Lectures on Random Matrices (Spring 2025)
Lecture 11: Asymptotics of Dyson Brownian Motion with an outlier

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Contents

*Course webpage • Live simulations • TeX Source • Updated at 15:38, Wednesday 26th March, 2025

1 Recap

1.1 Dyson Brownian Motion (DBM)

We introduced a time-dependent model of random matrices by letting an $N \times N$ Hermitian matrix $\mathcal{M}(t)$ evolve in time so that each off-diagonal entry follows independent Brownian increments (real or complex depending on the symmetry class). Setting

$$\mathcal{M}(t) \;=\; \frac{1}{\sqrt{2}}\big(X(t) + X^{\dagger}(t)\big),$$

where $X(t)$ is an $N \times N$ matrix of i.i.d. Brownian motions, produces a self-adjoint matrix with a stochastically evolving spectrum. This model is full-rank matrix Brownian motion, and works well for $\beta = 1, 2, 4$. For other β, we need an SDE to describe the evolution of the eigenvalues (particles).

1.2 Eigenvalue SDE

Denote by $\lambda_1(t) \geq \cdots \geq \lambda_N(t)$ the ordered eigenvalues of $\mathcal{M}(t)$. Dyson showed that these eigenvalues form a continuous-time Markov process satisfying the SDE

$$d\lambda_i(t) \;=\; \frac{\beta}{2} \sum_{j \neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)} \;+\; dW_i(t), \quad i = 1, \ldots, N,$$

where $\beta > 0$ and $W_i(t)$ are independent standard real Brownian motions. For classical random matrix ensembles ($\beta = 1, 2, 4$), this SDE describes how the eigenvalues evolve under real symmetric (GOE), Hermitian (GUE), or quaternionic (GSE) Brownian motion — in the last Lecture 10 we discussed the cases $\beta = 1, 2$ in detail. A key feature is the *repulsion* term $\frac{1}{\lambda_i - \lambda_j}$, which prevents collisions (and ensures the ordering remains intact).

1.3 Preservation of GβE density

A fundamental result is that starting from all eigenvalues at 0, the distribution of $\lambda(t)$ at time t has the joint density proportional to

$$\prod_{i<j} |\lambda_i - \lambda_j|^{\beta} \; \exp\Big\{-\tfrac{1}{2t} \sum_i \lambda_i^2\Big\},$$

matching the Gaussian β-Ensemble (GβE) law. Hence DBM provides a dynamical realization of GβE. Invariance can be checked by verifying that this density is annihilated by the generator of the SDE.

1.4 Harish–Chandra–Itzykson–Zuber (HCIZ) integral

The HCIZ integral is a key tool for computing matrix integrals involving traces. For two Hermitian matrices A and B with eigenvalues $(a_1, \ldots, a_N)$ and $(b_1, \ldots, b_N)$, it states (in one common normalization):

$$\int_{U(N)} \exp\bigl(\mathrm{Tr}(A\,U\,B\,U^\dagger)\bigr)\, dU \;=\; \prod_{k=1}^{N-1} k!\; \frac{\det\bigl[e^{\,a_i b_j}\bigr]_{i,j=1}^{N}}{\prod_{1\le i<j\le N}(a_j - a_i)\,\prod_{1\le i<j\le N}(b_j - b_i)}\,.$$

This formula is instrumental in deriving transition densities for $\beta = 2$ Dyson Brownian Motion.

2 Optional: proof of HCIZ integral via representation theory

In this section, we outline a standard argument (adapted from the theory of symmetric functions and representation theory of the unitary group) that leads to a proof of the Harish–Chandra–Itzykson–Zuber formula. It is often referred to as the "orbital integral" or "character expansion" approach.

Step 1. Setting up the integral and Schur expansions. Let A and B be two $N \times N$ diagonalizable matrices, with eigenvalues $a_1, \ldots, a_N$ and $\lambda_1, \ldots, \lambda_N$ respectively. Denote by $D_a = \mathrm{diag}(a_1, \ldots, a_N)$ and $D_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. We want to evaluate the integral

$$I \;=\; \int_{U(N)} \exp\bigl(\mathrm{Tr}(D_a\,U\,D_\lambda\,U^\dagger)\bigr)\, dU$$

over the Haar measure on $U(N)$.

