Lectures on Random Matrices (Spring 2025)
Lecture 6: Double contour integral kernel. Steepest descent and local statistics

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Contents

Notes for the lecturer

- GUE det structure
- Formulate Cauchy–Binet and Andreief
- Recall that $\rho_n = P_n$ and it is $(\det[\psi_i(x_j)]_{n\times n})^2$, then reproduce the proofs here.

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- Recall the Christoffel–Darboux formula:
$$K_n(x,y) = \frac{e^{-\frac{x^2+y^2}{4}}}{\sqrt{2\pi}h_{n-1}} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y},$$
here $h_{n-1} = \sqrt{2\pi}(n-1)!$.

1 Recap: Determinantal structure of the GUE

Last time, we proved the following result:

Theorem 1.1. *The GUE correlation functions are given by*
$$\rho_k(x_1,\ldots,x_k) = \det\Big[K_n(x_i,x_j)\Big]_{i,j=1}^k,$$
with the correlation kernel
$$K_n(x,y) = \sum_{j=0}^{n-1} \psi_j(x)\psi_j(y).$$
Here
$$\psi_j(x) = \frac{1}{\sqrt{h_j}} p_j(x)\, e^{-x^2/4},$$
where $p_j(x)$ are the monic Hermite polynomials, and h_j are the normalization constants so that $\psi_j(x)$ are orthonormal in $L^2(\mathbb{R})$.

For this theorem, we need Cauchy–Binet summation formula and Andreief identity (which is essentially the same as Cauchy–Binet, but when summation is replaced by integration). Having these, we can write
$$\begin{aligned}
\rho_k(x_1,\ldots,x_k) &= \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} p(x_1,\ldots,x_n)\,dx_{k+1}\cdots dx_n \\
&= \frac{1}{(n-k)!\,\widehat{Z}_{n,2}} \sum_{\substack{\sigma,\tau\in S_n \\ \sigma(k+1)=\tau(k+1),\ldots,\sigma(n)=\tau(n)}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{i=1}^k \psi_{\sigma(i)-1}(x_i)\psi_{\tau(i)-1}(x_i) \\
&= \mathrm{const}_n \sum_{I\subseteq[n],\,|I|=k}\ \sum_{\sigma',\tau'\in S(I)} \operatorname{sgn}(\sigma')\operatorname{sgn}(\tau')\prod_{i=1}^k \psi_{\sigma'(i)-1}(x_i)\psi_{\tau'(i)-1}(x_i) \\
&= \mathrm{const}_n \sum_{I\subseteq[n],\,|I|=k} \det\left[\psi_{i_\alpha}(x_j)\right]_{\alpha,j=1}^k \det\left[\psi_{i_\alpha}(x_j)\right]_{\alpha,j=1}^k,
\end{aligned}$$
where $I = \{i_1,\ldots,i_k\}$ is a subset of $[n]$ of size k, and $S(I)$ is the set of permutations of I. The last sum of products of two determinants is written by the Cauchy–Binet formula as
$$\mathrm{const}_n \cdot \det\left[\sum_{j=0}^{n-1}\psi_j(x_\alpha)\psi_j(x_\beta)\right]_{\alpha,\beta=1}^k,$$
and finally the constant is equal to 1 by Andreief identity.

2 Double Contour Integral Representation for the GUE Kernel

2.1 One contour integral representation for Hermite polynomials

Recall that the GUE kernel is defined by

$$K_N(x,y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y),$$

with the orthonormal functions

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\, p_n(x)\, e^{-x^2/4},$$

where the (monic, probabilists') Hermite polynomials are given by

$$p_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\, e^{-x^2/2}. \tag{2.1}$$

Note that the monic Hermite polynomials are uniquely defined by the orthogonality property. We are not proving (2.1) here, it is an exercise.

Lemma 2.1 (Generator function for Hermite polynomials)**.** *We have*

$$\exp\Bigl(xt - \frac{t^2}{2}\Bigr) = \sum_{n\geq 0} p_n(x)\frac{t^n}{n!}.$$

The series converges for all t since the left-hand side is an entire function of t.

