

# Lectures on Random Matrices (Spring 2025)

## Lecture 9: Loop equations and asymptotics

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### 1 Recap

#### 1.1 (Dynamical) loop equations

**Theorem 1.1.** We fix  $n = 1, 2, \dots$  and  $n+1$  real numbers  $\lambda_1 \geq \dots \geq \lambda_{n+1}$ . For  $\beta > 0$ , consider  $n+1$  i.i.d.  $\chi^2_\beta$  random variables  $\xi_i$  and set

$$w_i = \frac{\xi_i}{\sum_{j=1}^{n+1} \xi_j}, \quad 1 \leq i \leq n+1.$$

We define  $n$  random points  $\{\mu_1, \dots, \mu_n\}$  as  $n$  solutions to the equation

$$\sum_{i=1}^{n+1} \frac{w_i}{z - \lambda_i} = 0. \quad (1.1)$$

Take any polynomial  $W(z)$  and consider the complex function:

$$f_W(z) = \mathbb{E} \left[ \prod_{j=1}^n \exp(W(\mu_j)) \frac{\prod_{i=1}^{n+1} (z - \lambda_i)}{\prod_{j=1}^n (z - \mu_j)} \left( W'(z) + \sum_{i=1}^{n+1} \frac{\beta/2 - 1}{z - \lambda_i} + \sum_{j=1}^n \frac{1}{z - \mu_j} \right) \right]. \quad (1.2)$$

Then  $f_W(z)$  is an entire function of  $z$ , in the following sense:

- For  $z \in \mathbb{C} \setminus [\lambda_{n+1}, \lambda_1]$ , the expectation in (1.2) defines a holomorphic function of  $z$ .
- This function has an analytic continuation to  $\mathbb{C}$ , which has no singularities.

#### 1.2 Loop equations for $W = 0$

When  $W = 0$ , the loop equation (1.2) becomes

$$f_0(z) = \frac{(n+1)\beta}{2} - 1,$$

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so

$$\mathbb{E} \left[ \frac{\prod_{i=1}^{n+1} (z - \lambda_i)}{\prod_{j=1}^n (z - \mu_j)} \left( \sum_{i=1}^{n+1} \frac{\beta/2 - 1}{z - \lambda_i} + \sum_{j=1}^n \frac{1}{z - \mu_j} \right) \right] = \frac{(n+1)\beta}{2} - 1.$$

Recall that we defined

$$G_\lambda(z) = \frac{1}{n} \sum_{i=1}^{n+1} \frac{1}{z - \lambda_i}, \quad G_\mu(z) = \frac{1}{n} \sum_{j=1}^n \frac{1}{z - \mu_j}.$$

We also define the “logarithmic potentials” (indefinite integrals of the Stieltjes transforms):

$$\int G_\lambda(z) dz = \frac{1}{n} \sum_{i=1}^{n+1} \ln(z - \lambda_i), \quad \int G_\mu(z) dz = \frac{1}{n} \sum_{j=1}^n \ln(z - \mu_j).$$

We understand the integrals up to the same integration constant (and branch), so the exponent of the difference yields the original product:

$$\frac{\prod_{i=1}^{n+1} (z - \lambda_i)}{\prod_{j=1}^n (z - \mu_j)} = \exp \left( n \left( \int G_\lambda(z) - \int G_\mu(z) \right) \right)$$

We can rewrite the loop equation as:

$$\mathbb{E} \left[ \exp \left( n \left( \int G_\lambda(z) dz - \int G_\mu(z) dz \right) \right) \left( \left( \frac{\beta}{2} - 1 \right) G_\lambda(z) + G_\mu(z) \right) \right] = \frac{\beta}{2} + \frac{1}{n} \left( \frac{\beta}{2} - 1 \right). \quad (1.3)$$

### 1.3 The full corners process

Assume  $n$  is going to infinity, and we fix a sequence of top-level eigenvalues  $\lambda_j^{(n)}$ ,  $1 \leq j \leq n$ , growing in some way. This sequence can be random (like  $G\beta E$  rescaled to have eigenvalues in a bounded interval) or deterministic (for example,  $\lambda^{(n)}$  has  $n/10$  points at 0,  $n/10$  points at 1, and  $8n/10$  points at 2, see Figure 1).

