Lectures on Random Matrices (Spring 2025)
Lecture 7: Double contour integral kernel. Steepest descent and local statistics

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Contents

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1 Steepest descent for the GUE kernel

1.1 Recap

We continue the asymptotic analysis of the GUE kernel.

The GUE correlation kernel is defined by

$$K_n(x, y) = \sum_{j=0}^{n-1} \psi_j(x)\psi_j(y),$$

where the functions

$$\psi_j(x) = \frac{1}{\sqrt{h_j}}\, p_j(x)\, e^{-x^2/4}$$

are built from the monic Hermite polynomials $p_j(x)$ with normalization constants h_j ensuring that the ψ_j's form an orthonormal system in $L^2(\mathbb{R})$.

Using the generating function

$$\exp\Bigl(xt - \frac{t^2}{2}\Bigr) = \sum_{n\geq 0} p_n(x)\frac{t^n}{n!},$$

one obtains by Cauchy's integral formula

$$p_n(x) = \frac{n!}{2\pi i}\oint_C \frac{\exp\Bigl(xt - \frac{t^2}{2}\Bigr)}{t^{n+1}}\, dt,$$

which leads to

$$\psi_n(x) = \frac{e^{-x^2/4}}{\sqrt{h_n}}\,\frac{n!}{2\pi i}\oint_C \frac{\exp\Bigl(xt - \frac{t^2}{2}\Bigr)}{t^{n+1}}\, dt.$$

Starting from the Fourier transform identity

$$\int_{-\infty}^{\infty} \exp\Bigl(-\frac{t^2}{2} + i\,t\,x\Bigr)\, dt = \sqrt{2\pi}\, e^{-x^2/2},$$

and differentiating with respect to x, then changing variables, one obtains

$$\psi_n(x) = \frac{i\, e^{x^2/4}}{\sqrt{2\pi\, h_n}}\int_{-i\infty}^{i\infty} s^n\, e^{s^2/2 - s\,x}\, ds.$$

By inserting the above representations for $\psi_n(x)$ into the kernel sum, one arrives at the double contour integral formula (after conjugation and the trick with removing $1/(s-t)$):

$$K_n(x, y) = \frac{1}{(2\pi)^2}\oint_C dt \int_{-i\infty}^{i\infty} ds\, \frac{\exp\Bigl\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\Bigr\}}{s - t}\,\Bigl(\frac{s}{t}\Bigr)^n.$$

The integration contour C is a small contour around 0, and s is passing to the right of C.

This representation is especially useful for performing asymptotic analysis (for example, via the steepest descent method) and for deriving results such as the semicircle law.

1.2 Scaling

Let us now consider the GUE kernel,

$$K_n(x,y) = \frac{1}{(2\pi)^2} \oint_C dt \int_{-i\infty}^{i\infty} ds \, \frac{\exp\left\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\right\}}{s-t} \left(\frac{s}{t}\right)^n .$$

We know from the Wigner semicircle law (established for real symmetric matrices with general iid entries in in Lecture 2, and for the GUE in Lecture 4) that the eigenvalues live on the scare $\sqrt{n}$. This means that to capture the local asymptotics, we need to scale

$$x = X\sqrt{n} + \frac{\Delta x}{\sqrt{n}}, \qquad y = Y\sqrt{n} + \frac{\Delta y}{\sqrt{n}}, \qquad \Delta x, \Delta y \in \mathbb{R}. \tag{1.1}$$

Moreover, if $X \neq Y$ (i.e., different global positions), one can check that the kernel vanishes. In other words, the local behaviors at different global positions are independent. In what follows, we take $Y = X$.

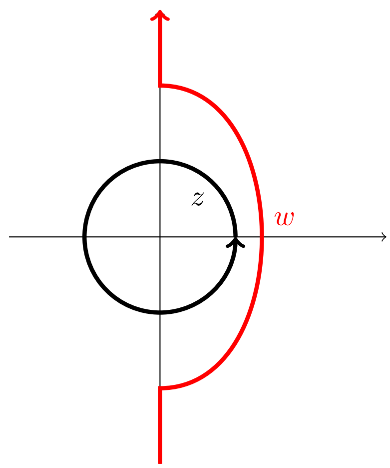


Figure 1: Integration contours for the GUE kernel.

Let us also make a change of the integration variables:

$$t = z\sqrt{n}, \qquad s = w\sqrt{n}.$$

The integration contours for z and w look the same as for t and s, up to a rescaling (Figure 1). However, as 0 and $t = s$ are the only singularities in the integrand, we can deform the z, w contours as we wish, while keeping $|z| < |w|$ and the general shape as in Figure 1.

