

Lectures on Random Matrices (Spring 2025)
Lecture 4: Semicircle law via tridiagonalization.
Orthogonal polynomial ensembles

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1 Recap

Note: I did some live random matrix simulations [here](#) and [here](#) — check them out. More simulations to come.

1.1 Gaussian ensembles

We introduced Gaussian ensembles, and for GOE ($\beta = 1$) we computed the joint eigenvalue density. The normalization is so that the off-diagonal elements have variance $\frac{1}{2}$ and the diagonal elements have variance 1. Then the joint eigenvalue density is

$$p(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_n} \prod_{i=1}^n e^{-\frac{1}{2}\lambda_i^2} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

1.2 Tridiagonalization

We showed that any real symmetric matrix A can be tridiagonalized by an orthogonal transformation Q :

$$Q^\top A Q = T,$$

where T is real symmetric tridiagonal, having nonzero entries only on the main diagonal and the first super-/subdiagonals:

$$T = \begin{pmatrix} d_1 & \alpha_1 & 0 & \cdots & 0 \\ \alpha_1 & d_2 & \alpha_2 & \cdots & 0 \\ 0 & \alpha_2 & d_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha_{n-1} \\ 0 & 0 & \cdots & \alpha_{n-1} & d_n \end{pmatrix}.$$

In the proof, each time we need to act in the orthogonal complement to the subspace e_1, \dots, e_{k-1} (starting from e_1), and apply a Householder reflection to zero out everything strictly below the subdiagonal. (We apply the transformations like $A \mapsto H A H^\top$, so that the first row transforms in the same way as the first column of A).

2 Tridiagonal random matrices

2.1 Distribution of the tridiagonal form of the GOE

Applying the tridiagonalization to GOE, we obtain the following random matrix model.

Theorem 2.1. Let W be an $n \times n$ GOE matrix (real symmetric) with variances chosen so that each off-diagonal entry has variance $1/2$ and each diagonal entry has variance 1. Then there exists an orthogonal matrix Q such that

$$W = Q^\top T Q,$$

where T is a real symmetric tridiagonal matrix

$$T = \begin{pmatrix} d_1 & \alpha_1 & 0 & \cdots \\ \alpha_1 & d_2 & \alpha_2 & \ddots \\ 0 & \alpha_2 & d_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad (2.1)$$

and the random variables $\{d_i, \alpha_j\}_{1 \leq i \leq n, 1 \leq j \leq n-1}$ are mutually independent, with

$$d_i \sim \mathcal{N}(0, 1), \quad \alpha_j = \sqrt{\frac{\chi_{n-j}^2}{2}},$$

where χ_ν^2 is a chi-square distribution with ν degrees of freedom.

Remark 2.2 (Chi-square distributions). The *chi-square distribution* with ν degrees of freedom, denoted by χ_ν^2 , is a fundamental distribution in statistics and probability theory. It arises naturally as the distribution of the sum of the squares of ν independent standard normal random variables. Formally, if Z_1, Z_2, \dots, Z_ν are independent random variables with $Z_i \sim \mathcal{N}(0, 1)$, then the random variable

$$Q = \sum_{i=1}^{\nu} Z_i^2$$

follows a chi-square distribution with ν degrees of freedom, i.e., $Q \sim \chi_\nu^2$. In the context of Theorem 2.1, the α_j 's can be called *chi random variables*.

The parameter ν does not need to be an integer, and the chi-square distribution is well defined for any positive real ν , for example, by continuation of the density formula. The probability density is

$$f(x) = \frac{1}{2^{\nu/2} \Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \quad x \geq 0.$$

Proof of Theorem 2.1. In the process of tridiagonalization, we apply Householder reflections. Note that the diagonal entries stay fixed, and we only change the off-diagonal entries. Let us consider these off-diagonal entries.

