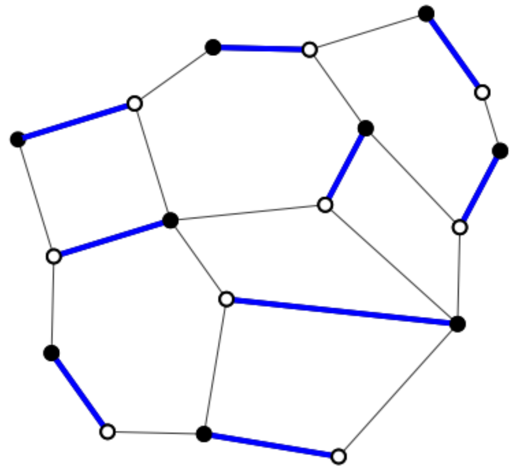
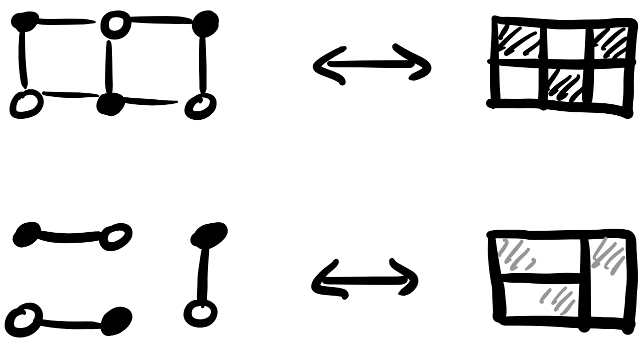
Lecture 1 — Kasteleyn theorem

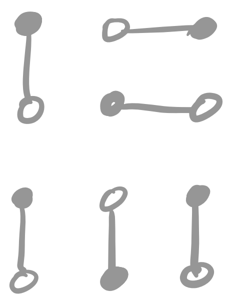
G - planar, bipartite graph

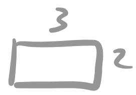


dimer configurations / perfect matchings

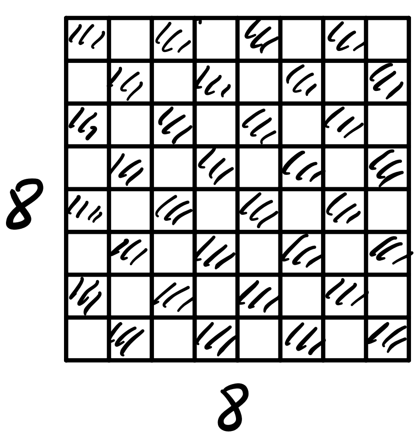
- Today: Dimer model on $\mathbb{Z}^2$ (domino tilings)

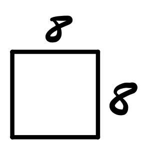




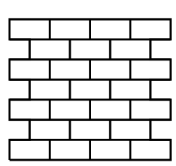
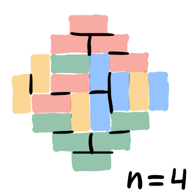
#[] = 3

Q: How many domino tilings?



(tilings of )

= 12 988 816

1		$2^{n(n+1)/2}$	

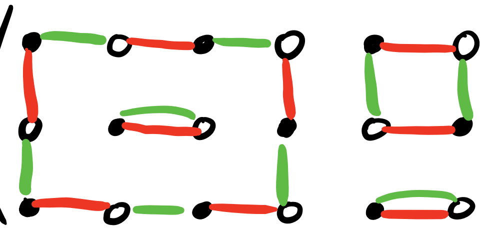
Thm: $(\#\text{ dimer configurations})^2 = \operatorname{Perm}\tilde{A}$,

where $\tilde{A}$ is an adjacency matrix of G

$$\tilde{A} = \begin{array}{c} \\ w_1 \\ \vdots \\ w_n \\ b_1 \\ \vdots \\ b_n \end{array} \begin{array}{c} w_1 \dots w_n \quad b_1 \dots b_n \\ \begin{pmatrix} \emptyset & \tilde{K} \\ \tilde{K}^T & \emptyset \end{pmatrix} \end{array} \qquad a_{ij} = \begin{cases} 1, & \text{if } i \sim j \\ 0, & \text{otherwise} \end{cases}$$

Reminder: $\operatorname{Perm}\tilde{A} = \sum_{\sigma \in S_{2n}} a_{1\sigma(1)} \cdots a_{2n\sigma(2n)}$

Proof: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 5 & 8 & 6 & 7 \end{pmatrix} = (1324)(5)(687)$