Proof. Write the generating function as

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = \sum_{n\geq 0} \frac{(-1)^n t^n}{n!}\, e^{x^2/2} \frac{d^n}{dx^n}\, e^{-x^2/2}.$$

Since the factor $e^{x^2/2}$ does not depend on n, we can factor it out:

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = e^{x^2/2} \sum_{n\geq 0} \frac{(-t)^n}{n!} \frac{d^n}{dx^n}\, e^{-x^2/2}.$$

Now, recall Taylor's theorem: for any analytic function f we have

$$f(x-t) = \sum_{n\geq 0} \frac{(-t)^n}{n!} f^{(n)}(x).$$

Applying this with $f(x) = e^{-x^2/2}$, we deduce that

$$\sum_{n\geq 0} \frac{(-t)^n}{n!} \frac{d^n}{dx^n}\, e^{-x^2/2} = e^{-(x-t)^2/2}.$$

Thus, our generating function becomes

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = e^{x^2/2}\, e^{-(x-t)^2/2},$$

as desired. $\square$

By Cauchy's integral formula we can write using Lemma 2.1:

$$p_n(x) = \frac{n!}{2\pi i} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{n+1}}\, dt, \tag{2.2}$$

where the contour C is a simple closed curve encircling the origin. Indeed, here we use the complex analysis property

$$\frac{1}{2\pi i} \oint_C \frac{1}{z^{k+1}}\, dz = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0, \end{cases}$$

so (2.2) is simply a complex analysis version of the operation of extracting the coefficient of t^n in the Taylor expansion.

Therefore,

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\, p_n(x)\, e^{-x^2/4} = \frac{e^{-x^2/4}}{\sqrt{h_n}} \frac{n!}{2\pi i} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{n+1}}\, dt.$$

2.2 Another contour integral representation for Hermite polynomials

We start with the Fourier transform identity

$$\int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2} + i\,t\,x\right) dt = \sqrt{2\pi}\, e^{-x^2/2}.$$

Differentiating both sides n times with respect to x yields

$$\frac{d^n}{dx^n}\left(e^{-x^2/2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i\,t)^n\, e^{-t^2/2 + i\,t\,x}\, dt.$$

Recalling the definition

$$p_n(x) = (-1)^n\, e^{x^2/2}\, \frac{d^n}{dx^n}\left(e^{-x^2/2}\right),$$

we obtain

$$p_n(x) = \frac{(-1)^n\, e^{x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i\,t)^n\, e^{-t^2/2 + i\,t\,x}\, dt.$$

Next, perform the change of variable

$$s = i\,t, \quad \text{so that} \quad t = -i\,s, \quad dt = -i\,ds.$$

Under this substitution the factors transform as follows:

$$(i\,t)^n = s^n,$$

and the exponent becomes

$$-\frac{t^2}{2} + i\,t\,x = -\frac{(-i\,s)^2}{2} + i\,(-i\,s)\,x = \frac{s^2}{2} + s\,x.$$

Thus, the integral transforms into

$$\int_{-\infty}^{\infty} (i\,t)^n \, e^{-t^2/2 + i\,t\,x} \, dt = -i \int_{-i\infty}^{i\infty} s^n \, e^{s^2/2 + s\,x} \, ds.$$

Substituting back we have

$$p_n(x) = \frac{(-1)^n \, e^{x^2/2}}{\sqrt{2\pi}} \, (-i) \int_{-i\infty}^{i\infty} s^n \, e^{s^2/2 + s\,x} \, ds.$$

That is,

$$p_n(x) = \frac{i\,(-1)^{n+1} \, e^{x^2/2}}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} s^n \, e^{s^2/2 + s\,x} \, ds.$$

Finally, change the sign of s, and we get:

$$p_n(x) = \frac{i\, e^{x^2/2}}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} s^n \, e^{s^2/2 - s\,x} \, ds.$$

Therefore,

$$\psi_n(x) = \frac{i\, e^{x^2/4}}{\sqrt{2\pi\, h_n}} \int_{-i\infty}^{i\infty} s^n \, e^{s^2/2 - s\,x} \, ds.$$

