Lectures on Random Matrices (Spring 2025)
Lecture 6: Double contour integral kernel. Steepest descent and local statistics

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Notes for the lecturer

- GUE det structure
- Formulate Cauchy–Binet and Andreief
- Recall that $\rho_n = P_n$ and it is $(\det[\psi_i(x_j)]_{n\times n})^2$, then reproduce the proofs here.
- Recall the Christoffel–Darboux formula:

$$K_n(x,y) = \frac{e^{-\frac{x^2+y^2}{4}}}{\sqrt{2\pi}h_{n-1}}\,\frac{p_n(x)p_{n-1}(y)-p_{n-1}(x)p_n(y)}{x-y},$$

  here $h_{n-1} = \sqrt{2\pi}(n-1)!$.

1 Recap: Determinantal structure of the GUE

Last time, we proved the following result:

Theorem 1.1. *The GUE correlation functions are given by*

$$\rho_k(x_1,\ldots,x_k) = \det\Big[K_n(x_i,x_j)\Big]_{i,j=1}^{k},$$

with the correlation kernel

$$K_n(x,y) = \sum_{j=0}^{n-1}\psi_j(x)\psi_j(y).$$

Here

$$\psi_j(x) = \frac{1}{\sqrt{h_j}}\,p_j(x)\,e^{-x^2/4},$$

where $p_j(x)$ are the monic Hermite polynomials, and h_j are the normalization constants so that $\psi_j(x)$ are orthonormal in $L^2(\mathbb{R})$.

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For this theorem, we need Cauchy–Binet summation formula and Andreief identity (which is essentially the same as Cauchy–Binet, but when summation is replaced by integration). Having these, we can write

$$\begin{aligned}
\rho_k(x_1,\ldots,x_k) &= \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} p(x_1,\ldots,x_n)\,dx_{k+1}\cdots dx_n \\
&= \frac{1}{(n-k)!\,\widehat{Z}_{n,2}} \sum_{\substack{\sigma,\tau\in S_n \\ \sigma(k+1)=\tau(k+1),\ldots,\sigma(n)=\tau(n)}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{i=1}^{k}\psi_{\sigma(i)-1}(x_i)\psi_{\tau(i)-1}(x_i) \\
&= \mathrm{const}_n \sum_{I\subseteq[n],\,|I|=k}\ \sum_{\sigma',\tau'\in S(I)} \operatorname{sgn}(\sigma')\operatorname{sgn}(\tau')\prod_{i=1}^{k}\psi_{\sigma'(i)-1}(x_i)\psi_{\tau'(i)-1}(x_i) \\
&= \mathrm{const}_n \sum_{I\subseteq[n],\,|I|=k} \det\left[\psi_{i_\alpha}(x_j)\right]_{\alpha,j=1}^{k} \det\left[\psi_{i_\alpha}(x_j)\right]_{\alpha,j=1}^{k},
\end{aligned}$$

where $I=\{i_1,\ldots,i_k\}$ is a subset of $[n]$ of size k, and $S(I)$ is the set of permutations of I. The last sum of products of two determinants is written by the Cauchy–Binet formula as

$$\mathrm{const}_n\cdot\det\left[\sum_{j=0}^{n-1}\psi_j(x_\alpha)\psi_j(x_\beta)\right]_{\alpha,\beta=1}^{k},$$

and finally the constant is equal to 1 by Andreief identity.

2 Double Contour Integral Representation for the GUE Kernel

2.1 One contour integral representation for Hermite polynomials

Recall that the GUE kernel is defined by

$$K_N(x,y)=\sum_{n=0}^{N-1}\psi_n(x)\psi_n(y),$$

with the orthonormal functions

$$\psi_n(x)=\frac{1}{\sqrt{h_n}}\,p_n(x)\,e^{-x^2/4},$$

where the (monic, probabilists') Hermite polynomials are given by

$$p_n(x)=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}\,e^{-x^2/2}. \tag{2.1}$$

Note that the monic Hermite polynomials are uniquely defined by the orthogonality property. We are not proving (2.1) here, it is an exercise.

Lemma 2.1 (Generator function for Hermite polynomials)**.** *We have*

$$\exp\Big(xt - \frac{t^2}{2}\Big) = \sum_{n\geq 0} p_n(x)\frac{t^n}{n!}.$$

The series converges for all t since the left-hand side is an entire function of t.