In the first step, we apply the reflection in \mathbb{R}^{n-1} to turn the column vector $(a_{2,1}, a_{3,1}, \dots, a_{n,1})$ into a vector parallel to $(1, 0, \dots, 0) \in \mathbb{R}^{n-1}$. Since the Householder reflection is orthogonal, it preserves lengths. So,

$$\alpha_1 = \sqrt{a_{2,1}^2 + a_{3,1}^2 + \cdots + a_{n,1}^2}, \quad a_{i1} \sim \mathcal{N}(0, \frac{1}{2}).$$

This implies that α_1 has the desired chi distribution. The distribution of the other entries is obtained similarly by the recursive application of the Householder reflections.

Note that α_j 's and d_i 's depend on nonintersecting subsets of the matrix entries, so they are independent. This completes the proof. \square

2.2 Dumitriu–Edelman $G\beta E$ tridiagonal random matrices

Let us define a general β extension of the tridiagonal model for the GOE.

Definition 2.3. Let $\beta > 0$ be a parameter. The tridiagonal $G\beta E$ is a random $n \times n$ tridiagonal real symmetric matrix T as in (2.1), where $d_i \sim \mathcal{N}(0, 1)$ are independent standard Gaussians, and

$$\alpha_j \sim \frac{1}{\sqrt{2}} \chi_{\beta(n-j)}, \quad 1 \leq j \leq n-1,$$

are chi-distributed random variables.

We showed that for $\beta = 1$, the $G\beta E$ is the tridiagonal form of the GOE random matrix model. The same holds for the two other classical betas:

Proposition 2.4 (Without proof). *For $\beta = 2$, the $G\beta E$ is the tridiagonal form of the GUE random matrix model, which is the random complex Hermitian matrix with Gaussian entries and maximal independence. Similarly, for $\beta = 4$, the $G\beta E$ is the tridiagonal form of the GSE random matrix model.*

Moreover, for all β , the joint eigenvalue density of $G\beta E$ is explicit:

Theorem 2.5 ([DE02]). *Let T be a $G\beta E$ matrix as in Definition 2.3. Then the joint eigenvalue density is given by*

$$p(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_{n,\beta}} e^{-\frac{1}{2} \sum_{i=1}^n \lambda_i^2} \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta, \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

This theorem is also given without proof. The proof involves linear algebra and computation of the Jacobians of the change of variables from the matrix entries to the eigenvalues in the tridiagonal setting. It can be found in the original paper [DE02].

2.3 The case $\beta = 2$

For many questions involving *local eigenvalue statistics*, the case $\beta = 2$ (the GUE, Gaussian Unitary Ensemble) is the most tractable. This is because the joint density of the eigenvalues admits a determinantal structure coming from a *square* Vandermonde factor $\prod_{i < j} (\lambda_i - \lambda_j)^2$ and the Gaussian exponential $\exp(-\frac{1}{2} \sum \lambda_j^2)$. Moreover, for $\beta = 2$, the random matrix model and its correlation functions can be expressed explicitly through determinants involving *orthogonal polynomials*, namely, the *Hermite polynomials*.

Proposition 2.6 (Joint density for GUE and orthogonal polynomials). *Consider the GUE (Gaussian Unitary Ensemble) random matrix model, i.e. an $n \times n$ complex Hermitian matrix whose entries are i.i.d. up to the Hermitian condition, with each off-diagonal entry distributed as $\mathcal{N}(0, \frac{1}{2}) + i\mathcal{N}(0, \frac{1}{2})$ and each diagonal entry $\mathcal{N}(0, 1)$. The ordered eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$ (or, without ordering, thought of as an unordered set) satisfy the joint probability density*

$$p(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_{n,2}} \prod_{j=1}^n e^{-\frac{1}{2} \lambda_j^2} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2, \quad (2.2)$$

where $Z_{n,2}$ is a normalization constant.

