

Lectures on Random Matrices (Spring 2025)

Lecture 8: Cutting corners and loop equations

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1 Cutting corners: polynomial equations and distribution

1.1 Recap: polynomial equations

Recall the polynomial equation we proved in the last [Lecture 7](#). Fix $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$. Let $H \in \text{Orbit}(\lambda)$ be a random Hermitian matrix defined as

$$H = U \text{diag}(\lambda_1, \dots, \lambda_n) U^\dagger,$$

where U is Haar-distributed unitary matrix from $U(n)$. This is the case $\beta = 2$, but the statement holds for the cases $\beta = 1, 4$ with appropriate modifications. Let μ_1, \dots, μ_{n-1} be the eigenvalues of the $(n-1) \times (n-1)$ corner $H^{(n-1)}$.

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Lemma 1.1. *The distribution of μ_1, \dots, μ_{n-1} is the same as the distribution of the roots of the polynomial equation*

$$\sum_{i=1}^n \frac{\xi_i}{z - \lambda_i} = 0, \quad (1.1)$$

where ξ_i are i.i.d. random variables with the distribution χ_β^2 .

Recall also that this passage from λ to μ works inductively, and the distribution of the next level eigenvalues $\nu = (\nu_1 \geq \dots \geq \nu_{n-2})$ is given by the same polynomial equation, but with λ replaced by μ . In this way, we can define a *Markov map* from λ to μ , which is then iterated to construct the full array of eigenvalues of the corners of H .

For $\beta = \infty$, this map is deterministic, and is equivalent to successive differentiating the characteristic polynomial of H .

1.2 Extension to general β

We extend the polynomial equations to general β , by *declaring* (defining) that the general β corners distribution is powered by the passage from $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ to $\mu = (\mu_1 \geq \dots \geq \mu_{n-1})$, where μ solves (1.1) with ξ_i i.i.d. χ_β^2 . In this way, μ interlaces with λ . For $\beta = 1, 2, 4$, this definition reduces to the one with invariant ensembles with fixed eigenvalues λ .

1.3 Distribution of the eigenvalues of the corners

Let μ be obtained from λ by the general β corners operation.

Theorem 1.2. *The density of μ with respect to the Lebesgue measure is given by*

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} \prod_{1 \leq i < j \leq n-1} (\mu_i - \mu_j) \prod_{i=1}^{n-1} \prod_{j=1}^n |\mu_i - \lambda_j|^{\beta/2-1} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^{1-\beta}.$$

Proof. Let $\varphi_i = \xi_i / \sum_{j=1}^n \xi_j$. It is well-known¹ the joint density of $(\varphi_1, \dots, \varphi_n)$ is the (symmetric) Dirichlet density

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} w_1^{\beta/2-1} \dots w_n^{\beta/2-1} dw_1 \dots dw_{n-1}$$

(note that the density is $(n-1)$ -dimensional).

We need to compute the Jacobian of the transformation from φ to μ , if we write

$$\sum_{i=1}^n \frac{\varphi_i}{z - \lambda_i} = \frac{\prod_{i=1}^{n-1} (z - \mu_i)}{\prod_{i=1}^n (z - \lambda_i)},$$

and compute (as a decomposition into partial fractions):

$$\varphi_a = \frac{\prod_{i=1}^{n-1} (\lambda_a - \mu_i)}{\prod_{i \neq a} (\lambda_a - \lambda_i)}.$$

¹See Problem [H.2](#).

Therefore,

$$\frac{\partial \varphi_a}{\partial \mu_b} = \frac{\prod_{i=1}^{n-1} (\lambda_a - \mu_i)}{\prod_{i \neq a} (\lambda_a - \lambda_i)} \frac{1}{\mu_b - \lambda_a}.$$

The Jacobian is essentially the determinant of the matrix $1/(\mu_b - \lambda_a)$, which is the Cauchy determinant (Problem H.1). The final density is obtained from the symmetric Dirichlet density, but we plug in $w = \varphi$, and also multiply by the Jacobian. This completes the proof. \square

Corollary 1.3 (Joint density of the corners). *The eigenvalues $\lambda^{(k)}_j$, $1 \leq j \leq k \leq n$, of a random matrix from $\text{Orbit}(\lambda)$ form an interlacing array, with the joint density*

$$\propto \prod_{k=1}^n \prod_{1 \leq i < j \leq k} \left(\lambda_j^{(k)} - \lambda_i^{(k)} \right)^{2-\beta} \prod_{a=1}^{k+1} \prod_{b=1}^k \left| \lambda_a^{(k+1)} - \lambda_b^{(k)} \right|^{\beta/2-1}.$$

For $\beta = 2$, all factors disappear, and we get the uniform distribution on the interlacing array. This is the *uniform Gibbs property* which is important for other models, including discrete ensembles.

2 Loop equations

Let us write down the *loop equations* for the passage from the eigenvalues λ to the eigenvalues μ . These loop equations are due to [GH24] by a limit from a discrete system (related to Jack symmetric polynomials). Note that despite the name, these are not **equations**, but rather a statement that some expectations are holomorphic.

2.1 Formulation

Theorem 2.1. *We fix $n = 1, 2, \dots$ and $n+1$ real numbers $\lambda_1 \geq \dots \geq \lambda_{n+1}$. For $\beta > 0$, consider $n+1$ i.i.d. χ^2_β random variables ξ_i and set*

$$w_i = \frac{\xi_i}{\sum_{j=1}^{n+1} \xi_j}, \quad 1 \leq i \leq n+1.$$

We define n random points $\{\mu_1, \dots, \mu_n\}$ as n solutions to the equation

$$\sum_{i=1}^{n+1} \frac{w_i}{z - \lambda_i} = 0. \quad (2.1)$$

Take any polynomial $W(z)$ and consider the complex function:

$$f_W(z) = \mathbb{E} \left[\prod_{j=1}^n \exp(W(\mu_j)) \frac{\prod_{i=1}^{n+1} (z - \lambda_i)}{\prod_{j=1}^n (z - \mu_j)} \left(W'(z) + \sum_{i=1}^{n+1} \frac{\beta/2 - 1}{z - \lambda_i} + \sum_{j=1}^n \frac{1}{z - \mu_j} \right) \right]. \quad (2.2)$$

Then $f_W(z)$ is an entire function of z , in the following sense:

- For $z \in \mathbb{C} \setminus [\lambda_{n+1}, \lambda_1]$, the expectation in (2.2) defines a holomorphic function of z .

- This function has an analytic continuation to \mathbb{C} , which has no singularities.

Remark 2.2. Note that for z in $[\lambda_{n+1}, \lambda_1]$, the integral determining (2.2) might be divergent, and, therefore, analytic continuation is the proper way to define $f_W(z)$, $z \in \Omega$.

2.2 Proof

3 Applications of loop equations

H Problems (due 2025-03-25)

H.1 Cauchy determinant

Prove the Cauchy determinant formula:

$$\det \left(\frac{1}{x_i - y_j} \right)_{1 \leq i, j \leq n} = \frac{\prod_{i < j} (x_i - x_j)(y_i - y_j)}{\prod_{i, j} (x_i - y_j)}.$$

H.2 Dirichlet density

Find or prove the first statement in the proof of Theorem 1.2 about the symmetric Dirichlet density arising from normalizing the ξ_i 's to φ_i 's.

References

[GH24] V. Gorin and J. Huang, *Dynamical loop equation*, Ann. Probab. **52** (2024), no. 5, 1758–1863. arXiv:2205.15785 [math.PR]. [↑3](#)

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