Lectures on Random Matrices (Spring 2025)
Lecture 5: Determinantal Point Processes and the GUE

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Contents

1 Recap

In Lecture 4 we discussed global spectral behavior of tridiagonal GβE random matrices, and obtained the Wigert semicircle law for the eigenvalue density.

In this lecture we shift our focus to another powerful technique in random matrix theory: the theory of *determinantal point processes* (DPPs). In the $\beta = 2$ (GUE) case the joint eigenvalue distributions can be written in determinantal form. We begin by discussing the discrete version of determinantal processes, and then derive the correlation kernel for the GUE using orthogonal polynomial methods. Finally, we show how the Christoffel–Darboux formula yields a compact representation of the kernel and indicate how one may represent it as a double contour integral—an expression well suited for steepest descent analysis in the large-n limit.

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2 Discrete determinantal point processes

2.1 Definition and basic properties

Let $\mathfrak{X}$ be a (finite or countably infinite) discrete set. A *point configuration* on $\mathfrak{X}$ is any subset $X \subset \mathfrak{X}$ (with no repeated points). A random point process is a probability measure on the space of such configurations.

Definition 2.1 (Determinantal Point Process)**.** A random point process P on $\mathfrak{X}$ is called *determinantal* if there exists a function (the *correlation kernel*) $K : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ such that for any n and every finite collection of distinct points $x_1, \dots, x_n \in \mathfrak{X}$, the joint probability that these points belong to the random configuration is

$$\mathbb{P}\{x_1, \dots, x_n \in X\} = \det\Big[K(x_i, x_j)\Big]_{i,j=1}^{n}.$$

Determinantal processes are very useful in probability theory and random matrices. They are a natural extension of Poisson processes, and have some parallel properties. Many properties of determinantal processes can be derived from "linear algebra" (broadly understood) applied to the kernel K. There are a few surveys on them: [Sos00], [HKPV06], [Bor11], [KT12]. Let us just mention two useful properties.

Proposition 2.2 (Gap Probability)**.** *If $I \subset \mathfrak{X}$ is a subset, then*

$$\mathbb{P}\{X \cap I = \varnothing\} = \det\Big[I - K_I\Big],$$

where K_I is the restriction of the kernel to I. If I is infinite, then the determinant is understood as a Fredholm determinant.

Remark 2.3. The Fredholm determinant might "diverge" (equal to 0 or 1).

Proposition 2.4 (Generating functions)**.** *Let $f : \mathfrak{X} \to \mathbb{C}$ be a function such that the support of $f - 1$ is finite. Then the generating function of the multiplicative statistics of the determinantal point process is given by*

$$\mathbb{E}\left[\prod_{x \in X} f(x)\right] = \det\Big[I + (\Delta_f - I)K\Big],$$

where the expectation is over the random point configuration $X \subseteq \mathfrak{X}$, Δ_f denotes the operator of multiplication by f (i.e., $(\Delta_f g)(x) = f(x)g(x)$) and the determinant is interpreted as a Fredholm determinant if $\mathfrak{X}$ is infinite.

Remark 2.5 (Fredholm Determinant — Series Definition)**.** The Fredholm determinant of an operator A on $\ell^2(\mathfrak{X})$ is given by the series

$$\det(I + A) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1, \dots, x_n \in \mathfrak{X}} \det\big[A(x_i, x_j)\big]_{i,j=1}^{n},$$

where the term corresponding to $n = 0$ is defined to be 1.

3 Determinantal structure in the GUE

3.1 Correlation functions as densities with respect to Lebesgue measure

In the discrete setting discussed above the joint probabilities of finding points in specified subsets of $\mathfrak{X}$ are given by determinants of the kernel evaluated at those points. When the underlying space is continuous (typically a subset of $\mathbb{R}$ or $\mathbb{R}^d$), one works instead with correlation functions which serve as densities with respect to the Lebesgue measure.

Let $X \subset \mathbb{R}$ be a random point configuration. The *n-point correlation function* $\rho_n(x_1, \ldots, x_n)$ is defined by the relation

$$\mathbb{P}\{\text{there is a point in each of the infinitesimal intervals } [x_i, x_i + dx_i],\ i = 1, \ldots, n\} \\ = \rho_n(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

For a determinantal point process the correlation functions take a determinantal form:

$$\rho_k(x_1, \ldots, x_k) = \det\Big[K(x_i, x_j)\Big]_{i,j=1}^k.$$

Remark 3.1. The reference measure does not necessarily have to be the Lebesgue measure. For example, in the discrete setting, we can also talk about the reference measure, it is the counting measure. The correlation kernel $K(x, y)$ is better understood not as a function of two variables, but as an operator on the Hilbert space $L^2(\mathfrak{X}, d\mu)$, where μ is the reference measure. One can also write $K(x, y)\,\mu(dy)$ or $K(x, y)\sqrt{\mu(dx)\,\mu(dy)}$ to emphasize this structure.