Moreover, if $\{\psi_k(\lambda)\}_{k=0}^\infty$ is the family of Hermite polynomials, orthonormal with respect to the measure $w(\lambda) d\lambda = e^{-\lambda^2/2} d\lambda$ on \mathbb{R} (i.e., $\int_{-\infty}^\infty \psi_k(\lambda) \psi_\ell(\lambda) w(\lambda) d\lambda = \mathbf{1}_{k=\ell}$), then one can also write

$$p(\lambda_1, \dots, \lambda_n) = \text{const} \cdot \det \left[\psi_{j-1}(\lambda_k) e^{-\frac{\lambda_k^2}{4}} \right]_{j,k=1}^n \det \left[\psi_{j-1}(\lambda_k) e^{-\frac{\lambda_k^2}{4}} \right]_{j,k=1}^n \quad (2.3)$$

(the two determinants are identical, but let us keep this notation for future convenience).

The square determinant structure is extremely useful. It is precisely the $\beta = 2$ counterpart of the squared Vandermonde factor $\prod_{i < j} (\lambda_i - \lambda_j)^2$.

Remark 2.7 (Hermite polynomials). There are various normalizations of Hermite polynomials. In random matrix theory for the Gaussian ensembles, we often use the *probabilists' Hermite polynomials* (sometimes called He_k , but we use the notation H_k). There are various normalizations due to the factor in the exponent of x^2 .

A convenient definition for use with the weight $e^{-x^2/2}$ is:

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k} \left(e^{-\frac{x^2}{2}} \right), \quad k = 0, 1, \dots, \quad (2.4)$$

whose leading term is x^k . Polynomials with the leading coefficient 1 are called *monic*. The first few monic Hermite polynomials are

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3.$$

The difference between H_k and ψ_k entering Proposition 2.6 is in a constant normalization, since H_k are monic but not orthonormal, while ψ_k are orthonormal but not monic.

Sketch of the determinantal representation. In brief, one observes that the factor $\prod_{i < j} (\lambda_i - \lambda_j)$ is exactly the Vandermonde determinant $\Delta(\lambda_1, \dots, \lambda_n) = \det [\lambda_k^{j-1}]_{j,k=1}^n$. Next, the Vandermonde determinant is also equal to the determinant built out of any monic family of polynomials of the corresponding degrees (by linear transformations), and so we get the desired representation. \square

We will work with Hermite polynomials and the determinantal structure in Proposition 2.6 in the next [Lecture 5](#)).

3 Wigner semicircle law via tridiagonalization

If W is an $n \times n$ real Wigner matrix with entries of mean zero and variance 1 on the off-diagonal, then as $n \rightarrow \infty$, the empirical spectral distribution (ESD) of W/\sqrt{n} converges weakly almost surely to the Wigner semicircle distribution:

$$\mu_{\text{sc}}(dx) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}_{|x| \leq 2} dx.$$

We already derived this in [Lecture 2](#) by a direct combinatorial argument on the trace. Now we present another proof by using the tridiagonal form of W . The argument is conceptually simpler in some steps, because the matrix is sparser (only tridiagonal). At the same time, we will establish the Wigner semicircle law for the general G β E case (but only Gaussian), and thus it will apply to GUE and GSE.

3.1 Moments for tridiagonal matrices

Consider the rescaled G β E matrix T/\sqrt{n} :

$$\frac{T}{\sqrt{n}} = \begin{pmatrix} d_1/\sqrt{n} & \alpha_1/\sqrt{n} & 0 & \cdots \\ \alpha_1/\sqrt{n} & d_2/\sqrt{n} & \alpha_2/\sqrt{n} & \ddots \\ 0 & \alpha_2/\sqrt{n} & d_3/\sqrt{n} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

where $d_i \sim \mathcal{N}(0, 1)$ and $\alpha_j \sim \frac{1}{\sqrt{2}}\chi_{\beta(n-j)}$. We want to show that the ESD of T/\sqrt{n} converges to the semicircle law. We will mostly consider expected traces of powers, and leave the analytic parts of the argument to the reader.

