Lectures on Random Matrices (Spring 2025)
Lecture 14: Matching Random Matrices to Random Growth II

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1 Recap

1.1 Main goal

In the previous Lecture 13, we began establishing a remarkable correspondence between two a priori different objects:

- The *spiked Wishart ensemble*: an $n \times n$ Hermitian random-matrix process $\{M(t)\}_{t \geq 0}$ whose entries come from columns of independent Gaussian random vectors of suitably chosen covariance.
- An *inhomogeneous last-passage percolation (LPP)* model: an array $\{W_{i,j}\}$ of exponential random weights on a portion of the two-dimensional lattice, whose last-passage times $L(t, n)$ match the largest eigenvalues of $M(t)$, jointly for all $t \in \mathbb{Z}_{\geq 0}$.

This equivalence, originally due to [DW08] (following [Def10], [FR06]; see also [Bar01], [Joh00] for earlier results of this kind), can be fully understood by passing to a *discrete* version of LPP with geometric site-weights and then applying the *Robinson–Schensted–Knuth* (RSK) correspondence.

*`Course webpage` • `Live simulations` • `TeX Source` • Updated at 10:55, Tuesday 15th April, 2025

1.2 Spiked Wishart ensembles and the largest eigenvalue process

We defined the *generalized* (or spiked) Wishart matrix $M(t)$ of size $n \times n$ by setting

$$M(t) \;=\; \sum_{m=1}^{t} A^{(m)}\big(A^{(m)}\big)^*$$

where $\{A^{(m)}\}_{m=1}^{\infty}$ are i.i.d. complex Gaussian column vectors of length n, with

$$\mathrm{Var}\big(A_i^{(m)}\big) \;=\; \frac{1}{\pi_i + \hat{\pi}_m}\,.$$

Here, $\pi = (\pi_1, \ldots, \pi_n)$ and $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \ldots)$ are positive and nonnegative parameters, respectively. Writing $\lambda_1(t) \ge \cdots \ge \lambda_n(t) \ge 0$ for the eigenvalues of $M(t)$, we then saw:

1. The vectors $\lambda(t) = \big(\lambda_1(t), \ldots, \lambda_n(t)\big)$ form a Markov chain in the *Weyl chamber* $\mathbb{W}^n = \{x_1 \ge \cdots \ge x_n \ge 0\}$.
2. There is an *interlacing* property: each update $M(t-1) \mapsto M(t)$ via the rank-one matrix $A^{(t)}\big(A^{(t)}\big)^*$ forces $\lambda(t)$ to interlace with $\lambda(t-1)$:

$$\lambda_1(t) \;\ge\; \lambda_1(t-1) \;\ge\; \lambda_2(t) \;\ge\; \cdots \;\ge\; \lambda_n(t-1) \;\ge\; \lambda_n(t).$$

In Lecture 13, we wrote down the transition kernel from $\lambda(t-1)$ to $\lambda(t)$:

Theorem 1.1 ([DW08])**.** *Fix an integer $n \ge 1$. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a strictly positive n-vector, and let $\widehat{\pi} = (\widehat{\pi}_1, \widehat{\pi}_2, \ldots)$ be any sequence of nonnegative real parameters. Under the probability measure $P^{\pi,\widehat{\pi}}$, the eigenvalues of the $n \times n$ generalized Wishart matrices $\{M(t)\}_{t\ge 0}$ form a time-inhomogeneous Markov chain $\{\mathrm{sp}(M(t))\}_{t\ge 0}$ in the Weyl chamber*

$$\mathbb{W}^n \;=\; \big\{\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\ge 0} : x_1 \ge x_2 \ge \cdots \ge x_n \big\}.$$

