Lectures on Random Matrices (Spring 2025)
Lecture 15: Random Matrices and Topology

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Wednesday, April 23, 2025*

Contents

*Course webpage • Live simulations • TeX Source • Updated at 05:12, Wednesday 23rd April, 2025

1 Introduction

In this wrap-up lecture, we go back to moments of random matrices, and outline their connection to topology (more precisely, to counting certain embedded graphs).

Remark 1.1. Throughout this lecture, to make an exact connection with the existing literature, the matrix size is denoted by N, and the small n is reserved to the order of the moment.

NOTE: course evaluations!

Todo on web: tables of contents in HTML + total PDF

2 Gluing polygons into surfaces

2.1 Gluing edges of a polygon

Consider a regular $2n$-gon with edges labeled by $1, \ldots, 2n$. We can glue the edges in pairs, so that the resulting surface is oriented.

Example 2.1. Consider a square. Recall that to obtain an orientable surface one must orient the square's boundary cyclically and then glue opposite sides with *opposite* orientations. There are three ways to glue the edges of a square. Note that in two cases, we get the sphere and in one case, the torus. The two spheres are obtained by gluing the edges in the same way, but this differs by a rotation — we consider these two cases as different.

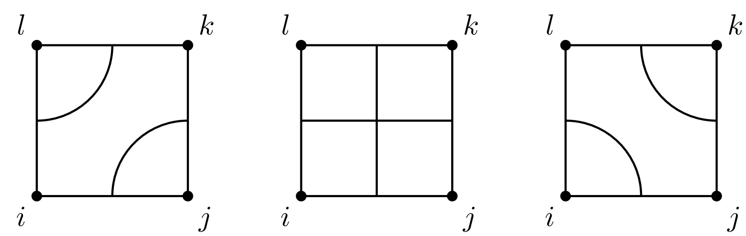

Figure 1: The three ways to glue edges of a square to make an orientable surface: two spheres (left and right) and one torus (center).

The boundary of the $2n$-gon becomes a graph embedded into the surface. It has exactly n edges and one face. It may have different number of vertices, and thus the number of vertices uniquely determines the genus of the surface:

$$V - E + F = 2 - 2g \implies g = \frac{n+1-V}{2}.$$

In the case of the square ($n = 2$), we have $V = 3$ and $g = 0$ for the sphere, and $V = 1$ and $g = 1$ for the torus.

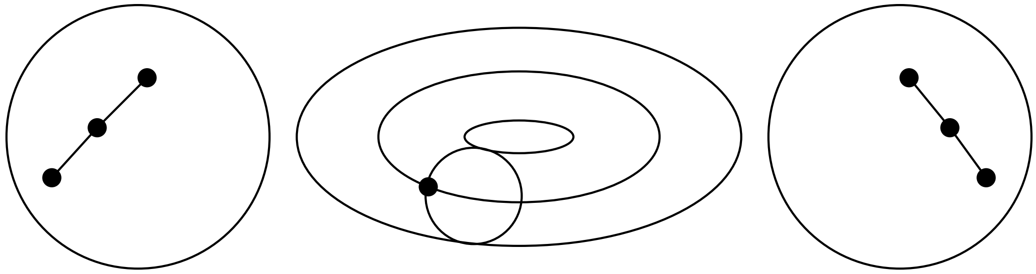

Figure 2: Surfaces corresponding to gluings: left and right show three-vertex trees (disk, sphere), center shows a one-vertex, one-face case (torus).

2.2 Starting to count

Proposition 2.2. *There is a total*

$$(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$$

ways to glue the edges of a $2n$*-gon into a surface.*

Proof. This is just the number of ways to pair $2n$ edges of the polygon. □

Proposition 2.3. *The following are equivalent:*

1. *The surface is a sphere;*
2. *The graph on the surface is a tree;*
3. *The identification of the opposite edges of the polygon is a* noncrossing pairing *of the edges of the polygon.*

Proof. See Problem O.1. □

There is $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ ways to get the sphere.

2.3 Dual picture

In the dual picture, we can consider a star with $2n$ half-edges. Then, we get a dual graph on the same surface. This graph has $V^* = 1$, $E^* = n$, but can have a variable number of faces (which corresponds to the genus):

$$F^* = n - 2g + 1.$$

When $n = 2$, for the sphere, we have $F^* = 3$, and for the torus, we have $F^* = 1$.