Denote the eigenvalues of the  $k \times k$  beta corner (that is, obtained by successively solving the polynomial equation (1.1)  $n - k$  times) by  $\lambda_j^{(k)}$ ,  $1 \leq j \leq k$ . As  $n \rightarrow \infty$ , we postulate that

The empirical distribution of  $\lambda_j^{(k)}$  converges to some deterministic probability measure  $\mathbf{m}_t$ , where  $k/n \rightarrow t \in [0, 1]$ . Consequently, the Stieltjes transform  $G_{\lambda^{(k)}}(z)$  converges to  $G_t(z)$ , for  $z$  in a complex domain outside of the support of  $\mathbf{m}_t$ .

Note that we do not assume the scaling of the  $\lambda_j^{(k)}$ 's, for convenience.

Denote by  $G_t(z) = \int_{\mathbb{R}} \frac{\mathbf{m}_t(dx)}{z - x}$  the Stieltjes transform of the measure  $\mathbf{m}_t$ .

**Proposition 1.2.** *The functions  $G_t(z)$  satisfy the complex Burgers equation*

$$\frac{\partial}{\partial t} G_t(z) + \frac{1}{G_t(z)} \frac{\partial}{\partial z} G_t(z) = 0.$$

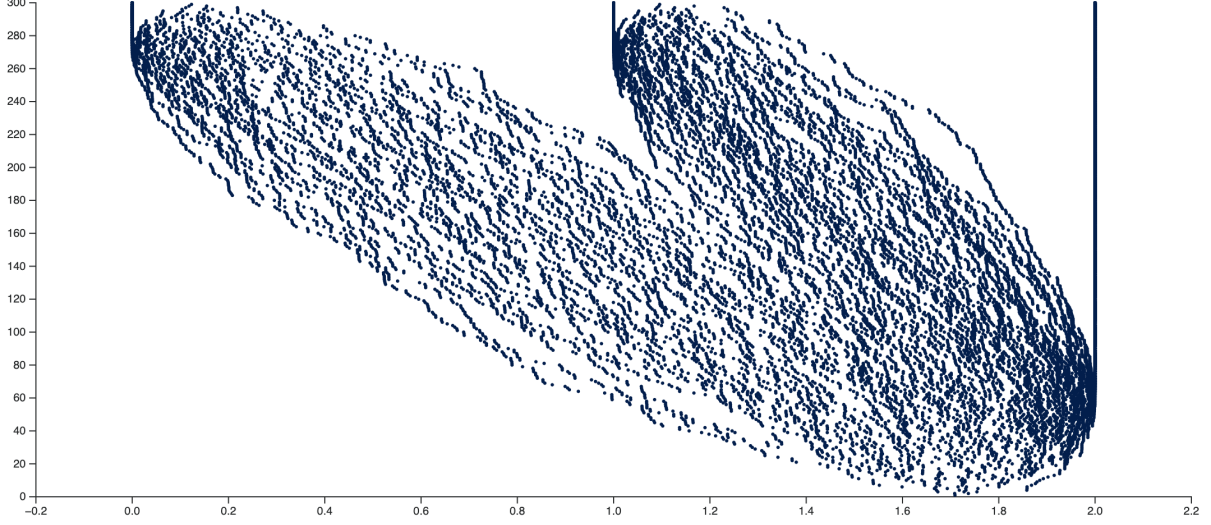


Figure 1: Corners process for  $n = 300$ ,  $\beta = 1$ , with  $n/10$  points at 0,  $n/10$  points at 1, and  $8n/10$  points at 2 on the top level.

*Proof.* We have in (1.3), if  $\lambda$  and  $\mu$  live on levels  $t$  and  $t - \frac{1}{n}$ , respectively:

$$G_\lambda(z) - G_\mu(z) \approx \frac{1}{n} \frac{\partial}{\partial t} G_t(z), \quad \left( \frac{\beta}{2} - 1 \right) G_\lambda(z) + G_\mu(z) \approx \frac{\beta}{2} G_t(z) - \frac{1}{n} \frac{\partial}{\partial t} G_t(z) \approx \frac{\beta}{2} G_t(z).$$

Due to the concentration assumption, we can ignore the expectation. Then, taking the logarithm of (1.3), and differentiating with respect to  $z$ , we get the Burgers equation.  $\square$

#### 1.4 Example: $G\beta E$ and the semicircle law

The Stieltjes transform of the semicircular law is given by:

$$G(z) = \int_{-2}^2 \frac{1}{z-x} \frac{\sqrt{4-x^2}}{2\pi} dx = \frac{1}{2} \left( z - \sqrt{z^2 - 4} \right).$$

We take this as the function  $G_t(z)$  for  $t = 1$ . Then, for each  $0 \leq t \leq 1$ , the  $G\beta E$  solution should be

$$\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{z - \lambda_i^{(\lfloor nt \rfloor)}} \rightarrow t G^{(\sqrt{t})}(z),$$

where

$$G^{(c)}(z) := \frac{z - \sqrt{z^2 - 4c^2}}{2c^2},$$

is the Stieltjes transform of the semicircular law on  $[-2c, 2c]$ .