$\#\Big($  $\Big) = (\#\text{ dimer configurations})^2$

$(\square)$

Rmk: $\operatorname{Perm}\tilde{A} = (\operatorname{Perm}\tilde{K})^2$

WANTED: $\det A \ \left(= (\det K)^2\right)$

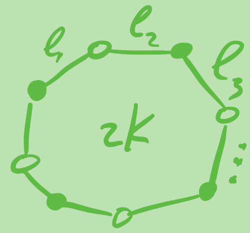
signs $\tau_{w_i b_j} \in \mathbb{C}$, $|\tau_{w_i b_j}| = 1$

$$K = \begin{array}{c} \\ w_1 \\ \vdots \\ w_n \end{array} \begin{array}{c} b_1 \dots b_n \\ \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{array} \qquad k_{w_i b_j} = \begin{cases} \tau(w_i b_j) & \text{if } w_i \sim b_j \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{\det K = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, k_{w_1 b_{\sigma(1)}} \cdots k_{w_n b_{\sigma(n)}}}$$

Kasteleyn sign condition:

For each face of degree $2k$



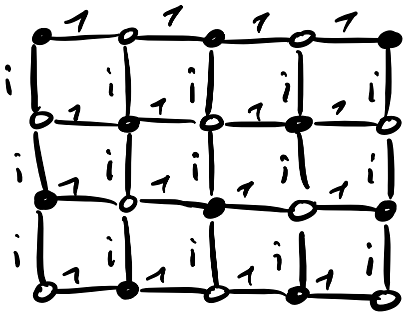
$$\frac{\tau(e_1)}{\tau(e_2)} \cdot \frac{\tau(e_3)}{\tau(e_4)} \cdot \ldots \cdot \frac{\tau(e_{2k-1})}{\tau(e_{2k})} = (-1)^{k-1}$$

Thm: (Percus, Kasteleyn)

$$\#\text{ dimer configurations } = |\det K|$$

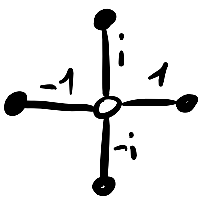
Two choices of Kasteleyn signs on $\mathbb{Z}^2$

$\boxed{\text{I}}$



$\rightsquigarrow$ formula for the # tilings of M  N

$\boxed{\text{II}}$



$\rightsquigarrow$ $K^{-1} \cdot K = Id$

$\Downarrow$

$F_{w_0}(b) := K^{-1}(b, w_0)$ is discrete holomorphic

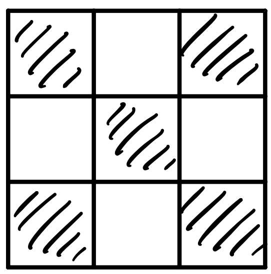
Next time

Proposition:

$$\left(\#\text{ tilings } M \; \square \; N\right)^4 = \prod_{p=1}^{M} \prod_{q=1}^{N} 4\left(\cos^2 \frac{\pi p}{M+1} + \cos^2 \frac{\pi q}{N+1}\right)$$

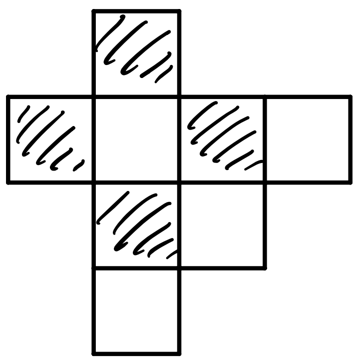
Thurston theorem

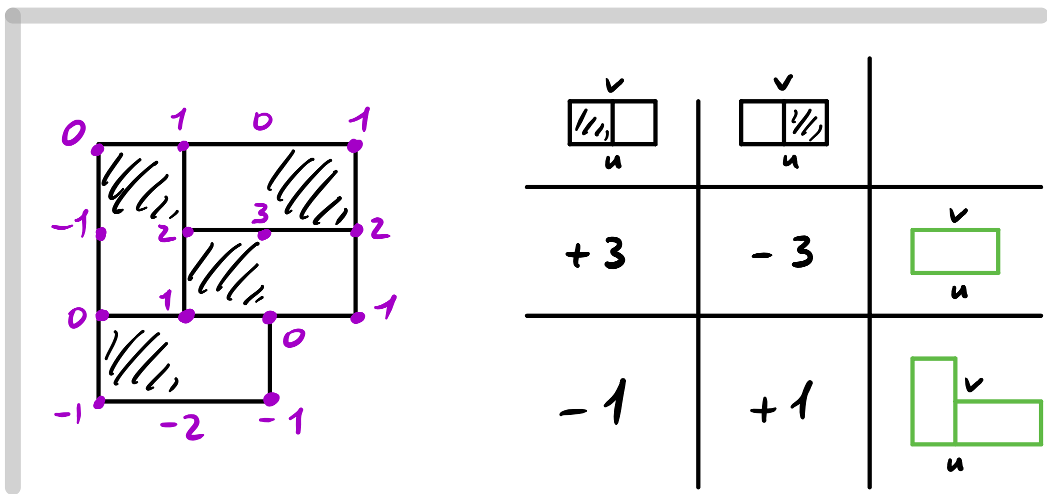
Q: When discrete domain is tileable?