More precisely, writing $x = \mathrm{sp}(M(t-1))$ and $y = \mathrm{sp}(M(t))$, the one-step transition law from time $(t-1)$ to t is absolutely continuous on the interior of $\mathbb{W}^n$ and can be factored as

$$Q^{\pi,\widehat{\pi}}_{t-1,t}(x,\, dy) \;=\; \Big[\prod_{i=1}^{n}(\pi_i + \widehat{\pi}_t)\Big] \cdot \frac{h_\pi(y)}{h_\pi(x)}\; \exp\Big(-(\widehat{\pi}_t - 1)\sum_{i=1}^{n}(y_i - x_i)\Big) \;\times\; Q^{(0)}(x,\, dy), \tag{1.1}$$

where

- *$Q^{(0)}\big(x,\, dy\big)$ is the* standard *(null-spike) Wishart transition kernel, given explicitly by*

$$Q^{(0)}(x,\, dy) \;=\; \frac{\Delta(y)}{\Delta(x)}\; \exp\Big(-\sum_{i=1}^{n}(y_i - x_i)\Big)\, \mathbf{1}_{\{x \prec y\}}\, dy, \tag{1.2}$$

  with $\Delta(z) = \prod_{1\le i<j\le n}(z_i - z_j)$ the Vandermonde determinant.

- *The function h_π is the (continuous) Harish-Chandra orbit integral factor*

$$h_\pi(z) \;=\; \frac{(-1)^{\binom{n}{2}}}{0!1!\cdots(n-1)!}\frac{\det\big(e^{-\pi_i\, z_j}\big)_{i,j=1}^{n}}{\Delta(\pi)\,\Delta(z)}.$$

  Note that $h_\pi(0) = 1$.

In particular, the chain starts from $\mathrm{sp}(M(0)) = 0$ (the zero matrix).

1.3 Inhomogeneous last-passage percolation

On the random growth side, we considered an array of site-weights $\{W_{i,j}\}_{i,j\geq 1}$ such that each $W_{i,j}$ is exponentially distributed with rate $\pi_i + \hat{\pi}_j$. For every integer $t \geq 1$, we define $L(t,n)$ to be the maximum total weight of all up-right paths from $(1,1)$ to (t,n):

$$L(t,n) \;=\; \max_{\Gamma\colon (1,1)\to(t,n)} \sum_{(i,j)\in\Gamma} W_{i,j}.$$

One checks that $L(\,\cdot\,,n)$ satisfies a simple additive recursion:

$$L(i,j) \;=\; W_{i,j} \;+\; \max\bigl\{\, L(i-1,j),\; L(i,j-1)\bigr\},$$

The main claim which we show in today's lecture is the equality in distribution:

$$\bigl(L(1,n),\, L(2,n),\, \dots,\, L(t,n)\bigr) \;\stackrel{d}{=}\; \bigl(\lambda_1(1),\, \lambda_1(2),\, \dots,\, \lambda_1(t)\bigr).$$

1.4 RSK via toggles: definitions and weight preservation

The *Robinson–Schensted–Knuth* correspondence (RSK) was the main new mechanism in Lecture 13. In our setup, we adopt a *toggle-based* viewpoint: we encode arrays by diagonals and successively *toggle* the diagonals to achieve a fully *ordered* array R. Concretely:

Definition 1.2 (Nonnegative and ordered arrays)**.** For integers $t, n \geq 1$:

- A *nonnegative array* W is a collection of integers $W_{i,j} \in \mathbb{Z}_{\geq 0}$, $1 \leq i \leq t$, $1 \leq j \leq n$.
- An *ordered array* R (of the same size) satisfies $R_{i,j} \leq R_{i,j+1}$ and $R_{i,j} \leq R_{i+1,j}$ for all valid i, j.

Definition 1.3 (Toggle operation)**.** Let $w \in \mathbb{Z}_{\geq 0}$ and let (λ, κ, μ) be three interlacing sequences of nonnegative integers, symbolically written as $\lambda \succ \kappa \prec \mu$. Then

$$\mathrm{T}\bigl(w;\, \lambda, \kappa, \mu\bigr) \;=\; (\lambda, \nu, \mu)$$

is defined by leaving λ, μ unchanged and setting

$$\nu_1 \;=\; w + \max(\lambda_1,\, \mu_1), \qquad \nu_i \;=\; \max(\lambda_i,\, \mu_i) \;+\; \min(\lambda_{i-1},\, \mu_{i-1}) \;-\; \kappa_{i-1}, \quad i \geq 2.$$

From a straightforward check (Problem 13.7), toggling preserves total weights in a precise sense, and we always end up with $\lambda \prec \nu \succ \mu$.

