Lectures on Random Matrices (Spring 2025)
Lecture 13: Matching Random Matrices to Random Growth I

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1 Recap

In the last lecture, we discussed various random growth models, and universal KPZ objects:

- **Airy line ensemble** which arises as the scaling limit of the Dyson Brownian motion.
- **KPZ Equation** as a universal continuous random growth model.
- **Polynuclear growth model** (PNG) as a discrete analogue of the KPZ equation.

Then we briefly mentioned how the PNG model matches to a last-passage percolation (LPP) model in $\mathbb{R}^2_{\geq 0}$ driven by the Poisson point process as noise. In this lecture, we are going to explore a different LPP model which is defined on cells of $\mathbb{Z}^2_{\geq 1}$, and match it *exactly* to the Wishart random matrix model which we have seen before in passing. This matching is due to Dieker and Warren (2009) [DW08], who proved it in the context of deformed random matrix spectra, as suggested in [BP08]. The key to this matching is a *dynamical* perspective on both the LPP and the random matrix models, which allows us to match Markov chains in the two models, and not simply the distributions.

Throughout the discussion, we will consider the "spiked", multiparameter models, which naturally include finite-rank deformations.

2 The spiked Wishart ensemble

2.1 Definition of the spiked Wishart process

Recall that a (complex) *Wishart matrix* M of dimension n with t degrees of freedom (and identity covariance) can be represented as $M = XX^*$, where X is an $n \times t$ random matrix with independent complex Gaussian entries. Clearly, M is a positive-semidefinite Hermitian matrix of size $n \times n$. The eigenvalues $(\lambda_1, \dots, \lambda_N)$ (with $\lambda_1 \geq \cdots \geq \lambda_N \geq 0$) have the joint density of the *Laguerre orthogonal polynomial ensemble* ($\beta = 2$). Now we introduce a more general model where the covariance of the underlying Gaussian matrix is not identity but has a perturbation (a "spike").

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Definition 2.1 (Generalized Wishart ensemble with parameters $(\pi, \hat{\pi})$)**.** Fix a positive integer n. Let $\pi = (\pi_1, \dots, \pi_n)$ be a fixed n-tuple of *positive* real parameters, and let $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \dots)$ be a sequence of *nonnegative* real parameters (possibly infinite in length). We define an array of complex random variables $\{A_{ij} : 1 \le i \le n, j \ge 1\}$ such that under the probability measure $P^{\pi,\hat{\pi}}$:

- The A_{ij} are independent for all $1 \le i \le n$ and $j \ge 1$.
- Each A_{ij} is a complex Gaussian with mean 0 and variance $\mathrm{Var}(A_{ij}) = \frac{1}{\pi_i + \hat{\pi}_j}$ (i.e. $\Re A_{ij}, \Im A_{ij} \sim N(0, \frac{1}{2(\pi_i+\hat{\pi}_j)})$ independent).

For each integer $t \ge 0$, let $A(t)$ denote the $n \times t$ sub-matrix consisting of the first t columns of A. We then define an $n \times n$ random Hermitian matrix

$$M(t) := A(t)\, A(t)^*, \qquad t \ge 0,$$

with the convention $M(0)$ is the zero matrix. We call $\{M(t) : t \ge 0\}$ the **generalized Wishart random-matrix process** with parameters $(\pi, \hat{\pi})$.

In particular, $M(t)$ has the form

$$M(t) = \sum_{m=1}^{t} A^{(m)} (A^{(m)})^*,$$

where $A^{(m)}$ denotes the m-th column of A (an n-dimensional complex random vector with independent entries of variance $1/(\pi_i + \hat{\pi}_m)$). When all $\pi_i = 1$ and all $\hat{\pi}_j = 0$, $M(t)$ reduces to the classical complex Wishart(n, t) with identity covariance.