Since $\mathrm{Tr}(B) = p_1(B)$ in the language of power sums (where $p_1(x_1, x_2, \ldots) = x_1 + x_2 + \ldots$), we have

$$\exp\bigl(\mathrm{Tr}(B)\bigr) \;=\; \exp\bigl(p_1(B)\bigr).$$

One can use a known expansion [Mac95]

$$e^{p_1(B)} \;=\; \sum_{m=0}^{\infty} \frac{p_1^m(B)}{m!} \;=\; \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu:|\mu|=m} \dim(\mu)\, s_\mu(B),$$

where the sum is over all partitions μ of size m, and $s_\mu(\cdot)$ is the Schur polynomial (or Schur function) indexed by μ. The coefficient $\dim(\mu)$ is the dimension of the corresponding representation of S_m.

We set $B = D_a\,U\,D_\lambda\,U^\dagger$ and write

$$I \;=\; \int_{U(N)} \exp\Bigl(\mathrm{Tr}(D_a\,U\,D_\lambda\,U^\dagger)\Bigr)\, dU \;=\; \int_{U(N)} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu:|\mu|=m} \dim(\mu)\; s_\mu\bigl(D_a\,U\,D_\lambda\,U^\dagger\bigr)\, dU.$$

One can exchange the integral and the sum (the series converges absolutely for all matrix arguments), giving

$$I \;=\; \sum_{m=0}^{\infty} \sum_{\mu:|\mu|=m} \frac{\dim(\mu)}{m!} \int_{U(N)} s_\mu(D_a\,U\,D_\lambda\,U^\dagger)\, dU. \qquad (2.1)$$

Step 2. Orthogonality of characters and the Unitary group. The Schur functions $s_\mu(\cdot)$ can be seen as irreducible characters of the unitary group $U(N)$ (up to a normalization factor) when restricted to N-tuples of eigenvalues.[1]

Proposition 2.1 (Functional equation for characters of compact groups)**.** *Let G be a compact group with normalized Haar measure dh, and let χ be an irreducible character of a finite-dimensional representation of G. Then for any elements $g_1, g_2 \in G$, the following relation holds:*

$$\int_G \chi(g_1 h g_2 h^{-1}) dh = \frac{\chi(g_1)\chi(g_2)}{\dim V}, \tag{2.2}$$

where $\dim V = \chi(e)$ is the dimension of the representation space.

Remark 2.2. A similar relation holds for characters of finite groups.

By Proposition 2.1, the integral over $U(N)$ in (2.1) can be evaluated as

$$\int_{U(N)} s_\mu(D_a\, U\, D_\lambda\, U^\dagger)\, dU \;=\; \frac{1}{\mathrm{Dim}_N(\mu)}\; s_\mu(a)\, s_\mu(\lambda),$$

where $\mathrm{Dim}_N(\mu)$ is the dimension of the corresponding irreducible representation of $U(N)$. Substituting back into (2.1) yields

$$I \;=\; \sum_{m=0}^{\infty} \sum_{\mu:|\mu|=m,\; \ell(\mu)\le N} \frac{\dim(\mu)}{m!}\, \frac{1}{\mathrm{Dim}_N(\mu)}\; s_\mu(a)\, s_\mu(\lambda),$$

where $\ell(\mu) \le N$ is needed for $s_\mu(\cdot)$ not to vanish on $U(N)$.

Step 3. Hook-length formulas and the final determinant. Next, one applies the hook-length formula and the hook-content formula to dimensions:

$$\dim \mu = \frac{|\mu|!}{\prod_{\square\in\mu} h(\square)}, \qquad \mathrm{Dim}_N(\mu) = \frac{\prod_{\square\in\mu}(N + c(\square))}{\prod_{\square\in\mu} h(\square)},$$

We have

$$\prod_{\square\in\mu}(N + c(\square)) = \prod_{i=1}^{N} \frac{(\mu_i + N - i)!}{(N-i)!},$$

so identifying $m_i = \mu_i + N - i$ gives

$$I = 0!1!\cdots(N-1)! \sum_{m_1>\ldots>m_N\ge 0} \frac{s_\mu(a)s_\mu(\lambda)}{m_1!\cdots m_N!},$$

which yields the HCIZ formula by the Cauchy-Binet summation.

[1] s_μ for $\ell(\mu) \le N$ can be viewed as the character of the corresponding polynomial representation of $GL(N,\mathbb{C})$, then restricted to $U(N)$. If $\ell(\mu) > N$, the function s_μ vanishes on $U(N)$. Thus, we need to impose the condition $|a_i| = |\lambda_i| = 1$ (so that $D_a, D_\lambda \in U(N)$) to ensure immediate applicability of representation theory of $U(N)$, then extend to general $\{a_i\}$ and $\{\lambda_i\}$ by analytic continuation.

