

Positivity everywhere

Lecture 2

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Themes (how we'll choose to interpret them)

Universality: same prob. structures arising in different contexts.

Integrability:

- Explicit formulas
- Algebraic/combinatorial underpinning

Positivity:
(non-negativity) When can a set of algebraic/combinatorial objects underpin probabilistic objects?

formal power series

Will enable us to discuss :

- symmetric polynomials / functions
- generating functions
- Continued fractions

Def

$R[[x]]$ denotes the algebra of **formal power series** over ring R

$\sum_{k \in \mathbb{N}_0} a_k x^k$ with $a_0, a_1, \dots \in R$ equipped with addition:

$$\sum_k a_k x^k + \sum_k b_k x^k = \sum_k (a_k + b_k) x^k$$

and multiplication $\left(\sum_k a_k x^k \right) \left(\sum_k b_k x^k \right) = \sum_k c_k x^k$

where $c_k = \sum_{l=0}^k a_l b_{k-l}$.

E.g. / Non-example in $\mathbb{C}[[x]]$?

$$1 + 2x + \sqrt{2}x^7,$$

$$\sum_{k=0}^{\infty} x^k,$$

$$\sum_{k=0}^{\infty} k! x^k,$$

$$\sum_{k=0}^{\infty} (1+x+x^2+\dots+x^k)$$

Equivalently: $\mathbb{R}[[x]]$ contains all functions $c: \mathbb{N}_0 \rightarrow \mathbb{R}$
 interpreted as $\sum_{k \in \mathbb{N}_0} c(k) x^k$
 with addition and multiplication as before.

Multivariate version (in countably many indeterminates):

$\mathbb{R}[[x_1, x_2, \dots]]$ contains all functions $c: \mathbb{N}_0^\infty \rightarrow \mathbb{R}$ s.t.
 $c(k_1, k_2, \dots) \neq 0 \Rightarrow k_n = 0 \text{ for all } n \text{ large enough}$, interpreted as

$$\sum_{\substack{(k_1, k_2, \dots) \\ \in \mathbb{N}_0^\infty}} c(k_1, k_2, \dots) x_1^{k_1} x_2^{k_2} \dots$$



finite degree
monomials

with addition and multiplication defined analogously to multivariate polynomials.

E.g. / Non-example?

$$\sum_{k=1}^{\infty} x_1^k, \quad \sum_{k=1}^{\infty} x_k, \quad \sum_{k=1}^{\infty} (x_k + x_{k+1})^k$$

$(1, 0, 0, 0, \dots)$

$\mathbf{1}_{\{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, 0, \dots), \dots\}}$

$$\sum_n \sum_{(k_1, \dots, k_n)} x_{k_1} \cdots x_{k_n}, \quad (1+x_1)(1+x_2)(1+x_3) \cdots$$

Addition, e.g.

$$\sum_{k=1}^{\infty} x_1^k + \sum_{k=1}^{\infty} k! x_k = \sum_{k=1}^{\infty} (x_1^k + k! x_k)$$

Multiplication, e.g.

$$\left(\sum_{k=1}^{\infty} x_1^k \right) \left(\sum_{k=1}^{\infty} k! x_k \right) = \sum_{m=1}^{\infty} m! x_1^k x_m$$

Differentiation, e.g.

$$\frac{\partial}{\partial x_1} \sum_{k=1}^{\infty} x_1^k x_2 = \sum_{k=1}^{\infty} k x_1^{k-1} x_2$$

Sometimes inverses, e.g.

$$\left(\sum_{k=0}^{\infty} x^k \right) (1-x) = 1$$

More on inverses (will be needed later, in univariate setting)

Fact (exercise): $K[[x]]$ where K is a field

For any fps $\sum_{k=0}^{\infty} a_k x^k$ with $a_0 \neq 0$, $\exists!$ fps $\sum_{k=0}^{\infty} b_k x^k$