Remark 2.2. The introduction of parameters π and $\hat{\pi}$ allows for **finite-rank deformations of the covariance**: one can think of the π_i's as baseline values (say $\pi_i = 1$ for all but a few coordinates), and a finite number of them being different from 1 corresponds to a finite-rank perturbation of the identity covariance matrix Σ (the directions in which $\pi_i \ne 1$ are "spiked" eigen-directions). Similarly, $\hat{\pi}_j$ can be viewed as adding a rank-one perturbation associated with each column; if only finitely many of the $\hat{\pi}_j$ are nonzero, that corresponds to having a finite number of distinguished samples (or boundary inhomogeneities in the equivalent percolation model, as we will see).

We emphasize that $M(t)$ depends on t in a way that $M(t)$ and $M(t-1)$ are not independent but are coupled through shared columns. Indeed $M(t) = M(t-1) + A^{(t)}(A^{(t)})^*$, which is a rank-1 update of $M(t-1)$.

Let us denote by $\lambda_1(t) \ge \lambda_2(t) \ge \cdots \ge \lambda_n(t) \ge 0$ the eigenvalues of $M(t)$ in non-increasing order (padded with zeros if $t < n$, since $\mathrm{rank}(M(t)) \le t$). We will use the notation $\mathrm{sp}(M(t)) = (\lambda_1(t), \dots, \lambda_n(t))$ for the *spectrum* of $M(t)$, viewed as a vector in the *Weyl chamber* $\mathbb{W}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \ge x_2 \ge \cdots \ge x_n\}$. We are particularly interested in the *largest eigenvalue process* $\{\lambda_1(t) : t \ge 0\}$, i.e. the sequence of the top eigenvalue as the number of samples t grows. Our goal is to describe the law of this process and to identify it with a combinatorial growth model.

Before stating the main result, we need a fundamental property of the eigenvalue sequence $\mathrm{sp}(M(t))$ as t increases, namely that it forms a *Markov chain* in $\mathbb{W}^n$. See Problem M.1.

We need another statement:

Lemma 2.3 (Interlacing; Problem M.2)**.** *For each $t \geq 1$, the eigenvalues of $M(t)$ and $M(t-1)$ satisfy the interlacing property:*

$$\lambda_1(t) \geq \lambda_1(t-1) \geq \lambda_2(t) \geq \lambda_2(t-1) \geq \cdots \geq \lambda_n(t-1) \geq \lambda_n(t) \geq 0. \tag{2.1}$$

We denote the relation (2.1) by

$$\lambda(t) \succ \lambda(t-1). \tag{2.2}$$

In other words, the eigenvalue *Markov processes* $\lambda(t)$, $t = 0, 1, 2, \ldots$ form an interlacing array, where at each step of the Markov process, a new row of the array is "revealed". The interlacing property is parallel to the uniform conditioning (Gibbs) property in the $\beta = 2$ corners. Moreover, one can check (Problem M.3) that in the null case $\pi_i = 1$ and $\hat{\pi}_j = 0$, the Wishart eigenvalue process satisfies the uniform Gibbs property as well.

2.2 Markov chain and transition kernel for eigenvalues

We say a random process $\{X(t) : t \geq 0\}$ taking values in $\mathbb{W}^n$ is an *inhomogeneous Markov chain* if for each $m < t$, the conditional law of $X(t)$ given $(X(t-1) = x_{t-1},\ X(t-2) = x_{t-2}, \ldots, X(m) = x_m)$ depends only on x_{t-1} (and possibly on t). In other words, the process has the Markov property but the transition kernel may depend on the time step t. In our case, since at each step t a new column $A^{(t)}$ with variance parameters $\{\pi_i + \hat{\pi}_t : 1 \leq i \leq n\}$ is added, the transition law from $M(t-1)$ to $M(t)$ will indeed depend on the index t through $\hat{\pi}_t$. We denote by $Q^{\pi,\hat{\pi}}_{t-1,t}(x, dy)$ the transition kernel: for $x \in \mathbb{W}^n$ given as the eigenvalue vector of $M(t-1)$, $Q^{\pi,\hat{\pi}}_{t-1,t}(x, \cdot)$ is the distribution of $\mathrm{sp}(M(t))$.

The null case $\pi_i = 1$ and $\hat{\pi}_j = 0$ of $Q^{\pi,\hat{\pi}}_{t-1,t}(x, dy)$ was computed in [Def10], see also [FR06].

