Lectures on Random Matrices (Spring 2025)
Lecture 14: Matching Random Matrices to Random Growth II

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1 Recap

1.1 Main goal

In the previous Lecture 13, we began establishing a remarkable correspondence between two a priori different objects:

- The *spiked Wishart ensemble*: an $n \times n$ Hermitian random-matrix process $\{M(t)\}_{t\geq 0}$ whose entries come from columns of independent Gaussian random vectors of suitably chosen covariance.
- An *inhomogeneous last-passage percolation (LPP)* model: an array $\{W_{i,j}\}$ of exponential random weights on a portion of the two-dimensional lattice, whose last-passage times $L(t,n)$ match the largest eigenvalues of $M(t)$, jointly for all $t \in \mathbb{Z}_{\geq 0}$.

*`Course webpage` • `Live simulations` • `TeX Source` • Updated at 13:51, Tuesday 15$^{\text{th}}$ April, 2025

This equivalence, originally due to [DW08] (following [Def10], [FR06]; see also [Bar01], [Joh00] for earlier results of this kind), can be fully understood by passing to a *discrete* version of LPP with geometric site-weights and then applying the *Robinson–Schensted–Knuth* (RSK) correspondence.

1.2 Spiked Wishart ensembles and the largest eigenvalue process

We defined the *generalized* (or spiked) Wishart matrix $M(t)$ of size $n \times n$ by setting

$$M(t) \;=\; \sum_{m=1}^{t} A^{(m)}\big(A^{(m)}\big)^*$$

where $\{A^{(m)}\}_{m=1}^{\infty}$ are i.i.d. complex Gaussian column vectors of length n, with

$$\mathrm{Var}\big(A_i^{(m)}\big) \;=\; \frac{1}{\pi_i + \hat{\pi}_m}.$$

Here, $\pi = (\pi_1, \dots, \pi_n)$ and $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \dots)$ are positive and nonnegative parameters, respectively. Writing $\lambda_1(t) \geq \cdots \geq \lambda_n(t) \geq 0$ for the eigenvalues of $M(t)$, we then saw:

1. The vectors $\lambda(t) = \big(\lambda_1(t), \dots, \lambda_n(t)\big)$ form a Markov chain in the *Weyl chamber* $\mathbb{W}^n = \{x_1 \geq \cdots \geq x_n \geq 0\}$.
2. There is an *interlacing* property: each update $M(t-1) \mapsto M(t)$ via the rank-one matrix $A^{(t)}\big(A^{(t)}\big)^*$ forces $\lambda(t)$ to interlace with $\lambda(t-1)$:

$$\lambda_1(t) \;\geq\; \lambda_1(t-1) \;\geq\; \lambda_2(t) \;\geq\; \cdots \;\geq\; \lambda_n(t-1) \;\geq\; \lambda_n(t).$$

In Lecture 13, we wrote down the transition kernel from $\lambda(t-1)$ to $\lambda(t)$:

Theorem 1.1 ([DW08])**.** *Fix an integer $n \geq 1$. Let $\pi = (\pi_1, \dots, \pi_n)$ be a strictly positive n-vector, and let $\widehat{\pi} = (\widehat{\pi}_1, \widehat{\pi}_2, \dots)$ be any sequence of nonnegative real parameters. Under the probability measure $P^{\pi,\widehat{\pi}}$, the eigenvalues of the $n \times n$ generalized Wishart matrices $\{M(t)\}_{t\geq 0}$ form a time-inhomogeneous Markov chain $\{\mathrm{sp}(M(t))\}_{t\geq 0}$ in the Weyl chamber*

$$\mathbb{W}^n \;=\; \big\{\, x = (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} : x_1 \geq x_2 \geq \cdots \geq x_n \big\}.$$