3 Determinantal structure for $\beta = 2$

3.1 Transition density

Theorem 3.1 ($\beta = 2$ Dyson Brownian Motion Transition Probabilities)**.** *For $\beta = 2$, let $\lambda(t) = (\lambda_1(t) \geq \cdots \geq \lambda_N(t))$ follow Dyson Brownian Motion starting at $\lambda(0) = \mathbf{a} = (a_1 \geq \cdots \geq a_N)$. Then for each fixed time $t > 0$,*

$$P\big(\lambda(t) = \mathbf{x} \mid \lambda(0) = \mathbf{a}\big) \;=\; N!\,\Big(\frac{1}{\sqrt{2\pi t}}\Big)^N \prod_{1 \leq i < j \leq N} \frac{x_i - x_j}{a_i - a_j}\, \det\Big[\exp\Big(-\frac{(x_i - a_j)^2}{2t}\Big)\Big]_{i,j=1}^N,$$

where $x_1 \geq \cdots \geq x_N$.

Proof. Consider an $N \times N$ Hermitian matrix process $X(t)$ whose entries perform independent complex Brownian motions (so that $X(t)$ is distributed as $A + \sqrt{t}\,\mathrm{GUE}$ at each fixed time, with $A = \mathrm{diag}(a_1, \ldots, a_N)$). Its eigenvalues $\lambda_1(t) \geq \cdots \geq \lambda_N(t)$ evolve exactly according to the $\beta = 2$ Dyson Brownian Motion.

The density of X at time t, viewed as a random matrix, is proportional to

$$\exp\Big(-\tfrac{1}{2t}\,\mathrm{Tr}\big(X - A\big)^2\Big).$$

If we replace A by $U\,A\,U^\dagger$ for any fixed unitary U, the law of X remains the same (this follows from the unitary invariance of the GUE). Thus the distribution of the eigenvalues of X is unchanged by such conjugation.

One writes

$$\int_{U(N)} \exp\Big(-\tfrac{1}{2t}\,\mathrm{Tr}\big(X - U\,A\,U^\dagger\big)^2\Big)\, dU \;=\; (\text{const.}) \times [\text{HCIZ integral in the variables } (X, A)],$$

which by the Harish–Chandra–Itzykson–Zuber formula leads to a product of determinants and a factor that is precisely

$$\exp\Big(-\tfrac{1}{2t}\sum_{i=1}^N x_i^2 - \tfrac{1}{2t}\sum_{i=1}^N a_i^2\Big)\, \frac{\det\Big[\exp\big(\frac{x_i\,a_j}{t}\big)\Big]}{\prod_{i<j}(x_i - x_j)\,(a_i - a_j)},$$

where $x_1, \ldots, x_N$ are the eigenvalues of X.

To convert this matrix distribution into a distribution on eigenvalues alone, we multiply by the usual Vandermonde Jacobian $\prod_{i<j}(x_i - x_j)^2$ (which comes from integrating out the unitary degrees of freedom). This produces exactly

$$N!\,\big(\tfrac{1}{\sqrt{2\pi t}}\big)^N \prod_{i<j} \frac{x_i - x_j}{a_i - a_j}\, \det\Big[\exp\Big(-\tfrac{(x_i - a_j)^2}{2t}\Big)\Big].$$

Hence we obtain the stated transition probability for the Dyson Brownian Motion at $\beta = 2$. $\square$

Remark 3.2. The factor $N!\,(\frac{1}{\sqrt{2\pi t}})^N$ arises naturally from normalizing the Gaussian increments and accounts for the ordering $\lambda_1 \geq \cdots \geq \lambda_N$. The determinant and product factors encode the eigenvalue "repulsion" characteristic of $\beta = 2$ random matrices.

3.2 Determinantal correlations

Theorem 3.3 (Determinantal structure for $\beta = 2$ DBM)**.** *Let $\{x_1(t), \ldots, x_n(t)\}$ be the eigenvalues at time $t > 0$ of the $\beta = 2$ Dyson Brownian Motion started at initial locations $(a_1, \ldots, a_n)$ at time 0. Equivalently, these $x_i(t)$ are the eigenvalues of*

$$A + \sqrt{t}\, G,$$

where $A = \mathrm{diag}(a_1, \ldots, a_n)$ and G is a random Hermitian matrix from the GUE. Then the (random) point configuration $\{x_i(t)\}$ forms a determinantal point process with correlation kernel

$$K_t(x, y) = \frac{1}{(2\pi)^2\, t} \int\!\!\int \exp\Bigl(\frac{w^2 - 2\,y\,w}{2\,t}\Bigr) \Big/ \exp\Bigl(\frac{z^2 - 2\,x\,z}{2\,t}\Bigr) \prod_{i=1}^{n} \frac{w - a_i}{z - a_i}\, \frac{dw\, dz}{w - z}.$$

Here z goes around all the points $a_1, \ldots, a_n$ in the positive direction, and the w contour passes from $-i\infty$ to $i\infty$, to the right of the z contour.

