Lectures on Random Matrices (Spring 2025)
Lecture 5: Determinantal Point Processes and the GUE

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Contents

1 Recap

In Lecture 4 we discussed global spectral behavior of tridiagonal GβE random matrices, and obtained the Wigert semicircle law for the eigenvalue density.

In this lecture we shift our focus to another powerful technique in random matrix theory: the theory of *determinantal point processes* (DPPs). In the $\beta = 2$ (GUE) case the joint eigenvalue distributions can be written in determinantal form. We begin by discussing the discrete version of determinantal processes, and then derive the correlation kernel for the GUE using orthogonal polynomial methods. Finally, we show how the Christoffel–Darboux formula yields a compact representation of the kernel and indicate how one may represent it as a double contour integral—an expression well suited for steepest descent analysis in the large-n limit.

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2 Discrete Determinantal Point Processes

2.1 Definition and Basic Properties

Let $\mathfrak{X}$ be a (finite or countably infinite) discrete set. A *point configuration* on $\mathfrak{X}$ is any subset $X \subset \mathfrak{X}$ (with no repeated points). A random point process is a probability measure on the space of such configurations.

Definition 2.1 (Determinantal Point Process)**.** A random point process P on $\mathfrak{X}$ is called *determinantal* if there exists a function (the *correlation kernel*) $K : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ such that for any n and every finite collection of distinct points $x_1, \ldots, x_n \in \mathfrak{X}$, the joint probability that these points belong to the random configuration is

$$\mathbb{P}\{x_1, \ldots, x_n \in X\} = \det\Big[K(x_i, x_j)\Big]_{i,j=1}^{n}.$$

Determinantal processes are very useful in probability theory and random matrices. They are a natural extension of Poisson processes, and have some parallel properties. Many properties of determinantal processes can be derived from "linear algebra" (broadly understood) applied to the kernel K. There are a few surveys on them: [Sos00], [HKPV06], [Bor11], [KT12]. Let us just mention two useful properties.

Proposition 2.2 (Gap Probability)**.** *If $I \subset \mathfrak{X}$ is a subset, then*

$$\mathbb{P}\{X \cap I = \varnothing\} = \det\Big[I - K_I\Big],$$

where K_I is the restriction of the kernel to I. If I is infinite, then the determinant is understood as a Fredholm determinant.

Remark 2.3. The Fredholm determinant might "diverge" (equal to 0 or 1).

Proposition 2.4 (Generating functions)**.** *Let $f : \mathfrak{X} \to \mathbb{C}$ be a function such that the support of $f - 1$ is finite. Then the generating function of the multiplicative statistics of the determinantal point process is given by*

$$\mathbb{E}\left[\prod_{x \in X} f(x)\right] = \det\Big[I + (\Delta_f - I)K\Big],$$

where the expectation is over the random point configuration $X \subseteq \mathfrak{X}$, Δ_f denotes the operator of multiplication by f (i.e., $(\Delta_f g)(x) = f(x)g(x)$) and the determinant is interpreted as a Fredholm determinant if $\mathfrak{X}$ is infinite.

Remark 2.5 (Fredholm Determinant — Series Definition)**.** The Fredholm determinant of an operator A on $\ell^2(\mathfrak{X})$ is given by the series

$$\det(I + A) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{x_1, \ldots, x_n \in \mathfrak{X}} \det\big[A(x_i, x_j)\big]_{i,j=1}^{n},$$

where the term corresponding to $n = 0$ is defined to be 1.

3 Determinantal Structure in the GUE

3.1 GUE Joint Density and Orthogonal Polynomials

Recall that the joint eigenvalue density for the Gaussian Unitary Ensemble (GUE) is given by

$$p(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_{n,2}} \prod_{j=1}^{n} e^{-\lambda_j^2/2} \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2. \tag{3.1}$$

This density, although seemingly complicated, can be rewritten in determinantal form by using the theory of orthogonal polynomials. Namely, if we denote by $\{p_j(x)\}_{j \ge 0}$ the family of monic Hermite polynomials orthogonal with respect to the weight

$$w(x) = e^{-x^2/2},$$

and define the corresponding orthonormal functions

$$\psi_j(x) = \frac{1}{\sqrt{h_j}} p_j(x) \sqrt{w(x)},$$

where h_j are the squared norms, then one obtains the famous determinantal representation for the k-point correlation functions:

$$\rho_k(x_1, \ldots, x_k) = \det \Big[K_n(x_i, x_j) \Big]_{i,j=1}^{k}, \tag{3.2}$$

with the correlation kernel

$$K_n(x, y) = \sum_{j=0}^{n-1} \psi_j(x) \psi_j(y). \tag{3.3}$$

3.2 Christoffel–Darboux Formula

A major advantage of the determinantal representation is that the sum in (3.3) can be rewritten in closed form using the Christoffel–Darboux formula. In our context, one obtains

$$K_n(x, y) = \sqrt{w(x)w(y)} \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x - y}, \tag{3.4}$$

where γ_n denotes the leading coefficient of $p_n(x)$ (for monic polynomials, $\gamma_n = 1$ but if one uses an alternate normalization this factor appears).

Remark 3.1. The derivation of the Christoffel–Darboux formula is standard in the theory of orthogonal polynomials; see, e.g., [?szego1975orthogonal]. The key idea is that the sum in (3.3) satisfies a three-term recurrence which then telescopes when writing the difference quotient.

3.3 Double Contour Integral Representation and Steepest Descent

For asymptotic analysis (for example, to derive the sine kernel in the bulk or the Airy kernel at the edge), it is extremely useful to represent the kernel in the form of a double contour integral. One classical route is as follows.

One first expresses the Hermite polynomials via a contour integral representation (see, e.g., the generating function or integral representation for Hermite polynomials):

$$H_n(z) = \frac{n!}{2\pi i} \oint_\Gamma e^{2zw - w^2} \frac{dw}{w^{n+1}},$$

with an appropriate choice of contour Γ. Inserting this representation into the Christoffel–Darboux formula (3.4) and interchanging summation and integration (justified by uniform convergence) one obtains a representation of the kernel as

$$K_n(x, y) = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} \frac{e^{n\Phi(x,\xi) - n\Phi(y,\eta)}}{\xi - \eta} \, d\xi \, d\eta, \tag{3.5}$$

where Φ is a certain phase function and the contours C_1, C_2 are chosen so that the integrals converge. (The precise form of Φ depends on the rescaling and normalization.) The representation (3.5) is well suited to a steepest descent (saddle point) analysis in the large-n limit, allowing one to derive universal kernels (such as the sine kernel in the bulk)

$$K_{\text{sine}}(x, y) = \frac{\sin \pi(x - y)}{\pi(x - y)},$$

or the Airy kernel at the spectral edge.

4 Summary and Outlook

In this lecture we:

- Recalled the resolvent (Stieltjes transform) method from Lecture 4 and noted that its analytic completion remains open.
- Introduced discrete determinantal point processes and outlined key properties such as the determinantal form of correlation functions and gap probabilities.
- Derived (via the orthogonal polynomial method) the determinantal kernel for the GUE, first as a finite sum (3.3) and then using the Christoffel–Darboux formula (3.4) for a more compact representation.
- Indicated how one can represent the kernel as a double contour integral (see (3.5)) and how steepest descent techniques are then used to obtain the universal limiting kernels.

In subsequent lectures we will use these results to study local eigenvalue statistics (the sine and Airy kernels) and discuss further universality aspects of random matrices.

E Problems (due DATE)

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