Proof. Write the generating function as

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = \sum_{n\geq 0} \frac{(-1)^n t^n}{n!}\, e^{x^2/2}\, \frac{d^n}{dx^n}\, e^{-x^2/2}.$$

Since the factor $e^{x^2/2}$ does not depend on n, we can factor it out:

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = e^{x^2/2} \sum_{n\geq 0} \frac{(-t)^n}{n!}\, \frac{d^n}{dx^n}\, e^{-x^2/2}.$$

Now, recall Taylor's theorem: for any analytic function f we have

$$f(x-t) = \sum_{n\geq 0} \frac{(-t)^n}{n!} f^{(n)}(x).$$

Applying this with $f(x) = e^{-x^2/2}$, we deduce that

$$\sum_{n\geq 0} \frac{(-t)^n}{n!}\, \frac{d^n}{dx^n}\, e^{-x^2/2} = e^{-(x-t)^2/2}.$$

Thus, our generating function becomes

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = e^{x^2/2}\, e^{-(x-t)^2/2},$$

as desired. $\square$

By Cauchy's integral formula we can write using Lemma 2.1:

$$p_n(x) = \frac{n!}{2\pi i} \oint_C \frac{\exp\Big(xt - \frac{t^2}{2}\Big)}{t^{n+1}}\, dt, \tag{2.2}$$

where the contour C is a simple closed curve encircling the origin. Indeed, here we use the complex analysis property

$$\frac{1}{2\pi i} \oint_C \frac{1}{z^{k+1}}\, dz = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0, \end{cases}$$

so (2.2) is simply a complex analysis version of the operation of extracting the coefficient of t^n in the Taylor expansion.

Therefore,

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\, p_n(x)\, e^{-x^2/4} = \frac{e^{-x^2/4}}{\sqrt{h_n}}\, \frac{n!}{2\pi i} \oint_C \frac{\exp\Big(xt - \frac{t^2}{2}\Big)}{t^{n+1}}\, dt.$$

2.2 Another contour integral representation for Hermite polynomials

We start with the Fourier transform identity

$$\int_{-\infty}^{\infty} \exp\Big(-\frac{t^2}{2} + i\,t\,x\Big)\,dt = \sqrt{2\pi}\,e^{-x^2/2}.$$

Differentiating both sides n times with respect to x yields

$$\frac{d^n}{dx^n}\Big(e^{-x^2/2}\Big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(i\,t)^n\,e^{-t^2/2+i\,t\,x}\,dt.$$

Recalling the definition

$$p_n(x) = (-1)^n\,e^{x^2/2}\,\frac{d^n}{dx^n}\Big(e^{-x^2/2}\Big),$$

we obtain

$$p_n(x) = \frac{(-1)^n\,e^{x^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(i\,t)^n\,e^{-t^2/2+i\,t\,x}\,dt.$$

Next, perform the change of variable

$$s = i\,t, \quad \text{so that} \quad t = -i\,s, \quad dt = -i\,ds.$$

Under this substitution the factors transform as follows:

$$(i\,t)^n = s^n,$$

and the exponent becomes

$$-\frac{t^2}{2} + i\,t\,x = -\frac{(-i\,s)^2}{2} + i\,(-i\,s)\,x = \frac{s^2}{2} + s\,x.$$

Thus, the integral transforms into

$$\int_{-\infty}^{\infty}(i\,t)^n\,e^{-t^2/2+i\,t\,x}\,dt = -i\int_{-i\infty}^{i\infty} s^n\,e^{s^2/2+s\,x}\,ds.$$

Substituting back we have

$$p_n(x) = \frac{(-1)^n\,e^{x^2/2}}{\sqrt{2\pi}}\,(-i)\int_{-i\infty}^{i\infty} s^n\,e^{s^2/2+s\,x}\,ds.$$

That is,

$$p_n(x) = \frac{i\,(-1)^{n+1}\,e^{x^2/2}}{\sqrt{2\pi}}\int_{-i\infty}^{i\infty} s^n\,e^{s^2/2+s\,x}\,ds.$$

Finally, change the sign of s, and we get:

$$p_n(x) = \frac{i\,e^{x^2/2}}{\sqrt{2\pi}}\int_{-i\infty}^{i\infty} s^n\,e^{s^2/2-s\,x}\,ds.$$

Therefore,

$$\psi_n(x) = \frac{i\,e^{x^2/4}}{\sqrt{2\pi\,h_n}}\int_{-i\infty}^{i\infty} s^n\,e^{s^2/2-s\,x}\,ds.$$

2.3 Normalization of Hermite polynomials

Lemma 2.2. *We have*

$$h_n = \int_{-\infty}^{\infty} p_n(x)^2 \, e^{-x^2/2} \, dx = n!\sqrt{2\pi}.$$

Proof. Multiply the generating function

$$\exp\Big(xt - \frac{t^2}{2}\Big) = \sum_{n \ge 0} p_n(x) \frac{t^n}{n!}$$

with a second copy (with parameter s):

$$\exp\Big(xs - \frac{s^2}{2}\Big) = \sum_{m \ge 0} p_m(x) \frac{s^m}{m!}.$$

Then,

$$\exp\Big(xt - \frac{t^2}{2}\Big) \exp\Big(xs - \frac{s^2}{2}\Big) = \sum_{n,m \ge 0} p_n(x) p_m(x) \frac{t^n s^m}{n! m!}.$$