The k -th (random) moment of the ESD $\frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i/\sqrt{n}}$ is

$$\frac{1}{n} \text{Tr} \left(\frac{T}{\sqrt{n}} \right)^k = \frac{1}{n^{1+\frac{k}{2}}} \sum_{i_1, \dots, i_k=1}^n t_{i_1, i_2} \cdots t_{i_k, i_1}, \quad (3.1)$$

where t_{ij} are the non-rescaled entries of T . But now t_{ij} is nonzero only if $|i - j| \leq 1$, i.e. the (i, j) entry is on the main or first super-/subdiagonal. In a closed product $t_{i_1, i_2} \cdots t_{i_k, i_1}$, we thus get a *closed walk* in a linear graph on the vertex set $\{1, 2, \dots, n\}$ with edges only between consecutive indices.

The relevant combinatorial objects encoding these walks are lattice walks in $\mathbb{Z}_{\geq 0}^2$ starting at $(0, m)$, ending at (k, m) , and consisting of steps $(1, 0)$, $(1, 1)$, and $(1, -1)$. The steps $(1, 0)$ correspond to picking the diagonal element; steps $(1, 1)$ correspond to picking $i_{\ell+1} = i_\ell + 1$, and steps $(1, -1)$ correspond to $i_{\ell+1} = i_\ell - 1$. See Figure 1 for an illustration of a path.

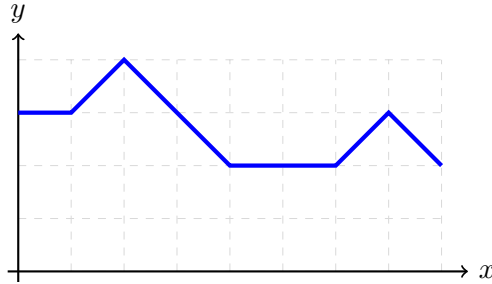


Figure 1: Example of a lattice path starting at height 3.

Now, each term in the sum in (3.1) corresponds to a path. Moreover, for each path shape, there are $O(n)$ summands corresponding to it. The number of paths of length k starting from a fixed m is finite (independent of n for $m \gg 1$), so we need to look more closely at the asymptotics of the product in (3.1). This product involves chi random variables which depend on n , too.

3.2 Asymptotics of chi random variables

One additional technical point in analyzing T/\sqrt{n} is to note that α_j is roughly $\sqrt{\beta(n-j)/2}$ for large n . Indeed, we have

$$\chi_\nu^2 = \sum_{i=1}^{\nu} Z_i^2, \quad \mathbb{E}[\chi_\nu^2] = \nu, \quad \text{Var}[\chi_\nu^2] = 2\nu.$$

Now, since we are dividing by \sqrt{n} , we have

$$\frac{\alpha_j}{\sqrt{n}} \sim \sqrt{\frac{\beta}{2}} \sqrt{1-\theta}, \quad \theta = \frac{j}{n} \in [0, 1].$$

This estimate is valid in the “bulk” region, that is, when θ is strictly between 0 and 1.

Let us make these estimates more precise. We have:

Proposition 3.1 (Pointwise asymptotics in the bulk). *Fix any $\delta \in (0, 1)$ and let j range so that $\theta_j := j/n \in [\delta, 1 - \delta]$. Then for each such j , we have¹*

$$\frac{\alpha_j}{\sqrt{n}} = \sqrt{\beta \left(1 - \frac{j}{n}\right)} + O_p\left(\frac{1}{\sqrt{n}}\right),$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{\alpha_j}{\sqrt{n}} = \sqrt{\beta(1 - \theta_j)} \quad \text{in probability.}$$

Remark 3.2. Outside the bulk region (i.e. very close to $j = 0$ or $j = n$), one would need a different statement to handle the case $\beta(n-j)$ is not large. In our application, we only need the bulk behavior.

