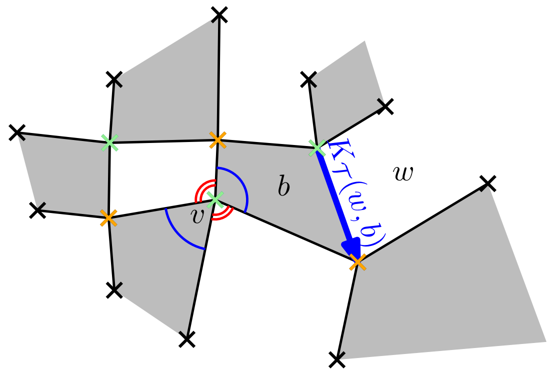
Reminder:

t-embedding $\mathcal{T}(\mathcal{G}^*)$:

[Chelkak, Laslier, R.]

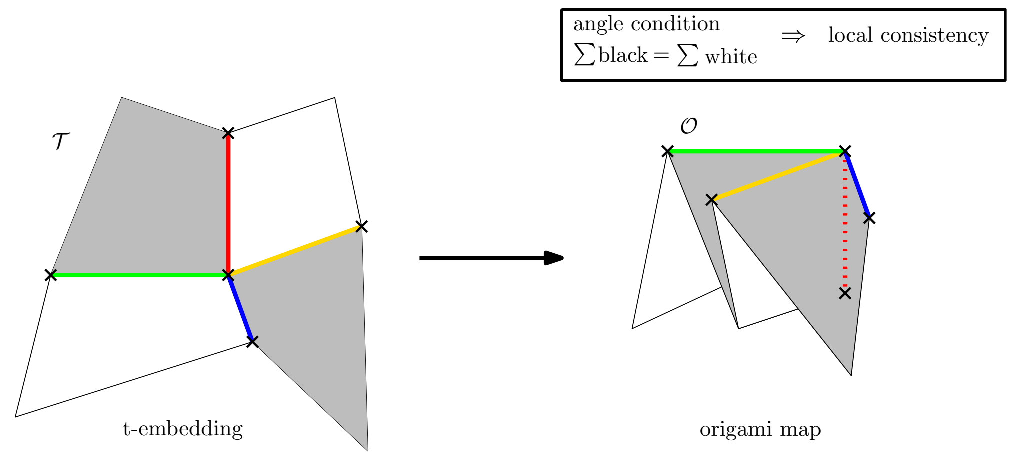


1) **lengths** are gauge equivalent to (given) dimer weights

2) **angles** at vertices are balanced:

$$\sum_{f \text{ white}} \theta(f,v) = \sum_{f \text{ black}} \theta(f,v) = \pi.$$

To get an origami map $\mathcal{O}(\mathcal{G}^*)$ from $\mathcal{T}(\mathcal{G}^*)$ one can fold the plane along every edge of the embedding.



t-embeddings: $(\mathcal{T},\mathcal{O}) \subset \mathbb{R}^{2+2}$

$|\mathcal{O}(z) - \mathcal{O}(z')| \leq |\mathcal{T}(z) - \mathcal{T}(z')|$

discrete space-like surfaces in Minkowski space $\mathbb{R}^{2+2}$

Results

Theorem (Kenyon, Lam, Ramassamy, R. '19)

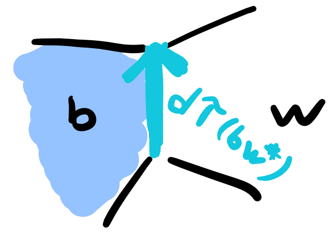
t-embeddings exist at least in the following cases:

- *If $\mathcal{G}^\delta$ is a bipartite finite graph with outer face of degree 4.*
- *If $\mathcal{G}^\delta$ is a biperiodic bipartite graph.*

Scaling limit results: [Chelkak, Laslier, R. '20-21]

- Develop new discrete complex analysis techniques on t-embeddings
- Perfect t-embeddings reveal the relevant conformal structure of the Dimer model