We thus have:

$$\begin{aligned} &K_n(X\sqrt{n} + \Delta x/\sqrt{n}, X\sqrt{n} + \Delta y/\sqrt{n}) \\ &\quad = \frac{\sqrt{n}}{(2\pi)^2} \oint_C dz \int_{-i\infty}^{i\infty} dw \, \frac{\exp\left\{n\left(\log w - \log z + \frac{w^2}{2} - \frac{z^2}{2} + X(z-w) + \frac{z\Delta x - w\Delta y}{n}\right)\right\}}{w-z}. \end{aligned} \tag{1.2}$$

Remark 1.1. The logarithms in the exponent are harmless, since for the estimates we only need the real parts of the logarithms, and for the main contributions, we will have $z \approx w$, so any phases of the logarithms would cancel.

The asymptotic analysis of double contour integrals like (1.2) in the context of determinantal point processes was pioneered in [Oko02, Section 3].

1.3 Critical points

Let us define

$$S(z) := \frac{z^2}{2} + \log z - Xz.$$

Then the exponent contains $n\,(S(w) - S(z))$. According to the steepest descent ideology, we should deform the integration contours to pass through the critical point(s) z_{cr} of $S(z)$. Moreover, the new w contour should maximize the real part of $S(z)$ at z_{cr}, and the new z contour should minimize it. If $S''(z_{cr}) \neq 0$, it is possible to locally choose such contours, they will be perpendicular to each other at z_{cr}.

Thus, we need to find the critical points of $S(z)$. They are found from the quadratic equation:

$$S'(z) = z + \frac{1}{z} - X = 0, \qquad z_{cr} = \frac{X \pm \sqrt{X^2 - 4}}{2}. \tag{1.3}$$

Depending on whether $|X| < 2$, there are three cases. Unless $|X| = 2$ (when equation (1.3) has a single root), we have $S''(z_{cr}) \neq 0$. We will consider the three cases in Sections 1.4 to 1.6 below.

1.4 Imaginary critical points: $|X| < 2$, "bulk"

When $|X| < 2$, the critical points are complex conjugate. Denote them by z_{cr} and $\overline{z_{cr}}$. Since $S(z)$ has real coefficients, we have

$$\operatorname{Re} S(z_{cr}) = \operatorname{Re} S(\overline{z_{cr}}).$$

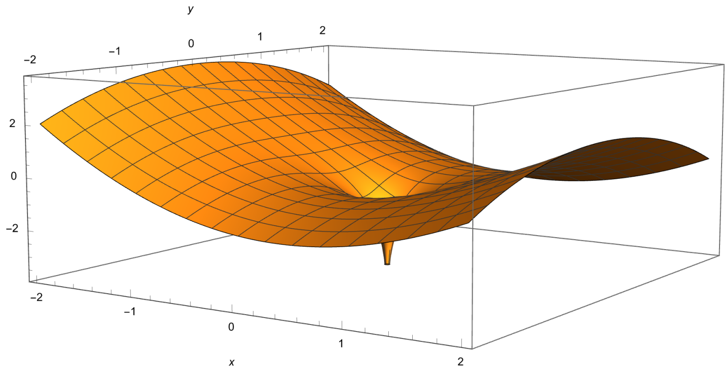
Thus, we need to consider the contribution from both points. For simplicity of the computations, let us consider only the case $X = 0$. See Problem G.1. We have

$$z_{cr} = i, \qquad S''(z_{cr}) = 2.$$

The behavior of $\operatorname{Re} S(z)$ on the complex plane can be illustrated by a 3D plot or by a region plot of the regions where $\operatorname{Re} S(z) - \operatorname{Re} S(z_{cr})$ has constant sign. See Figure 2 for an illustration in the case $X = \frac{1}{2}$. (We take $X \neq 0$ to break symmetry, for a better intuition.)

From the region plot, we see that the new z contour should pass through the shaded region $\operatorname{Re} S(z) - \operatorname{Re} S(z_{cr}) > 0$, and the new w contour should pass through the unshaded region $\operatorname{Re} S(z) - \operatorname{Re} S(z_{cr}) < 0$.

