_j}{1 - z_j/q} \right)^{k_1} \left(\frac{1 - w_j}{1 - w_j/q} \right)^{k_2 - k_1} \Psi_{\vec{x}}^r(\vec{z}). \end{aligned}$$

All integration contours are small positively oriented circles around 1, with $|z_i - 1| < |w_i - 1|$ for all z_i, w_j on the contours.

Asymptotics,
anyone?

Single-time asymptotic analysis - [Tracy-Widom 2008], ...

Multipoint analysis (six vertex model) - [Dimitrov 2020]

Determinantal models (TASEP):

- multipoint results for TASEP are available via Schur measures, as well as multitime results along **space-like** paths
- best general (multitime and multipoint) results are on a ring [Baik, Liu 2016+];
- Multitime results on the line - [Johansson, Rahman 2015+]

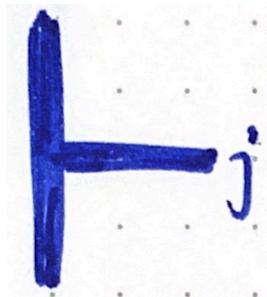
Part 2

Spin Whittaker functions

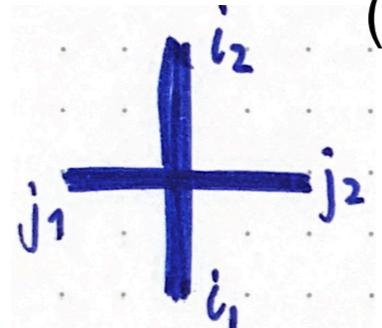
j.w. Matteo Mucciconi

Spin q -Whittaker weights = fusion of the spin Hall-Littlewood ones

(most weights become polynomial in x or y)



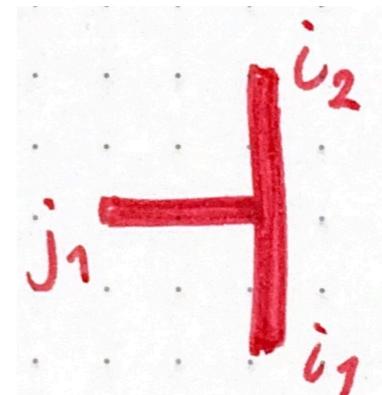
$$x^j \frac{(-s/x; q)_j}{(q; q)_j}$$



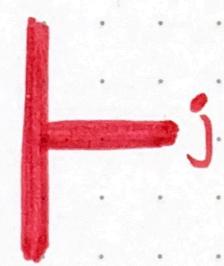
$$\mathbf{1}_{i_1+j_1=i_2+j_2} \mathbf{1}_{i_1 \geq j_2} x^{j_2} \frac{(-s/x; q)_{j_2} (-sx; q)_{i_1-j_2} (q; q)_{i_2}}{(q; q)_{j_2} (q; q)_{i_1-j_2} (s^2; q)_{i_2}}$$



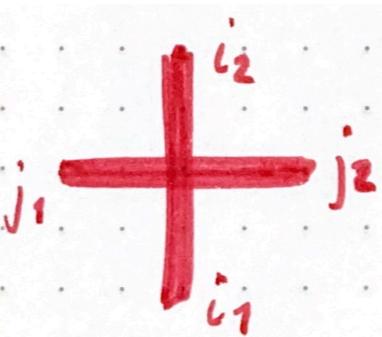
$$\frac{(q; q)_j}{(-s/x; q)_j}.$$



$$\mathbf{1}_{i_1=j_1+i_2}$$

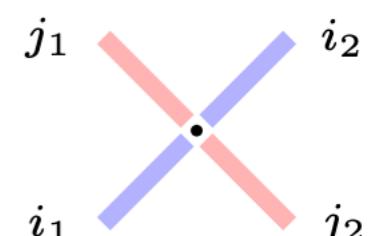


$$x^j \frac{(-s/x; q)_j}{(q; q)_j}$$



$$\mathbf{1}_{i_2+j_1=i_1+j_2} \mathbf{1}_{i_2 \geq j_2} \frac{y^{j_2} (q; q)_{i_2} (-s/y; q)_{j_2} (-sy; q)_{i_2-j_2}}{(q; q)_{i_2-j_2} (q; q)_{j_2} (s^2; q)_{i_2}}$$

$$\begin{aligned} \mathbb{R}_{x,y,s}(i_1, j_1; i_2, j_2) &\coloneqq \mathbf{1}_{i_2+j_1=i_1+j_2} \frac{q^{i_2 i_1 + \frac{1}{2} j_2(j_2-1)} (sx)^{j_2} (q; q)_{j_1}}{(s^2; q)_{j_1+i_2} (q; q)_{j_2} (q; q)_{i_2} (-q/(sx); q)_{i_1-j_1}} \\ &\quad \times {}_4\bar{\phi}_3 \left(\begin{matrix} q^{-n} & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & \end{matrix} ; q, z \right) \\ &= \sum_{j=0}^n z^j \frac{(q^{-n}; q)_j}{(q; q)_j} (a_1, a_2, a_3; q)_j (q^j b_1, q^j b_2, q^j b_3; q)_{n-j}. \end{aligned}$$