**Lemma 1.3.** *The function  $G_t(z) := t G^{(\sqrt{t})}(z)$  satisfies the Burgers equation.*

*Proof.* Straightforward verification.  $\square$

## 2 Gaussian Free Field

The *Gaussian Free Field* (GFF) is a fundamental object in probability theory and mathematical physics. Roughly speaking, it can be viewed as a multi-dimensional analog of Brownian motion: instead of one-dimensional “time,” the underlying parameter space is a multi-dimensional domain (often two-dimensional). In one dimension, the GFF reduces to an ordinary Brownian bridge (or motion). In higher dimensions, it becomes a random generalized function (a “distribution”) whose covariance structure is governed by the appropriate Green’s function of the Laplacian. Below we provide an introduction, starting from finite-dimensional Gaussian vectors and culminating in the GFF as a random distribution.

### 2.1 Gaussian correlated vectors and random fields

Recall that an  $n$ -dimensional real-valued random vector  $X = (X_1, \dots, X_n)$  is called *Gaussian* if every linear combination

$$\alpha_1 X_1 + \dots + \alpha_n X_n$$

of its components is a univariate Gaussian random variable. The law of such a vector is completely determined by its mean vector  $m \in \mathbb{R}^n$  and its covariance matrix  $\Sigma \in \mathbb{R}^{n \times n}$ . The density function, for invertible  $\Sigma$ , is

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2} (x - m)^\top \Sigma^{-1} (x - m)\right).$$

For simplicity, we will assume that  $m = 0$  (the centered case).

### 2.2 Gaussian fields as random generalized functions

A natural extension from finite-dimensional Gaussian vectors to infinite-dimensional settings leads us to Gaussian fields. Informally, a Gaussian field is a collection of Gaussian random variables indexed by points in some space.

For a domain  $D \subset \mathbb{R}^d$ , we might wish to define a random function  $\Phi : D \rightarrow \mathbb{R}$  such that for any finite collection of points  $x_1, \dots, x_n \in q$ , the vector  $(\Phi(x_1), \dots, \Phi(x_n))$  is a Gaussian vector. However, such a random function may not exist as a proper function in the usual sense. The reason is that we would like to consider analogues of linear combinations of the form

$$\Phi(f) = \int_D \Phi(x) f(x) dx, \tag{2.1}$$

For example, if we wish the vector  $(\Phi(x_1), \dots, \Phi(x_n))$  to have independent components, we would need to assign a value to each point in  $D$ . This means that the hypothetical function  $\Phi$  would be too irregular, and even non-measurable, and the integral (2.1) would not be well-defined.

Instead, for the field with independent values at all points, we would like  $\Phi(f)$  to be normal with mean zero and variance (paralleling the finite-dimensional story)

$$\text{Var}(\Phi(f)) = \|f\|_{L^2(D)}^2 = \int_D f(x)^2 dx.$$

So, Gaussian fields (in particular, our topic, the *Gaussian Free Field*) are defined as random distributions, not as functions. That is, rather than assigning a value to each point, we assign a random value to each test function  $f$  in some appropriate space via (2.1).

The covariance structure of the mean zero Gaussian random variables  $\Phi(f_1), \dots, \Phi(f_n)$  is given by a certain bilinear form determined by the domain  $D$ .

### 2.3 Concrete treatment via orthogonal functions

Let us now construct the Gaussian Free Field more concretely. Consider a bounded domain  $D \subset \mathbb{R}^d$  with smooth boundary. Let  $\{f_n\}_{n=1}^\infty$  be an orthonormal basis of  $L^2(D)$  consisting of eigenfunctions of the Laplacian with Dirichlet boundary conditions:

$$\begin{cases} -\Delta f_n = \lambda_n f_n & \text{in } D, \\ f_n = 0 & \text{on } \partial D, \end{cases} \quad (2.2)$$

where  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  are the corresponding eigenvalues.