Lecture 4 — Origami map

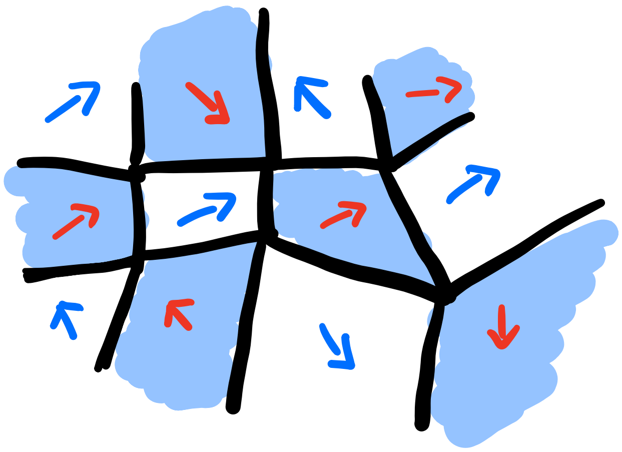


Def: Origami square root function

$$\eta : B \sqcup W \to \mathbb{T}$$

For any $b \sim w$

$$\bar{\eta}_b \bar{\eta}_w = \frac{d\mathcal{T}(bw^*)}{|d\mathcal{T}(bw^*)|}$$



Rmk: Note that η^2 is well-defined but the function η itself has to branch over every vertex v of G^* such that $\deg v \in 4\mathbb{Z}$.

Def: The origami differential form associated to η is defined as

$$d\theta(z) := \begin{cases} \eta_w^2 \, dz & \text{if } z \in \mathcal{T}(w) \\ \bar{\eta}_b^2 \, d\bar{z} & \text{if } z \in \mathcal{T}(b) \end{cases}$$

closed $\Rightarrow \exists \theta : \mathbb{C} \to \mathbb{C}$

Rmk: $d\theta(z)$ is a piecewise constant differential form defined inside the whole discrete domain of the t-embedding. But we can also view it as a 1-form on edges of $\mathcal{T}$ by setting

$$d\theta(bw^*) := \eta_w^2 \, d\mathcal{T}(bw^*) = \bar{\eta}_b \eta_w |d\mathcal{T}(bw^*)| = \bar{\eta}_b^2 \, \overline{d\mathcal{T}(bw^*)}$$

t-holomorphicity

Def: t-white-holomorphicity

A function $F: B \to \mathbb{C}$ is called t-holomorphic at $w \in W$ if

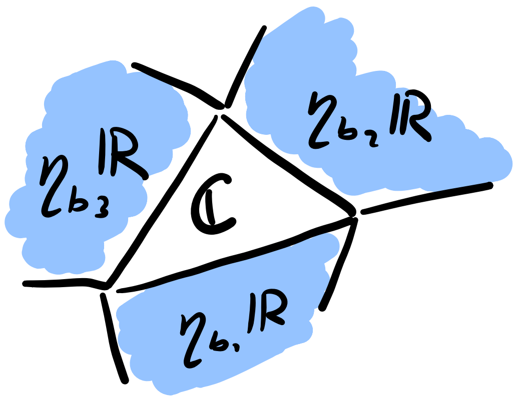
$$\begin{cases} F(b) \in \eta_b \mathbb{R} & \forall\, b \sim w, \\ \oint_{\partial w} F\, dT = 0 & (\Leftrightarrow K_T F = 0). \end{cases}$$

Lemma: Assume that all white faces of the t-emb. are triangles. Then any t-white-holomorphic function can be extended to W s.t. $\forall\, b \sim w$

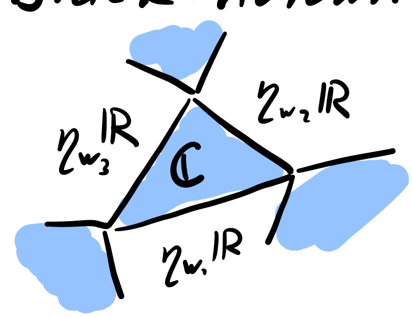
$$\mathrm{Proj}\,(F(w);\, \eta_b \mathbb{R}) = F(b)$$

where $\mathrm{Proj}\,(F, \eta\mathbb{R}) := \frac{1}{2}(F + \eta^2 \overline{F})$

Rmk: A typical example of a t-white-holom. fct. is given by $F_{w_0}(b) := \overline{\eta}_{w_0} \cdot K^{-1}(w_0, b)$ for a fixed $w_0 \in W$.