2.3 Normalization of Hermite polynomials

Lemma 2.2. *We have*

$$h_n = \int_{-\infty}^{\infty} p_n(x)^2 \, e^{-x^2/2} \, dx = n!\sqrt{2\pi}.$$

Proof. Multiply the generating function

$$\exp\Big(xt - \frac{t^2}{2}\Big) = \sum_{n\geq 0} p_n(x) \frac{t^n}{n!}$$

with a second copy (with parameter s):

$$\exp\Big(xs - \frac{s^2}{2}\Big) = \sum_{m\geq 0} p_m(x) \frac{s^m}{m!}.$$

Then,

$$\exp\Big(xt - \frac{t^2}{2}\Big) \exp\Big(xs - \frac{s^2}{2}\Big) = \sum_{n,m\geq 0} p_n(x) p_m(x) \frac{t^n s^m}{n! m!}.$$

Integrate both sides against $e^{-x^2/2}\,dx$. Using the orthogonality

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) e^{-x^2/2} dx = h_n \delta_{nm},$$

the right-hand side becomes

$$\sum_{n\geq 0} \frac{h_n}{(n!)^2}(ts)^n.$$

On the left-hand side, we have

$$\int_{-\infty}^{\infty} e^{-x^2/2} \exp\Big(x(t+s) - \frac{t^2+s^2}{2}\Big) dx.$$

Completing the square in x or recalling the standard Gaussian integral yields

$$\sqrt{2\pi} \exp\Big(\frac{(t+s)^2 - (t^2+s^2)}{2}\Big) = \sqrt{2\pi} \exp(ts).$$

Thus, we obtain

$$\sqrt{2\pi} \exp(ts) = \sum_{n\geq 0} \frac{h_n}{(n!)^2}(ts)^n.$$

Expanding the left side as

$$\sqrt{2\pi} \sum_{n\geq 0} \frac{(ts)^n}{n!},$$

and comparing coefficients, we conclude that

$$\frac{h_n}{(n!)^2} = \frac{\sqrt{2\pi}}{n!} \quad \Longrightarrow \quad h_n = n!\sqrt{2\pi}.$$

This completes the proof. $\square$

2.4 Double contour integral representation for the GUE kernel

We can sum up the kernel (essentially, this is another proof of the Christoffel–Darboux formula):

$$\begin{aligned} K_n(x,y) &= \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y) \\ &= \frac{e^{\frac{x^2-y^2}{4}}}{(2\pi)^2} \oint_C dt \int_{-i\infty}^{i\infty} ds \exp\left\{-\frac{t^2}{2} + xt + \frac{s^2}{2} - ys\right\} \underbrace{\sum_{k=0}^{n-1} s^k t^{-k-1}}_{\frac{1-(s/t)^n}{t-s}}. \end{aligned} \tag{2.3}$$

Here we used the two contour integral representations for Hermite polynomials, and the explicit norm (Lemma 2.2). At this point, the t contour is a small circle around 0, and the s contour is a vertical line in the complex plane. Their mutual position can be arbitrary at this point — the s contour goes along the imaginary line. Indeed, the fraction $\frac{1-(s/t)^n}{t-s}$ does not have a singularity at $s = t$ due to the cancellation.

Let us now move the s contour to be to the left of the t contour, as in Figure 1. On the new contours, we have $|s| > |t|$. Now we can add the summands $s^k t^{-k-1}$ for all $k \leq -1$ into the sum

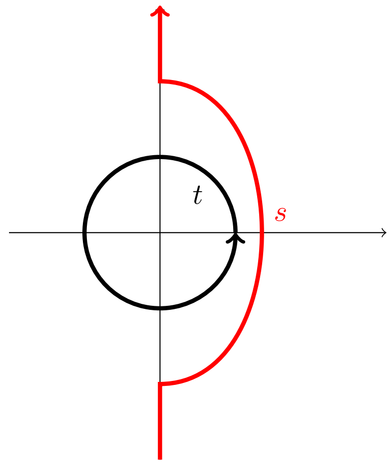

Figure 1: Integration contours for the GUE kernel (2.4).

in (2.3). Indeed, for $|s| > |t|$, the series in k converges, while the summand $s^k t^{-k-1}$ has zero residue at 0 and thus adding the summands does not change the value of the integral.