Deforming the contours from Figure 1 to the new contours is impossible without passing through the residue at $w = z$. Moreover, this residue appears only for certain values of z. Namely, for $X = 0$, let us first make the z contour to be the positively (counterclockwise) oriented unit circle. It passes through the critical points $z_{cr} = i$ and $\overline{z_{cr}} = -i$. Since the original w contour



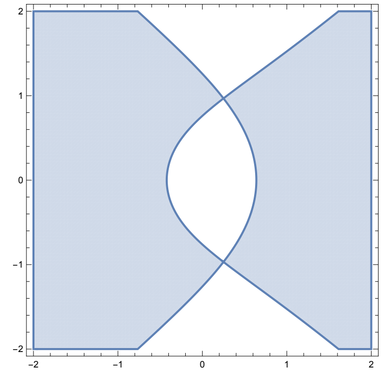


Figure 2: A 3D plot and a region plot of the regions where $\operatorname{Re} S(z) - \operatorname{Re} S(z_{cr})$ is positive (highlighted) or negative, in the case $X = \frac{1}{2}$. In this case, $z_{cr} \approx 0.25 + 0.96i$.

is to the right of the z contour, we only encounter the residue when z is in the right half of the circle.

Thus, we can write

$$\oiint_{\text{old contours}} = \oiint_{\text{new contours}} + \int_{-i}^{i} 2\pi i \operatorname{Res}_{w=z} dz, \tag{1.4}$$

where in the single integral, the z contour passes to the right of the origin, along the right half of the unit circle.

It remains to consider the two integrals in the right-hand side of (1.4). Recall that the correlation functions are defined relative to a reference measure, and the right object to scale is

$$K_n(x,y)dy = \frac{1}{\sqrt{n}} d\left(\Delta y\right).$$

The extra factor $n^{-1/2}$ compensates the prefactor $\sqrt{n}$ in (1.2).

The single integral takes the form

$$\frac{-i}{2\pi}\int_{-i}^{i} e^{z(\Delta x - \Delta y)}\, dz = \frac{\sin\left(\Delta x - \Delta y\right)}{\pi(\Delta x - \Delta y)}, \qquad \Delta x, \Delta y \in \mathbb{R}. \tag{1.5}$$

Definition 1.2. The *sine kernel* is defined as

$$K_{\text{sine}}(x,y) \coloneqq \begin{cases} \dfrac{\sin(x-y)}{\pi(x-y)}, & x \neq 0, \\ \dfrac{1}{\pi}, & x = 0. \end{cases}$$

(The value at $x = y$ is defined by continuity.)

This kernel is translation invariant, and is often defined with a single argument, as $K_{\text{sine}}(x-y)$.

The double integral has both contours in the "steepest descent" regime, which means that the main contribution is

$$\text{const} \cdot \frac{e^{n(\operatorname{Re} S(z_{cr}) - \operatorname{Re} S(z_{cr}))}}{\sqrt{n}} \sim \frac{\text{const}}{\sqrt{n}}.$$

At this rate, the double integral over the new contours *does not* contribute to the asymptotics of the correlation functions. Recall that the correlation functions are expressed as finite-dimensional determinants of the kernel $K_n(x, y)$, and the error $O(n^{-1/2})$ is negligible in the limit $n \to +\infty$. This is because the main term comes from the single integral, which does not vanish.

We have established the following result:

Proposition 1.3 (Bulk asymptotics at $X = 0$)**.** *The correlation kernel K_n of the GUE has the following asymptotics close to zero as $n \to +\infty$:*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} K_n\left(\frac{\Delta x}{\sqrt{n}}, \frac{\Delta y}{\sqrt{n}}\right) = K_{\text{sine}}\left(\Delta x, \Delta y\right), \qquad \Delta x, \Delta y \in \mathbb{R}.$$

Consequently, the eigenvalues of the GUE converge to the sine process determined by the sine kernel (Definition 1.2), in the sense of finite-dimensional distributions.

Remark 1.4. Beyond $X = 0$, the local correlations are essentially the same, up to rescaling of the real line by a constant factor (depending on the semicircle density). See Problem G.1.

1.5 Real critical points: $|X| > 2$, "large deviations"

For $X^2 > 4$, both solutions (1.3) are real. Let us assume $X > 2$, the case $X < 2$ is similar. For $X > 2$, both solutions are positive. Label these solutions as

$$z_+ = \frac{X + \sqrt{X^2 - 4}}{2}, \qquad z_- = \frac{X - \sqrt{X^2 - 4}}{2}, \quad \text{so that} \quad z_+ z_- = 1.$$

A straightforward check reveals that $z_+ > 1$ and $z_- < 1$ (for $X > 2$). Note that $S''(z) = 1 - z^{-2}$, which is positive for $z_+ > 1$ and negative for $z_- < 1$. Thus, the critical points z_+ and z_- are a local minimum and a local maximum. A crucial observation is that

$$S(z_+) < S(z_-).$$

One can deform the z integration contour to pass through z_- and the w contour to pass through z_+. Then, on these contours, one can show that

$$\operatorname{Re} S(w) - \operatorname{Re} S(z) < 0.$$

According to the steepest descent ideology, we see that the main exponential behavior of the double contour integral is

$$\exp\left\{n\left(\operatorname{Re} S(z_+) - \operatorname{Re} S(z_-)\right)\right\} = O(e^{-\delta(X)n}), \qquad |X| > 2. \tag{1.6}$$

Here $\delta(X) > 0$ for $|X| > 2$, and $\delta(X) \to 0$ when $|X| \to 2$.