Meanwhile, on the diagonal, d_i/\sqrt{n} almost surely vanishes in the limit as $n \rightarrow \infty$, because d_i is standard Gaussian and does not depend on n .

3.3 Completing the proof: global semicircle behavior

Putting the above pieces together, we see that

$$\frac{T}{\sqrt{n}} = \frac{1}{n} \sum_{i_1, \dots, i_k=1}^n \prod_{\ell=1}^k \frac{t_{i_\ell i_{\ell+1}}}{\sqrt{n}}, \quad i_{k+1} = i_1 \text{ by agreement.} \quad (3.2)$$

The terms in the sum have all i_ℓ ’s close together (there are k indices, and they differ by ± 1 from each other). We may think that they are close to some θn , where $\theta \in [0, 1]$. We can consider only the case when $\delta < \theta < 1 - \delta$ for some fixed small $\delta > 0$; the case of edges does not contribute (see Problem D.3).

¹Here and below, $O_p(\cdot)$ denotes a term that is stochastically bounded at the indicated order as $n \rightarrow \infty$. That is, $X_n = O_p(a_n)$ means that for any $\epsilon > 0$, there exists $M > 0$ such that $\mathbb{P}(|X_n/a_n| > M) < \epsilon$ for all sufficiently large n .

If at least one of the t_{ij} 's in (3.2) is on the diagonal, the term vanishes in the limit. Therefore, it suffices to consider only the off-diagonal α_j 's. The number of length k walks starting from $m = \theta n$ for $\theta > \delta$ is just the number of lattice walks with steps $(1, \pm 1)$. This number is $\binom{k}{k/2}$.² (From now on till the end of the section, we assume that k is even — the moments become zero for odd k).

Fixing the starting location $\theta = \frac{i_\ell}{n} \in (\delta, 1 - \delta)$, we have

$$\prod_{\ell=1}^k \frac{t_{i_\ell i_{\ell+1}}}{\sqrt{n}} \rightarrow \beta^{k/2} (1 - \theta)^{k/2}.$$

There is an extra factor $1/n$ in front in (3.2), which is interpreted as transforming the sum over i_1, \dots, i_k into an integral in θ . We thus see that the moments converge to

$$\beta^{k/2} \binom{k}{k/2} \int_0^1 (1 - \theta)^{k/2} d\theta = \beta^{k/2} \binom{k}{k/2} \cdot \frac{1}{1 + k/2},$$

and we recover our favorite Catalan moments of the semicircle distribution.

This completes the proof.

Remark 3.3 (The factor $\beta^{k/2}$). Note that the factor $\beta^{k/2}$ refers just to the scaling of the Wigner semicircle law, and does not affect the semicircle shape. More precisely, the limiting semicircle distribution lies from $[-2\sqrt{\beta}, 2\sqrt{\beta}]$.

4 Wigner semicircle law via Stieltjes transform

Let us stay in the tridiagonal setting, and explore a more analytic method to derive the Wigner semicircle law.

4.1 Tridiagonal Structure and Characteristic Polynomials

We let

$$T - \lambda I = \begin{pmatrix} d_1 - \lambda & \alpha_1 & 0 & \cdots \\ \alpha_1 & d_2 - \lambda & \alpha_2 & \ddots \\ 0 & \alpha_2 & d_3 - \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

We want to understand eigenvalues, that is, zeros of the characteristic polynomial $\det(T - \lambda I)$.

4.1.1 Three-Term Recurrence for the Characteristic Polynomial

For each $k = 1, \dots, n$, denote by T_k the top-left $k \times k$ submatrix of T . Define the *characteristic polynomial* of that block:

$$p_k(\lambda) = \det(T_k - \lambda I_k).$$

By convention, set $p_0(\lambda) := 1$. Then a basic determinant expansion argument along the last row or column shows:

²Not Catalan yet!