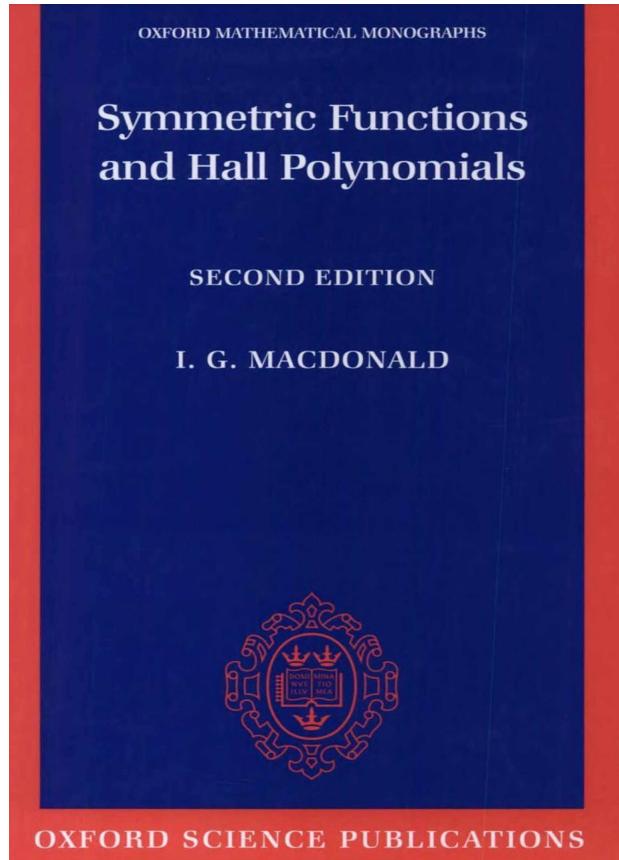
s.t. $\left(\sum_{k=0}^{\infty} a_k x^k \right) \left(\sum_{k=0}^{\infty} b_k x^k \right) = 1.$

In fact,

$$b_0 = a_0^{-1}, \quad b_k = \frac{(-1)^k}{a_0^{k+1}} \det$$

$$\begin{bmatrix} a_1 & a_0 \\ a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ \vdots & & & \\ a_k & a_{k-1} & a_{k-2} & a_{k-3} & \dots & a_1 & a_0 \end{bmatrix}$$

Positivity everywhere (continued)



Symmetric Functions and Hall Polynomials

Second Edition

I. G. MACDONALD

*Queen Mary and Westfield College
University of London*

and their many occurrences/uses in integrable probability

Degree of a monomial: $x_{k_1}^{d_1} x_{k_2}^{d_2} \cdots x_{k_n}^{d_n} \mapsto d_1 + d_2 + \cdots + d_n$

Projection maps: $\pi_n: R[[x_1, x_2, \dots]] \rightarrow R[[x_1, x_2, \dots]]$

$$c(k_1, k_2, \dots) \xrightarrow{\pi_n} c(k_1, k_2, \dots) 1_{\{0 = k_{n+1} = k_{n+2} = \dots\}}$$

$p_d: R[[x_1, x_2, \dots]] \rightarrow R[[x_1, x_2, \dots]]$

$$c(k_1, k_2, \dots) \xrightarrow{p_d} c(k_1, k_2, \dots) 1_{\{k_1 + k_2 + \dots = d\}}$$

Together allow us to recover polynomials in x_1, x_2, \dots, x_n .

An element $c \in R[[x_1, x_2, \dots]]$ is homogeneous of degree d

if $p_d(c) = c$, i.e. if c is of the form

$$\sum_{\substack{k_1, k_2, \dots \in \mathbb{N}_0 \\ k_1 + k_2 + \dots = d}} c(k_1, k_2, \dots) \underbrace{x_1^{k_1} x_2^{k_2} \cdots}_{\text{total degree } = d}$$

S_n : group of bijections $[n] \rightarrow [n]$ $([n] := \{1, 2, \dots, n\})$

Write $\sigma \in S_n$ in "one-line notation": $\sigma(1) \sigma(2) \dots \sigma(n)$

S_∞ : group of bijections $\mathbb{N} \rightarrow \mathbb{N}$ of the form

$\sigma(1) \sigma(2) \dots \sigma(n) (n+1) (n+2) (n+3) \dots$ ($n \in \mathbb{N}$)