More precisely, writing $x = \mathrm{sp}(M(t-1))$ and $y = \mathrm{sp}(M(t))$, the one-step transition law from time $(t-1)$ to t is absolutely continuous on the interior of $\mathbb{W}^n$ and can be factored as

$$Q^{\pi,\widehat{\pi}}_{t-1,t}(x, dy) \;=\; \Big[\prod_{i=1}^{n}\big(\pi_i + \widehat{\pi}_t\big)\Big] \cdot \frac{h_\pi(y)}{h_\pi(x)}\; \exp\Big(-(\widehat{\pi}_t - 1)\sum_{i=1}^{n}(y_i - x_i)\Big) \;\times\; Q^{(0)}(x, dy), \tag{1.1}$$

where

- *$Q^{(0)}\big(x, dy\big)$ is the* standard *(null-spike) Wishart transition kernel, given explicitly by*

$$Q^{(0)}(x, dy) \;=\; \frac{\Delta(y)}{\Delta(x)}\; \exp\Big(-\sum_{i=1}^{n}(y_i - x_i)\Big)\, \mathbf{1}_{\{x \prec y\}}\, dy, \tag{1.2}$$

  with $\Delta(z) = \prod_{1\leq i<j\leq n}(z_i - z_j)$ the Vandermonde determinant.

- *The function h_π is the (continuous) Harish-Chandra orbit integral factor*

$$h_\pi(z) \;=\; \frac{(-1)^{\binom{n}{2}}}{0!1!\cdots(n-1)!}\,\frac{\det\bigl(e^{-\pi_i\, z_j}\bigr)_{i,j=1}^{n}}{\Delta(\pi)\,\Delta(z)}.$$

Note that $h_\pi(0) = 1$.

In particular, the chain starts from $\mathrm{sp}(M(0)) = 0$ (the zero matrix).

1.3 Inhomogeneous last-passage percolation

On the random growth side, we considered an array of site-weights $\{W_{i,j}\}_{i,j\ge 1}$ such that each $W_{i,j}$ is exponentially distributed with rate $\pi_i + \hat{\pi}_j$. For every integer $t \ge 1$, we define $L(t,n)$ to be the maximum total weight of all up-right paths from $(1,1)$ to (t,n):

$$L(t,n) \;=\; \max_{\Gamma\colon (1,1)\to(t,n)} \sum_{(i,j)\in\Gamma} W_{i,j}.$$

One checks that $L(\,\cdot\,,n)$ satisfies a simple additive recursion:

$$L(i,j) \;=\; W_{i,j} \;+\; \max\bigl\{\, L(i-1,j),\; L(i,j-1)\bigr\},$$

The main claim which we show in today's lecture is the equality in distribution:

$$\bigl(L(1,n),\, L(2,n),\, \ldots,\, L(t,n)\bigr) \;\stackrel{d}{=}\; \bigl(\lambda_1(1),\, \lambda_1(2),\, \ldots,\, \lambda_1(t)\bigr). \tag{1.3}$$

1.4 RSK via toggles: definitions and weight preservation

The *Robinson–Schensted–Knuth* correspondence (RSK) was the main new mechanism in Lecture 13. In our setup, we adopt a *toggle-based* viewpoint: we encode arrays by diagonals and successively *toggle* the diagonals to achieve a fully *ordered* array R. The key to how RSK links LPP and random matrices is its *weight preservation* property.

We work with arrays $W = \{W_{ij}\}_{1\le i\le t,\ 1\le j\le n}$ and $R = \{R_{ij}\}_{1\le i\le t,\ 1\le j\le n}$, where W is a nonnegative integer array and R is an ordered array, that is, $R_{i,j} \le R_{i,j+1}$ and $R_{i,j} \le R_{i+1,j}$ for all i,j. Using RSK, we showed in Lecture 13 that there is a bijection which maps W to R.

