Lectures on Random Matrices (Spring 2025)
Lecture 8: Cutting corners and loop equations

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1 Cutting corners: polynomial equations and distribution

1.1 Recap: polynomial equations

Recall the polynomial equation we proved in the last Lecture 7. Fix $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$. Let $H \in \operatorname{Orbit}(\lambda)$ be a random Hermitian matrix defined as

$$H = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n) U^{\dagger},$$

where U is Haar-distributed unitary matrix from $U(n)$. This is the case $\beta = 2$, but the statement holds for the cases $\beta = 1, 4$ with appropriate modifications. Let $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of the $(n-1) \times (n-1)$ corner $H^{(n-1)}$.

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Lemma 1.1. *The distribution of $\mu_1, \ldots, \mu_{n-1}$ is the same as the distribution of the roots of the polynomial equation*

$$\sum_{i=1}^{n} \frac{\xi_i}{z - \lambda_i} = 0, \tag{1.1}$$

where ξ_i are i.i.d. random variables with the distribution χ^2_β.

Recall also that this passage from λ to μ works inductively, and the distribution of the next level eigenvalues $\nu = (\nu_1 \geq \ldots \geq \nu_{n-2})$ is given by the same polynomial equation, but with λ replaced by μ. In this way, we can define a *Markov map* from λ to μ, which is then iterated to construct the full array of eigenvalues of the corners of H.

For $\beta = \infty$, this map is deterministic, and is equivalent to successive differentiating the characteristic polynomial of H.

1.2 Extension to general β

We extend the polynomial equations to general β, by *declaring* (defining) that the general β corners distribution is powered by the passage from $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ to $\mu = (\mu_1 \geq \ldots \geq \mu_{n-1})$, where μ solves (1.1) with ξ_i i.i.d. χ^2_β. In this way, μ interlaces with λ. For $\beta = 1, 2, 4$, this definition reduces to the one with invariant ensembles with fixed eigenvalues λ.

1.3 Distribution of the eigenvalues of the corners

Let μ be obtained from λ by the general β corners operation.

Theorem 1.2. *The density of μ with respect to the Lebesgue measure is given by*

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} \prod_{1 \leq i < j \leq n-1} (\mu_i - \mu_j) \prod_{i=1}^{n-1} \prod_{j=1}^{n} |\mu_i - \lambda_j|^{\beta/2-1} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^{1-\beta}.$$

Proof. Let $\varphi_i = \xi_i / \sum_{j=1}^{n} \xi_j$. It is well-known[1] the joint density of $(\varphi_1, \ldots, \varphi_n)$ is the (symmetric) Dirichlet density

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} w_1^{\beta/2-1} \ldots w_n^{\beta/2-1} dw_1 \ldots dw_{n-1}$$

(note that the density is $(n-1)$-dimensional).

We need to compute the Jacobian of the transformation from φ to μ, if we write

$$\sum_{i=1}^{n} \frac{\varphi_i}{z - \lambda_i} = \frac{\prod_{i=1}^{n-1}(z - \mu_i)}{\prod_{i=1}^{n}(z - \lambda_i)},$$

and compute (as a decomposition into partial fractions):

$$\varphi_a = \frac{\prod_{i=1}^{n-1}(\lambda_a - \mu_i)}{\prod_{i \neq a}(\lambda_a - \lambda_i)}.$$

[1] See Problem H.2.

Therefore,

$$\frac{\partial \varphi_a}{\partial \mu_b} = \frac{\prod_{i=1}^{n-1}(\lambda_a - \mu_i)}{\prod_{i\neq a}(\lambda_a - \lambda_i)} \frac{1}{\mu_b - \lambda_a}.$$

The Jacobian is essentially the determinant of the matrix $1/(\mu_b - \lambda_a)$, which is the Cauchy determinant (Problem H.1). The final density is obtained from the symmetric Dirichlet density, but we plug in $w = \varphi$, and also multiply by the Jacobian. This completes the proof. $\square$