Action of S_∞ on $R[[x_1, x_2, \dots]]$:

For $\sigma \in S_\infty$, $f \in R[[x_1, x_2, \dots]]$:

$$\sigma f = \sigma \sum_{k_1, k_2, \dots \in \mathbb{N}_0} f(k_1, k_2, \dots) x_1^{k_1} x_2^{k_2} \dots$$

$$= \sum_{k_1, k_2, \dots \in \mathbb{N}_0} f(k_1, k_2, \dots) x_{\sigma(1)}^{k_1} x_{\sigma(2)}^{k_2} \dots$$

When $\sigma f = f$ & $\sigma \in S_\infty$, f is a symmetric function.

Check (exercise) : $\sigma, \pi \in S_\infty$, $\alpha \in \mathbb{R}$

$$\sigma(f+g) = \sigma f + \sigma g$$

$$\sigma(\alpha f) = \alpha \sigma f$$

$$\sigma(f \cdot g) = (\sigma f) \cdot (\sigma g)$$

$$(\sigma\pi)f = \sigma(\pi f)$$

let Λ = symmetric elements of $\mathbb{R}[x_1, x_2, \dots]$

Check (exercise) : Λ is a subalgebra of $\mathbb{R}[x_1, x_2, \dots]$

From now on, work with symmetric functions over \mathbb{C} .

Special families in Λ :

Elementary sym func : $e_n = \sum_{k_1 < k_2 < \dots < k_n} x_{k_1} x_{k_2} \dots x_{k_n}$

(Complete)

Homogeneous sym func : $h_n = \sum_{k_1 \leq k_2 \leq \dots \leq k_n} x_{k_1} x_{k_2} \dots x_{k_n}$

Power sum sym func : $p_n = \sum_k x_k^n$

Fact (exercise / see Macdonald) :

Any one of these families generates Λ

$$\begin{aligned} \text{i.e. } \Lambda &= \text{alg}\{e_n : n \in \mathbb{N}\} = \text{alg}\{h_n : n \in \mathbb{N}\} \\ &= \text{alg}\{p_n : n \in \mathbb{N}\} \end{aligned}$$

Generating functions:

$$E_n(z) = \sum_{n \geq 0} e_n z^n = \prod_{i \geq 1} \frac{1}{1 - x_i z}$$

Swap

$$H_n(z) = \sum_{n \geq 0} h_n z^n = \prod_{i \geq 1} (1 + x_i z)$$

$$P_n(z) = \sum_{n \geq 0} p_{n+1} z^n = \frac{d}{dz} \log \prod_{i \geq 1} \frac{1}{1 - x_i z}$$

(Exercise)

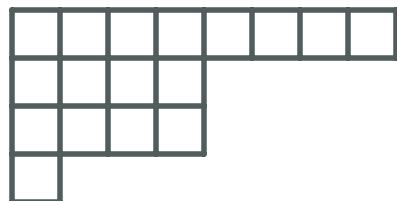
Toward another interesting basis:

A Young diagram (Ferrers tableau) $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ is a non-increasing sequence in \mathbb{N}_0^∞ with finitely many non-zero terms.

It has size $|\lambda| = \sum_{k=1}^{\infty} \lambda_k$ and

length $l(\lambda) = \max \{ k \mid \lambda_k > 0 \}$.

E.g.



$$\lambda_1 = 5$$

$$\lambda_2 = 4$$

$$\lambda_3 = 4$$

$$\lambda_4 = 1$$

$$\lambda_5 = 0$$

⋮

$$|\lambda| = 17$$

$$l(\lambda) = 4$$

Denote by \mathbb{Y}_n the set of diag's with $|\lambda| = n$ and \mathbb{Y} the set of all Young diag's.