2.4 Notation

Let us denote

$$\varepsilon_g(n) \coloneqq \text{number of ways to glue the edges of a } 2n\text{-gon into a surface of genus } g,$$

$$T_n(N) \coloneqq \sum_{\text{gluings } \sigma} N^{V(\sigma)} = \sum_{g=0}^{\infty} \varepsilon_g(n) N^{n+1-2g},$$

that is, this is the generating function of the gluings of the edges of a $2n$-gon, where N is the generating function variable.

Remark 2.4. The polynomial $T_n(N)$ has only powers of N of the same parity as n.

We have the first few polynomials (the case $n = 2$ corresponds to the square):

$$\begin{aligned}
T_1(N) &= N^2;\\
T_2(N) &= 2N^3 + N;\\
T_3(N) &= 5N^4 + 10N^2;\\
T_4(N) &= 14N^5 + 70N^3 + 21N;\\
T_5(N) &= 42N^6 + 420N^4 + 483N^2.
\end{aligned}$$

3 Harer–Zagier formula (statement)

Introduce the *exponential generating function* for the sequence $\{T_n(N)\}_{n\geq 0}$:

$$\begin{aligned}
T(N,s) \ &= 1 + 2Ns + 2s \sum_{n\geq 1} \frac{T_n(N)}{(2n-1)!!}\, s^n \\
&= 1 + 2Ns + 2N^2 s^2 + \frac{2}{3}\big(2N^3 + N\big)s^3 + \frac{2}{15}\big(5N^4 + 10N^2\big)s^4 + \dots
\end{aligned} \tag{3.1}$$

One of the goals of today's lecture is to prove the following:

Theorem 3.1 (Harer–Zagier formula [HZ86])**.** *For every $N \in \mathbb{Z}_{>0}$ one has the closed form*

$$T(N,s) \ = \ \left(\frac{1+s}{1-s}\right)^N. \tag{3.2}$$

Let us at least verify that the first few Taylor coefficients of (3.2) indeed coincide with those in (3.1). Write

$$\begin{aligned}
\left(\frac{1+s}{1-s}\right)^N &= (1+s)^N (1-s)^{-N} \\
&= \left(1 + Ns + \frac{N(N-1)}{2!}s^2 + \frac{N(N-1)(N-2)}{3!}s^3 + \dots\right) \\
&\qquad \times \left(1 + Ns + \frac{N(N+1)}{2!}s^2 + \frac{N(N+1)(N+2)}{3!}s^3 + \dots\right).
\end{aligned}$$

Multiplying the two series and collecting terms up to s^3, we find

$$1 + 2Ns + 2N^2 s^2 + \frac{2}{3}(2N^3 + N)s^3 + \dots,$$

which matches the expansion (3.1) exactly.

Corollary 3.2. *For all $g \geq 0$ and $n \geq 0$, the numbers $\varepsilon_g(n)$ obey*

$$(n+2)\,\varepsilon_g(n+1) \;=\; (4n+2)\,\varepsilon_g(n) \;+\; (4n^3-n)\,\varepsilon_{g-1}(n-1), \tag{3.3}$$

with the initial condition

$$\varepsilon_g(0) \;=\; \begin{cases} 1, & g=0, \\ 0, & g \geq 1. \end{cases}$$

Proof. Follows from the identity

$$\left(\frac{1+s}{1-s}\right)^N = (1+s)(1+s+s^2+\dots)\left(\frac{1+s}{1-s}\right)^{N-1}.$$

□

Corollary 3.3. *The number $\varepsilon_g(n)$ can be written as*

$$\varepsilon_g(n) \;=\; \frac{(2n)!}{(n+1)!\,(n-2g)!}\left[s^{2g}\right]\left(\frac{s/2}{\tanh(s/2)}\right)^{n+1},$$

where $\left[s^{2g}\right]f(s)$ denotes the coefficient of s^{2g} in the power-series expansion of $f(s)$.

One can define another family of coefficients:

$$C_g(n) \coloneqq \frac{2^g\varepsilon_g(n)}{\mathrm{Cat}_n}.$$

Then, (3.3) can be rewritten as

$$C_g(n+1) = C_g(n) + \binom{n+1}{2}C_{g-1}(n-1).$$

In particular, $C_g(n)$ is a positive integer, which is not straightforward from the definition of $\varepsilon_g(n)$.