Integrate both sides against $e^{-x^2/2} \, dx$. Using the orthogonality

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) e^{-x^2/2} dx = h_n \delta_{nm},$$

the right-hand side becomes

$$\sum_{n \ge 0} \frac{h_n}{(n!)^2} (ts)^n.$$

On the left-hand side, we have

$$\int_{-\infty}^{\infty} e^{-x^2/2} \exp\Big(x(t+s) - \frac{t^2+s^2}{2}\Big) dx.$$

Completing the square in x or recalling the standard Gaussian integral yields

$$\sqrt{2\pi} \exp\Big(\frac{(t+s)^2 - (t^2+s^2)}{2}\Big) = \sqrt{2\pi} \exp(ts).$$

Thus, we obtain

$$\sqrt{2\pi} \exp(ts) = \sum_{n \ge 0} \frac{h_n}{(n!)^2} (ts)^n.$$

Expanding the left side as

$$\sqrt{2\pi} \sum_{n \ge 0} \frac{(ts)^n}{n!},$$

and comparing coefficients, we conclude that

$$\frac{h_n}{(n!)^2} = \frac{\sqrt{2\pi}}{n!} \quad \Longrightarrow \quad h_n = n!\sqrt{2\pi}.$$

This completes the proof. $\square$

2.4 Double contour integral representation for the GUE kernel

We can sum up the kernel (essentially, this is another proof of the Christoffel–Darboux formula):

$$\begin{aligned} K_n(x,y) &= \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y) \\ &= \frac{e^{\frac{x^2-y^2}{4}}}{(2\pi)^2} \oint_C dt \int_{-i\infty}^{i\infty} ds \exp\left\{-\frac{t^2}{2}+xt+\frac{s^2}{2}-ys\right\} \underbrace{\sum_{k=0}^{n-1} s^k t^{-k-1}}_{\frac{1-(s/t)^n}{t-s}}. \end{aligned} \tag{2.3}$$

Here we used the two contour integral representations for Hermite polynomials, and the explicit norm (Lemma 2.2). At this point, the t contour is a small circle around 0, and the s contour is a vertical line in the complex plane. Their mutual position can be arbitrary at this point — the s contour goes along the imaginary line. Indeed, the fraction $\frac{1-(s/t)^n}{t-s}$ does not have a singularity at $s=t$ due to the cancellation.

Let us now move the s contour to be to the left of the t contour, as in Figure 1. On the new contours, we have $|s|>|t|$. Now we can add the summands $s^k t^{-k-1}$ for all $k \le -1$ into the sum in (2.3). Indeed, for $|s|>|t|$, the series in k converges, while the summand $s^k t^{-k-1}$ has zero residue at 0 and thus adding the summands does not change the value of the integral.

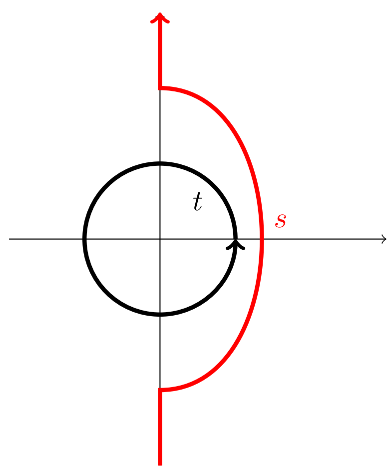

Figure 1: Integration contours for the GUE kernel (2.4).

With this extension of the sum, formula (2.3) becomes

$$K_n(x,y) = \frac{e^{(y^2-x^2)/4}}{(2\pi)^2} \oint_C dt \int_{-i\infty}^{i\infty} ds \frac{\exp\left\{\frac{s^2}{2}-sy-\frac{t^2}{2}+tx\right\}}{s-t}\left(\frac{s}{t}\right)^n. \tag{2.4}$$

Remark 2.3. The s contour passes to the right of the t contour, but it might as well pass to the left of it. Indeed, one can deform the s contour to the left while picking the residue at $s=t$:

$$\operatorname{Res}_{s=t} \frac{\exp\left\{\frac{s^2}{2}-sy-\frac{t^2}{2}+tx\right\}}{s-t}\left(\frac{s}{t}\right)^n = -e^{t(x-y)}.$$

This function is entire in t, and its integral over the t contour is zero. Therefore, there is no difference where the s contour passes with respect to the t contour.

2.5 Extensions

Many other versions of the GUE / unitary invariant ensembles admit determinantal structure:

1. The GUE corners process [JN06]
2. The Dyson Brownian motion (e.g., add a GUE to a diagonal matrix) [NF98]
3. GUE corners plus a fixed matrix [FF14]
4. Corners invariant ensembles with fixed eigenvalues $UDU^{\dagger}$, where D is a fixed diagonal matrix and U is Haar distributed on the unitary group [Met13]

We will discuss the corners process structure in the next Lecture 7.

F Problems (due 2025-03-12)

References

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