With this extension of the sum, formula (2.3) becomes

$$K_n(x,y) = \frac{e^{(y^2-x^2)/4}}{(2\pi)^2} \oint_C dt \int_{-i\infty}^{i\infty} ds \, \frac{\exp\left\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\right\}}{s-t} \left(\frac{s}{t}\right)^n. \tag{2.4}$$

Remark 2.3. The s contour passes to the right of the t contour, but it might as well pass to the left of it. Indeed, one can deform the s contour to the left while picking the residue at $s = t$:

$$\operatorname{Res}_{s=t} \frac{\exp\left\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\right\}}{s-t} \left(\frac{s}{t}\right)^n = -e^{t(x-y)}.$$

This function is entire in t, and its integral over the t contour is zero. Therefore, there is no difference where the s contour passes with respect to the t contour.

2.5 Extensions

Many other versions of the GUE / unitary invariant ensembles admit determinantal structure:

1. The GUE corners process [JN06]
2. The Dyson Brownian motion (e.g., add a GUE to a diagonal matrix) [NF98]
3. GUE corners plus a fixed matrix [FF14]
4. Corners invariant ensembles with fixed eigenvalues $UDU^\dagger$, where D is a fixed diagonal matrix and U is Haar distributed on the unitary group [Met13]

We will discuss the corners process structure in the next Lecture 7.

3 Steepest descent — generalities

In many problems arising in random matrix theory—as well as in asymptotic analysis more generally—it is necessary to evaluate integrals of the form

$$I(\lambda) = \int_\gamma e^{\lambda f(z)} \phi(z)\, dz, \tag{3.1}$$

where

- $\lambda > 0$ is a large parameter,
- $f(z)$ and $\phi(z)$ are analytic functions in a neighborhood of the contour $\gamma \subset \mathbb{C}$,
- and the contour γ is chosen in such a way that the integral converges.

The *method of steepest descent* (also known as the *saddle point method*) provides a systematic procedure for obtaining the asymptotic behavior of $I(\lambda)$ as $\lambda \to +\infty$.

3.1 Motivation and setup

The key observation is that for large λ, the exponential term $e^{\lambda f(z)}$ is highly oscillatory or decaying, so that the main contributions to the integral come from small neighborhoods of points where the real part of $f(z)$ is maximal. When f is analytic, these critical points are found by solving

$$f'(z) = 0.$$

Any solution $z_0 \in \mathbb{C}$ of $f'(z) = 0$ is called a *saddle point* of f. (In our context, the term "saddle point" is used because, in the complex plane, the level sets of $\Re(f)$ near z_0 have a saddle-like geometry.)

Once the saddle points are identified, one deforms the contour γ (using Cauchy's theorem) so that it passes through the saddle points along the *steepest descent paths* (which we now define).

3.2 Saddle points and steepest descent paths

Definition 3.1 (Saddle Point)**.** A point $z_0 \in \mathbb{C}$ is called a *saddle point* of $f(z)$ if

$$f'(z_0) = 0.$$

We shall assume in what follows that at every saddle point under consideration the second derivative satisfies

$$f''(z_0) \neq 0.$$

Definition 3.2 (Steepest Descent Path)**.** Let z_0 be a saddle point of $f(z)$. A curve $\Gamma \subset \mathbb{C}$ passing through z_0 is called a *steepest descent path* for $f(z)$ if along Γ the imaginary part of $f(z)$ is constant (i.e.,

$$\Im\big(f(z)\big) = \Im\big(f(z_0)\big)$$

for all $z \in \Gamma$) and the real part $\Re\big(f(z)\big)$ decreases most rapidly away from z_0.

A standard local analysis shows that in a neighborhood of a saddle point z_0, if we write

$$z = z_0 + w,$$

then by the Taylor expansion

$$f(z) = f(z_0) + \frac{1}{2} f''(z_0) w^2 + O(w^3).$$

If we denote

$$f''(z_0) = |f''(z_0)| e^{i\theta_0},$$

then writing $w = r\, e^{i\varphi}$, we obtain

$$f(z) = f(z_0) + \frac{1}{2} |f''(z_0)| r^2 e^{i(2\varphi+\theta_0)} + O(r^3).$$

For the imaginary part to remain constant (that is, for the phase of the quadratic term to be π modulo 2π), one must choose φ so that

$$2\varphi + \theta_0 = \pi \pmod{2\pi}. \tag{3.2}$$

There are two such directions through z_0. Along these directions, one finds that

$$\Re\big(f(z)\big) = \Re\big(f(z_0)\big) - \frac{1}{2} |f''(z_0)| r^2 + O(r^3),$$

so that $\Re(f(z))$ is maximal at $z = z_0$ and decays quadratically as one moves away from z_0 along the steepest descent paths.