Theorem 1.4 (RSK is a bijection, cf. Lecture 13)**.** *Given a nonnegative array W of size $t \times n$, the* RSK map *outputs an ordered array $R = \mathrm{RSK}(W)$ by the following procedure:*

- *Process the cells (i,j) of W in an arbitrary order (e.g. row by row from bottom to top).*
- *For each cell (i,j),* toggle *the diagonal containing that cell in the partial R, inserting weight $w = W_{i,j}$.*

All toggles commute on different diagonals, so the final ordered array R does not depend on the insertion order. Moreover, $W \mapsto R$ is a bijection between nonnegative arrays and ordered arrays.

The key to how RSK links LPP and random matrices is its *weight preservation* property, which we restate in a concise form here:

Theorem 1.5 (Weight preservation, cf. Proposition 13.25)**.** *Let $W = \{W_{i,j}\}$ be a nonnegative integer array, and $R = \mathrm{RSK}(W)$. Denote*

$$\mathrm{row}_i \;=\; \sum_{j=1}^{n} W_{i,j}, \quad \mathrm{col}_j \;=\; \sum_{i=1}^{t} W_{i,j},$$

and for R define the diagonal sums

$$\mathrm{diag}_{i,j} \;=\; \sum_{k=0}^{\min(i,j)-1} R_{i-k,\,j-k}.$$

Then for each $1 \le j \le n$ and $1 \le i \le t$, we have

$$\mathrm{diag}_{t,j} \;=\; \sum_{m=1}^{j} \mathrm{col}_m, \qquad \mathrm{diag}_{i,n} \;=\; \sum_{m=1}^{i} \mathrm{row}_m, \tag{1.3}$$

ensuring that the total sum of W over all cells equals the total sum of R over all cells.

Proof (sketch). One inductively builds R by adding the sites (i,j) one at a time. Each toggle modifies exactly one diagonal and preserves an inclusion–exclusion count on neighboring diagonals. Concretely, after adding a box (i,j), the diagonal-sum identity

$$\mathrm{diag}_{i,j} \;=\; \mathrm{diag}_{i-1,j} + \mathrm{diag}_{i,j-1} \;-\; \mathrm{diag}_{i-1,j-1} \;+\; W_{i,j}$$

holds, expressing that R captures the discrete "second difference" of W. Since toggles commute on disjoint diagonals, the partial sums assemble to match the row and column sums of W regardless of the order of addition. $\square$

Thus, applying RSK to random arrays W (in particular, to a geometric LPP environment) yields an ordered array R whose interlacing diagonals reflect precisely the combinatorial structure of the LPP. By interpreting each diagonal as encoding eigenvalue increments, one connects R to the same interlacing patterns arising in Hermitian random matrices of $\beta = 2$ type. This observation is what ultimately shows the distributional identity between $\big(L(1,n),\dots,L(t,n)\big)$ and $\big(\lambda_1(1),\dots,\lambda_1(t)\big)$ under appropriate limiting and deformation parameters.

Outline of Next Steps

In the upcoming lecture, we will:

- Translate the RSK interlacing arrays directly into a form resembling eigenvalue distributions.

- Show how the parameter choices $(\pi, \hat{\pi})$ in the geometric version correspond to the spiked Wishart setup.
- Conclude the proof of Theorem 13.10 (the exact matching of the largest eigenvalues of spiked Wishart and the last-passage times in the exponential LPP).

These steps will complete our new perspective on why matrix spectra in the Wishart class align so precisely with the maximum-weight growth in an LPP model.

N Problems (due 2025-04-29)

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