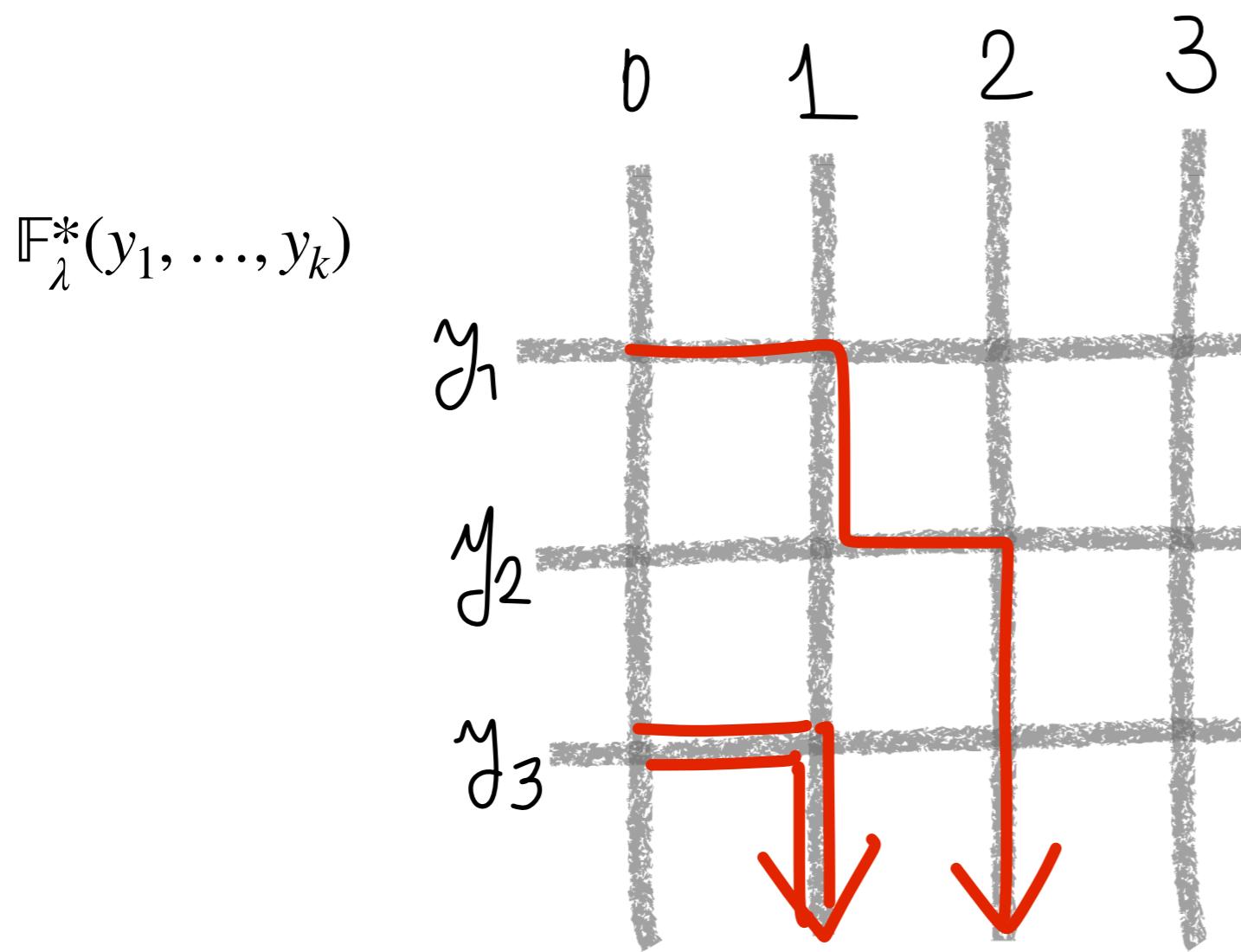
$$\times {}_4\bar{\phi}_3 \left(\begin{matrix} q^{-i_2}; q^{-i_1}, -sy, -q/(sx) \\ -s/x, q^{1+j_2-i_2}, -yq^{1-i_1-j_2}/s \end{matrix} \middle| q, q \right).$$

Spin q -Whittaker symmetric polynomials

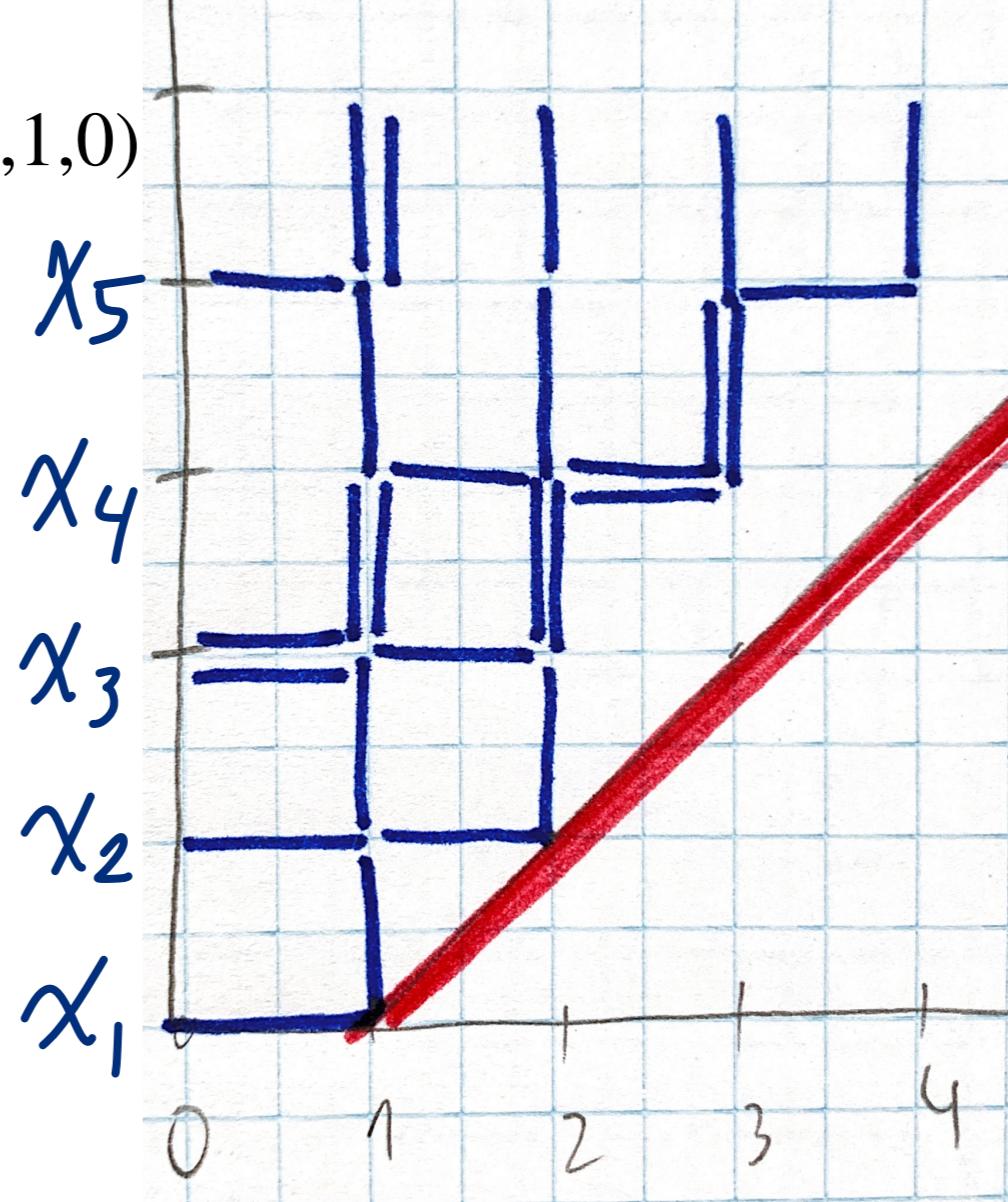
$$\lambda = (5, 3, 2, 1, 0)$$

$$\mathbb{F}_\lambda(x_1, \dots, x_N)$$

$\lambda_i - \lambda_{i+1} = \#\{\text{paths through the } i\text{-th vertical edge}\}.$



$$\lambda = (3, 1, 0)$$



Spin q -Whittaker polynomials

$$\mathbb{F}_{\lambda/\mu}(x) := x^{|\lambda|-|\mu|} \prod_{i=1}^k \frac{(-s/x; q)_{\lambda_i - \mu_i} (-sx; q)_{\mu_i - \lambda_{i+1}} (q; q)_{\lambda_i - \lambda_{i+1}}}{(q; q)_{\lambda_i - \mu_i} (q; q)_{\mu_i - \lambda_{i+1}} (s^2; q)_{\lambda_i - \lambda_{i+1}}} \quad (*)$$