We can now define the Gaussian Free Field on  $D$  as:

$$\Phi = \sum_{n=1}^{\infty} \frac{\alpha_n}{\sqrt{\lambda_n}} f_n, \quad (2.3)$$

where  $\{\alpha_n\}_{n=1}^\infty$  are independent standard Gaussian random variables. This series does not converge pointwise, but it does converge in the space of distributions almost surely.

For any test function  $g \in C_0^\infty(D)$ , we have:

$$\Phi(g) = \int_D \Phi(x)g(x) dx = \sum_{n=1}^{\infty} \frac{\alpha_n}{\sqrt{\lambda_n}} \int_D f_n(x)g(x) dx, \quad (2.4)$$

which is a well-defined Gaussian random variable.

### 2.4 Connection to Brownian bridge

The Gaussian Free Field in one dimension is closely related to the Brownian bridge. Consider the interval  $[0, 1]$  with the Dirichlet Laplacian. The eigenfunctions are  $f_n(x) = \sqrt{2} \sin(n\pi x)$  with eigenvalues  $\lambda_n = n^2\pi^2$ . The Gaussian Free Field on  $[0, 1]$  can be expressed as:

$$\Phi(x) = \sqrt{2} \sum_{n=1}^{\infty} \frac{\alpha_n}{n\pi} \sin(n\pi x). \quad (2.5)$$

This series representation converges to a continuous function, which is precisely the Brownian bridge on  $[0, 1]$ . The Brownian bridge is a Gaussian process  $B_t$  with mean zero and covariance function:

$$\mathbb{E}[B_s B_t] = \min(s, t) - st. \quad (2.6)$$

The key difference between the one-dimensional and higher-dimensional cases is that in one dimension, the Gaussian Free Field is a continuous function, whereas in dimensions two and higher, it is a genuine distribution (not a function). This reflects the fact that Brownian motion is a continuous path in one dimension but becomes increasingly irregular in higher dimensions.

## 2.5 Covariance structure and Green's function

The covariance structure of the Gaussian Free Field is intimately connected to the Green's function of the Laplacian. For test functions  $f, g \in C_0^\infty(D)$ , we have:

$$\mathbb{E}[\Phi(f)\Phi(g)] = \mathbb{E}\left[\sum_{n,m=1}^{\infty} \frac{\alpha_n \alpha_m}{\sqrt{\lambda_n \lambda_m}} \int_D f_n(x) f(x) dx \int_D f_m(y) g(y) dy\right] \quad (2.7)$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_D f_n(x) f(x) dx \int_D f_n(y) g(y) dy. \quad (2.8)$$

Define the Green's function  $G_D(x, y)$  for the Dirichlet Laplacian on  $D$  as the solution to:

$$\begin{cases} -\Delta_x G_D(x, y) = \delta(x - y) & \text{for } x, y \in D, \\ G_D(x, y) = 0 & \text{for } x \in \partial D \text{ or } y \in \partial D. \end{cases} \quad (2.9)$$

The Green's function has the eigenfunction expansion:

$$G_D(x, y) = \sum_{n=1}^{\infty} \frac{f_n(x) f_n(y)}{\lambda_n}. \quad (2.10)$$

Using this, we can rewrite the covariance as:

$$\mathbb{E}[\Phi(f)\Phi(g)] = \int_D \int_D G_D(x, y) f(x) g(y) dx dy. \quad (2.11)$$

This relationship between the covariance of the GFF and the Green's function is fundamental. It shows that the GFF can be viewed as a random solution to the equation  $-\Delta\Phi = W$ , where  $W$  is white noise. Here the white noise is the Gaussian field with covariance  $\delta(x - y)$  — the object which is the correct way of constructing a Gaussian field with i.i.d. values at all points.

## 2.6 The GFF on the upper half-plane

In the complex upper half-plane  $\{\text{Im } z > 0\}$  with  $\mathbb{R}$  as the boundary, the Green function has the form

$$G(z, w) = -\frac{1}{\pi} \ln |z - w| + \frac{1}{\pi} \ln |z - \bar{w}|.$$

The covariance is

$$\mathbb{E}[\Phi(f)\Phi(g)] = \int \int |dz|^2 |dw|^2 f(z) g(w) G(z, w).$$

here

## 3 Introduction to fluctuations

In the previous lecture, we derived the loop equations for corners processes and explored how they lead to deterministic limiting laws for the eigenvalue density. Today, we'll extend this investigation to study fluctuations around these limiting laws.