- If $a_1 = \cdots = a_n = 0$ and $t = 1$, this kernel reduces to the familiar correlation kernel of the GUE (see Lecture 6).
- One can use this formula to study the Baik–Ben Arous–Péché (BBP) [BBP05] phase transition for $\beta = 2$, which deals with finite rank perturbations of the GUE random matrix ensemble. Indeed, rank r perturbation corresponds to taking $a_1, \ldots, a_r \neq 0$, and $a_{r+1} = \cdots = a_n = 0$.

3.3 On the proof of determinantal structure

The idea of the proof of Theorem 3.3 is to represent the measure (the transition density) as a product of determinants. In general, if a measure is given as a product of determinants, there is a well-studied method (biorthogonal ensembles and, more generally, the Eynard–Mehta theorem) to compute the determinantal correlation kernel. We refer to [BR05], [Bor11] for a detailed exposition in the discrete case (which is arguably more transparent). The first step for the Dyson Brownian Motion is as follows.

Lemma 3.4 (Density representation)**.** *Let $P_t(x \to y)$ be the transition probability kernel of standard Brownian motion,*

$$P_t(x \to y) \;=\; \frac{1}{\sqrt{2\pi\, t}}\, \exp\Bigl(-\tfrac{(x-y)^2}{2\,t}\Bigr).$$

Then the density of the eigenvalues $(x_1, \ldots, x_N)$ of DBM started at $(a_1, \ldots, a_N)$ at time 0 admits the representation

$$\lim_{s \to \infty} \Bigl(\frac{1}{Z}\Bigr)\, \det\Bigl[P_t\bigl(a_i \to x_j\bigr)\Bigr]_{i,j=1}^{N}\, \det\Bigl[P_s\bigl(x_i \to k-1\bigr)\Bigr]_{i,k=1}^{N}. \tag{3.1}$$

Remark 3.5. This representation (3.1) is related to an alternative description of the $\beta = 2$ Dyson Brownian Motion as an ensemble of noncolliding Brownian motions (that is, independent Brownian motions, conditioned to never collide).

Proof of Lemma 3.4. The first determinant (as $s \to \infty$) matches the determinant we have in Theorem 3.1. It remains to analyze the second determinant

$$\det\Big[P_s\big(x_j \to k-1\big)\Big]_{j,k=1}^{N} \;=\; \det\Big[\tfrac{1}{\sqrt{2\pi\, s}}\, \exp\Big(-\tfrac{\big((k-1)-x_j\big)^2}{2\, s}\Big)\Big]_{j,k=1}^{N}.$$

We may ignore the factor $\frac{1}{\sqrt{2\pi\, s}}$ in each entry since it does not depend on x_j. Inside the exponential,

$$-\,\frac{\big((k-1)-x_j\big)^2}{2\, s} \;=\; -\,\frac{x_j^2}{2\, s} \;+\; \frac{x_j\,(k-1)}{s} \;-\; \frac{(k-1)^2}{2\, s}.$$

Thus, up to the factor $\exp\big(-\,\frac{(k-1)^2}{2\, s}\big)$ (which depends only on k and hence is independent of each x_j), we can factor out $\exp\big(-\,\frac{x_j^2}{2\, s}\big)$ from row j. Consequently, the nontrivial part of the determinant becomes

$$\det\Big[e^{\,\frac{x_j\,(k-1)}{s}}\Big]_{j,k=1}^{N}.$$

Recognize this as a Vandermonde-type determinant in the variables $e^{\,x_j/s}$. Indeed,

$$\det\Big[e^{\,\frac{x_j\,(k-1)}{s}}\Big]_{j,k=1}^{N} \;=\; \prod_{1\le i<j\le N}\Big(e^{\,\frac{x_i}{s}} - e^{\,\frac{x_j}{s}}\Big).$$