(this is a polynomial in x)

$$0 \leq \lambda_{k+1} \leq \mu_k \leq \lambda_k \leq \dots \mu_1 \leq \lambda_1, \quad \lambda_i, \mu_i \in \mathbb{Z}$$

$$\mathbb{F}_\nu(x_1, \dots, x_n) = \sum_{\varkappa} \mathbb{F}_\varkappa(x_1, \dots, x_{n-1}) \mathbb{F}_{\nu/\varkappa}(x_n)$$

This is a symmetric polynomial in x_1, \dots, x_n , which follows from YBE

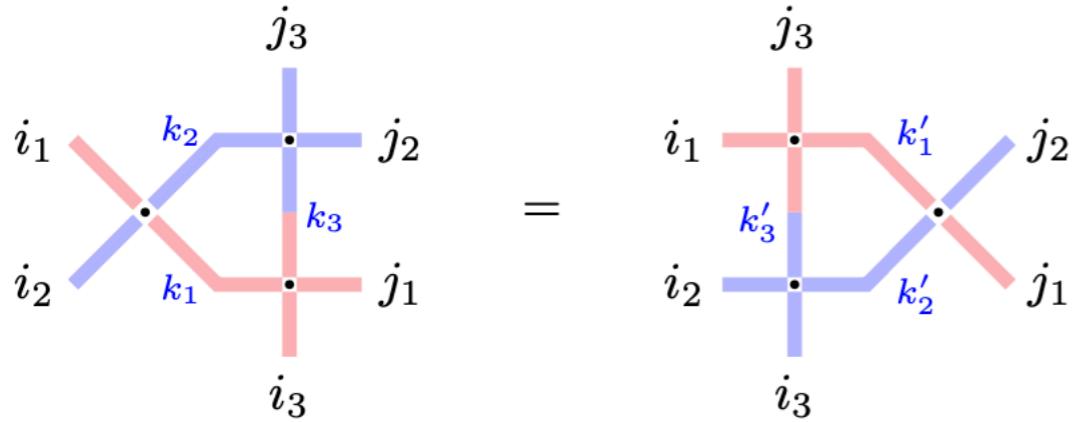
Borodin-Wheeler's version (2017)

$$\mathbb{F}_{\lambda/\mu}^{BW}(x) = \frac{(-s/x; q)_{\lambda_{k+1}}}{(s^2; q)_{\lambda_{k+1}}} \mathbb{F}_{\lambda/\mu}(x).$$

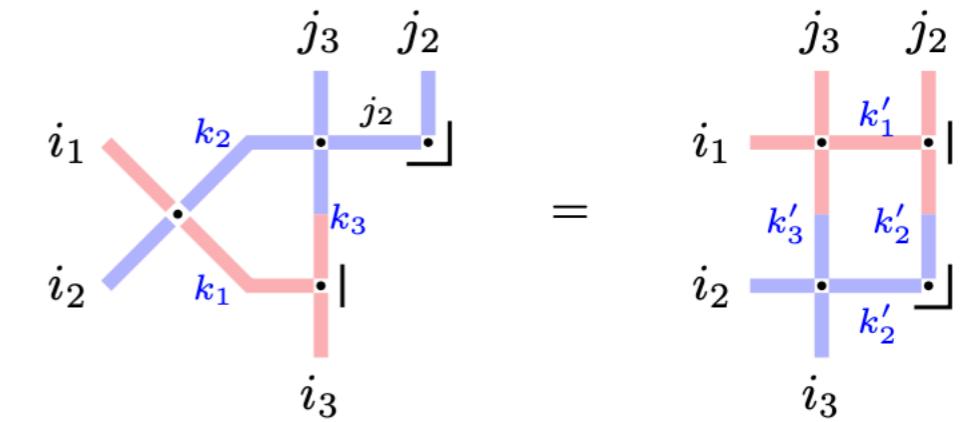
$$\mathbb{F}_\lambda(0, x_2, \dots, x_n) = \mathbb{F}_\lambda^{BW}(x_2, \dots, x_n).$$

We made a typo implementing $\mathbb{F}_{\lambda/\mu}^{BW}$ and wrote **(*)** instead of the correct expression. But surprisingly **(*)** leads to symmetric polynomials satisfying nice properties - that is how our sqW polynomials were discovered. 😊

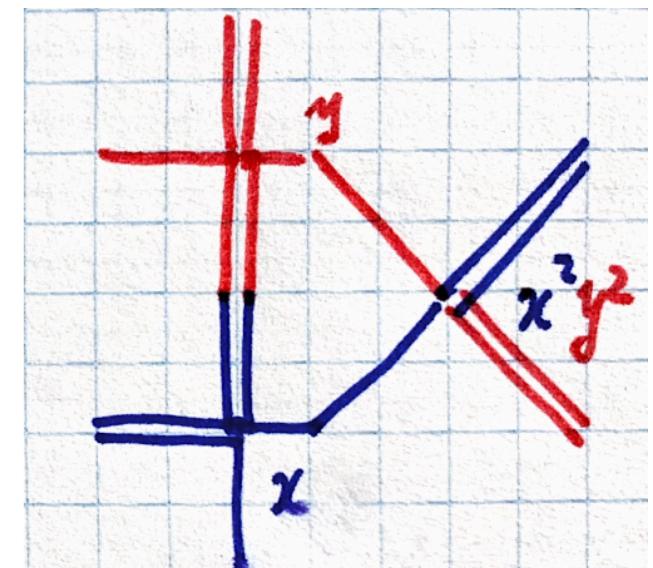
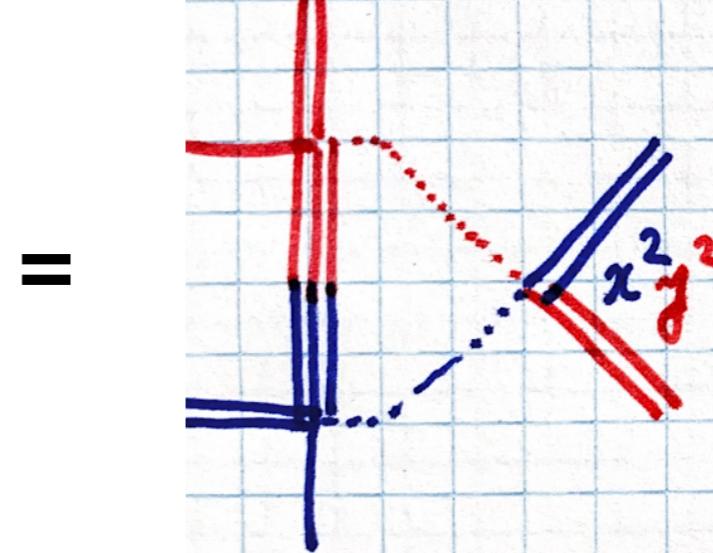
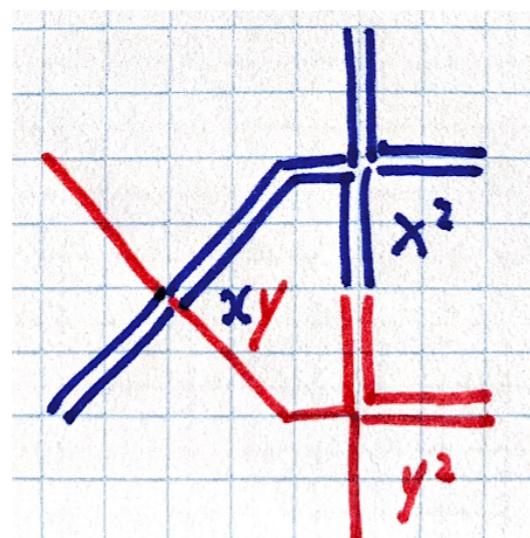
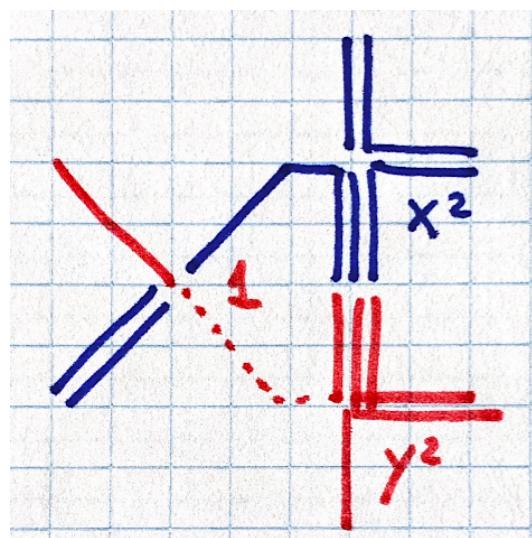
Yang-Baxter equation example ($q = s = 0$)