Lemma 4.1 (Three-Term Recurrence). *For each $k \geq 1$, we have*

$$p_{k+1}(\lambda) = (d_{k+1} - \lambda)p_k(\lambda) - \alpha_k^2 p_{k-1}(\lambda).$$

See also Problem D.4.

4.1.2 Spectral Connection and Eigenvalues

The eigenvalues $\lambda_1, \dots, \lambda_n$ of T are exactly the roots of $p_n(\lambda)$. For any $\lambda \in \mathbb{C}$, if λ is not an eigenvalue, then $(T - \lambda I)$ is invertible.

When λ is close to a real eigenvalue, the behavior of the resolvent $(T - \lambda I)^{-1}$ becomes large. Tracking these poles in the complex plane is the key to the resolvent or Stieltjes transform approach.

4.2 Stieltjes Transform / Resolvent

Recall that for a matrix A with real eigenvalues $\lambda_1, \dots, \lambda_n$, the *Stieltjes transform* (or Green's function, or resolvent trace) is

$$G_n(z) = \frac{1}{n} \text{Tr}[(A - zI)^{-1}], \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

If $z = x + iy$ is in the upper half-plane ($y > 0$), this $G_n(z)$ can be seen as

$$G_n(z) = \int_{\mathbb{R}} \frac{d\mu_n(\lambda)}{\lambda - z},$$

where $\mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k}$ is the empirical spectral measure. Equivalently, $\text{Im } G_n(x + i0^+)$ encodes the density of eigenvalues around x . Thus, understanding $G_n(z)$ for large n pinpoints the limiting spectral distribution.

4.2.1 Resolvent of a Tridiagonal Matrix: Recurrence Relations

Let us apply this to $A = T/\sqrt{n}$ (an $n \times n$ tridiagonal matrix). We want

$$(T/\sqrt{n} - zI)^{-1},$$

for complex z . Since T/\sqrt{n} has nonzero entries only on the main and first off-diagonals, one can write down a linear recurrence for the entries of the resolvent.

Concretely, let us denote $(T/\sqrt{n} - zI)^{-1} = (R_{ij})$. Then from the definition

$$(T/\sqrt{n} - zI)(R_{ij}) = I,$$

one gets for each row k :

to continue improvement from here

$$\frac{d_k}{\sqrt{n}} R_{kj} - z R_{kj} + \frac{\alpha_{k-1}}{\sqrt{n}} R_{k-1,j} \mathbf{1}_{(k>1)} + \frac{\alpha_k}{\sqrt{n}} R_{k+1,j} \mathbf{1}_{(k<n)} = \delta_{k=j}.$$

This can be reorganized into a second-order *linear difference equation* in k for each fixed column index j . Similarly, one obtains boundary conditions.

Because the α_k and d_k are random but *independent*, the coefficients in this difference equation become “stationary” for large n except near the boundaries. In fact, up to small fluctuations, α_k/\sqrt{n} stays near α for some constant α (like $\alpha \approx 1/\sqrt{2}$ if off-diagonal variance is $\frac{1}{2}$), and $d_k/\sqrt{n} \rightarrow 0$ with k . One can then guess that

$$G_n(z) = \frac{1}{n} \sum_{k=1}^n R_{kk}$$

has a continuum limit satisfying a certain *algebraic* equation.

4.3 Functional Equation for the Limit and the Semicircle Law

Let us sketch more precisely how $G_n(z)$ converges to the semicircle transform.

- (1) **Asymptotic stationarity.** By a law of large numbers or a concentration argument on χ_{n-j}^2 (when we do the Dumitriu–Edelman version), the subdiagonal α_j is close to $\sqrt{(n-j)/2}$, hence $\alpha_j/\sqrt{n} \approx \sqrt{\frac{1-j/n}{2}}$. In the bulk of the index range (where $j \approx \theta n$ for some $\theta \in (0, 1)$), this is about $\sqrt{\frac{1-\theta}{2}}$. Meanwhile $d_i/\sqrt{n} \approx 0$.