Def The Schur polynomial in n variables parametrized by a Young diagram $\lambda \neq \emptyset$ with $l(\lambda) \leq n$ is:

$$S_\lambda(x_1, x_2, \dots, x_n) = \frac{\det \begin{bmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-1} & \dots & x_1^{\lambda_n+n-1} \\ x_2^{\lambda_1+n-2} & x_2^{\lambda_2+n-2} & \dots & x_2^{\lambda_n+n-2} \\ \vdots & & & \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \dots & x_n^{\lambda_n} \end{bmatrix}}{\det \begin{bmatrix} x_1^{n-1} & x_1^{n-1} & \dots & x_1^{n-1} \\ x_2^{n-2} & x_2^{n-2} & \dots & x_2^{n-2} \\ \vdots & & & \\ 1 & 1 & \dots & 1 \end{bmatrix}}$$

} skew-symmetrization of $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$

$$= \frac{\det [x_i^{\lambda_j + n - i}]}{\prod_{i < j} (x_i - x_j)}_{i,j=1,\dots,n}$$

This is a polynomial. (Why?) It is symmetric. (Why?)

Equivalently, the Jacobi-Trudi identities : for $\lambda \in \mathbb{Y}_n$,

$$S_\lambda(x_1, \dots, x_n) = \det \begin{bmatrix} h_{\lambda_1}(x_1, \dots, x_n) & h_{\lambda_1+1}(x_1, \dots, x_n) & \dots & h_{\lambda_1-1+n}(x_1, \dots, x_n) \\ h_{\lambda_2-1}(x_1, \dots, x_n) & h_{\lambda_2}(x_1, \dots, x_n) & \dots & h_{\lambda_2-2+n}(x_1, \dots, x_n) \\ \vdots & & & \\ h_{\lambda_{n-1}+1}(x_1, \dots, x_n) & h_{\lambda_{n-1}+2}(x_1, \dots, x_n) & \dots & h_{\lambda_n}(x_1, \dots, x_n) \end{bmatrix}$$

(Exercise. Also available in the basis of e_n , p_n . See Macdonald.)

To each $\lambda \in \mathbb{Y}$, we will associate the fgs

$$S_\lambda = S_\lambda(x_1, x_2, \dots) \quad (x_1, x_2, \dots)$$

$$= \det \begin{bmatrix} h_{\lambda_1}(x_1, x_2, \dots) & h_{\lambda_1+1}(x_1, x_2, \dots) & \dots & h_{\lambda_1-1+n}(x_1, x_2, \dots) \\ h_{\lambda_2-1}(x_1, x_2, \dots) & h_{\lambda_2}(x_1, x_2, \dots) & \dots & h_{\lambda_2-2+n}(x_1, x_2, \dots) \\ \vdots & & & \\ h_{\lambda_{n-1}+1}(x_1, x_2, \dots) & h_{\lambda_{n-1}+2}(x_1, x_2, \dots) & \dots & h_{\lambda_n}(x_1, x_2, \dots) \end{bmatrix}$$

Frequently asked positivity questions:

(1) When does a symmetric function expand positively in a given basis?

E.g. Thm (Kostka) $s_\lambda(x_1, x_2, \dots, x_n) = \sum_{\mu \in \mathbb{Y}_n} \underbrace{k_{\lambda, \mu}}_{\text{Combinatorial}} \underbrace{m_\mu(x_1, x_2, \dots, x_n)}_{\text{monomials}}$

$$\geq 0 \quad (\text{Exercise})$$

E.g. When does a symmetric function expand positively in the Schur basis?

(Consequences for algebraic geometry)

(2) $s_\lambda(x_1, \dots, x_n)$ "evaluated" on $\mathbb{R}_{\geq 0}$ is ≥ 0 . (Substitution $x_i \mapsto a_i, i=1, \dots, n$)

More generally:

Def A **specialization** is an algebra homomorphism $\rho: \Lambda \rightarrow \mathbb{C}$, $f \mapsto f(\rho)$

i.e. $(f+g)(\rho) = f(\rho) + g(\rho)$, $(\alpha f)(\rho) = \alpha f(\rho)$

$$(fg)(\rho) = f(\rho)g(\rho) \quad \forall \alpha \in \mathbb{C}, f, g \in \Lambda$$

$$\rho: \Lambda \rightarrow \mathbb{C}, f \mapsto f(\rho)$$

Notice: ρ is defined by its values on elements of an alg. basis of Λ .
(Can view ρ as any element of \mathbb{C}^∞ .)