Corollary 1.3 (Joint density of the corners)**.** *The eigenvalues* $\lambda^(k)_j$*,* $1 \le j \le k \le n$*, of a random matrix from* $\mathrm{Orbit}(\lambda)$ *form an interlacing array, with the joint density*

$$\propto \prod_{k=1}^{n} \prod_{1\le i<j\le k} \left(\lambda_j^{(k)} - \lambda_i^{(k)}\right)^{2-\beta} \prod_{a=1}^{k+1}\prod_{b=1}^{k} \left|\lambda_a^{(k+1)} - \lambda_b^{(k)}\right|^{\beta/2-1}.$$

For $\beta = 2$, all factors disappear, and we get the uniform distribution on the interlacing array. This is the *uniform Gibbs property* which is important for other models, including discrete ensembles.

2 Loop equations

Let us write down the *loop equations* for the passage from the eigenvalues λ to the eigenvalues μ. These loop equations are due to [GH24] by a limit from a discrete system (related to Jack symmetric polynomials). Note that despite the name, these are not **equations**, but rather a statement that some expectations are holomorphic. We stick to the random matrix setting, and present a formulation and a proof given by [Gor25].

2.1 Formulation

Theorem 2.1. *We fix* $n = 1, 2, \ldots$ *and* $n+1$ *real numbers* $\lambda_1 \ge \cdots \ge \lambda_{n+1}$*. For* $\beta > 0$*, consider* $n+1$ *i.i.d.* χ^2_β *random variables* ξ_i *and set*

$$w_i = \frac{\xi_i}{\sum_{j=1}^{n+1}\xi_j}, \qquad 1 \le i \le n+1.$$

We define n *random points* $\{\mu_1, \ldots, \mu_n\}$ *as* n *solutions to the equation*

$$\sum_{i=1}^{n+1} \frac{w_i}{z - \lambda_i} = 0. \tag{2.1}$$

Take any polynomial $W(z)$ *and consider the complex function:*

$$f_W(z) = \mathbb{E}\left[\prod_{j=1}^{n} \exp\bigl(W(\mu_j)\bigr) \frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} \left(W'(z) + \sum_{i=1}^{n+1}\frac{\beta/2-1}{z-\lambda_i} + \sum_{j=1}^{n}\frac{1}{z-\mu_j}\right)\right]. \tag{2.2}$$

Then $f_W(z)$ *is an* entire function *of* z*, in the following sense:*

- *For $z \in \mathbb{C} \setminus [\lambda_{n+1}, \lambda_1]$, the expectation in (2.2) defines a holomorphic function of z.*
- *This function has an analytic continuation to $\mathbb{C}$, which has no singularities.*

Remark 2.2. Note that for z in $[\lambda_{n+1}, \lambda_1]$, the integral determining (2.2) might be divergent, and, therefore, analytic continuation is the proper way to define $f_W(z)$, $z \in [\lambda_{n+1}, \lambda_1]$.

Corollary 2.3. *We have*

$$f_0(z) = \frac{(n+1)\beta}{2} - 1.$$

Here f_0 means f_W with $W \equiv 0$.

Proof. This is obtained by sending $z \to \infty$ in (2.2). $\square$

2.2 Proof of Theorem 2.1 for $\beta > 2$

Theorem 2.1 remains valid for $\beta > 0$, but we only prove it for $\beta > 2$ here. We also assume that $\lambda_1 > \ldots > \lambda_n$.

We begin by observing that for $z \in \mathbb{C} \setminus [\lambda_{n+1}, \lambda_1]$, the expectation in (2.2) is well-defined and holomorphic in z. This follows since for such z, the denominators $z - \lambda_i$ and $z - \mu_j$ are bounded away from zero with probability 1. The key challenge is to show that $f_W(z)$ can be analytically continued to an entire function. Potential singularities of $f_W(z)$ are inside the intervals $(\lambda_{i+1}, \lambda_1)$. We will show that these singularities do not actually occur.