This formulation is particularly useful in the continuous setting, as it allows one to express statistical properties of the point process in terms of integrals over the kernel. For example, the expected number of points in a measurable set $A \subset \mathbb{R}$ is given by

$$\mathbb{E}[\#(X \cap A)] = \int_A \rho_1(x)\, dx,$$

while higher order joint intensities provide information about correlations between points.

3.2 The GUE eigenvalues as DPP

3.2.1 Setup

We start from the joint eigenvalue density for the Gaussian Unitary Ensemble (GUE)

$$p(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = \frac{1}{Z_{n,2}} \prod_{j=1}^n e^{-x_j^2/2} \prod_{1 \le i < j \le n} (x_i - x_j)^2\, dx_1 \cdots dx_n. \tag{3.1}$$

We will show step by step why this is a determinantal point process,

$$\rho_k(x_1, \ldots, x_k) = \det\Big[K_n(x_i, x_j)\Big]_{i,j=1}^k, \qquad k \ge 1,$$

with the kernel defined as

$$K_n(x, y) = \sum_{j=0}^{n-1} \psi_j(x)\psi_j(y),$$

where the functions

$$\psi_j(x) = \frac{1}{\sqrt{h_j}}\, p_j(x)\sqrt{w(x)}, \qquad w(x) = e^{-x^2/2},$$

are constructed from the monic Hermite polynomials $\{p_j(x)\}$ which are orthogonal with respect to the weight $w(x)$:

$$\int_{-\infty}^{\infty} p_j(x)p_k(x)e^{-x^2/2}\, dx = h_j\, \delta_{jk}.$$

Recall that "monic" means that the leading coefficient of $p_j(x)$ is 1, and we divide by the norm to make the polynomials orthonormal.

3.2.2 Writing the Vandermonde as a determinant

The product

$$\prod_{1\le i<j\le n} (x_i - x_j)^2$$

is the square of the Vandermonde determinant. Recall that the Vandermonde determinant is given by

$$\Delta(x_1, \dots, x_n) = \prod_{1\le i<j\le n} (x_j - x_i) = \det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}.$$

Thus, we have

$$\prod_{1\le i<j\le n} (x_i - x_j)^2 = \left(\det\left[x_i^{j-1}\right]_{i,j=1}^{n}\right)^2.$$

3.2.3 Orthogonalization by linear operations

Since determinants are invariant under elementary row or column operations, we can replace the monomials x^{j-1} by any sequence of monic polynomials of degree $j-1$. In particular, we choose the monic Hermite polynomials $p_{j-1}(x)$ and obtain

$$\det\left[x_i^{j-1}\right]_{i,j=1}^{n} = \det\left[p_{j-1}(x_i)\right]_{i,j=1}^{n}.$$

The orthogonality condition for these polynomials is

$$\int_{-\infty}^{\infty} p_j(x)p_k(x)e^{-x^2/2}\, dx = h_j\, \delta_{jk}.$$

We define the functions

$$\phi_j(x) = p_j(x)e^{-x^2/4},$$

and then introduce the orthonormal functions

$$\psi_j(x) = \frac{1}{\sqrt{h_j}}\phi_j(x) = \frac{1}{\sqrt{h_j}}\, p_j(x)e^{-x^2/4}.$$

Note that here the weight splits as $e^{-x^2/2} = e^{-x^2/4}e^{-x^2/4}$, which is useful in the next step.

3.2.4 Rewriting the density in determinantal form

Substituting the determinant form into the joint density (3.1), we have

$$p(x_1, \ldots, x_n) = \frac{1}{Z_{n,2}} \prod_{j=1}^{n} e^{-x_j^2/2} \Big[\det\Big[p_{j-1}(x_i)\Big]_{i,j=1}^{n}\Big]^2.$$

Incorporate the weight factors into the determinant by writing

$$\prod_{i=1}^{n} e^{-x_i^2/2} = \prod_{i=1}^{n} \left(e^{-x_i^2/4} \cdot e^{-x_i^2/4}\right),$$

so that

$$\prod_{i=1}^{n} e^{-x_i^2/4} \det\Big[p_{j-1}(x_i)\Big]_{i,j=1}^{n} = \det\Big[\phi_{j-1}(x_i)\Big]_{i,j=1}^{n}.$$

Thus, the joint density becomes

$$p(x_1, \ldots, x_n) = \frac{1}{\tilde{Z}_{n,2}} \Big[\det\Big[\phi_{j-1}(x_i)\Big]_{i,j=1}^{n}\Big]^2.$$

This squared-determinant structure is characteristic of determinantal point processes.

