Lectures on Random Matrices (Spring 2025)
Lecture 8: Cutting corners and loop equations

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Contents

1 Cutting corners: polynomial equation and distribution

1.1 Recap: polynomial equation

Recall the polynomial equation we proved in the last Lecture 7. Fix $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$. Let $H \in \mathrm{Orbit}(\lambda)$ be a random Hermitian matrix defined as

$$H = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n) U^{\dagger},$$

*Course webpage • Live simulations • TeX Source • Updated at 02:33, Friday 28th February, 2025 where U is Haar-distributed unitary matrix from $U(n)$. This is the case $\beta = 2$, but the statement holds for the cases $\beta = 1, 4$ with appropriate modifications. Let $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of the $(n-1) \times (n-1)$ corner $H^{(n-1)}$.

Lemma 1.1. *The distribution of $\mu_1, \ldots, \mu_{n-1}$ is the same as the distribution of the roots of the polynomial equation*

$$\sum_{i=1}^{n} \frac{\xi_i}{z - \lambda_i} = 0, \tag{1.1}$$

where ξ_i are i.i.d. random variables with the distribution χ^2_β.

Recall also that this passage from λ to μ works inductively, and the distribution of the next level eigenvalues $\nu = (\nu_1 \geq \ldots \geq \nu_{n-2})$ is given by the same polynomial equation, but with λ replaced by μ. In this way, we can define a *Markov map* from λ to μ, which is then iterated to construct the full array of eigenvalues of the corners of H.

For $\beta = \infty$, this map is deterministic, and is equivalent to successive differentiating the characteristic polynomial of H.

1.2 Extension to general β

We extend the polynomial equation to general β, by *declaring* (defining) that the general β corners distribution is powered by the passage from $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ to $\mu = (\mu_1 \geq \ldots \geq \mu_{n-1})$, where μ solves (1.1) with ξ_i i.i.d. χ^2_β. In this way, μ interlaces with λ. For $\beta = 1, 2, 4$, this definition reduces to the one with invariant ensembles with fixed eigenvalues λ.

1.3 Distribution of the eigenvalues of the corners

Let μ be obtained from λ by the general β corners operation.

Theorem 1.2. *The density of μ with respect to the Lebesgue measure is given by*

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} \prod_{1 \leq i < j \leq n-1} (\mu_i - \mu_j) \prod_{i=1}^{n-1} \prod_{j=1}^{n} |\mu_i - \lambda_j|^{\beta/2-1} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^{1-\beta}.$$

Proof. Let $\varphi_i = \xi_i / \sum_{j=1}^{n} \xi_j$. It is well-known[1] the joint density of $(\varphi_1, \ldots, \varphi_n)$ is the (symmetric) Dirichlet density

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} w_1^{\beta/2-1} \ldots w_n^{\beta/2-1} dw_1 \ldots dw_{n-1}$$

(note that the density is $(n-1)$-dimensional).

We need to compute the Jacobian of the transformation from φ to μ, if we write

$$\sum_{i=1}^{n} \frac{\varphi_i}{z - \lambda_i} = \frac{\prod_{i=1}^{n-1}(z - \mu_i)}{\prod_{i=1}^{n}(z - \lambda_i)},$$

[1] See Problem H.3.

and compute (as a decomposition into partial fractions):

$$\varphi_a = \frac{\prod_{i=1}^{n-1}(\lambda_a - \mu_i)}{\prod_{i\neq a}(\lambda_a - \lambda_i)}.$$

Therefore,

$$\frac{\partial \varphi_a}{\partial \mu_b} = \frac{\prod_{i=1}^{n-1}(\lambda_a - \mu_i)}{\prod_{i\neq a}(\lambda_a - \lambda_i)} \frac{1}{\mu_b - \lambda_a}, \qquad a = 1, \ldots, n, \quad b = 1, \ldots, n-1. \tag{1.2}$$

The Jacobian is essentially the determinant of the matrix $1/(\mu_b - \lambda_a)$, which is the Cauchy determinant (Problems H.1 and H.2). The final density is obtained from the symmetric Dirichlet density, but we plug in $w = \varphi$, and also multiply by the inverse of the Jacobian determinant (1.2). After the necessary simplifications, this completes the proof. □