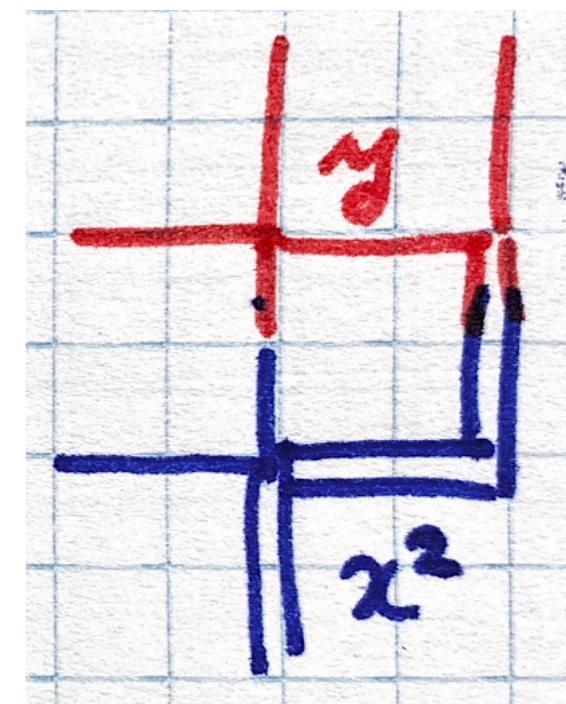
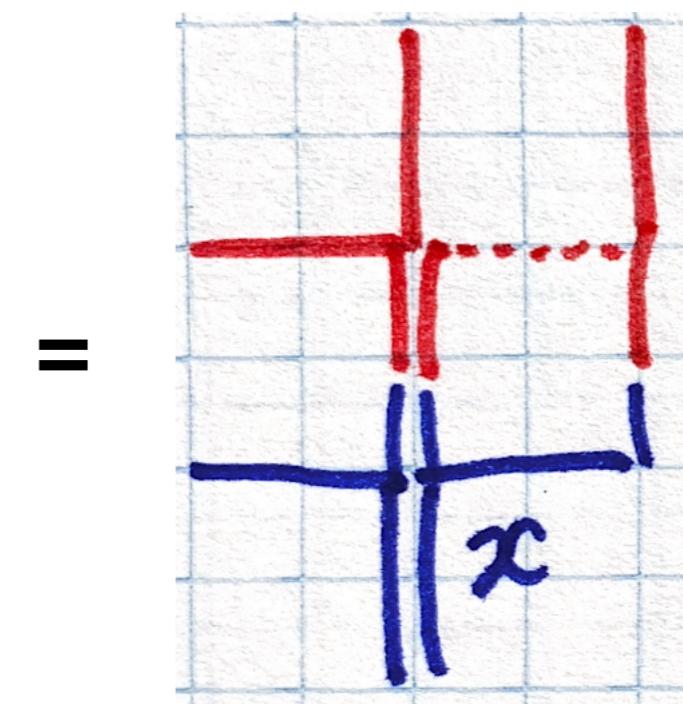
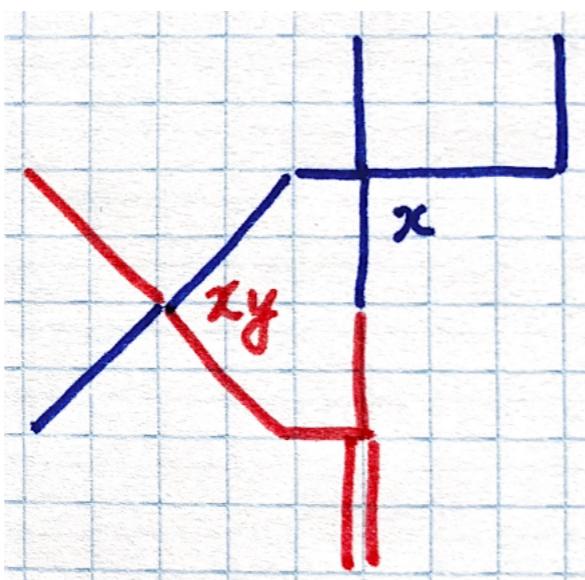
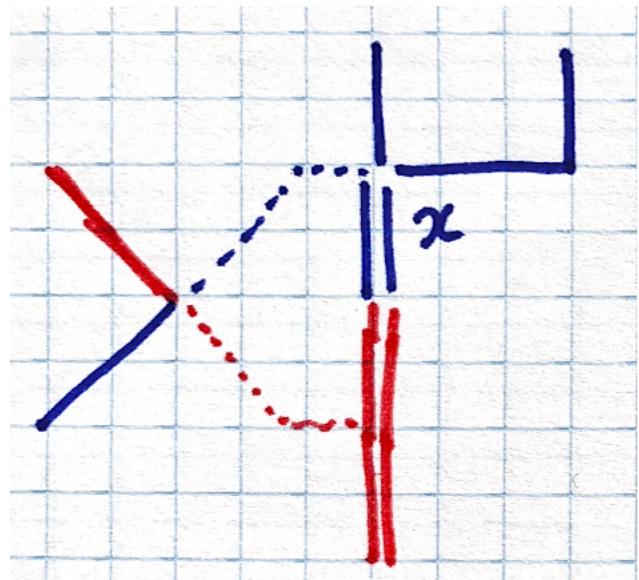
(a)