We now compute the k-point correlation function by integrating out the remaining $n-k$ variables:

$$\rho_k(x_1, \ldots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p(x_1, \ldots, x_n)\, dx_{k+1} \cdots dx_n.$$

Remark 3.2. When defining the k-point correlation function, one might initially expect a combinatorial factor corresponding to the number of ways of choosing k variables out of n, namely $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. The absence of an extra $k!$ in the denominator is due to the fact that $x_1, \ldots, x_k$ are fixed, and we are not integrating over all permutations of these variables.

Theorem 3.3 (Determinantal structure for squared-determinant densities)**.** *We have*

$$\rho_k(x_1, \ldots, x_k) = \det\Big[K_n(x_i, x_j)\Big]_{i,j=1}^{k},$$

with the correlation kernel given by

$$K_n(x, y) = \sum_{j=0}^{n-1} \psi_j(x)\psi_j(y).$$

Proof. We begin by writing the joint density as

$$p(x_1, \ldots, x_n) = \frac{1}{\tilde{Z}_{n,2}} \left[\det\Big[\phi_{j-1}(x_i)\Big]_{i,j=1}^{n}\right]^2.$$

Expanding the square of the determinant, we have

$$\left[\det\Big[\phi_{j-1}(x_i)\Big]_{i,j=1}^{n}\right]^2 = \sum_{\sigma,\tau \in S_n} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \prod_{i=1}^{n} \phi_{\sigma(i)-1}(x_i)\phi_{\tau(i)-1}(x_i),$$

where S_n denotes the symmetric group on n elements.

Next, to obtain the k-point correlation function $\rho_k(x_1, \ldots, x_k)$, we integrate out the remaining $n-k$ variables:

$$\rho_k(x_1, \ldots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p(x_1, \ldots, x_n)\, dx_{k+1} \cdots dx_n.$$

Since the joint density is symmetric under permutations of the variables, we may assume without loss of generality that the first k variables are the ones being fixed.

Substituting the expansion of the squared determinant into the expression for ρ_k, we have

$$\rho_k(x_1, \ldots, x_k) = \frac{n!}{(n-k)!\, \tilde{Z}_{n,2}} \sum_{\sigma,\tau \in S_n} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) \\ \left\{ \prod_{i=1}^{k} \phi_{\sigma(i)-1}(x_i)\phi_{\tau(i)-1}(x_i) \prod_{j=k+1}^{n} \int_{\mathbb{R}} \phi_{\sigma(j)-1}(x)\phi_{\tau(j)-1}(x)\, dx \right\}.$$

Now, change the functions $\phi_j(x)$ to the orthonormal functions $\psi_j(x)$ using the relation

$$\phi_j(x) = \sqrt{h_j}\, \psi_j(x).$$

This substitution yields

$$\int_{\mathbb{R}} \phi_{\sigma(j)-1}(x)\phi_{\tau(j)-1}(x)\, dx = \sqrt{h_{\sigma(j)-1}h_{\tau(j)-1}} \int_{\mathbb{R}} \psi_{\sigma(j)-1}(x)\psi_{\tau(j)-1}(x)\, dx.$$

By the orthonormality of the ψ_j's, we have

$$\int_{\mathbb{R}} \psi_{\sigma(j)-1}(x)\psi_{\tau(j)-1}(x)\, dx = \delta_{\sigma(j),\tau(j)}.$$

Therefore, for the indices $j = k+1, \ldots, n$, the integrals enforce the condition $\sigma(j) = \tau(j)$. As a result, the double sum over σ and τ reduces to a single sum over permutations on the first k indices, and the factors for the remaining indices simply contribute to the normalization constant.

Collecting these results, one deduces that

$$\rho_k(x_1, \ldots, x_k) = \text{const} \cdot \det\Big[K_n(x_i, x_j)\Big]_{i,j=1}^{k},$$

where the kernel is given by

$$K_n(x, y) = \sum_{j=0}^{n-1} \psi_j(x)\psi_j(y).$$

To complete the proof, one must verify that the normalization constant is indeed 1. We can achieve this by using the fact that p_n is the same as ρ_n. Then, integrating ρ_n over all variables gives the normalization constant, and we have

$$\int_{\mathbb{R}^n} \det\Big[\sum_{\ell=0}^{n-1} \psi_\ell(x_i)\psi_\ell(x_j)\Big]_{i,j=1}^{n} dx_1 \cdots dx_n = n!, \tag{3.2}$$

and the integral over $x_1 > \cdots > x_n$ is equal to 1, as it should be.