Corollary 1.3 (Joint density of the corners)**.** *The eigenvalues* $\lambda^(k)_j$*,* $1 \le j \le k \le n$*, of a random matrix from* Orbit(λ) *form an interlacing array, with the joint density*

$$\propto \prod_{k=1}^{n} \prod_{1\le i<j\le k} \left(\lambda_j^{(k)} - \lambda_i^{(k)}\right)^{2-\beta} \prod_{a=1}^{k+1}\prod_{b=1}^{k} \left|\lambda_a^{(k+1)} - \lambda_b^{(k)}\right|^{\beta/2-1}.$$

For $\beta = 2$, all factors disappear, and we get the uniform distribution on the interlacing array. This is the *uniform Gibbs property* which is important for other models, including discrete ensembles.

2 Loop equations

Let us write down the *loop equations* for the passage from the eigenvalues λ to the eigenvalues μ. These loop equations are due to [GH24] by a limit from a discrete system (related to Jack symmetric polynomials). Note that despite the name, these are not **equations**, but rather a statement that some expectations are holomorphic. We stick to the random matrix setting, and present a formulation and a proof given by [Gor25].

2.1 Formulation

Theorem 2.1. *We fix* $n = 1, 2, \ldots$ *and* $n+1$ *real numbers* $\lambda_1 \ge \cdots \ge \lambda_{n+1}$*. For* $\beta > 0$*, consider* $n+1$ *i.i.d.* χ^2_β *random variables* ξ_i *and set*

$$w_i = \frac{\xi_i}{\sum_{j=1}^{n+1} \xi_j}, \qquad 1 \le i \le n+1.$$

We define n *random points* $\{\mu_1, \ldots, \mu_n\}$ *as* n *solutions to the equation*

$$\sum_{i=1}^{n+1} \frac{w_i}{z - \lambda_i} = 0. \tag{2.1}$$

Take any polynomial $W(z)$ *and consider the complex function:*

$$f_W(z) = \mathbb{E}\left[\prod_{j=1}^{n} \exp\bigl(W(\mu_j)\bigr) \frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} \left(W'(z) + \sum_{i=1}^{n+1} \frac{\beta/2-1}{z-\lambda_i} + \sum_{j=1}^{n} \frac{1}{z-\mu_j}\right)\right]. \tag{2.2}$$

Then $f_W(z)$ *is an* entire function *of* z*, in the following sense:*

- *For* $z \in \mathbb{C} \setminus [\lambda_{n+1}, \lambda_1]$*, the expectation in* (2.2) *defines a holomorphic function of* z*.*
- *This function has an analytic continuation to* $\mathbb{C}$*, which has no singularities.*

Remark 2.2. Note that for z in $[\lambda_{n+1}, \lambda_1]$, the integral determining (2.2) might be divergent, and, therefore, analytic continuation is the proper way to define $f_W(z)$, $z \in [\lambda_{n+1}, \lambda_1]$.

Corollary 2.3. *We have*

$$f_0(z) = \frac{(n+1)\beta}{2} - 1.$$

Here f_0 *means* f_W *with* $W \equiv 0$*.*

Proof. This is obtained by sending $z \to \infty$ in (2.2). □

2.2 Proof of Theorem 2.1 for $\beta > 2$

Theorem 2.1 remains valid for $\beta > 0$, but we only prove it for $\beta > 2$ here. We also assume that $\lambda_1 > \ldots > \lambda_n$.

We begin by observing that for $z \in \mathbb{C} \setminus [\lambda_{n+1}, \lambda_1]$, the expectation in (2.2) is well-defined and holomorphic in z. This follows since for such z, the denominators $z - \lambda_i$ and $z - \mu_j$ are bounded away from zero with probability 1. The key challenge is to show that $f_W(z)$ can be analytically continued to an entire function. Potential singularities of $f_W(z)$ are inside the intervals $(\lambda_{i+1}, \lambda_1)$. We will show that these singularities do not actually occur.