(b)



Example (b)



Cauchy identity

$$\sum_{\lambda \in \text{Sign}_N} \mathbb{F}_\lambda(x_1, \dots, x_N) \mathbb{F}_\lambda^*(y_1, \dots, y_k) = \prod_{j=1}^k \left(\frac{(-sy_j; q)_\infty}{(s^2; q)_\infty} \right)^{N-1} \prod_{i=1}^N \prod_{j=1}^k \frac{(-sx_i; q)_\infty}{(x_i y_j; q)_\infty}.$$

q -difference operators ($T_{q,x}$ maps $f(x)$ to $f(qx)$)

$$\mathfrak{D}_1 := \sum_{i=1}^N \prod_{\substack{j=1 \\ j \neq i}}^N \frac{(1 + sx_i)}{1 - x_i/x_j} T_{q,x_i},$$

$$\mathfrak{D}_1 \mathbb{F}_\lambda(x_1, \dots, x_N) = q^{\lambda_N} \mathbb{F}_\lambda(x_1, \dots, x_N).$$

$$\overline{\mathfrak{D}}_1 := \sum_{i=1}^N \prod_{\substack{j=1 \\ j \neq i}}^N \frac{(1 + s/x_i)}{1 - x_j/x_i} T_{q^{-1},x_i}.$$

$$\overline{\mathfrak{D}}_1 \mathbb{F}_\lambda(x_1, \dots, x_N) = q^{-\lambda_1} \mathbb{F}_\lambda(x_1, \dots, x_N).$$

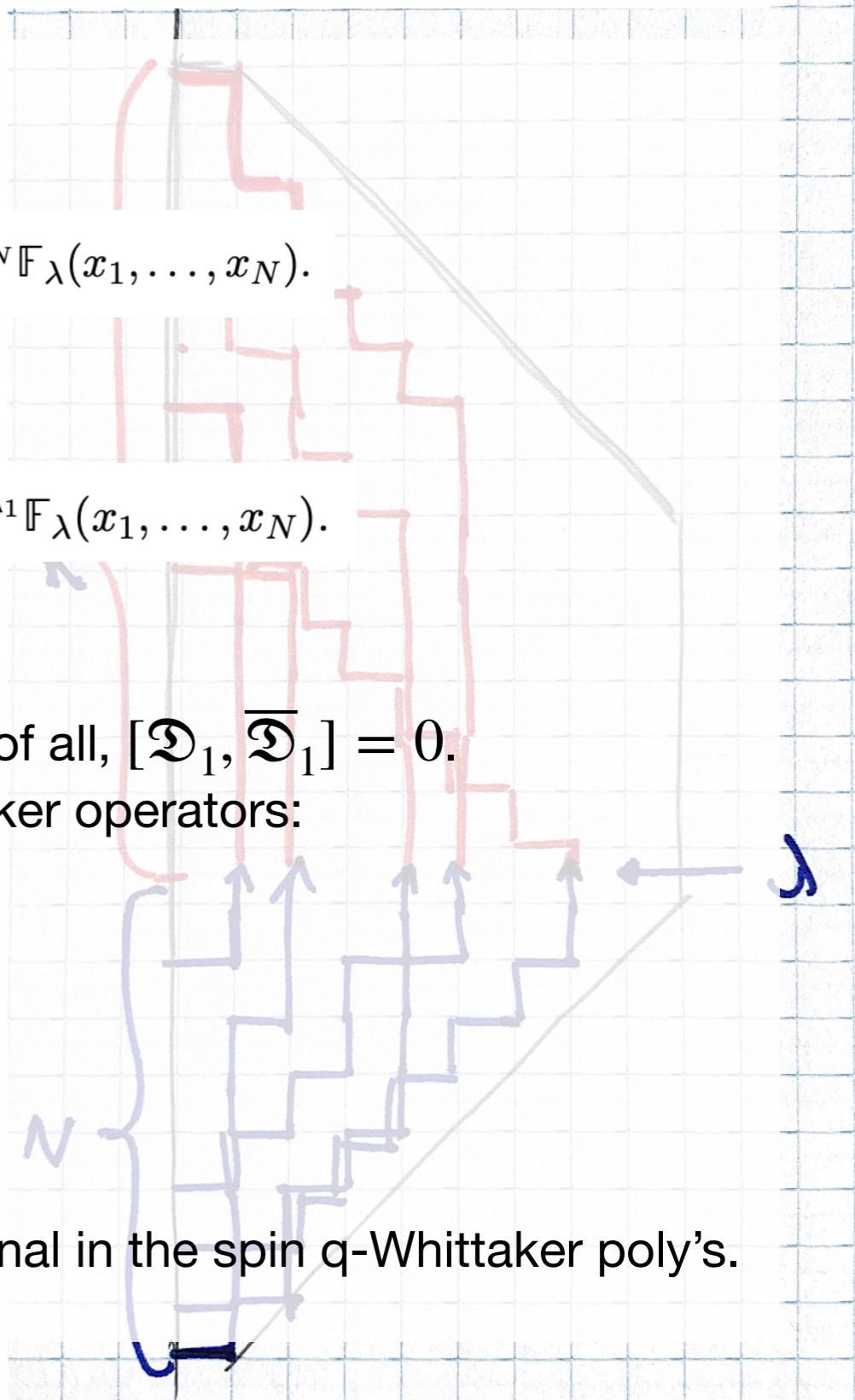
In the spin deformation the situation is more mysterious. First of all, $[\mathfrak{D}_1, \overline{\mathfrak{D}}_1] = 0$.

Next, both of them are conjugations of the first order q -Whittaker operators:

$$\mathbb{U}_N := \prod_{i=1}^N \frac{1}{(-sx_i; q)_\infty^{N-1}}, \quad \mathbb{V}_N := \prod_{i=1}^N \frac{1}{(-s/x_i; q)_\infty^{N-1}}.$$

$$\mathfrak{D}_1 = \mathbb{U}_N^{-1} W_N^1 \mathbb{U}_N, \quad \overline{\mathfrak{D}}_1 = \mathbb{V}_N^{-1} \tilde{W}_N^1 \mathbb{V}_N,$$

Same conjugations of the q -Whittaker operators are not diagonal in the spin q -Whittaker poly's.



Limit transitions $\text{sqW} \rightarrow \text{sW} \rightarrow \text{W}$

$$F_\lambda(x_1, \dots, x_N \mid q, s)$$

$0 < q < 1$
 $-1 < s < 0$

$$q \rightarrow 1, \quad x_i = q^{X_i}, \quad L_i = q^{-\lambda_i}$$

$$s = -q^S$$

$$s > 0, \quad |x_i| < s, \quad 1 \leq L_N \leq L_{N-1} \leq \dots \leq L_1$$

Theorem (Mucciconi-P. 2020)

$$\lim_{q \rightarrow 1} \frac{\mathbb{F}_\lambda(x_1, \dots, x_N)}{(-\log q)^{N(N-1)/2}} = \mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N)$$

$$1 \leq L_N \leq \dots \leq L_1$$

Here $\mathfrak{f}_{X_1, \dots, X_N}$ is the *spin Whittaker* function, which is symmetric in X_i , depends on \underline{L}_N and on a parameter S .