We also started to prove the following result, which we now complete:

Theorem 1.2 (Weight preservation)**.** *Let $W = \{W_{i,j}\}$ be a nonnegative integer array, and $R = \mathrm{RSK}(W)$. Denote*

$$\mathrm{row}_i \;=\; \sum_{j=1}^{n} W_{i,j}, \quad \mathrm{col}_j \;=\; \sum_{i=1}^{t} W_{i,j}$$

(which are essentially the cdf's of the array W), and for R define the diagonal sums starting at each (i,j) and going diagonally down and to the right:

$$\mathrm{diag}_{i,j} \;=\; \sum_{k=0}^{\min(i,j)-1} R_{i-k,\,j-k}.$$

Then for each $1 \le j \le n$ and $1 \le i \le t$, we have

$$\mathrm{diag}_{t,j} \;=\; \sum_{m=1}^{j} \mathrm{col}_m, \qquad \mathrm{diag}_{i,n} \;=\; \sum_{m=1}^{i} \mathrm{row}_m. \tag{1.4}$$

In particular, the total sum of W over all cells equals the total sum of R over all cells.

Proof (sketch). One inductively builds R by adding the sites (i,j) one at a time. Each toggle modifies exactly one diagonal. After adding a box (i,j), the diagonal-sum identity

$$\mathrm{diag}_{i,j} \;=\; \mathrm{diag}_{i-1,j} + \mathrm{diag}_{i,j-1} \;-\; \mathrm{diag}_{i-1,j-1} \;+\; W_{i,j}$$

holds, expressing that W captures the discrete "mixed second differences" of the diagonal sums in R. Thus, the cdf's of W must coincide with the diagonal sums of R, as desired. □

2 Distributions of last-passage times in geometric LPP

2.1 Matching RSK to last-passage percolation

Recall that we are working with the independent geometric random variables

$$\mathrm{Prob}\,(W_{ij} = k) = (a_i b_j)^k (1 - a_i b_j), \qquad k = 0, 1, \ldots .$$

The parameters $a_1, \ldots, a_t$ and $b_1, \ldots, b_n$ are positive real numbers, and we assume that $a_i b_j < 1$ for all i, j, so that the random variables W_{ij} are well-defined. Let $R = \mathrm{RSK}(W)$.

Lemma 2.1. *The distribution of the top row of the array R, $R_{t,1}, \ldots, R_{t,n}$, is the same as the distribution of the last-passage times $L(t,1), \ldots, L(t,n)$, defined in the same environment $W = \{W_{ij}\}$.*

Note that this statement does not rely on the exact distribution of W, and holds for any fixed or random nonnegative integer array W.

Proof of Lemma 2.1. The values in R update according to the toggle rule. Denote by $R^{(i)}$ the array obtained after toggling the i-th row (and all previous rows) of W. Then, the top row of $R^{(i)}$ updates as

$$R^{(i)}_{i,j} = W_{i,j} + \max\left\{R^{(i-1)}_{i-1,j}, R^{(i)}_{i,j-1}\right\}.$$

By the induction hypothesis, we have

$$R^{(i-1)}_{i-1,j} = L(i-1,j), \qquad R^{(i)}_{i,j-1} = L(i,j-1).$$

This implies that $L(i,j) = R^{(i)}_{i,j}$, and we may proceed by induction on j and then on i. □

Remark 2.2. The correspondence between $R_{t,j}$ and $L(t,j)$ holds only for the top row of the final array $R = R^{(t)}$. For rows below the top row (i.e., for $R_{k,j}$ with $k < t$), there is no such direct correspondence with one-path last-passage times. On the other hand, the whole array R can be defined through multipath last-passage times. This is known as *Greene's theorem* [Sag01] for RSK, and falls outside the scope of this course.

2.2 Distributions in RSK

Fix t, n, and consider the following quantities in a diagonal of the array $R = \mathrm{RSK}(W)$:

$$\lambda_1 \coloneqq R_{t,n}, \lambda_2 \coloneqq R_{t-1,n-1}, \ldots, \lambda_n \coloneqq R_{t-n+1,1}.$$

Clearly, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (we pad diag's by zeroes if necessary), and these are integers. We regard $\lambda = (\lambda_1, \ldots, \lambda_n)$ as an integer partition, or a Young diagram. Denote by $T(\lambda)$ the space of all *semistandard Young tableaux* (SSYT) of shape λ, that is, all collections of numbers r_{ij} which interlace as

$$r_{i,j} \leq r_{i,j+1}, \quad r_{i,j} \leq r_{i+1,j}, \quad i = 1, \ldots, t, \ j = 1, \ldots, n; \qquad r_{t-k+1,n-k+1} = \lambda_k, \quad k = 1, \ldots, n.$$

We are after the distribution of the random Young diagram λ.