The outcome (1.6) reflects the fact that the Wigner semicircle law places all eigenvalues inside the interval $|X| \le 2$. The probability to see even a single eigenvalue outside $[-2, 2]$ is exponentially small.

This exponential decay corresponds to a large deviation regime. Indeed, if at least one of the diagonal entries of the matrix is unusually large, this corresponds to the maximal eigenvalue to get outside the interval $[-2, 2]$. See also Problem G.2.

1.6 Double critical point: $|X| = 2$, "edge"

Throughout the subsection, we assume that $X = 2$. The case $X = -2$ is symmetric.

When $X = 2$, the two solutions in (1.3) merge into a double critical point $z_{cr} = 1$. We have

$$S'(1) = 0, \qquad S''(1) = 0, \qquad S'''(1) = 2.$$

Thus, the usual quadratic approximation fails and one must expand to third order. Writing

$$z = 1 + u, \qquad w = 1 + v,$$

with u, v small, we have

$$S(1+u) = S(1) + \frac{S'''(1)}{6}\, u^3 + O(u^4) = S(1) + \frac{u^3}{3} + O(u^4),$$

and similarly for $S(1+v)$. Hence, the difference in the exponents becomes

$$S(1+v) - S(1+u) = \frac{v^3 - u^3}{3} + O(u^4 + v^4).$$

To capture the correct asymptotics, we rescale the local variables by setting

$$u = \frac{U}{n^{1/3}}, \qquad v = \frac{V}{n^{1/3}},$$

so that

$$n\Big[S(1+v) - S(1+u)\Big] = \frac{V^3 - U^3}{3} + O\Big(n^{-1/3}\Big).$$

Moreover, the correct edge scaling for the spatial variables is obtained by writing

$$x = 2\sqrt{n} + \frac{\xi}{n^{1/6}}, \qquad y = 2\sqrt{n} + \frac{\eta}{n^{1/6}}, \qquad \xi, \eta \in \mathbb{R}.$$

We have

$$n\left(S(w) - S(z)\right) = n^{1/3}(\xi - \eta) + \frac{V^3 - U^3}{3} + \xi U - \eta V + O\Big(n^{-1/3}\Big).$$

The terms $n^{1/3}(\xi - \eta)$ are harmless as they can be removed by conjugation.

The region plot of $\operatorname{Re} S(z) - \operatorname{Re} S(1)$ (shown in Figure 3) makes sure that we can deform the z contour so that it passes through $z_{cr} = 1$ as the new U contour at the angles $\pm\frac{2\pi}{3}$ (where $\operatorname{Re} U^3 > 0$), we can deform the w contour so that it passes through $z_{cr} = 1$ as the new V contour at the angles $\pm\frac{\pi}{3}$ (where $\operatorname{Re} V^3 < 0$). This will ensure the convergence of the new double integral.

Thus, we have shown that under the rescaling, the GUE correlation kernel $K_n(x, y)\,dy$ converges to a new kernel.

Definition 1.5. Define the *Airy kernel* on $\mathbb{R}$ by

$$K_{\mathrm{Ai}}(\xi, \eta) = \frac{1}{(2\pi i)^2} \int_{e^{-\frac{\pi i}{3}}\infty}^{e^{\frac{\pi i}{3}}\infty} dV \int_{e^{-\frac{2\pi i}{3}}\infty}^{e^{2\frac{\pi i}{3}}\infty} dU\, \frac{\exp\left\{\frac{V^3 - U^3}{3} + U\,\xi - V\,\eta\right\}}{V - U}.$$

For another formula for the Airy kernel which does not involve integrals, see Problem G.3.

