Lectures on Random Matrices (Spring 2025)
Lecture 6: Steepest descent and local statistics

Leonid Petrov

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1 Recap

1.1 Determinantal structure of the GUE

Last time, we proved the following result:

Theorem 1.1.

2 Double Contour Integral Representation for the GUE Kernel

2.1 One contour integral representation for Hermite polynomials

Recall that the GUE kernel is defined by

$$K_N(x,y) = \sum_{n=0}^{N-1} \psi_n(x)\psi_n(y),$$

with the orthonormal functions

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\, p_n(x)\, e^{-x^2/4},$$

where the (monic, probabilists') Hermite polynomials are given by

$$p_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\, e^{-x^2/2},$$

and satisfy the generating function

$$\exp\Big(xt - \frac{t^2}{2}\Big) = \sum_{n\ge 0} p_n(x) \frac{t^n}{n!}.$$

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By Cauchy's integral formula we can write

$$p_n(x) = \frac{n!}{2\pi i} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{n+1}}\, dt,$$

where the contour C is a simple closed curve encircling the origin. Therefore,

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\, p_n(x)\, e^{-x^2/4} = \frac{e^{-x^2/4}}{\sqrt{h_n}} \frac{n!}{2\pi i} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{n+1}}\, dt.$$

2.2 Another contour integral representation for Hermite polynomials

Note also that

$$\int_{-\infty}^{\infty} e^{-t^2+\sqrt{2}i\,t\,x}\, dt = \sqrt{\pi}\, e^{-x^2/2}.$$

Differentiating both sides n times with respect to x (and using the fact that in our convention the Gaussian appears with $x^2/2$) yields

$$\frac{d^n}{dx^n}\left(e^{-x^2/2}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\sqrt{2}i\,t\right)^n e^{-t^2+\sqrt{2}i\,t\,x}\, dt.$$

Changing variables via $s = i\,t$ (so that $t = -i\,s$ and $dt = -i\,ds$) one obtains

$$\frac{d^n}{dx^n}\left(e^{-x^2/2}\right) = \frac{(\sqrt{2})^n}{i\sqrt{\pi}} \int_{-i\infty}^{i\infty} s^n\, e^{s^2+\sqrt{2}\,s\,x}\, ds.$$

Multiplying by $(-1)^n e^{x^2/2}$ we deduce that

$$p_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right) = \frac{i\,(\sqrt{2})^n\, e^{x^2/2}}{\sqrt{\pi}} \int_{-i\infty}^{i\infty} s^n\, e^{s^2-\sqrt{2}\,s\,x}\, ds. \tag{2.1}$$

Now, recall that the orthonormal functions are defined as

$$\psi_n(x) = \frac{1}{\sqrt{h_n}}\, p_n(x)\, e^{-x^2/4},$$

so that by (2.1)

$$\psi_n(x) = \frac{i\, e^{x^2/4}}{\sqrt{\pi\, h_n}} \int_{-i\infty}^{i\infty} (\sqrt{2}s)^n\, e^{s^2-\sqrt{2}\,s\,x}\, ds = \frac{i\, e^{x^2/4}}{\sqrt{2\pi\, h_n}} \int_{-i\infty}^{i\infty} s^n\, e^{s^2/2-s\,x}\, ds.$$

2.3 Double contour integral representation for the GUE kernel

We have (Problem **??**)

$$h_n = \int_{-\infty}^{\infty} p_n(x)^2\, e^{-x^2/2}\, dx = n!\sqrt{2\pi}.$$

Therefore, we can sum up the kernel (another proof of the Christoffel–Darboux formula):

$$\begin{aligned} K_n(x,y) &= \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y) \\ &= \sum_{k=0}^{n-1} \frac{e^{-x^2/4}}{\sqrt{h_k}} \frac{k!}{2\pi i} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{k+1}}\, dt \frac{i\, e^{y^2/4}}{\sqrt{2\pi\, h_k}} \int_{-i\infty}^{i\infty} s^k\, e^{s^2/2 - s\, y}\, ds \\ &= e^{(y^2-x^2)/4} \sum_{k=0}^{n-1} \frac{1}{4\pi^2} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{k+1}}\, dt \int_{-i\infty}^{i\infty} s^k\, e^{s^2/2 - s\, y}\, ds. \end{aligned}$$

We can now extend the sum to $k = -\infty$, and get a formula for the GUE kernel as a double contour integral:

$$K_n(x,y) = \frac{e^{(y^2-x^2)/4}}{4\pi^2} \oint_C \int_{-\mathbf{i}\infty}^{\mathbf{i}\infty} \frac{\exp\left\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\right\}}{s-t} \left(\frac{s}{t}\right)^n ds\, dt.$$

Details will be in the next Lecture 6.

Remark 2.1. Many other versions of the GUE / unitary invariant ensembles admit determinantal structure:

1. The GUE corners process [JN06]
2. The Dyson Brownian motion (e.g., add a GUE to a diagonal matrix) [NF98]
3. GUE corners plus a fixed matrix [FF14]
4. Corners invariant ensembles with fixed eigenvalues $UDU^{\dagger}$, where D is a fixed diagonal matrix and U is Haar distributed on the unitary group [Met13]

F Problems (due DATE)

References

[FF14] P. Ferrari and R. Frings, *Perturbed GUE minor process and Warren's process with drifts*, J. Stat. Phys **154** (2014), no. 1-2, 356–377. arXiv:1212.5534 [math-ph]. ↑3

[JN06] K. Johansson and E. Nordenstam, *Eigenvalues of GUE minors*, Electron. J. Probab. **11** (2006), no. 50, 1342–1371. arXiv:math/0606760 [math.PR]. ↑3

[Met13] A. Metcalfe, *Universality properties of Gelfand-Tsetlin patterns*, Probab. Theory Relat. Fields **155** (2013), no. 1-2, 303–346. arXiv:1105.1272 [math.PR]. ↑3

[NF98] T. Nagao and P.J. Forrester, *Multilevel dynamical correlation functions for Dyson's Brownian motion model of random matrices*, Physics Letters A **247** (1998), no. 1-2, 42–46. ↑3

L. Petrov, University of Virginia, Department of Mathematics, 141 Cabell Drive, Kerchof Hall, P.O. Box 400137, Charlottesville, VA 22904, USA

E-mail: `lenia.petrov@gmail.com`