Similarly, one can define t-black-holom. functions



Closed forms associated to t-holom. functions

$\boxed{\mathrm{I}}$ On edges:

$$2F(b)\,d\mathcal{T} = F(w)\,d\mathcal{T} + \overline{F}(w)\,d\overline{\theta}$$

($2F(w)\,d\mathcal{T} = F(b)\,d\mathcal{T} + \overline{F}(b)\,d\theta$ for t-black-holom.)

Rmk: One can view it as a closed piecewise const. differential form in the plane:

$$F(z)\,dz + \overline{F}(z)\,d\overline{\theta(z)}$$

($F(z)\,dz + \overline{F(z)}\,d\theta(z)$ for t-black-holom.)

where $\begin{cases} F(z) := F(w), & \text{if } z \in \mathcal{T}(w) \\ F(z) := F(w) \text{ for any } w \sim b, & \text{if } z \in \mathcal{T}(b) \end{cases}$

$\boxed{\mathrm{II}}$ Let $F^{\bullet}$ be t-black holomorphic and F° be t-white lolomorphic, then on edges

$$F^{\circ}(b)F^{\bullet}(w)\,d\mathcal{T} = \tfrac{1}{2}\mathrm{Re}\left(F^{\circ}(w)F^{\bullet}(b)\,d\mathcal{T} + F^{\circ}(w)\overline{F^{\bullet}(b)}\,d\theta\right)$$

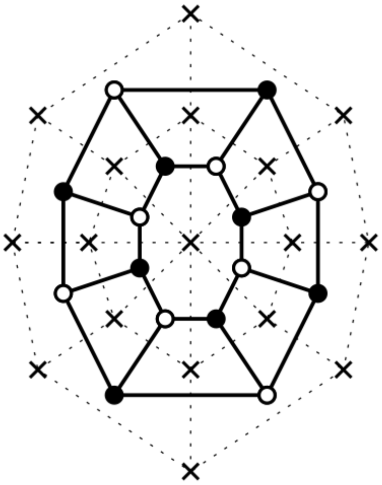
and the form $F^{\circ}(b)F^{\bullet}(w)\,d\mathcal{T}$ is closed

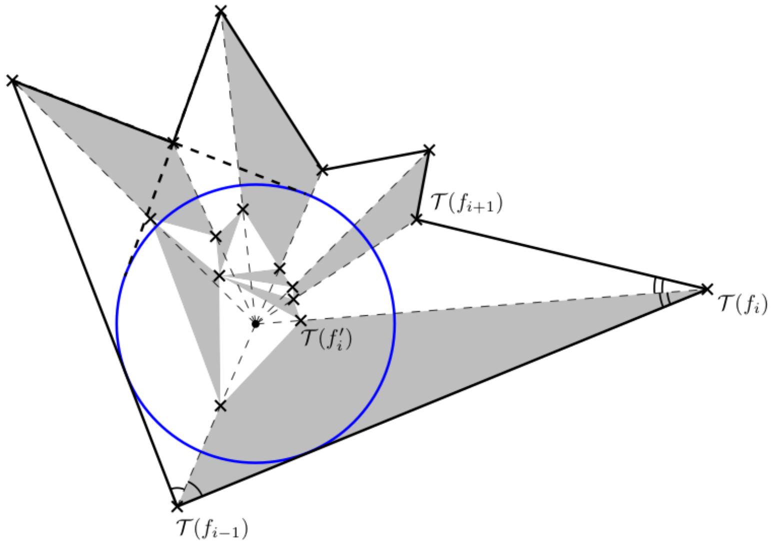
Rmk: Similarly, can be extended to

$$\tfrac{1}{2}\mathrm{Re}\left(F^{\circ}(z)F^{\bullet}(z)\,dz + F^{\circ}(z)\overline{F^{\bullet}(z)}\,d\theta(z)\right)$$

- Height correlations are linear combinations of $\int \mathrm{Re}\left(F^{\circ}F^{\bullet}dz + F^{\circ}\overline{F^{\bullet}}\,d\theta(z)\right)$.