(Exercise: is any substitution $x_i \mapsto a_i \in \mathbb{C}$ an evaluation?)

Def ρ is **Schur-positive** if $s_\lambda(\rho) \geq 0 \quad \forall \lambda \in \Lambda$

Allows one to define a measure:

Def (Okounkov) let $\rho_1, \rho_2: \Lambda \rightarrow \mathbb{C}$ be two Schur-positive

specializations s.t. $\sum_{\lambda \in \Lambda} s_\lambda(\rho_1) s_\lambda(\rho_2) < \infty$.

The **Schur measure** on Λ is the probability measure

$$P_{\rho_1, \rho_2}(\lambda) \sim s_\lambda(\rho_1) s_\lambda(\rho_2)$$

See Borodin-Gorin lecture notes for examples, also properties / uses of Schur measures.

Also for the definition of the **Schur process**, a prob measure on $\lambda^{(1)}, \mu^{(1)}, \lambda^{(2)}, \mu^{(2)}, \dots, \lambda^{(N)}, \mu^{(N)} \in \Lambda$ parametrized by $2N$ Schur-positive spec's $\rho_0^+, \dots, \rho_{N-1}^+, \rho_1^-, \dots, \rho_N^-$

→ general framework giving models that can be analyzed

e.g. measure on plane partitions $\propto \frac{\text{vol}(\pi)}{g}$, long range TASEP

When is a spec. $\rho: \Lambda \rightarrow \mathbb{C}$ Schur positive?

Recall: $S_\lambda = \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \dots & h_{\lambda_1 + l(\lambda)-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \dots & h_{\lambda_2 + l(\lambda)-2} \\ \vdots & \vdots & & \vdots \\ h_{\lambda_{l(\lambda)} - l(\lambda) + 1} & h_{\lambda_{l(\lambda)} - l(\lambda) + 2} & \dots & h_{\lambda_{l(\lambda)}} \end{bmatrix}$

ρ is Schur positive $\iff \rho(h_1), \rho(h_2), \dots$ is Toeplitz totally positive

Corollary Schur-positive specializations are parametrized by

$$\alpha \geq 0, \beta_1 \geq \beta_2 \geq \dots \geq 0, \gamma_1 \geq \gamma_2 \geq \dots \geq 0 \text{ s.t. } \sum_i \beta_i + \sum_i \gamma_i < \infty$$

and given by

$$\sum_{n \geq 0} P_{\alpha, \beta, \gamma}(h_n) z^n = e^{z\alpha} \prod_{i \geq 0} \frac{(1 + \beta_i z)}{(1 - \gamma_i z)}.$$

(Analogue for Macdonald functions conj' kerov '92, proof Matveev '17)

Part II - A moment sequencer's toolkit

Recall :

Algebraic / combinatorial object \leadsto Representing matrix \sim Positivity of the matrix

Prototypical example in probability:

combinatorial seq $(a_n)_{n \geq 0}$ ($a_0 = 1$)

When is there some prob. measure μ on \mathbb{R}
s.t. $\forall n \in \mathbb{N}$

$$a_n = \int_{\mathbb{R}} x^n d\mu(x) ?$$

Hamburger: positive semi-definiteness of the Hankel matrices $[a_{i+j}]_{i,j \geq 0}$

(1) Continued fractions

Ex. 1 Let $\varphi_0 = 1, \varphi_1 = 2, \varphi_n = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}$ } depth:
n levels

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \dots = 1+0, 1+1, 1+\frac{1}{2}, 1+\frac{2}{3}, 1+\frac{3}{5}, \dots$$

Exercise: Show that

- $\varphi_n = 1 + \frac{f_n}{f_{n+1}}$ where $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$

$$0, 1, 1, 2, 3, \underline{5}, 8, \dots$$

- $\varphi_n \rightarrow \varphi = \frac{1+\sqrt{5}}{2}$

Ex. 2 As formal power series

$$\frac{1}{1 - \frac{2}{1 - \frac{2z}{1 - \frac{3z}{\dots}}}} = \sum_{n \geq 0} \underbrace{(2n-1)(2n-3)\dots 3 \cdot 1}_{=: (2n-1)!!} z^n \quad (\text{Euler})$$

Def

A Motzkin path of length n is a walk in $\mathbb{N}_0 \times \mathbb{N}_0$

that starts at $(0,0)$, ends at $(n,0)$, consists of :

- level steps



- upsteps



- downsteps



and remains positive (i.e. $j \geq 0$ at each step).