Theorem 2.4. *Fix an integer $n \geq 1$. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a strictly positive n-vector, and let $\widehat{\pi} = (\widehat{\pi}_1, \widehat{\pi}_2, \ldots)$ be any sequence of nonnegative real parameters. Under the probability measure $P^{\pi,\widehat{\pi}}$, the eigenvalues of the $n \times n$ generalized Wishart matrices $\{M(t)\}_{t\geq 0}$ form a time-inhomogeneous Markov chain $\{\mathrm{sp}(M(t))\}_{t\geq 0}$ in the Weyl chamber*

$$\mathbb{W}^n \ = \ \big\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} : x_1 \geq x_2 \geq \cdots \geq x_n \big\}.$$

More precisely, writing $x = \mathrm{sp}(M(t-1))$ and $y = \mathrm{sp}(M(t))$, the one-step transition law from time $(t-1)$ to t is absolutely continuous on the interior of $\mathbb{W}^n$ and can be factored as

$$Q^{\pi,\widehat{\pi}}_{t-1,t}(x,\, dy) \ = \ \Big[\prod_{i=1}^{n}(\pi_i + \widehat{\pi}_t)\Big] \cdot \frac{h_\pi(y)}{h_\pi(x)} \ \exp\Big(-(\widehat{\pi}_t - 1)\sum_{i=1}^{n}(y_i - x_i)\Big) \ \times \ Q^{(0)}\big(x,\, dy\big), \tag{2.3}$$

where

- $Q^{(0)}\big(x,\, dy\big)$ *is the* standard *(null-spike) Wishart transition kernel, given explicitly by*

$$Q^{(0)}(x,\, dy) \ = \ \frac{\Delta(y)}{\Delta(x)} \ \exp\Big(-\sum_{i=1}^{n}(y_i - x_i)\Big)\, \mathbf{1}_{\{x \prec y\}}\, dy, \tag{2.4}$$

  with $\Delta(z) = \prod_{1\leq i<j\leq n}(z_i - z_j)$ the Vandermonde determinant.

- *The function h_π is the (continuous) Harish-Chandra orbit integral factor*

$$h_\pi(z) \;=\; \frac{(-1)^{\binom{n}{2}}}{0!1!\cdots(n-1)!}\,\frac{\det\bigl(e^{-\pi_i z_j}\bigr)_{i,j=1}^{n}}{\Delta(\pi)\,\Delta(z)}.$$

  Note that $h_\pi(0) = 1$.

In particular, the chain starts from $\mathrm{sp}(M(0)) = 0$ (the zero matrix).

Sketch of proof; see [DW08]. First of all, random-matrix arguments [Def10], [FR06] show that the theorem holds for the null case $\pi_i = 1$ and $\hat{\pi}_j = 0$. The Radon-Nikodym derivative of the transition kernel factors through the diagonal entries of the matrix, and can be written in terms of the eigenvalues via the HCIZ integral. This yields an explicit factor multiplying the null-case transition density. □

Remark 2.5 (Problem M.4)**.** In order to see directly that the family $\bigl\{Q^{\pi,\hat{\pi}}_{t-1,t}\bigr\}$ of transition kernels does indeed define Markov transitions (that is, each $Q^{\pi,\hat{\pi}}_{t-1,t}(x,\cdot)$ is a probability measure for every x), one can use the fact that

$$\mathbf{1}_{z \prec z'} \;=\; \det\bigl[\mathbf{1}_{z_i < z'_j}\bigr],$$

along with the Cauchy–Binet (or Andréief) identity:

$$\int_{\mathbb{W}^N} \det[\xi_i(z_j)]\,\det[\psi_j(z_i)]\,dz \;=\; \det\Bigl[\int_{\mathbb{R}} \xi_i(z)\,\psi_j(z)\,dz\Bigr].$$

Applying this to (2.3)–(2.4) yields a sequence of integrals of the exponential densities of the form $e^{-(\pi_i+\hat{\pi}_t)y}$. This yields the normalizing factor $\prod_{i,j}(\pi_i + \hat{\pi}_j)$, and confirms that each transition kernel integrates to one, in line with the notation and factorization in Theorem 2.4.