As $s \to \infty$, we expand $e^{\,\frac{x_i}{s}} = 1 + \frac{x_i}{s} + O\big(\frac{1}{s^2}\big)$, so each difference $\big(e^{\,\frac{x_i}{s}} - e^{\,\frac{x_j}{s}}\big) \sim \frac{x_i - x_j}{s}$. Hence,

$$\prod_{1\le i<j\le N}\Big(e^{\,\frac{x_i}{s}} - e^{\,\frac{x_j}{s}}\Big) \;\sim\; \frac{1}{s^{\,\frac{N(N-1)}{2}}}\prod_{1\le i<j\le N}(x_i - x_j).$$

Combining all these factors and matching with the first determinant (as $s \to \infty$) verifies the claimed product form, up to overall constants that do not depend on the variables x_j. This completes the proof. □

Then, the product of determinants idea (biorthogonal ensembles) applies to the density (3.1) before the limit $s \to \infty$, and simplifies after taking the limit. We omit the details here, see Problem K.1.

4 Asymptotic analysis: signal plus noise

4.1 Setup

In this section, we provide a detailed derivation of how the rank-1 spike

$$A + \sqrt{G}, \qquad A = \mathrm{diag}(a, 0, \ldots, 0) \quad \text{with } a \in \mathbb{R},$$

affects the large-n and large-time behavior of the Dyson Brownian Motion at $\beta = 2$. See the simulation at `https://lpetrov.cc/simulations/2025-01-28-bbp-transition/`.[2]

[2]Note that the simulation has $\beta = 1$ (real matrices), so the edge is at $\sqrt{2}$, and the critical value of the spike is at $1/\sqrt{2}$.

We set $a_1 = a\sqrt{n}$ and $a_2 = a_3 = \cdots = a_n = 0$, which simplifies the product:

$$\prod_{i=1}^{n}(w - a_i) \;=\; (w - a\sqrt{n})\, w^{\,n-1}, \qquad \prod_{i=1}^{n}(z - a_i) \;=\; (z - a\sqrt{n})\, z^{\,n-1}.$$

Let us also take $t = 1$ for simplicity, so that the limit shape (at least in the case $a = 0$, but also in general) is supported by $[-2\sqrt{n}, 2\sqrt{n}]$. Let us also make the change of the integration variables $w \to w\sqrt{n}$, $z \to z\sqrt{n}$.

Hence, the correlation kernel becomes

$$K_t(x,y) = \frac{\sqrt{n}}{(2\pi)^2} \int\!\!\int \exp\Bigl(\frac{nw^2 - 2\,y\,w\sqrt{n}}{2}\Bigr) \Big/ \exp\Bigl(\frac{nz^2 - 2\,x\,z\sqrt{n}}{2}\Bigr)\, \frac{w-a}{z-a}\, \Bigl(\frac{w}{z}\Bigr)^{n-1}\, \frac{dw\,dz}{w-z}. \tag{4.1}$$

Here:

- The z-contour is a small positively oriented loop around $z = a$, and also around $z = 0$, so that it encircles these two singularities but excludes w.
- The w-contour is a vertical line (or an equivalent contour from $-i\infty$ to $i\infty$) passing to the right of all singularities (i.e. to the right of z).

Note that to capture the edge behavior, we need to set $x = y = 2$ plus lower order terms. Let us make this substitution $x = 2\sqrt{n} + x'$, $y = 2\sqrt{n} + y'$, and the scale of x', y' will be determined later (but for now we assume that they are $o(\sqrt{n})$).

4.2 Outline of the steepest descent approach

We aim to understand the behavior of (4.1) in the regime $n \to \infty$, especially near the largest eigenvalue $\lambda_1(t)$. Recall from standard GUE (i.e. $a = 0$) that the top of the spectrum is about $2\sqrt{n}$. The presence of the rank-1 spike a can drastically modify the top eigenvalue if a is large enough to produce an "outlier." Our goal is to detect precisely how this occurs by analyzing the double contour integral via steepest descent.

For large n, the integral localizes around these double critical point. Any crossing from z- to w-contour may pick up residues, which account for separate contributions (leading, for instance, to the Airy kernel in the unperturbed GUE). We track how the spike a changes these deformations.

4.3 Asymptotics

Set

$$S(w; y') \;=\; \tfrac{w^2}{2} - 2\,w - y'w/\sqrt{n} \;+\; \frac{n-1}{n}\ln(w).$$

Then the integrand in (4.1) is

$$\frac{\exp\bigl\{n\bigl[S(w;y') - S(z;x')\bigr]\bigr\}}{w - z}\,\frac{w-a}{z-a}.$$

To capture the Airy behavior, we can ignore y', and find the double critical point of $S(w;0)$. It is equal to $w_c = 1$, and we would like to bring the z and w contours to intersect at $w_c = 1$. Note however that the old z contour must encircle $z = a$ and $z = 0$, and $z = a$ is a pole of the integrand. The w contour must always be to the right of the z contour.