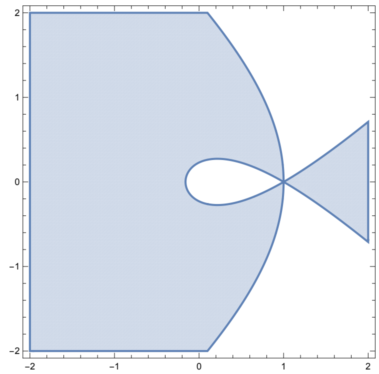


Figure 3: The plot of the region $\operatorname{Re} S(z) - \operatorname{Re} S(1) > 0$ for $X = 2$.

Proposition 1.6. *We have*

$$\lim_{n\to\infty} \frac{1}{n^{1/6}} K_n\Big(2\sqrt{n} + \frac{\xi}{n^{1/6}}, 2\sqrt{n} + \frac{\eta}{n^{1/6}}\Big) \to K_{\mathrm{Ai}}(\xi, \eta).$$

Consequently, the eigenvalue statistics at the edge of the spectrum converge to the Airy point process, in the sense of fine-dimensional distributions.

1.7 Airy kernel, Tracy–Widom distribution, and convergence of the maximal eigenvalue

Let us make a few remarks on the asymptotic results of Propositions 1.3 and 1.6. First, a rigorous justification of convergence of contour integrals requires some estimates on the error terms in the steepest descent analysis, but these estimates are mild and not hard to obtain.

Second, the GUE has the maximal eigenvalue λ_{max}. It is reasonable to assume that the Airy process also (almost surely) admits a maximal point (usually denoted by $\mathfrak{a}_1$), and that λ_{max} converges to $\mathfrak{a}_1$ under appropriate rescaling:

$$\lim_{n\to\infty} n^{\frac{1}{6}} \big(\lambda_{max} - 2\sqrt{n}\big) = \mathfrak{a}_1. \tag{1.7}$$

This is indeed the case, but to show (1.7), one needs to show the convergence in distribution:

$$\lim_{n\to\infty} \mathbb{P}\Big(n^{1/6}(\lambda_{max} - 2\sqrt{n}) \le x\Big) \to \mathbb{P}(\mathfrak{a}_1 \le x). \tag{1.8}$$

Both events (1.8) are so-called *gap probabilities*, for example,

$$\mathrm{P}(\mathfrak{a}_1 \le x) = \mathrm{P}(\text{there are no eigenvalues in the interval } (x, \infty)),$$

which is expressed as the Fredholm determinant

$$\det\,(1 - K_{\mathrm{Ai}})_{(x,\infty)} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int_x^\infty dy_1 \int_x^\infty dy_2 \cdots \int_x^\infty dy_m \det_{i,j=1}^{m} K_{\mathrm{Ai}}(y_i, y_j). \tag{1.9}$$

Thus, to get (1.8)), one needs to show the convergence of sums like this for the GUE kernel to the corresponding sums for the Airy kernel. This is doable, but tedious.

Moreover, to get convergence in distribution of random variables, one would also have to argue either *tightness*, or independently show that (1.9) defines a cumulative probability distribution function in x:

$$F_2(x) = \det\left(1 - K_{\mathrm{Ai}}\right)_{(x,\infty)}. \tag{1.10}$$

The distribution (1.10) is known as the *GUE Tracy–Widom distribution*. The subscript 2 indicates that $\beta = 2$. There are distributions F_β for all beta, most notably, the GOE and GSE distributions. The classical distributions F_1, F_2, F_4 also appear as fluctuation distributions in interacting particle systems, while other beta values do not quite appear in the particle systems domain.

More details may be found in the original papers [TW93], [For93], [TW94].

1.8 Remark: what happens for general β?

- The determinantal structure exploited above is special to the $\beta = 2$ case. In contrast, for $\beta = 1$ (GOE) and $\beta = 4$ (GSE) the eigenvalue correlations are expressed in terms of *Pfaffians* rather than determinants. This happens before and after the scaling limit.

- Earlier attempts to extend the $\beta = 2$ techniques were determinantal. For example, one can replace the squared Vandermonde $\prod_{i<j}(x_i - x_j)^2$ with

$$\prod_{i<j}(x_i - x_j)(x_i^{\beta/2} - x_j^{\beta/2}).$$

  This is known as the *Muttalib–Borodin* ensemble [FW17], and the kernel can be computed in a similar way using (bi)orthogonalization.

- Local eigenvalue statistics of general β-ensembles converge to the so-called *general β sine process* and *general β Airy process* in the bulk and at the edge, respectively. Detailed analyses of this convergence can be found in [RRV11], [VV09], [GS18], and the literature referenced in the recent work [GXZ24].

2 Cutting corners: setup

We begin a new topic, which will be the main focus for this and the next week.