The fixed-time distribution of the eigenvalues in the null case $\pi_i = 1$ and $\hat{\pi}_j = 0$ is given by the Laguerre orthogonal polynomial ensemble. For example, for $t \geq n$, we have

$$\mathrm{Prob}(\mathrm{sp}(M(t)) \in dy) = \frac{1}{Z}\prod_{i<j}(y_i - y_j)^2 \prod_{i=1}^{n} y_i^{t-n} e^{-y_i}. \tag{2.5}$$

For the non-null case, see Problem M.5.

3 The exponential LPP model

We now turn to a seemingly different probabilistic model: a model of random paths in a grid with random weights. Fix an integer n. Consider an infinite array of independent, nonnegative random weights $\{W_{ij} : i \geq 1,\ 1 \leq j \leq n\}$ defined under the probability measure $P^{\pi,\hat{\pi}}$, where each W_{ij} is an independent random variable with an *exponential* distribution of rate $(\pi_i + \hat{\pi}_j)$. Note that $\mathbb{E}[W_{ij}] = \frac{1}{\pi_j + \hat{\pi}_i}$. These rates $(\pi_j + \hat{\pi}_i)$ are chosen deliberately to mirror the variance parameters of A_{ij} in the generalized Wishart model (Definition 2.1).

We interpret $\{W_{ij}\}$ as random weights on the vertices of a directed lattice in the first quadrant. Specifically, consider the set of lattice points

$$\{(i,j) : i = 1, \dots, t, \dots,\ j = 1, \dots, n\}.$$

We say a path Γ is an *up-right path* from $(1,1)$ to (t,n) if it is a sequence of lattice points starting at $(1,1)$ and ending at (t,n), with steps either one step to the right or one step up. Since each step either increases the column index by 1 or the row index by 1, any such path from $(1,1)$ to (t,n) must consist of $(t-1)$ right-steps and $(n-1)$ up-steps, for a total of $(t+n-2)$ steps. We define the *weight* of a path Γ to be the sum of the W_{ij} along its vertices:

$$\mathcal{W}(\Gamma) := \sum_{(i,j)\in\Gamma} W_{ij},$$

where by $(i,j) \in \Gamma$ we mean that the vertex (i,j) is visited by the path Γ. The random variable of interest is the *maximum total weight achievable among all such paths*, i.e.

$$L(t,n) := \max_{\Gamma:\,(1,1)\to(t,n)} \mathcal{W}(\Gamma)\,. \tag{3.1}$$

We call $L(t,n)$ the *last-passage time* to (t,n), in analogy with the usual terminology of growth models (if we interpret W_{ij} as random passage times on a lattice, then the longest time to reach a certain site is given by the maximal weight path).

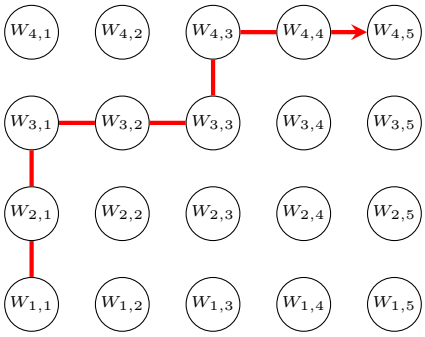

Figure 1: A portion of the lattice with vertex-weights $W_{i,j}$ and one up-right path.

Indeed, it is immediate from the definition that the random variables $L(t,n)$ satisfy the following random recursion:

$$L(i,j) = W_{ij} + \max\{\, L(i,j-1),\ L(i-1,j)\,\}\,, \tag{3.2}$$

for $i > 1, j > 1$, with boundary conditions $L(t,1) = \sum_{k=1}^{t} W_{k,1}$ and $L(1,i) = \sum_{\ell=1}^{i} W_{1,\ell}$. The recursion (3.2) expresses that the optimal path to (i,j) either comes from below (then last step is down, contributing W_{ij} plus the optimal weight to $(i,j-1)$) or from the left (last step is right from $(i-1,j)$). It is the fundamental equation of growth models, which is a part of the *Robinson–Schensted–Knuth insertion algorithm* in combinatorics.