4 Gaussian integrals and Wick formula

4.1 The standard one–dimensional Gaussian measure

Denote by

$$d\mu(x) \;=\; \frac{1}{\sqrt{2\pi}}\,e^{-\frac{x^2}{2}}\,dx, \qquad x \in \mathbb{R},$$

the *standard centred Gaussian measure.* We record the elementary facts that will be used repeatedly:

(i) **Normalization:** $\displaystyle\int_{\mathbb{R}} d\mu(x) = 1.$

(ii) **Odd moments vanish:** $\langle x^{2n+1}\rangle = 0.$

(iii) **Even moments:**

$$\langle x^{2n}\rangle \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^{2n}e^{-\frac{x^2}{2}}\,dx \;=\; (2n-1)!!, \qquad n\in\mathbb{N}.$$

(iv) **Characteristic (Fourier–Laplace) transform:**

$$\varphi(t) \;=\; \int_{\mathbb{R}} e^{itx}\,d\mu(x) \;=\; e^{-\frac{t^2}{2}}, \qquad t\in\mathbb{R}.$$

Here and below we use the convenient bracket notation $\langle f\rangle := \int_{\mathbb{R}} f(x)\,d\mu(x)$ for expectations.

Example 4.1. For $k=1$ with variance 1 we have $\langle x^4\rangle = 3\,\langle x^2\rangle^2 = 3$. For degree 6 one finds $\langle x^6\rangle = 15$. More generally, $\langle x^{2n}\rangle = (2n-1)!!$. This can be computed by a simple induction.

4.2 Gaussian measures on $\mathbb{R}^k$

Fix a positive–definite symmetric matrix $B\in \mathrm{Sym}_k^+(\mathbb{R})$ and set $C := B^{-1}$. The centred Gaussian measure with covariance C is

$$d\mu_B(x) \;=\; \underbrace{\left[(2\pi)^{-k/2}(\det B)^{1/2}\right]}_{=:\; Z_B^{-1}}\exp\!\Big(-\tfrac12\langle Bx,x\rangle\Big)\,d^kx, \qquad x\in\mathbb{R}^k. \tag{4.1}$$

Orthogonal diagonalisation of B shows that the normalising prefactor indeed gives $\int_{\mathbb{R}^k} d\mu_B = 1$.

Basic facts.

$$\langle x_i\rangle = 0, \quad 1\le i\le k; \tag{4.2}$$

$$\langle x_i x_j\rangle = C_{ij}, \quad 1\le i,j\le k. \tag{4.3}$$

All higher moments are expressed in terms of the matrix C via Wick's formula in Section 4.3 below.

Remark 4.2. In this lecture, we consider only *centered* (mean zero) Gaussian measures.

4.3 Wick (Isserlis) formula

The essence of Wick's formula is that *every* moment of a centred Gaussian vector is a sum over pairwise contractions governed solely by the covariance matrix.

Theorem 4.3 (Wick's (or Isserlis') formula)**.** *Let $x=(x_1,\dots,x_k)$ be distributed according to (4.1). For an integer $n\ge 1$ and indices $i_1,\dots,i_{2n}\in\{1,\dots,k\}$,*

$$\big\langle x_{i_1}\cdots x_{i_{2n}}\big\rangle \;=\; \sum_{p\in \mathrm{Pair}(2n)}\;\prod_{\{a,b\}\in p} C_{i_a i_b}, \tag{4.4}$$

where $\mathrm{Pair}(2n)$ *is the set of all* $(2n-1)!!$ *perfect pairings of* $\{1,\dots,2n\}$*. If the degree is* odd*, then the expectation vanishes.*

More generally, for any linear *functions (not necessarily distinct)* $f_1, \ldots, f_{2n}$ *of the variables* $x_1, \ldots, x_k$*, we have*

$$\langle f_1 \cdots f_{2n} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \ldots \langle f_{p_n} f_{q_n} \rangle, \tag{4.5}$$

where the sum is over all pairings of the indices $1, \ldots, 2n$*, and* $p_1 < p_2 < \ldots < p_n$, $q_1 < q_2 < \ldots < q_n$ *are the indices encoding the pairing.*

Sketch of proof. When $C = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_k^2)$, mixed covariances vanish and Wick's formula factorizes:

$$\left\langle x_1^{2n_1} \cdots x_k^{2n_k} \right\rangle \;=\; \prod_{i=1}^{k} (2n_i - 1)!!\, \sigma_i^{2n_i}, \qquad n_1, \ldots, n_k \in \mathbb{N}.$$

Indeed, pairings are allowed only between indices of the same variable, and then the number of pairings within one variable x_i is $(2n_i - 1)!!$.