In random matrix theory, one often studies the entire spectrum of an $n \times n$ matrix ensemble such as the Gaussian Unitary Ensemble (GUE), the Gaussian Orthogonal Ensemble (GOE), or, more generally, β-ensembles. However, it is also natural to examine the spectra of *principal minors* of such matrices.

When we say "cutting corners," we typically refer to extracting a top-left $k \times k$ submatrix (or *corner*) out of an $n \times n$ random matrix H and then looking at the interplay among the eigenvalues of all corners $k = 1, \ldots, n$. This forms a *nested* family of spectra, often described by interlacing (or Gelfand–Tsetlin) patterns.

The *GUE corners process* is a classical example of this phenomenon. If H is an $n \times n$ GUE matrix, then the top-left $k \times k$ corners (for $1 \le k \le n$) have jointly distributed eigenvalues that exhibit a determinantal structure. We will employ the technique of *polynomial equations* and then *loop equations* to study global limits (note that they are not suitable to get local limits like sine and Airy processes).

So far, we have the following access to eigenvalues and corners:

1. For $\beta = 1, 2, 4$, we have the actual matrices, and can cut the corners in the usual way.
2. For general β, we have the joint eigenvalue distribution with the interaction term $\prod_{i<j} |x_i - x_j|^\beta$, which is an interpolation.
3. For general β, we also have the Dumitriu–Edelman tridiagonal model [DE02].

Cutting corners from the tridiagonal matrix is not a good idea, for many reasons. The simplest might be that the $(n-1) \times (n-1)$ corner eigenvalues do not have the same distribution (up to changing n) as the general β ensemble eigenvalues. Maybe we might cut the lower right corners? Well, this is not a good idea either, because the total number of random variables (the "noise") in the tridiagonal matrix is $O(n)$, while the number of eigenvalues of all corners is $O(n^2)$.

3 Corners of Hermitian matrices

3.1 Principal corners

Let H be an $n \times n$ Hermitian matrix. For each $1 \le k \le n$, define the *top-left $k \times k$ corner* $H^{(k)}$ by

$$H^{(k)} \;=\; \bigl[H_{ij}\bigr]_{1 \le i,j \le k}.$$

Since H is Hermitian, each $H^{(k)}$ is also Hermitian. Let

$$\lambda_1^{(k)} \;\ge\; \lambda_2^{(k)} \;\ge\; \cdots \;\ge\; \lambda_k^{(k)}$$

denote the eigenvalues of $H^{(k)}$. Then the collection

$$\bigl\{\lambda_j^{(k)} : 1 \le j \le k \le n\bigr\}$$

is called the *corners spectrum* (or *minor spectrum*) of H. When H is random, this triangular array of eigenvalues becomes a random point configuration in the two-dimensional set $\{1, \ldots, n\} \times \mathbb{R}$.

3.2 Interlacing

A fundamental feature of Hermitian matrices is that the eigenvalues of corners interlace with the eigenvalues of the full matrix:

Proposition 3.1. *If $\nu_1 \ge \cdots \ge \nu_n$ are the eigenvalues of H itself (i.e., the full $n \times n$ matrix), and $\mu_1 \ge \cdots \ge \mu_{n-1}$ are the eigenvalues of $H^{(n-1)}$, then we have:*

$$\nu_1 \;\ge\; \mu_1 \;\ge\; \nu_2 \;\ge\; \mu_2 \;\ge\; \ldots \;\ge\; \mu_{n-1} \;\ge\; \nu_n.$$

Proof. One can prove the statement using the Courant–Fischer (min–max) characterization of eigenvalues, often referred to as the variational principle. Recall that for an $n \times n$ Hermitian matrix H with ordered eigenvalues $\nu_1 \ge \nu_2 \ge \cdots \ge \nu_n$, the j-th largest eigenvalue ν_j admits the variational characterization

$$\nu_j = \max_{\substack{V \subset \mathbb{F}^n \\ \dim(V)=j}} \min_{\substack{x \in V \\ x \neq 0}} \frac{x^* H\, x}{x^* x} = \min_{\substack{W \subset \mathbb{F}^n \\ \dim(W)=n-j+1}} \max_{\substack{x \in W \\ x \neq 0}} \frac{x^* H\, x}{x^* x},$$

where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$, or the quaternions (depending on $\beta = 1, 2, 4$, respectively). We leave this as Problem G.4. $\square$

The same interlacing property holds for real symmetric matrices ($\beta = 1$), and in the case $\beta = 4$. Therefore, it is natural to require this property for all β-ensembles.

