Mini Course: Dimers and Embeddings

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Exercise Session Four: t-embeddings and t-holomorphicity

1 Review

Definition: *gauge equivalence*
Two weight functions ν_1, ν_2 are said to be gauge equivalent if there is a function $F : V(G) \to \mathbb{R}$ such that for any edge vv', $\nu_1(vv') = F(v)F(v')\nu_2(vv')$. Gauge equivalent weights define the same probability measure μ.

Definition: *origami square root function*
A function $\eta : V(G) = B \cup W \to \mathbb{T}$ is said to be an *origami square root function* if it satisfies the identity

$$\overline{\eta_b}\,\overline{\eta_w} = \frac{dT(bw^*)}{|dT(bw^*)|}$$

for all pairs (b,w) of white and black neighbouring faces of G^*.

Definition: *t-white-holomorphicity*
Let η be an origami square root function. A function $F : B \to \mathbb{C}$ is called t-white-holomorphic at $w \in W$ if

$$\begin{cases} F(b) \in \eta_b\mathbb{R} & \forall b \in B, b \sim w, \\ \oint_{\partial w} F\,dT = 0 & (\iff (K_T F)(w) = \sum_{b\sim w} K_T(w,b)F(b) = 0). \end{cases}$$

2 Questions

1. **Gauge equivalence.** For a planar bipartite graph, show that two weight functions are gauge equivalent if and only if their face weights are all equal, where the face weight X_f of a face f with vertices w_1, b_1, …, w_k, b_k is the "alternating product" of its edge weights:

$$X_f := \prod_{j=1}^{k} \frac{\nu(w_1b_1)\cdot \ldots \cdot \nu(w_kb_k)}{\nu(b_1w_2)\cdot \ldots \cdot \nu(w_kb_{k+1})}$$

(here we index cyclically so we think of $k+1 = 1$).

Hint: For the difficult direction, use some basic homology theory.

2. **Origami square root.** Recall that another definition of the origami map O is

$$dO(bw^*) = \eta_w^2 dT(bw^*)$$

where η is any origami square root function.

a) Show that

$$\eta_w^2 dT(bw^*) = \eta_w\overline{\eta_b}|dT(bw^*)| = \overline{\eta_b}^2\overline{dT(bw^*)}$$

b) Show that we can "extend" dO to the interior of each face of T, i.e. we can define a smooth differential form dO on the union of faces of T by

$$dO(z) := \begin{cases} \eta_w^2 dz & \text{if } z \text{ in a white face of } T \\ \overline{\eta_b}^2 d\bar{z} & \text{if } z \text{ in a black face of } T \end{cases}.$$

Show that this definition is consistent along edges of T, and that dO is a closed form (by which we mean it integrates to 0 around closed loops) inside the domain covered by T.

3. t-holomorphicity. Assume that all white faces of the t-embedding T are triangles. Let $F : B \to \mathbb{C}$ be a function such that $\forall b \in B$, $F(b) \in \eta_b\mathbb{R}$. Show that F can be extended to a white vertex w such that $Proj(F(w);\eta_b\mathbb{R}) = F(b)$ for all $b \sim w$, if and only if

$$\oint_{\partial w} F dT = 0.$$

4. Closed forms. Below, if we multiply a function on (primal) vertices by a discrete one form on G^*, we will use the upper subscript to indicate which primal vertex adjacent to the edge we use to evaluate the function. So if F is a function on all primal vertices, the form $(F^\bullet dT)(bw^*) := F^\bullet(b)dT(bw^*)$, and $(F^\circ dT)(bw^*) := F^\circ(w)dT(bw^*)$.

a) Let U be a simply connected region in the domain of a tembedding and F be a t-white-holomorphic function on a punctured region U. Then, on edges not adjacent to boundary white faces,

$$2F^\bullet dT = F^\circ dT + \overline{F^\circ dO}$$

and $F_w^\bullet dT$ is a closed form in U away from the boundary (i.e., the integral over any closed contour γ running over interior edges vanishes).

b) If F_b and F_w are respectively t-black- and t-white-holomorphic functions on some region U, then, on edges not adjacent to boundary faces, the identity

$$F_w^\bullet F_b^\circ dT = \frac{1}{2}\mathrm{Re}\left(F_w^\circ F_b^\bullet dT + F_w^\circ \overline{F_b^\bullet} dO\right)$$

holds and the form $F_w^\bullet F_b^\circ dT$ is closed in U away from the boundary.

5. Perfect t-embeddings. Check that the origami map of a perfect t-embedding is real valued *on the boundary edges* of the embedding.