The general case of Wick's formula follows from the diagonal case by making a linear change of variables which diagonalizes the covariance matrix, and using the linearity of (4.5). □

Example 4.4. The one-dimensional integral $\langle x^4 \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}} dx$ can be computed using Wick's formula:

$$\langle f_1 f_2 f_3 f_4 \rangle = \langle f_1 f_2 \rangle \langle f_3 f_4 \rangle + \langle f_1 f_3 \rangle \langle f_2 f_4 \rangle + \langle f_1 f_4 \rangle \langle f_2 f_3 \rangle,$$

where $f_i(x) = x$ for $i = 1, 2, 3, 4$. We know this integral is equal to 3.

Remark 4.5. Note that in the second part of Theorem 4.3, the linear functions f_j must be not *affine*, but truly *linear*, that is, $f_j(0, \ldots, 0) = 0$. See Problem O.2.

5 GUE integrals and gluing polygons

We will now apply Wick's formula to compute the moments of traces of GUE matrices. Recall that in Lecture 1 and Lecture 2 we worked with general Wigner matrices (real symmetric or Hermitian), and now we will deal with the special case of GUE, Gaussian Hermitian matrices. Here, the Gaussian distribution will allow us to connect the moments of traces of GUE matrices to the topology of surfaces.

5.1 Traces of powers, again

Let $\mathcal{H}_N$ be the space of $N \times N$ Hermitian matrices, and μ on $\mathcal{H}_N$ be the GUE measure, with complex variances 1 for the diagonal and off-diagonal entries. Let us begin by an example with $n = 2$.

Consider the integral

$$\int_{\mathcal{H}_N} \operatorname{tr}(H^4)\, d\mu(H).$$

Here the integrand is a sum of monomials,

$$\operatorname{tr}(H^4) \;=\; \sum_{i,j,k,l=1}^{N} h_{ij}\, h_{jk}\, h_{kl}\, h_{li}.$$

Since each entry h_{pq} is a linear function of the real and imaginary parts of H, we may apply Wick's formula:

$$\langle h_{ij}h_{jk}h_{kl}h_{li}\rangle = \langle h_{ij}h_{jk}\rangle \langle h_{kl}h_{li}\rangle + \langle h_{ij}h_{kl}\rangle \langle h_{jk}h_{li}\rangle + \langle h_{ij}h_{li}\rangle \langle h_{jk}h_{kl}\rangle. \tag{5.1}$$

Lemma 5.1. *We have $\langle h_{ij}h_{ji}\rangle = 1$, and all other second moments are zero.*

Proof. This is straightforward from the independence of real and imaginary parts of the entries of H. □

Let us inspect each term in (5.1) separately:

- In the first product $\langle h_{ij}h_{jk}\rangle$ is nonzero only when $i = k$, and then equals 1. Likewise $\langle h_{kl}h_{li}\rangle = 1$ only when $k = i$. Summing over all i, j, k, l with $i = k$ gives N^3.
- In the second product $\langle h_{ij}h_{kl}\rangle \langle h_{jk}h_{li}\rangle$ is nonzero only if $i = j = k = l$, and then each factor equals 1. Hence this term contributes N.
- The third product is identical in structure to the first and therefore contributes another N^3.

There is a one-to-one correspondence between these three terms in (5.1) and the three pairings of the edges of a square (see Figure 1). Each pairing contributes $N^{V(\sigma)}$, where $V(\sigma)$ is the number of vertices in the glued graph.

Putting everything together, we get

$$\int_{\mathcal{H}_N} \mathrm{tr}(H^4)\, d\mu(H) \;=\; 2\,N^3 + N \;=\; T_2(N).$$

6 Multi-matrix models

7 Two-matrix models and the Ising model

O Problems (due 2025-04-29)

O.1 Gluing a Sphere

Show that for a connected, orientable surface formed by gluing the edges of a $2n$-gon in pairs, the following are equivalent:

1. The resulting surface is a sphere.
2. The embedded graph formed by the identification is a tree.
3. The pairing of edges corresponds to a *noncrossing pairing* (i.e., when the edges are arranged around the polygon in order, the identifications can be drawn inside the disk without crossings).

(This is the proof of Proposition 2.3.)

O.2 Wick's formula for affine functions

Consider the integrals of the form

$$I(a_1, \ldots, a_k) := \int_{-\infty}^{\infty} \prod_{i=1}^{k} (x - a_i) \, \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx,$$

where $a_1, \ldots, a_k \in \mathbb{R}$ are fixed parameters.

Compute $I(a_1, a_2)$ and $I(a_1, a_2, a_3, a_4)$ explicitly as polynomials in $a_1, \ldots, a_4$, and compare $I(a_1, a_2, a_3, a_4)$ with the Wick-like expansion.

References

[HZ86] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), 457–485. ↑4

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