To prove (3.2), define the $n \times n$ matrix

$$A = \Big[\psi_{j-1}(x_i)\Big]_{i,j=1}^{n}.$$

Then, by the Cauchy–Binet formula,

$$\det\Big[K_n(x_i, x_j)\Big]_{i,j=1}^{n} = \det\Big[AA^\top\Big] = \det\Big[A\Big]^2.$$

The Andreief integration formula tells us that

$$\int_{\mathbb{R}^n} \det\Big[A\Big]^2 dx_1 \cdots dx_n = n! \det\Big[\int_{\mathbb{R}} \psi_{i-1}(x)\psi_{j-1}(x)\, dx\Big]_{i,j=1}^{n}.$$

Since the ψ_j's are orthonormal,

$$\int_{\mathbb{R}} \psi_{i-1}(x)\psi_{j-1}(x)\, dx = \delta_{ij},$$

and hence

$$\det\Big[\delta_{ij}\Big]_{i,j=1}^{n} = 1.$$

This completes the proof of the theorem. □

3.3 Christoffel–Darboux formula

Theorem 3.4 (Christoffel–Darboux Formula)**.** *Let $\{p_j(x)\}_{j\geq 0}$ be a family of orthogonal polynomials with respect to a weight function $w(x)$ on an interval $I \subset \mathbb{R}$. Denote by γ_j the leading coefficient of $p_j(x)$ (so that for monic polynomials, $\gamma_j = 1$) and by h_j their squared norms,*

$$\int_I p_j(x)p_k(x)w(x)\, dx = h_j\, \delta_{jk}.$$

Define the orthonormal functions

$$\psi_j(x) = \frac{1}{\sqrt{h_j}}\, p_j(x)\sqrt{w(x)}.$$

Then the kernel

$$K_n(x, y) = \sum_{j=0}^{n-1} \psi_j(x)\psi_j(y) = \sqrt{w(x)w(y)} \sum_{j=0}^{n-1} \frac{p_j(x)p_j(y)}{h_j},$$

admits the closed-form representation

$$K_n(x, y) = \sqrt{w(x)w(y)}\, \frac{\gamma_{n-1}}{\gamma_n}\, \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x - y}, \tag{3.3}$$

with the obvious continuous extension when $x = y$.

Proof. We begin by defining

$$S_n(x,y) = \sum_{j=0}^{n-1} \frac{p_j(x)p_j(y)}{h_j},$$

so that

$$K_n(x,y) = \sqrt{w(x)w(y)}\, S_n(x,y).$$

Our goal is to show that

$$(x-y)S_n(x,y) = \frac{1}{h_{n-1}}\Big[p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)\Big]. \tag{3.4}$$

To establish (3.4), we use the three-term recurrence relation satisfied by the orthogonal polynomials. In its standard form (with $p_j(x)$ monic), the recurrence is

$$x\, p_j(x) = p_{j+1}(x) + \alpha_j\, p_j(x) + \beta_j\, p_{j-1}(x), \qquad j \geq 0,$$

with the convention $p_{-1}(x) = 0$. (One has $\beta_j = \frac{h_j}{h_{j-1}}$ for monic orthogonal polynomials.) This recurrence comes from the two facts:

1. The polynomials are orthogonal with respect to the weight function $w(x)$ supported on the real line;
2. The operator of multiplication by x is self-adjoint with respect to the inner product induced by $w(x)$.

Write the recurrence for both $p_j(x)$ and $p_j(y)$:

$$\begin{aligned} x\, p_j(x) &= p_{j+1}(x) + \alpha_j\, p_j(x) + \beta_j\, p_{j-1}(x), \\ y\, p_j(y) &= p_{j+1}(y) + \alpha_j\, p_j(y) + \beta_j\, p_{j-1}(y). \end{aligned}$$

Multiplying the first equation by $p_j(y)$ and the second by $p_j(x)$ and then subtracting, we obtain

$$(x-y)\, p_j(x)p_j(y) = p_{j+1}(x)p_j(y) - p_j(x)p_{j+1}(y) + \beta_j\Big[p_{j-1}(x)p_j(y) - p_j(x)p_{j-1}(y)\Big].$$

Dividing by h_j and summing over $j = 0, \dots, n-1$ yields

$$(x-y)S_n(x,y) = \sum_{j=0}^{n-1} \frac{1}{h_j}\Big[p_{j+1}(x)p_j(y) - p_j(x)p_{j+1}(y)\Big] + \sum_{j=0}^{n-1} \frac{\beta_j}{h_j}\Big[p_{j-1}(x)p_j(y) - p_j(x)p_{j-1}(y)\Big].$$

A reindexing of the sums shows that they telescope, leaving only the boundary term at $j = n-1$. In particular, one finds

$$(x-y)S_n(x,y) = \frac{1}{h_{n-1}}\Big[p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)\Big].$$

This proves (3.4).

Finally, the factor $1/h_{n-1}$ may be expressed in terms of the leading coefficients by the relation

$$\frac{1}{h_{n-1}} = \frac{\gamma_{n-1}}{\gamma_n},$$

which completes the derivation of (3.3).

The continuous extension to $x = y$ is obtained via l'Hôpital's rule. □

E Problems (due 2025-03-09)

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