\# black faces = \# white faces

NOT enough:





Def: [Thurston height function]

Given a tiling of a simply connected domain Ω with $\partial\Omega$ on $\mathbb{Z}^2$, the height function h is a $\mathbb{Z}$-valued fct on the set of vertices of Ω defined by local rules: $\forall\ u \sim v$

$$h(v)-h(u) = \begin{cases} \pm 1 & \text{if } uv \text{ does not cross domino} \\ \mp 3 & \text{if } uv \text{ crosses domino} \end{cases}$$

Rmk: h is defined up to a global additive constant by local rules.

Rmk: Ω - simply connected $\Rightarrow$ $h|_{\partial\Omega}$ does not depend on a tiling

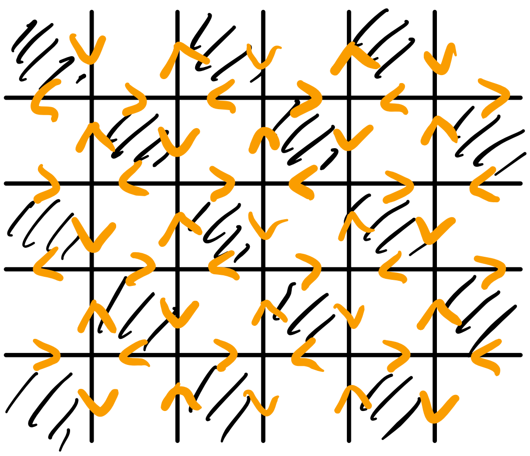
Thm: Simply-connected Ω is tileable iff both conditions hold:

1) h is well-defined on $\partial\Omega$
2) $\forall\ u,v \in \partial\Omega$

$$h(v)-h(u) \leq d(u,v)$$

Def: Oriented graph distance:

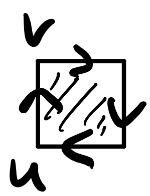
""-orientation

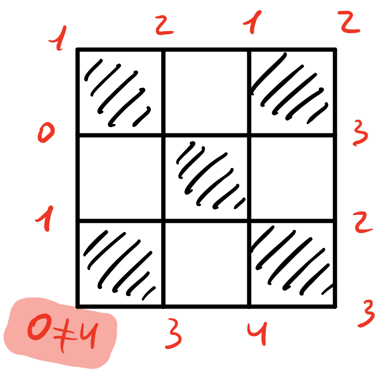


$d(u,v)$ = edge length of shortest positive oriented path from u to v (within $\bar{\Omega} = \Omega \cup \partial\Omega$)

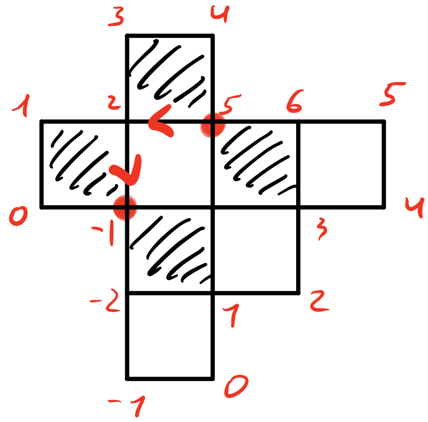
Rmk: $d(u,v)$ NOT symmetric

$d(u,v)=1$ & $d(v,u)=3$





not well-def on ∂R (1)



$h(v)-h(u)=6$

$> d(u,v)=2$

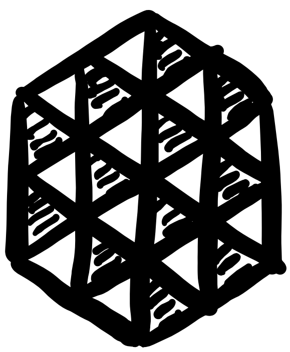
(2)

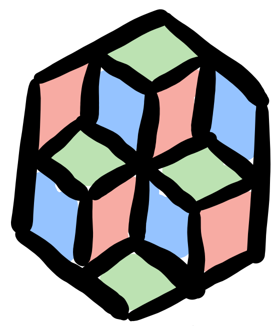
Height function of lozenge tilings:

$$h(v)-h(u)=\begin{cases}\pm 1\\ \mp 2\end{cases}$$

$\Updownarrow$

distance from surface to (1,1,1)-plane





"3D picture"