Recall that we established a law of large numbers for the eigenvalue density: as  $n \rightarrow \infty$ , the empirical density  $\rho(s; x(t/\varepsilon))$  converges in probability to a deterministic limit  $\rho_t(s)$ . This convergence can be formulated as:

$$\sup_{(t,x) \in P} |\varepsilon h(\varepsilon^{-1}t, \varepsilon^{-1}x) - h(t, x)| \rightarrow 0,$$

where  $h$  is the height function. However, this description doesn't capture the random fluctuations around the limiting shape. Intuitively, we expect these fluctuations to exist on a scale of  $\varepsilon^{1/2}$ , which is a common phenomenon in many random matrix and random tiling models.

Our goal is to understand the asymptotic behavior of the centered and rescaled height function

$$\sqrt{\pi}(h(\varepsilon^{-1}t, \varepsilon^{-1}x) - \mathbb{E}[h(\varepsilon^{-1}t, \varepsilon^{-1}x)]).$$

As we'll see, these fluctuations converge to a Gaussian random field, specifically the Gaussian Free Field with an appropriate complex structure.

## 4 The scale of fluctuations

Let's begin with a heuristic argument for why the fluctuations should scale with  $\varepsilon^{1/2}$ . Consider a single time-slice of our corners process. The empirical density at time  $t$  is defined as:

$$\rho(s; x(t)) = \frac{1}{\theta} \sum_{i=1}^n 1(\varepsilon x_i \leq s \leq \varepsilon x_i + \varepsilon \theta).$$

For large  $n$ , if we integrate this density against a smooth test function  $f$ , we get

$$\int_{l(t)}^{r(t)} f(s) \rho(s; x(t)) ds \approx \frac{1}{n} \sum_{i=1}^n f(\varepsilon x_i).$$

This is a sum of  $n$  terms, each of order  $1/n$ , and we can view it as an average of  $n$  weakly dependent random variables. By a central limit theorem type of argument, we expect that the fluctuations of this sum around its mean should be of order  $1/\sqrt{n} \approx \varepsilon^{1/2}$ .

This intuition aligns with the results from random matrix theory. For example, in Gaussian Unitary Ensembles and similar models, the centered linear statistics (integrals of the empirical spectral density against test functions) exhibit Gaussian fluctuations of order  $1/\sqrt{n}$ .

## 5 Dynamical loop equations for fluctuations

To analyze these fluctuations rigorously, we'll use the dynamical loop equations developed in the previous lecture. Recall that we derived the following loop equation:

$$\mathbb{E} \left[ \phi_+(z) \prod_{j=1}^n \frac{b(z + \theta) - b(x_j + \theta e_j)}{b(z) - b(x_j)} + \phi_-(z) \prod_{j=1}^n \frac{b(z) - b(x_j + \theta e_j)}{b(z) - b(x_j)} \right]$$

is a holomorphic function of  $z$ .

To study fluctuations, we introduce the random field:

$$G_t(z) = \int_{l(t)}^{r(t)} \frac{b'_t(z)\rho(s; x(t))}{b_t(z) - b_t(s)} ds - \mathbb{E} \left[ \int_{l(t)}^{r(t)} \frac{b'_t(z)\rho(s; x(t))}{b_t(z) - b_t(s)} ds \right].$$

This field  $G_t(z)$  captures the fluctuations of a modified Stieltjes transform of the empirical density. By analyzing the evolution of this field through the dynamical loop equations, we can derive its asymptotic behavior.

Let's introduce the martingale difference:

$$\Delta M_t(z) = \frac{1}{\varepsilon} [G_{t+\varepsilon}(z) - G_t(z) - \text{deterministic drift}].$$

Under appropriate conditions, these martingale differences converge to a Gaussian random field as  $\varepsilon \rightarrow 0$ . The covariance structure of this field is given by:

$$\mathbb{E}[\Delta M_t(z_1)\Delta M_t(z_2)] = \frac{1}{2\pi i\theta} \oint_{\omega_-} \frac{\tilde{f}_t(w)}{\tilde{f}_t(w) - 1} \frac{b'_t(w)b'_t(z_1)}{(b_t(w) - b_t(z_1))^2} \frac{b'_t(w)b'_t(z_2)}{(b_t(w) - b_t(z_2))^2} dw,$$

where  $\omega_-$  is a contour enclosing  $[l(t), r(t)]$  but not  $z_1$  or  $z_2$ , and  $\tilde{f}_t$  is related to the complex slope of the limiting shape.