3.3 Local asymptotic evaluation near a saddle point

Assume now that the contour γ in (3.1) has been deformed so that it passes through a saddle point z_0 along a steepest descent path. In a small neighborhood of z_0, we write

$$z = z_0 + w,$$

and expand

$$f(z) = f(z_0) + \frac{1}{2} f''(z_0) w^2 + O(w^3), \qquad \phi(z) = \phi(z_0) + O(w).$$

Thus, the local contribution of a neighborhood of z_0 to the integral is

$$I_{z_0}(\lambda) = e^{\lambda f(z_0)} \phi(z_0) \int_{-\infty}^{\infty} e^{\lambda \frac{1}{2} f''(z_0) w^2}\, dw \Big(1 + O\Big(\frac{1}{\lambda}\Big)\Big). \tag{3.3}$$

Here the integration is taken along the steepest descent direction, so that the quadratic term in the exponent is real and negative. (That is, by the choice (3.2), we have $\Re\big(f''(z_0) w^2\big) = -|f''(z_0)| r^2$.) Then the Gaussian integral evaluates to

$$\int_{-\infty}^{\infty} e^{-\lambda \frac{|f''(z_0)|}{2} r^2}\, dr = \sqrt{\frac{2\pi}{\lambda |f''(z_0)|}},$$

up to a phase factor which is determined by the precise orientation of the steepest descent contour. Hence, we arrive at the following fundamental result.

Theorem 3.3 (Local Asymptotics via Steepest Descent)**.** *Let z_0 be a saddle point of $f(z)$ with $f'(z_0) = 0$ and $f''(z_0) \neq 0$, and assume that $\phi(z)$ is analytic in a neighborhood of z_0. Then, as $\lambda \to +\infty$, the contribution of a small neighborhood of z_0 to the integral (3.1) is given by*

$$I_{z_0}(\lambda) \sim e^{\lambda f(z_0)} \phi(z_0) \sqrt{\frac{2\pi}{\lambda f''(z_0)}}, \tag{3.4}$$

where the square root is defined by choosing the branch so that the integration contour coincides with the path of steepest descent.

Proof. As outlined above, write $z = z_0 + w$, expand

$$f(z) = f(z_0) + \frac{1}{2} f''(z_0) w^2 + O(w^3), \qquad \phi(z) = \phi(z_0) + O(w).$$

By restricting the integration to a small neighborhood of z_0 (which contributes the main term as $\lambda \to \infty$), one obtains

$$I_{z_0}(\lambda) = e^{\lambda f(z_0)} \phi(z_0) \int_{-\delta}^{\delta} e^{\lambda \frac{1}{2} f''(z_0) w^2}\, dw + O\Big(e^{\lambda f(z_0)} \lambda^{-3/2}\Big).$$

By the steepest descent property the contour can be locally rotated so that the quadratic form in the exponent is real and negative. After this change of variable, the standard Gaussian integral yields the stated result. $\square$

3.4 Global asymptotic evaluation and deformation of the contour

In many applications, the original contour γ does not pass through any saddle points. However, because f is analytic, one may continuously deform γ (without crossing any singularities) to a new contour $\widetilde{\gamma}$ that does pass through one or several saddle points along the corresponding steepest descent paths. If there are several saddle points, one must check that the contribution of each (and any possible end-point contributions) is asymptotically negligible except for the saddle point(s) that maximize $\Re(f(z))$. Then one sums the contributions from each relevant saddle point.

3.5 Error estimates

Under appropriate assumptions (typically, if f and ϕ are analytic on a neighborhood that can be reached by the deformed contour and if the contributions away from the saddle points are exponentially small), one may show that the error in approximating the full integral by the sum of the local contributions is itself exponentially small relative to the leading order terms. In many cases, the next-order corrections can be computed by carrying the expansion in (3.3) to higher order in w. (See, e.g., [Olv74] for a systematic treatment.)

F Problems (due 2025-03-12)

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