Random Fibonacci Words

Leonid Petrov

Talk at University of Illinois at Urbana-Champaign, Integrability & Representation Theory (IRT) Seminar, February 20, 2025

1 Motivation 1. De Finetti's theorem and Pascal triangle

1.1

Definition 1.1. A sequence $X_1, X_2, \ldots$ of binary random variables (taking values in $\{0, 1\}$) is called *exchangeable* if for any n and any permutation σ of $\{1, 2, \ldots, n\}$ the joint distribution of $X_1, X_2, \ldots, X_n$ is the same as the joint distribution of $X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}$.

Exchangeable sequences are more than just Bernoulli iid sequences with some parameter $p \in [0, 1]$. Consider the Polya urn scheme.

Start with an urn containing b black and w white balls. At each step, draw a ball uniformly at random from the urn and put it back along with another ball of the same color.

Exercise 1.2. The sequence of ball colors drawn from the urn is exchangeable.

At time n, there are n new balls in the urn, and the distribution of the number of, say, black balls,

$$\mathbb{P}\left(\text{black} = k\right) = M_n(k), \qquad k = 0, 1, \ldots, n,$$

is called the (n-th) *coherent measure*. We can talk about S_n, the random variable which is the number of black balls drawn by time n. The coherent measures M_n for various n satisfy certain linear recurrence relations.

One can convince oneself that the space of coherent measures is the same as the space of exchangeable random sequences of 0's and 1's. This space is a convex set, moreover, it is a simplex.

Extreme points of this simplex are given by iid sequences, that is, Bernoulli product measures on $\{0, 1\}^{\infty}$. This is de Finetti's theorem.

1.2

Coherent measures on Pascal triangle are related to exchangeable sequences of 0's and 1's. The *boundary* of the Pascal triangle encodes all possible coherent measures via the law of large numbers,

$$\frac{S_n}{n} \to \mu \quad \text{on} \quad [0, 1].$$

Extreme measures correspond to delta point masses. For example, the Polya urn for $a = b = 1$ corresponds to μ being the uniform measure on $[0, 1]$.

1.3 Lonely paths

There are two distinguished paths in the Pascal triangle, the *lonely paths* $0 \to 00 \to 000 \to \dots$ and $1 \to 11 \to 111 \to \dots$, which are characterized by the property that [GK00]

> All but finitely many vertices in the path have a single immediate predecessor.

These paths correspond to the extreme measures with $\mu = \delta_0$ and $\mu = \delta_1$, respectively.

2 Motivation 2. Young lattice

The Young lattice $\mathbb{Y}$ of integer partitions ordered by the relation "adding a box" encodes another meaningful structure — irreducible representations of the symmetric groups. The boundary encodes the irreducible representations of the infinite symmetric group $S(\infty)$.

2.1

The Young lattice is a *differential poset* [Sta88], [Fom94], in the sense that

> for each λ, there is one more element in the set $\{\nu\colon \nu = \lambda + \square\}$ than in the set $\{\mu\colon \mu = \lambda - \square\}$.

Differential poset property implies that for f^λ the number of paths from $\varnothing$ to λ, we have

$$\sum_{|\lambda|=n} (f^\lambda)^2 = n!, \qquad \text{define} \qquad M_n(\lambda) \coloneqq \frac{(f^\lambda)^2}{n!}.$$

The measure M_n is called *Plancherel*, it is coherent and extremal. It corresponds to the regular representation of $S(\infty)$, which is irreducible.

2.2

There are two lonely paths here, as well — corresponding to growing one-row and one-column partitions.

2.3

All extreme coherent measures on the Young lattice are given by specializations of Schur symmetric functions, and have the form

$$M_n(\lambda) = s_\lambda(\vec{\alpha}; \vec{\beta}; \gamma) \cdot f^\lambda.$$

The problem of describing the boundary of $\mathbb{Y}$ is equivalent to the problem of finding parameters $\vec{\alpha}, \vec{\beta}, \gamma$ such that the Schur functions $s_\lambda(\vec{\alpha}; \vec{\beta}; \gamma)$ are nonnegative for all λ.

Schur functions are (essentially) determinants, and for the Young lattice, we have a great match between these multiparameter functions and extreme coherent measures.

3 Another differential poset — the Young–Fibonacci lattice

3.1

Are there any other differential posets?

References

[Fom94] S. Fomin, *Duality of graded graphs*, J. Algebr. Comb. **3** (1994), no. 4, 357–404. ↑2

[GK00] F. Goodman and S. Kerov, *The Martin Boundary of the Young-Fibonacci Lattice*, Jour. Alg. Comb. **11** (2000), 17–48. arXiv:math/9712266 [math.CO]. ↑2

[Sta88] R. Stanley, *Differential posets*, Jour. AMS **1** (1988), no. 4, 919–961. ↑2