- (2) **Discrete difference to continuous ODE.** The second-order difference equation for $R_{k+1,k+1}$ is roughly

$$-z R_{k,j} + (\alpha_k/\sqrt{n}) R_{k+1,j} + (\alpha_{k-1}/\sqrt{n}) R_{k-1,j} = \delta_{k=j},$$

neglecting the small diagonal d_k/\sqrt{n} term. In the limit $n \rightarrow \infty$, if we interpret k/n as $x \in [0, 1]$, then $R_{k \pm 1, j}$ becomes $R(x \pm \frac{1}{n})$, and $\alpha_k/\sqrt{n} \approx \alpha(x)$ for some function α . This difference equation becomes an approximate differential equation in x , from which one solves for $R(x, \theta)$ (thinking of j as θn).

- (3) **Solving for $G(z)$.** Summing $R_{k,k}$ over k becomes an integral of $R(x, x)$ over $x \in [0, 1]$. One obtains an algebraic equation in $G(z)$ akin to

$$G(z)^2 + z G(z) + 1 = 0.$$

(Here we used normalizations that match the classical semicircle radius 2 or $\pm\sqrt{z^2 - 4}$ form.) Solving for $G(z)$, we get

$$G(z) = \frac{-z \pm \sqrt{z^2 - 4}}{2},$$

and the negative branch is selected by the imaginary part condition (i.e. $G(z) \sim -1/z$ for large $|z|$).

- (4) **Back to the real axis.** From $G(z)$ in the upper half-plane, one extracts the limit distribution by looking at $\text{Im}(G(x + i0^+))$ for real x . The square-root $\sqrt{x^2 - 4}$ is purely imaginary for $|x| < 2$, giving the semicircle density on $[-2, 2]$.

Thus, in the limit, the spectral measure is forced to match the one whose Stieltjes transform is $G(z)$. Since that is exactly the Wigner semicircle distribution, we conclude:

Theorem 4.2 (Semicircle Law via Resolvent). *Let T be the Dumitriu–Edelman $G\beta E$ tridiagonal matrix (in particular, the GOE case when $\beta = 1$) with diagonal d_i/\sqrt{n} and subdiagonals α_j/\sqrt{n} . Then as $n \rightarrow \infty$, its empirical spectral distribution converges weakly (almost surely) to the Wigner semicircle law on $[-2, 2]$. Equivalently, its Stieltjes transform $G_n(z)$ converges for each $z \in \mathbb{C}^+$ to the unique solution of $G(z)^2 + zG(z) + 1 = 0$, which yields the semicircle density.*

Remark 4.3. One can make this rigorous by carefully bounding the difference between (α_j/\sqrt{n}) and its limiting profile, as well as controlling the diagonal (d_i/\sqrt{n}) . The approach is an archetype for many *local laws* in random matrix theory, where one obtains more precise estimates on the resolvent entries for z in small neighborhoods near the real axis.

4.4 Concluding Remarks

We have now seen *two main proof strategies* for deriving the Wigner semicircle law:

- **Moment expansions and Catalan counting:** Expand $\text{Tr}((W/\sqrt{n})^k)$, interpret it as a sum over closed walks, identify the main combinatorial configurations, show that the expected moments converge to the Catalan numbers that match the semicircle distribution.
- **Stieltjes transform and resolvent analysis:** Tridiagonalize W , then exploit the near one-dimensional recurrence for the inverse $(T/\sqrt{n} - zI)^{-1}$. In the large- n limit, show that $G_n(z)$ satisfies a deterministic algebraic equation forcing the semicircle $G(z)$.

Both lead to the same final statement of the global spectral distribution being the semicircle law. The resolvent method paves the way for refined *local* analysis (spectrum on small intervals) and *universality proofs*.

5 Determinantal point processes (discrete)

We are now going to start the discussion of the local eigenvalue behavior at $\beta = 2$, started in Section 2.3. We begin with a general discussion of *determinantal point processes* (DPPs), starting in discrete world. The continuous world is going to be considered in the next [Lecture 5](#).