Consider a specific interval (λ_2, λ_1). We need to show that $f_W(z)$ has no singularities in this interval. From Theorem 1.2, the probability distribution of $\mu = (\mu_1, \ldots, \mu_n)$ has density proportional to:

$$\prod_{1 \le i < j \le n-1} (\mu_i - \mu_j) \prod_{i=1}^{n} \prod_{j=1}^{n+1} |\mu_i - \lambda_j|^{\beta/2-1}.$$

Let us analyze the function in (2.2). For $z \in (\lambda_2, \lambda_1)$, we need to demonstrate that the expectation

$$\mathbb{E}\left[\prod_{j=1}^{n} \exp\bigl(W(\mu_j)\bigr) \frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} \left(W'(z) + \sum_{i=1}^{n+1} \frac{\beta/2-1}{z-\lambda_i} + \sum_{j=1}^{n} \frac{1}{z-\mu_j}\right)\right]$$

is holomorphic. This expectation is an $(n-1)$-fold integral over $\mu_1, \ldots, \mu_n$. For $z \in (\lambda_2, \lambda_1)$, we will show that the one-dimensional integral over μ_1 is already holomorphic, and the remaining integrals are over domains which do not encounter singularities in z. We need to consider the integral

$$\int_{\lambda_2}^{\lambda_1} \prod_{i<j}(\mu_i - \mu_j) \prod\prod(\mu_j - \lambda_i)^{\beta/2-1} \prod e^{W(\mu_j)} \frac{\prod(z-\lambda_i)}{\prod(z-\mu_j)} \times \left(W'(z) + \sum \frac{\beta/2-1}{z-\lambda_i} + \sum \frac{1}{z-\mu_j}\right) d\mu_2. \tag{2.3}$$

Note that (here we are using the fact that $\beta > 2$)

$$0 = \int_{\lambda_2}^{\lambda_1} d\mu_1 \frac{\partial}{\partial \mu_1} \left(\underbrace{\prod_{i<j}(\mu_i - \mu_j) \prod\prod(\mu_j - \lambda_i)^{\beta/2-1} \prod e^{W(\mu_j)} \frac{\prod(z-\lambda_i)}{\prod(z-\mu_j)}}_{(*)} \right)$$
$$= \int_{\lambda_2}^{\lambda_1} d\mu_1 (*) \cdot \left[\sum_{j=2}^{n} \frac{1}{\mu_1 - \mu_j} + \sum_{i=1}^{n+1} \frac{\beta/2-1}{\mu_1 - \lambda_i} + W'(\mu_1) + \frac{1}{z-\mu_1} \right].$$

Subtracting this expression from our original integral (2.3) and noting that

$$\left(W'(z) + \sum \frac{\beta/2-1}{z-\lambda_i} + \sum \frac{1}{z-\mu_j}\right) - \left(\sum_{j=2}^{n} \frac{1}{\mu_1 - \mu_j} + \sum_{i=1}^{n+1} \frac{\beta/2-1}{\mu_1 - \lambda_i} + W'(\mu_1) + \frac{1}{z-\mu_1} \right)$$

has zero at $z = \mu_1$, we conclude that our integral has no singularity at μ_1, and therefore no singularities in the $[\lambda_2, \lambda_1]$ interval. This completes the proof of Theorem 2.1 for $\beta > 2$.

3 Applications of loop equations

The loop equations provide a powerful tool for analyzing the spectral properties of random matrices through their eigenvalue distributions. Let us derive an equation for the Stieltjes transform of the empirical measures.

3.1 Stieltjes transform equations

Starting from Theorem 2.1 with $W = 0$, we have:

$$\mathbb{E}\left[\frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} \left(\sum_{i=1}^{n+1} \frac{\beta/2-1}{z-\lambda_i} + \sum_{j=1}^{n} \frac{1}{z-\mu_j} \right) \right] = \frac{(n+1)\beta}{2} - 1. \tag{3.1}$$

Let us introduce the empirical Stieltjes transforms:

$$G_\lambda(z) = \frac{1}{n} \sum_{i=1}^{n+1} \frac{1}{z-\lambda_i},$$

$$G_\mu(z) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{z-\mu_j}.$$

We also define the logarithmic potentials (indefinite integrals of the Stieltjes transforms):

$$\int G_\lambda(z)dz = \frac{1}{n}\sum_{i=1}^{n+1}\ln(z-\lambda_i),$$
$$\int G_\mu(z)dz = \frac{1}{n}\sum_{j=1}^{n}\ln(z-\mu_j).$$

Noting that

$$\frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} = \exp\left(n\left(\int G_\lambda(z) - \int G_\mu(z)\right)\right),$$

we can rewrite equation (3.1) as:

$$\mathbb{E}\left[\exp\left(n\left(\int G_\lambda(z)\,dz - \int G_\mu(z)\,dz\right)\right)\left(\left(\frac{\beta}{2}-1\right)G_\lambda(z) + G_\mu(z)\right)\right] = \frac{\beta}{2} + \frac{1}{n}\left(\frac{\beta}{2}-1\right). \tag{3.2}$$

3.2 Asymptotic behavior

Equation (3.2) can be reinterpreted in terms of a time evolution of eigenvalue distributions. This perspective offers significant insights into the asymptotic behavior of the corners process.