## 6 Connection to the Gaussian Free Field

The fluctuations of the height function can be expressed in terms of the fluctuations of the integrated empirical density. For a test function  $f(x) = \partial_x[F(b_t(x))]$ , where  $F$  is analytic, we have:

$$\sqrt{\pi} \int_{l(t)}^{r(t)} f(x)(h(\varepsilon^{-1}t, \varepsilon^{-1}x) - \mathbb{E}[h(\varepsilon^{-1}t, \varepsilon^{-1}x)])dx = \frac{\varepsilon^{-1}}{2\pi i} \oint_{\Upsilon} F(b_t(z))G_{\varepsilon^{-1}t}(z)dz,$$

where  $\Upsilon$  is an appropriate contour enclosing  $[l(t), r(t)]$ .

One of the remarkable results is that these fluctuations converge to the Gaussian Free Field (GFF) in an appropriate complex structure. To make this precise, we introduce a bijection  $\Omega$  between the liquid region  $L(P)$  and the upper half-plane  $\mathbb{H}_+$ , defined via the non-real solutions to equation (19).

The main theorem states that inside the liquid region  $(t, x) \in L(P)$ , we have:

$$\lim_{\varepsilon \rightarrow 0} \sqrt{\pi}(h(\varepsilon^{-1}t, \varepsilon^{-1}x) - \mathbb{E}[h(\varepsilon^{-1}t, \varepsilon^{-1}x)]) = \bar{\Omega}\text{-pullback of GFF in } \mathbb{H}_+,$$

in the sense of convergence of joint moments for pairings with appropriate test measures.



## 7 Computation of the covariance structure

Let's now calculate the covariance structure of the limiting Gaussian field more explicitly. For test functions  $f_1, f_2$  satisfying the conditions above, the covariance between the corresponding linear statistics of the height function is given by:

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sqrt{\pi} \int f_1(h - \mathbb{E}[h]) \cdot \sqrt{\pi} \int f_2(h - \mathbb{E}[h]) \right] = -\frac{1}{4\pi} \iint_{C_1 \times C_2} \partial_{w_1} \partial_{w_2} \log(w_1 - w_2) F_1(w(t_1, w_1)) F_2(w(t_2, w_2)) dw$$

where  $C_1, C_2$  are appropriate contours.

This can be rewritten in terms of the Gaussian Free Field. If we define the GFF in the upper half-plane  $\mathbb{H}_+$  as a generalized Gaussian field with covariance kernel

$$K(z, w) = -\frac{1}{2\pi} \log \frac{z - w}{z - \bar{w}},$$

then the covariance above is precisely the covariance of the pullback of this GFF under the map  $\bar{\Omega}$ .

## 8 Example: Gaussian $\beta$ -ensemble

Let's illustrate these results with the Gaussian  $\beta$ -ensemble ( $G\beta E$ ). In this case, the limiting density follows the semicircle law:

$$\rho_{sc}(x) = \frac{\sqrt{4 - x^2}}{2\pi}, \quad x \in [-2, 2].$$

The Stieltjes transform of this density is  $G(z) = \frac{1}{2}(z - \sqrt{z^2 - 4})$ , and we can compute the complex slope  $f_t(z)$  explicitly.

For the  $G\beta E$ , the fluctuations of the height function converge to the Gaussian Free Field with covariance structure determined by the complex slope. This provides a concrete example of how the abstract machinery we've developed applies to a well-studied random matrix ensemble.

## 9 Outlook and open problems

The connection between dynamical loop equations and Gaussian Free Field fluctuations extends beyond the models we've discussed. A natural conjecture is that for many integrable probability models, including  $(q, \kappa)$ -distributed random lozenge tilings of arbitrary domains, the macroscopic fluctuations in the liquid region are asymptotically described by the Gaussian Free Field in a complex structure given by the first integrals of the corresponding complex Burgers equation.

This perspective unifies many seemingly different models under a common framework, suggesting deep connections between random matrix theory, statistical mechanics, and conformal field theory.

In the next lecture, we'll explore further applications of loop equations to specific models and discuss more detailed properties of the fluctuation fields.

## I Problems (due 2025-04-29)

### I.1 Brownian bridge

Derive the covariance structure of the Brownian bridge (2.6) from the series representation (2.5).

## References

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