Lectures on Random Matrices (Spring 2025)
Lecture 1: Moments of random variables and random matrices

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Monday, January 13, 2025*

1 Recall Central Limit Theorem

We begin by establishing the necessary groundwork for understanding and proving the Central Limit Theorem. The theorem's power lies in its remarkable universality: it applies to a wide variety of probability distributions under mild conditions.

Definition 1.1. A sequence of random variables $\{X_i\}_{i=1}^{\infty}$ is said to be *independent and identically distributed (i.i.d.)* if:

- Each X_i has the same probability distribution as every other X_j, for all i, j.
- The variables are mutually independent, meaning that for any finite subset $\{X_1, X_2, \ldots, X_n\}$, the joint distribution factors as the product of the individual distributions:

$$\mathbb{P}(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) = \mathbb{P}(X_1 \le x_1)\,\mathbb{P}(X_2 \le x_2) \cdots \mathbb{P}(X_n \le x_n).$$

Theorem 1.2 (Classical Central Limit Theorem)**.** *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with finite mean $\mu = \mathbb{E}[X_i]$ and finite variance $\sigma^2 = \operatorname{Var}(X_i)$. Define the normalized sum*

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu)\,.$$

Then, as $n \to \infty$, the distribution of Z_n converges in distribution to a normal random variable with mean 0 and variance σ^2, i.e.,

$$Z_n \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Convergence in distribution means

$$\lim_{n\to\infty} \mathbb{P}(Z_n \le x) = \mathbb{P}(Z \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{t^2}{2\sigma^2}}\, dt \qquad \text{for all } x \in \mathbb{R}, \tag{1.1}$$

where $Z \sim \mathcal{N}(0, \sigma^2)$ is the Gaussian random variable.

*Course webpage • TeX Source • Updated at 07:24, Sunday 12th January, 2025

Remark 1.3. For a general random variable instead of $Z \sim \mathcal{N}(0, \sigma^2)$, the convergence in distribution (1.1) holds only for x at which the cumulative distribution function of Z is continuous. Since the normal distribution is absolutely continuous (has density), the convergence holds for all x.

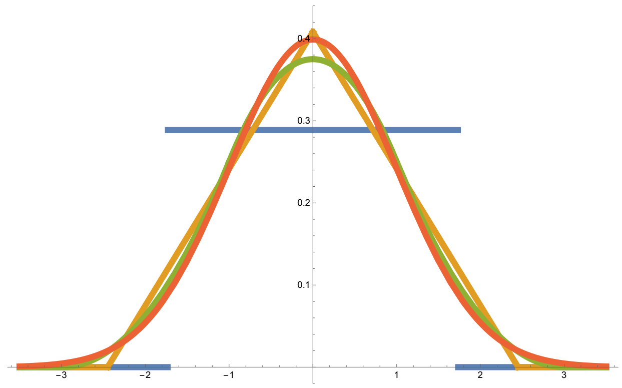

Figure 1: Densities of U_1, $U_1 + U_2$, $U_1 + U_2 + U_3$ (where U_i are iid uniform on $[0, 1]$), and $\mathcal{N}(0, 1)$, normalized to have the same mean and variance.

Example 1.4. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. Bernoulli random variables with parameter p, meaning that each X_i takes the value 1 with probability p and 0 with probability $1 - p$. The mean and variance of each X_i are given by:

$$\mu = \mathbb{E}[X_i] = p, \quad \sigma^2 = \operatorname{Var}(X_i) = p(1 - p).$$

We also have the distribution of $X_1 + \cdots + X_n$:

$$\mathbb{P}\left(X_1 + \cdots + X_n = k\right) = \binom{n}{k} p^k (1 - p)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

local CLT via Stirling

References

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