Reduction to the usual \mathfrak{gl}_N Whittaker functions, $S \rightarrow +\infty$

$$L_i = S^{N+1-2i} e^{u_i}, \quad X_k = -i\lambda_k,$$

Whittaker symmetric functions

(Kostant, Givental, Bump, Stade, Gerasimov-Lebedev-Oblezin, Corwin-O'Connell-Seppalainen-Zygouras,...)

$$\psi_{\lambda_1, \dots, \lambda_N}(\underline{u}_N) = \int_{\mathbb{R}^{N-1}} \psi_{\lambda_1, \dots, \lambda_{N-1}}(\underline{u}_{N-1}) Q_{\lambda_N}^{N \rightarrow N-1}(\underline{u}_N, \underline{u}_{N-1}) \prod_{k=1}^{N-1} du_{N-1,k},$$

where

$$Q_{\lambda}^{N \rightarrow N-1}(\underline{u}_N, \underline{u}_{N-1}) = e^{i\lambda(\sum_{i=1}^N u_{N,i} - \sum_{i=1}^{N-1} u_{N-1,i})} \prod_{i=1}^{N-1} \exp \left\{ -e^{u_{N-1,i} - u_{N,i}} - e^{u_{N,i+1} - u_{N-1,i}} \right\}$$

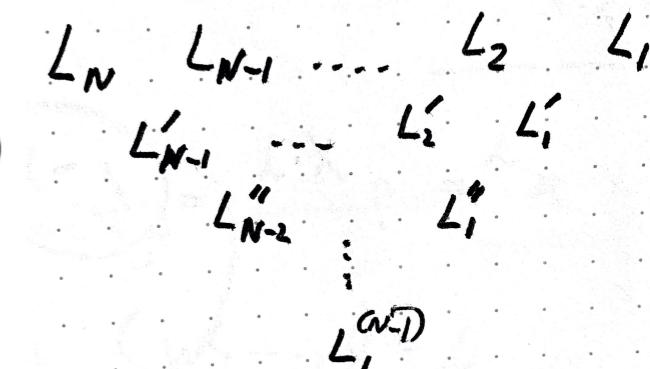
Conjecture (Mucciconi-P. 2020, holds modulo decay estimates)

$$\left(\frac{4\pi}{S 16^S} \right)^{\frac{N(N-1)}{4}} \mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N) \xrightarrow{S \rightarrow \infty} \psi_{\lambda_1, \dots, \lambda_N}(u_1, \dots, u_N)$$

Definition of spin Whittaker functions via “combinatorial formula” (or “Givental integral”)

$$\mathcal{A}_{S,X}(u,v,z) := \frac{1}{B(S+X, S-X)} \left(1 - \frac{v}{z}\right)^{S-X-1} \left(1 - \frac{u}{v}\right)^{S+X-1} \left(1 - \frac{u}{z}\right)^{1-2S}$$

$$\mathfrak{f}_X(\underline{L}_k; \underline{L}_{k+1}) := \mathbf{1}_{\underline{L}_k \prec \underline{L}_{k+1}} \left(\frac{L_{k+1,k+1} \cdots L_{k+1,1}}{L_{k,k} \cdots L_{k,1}} \right)^{-X} \prod_{i=1}^k \mathcal{A}_{S,X}(L_{k+1,i+1}, L_{k,i}, L_{k+1,i})$$



Interlacing $1 \leq L_{k,k} \leq L_{k-1,k-1} \leq L_{k,k-1} \leq \dots \leq L_{k-1,1} \leq L_{k,1}$

Definition. $\mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N) := \int_{\underline{L}_{N-1} \prec \underline{L}_N} \mathfrak{f}_{X_1, \dots, X_{N-1}}(\underline{L}_{N-1}) \mathfrak{f}_{X_N}(\underline{L}_{N-1}; \underline{L}_N) \frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}}$

Examples. $\mathfrak{f}_{X_1}(L_{1,1}) = L_{1,1}^{-X_1}$

$$\mathfrak{f}_{X,Y}(u, z) = (z/u)^S u^{-X-Y} {}_2F_1 \left(\begin{matrix} S+X, S+Y \\ 2S \end{matrix} \middle| 1 - \frac{z}{u} \right).$$

“Dual” functions.

$$\mathfrak{g}_Y(\tilde{\underline{L}}_k; \underline{L}_k) = \frac{L_{k,1}^{-Y}}{\Gamma(S-Y)} \left(1 - \frac{\tilde{L}_{k,1}}{L_{k,1}}\right)^{S-Y-1} \mathfrak{f}_{-Y}(\ell_{k-1}; \tilde{\underline{L}}_k),$$

$$\mathfrak{g}_{Y_1, \dots, Y_M}(\underline{L}_N) = \begin{cases} \int \mathfrak{g}_{Y_1, \dots, Y_{M-1}}(\tilde{\underline{L}}_N) \mathfrak{g}_{Y_M}(\tilde{\underline{L}}_N; \underline{L}_N) \frac{d\tilde{\underline{L}}_N}{\tilde{\underline{L}}_N} & \text{if } N < M, \\ \int \mathfrak{g}_{Y_1, \dots, Y_{N-1}}(\tilde{\underline{L}}_{N-1}) \mathfrak{g}_{Y_N}(\tilde{\underline{L}}_{N-1}; \underline{L}_N) \frac{d\tilde{\underline{L}}_{N-1}}{\tilde{\underline{L}}_{N-1}} & \text{if } N = M. \end{cases}$$

$$x^{|\lambda|-|\mu|} \prod_{i=1}^k \frac{(-s/x; q)_{\lambda_i-\mu_i} (-sx; q)_{\mu_i-\lambda_{i+1}} (q; q)_{\lambda_i-\lambda_{i+1}}}{(q; q)_{\lambda_i-\mu_i} (q; q)_{\mu_i-\lambda_{i+1}} (s^2; q)_{\lambda_i-\lambda_{i+1}}}.$$

$$\mathfrak{g}_Y(L) = \mathfrak{g}_Y(1; L) = \frac{L^{-Y} (1 - L^{-1})^{S-Y-1}}{\Gamma(S-Y)}.$$

Properties

(1) $\mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N) = a^{-X_1 - \dots - X_N} \mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N), \quad a > 1.$