3.3 Orbital measure

It is natural to consider an extended setup, and take the matrix H to not just be GUE, but instead fix its eigenvalues. Let

$$H = U \Lambda U^\dagger, \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n),$$

where Λ is fixed and $U \in U(n)$ is Haar (uniformly) distributed. Denote the set of all such H by $\operatorname{Orbit}(\lambda)$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $\lambda_1 \ge \cdots \ge \lambda_n$.

Then, if we understand the distribution structure of all corners of a random $H \in \operatorname{Orbit}(\lambda)$, we can then "average over" the GUE eigenvalue ensemble distribution of λ to get the GUE corners process.

Remark 3.2. The setting with orbits presents a bridge into "asymptotic representation theory". Namely, as $n \to \infty$, how does the corners distribution look like? We may ask for a characterization of *all the ways* how $\lambda^{(n)} = (\lambda_1^{(n)} \ge \ldots \lambda_n^{(n)})$ goes to infinity, in such a way that the corners spectrum converges on all levels $k = 1, \ldots, K$ for arbitrary K (independent of n). This problem was solved in [OV96]. More direct formulas for projections of orbital measures were obtained in [Ols13].

4 Polynomial equations and joint distribution

Fix $\lambda = (\lambda_1 \ge \ldots \ge \lambda_n)$. Let $H \in \operatorname{Orbit}(\lambda)$ be a random matrix (in the case $\beta = 2$, but the proof works for $\beta = 1, 4$ as well). Let $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of the $(n-1) \times (n-1)$ corner $H^{(n-1)}$.

Lemma 4.1. *The distribution of $\mu_1, \ldots, \mu_{n-1}$ is the same as the distribution of the roots of the polynomial equation*

$$\sum_{i=1}^{n} \frac{\xi_i}{z - \lambda_i} = 0, \tag{4.1}$$

where ξ_i are i.i.d. random variables with the distribution χ^2_β.

Proof. $\mu_1, \dots, \mu_{n-1}$ are the roots of the following equation with the determinant of order $n+1$:

$$\det \begin{pmatrix} U\operatorname{diag}(\lambda)U^\dagger - zI_N & v^\top \\ v & 0 \end{pmatrix} = 0, \qquad v = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Indeed, expanding the determinant along the last row, we get the $(n-1)$th determinant, which corresponds to cutting the corner.

Next, multiply the determinant by $\begin{pmatrix} U^\dagger & 0 \\ 0 & 1 \end{pmatrix}$ on the left and $\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$ on the right:

$$\det \begin{pmatrix} \operatorname{diag}(\lambda) - zI_N & u^\dagger \\ u & 0 \end{pmatrix} = 0,$$

where $u^\dagger = U^\dagger v^\top$ is the last row of $U^\dagger$. The determinant now can be expressed as

$$\det = -\prod_{i=1}^{n}(\lambda_i - z) \sum_{i=1}^{n} \frac{|u_i|^2}{\lambda_i - z}.$$

Since u is a row of a Haar unitary matrix, it is distributed uniformly on the unit sphere in $\mathbb{C}^n$. However, we can identify it with a normalized vector from a rotationally invariant measure on $\mathbb{C}^n$, the best of which is Gaussian. This completes the proof. $\square$

Remark 4.2. Lemma 4.1 provides another proof of the eigenvalue interlacing property. Indeed, assume that all ξ_i are rational. Then equation (4.1) is essentially $P'(z) = 0$, where $P(z)$ is a product of powers of the $(z - \lambda_i)$'s (the powers depend on the ξ_i's). As the roots of the derivative of a polynomial interlace with the roots of the polynomial, we get the interlacing property.

Let us now compute the distribution of μ_i explicitly:

Theorem 4.3. *The density of μ with respect to the Lebesgue measure is given by*

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} \prod_{1 \le i < j \le n-1} (\mu_i - \mu_j) \prod_{i=1}^{n-1} \prod_{j=1}^{n} |\mu_i - \lambda_j|^{\beta/2-1} \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^{1-\beta}.$$

Proof. Let $\varphi_i = \xi_i / \sum_{j=1}^n \xi_j$. The joint density of $(\varphi_1, \dots, \varphi_n)$ is the (symmetric) Dirichlet density

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} w_1^{\beta/2-1} \dots w_n^{\beta/2-1} \, dw_1 \dots dw_{n-1}$$

(note that the density is $(n-1)$-dimensional).