If we think of λ as configuration at time $t = 1$ and μ as configuration at time $t = 1 - \frac{1}{n}$, then denoting the general time parameter as t and setting $G_\lambda = G_1$, $G_\mu = G_{1-\frac{1}{n}}$, we obtain a continuous time evolution of Stieltjes transforms. (And similarly for all t, of course.)

As $n \to \infty$, equation (3.2) transforms into:

$$\frac{\beta}{2}\exp\left(\frac{\partial}{\partial t}\int G_t(z)\,dz\right)\cdot G_t(z) = \frac{\beta}{2}.$$

This implies

$$\frac{\partial}{\partial t}\int G_t(z)\,dz + \ln G_t(z) = 0.$$

Taking the derivative with respect to z, we get:

$$\frac{\partial}{\partial t}G_t(z) + \frac{1}{G_t(z)}\frac{\partial}{\partial z}G_t(z) = 0. \tag{3.3}$$

This is precisely the inviscid Burgers equation, a fundamental nonlinear PDE in fluid dynamics. The appearance of this equation indicates that the eigenvalue distributions evolve according to a hydrodynamic flow as we move through the corners of the random matrix from full size down to zero.

Remark 3.1. We see that the Burgers equation (3.3) does not depend on β, which is expected. Indeed, for example, GβE eigenvalues have the same Wigner semicircle law as $\beta = 2$, up to an overall rescaling.

3.3 Example with semicircle law

The Stieltjes transform of the semicircular law is given by:

$$G(z) = \int_{-2}^{2} \frac{1}{z-x} \frac{\sqrt{4-x^2}}{2\pi} dx = \frac{1}{2}\left(z - \sqrt{z^2-4}\right).$$

Let us define

$$G^{(c)}(z) := \frac{z - \sqrt{z^2 - 4c^2}}{2c^2},$$

this is the Stieltjes transform of the semicircular law on $[-2c, 2c]$.

H Problems (due 2025-03-25)

H.1 Cauchy determinant

Prove the Cauchy determinant formula:

$$\det\left(\frac{1}{x_i - y_j}\right)_{1 \le i,j \le n} = \frac{\prod_{i<j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(x_i - y_j)}.$$

H.2 Jacobian from $n-1$ to n dependent variables

Explain how the factor $\prod_{i=1}^{n-1}\prod_{j=1}^{n} |\mu_i - \lambda_j|$ appears from the Jacobian of the transformation from φ to μ (1.2), even though $\partial\varphi_a/\partial\mu_b$ is defined for $a = 1, \ldots, n$, $b = 1, \ldots, n-1$, but the φ_i's are not independent.

H.3 Dirichlet density

Find in the literature or prove on your own the first statement in the proof of Theorem 1.2 about the symmetric Dirichlet density arising from normalizing the ξ_i's to φ_i's.

H.4 General β Corners Process Simulation

This problem explores computational aspects of the general β corners process.

(a) Write code for generating a sample from the distribution of $\mu = (\mu_1, \ldots, \mu_{n-1})$ given $\lambda = (\lambda_1, \ldots, \lambda_n)$ for arbitrary $\beta > 0$, using the polynomial equation characterization.

(b) Let $\lambda = (n, n-1, \ldots, 2, 1)$. For $n = 7$, compute (numerically) the expected values $\mathbb{E}[\mu_i]$ for each i, when $\beta = 1, 2, 4$, and 10. Describe the behavior as β increases.

References

[GH24] V. Gorin and J. Huang, *Dynamical loop equation*, Ann. Probab. **52** (2024), no. 5, 1758–1863. arXiv:2205.15785 [math.PR]. ↑3

[Gor25] V. Gorin, *Private communication*, 2025. ↑3

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