(2) Cauchy identity, $M \geq N$

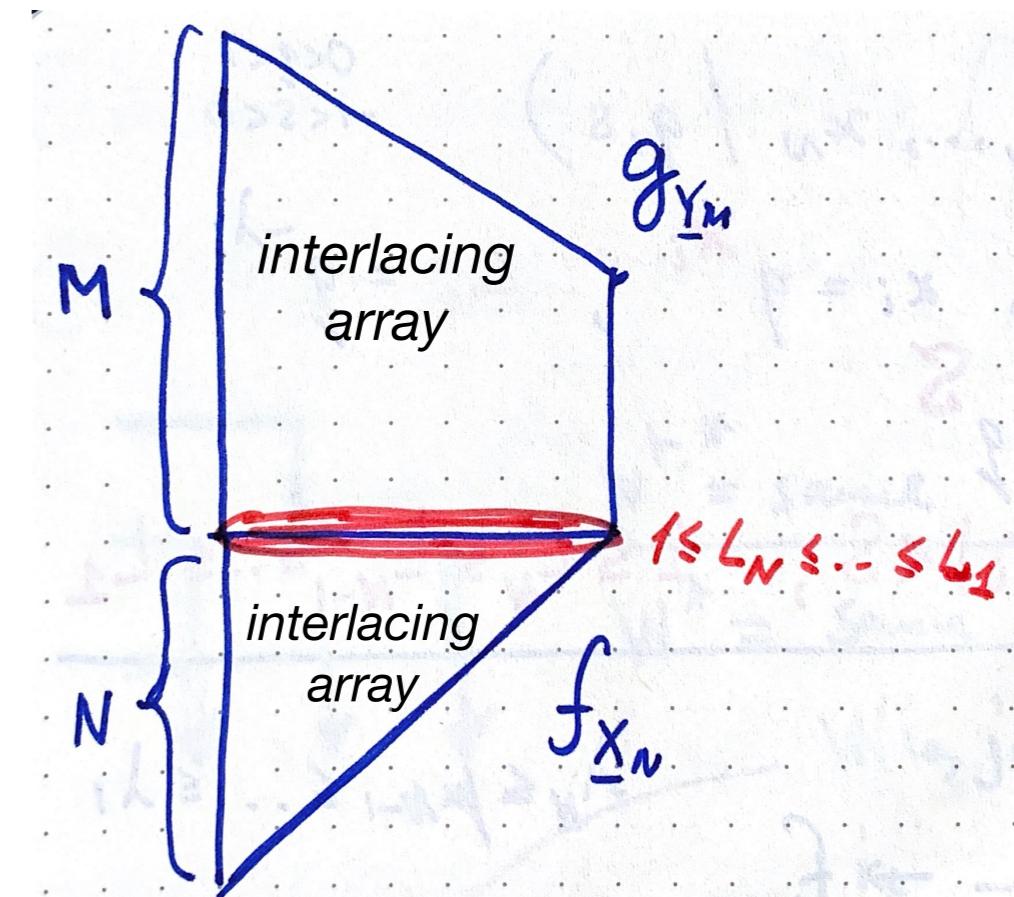
$$\int \mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N) \mathfrak{g}_{Y_1, \dots, Y_M}(\underline{L}_N) \frac{d\underline{L}_N}{\underline{L}_N} = \prod_{j=1}^M \frac{\Gamma(X_1 + Y_j)}{\Gamma(S + X_1)} \left(\prod_{i=2}^N \frac{\Gamma(X_i + Y_j)\Gamma(2S)}{\Gamma(S + X_i)\Gamma(S + Y_j)} \right).$$

(3) Difference eigenoperators (\mathcal{T}_X - shift by 1)

$$\mathcal{D}_1 := \sum_{i=1}^N \prod_{\substack{j=1 \\ j \neq i}}^N \frac{X_i + S}{X_i - X_j} \mathcal{T}_{X_i}, \quad \overline{\mathcal{D}}_1 := \sum_{i=1}^N \prod_{\substack{j=1 \\ j \neq i}}^N \frac{X_i - S}{X_i - X_j} \mathcal{T}_{X_i}^{-1}.$$

$$\mathcal{D}_1 \mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N) = L_{N,N}^{-1} \mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N),$$

$$\overline{\mathcal{D}}_1 \mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N) = L_{N,1} \mathfrak{f}_{X_1, \dots, X_N}(\underline{L}_N).$$



(4) Deformed quantum Toda (scaling limit of Pieri rules, similar to
 [Gerasimov-Lebedev-Oblezin 2011-12])

$$\mathcal{H}_2 := -\frac{1}{2} \sum_{i=1}^N \partial_{u_i}^2 + \sum_{1 \leq i < j \leq N} S^{-2(j-i)} e^{u_j - u_i} (S - \partial_{u_i})(S + \partial_{u_j}).$$

Additive variables u_i

$$L_i = S^{N+1-2i} e^{u_i}.$$

Theorem.

$$\mathcal{H}_2 f_{\underline{X}}^{add}(u_1, \dots, u_N) = -\frac{1}{2} (X_1^2 + \dots + X_N^2) f_{\underline{X}}^{add}(u_1, \dots, u_N).$$

Remark. For $S \rightarrow +\infty$ we get the usual \mathfrak{gl}_N quantum Toda Hamiltonian

$$\mathcal{H}_2^{\text{Toda}} := -\frac{1}{2} \sum_{i=1}^N \partial_{u_i}^2 + \sum_{i=1}^{N-1} e^{u_{i+1} - u_i}.$$

(5) Conjectural “weak” orthogonality with deformed Sklyanin measure

$$\int_{(\mathbb{i}\mathbb{R})^N} f_{\underline{Z}}(\underline{L}_N) f_{-\underline{Z}}(\underline{L}'_N) \mathfrak{M}_S^N(\underline{Z}) dZ_1 \dots dZ_N = \prod_{i=1}^{N-1} \left(1 - \frac{L_{N,i+1}}{L_{N,i}}\right)^{1-2S} \delta_{\underline{L}_N - \underline{L}'_N},$$

$$\mathfrak{M}_S^N(\underline{Z}) = \frac{1}{N!(2\pi\mathbb{i})^N} \prod_{1 \leq i \neq j \leq N} \frac{\Gamma(S + Z_i)\Gamma(S - Z_i)}{\Gamma(2S)\Gamma(Z_i - Z_j)},$$

Spin Whittaker processes

Define a process based on the spin Whittaker functions (like Macdonald processes)

[Borodin-Corwin 2011]

$$\mathfrak{P}_{\mathbf{X}; \mathbf{Y}}(\underline{\underline{L}}_N) = \frac{\mathfrak{f}_{X_1}(\underline{L}_1)\mathfrak{f}_{X_2}(\underline{L}_1; \underline{L}_2) \cdots \mathfrak{f}_{X_N}(\underline{L}_{N-1}; \underline{L}_N)\mathfrak{g}_{\mathbf{Y}}(\underline{L}_N)}{\Pi(\mathbf{X}; \mathbf{Y})}.$$

$$\Pi(\mathbf{X}; \mathbf{Y}) = \prod_{j=1}^T \frac{\Gamma(X_1 + Y_j)}{\Gamma(S + X_1)} \left(\prod_{i=2}^N \frac{\Gamma(X_i + Y_j)\Gamma(2S)}{\Gamma(S + X_i)\Gamma(S + Y_j)} \right).$$

$\mathbf{X} = (X_1, \dots, X_N)$
 $\mathbf{Y} = (Y_1, \dots, Y_T)$

We also have Markov dynamics on spin Whittaker processes which increase the parameter T .