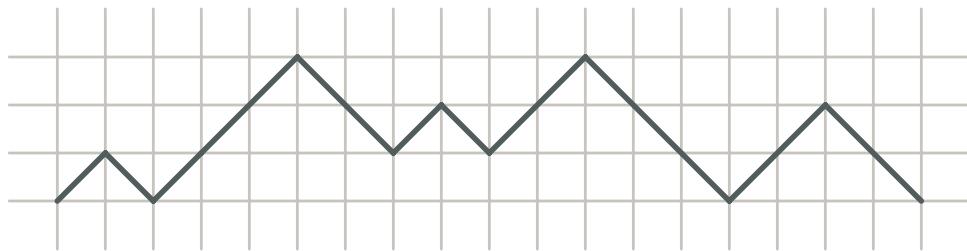


Exercise : Let $m_n = \#$ Motzkin paths with n steps

Then $\sum_{n \geq 0} m_n z^n = \frac{1 - 2\sqrt{1 - 2z - 3z^2}}{2z^2}$

Def

A **Dyck path** is a Motzkin path consisting only of upsteps and downsteps.



Exercise: # Dyck paths with n steps = $C_n := \frac{1}{n+1} \binom{2n}{n}$

Recall: $\mathbb{R}[[x]]$ denotes the set of formal power series

$\sum_{k \geq 0} a_k x^k$ with $a_0, a_1, \dots \in \mathbb{R}$ equipped with addition:

$$\sum_k a_k x^k + \sum_n b_n x^n = \sum_k (a_k + b_k) x^k$$

and multiplication $\left(\sum_k a_k x^k \right) \left(\sum_l b_l x^l \right) = \sum_m c_m x^m$

where $c_m = \sum_{l=0}^k a_l b_{m-l}$.

The topology on $\mathbb{R}[[x]]$ is the product topology,
with the discrete topology on \mathbb{R} .

i.e.

$$\sum_k a_k^{(n)} x^k \xrightarrow{n \rightarrow \infty} \sum_k a_k x^k \quad \text{iff} \quad a_k^{(n)} = a_k \text{ for } n \text{ large enough}$$

From now on, take

$$R = \mathbb{C} \quad \text{or}$$

R = quotients of multivariate polynomials

(when working with "combinatorial statistics")

Recall: When $a_0 \neq 0$, $\sum_k a_k x^k$ has a multiplicative inverse.

$$\text{E.g. } \frac{1}{1-x} = \sum_{k \geq 0} x^k$$

Thom (Flajolet '80)

The sequence

$$\left(\frac{1}{1-\alpha_0 z - \frac{\beta_1 z^2}{1-\alpha_1 z - \frac{\beta_2 z^2}{\dots \frac{1}{1-\alpha_{n-1} z - \beta_n z^2}}}} \right)_{n \geq 0}$$

converges as formal power series. Its limit is denoted:

