UVA Summer School Invariant Measures PS 1

July 15, 2024

The number of stars next to an exercise indicates the expected time needed to fully solve the exercise. A sample solution to all the exercises will be provided at the end of the class.

Exercise 1 [***]

Recall that $G = (V, E)$ is a locally finite, connected graph. Let P_p be the Bernoulli-p-product measure on $\{0,1\}^E$ where each edge e is open with probability p independently. Let $p_c \in [0,1]$ denote the critical value for bond percolation on G, i.e.,

$$p_c := \sup\{p \geq 0 : P_p(\text{ there exists no infinite open cluster}) = 0\}.$$

(1) Give a proof of Theorem 2.2, i.e., show that the simple exclusion process on any graph with $p_c > 0$ is a Feller process.

(2) Find an example of a graph G such that assumptions of Theorem 2.1 do not hold, i.e.,

$$\sup_{x \in V} \sum_{y \in V : y \neq x} [p(x,y) + p(y,x)] = \infty$$

but the simple exclusion process on G is a Feller process.

(3) Find an example of a graph G' such that the simple exclusion process on G' is not a Feller process.

Hint: You may for part (3) consider an asymmetric simple exclusion process on G'.

Exercise 2 [**]

(1) Give a proof of Theorem 2.6, i.e., show that the Bernoulli–ρ–product measures ν_ρ for $\rho \in [0,1]$ are invariant measures for the simple exclusion process whenever the transition rates satisfy a flow rule

$$\sum_{v \in V} p(x,v) = \sum_{w \in V} p(w,x) \quad \text{for all } x \in V.$$

(2) Prove Remark 3.2, i.e., when $AC = 1$ and $C > 1$ for the open ASEP, its unique invariant distribution is a Bernoulli-$(1+C)^{-1}$-product measure.

Exercise 3 [*]

Show that the definition of current of open ASEP in equation (3.14), i.e., for some $i \in [N-1]$,

$$\mathcal{J}^N := \mu(\eta(i) = 1 \text{ and } \eta(i+1) = 0) - q\mu(\eta(i) = 0 \text{ and } \eta(i+1) = 1),$$

does not depend on the choice of the site i.

Exercise 4 [**]

Let ν_ρ^* denote the Bernoulli-ρ-product measure for some $\rho \in (0,1]$, conditioned to have a particle at 0. Let $(\eta_t)_{t\geq 0}$ be a simple exclusion process on $\mathbb{Z}$ with $q \in [0,1]$ started from ν_ρ^*. Let $(X_t)_{t\geq 0}$ denote the position of the particle starting from the origin, called the **tagged particle**.

(1) Show that ν_ρ^* is an invariant measure for $(\theta_{X_t}\eta_t)_{t\geq 0}$, where we recall that

$$\theta_x \eta(y) = \eta(x+y)$$

for all $\eta \in \{0,1\}^{\mathbb{Z}}$ and $x, y \in \mathbb{Z}$.

(2) Deduce that the current J_t (i.e., the net number of particles passing through the origin until time t) of the simple exclusion process on $\mathbb{Z}$ started from ν_ρ satisfies for all $t \geq 0$

$$\mathbb{E}[J_t] = t(1-q)\rho(1-\rho).$$

(3) Do item (1) hold true if we replace $(X_t)_{t\geq 0}$ by a second class particle?