Remark 3.1. The quantity $L(t,n)$ appears in many contexts besides the LPP. Namely, it is also the total service time in a series of n exponential queueing servers with t customers (the *Jackson network* interpretation [Bar01]), and it is a prototype of models in the KPZ universality class (often called the *exponential corner growth model*). For the random growth interpretation [Joh00], define the growing percolation cluster as

$$F_\tau := \{(i,j)\colon L(j,i) \le t\} \subseteq \mathbb{Z}^2_{\ge 1}, \qquad \tau \in \mathbb{R}_{\ge 0}.$$

Then this cluster grows by adding 1×1 boxes after exponential random times, when each rate $\pi_j + \hat{\pi}_i$ exponential clock starts ticking when the cluster reaches the two adjacent vertices to (i,j).

Let us define the whole *vector of last-passage times to the bottom row* at column t as

$$Z(t) := \big(L(t,1),\, L(t,2),\, \dots,\, L(t,n)\big) \in \mathbb{W}^n,$$

where we list the values in increasing order $L(t,1) \le L(t,2) \le \cdots \le L(t,n)$.[1] In particular, $L(t,n)$ is the largest component of $Z(t)$. One readily sees from the recursion (3.2) that the sequence $\{Z(t) : t \ge 0\}$ is a Markov process in $\mathbb{W}^n$.

Remark 3.2. The process $Z(t)$ is **not** the same as the Markov process of the spectra of the Wishart matrices $M(t)$.

In the next section, we will consider a discrete version of the LPP model, and consider a crucial bijection — the celebrated *Robinson–Schensted–Knuth (RSK) correspondence*. In the next Lecture 14, we will use this to complete the proof of the matching between the Wishart process and the LPP with exponential weights. That is, we are after the following result (cf. Remark 3.2):

Theorem 3.3 ([DW08])**.** *The joint distribution of the last-passage times*

$$L(1,n),\ L(2,n), \dots, L(t,n) \tag{3.3}$$

is the same as the joint distribution of the largest eigenvalues of the $n \times n$ Wishart matrices

$$\mathrm{sp}(M(1))_{\max},\ \mathrm{sp}(M(2))_{\max}, \dots, \mathrm{sp}(M(t))_{\max}. \tag{3.4}$$

Remark 3.4. It is important to note that neither sequence (3.3) nor (3.4) is a Markov process.

4 Geometric LPP and Robinson-Schensted-Knuth correspondence

4.1 Geometric LPP

Throughout this section, we are interested in the last-passage percolation with discrete weights $W_{ij} \in \mathbb{Z}_{\ge 0}$, which have the geometric distribution

$$\mathrm{Prob}(W_{ij} = k) = \frac{(a_i b_j)^k}{(1 - a_i b_j)^{k+1}}, \qquad k = 0, 1, \dots \tag{4.1}$$

The last-passage times are defined by (3.2), the same as in the exponential case.

[1] We have $L(t,1) \le \cdots \le L(t,n)$ almost surely because giving the path more freedom to move down can only increase the maximum weight. This is easily checked from (3.2). Thus $Z(t) \in \mathbb{W}^n$ indeed.

4.2 Bijective mapping of arrays via toggles

We are now going to present the Robinson-Schensted-Knuth correspondence via the operation called *toggle*. This exposition is different from the usual discussions in e.g., [Sag01], [Ful97], and follows these notes by Sam Hopkins. Fix t, n, and consider the array $W = \{W_{ij}\}_{1\le i\le t, 1\le j\le n}$ of nonnegative integers. We can think of W as a realization of the geometric environment, but for now let us assume that W is a fixed array. See Figure 1 for the order of indices.

We are going to inductively construct a bijection RSK between the array W and another array $R = \{R_{ij}\}_{1\le i\le t, 1\le j\le n}$ of nonnegative integers, which is *ordered*:

$$R_{i,j} \le R_{i,j+1}, \qquad R_{i,j} \le R_{i+1,j}, \qquad \text{for all } i, j.$$

Note that this ordering means that the diagonals in R interlace.

To define RSK, we first define an elementary operation called *toggle* and denoted by T.