Definition 2.3 (Schur polynomial)**.** For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ in n variables is defined as:

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\det(x_i^{n-j})_{i,j=1}^n} = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\prod_{1\leq i<j\leq n}(x_i - x_j)}. \tag{2.1}$$

Alternatively, the Schur polynomial has a combinatorial interpretation as a sum over semistandard Young tableaux:

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T\in T(\lambda)} x_n^{\lambda_1+\ldots+\lambda+n} \left(\frac{x_{n-1}}{x_n}\right)^{r_{t,n-1}+r_{t-1,n-2}+\ldots+r_{t-n+2,1}} \cdots \left(\frac{x_2}{x_3}\right)^{r_{t,2}+r_{t-1,1}} \left(\frac{x_1}{x_2}\right)^{r_{t,1}}, \tag{2.2}$$

where $T(\lambda)$ is the set of all semistandard Young tableaux of shape λ, as defined above.

From (2.1), it is evident that $s_\lambda(x_1, \ldots, x_n)$ is a symmetric polynomial in $x_1, \ldots, x_n$. This is highly non-obvious from the combinatorial definition (2.2). See Problem N.2 for a proof of the equivalence of the two definitions.

The Schur polynomials satisfy the stability property:

$$s_\lambda(x_1, \ldots, x_{n-1}, x_n)\big|_{x_n=0} = \begin{cases} s_\lambda(x_1, \ldots, x_{n-1}) & \text{if } \lambda_n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Theorem 2.4. *Let $\mu = (\mu_1, \ldots, \mu_n)$ be a fixed Young diagram. Then, for $R = \mathrm{RSK}(W)$, where W is the array of independent geometric random variables, we have*

$$\mathrm{Prob}\left(R_{t,n} = \mu_1, \ldots, R_{t-n+1,1} = \mu_n\right) = \prod_{i=1}^{t}\prod_{j=1}^{n}(1 - a_i b_j) \cdot s_\mu(a_1, \ldots, a_t) s_\mu(b_1, \ldots, b_n). \tag{2.4}$$

Note that if $t < n$, then $\mu_{t+1} = \ldots = \mu_n = 0$, as it should be. Note also that the statement of the theorem implies that the expressions in the right-hand side of (2.4) sum to one over all $\mu_1 \geq \ldots \geq \mu_n \geq 0$, which is the celebrated *Cauchy identity* for Schur polynomials. One can alternatively establish the Cauchy identity from the Cauchy-Binet formula, using the determinantal formulas (2.1). See Problem N.4.

Proof of Theorem 2.4. To get the probability (2.4), we need to sum the probability weights of all ordered arrays $R = (R_{ij})_{1\le i\le t,\ 1\le j\le n}$, such that

$$R_{t,j} = \mu_1, \quad R_{t-1,j-1} = \mu_2, \ldots, R_{t-n+1,1} = \mu_n.$$

Denote the set of such arrays by $\mathcal{R}(\mu)$. Each $R \in \mathcal{R}(\mu)$ has a probability weight which we can express (thanks to the RSK bijection) in terms of the original array W, so in terms of the parameters a_i and b_j.