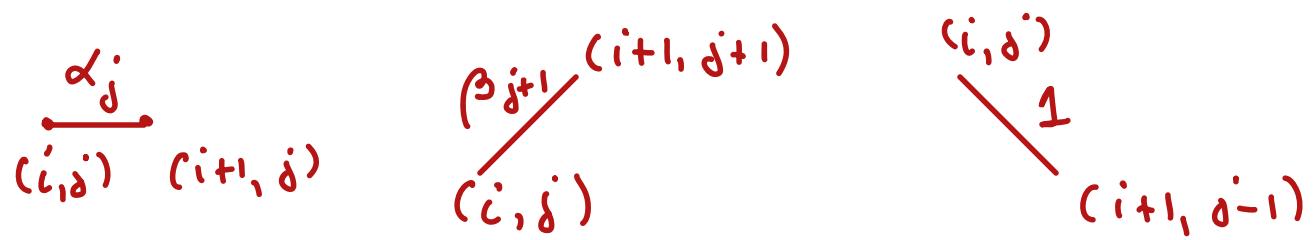
$$\frac{1}{1-\alpha_0 z - \frac{\beta_1 z^2}{1-\alpha_1 z - \frac{\beta_2 z^2}{\dots}}}$$

(Continued ...)

Thm (Flajolet '80) ... Moreover

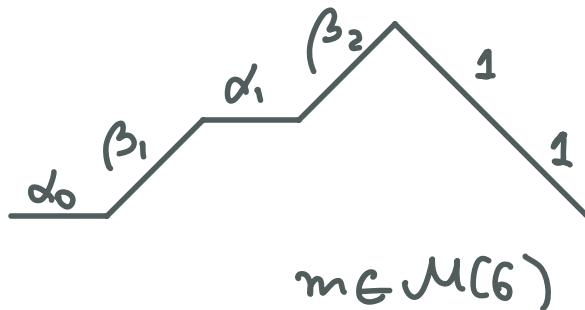
$$\frac{1}{1 - \alpha_0 z - \frac{\beta_1 z^2}{1 - \alpha_1 z - \frac{\beta_2 z^2}{\dots}}} = \sum_{n \geq 0} \sum_{m \in M(n)} \text{wt}(m) z^n$$

where $M(n)$ is the set of Motzkin paths with n steps labeled as :



with $\text{wt}(m) = \text{product of the labels.}$

Example :



$$\text{wt}(m) = \alpha_0 \beta_1 \alpha_1 \beta_2$$

Proof by example: Start expanding

$$\begin{aligned} C_n(z) &= \frac{1}{1 - \alpha_0 z - \frac{\beta_1 z^2}{1 - \alpha_1 z - \frac{\beta_2 z^2}{\dots}}} = \frac{1}{1 - \alpha_0 z - \beta_1 z^2 C'(z)} \\ &= 1 + (\alpha_0 z + \beta_1 z^2 C'(z)) + (\alpha_0 z + \beta_1 z^2 C'(z))^2 + (\dots)^3 + \dots \end{aligned}$$

$$[z^0] C_n(z) = 1$$

$$[z] C_n(z) = \alpha_0$$

$$[z^2] C_n(z) = \alpha_0^2 + \beta_1 [z^0] C'(z) = \alpha_0^2 + \beta_1$$

$$\begin{aligned} [z^3] C_n(z) &= \alpha_0^3 + 2 \alpha_0 \beta_1 [z^0] C'(z) + \beta_1 [z] C'(z) \\ &= \alpha_0^3 + 2 \alpha_0 \beta_1 + \beta_1 \alpha_1 \end{aligned}$$

Exercise: write down a proof

More examples:

$$\frac{1}{1 - \frac{z}{1 - \frac{2z}{1 - \frac{2z}{1 - \frac{3z}{\dots}}}}} = \sum_{n \geq 0} (2n-1)(2n-3)\cdots 3 \cdot 1 \cdot z^n$$

$$\frac{1}{1 - \frac{z}{1 - \frac{2z}{1 - \frac{z}{1 - \frac{z}{\dots}}}}} = \sum_{n \geq 0} c_n z^n$$

$$\frac{1}{1-z - \frac{z^2}{1-2z - \frac{2z^2}{1-\dots}}} = \sum_{n \geq 0} b_n z^n$$

$$\frac{1}{1-z - \frac{1^2 \cdot 2^2}{1-3z - \frac{2^2 \cdot 2^2}{1-\dots}}} = \sum_{n \geq 0} n! z^n$$

\uparrow \uparrow
 $2n+1$ n^2

Q: What does $\alpha_n \geq 0$, $\beta_n \geq 0$ imply?

Next : orthogonal polynomials