Definition 4.1 (Toggle)**.** The toggle operation is a map T which takes in a nonnegative integer w and a triple (λ, κ, μ) of sequences of nonnegative integers satisfying interlacing

$$\lambda \succ \kappa \prec \mu.$$

The lengths of the sequences are differ by 0 or 1, and if necessary, we pad the sequences with 0's to make the interlacing make sense. The output of T is a triple (λ, ν, μ), where λ and μ are not changed, and ν is obtained from λ, μ, κ, and w as follows:

$$\nu_1 = w + \max(\lambda_1, \mu_1), \qquad \nu_i = \max(\lambda_i, \mu_i) + \min(\lambda_{i-1}, \mu_{i-1}) - \kappa_{i-1}, \qquad \text{for } i \ge 2.$$

Define $|\kappa| := \kappa_1 + \kappa_2 + \cdots$, and similarly for λ, μ, ν.

Proposition 4.2. *If $(\lambda, \nu, \mu) = \mathrm{T}(w; \lambda, \kappa, \mu)$, then $\lambda \prec \nu \succ \mu$, and*

$$|\nu| = w + |\lambda| + |\mu| - |\kappa|.$$

Proof. See Problem M.6. □

Now, define $R = \mathrm{RSK}(W)$ as follows. We will build R by sequentially modifying the array W, starting from the bottom-left corner $(1, 1)$ and moving by adding one box at a time. We represent the partially filled R array as a collection of interlacing sequences. Let $R^{(i,j)}$ denote the already constructed part of R, where we are adding a box (i, j). Then, we modify the diagonal containing (i, j) by applying the toggle operation to the weight $w = W_{i,j}$ and the three diagonals

$$\lambda^{(i,j)} = \{R^{(i,j)}_{i-k+1,j-k}\}_{k\ge 1}, \quad \mu^{(i,j)} = \{R^{(i,j)}_{i-k,j-k+1}\}_{k\ge 1}, \quad \kappa^{(i,j)} = \{R^{(i,j)}_{i-k,j-k}\}_{k\ge 1}.$$

The next statement is straightforward:

Proposition 4.3. *The toggle operation* T *is a bijection*

$$\mathbb{Z}_{\ge 0} \times \{(\lambda, \kappa, \mu) \colon \lambda \succ \kappa \prec \mu\} \leftrightarrow \{(\nu, \lambda, \mu) \colon \lambda \prec \nu \succ \mu\}.$$

Consequently, the map RSK *is a bijection between nonnegative arrays W and ordered nonnegative arrays R.*

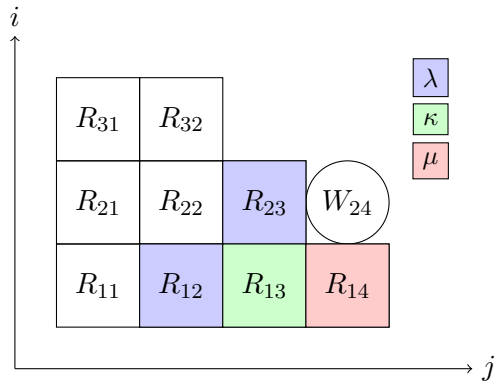

Figure 2: Illustration of the RSK toggle operation, with $w = W_{24}$ being added to the array R, and $\lambda = (R_{23}, R_{12})$, $\kappa = (R_{13})$, $\mu = (R_{14})$.

Proposition 4.4. *The bijection* RSK *does not depend on the order of adding the boxes to the array R.*

Proof. See Problem M.7. □

Remark 4.5. The map RSK is transposition-equivariant, meaning that if $R = \mathrm{RSK}(W)$, then $R^\top = \mathrm{RSK}(W^\top)$.