Consider a specific interval (λ_2, λ_1). We need to show that $f_W(z)$ has no singularities in this interval. From Theorem 1.2, the probability distribution of $\mu = (\mu_1, \ldots, \mu_n)$ has density proportional to:

$$\prod_{i<j}(\mu_i - \mu_j) \prod_{i=1}^{n} \prod_{j=1}^{n+1} |\mu_i - \lambda_j|^{\beta/2-1}.$$

Let us analyze the function in (2.2). For $z \in (\lambda_2, \lambda_1)$, we need to demonstrate that the expectation

$$\mathbb{E}\left[\prod_{j=1}^{n} \exp\big(W(\mu_j)\big) \frac{\prod_{i=1}^{n+1}(z - \lambda_i)}{\prod_{j=1}^{n}(z - \mu_j)} \left(W'(z) + \sum_{i=1}^{n+1} \frac{\beta/2 - 1}{z - \lambda_i} + \sum_{j=1}^{n} \frac{1}{z - \mu_j}\right)\right]$$

is holomorphic. We need to show that the integral

$$\int\limits_{\mu_i \in [\lambda_{i+1}, \lambda_i]} \prod_{i<j}(\mu_i - \mu_j) \prod\prod(\mu_j - \lambda_i)^{\beta/2-1} \prod e^{W(\mu_j)} \frac{\prod(z - \lambda_i)}{\prod(z - \mu_j)}$$
$$\times \left(W'(z) + \sum \frac{\beta/2 - 1}{z - \lambda_i} + \sum \frac{1}{z - \mu_j}\right) d\mu_1 \ldots d\mu_n$$

is holomorphic for $z \in (\lambda_2, \lambda_1)$. Note that (here we are using the fact that $\beta > 2$)

$$0 = \int\limits_{\mu_i \in [\lambda_{i+1}, \lambda_i]} \frac{\partial}{\partial \mu_1} \left(\underbrace{\prod_{i<j} (\mu_i - \mu_j) \prod \prod (\mu_j - \lambda_i)^{\beta/2-1} \prod e^{W(\mu_j)} \frac{\prod (z - \lambda_i)}{\prod (z - \mu_j)}}_{(*)} \right) d\mu_1 \dots d\mu_n$$

$$= \int\limits_{\mu_i \in [\lambda_{i+1}, \lambda_i]} (*) \cdot \left[\sum_{j=2}^{n} \frac{1}{\mu_1 - \mu_j} + \sum_{i=1}^{n+1} \frac{\beta/2 - 1}{\mu_1 - \lambda_i} + W'(\mu_1) + \frac{1}{z - \mu_1} \right] d\mu_1 \dots d\mu_n$$

Subtracting this expression from our original integral and noting that

$$\left(W'(z) + \sum \frac{\beta/2 - 1}{z - \lambda_i} + \sum \frac{1}{z - \mu_j} \right) - \left(\sum_{j=2}^{n} \frac{1}{\mu_1 - \mu_j} + \sum_{i=1}^{n+1} \frac{\beta/2 - 1}{\mu_1 - \lambda_i} + W'(\mu_1) + \frac{1}{z - \mu_1} \right)$$

has zero at $z = \mu_1$, we conclude that our integral has no singularity at μ_1, and therefore no singularities in the $[\lambda_2, \lambda_1]$ interval. This completes the proof of Theorem 2.1 for $\beta > 2$.

3 Applications of loop equations

H Problems (due 2025-03-25)

H.1 Cauchy determinant

Prove the Cauchy determinant formula:

$$\det \left(\frac{1}{x_i - y_j} \right)_{1 \le i,j \le n} = \frac{\prod_{i<j} (x_i - x_j)(y_i - y_j)}{\prod_{i,j} (x_i - y_j)}.$$

H.2 Dirichlet density

Find or prove the first statement in the proof of Theorem 1.2 about the symmetric Dirichlet density arising from normalizing the ξ_i's to φ_i's.

References

[GH24] V. Gorin and J. Huang, *Dynamical loop equation*, Ann. Probab. **52** (2024), no. 5, 1758–1863. arXiv:2205.15785 [math.PR]. ↑3

[Gor25] V. Gorin, *Private communication*, 2025. ↑3

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