Theorem (Mucciconi-P. 2020).

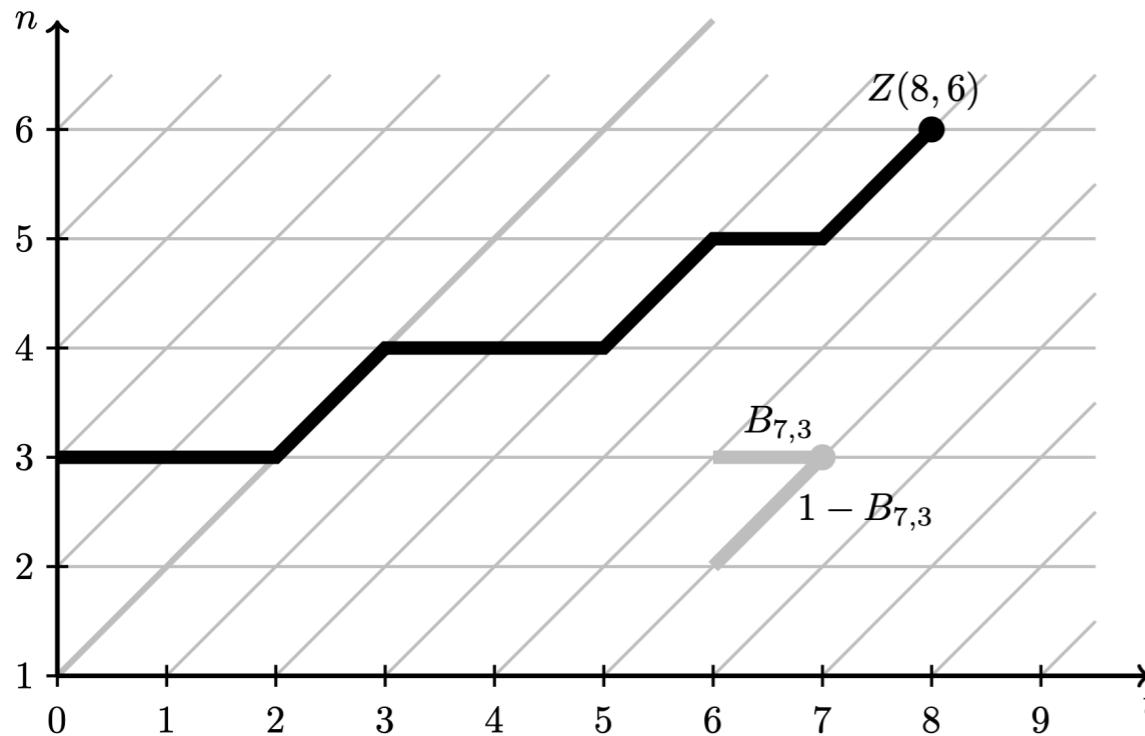
The marginals $L_{k,k}(T)^{-1}$ have the same distribution as the strict-weak beta polymer model $Z(k, T)$ of [Barraquand-Corwin 2015].

The marginals $L_{k,1}(T)^{-1}$ have the same distribution as the “weird” beta polymer model $\tilde{Z}(k, T)$ of [Corwin-Matveev-P. 2018].

Both polymer models arise as $q \rightarrow 1$ limits of the q-Hahn particle systems.

The “weird” model as $S \rightarrow \infty$ also reduces to the more usual **log-gamma polymer model**. The strict-weak beta polymer reduces to the strict-weak log-gamma polymer.

Beta polymers



Strict-weak beta polymer

$$\begin{cases} Z(i, j) = Z(i, j-1)B_{i,j} + Z(i-1, j-1)(1-B_{i,j}) & \text{for } 1 < i \leq j; \\ Z(1, j) = Z(1, j-1)B_{1,j} & \text{for } j > 0; \\ Z(i, 0) = 1 & \text{for } i > 0. \end{cases}$$

$$B_{i,j} \sim \text{Beta}(X_i + Y_j, S - Y_j)$$

$$\mathcal{B}(m, n)[x] = \frac{x^{m-1}(1-x)^{n-1}}{\text{B}(n, m)} \quad \text{for } x \in (0, 1).$$

Another beta polymer-type model - a random recursion

$$\tilde{Z}(i, j) = \begin{cases} 1 & \text{for } j = 0, \\ \tilde{Z}(1, j-1)\tilde{B}_{1,j} & \text{for } i = 1, \\ W_{i,j}^>\tilde{Z}(i, j-1) + (1 - W_{i,j}^>)\tilde{Z}(i-1, j) & \text{if } \tilde{Z}(i, j-1) > \tilde{Z}(i-1, j), \\ (1 - W_{i,j}^<)\tilde{Z}(i, j-1) + W_{i,j}^<\tilde{Z}(i-1, j) & \text{if } \tilde{Z}(i, j-1) < \tilde{Z}(i-1, j), \end{cases} \quad \tilde{B}_{1,j} \sim \text{Beta}^{-1}(X_1 + Y_j, S - Y_j)$$

$$W_{i,j}^> \sim \mathcal{NBB}^{-1} \left(2S - 1, \frac{\tilde{Z}(i-1, j) - \tilde{Z}(i-1, j-1)}{\tilde{Z}(i, j-1) - \tilde{Z}(i-1, j-1)}, X_i + Y_j, S - Y_j \right),$$

$$W_{i,j}^< \sim \mathcal{NBB}^{-1} \left(2S - 1, \frac{\tilde{Z}(i, j-1) - \tilde{Z}(i-1, j-1)}{\tilde{Z}(i-1, j) - \tilde{Z}(i-1, j-1)}, X_i + Y_j, S - X_i \right).$$

Where NBB is a random variable on $[0, 1]$ with density

$$\mathcal{NBB}(r, p, m, n)[x] = \frac{(1-p)^r x^{m-1} (1-x)^{n-1}}{\text{B}(n, m)} {}_2F_1 \left(\begin{matrix} r, n+m \\ n \end{matrix} \middle| p(1-x) \right),$$

Conclusions and further problems

- Symmetric functions continue to be interesting and very useful in integrable probability. We discussed the ones coming from integrable \mathfrak{sl}_2 vertex models.
- New identities relating Izergin-Korepin type determinants and expectations.
- A multitime ASEP formula; **asymptotics unclear**



- New hierarchy stemming from spin q-Whittaker polynomials.
They satisfy the “usual” properties of symmetric functions, including Cauchy identity, some eigenoperators, etc.
- **How to prove the conjectural orthogonalities?**
- **Are there higher order eigenoperators for sqW or sW?**
- There are probabilistic applications (our initial motivation), and we have added an extra parameter to q-Whittaker / Whittaker setup ([COSZ], [BC] 2010+). We also get connections to random polymers, etc.
- **Polymer interpretation of multilayer distributions? Multilayer beta polymers? “gRSK” for beta polymers and spin Whittaker processes?**
- **Representation theory behind spin Whittaker functions?**