We see that there are three regimes, which we consider in the next three subsections.

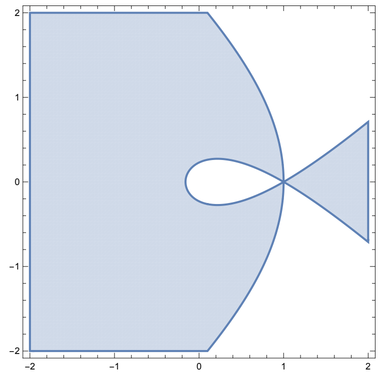

Figure 1: The plot of the region $\operatorname{Re} S(z) - \operatorname{Re} S(1) > 0$ at the edge, in the neighborhood of the double critical point $w_c = 1$. The new w contour should pass through the shaded region, and the new z must stay in the non-shaded region.

4.4 Airy kernel

If $a < 1$, we can deform the z contour to encircle $z = 0$ and $z = a$, and the w contour to pass through $w = 1$. This will lead to the Airy kernel, and the derivation is the same as in Lecture 7. We obtain[3]

$$z = 1 + \frac{Z}{n^{1/3}}, \quad w = 1 + \frac{W}{n^{1/3}}, \qquad x' = \frac{\xi}{n^{1/6}}, \quad y' = \frac{\eta}{n^{1/6}}, \qquad \frac{1}{n^{1/6}} K_n \to K_{\mathrm{Airy}}(\xi, \eta).$$

Here

$$K_{\mathrm{Airy}}(\xi, \eta) = \frac{1}{(2\pi i)^2} \iint \frac{\exp\left\{\frac{W^3}{3} - \xi W - \frac{Z^3}{3} + \eta Z\right\}}{W - Z} \, dW \, dZ.$$

Indeed, the only one new thing that happens here is that $a < 1$, and so

$$\frac{w - a}{z - a} = \frac{1 - a + W/n^{1/3}}{1 - a + Z/n^{1/3}} = 1 + O(n^{-1/3}), \tag{4.2}$$

so this term does not contribute to the asymptotics of the kernel.

4.5 BBP transition and the deformed Airy kernel

If $a = 1$, the behavior is going to be critical — we still will be able to get the same scaling, but the limiting kernel will be different. Moreover, looking at (4.2), we see that we need to critically rescale a, so

$$a = 1 + A n^{-1/3}, \qquad \frac{w - a}{z - a} = \frac{W - A}{Z - A} + O(n^{-4/3}), \qquad \frac{1}{n^{1/6}} K_n \to \tilde{K}_{\mathrm{Airy}}(\xi, \eta),$$

where

$$\tilde{K}_{\mathrm{Airy}}(\xi, \eta) = \frac{1}{(2\pi i)^2} \iint \frac{\exp\left\{\frac{W^3}{3} - \xi W - \frac{Z^3}{3} + \eta Z\right\}}{W - Z} \frac{W - A}{Z - A} \, dW \, dZ.$$

[3]Here and below, we understand the convergence of the kernels is up to a gauge transformation of the form $K(x, y) \mapsto \frac{f(x)}{f(y)} K(x, y)$.

This kernel is the BBP transition kernel, first obtained in the seminal paper by Baik–Ben Arous–Péché [BBP05]. The spiked top eigenvalue distribution (and the Tracy–Widom distribution) are widely used in statistics of high-dimensional, highly correlated data.

4.6 Gaussian regime

Finally, for $a > 1$, we cannot deform the integration contours so that they pass through the double critical point $w_c = 1$. Instead, we can make the contours pass through the point a itself, and scale the integration variables w, z around a on the scale $n^{-1/2}$ and not $n^{-1/3}$.

