① Declan Stacy — Doob's h-transform & Dyson Brownian motion

- $\beta=2$ DBM, special construction (noncolliding)
- condition @ prob 0 event

DBM $\quad d\lambda_i(t) = dW_i(t) + \sum_{j\neq i} \frac{dt}{\lambda_i(t)-\lambda_j(t)}$

$\lambda(t) \in$ Weyl chamber $\quad M = \{x \in \mathbb{R}^n : \underbrace{x_1 \geq \dots \geq x_n}\}$ $\quad (x_1 \leq \dots \leq x_n)$

→ Markov process reminder

- $X_0 = x$ a.s.
- $E[f(X_{t+s}) \mid \mathcal{F}_t] = (P_s f)(X_t)$
- $P_s f(x) = E^{P_x}(f(X_s))$

<u>Ex.</u> n-dim. Brownian motion

$C = \{x \in M \mid x_i = x_j \text{ for some } i \neq j\}$

$\tau = \inf\{t > 0 \mid W_t \in C\}$ ← first collision time

$\tilde{X}_t = W_{t \wedge \tau}$ $\left(\begin{array}{c}\text{stopped at}\\ \text{bdry forever}\end{array}\right)$ $P(\tau=\infty)=0$

$P(A|B) = \frac{P(AB)}{P(B)}$ $\qquad dP(\cdot|B) = \frac{1_B}{P(B)}\, dP$

(density version)

② $h : M \mapsto [0,\infty)$ ← will be densities

s.t. $h(x) = 0 \iff x \in C$

In our example, $h(x) = \prod_{i<j} (x_j - x_i)$.

$\{T = \infty\} = \{h(X_t) > 0 \text{ for all } t\}$

$P_x \rightsquigarrow Q_x$, define

$$dQ_x = \frac{h(X_t)}{h(x)} \, dP_x$$

Q_x – how to define, s.t. dQ_x is a density?

[we want dQ_x indep. of t
(so, RHS needs indep. of t)

if $A \in \mathcal{F}_t$, and $s > 0$
– need $Q_x(A) = E^{P_x}\left(1_A \frac{h(X_t)}{h(x)}\right)$

$$\overset{\text{want}}{=} E^{P_x}\left[1_A \frac{h(X_{t+s})}{h(x)}\right] \quad (*)$$

so, do cond. expectation from $(*)$

$$E^{P_x}\left[E^{P_x}\left[1_A \frac{h(X_{t+s})}{h(x)} \,\middle|\, \mathcal{F}_t\right]\right] = (*)$$

③ So, need

$$E^{P_x}\left(h(X_{t+s}) \mid \mathcal{F}_t \right) = E\left(h(X_t)\right)$$

~~e.g.~~ i.e. $P_s h = h \quad \forall s$

(harmonic)

$\iff$ $\boxed{h(X_t) \text{ is a martingale}}$

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we man Q_x exists for individ. $\mathcal{F}_t$

harmonic $\implies$ extend Q_x to all of $\mathcal{F}$.

- need to check, h — harmonic for BM.

$P_s h = h \quad \forall s \rightarrow$ derivative in s equals 0 (generator)

$Lh = 0$

$\Delta h = 0.$ (harmonic) $\leftarrow$ can check.

$\rightarrow$ constructed Q_x

$\rightarrow \{h(X_t) > 0\}$ has measure 0 under P_x

$\rightarrow$ but has meas. 1 under Q_x

$\rightarrow$ can compute the semigroup now

④ $Q_s = E^{Q_x}(f(X_s))$ ← we have density!

$= E^{P_x}\left[f(X_s)\frac{h(X_s)}{h(x)}\right]$

$= \frac{1}{h(x)} P_s(fh)$, conjugated semigroup

$\Rightarrow$ generator is also, $\frac{1}{h}\left(\frac{1}{2}\Delta(hf)\right)$

$$\boxed{\tilde{L} = \frac{1}{2}\Delta + \sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{x_i - x_j}\frac{\partial}{\partial x_i}}$$

new generator

$$\left(\frac{h\Delta f + \overbrace{f\Delta h}^{=0} + 2(\nabla f\cdot\nabla h)}{2h}\right)$$

$\rightarrow \langle \nabla f, \frac{\nabla h}{h}\rangle$

drift terms, agrees with DBM.

$\Rightarrow$ under Q_x, X_t satisfies DBM SDE

⑤ h-transform

$\Rightarrow$ transition densities

$$q_s(x,y) = \frac{h(y)}{h(x)} \underbrace{p_s(x,y)}_{\text{not the}}$$

not the Gaussian kernel, but a det of Gaussian kernels because of noncolliding rules.