4.3 Weight preservation

Define the row and column sums in $W = (W_{ij})_{1 \le i \le t, 1 \le j \le n}$ by

$$\mathrm{row}_i := \sum_{j=1}^{n} W_{ij}, \qquad \mathrm{col}_j := \sum_{i=1}^{t} W_{ij}.$$

Also define the diagonal sums in R by

$$\mathrm{diag}_{i,j} := \sum_{k=0}^{\min(i,j)-1} R_{i-k,j-k}.$$

Proposition 4.6 (RSK weight preservation)**.** *Under the bijection* RSK*, we have*

$$\mathrm{diag}_{t,j} = \sum_{j'=1}^{j} \mathrm{col}_{j'}, \qquad \mathrm{diag}_{i,n} = \sum_{i'=1}^{i} \mathrm{row}_{i'}.$$

In particular, the last diagonal sum $\mathrm{diag}_{t,n}$ is equal to the total weight of the array W, which is the aggregate of the row sums or the column sums, as it should be.

Proof of Proposition 4.6. We can prove this by induction by adding one box at a time. Define the partial row and column sums as

$$\mathrm{row}_{i,k} := \sum_{j=1}^{k} W_{ij}, \qquad \mathrm{col}_{k,j} := \sum_{i=1}^{k} W_{ij}.$$

We claim that for partial arrays in the process of RSK, the equalities between aggregated partial row/column sums and diagonal sums hold, where we take the column sum if the boundary of a partial array is horizontal, and the row sum if the boundary is vertical. This is our induction claim, and it clearly holds for the empty array.

Now, consider adding a box (i, j) to the array R. For the diagonal sums, we have the identity due to Proposition 4.2:

$$\operatorname{diag}_{i,j} = \operatorname{diag}_{i,j-1} + \operatorname{diag}_{i-1,j} - \operatorname{diag}_{i-1,j-1} + W_{ij}. \tag{4.2}$$

Now, using the induction hypothesis for $\operatorname{diag}_{i,j-1}$, $\operatorname{diag}_{i-1,j}$, and $\operatorname{diag}_{i-1,j-1}$, we see that identity (4.2) simply represents the fact that diag_{ij} is the "cumulative distribution function" of the array W, or, which is the same, that W is the second mixed partial (discrete) derivative of the diagonal sums. This completes the proof. □

In the next Lecture 14, we will consider the effect of applying the RSK to the array W of independent geometric random variables.

M Problems (due 2025-04-29)

M.1 Wishart Markov chain

In the null case $\pi_i = 1$ and $\hat{\pi}_j = 0$, show that the process $\operatorname{sp}(M(t))$ defined in Section 2.1 is a Markov chain.

Hint: Use diagonalization and the fact that the Wishart matrix distribution is invariant under conjugations by unitary matrices, similarly to how we did it for the Dyson Brownian motion in Lecture 10.

M.2 Interlacing

Prove Lemma 2.3.

Hint: You can use the minimax definition of the eigenvalues to show the interlacing.

M.3 Gibbs property

Show that in the null case $\pi_i = \hat{\pi}_j = 0$, the Wishart eigenvalue process from Section 2.1 has the Gibbs conditioning property: when conditioned on the values of $\lambda(t)$, the joint distribution of all the eigenvalues $\{\lambda(s)\colon s = 0, 1, \ldots, t-1\}$ is uniform in the Gelfand–Tsetlin polytope determined by $\lambda(t)$ and the interlacing.

M.4 Transition kernels integrate to one

Complete the argument outlined in Remark 2.5 that the transition densities $Q^{\pi,\hat{\pi}}_{t-1,t}(x, dy)$ integrate to one in y.

M.5 Distribution of the eigenvalues

Find the density $\mathrm{Prob}\left(\mathrm{sp}(M(t)) \in dy\right)/dy$ of the spiked Wishart ensemble at an arbitrary fixed time t. For this, you can multiply the transition operators $Q^{\pi,\hat{\pi}}_{t-1,t}$ from Theorem 2.4.

M.6 Weight preservation under toggle

Prove Proposition 4.2.

M.7 RSK independence of order

Prove Proposition 4.4, which states that the bijection RSK does not depend on the order of adding the boxes to the array R.

Hint: Toggle operations commute when they act on non-overlapping diagonals.

M.8 Asymptotics: BBP phase transition

Review the proof of the BBP transition for a rank-1 spiked Wishart matrix (or the rank-1 inhomogeneous corner-growth model). Show how to compute the large-n limiting distribution of the top eigenvalue in the critical case Identify the limit law as a *deformed Airy kernel* (or equivalently a shifted Airy_2 process).

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