Moreover, we need to make x, y to scale around a different location instead of $2\sqrt{n}$. We can find this location by first considering $x = c\sqrt{n}$ and expanding as $n \to \infty$:

$$
\begin{aligned}
&n\left(\frac{w^2}{2} - yw/\sqrt{n} + \log w\right)\Big|_{w=a+W/\sqrt{n},\ y=c\sqrt{n}+\eta} \\
&\qquad = n\left(\frac{a^2}{2} - ac + \log(a)\right) + \sqrt{n}\left(-a\eta + aW + \frac{W}{a} - cW\right) - \frac{W^2}{2a^2} + \frac{W^2}{2} - \eta W.
\end{aligned}
$$

The term by n is the same in $S(w)$ and $S(z)$ and thus cancels out. The term by $\sqrt{n}$ depends on W and cannot be simply removed by a gauge transformation, so we need to match c. We have

$$
c = a + \frac{1}{a}.
$$

Remark 4.1. You can go to `https://lpetrov.cc/simulations/2025-01-28-bbp-transition/` and set the parameter θ (which is the same as a) to an integer, make N large, and check that the location of the top or bottom eigenvalue becomes exactly $a + 1/a$. (Despite the fact that the simulation at the link is for $\beta = 1$.)

Setting $c = a + 1/a$, we have

$$
nS \sim -\frac{W^2}{2a^2} + \frac{W^2}{2} - \eta W,
$$

and thus the distribution of the top eigenvalue is given by a Fredholm determinant with the kernel

$$
K_G(\xi, \eta) = \frac{1}{(2\pi)^2} \int\int \exp\left\{\frac{a^{-2}-1}{2}(Z^2 - W^2) - \eta W + \xi Z\right\} \cdot \frac{W}{Z} \cdot \frac{dW\, dZ}{W - Z}
$$

Note that the factor $\sqrt{n}$ in front of K_t is precisely removed by the scaling of w, z, and there is no additional scaling coming from the map $(x, y) \mapsto (\xi, \eta)$. The contribution from $(w - a)/(z - a)$ becomes W/Z.

The integration contours in K_G are such that $Re(W^2) > 0$ and $Re(Z^2) < 0$ on them, and this can be achieved by the contour deformation. Indeed, in the new variables, the behavior at $W = Z = 0$ is quadratic, so the Z contour must pass on the left, and the W contour must be on the right. One can check that this contour deformation is possible.

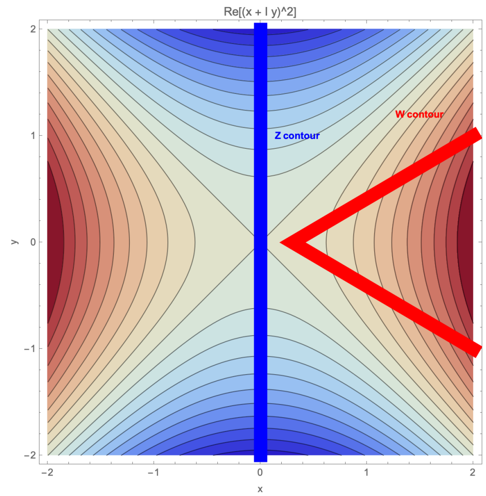

Figure 2: The contour plot of $\Re(Z^2)$ around zero. Blue shades correspond to negative values, and yellow to positive. The Z contour must pass through the blue region and becomes vertical, and the W contour must stay in the yellow region, and becomes a union of two half-lines, which are at the angle $< \frac{\pi}{4}$ from the real line.

4.7 Matching Fredholm determinant to the Gaussian distribution

Let us renormalize the integration variables to remove the factor $a^{-2} - 1$ in front of the squares, and match $\det(1 - K_G)_{\geq x}$ to the Gaussian distribution (see also Problem K.5 for another way to match). We will work with

$$K_G(\xi, \eta) = \frac{1}{(2\pi)^2} \int \int \exp\left\{ \frac{1}{2}(Z^2 - W^2) - \eta W + \xi Z \right\} \cdot \frac{W}{Z} \cdot \frac{dW\, dZ}{W - Z}$$

The discussion below is informal, but can be easily made rigorous.

Step 1. Partial fractions and decomposition. Observe that

$$\frac{W}{Z(W - Z)} = \frac{1}{Z} + \frac{1}{W - Z}.$$

Thus we can write

$$K_G(\xi, \eta) = K^{(1)}(\xi, \eta) + K^{(2)}(\xi, \eta),$$

where

$$K^{(1)}(\xi, \eta) = \frac{1}{(2\pi)^2} \int \int \exp\Big(\tfrac{1}{2}(Z^2 - W^2) + \xi Z - \eta W \Big) \frac{1}{Z}\, dW\, dZ,$$
$$K^{(2)}(\xi, \eta) = \frac{1}{(2\pi)^2} \int \int \exp\Big(\tfrac{1}{2}(Z^2 - W^2) + \xi Z - \eta W \Big) \frac{1}{W - Z}\, dW\, dZ.$$

The term $K^{(1)}$ has a factor $\frac{1}{Z}$ independent of $W - Z$, while $K^{(2)}$ contains the remaining part $\frac{1}{W-Z}$.