We need to compute the Jacobian of the transformation from φ to μ, if we write

$$\sum_{i=1}^{n} \frac{\varphi_i}{z - \lambda_i} = \frac{\prod_{i=1}^{n-1}(z - \mu_i)}{\prod_{i=1}^{n}(z - \lambda_i)},$$

and compute (as a decomposition into partial fractions):

$$\varphi_a = \frac{\prod_{i=1}^{n-1}(\lambda_a - \mu_i)}{\prod_{i\neq a}(\lambda_a - \lambda_i)}.$$

Therefore,

$$\frac{\partial \varphi_a}{\partial \mu_b} = \frac{\prod_{i=1}^{n-1}(\lambda_a - \mu_i)}{\prod_{i\neq a}(\lambda_a - \lambda_i)} \frac{1}{\mu_b - \lambda_a}.$$

The Jacobian is essentially the determinant of the matrix $1/(\mu_b - \lambda_a)$, which is the Cauchy determinant (Problem G.5). The final density is obtained from the symmetric Dirichlet density, but we plug in $w = \varphi$, and also multiply by the Jacobian. This completes the proof. $\square$

Corollary 4.4 (Joint density of the corners)**.** *The eigenvalues $\lambda(k)_j$, $1 \le j \le k \le n$, of a random matrix from* Orbit(λ) *form an interlacing array, with the joint density*

$$\propto \prod_{k=1}^{n} \prod_{1\le i<j\le k} \left(\lambda_j^{(k)} - \lambda_i^{(k)}\right)^{2-\beta} \prod_{a=1}^{k+1}\prod_{b=1}^{k} \left|\lambda_a^{(k+1)} - \lambda_b^{(k)}\right|^{\beta/2-1}.$$

For $\beta = 2$, all factors disappear, and we get the uniform distribution on the interlacing array. This is the *uniform Gibbs property* which is important for other models, including discrete ensembles.

G Problems (due 2025-03-25)

G.1 General bulk case

Perform the asymptotic analysis of the correlation kernel as in Section 1.4, but in the general case $-2 < X < 2$.

G.2 Large deviations

Let W_n be an $n \times n$ Wigner real or Hermitian matrix with finite variance entries. Assume that the matrix is normalized so that the variance of each diagonal entry is 1.

Assumption [BBP05]**.** *If a Wigner matrix is normalized to have diagonal variance 1, then a rank 1 perturbation of magnitude $c > 0$ is sufficient to shoot the maximum eigenvalue outside the support of the Wigner semicircle law. (For a simulation of this phenomenon, see here.)*

Consider the following large deviation event. For a fixed $\eta > 0$, let

$$E_{n,\eta} := \left\{ \exists\, i \in \{1, \ldots, n\} \text{ such that } W_{ii} \ge \eta \right\}.$$

Under the above assumption, if for some i the diagonal entry W_{ii} is unusually large, it will push the maximal eigenvalue of W_n outside the bulk.

1. Assuming that the entries are Gaussian, *lower bound* the probability of the event $E_{n,\eta}$ for large n.

2. Assuming another tail behavior of the diagonal entries (exponential or power-law tails), use the limit theorems for maxima of independent random variables to generalize the *lower bound* of $\mathbb{P}(E_{n,\eta})$.

G.3 Airy kernel

Define the Airy function by

$$Ai(\xi) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iU^3/3+i\xi U}dU = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{U^3}{3}+\xi U\right) dU.$$

This integral converges, but only conditionally. To improve convergence, one should instead integrate along a complex contour, from $e^{\frac{5\pi i}{6}}\infty$ to 0 to $e^{\frac{\pi i}{6}}\infty$.

Show that

$$K_{\mathrm{Ai}}(\xi,\eta) = \frac{Ai(\xi)\,Ai'(\eta) - Ai(\eta)\,Ai'(\xi)}{\xi-\eta}.$$

Note that this expression is parallel to the sine kernel,

$$\frac{\sin(x-y)}{\pi(x-y)} = \frac{\sin x\cos y - \cos x \sin y}{\pi(x-y)}, \qquad \cos x = (\sin x)'.$$

These correlation kernels are called *integrable* [IIKS90].

Hint for the problem: observe that

$$\exp\{-izx+iwy\} = \frac{i}{x-y}\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial w}\right)\exp\{-izx+iwy\},$$

and use integration by parts in $K_{\mathrm{Ai}}(\xi,\eta)$ from Definition 1.5.

G.4 Interlacing proof

Finish the proof of Proposition 3.1.

G.5 Cauchy determinant

Prove the Cauchy determinant formula:

$$\det\left(\frac{1}{x_i-y_j}\right)_{1\le i,j\le n} = \frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j}(x_i-y_j)}.$$

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