Perfect t-embeddings



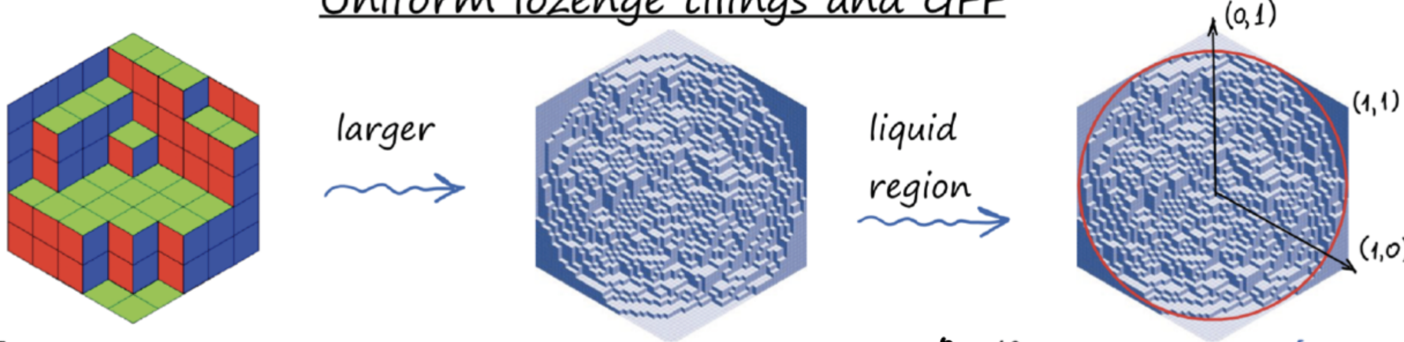


A t-embedding is a perfect t-embedding if

1) the bdry forms a tangental polygon
2) inner edges adjacent to the bdry lie on bisectors

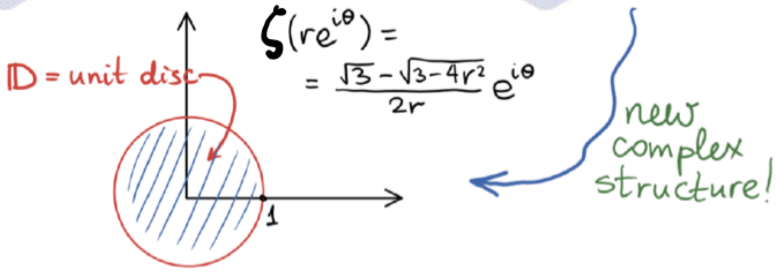
Reminder:

Uniform lozenge tilings and GFF



[Petrov]

Theorem As mesh goes to zero, Fluctuations of height ⇒ Gaussian Free Field on $\mathbb{D}$ with zero boundary conditions.



Thm: (Chelkak, Laslier, R.)

Assume G^δ are perfectly t-embedded

a) Assumption $\mathrm{Lip}(k,\delta)$, $k<1$:

$$|\theta^\delta(x) - \theta^\delta(y)| \leq k \cdot \underbrace{|\mathcal{T}^\delta(x) - \mathcal{T}^\delta(y)|}_{\text{provided that } \geq \delta}$$

b) Assumption $\text{Exp-Fat}(\delta)$, $\delta \to 0$

[triangulations] $\forall \beta > 0$, if we remove all $\exp(-\beta\delta^{-1})$-fat triangles from $\mathcal{T}^\delta$, then the size of remaining vertex-connected components $\xrightarrow[\delta\to 0]{} 0$

c) The origami maps converge to a maximal surface S in the Minkowski space $\mathbb{R}^{2,1}$.

$\Rightarrow$ convergence to the GFF in the conformal parametrization of this surface.

Rmk:
- In general, the existence of perfect t-emb. is an open question.
- Not easy to check this assumptions.
- Few examples are known.

A priori regularity of t-holom. functions

① Assumption $Lip(k,\delta)$ $\Rightarrow$ t-hol. fcts on T^δ are Hölder above scale δ

② Assumption $Exp\text{-}Fat(\delta)$ $\Rightarrow$ Harnack-type control of t-holom. functions via primitives

Under ① + ②, if primitives of F^δ's are bounded on compacts, then $\exists$ subseq. limits f_w, f_b

PROVIDED $\{(T,\theta)\} \to \{(z,\theta(z))\}$

$f_w\, dz + \overline{f_w}\, d\theta$ & $f_b\, dz + \overline{f_w}\, d\overline{\theta}$ are closed diff. forms