Our first observation is that the probability weight of $R = \mathrm{RSK}(W)$ depends only on its diagonal sums $\mathrm{diag}_{1,n}, \ldots, \mathrm{diag}_{t,n}, \mathrm{diag}_{t,n-1}, \ldots, \mathrm{diag}_{t,1}$ along the right and the top borders. Indeed, knowing these diagonal sums, we know (by the weight-preservation property of RSK, Theorem 1.2) the row and column sums of W. However, the joint distribution of all elements of W has the following form:

$$\begin{aligned}\operatorname{Prob}\left(W_{ij} = k_{ij} \text{ for all } i,j\right) &= \prod_{i=1}^{t}\prod_{j=1}^{n}(1-a_ib_j)\cdot(a_ib_j)^{k_{ij}} \\ &= \left(\prod_{i=1}^{t}\prod_{j=1}^{n}(1-a_ib_j)\right)\cdot\prod_{i=1}^{t}a_i^{k_{i1}+\ldots+k_{in}}\prod_{j=1}^{n}b_j^{k_{1j}+\ldots+k_{tj}}.\end{aligned} \tag{2.5}$$

Thus, we now need to sum expressions (2.5) over all $R \in \mathcal{R}(\mu)$, and we use the fact that the row/column sums in W are differences of diagonal sums in R, to get the Schur polynomials in the combinatorial form (2.2). This completes the proof of Theorem 2.4. $\square$

N Problems (due 2025-04-29)

N.1 Non-Markovianity

Show that the sequence of random variables defined in the exponential LPP model,

$$L(1,n), L(2,n), \ldots, L(t,n),$$

is **not** a Markov chain. By virtue of the equivalence with the spiked Wishart ensemble (1.3), you may alternatively show that the sequence of maximal eigenvalues

$$\lambda_1(1), \lambda_1(2), \ldots, \lambda_1(t)$$

of successive Wishart matrices $M(1), M(2), \ldots, M(t)$ is **not** a Markov chain either.

N.2 Schur polynomials — equivalence of definitions

Show the equivalence of the two definitions of Schur polynomials (2.1) and (2.2).

Hint: Substitute $x_n = 1$ and consider how both formulas expand as linear combinations of Schur polynomials $s_\mu(x_1, \ldots, x_{n-1})$ in $n-1$ variables. This induction (together with the fact that Schur polynomials are a linear basis in the ring of symmetric polynomials in a given fixed number of variables) will show that the two definitions are equivalent.

N.3 Schur polynomials — stability property

Show the stability property of Schur polynomials (2.3).

N.4 Cauchy identity for Schur polynomials

Let $a_1, \ldots, a_t$ and $b_1, \ldots, b_n$ be positive parameters satisfying $a_i b_j < 1$ for all pairs (i, j). Prove the Cauchy identity for Schur polynomials:

$$\sum_{\mu:\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n \ge 0} s_\mu(a_1, \ldots, a_t) s_\mu(b_1, \ldots, b_n) = \prod_{i=1}^{t} \prod_{j=1}^{n} \frac{1}{1 - a_i b_j}.$$

References

[Bar01] Yu. Baryshnikov, *GUEs and queues*, Probab. Theory Relat. Fields **119** (2001), 256–274. ↑2

[Def10] M. Defosseux, *Orbit measures, random matrix theory and interlaced determinantal processes*, Ann. Inst. H. Poincaré Probab. Statist. **46** (2010), no. 1, 209–249. arXiv:0810.1011 [math.PR]. ↑2

[DW08] A. B. Dieker and J. Warren, *On the largest-eigenvalue process for generalized Wishart random matrices*, arXiv preprint (2008). arXiv:0812.1504 [math.PR]. ↑2

[FR06] P. J. Forrester and E. M. Rains, *Jacobians and rank 1 perturbations relating to unitary Hessenberg matrices*, Int. Math. Res. Not. **2006** (2006), Art. ID 48306. arXiv:math/0505552 [math.PR]. ↑2

[Joh00] K. Johansson, *Shape fluctuations and random matrices*, Commun. Math. Phys. **209** (2000), no. 2, 437–476. arXiv:math/9903134 [math.CO]. ↑2

[Sag01] B.E. Sagan, *The symmetric group: representations, combinatorial algorithms, and symmetric functions*, Springer Verlag, 2001. ↑4

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