Step 2. Analysis of $K^{(1)}$. Focus on

$$K^{(1)}(\xi,\eta) = \frac{1}{(2\pi)^2}\Big(\int e^{\frac{1}{2}Z^2+\xi Z}\,\frac{dZ}{Z}\Big)\Big(\int e^{-\frac{1}{2}W^2-\eta W}\,dW\Big).$$

The operator $K^{(1)}$ is a rank-1 operator in the variables ξ,η:

$$K^{(1)}(\xi,\eta) = u(\xi)\,v(\eta)$$

for some functions $u(\cdot)$ and $v(\cdot)$ of one variable each. Hence $K^{(1)}$ has at most one nonzero eigenvalue (its trace).

Step 3. Representation of $K^{(2)}$ and the key identity. For $K^{(2)}$, we use

$$\frac{1}{W-Z} = \int_0^\infty e^{-t(W-Z)}\,dt$$

(again justified by the choice of integration contours). Then

$$K^{(2)}(\xi,\eta) = \int_0^\infty \Big[\frac{1}{2\pi i}\int e^{\frac{1}{2}Z^2+(\xi+t)Z}\,dZ\Big]\Big[\frac{1}{2\pi i}\int e^{-\frac{1}{2}W^2-(\eta+t)W}\,dW\Big]dt.$$

Denote

$$A(\xi,t) = \frac{1}{2\pi i}\int e^{\frac{1}{2}Z^2+(\xi+t)Z}\,dZ, \quad B(t,\eta) = \frac{1}{2\pi i}\int e^{-\frac{1}{2}W^2-(\eta+t)W}\,dW.$$

Hence $K^{(2)}(\xi,\eta) = \int_0^\infty A(\xi,t)\,B(t,\eta)\,dt$. In operator form on suitable spaces, this reads $K^{(2)} = A\,B$, and one checks $B\,A = I$ (*identity on the t-variable space*), so $A\,B$ and $B\,A$ share the same nonzero spectrum. Indeed,

$$BA(s,t) = \frac{1}{(2\pi i)^2}\int_{\mathbb{R}} d\xi \int dW\,dZ e^{-\frac{1}{2}W^2+\frac{1}{2}Z^2}e^{-(s+\xi)W+(\xi+t)Z}.$$

Integrating over ξ in the Fourier sense yields the delta:

$$\int_{\mathbb{R}} d\xi e^{\xi(Z-W)} = 2\pi\delta(Z-W).$$

Integrating in W is again an integral of $e^{(t-s)W}$, and thus, the second 2π disappears, and we arrive at $BA(s,t) = \delta(s-t)$, which is the kernel of the identity operator.

We conclude that AB is a projection, since $(AB)^2 = ABAB = AB$.

For the rest of the analysis, continue to Problem K.4.

K Problems (due 2025-04-29)

K.1 Biorthogonal ensembles

Derive Theorem 3.3 from Lemma 3.4 using the orthogonalization process similar to Lecture 5, and then taking the limit as $s \to \infty$.

K.2 Scaling of the kernel

Let $a_i = 0$ in Theorem 3.3. Find α such that $t^\alpha K_t(x/\sqrt{t}, y/\sqrt{t})$ is independent of t. Can you explain this value of α?

K.3 Gaussian regime and integration contours

Check that the contour deformation from (z, w) to (Z, W) passing through a described in Section 4.6 is valid.

K.4 Gaussian kernel

Finish the proof of the Fredholm determinant representation of the Gaussian cumulative distribution function by manipulation with Fredholm determinants, which was started in Section 4.7.

K.5 GUE kernel

Consider the following generalization of the kernel K_G from Section 4.6:

$$K_G^m(\xi, \eta) = \frac{1}{(2\pi)^2} \int\int \exp\left\{\frac{1}{2}(Z^2 - W^2) - \eta W + \xi Z\right\} \cdot \left(\frac{W}{Z}\right)^m \frac{dW\, dZ}{W - Z},$$

where $m \geq 1$ is an integer and the contours are as in Figure 2. Show that the Fredholm determinant $\det(1 - K_G^m)_{L^2(x,+\infty)}$ is the cumulative distribution function of the largest eigenvalue of the $m \times m$ GUE matrix, that is, $\mathbb{P}(\lambda_{\max}^{(m\times m)} \leq x)$.

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