In this section, we introduce *determinantal point processes* (DPPs) over a discrete state space and explore some of their properties. Our main reference is [\[Bor11\]](#).

Setup. Let \mathfrak{X} be a (finite or countably infinite) discrete set endowed with the counting measure μ . A *point configuration* on \mathfrak{X} is any subset $X \subset \mathfrak{X}$, finite or infinite, with no repeated points.³ We write $\text{Conf}(\mathfrak{X})$ for the set of all point configurations, which carries the natural σ -algebra generated by the functions $\mathbf{1}_{\{x \in X\}}$, $x \in \mathfrak{X}$. A *random point process* P on \mathfrak{X} is a probability measure on $\text{Conf}(\mathfrak{X})$.

³Some texts allow multiplicities, but we disallow them here.

Definition 5.1 (Determinantal point process). A random point process P on a discrete set \mathfrak{X} is *determinantal* if there exists a kernel function $K : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathbb{C}$ such that for every finite collection of pairwise distinct points $x_1, \dots, x_n \in \mathfrak{X}$,

$$\mathbb{P}\{x_1, \dots, x_n \in X\} = \det[K(x_i, x_j)]_{i,j=1}^n. \quad (5.1)$$

That is, all finite-dimensional distributions of P take a determinantal form. The function K is called a *correlation kernel* for P .

Correlation functions and the kernel. The condition (5.1) captures all finite-dimensional distributions of P . Equivalently, let

$$\rho_n(x_1, \dots, x_n) := \mathbb{P}\{\text{there is a particle at each } x_i\}$$

for distinct x_1, \dots, x_n . In the discrete setting, ρ_n is sometimes called the *(unordered) correlation function*. The process is determinantal if and only if

$$\rho_n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n \quad \text{for each } n \geq 1.$$

Basic properties. If P is a DPP with correlation kernel $K : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathbb{C}$, then for any subset $I \subset \mathfrak{X}$,

$$\mathbb{P}\{X \cap I = \emptyset\} = \det[\mathbf{1} - K_I], \quad (5.2)$$

where K_I is the operator $[K(x, y)]_{x, y \in I}$ (viewed as a matrix if \mathfrak{X} is finite, or an infinite matrix if \mathfrak{X} is countably infinite with convergent sums). More generally, if $I_1, \dots, I_m \subset \mathfrak{X}$ are disjoint subsets, then the joint event $\{|X \cap I_k| = n_k \text{ for } 1 \leq k \leq m\}$ can be expressed via the determinant $\det[\mathbf{1} - \sum_{k=1}^m z_k K_{I_k}]$ and its derivatives.

Remark 5.2. For any function $\phi : \mathfrak{X} \rightarrow \mathbb{C}$ such that the operator $[(1 - \phi(x))K(x, y)]_{x, y \in \mathfrak{X}}$ is trace class, the exponential generating function for ϕ is

$$\mathbb{E}\left[\prod_{x \in X} \phi(x)\right] = \det[\mathbf{1} - (1 - \phi)K].$$

This identity makes determinantal point processes more tractable than general processes.

A key example: one-dependent processes on \mathbb{Z}

We highlight an important application from [BDF10] that connects *1-dependent* processes on an integer segment (or a finite subset of \mathbb{Z}) to determinantal processes. A point process P on \mathbb{Z} is *1-dependent* if, for any two disjoint finite sets $A, B \subset \mathbb{Z}$ with $\text{dist}(A, B) \geq 2$, the correlation function factorizes:

$$\rho_{|A|+|B|}(A \cup B) = \rho_{|A|}(A) \rho_{|B|}(B).$$

Theorem 5.3 ([BDF10, Thm. 1.1]). *Any one-dependent point process on a finite segment of \mathbb{Z} is a determinantal process. Moreover, its correlation kernel K can be explicitly computed.*

Example 5.4 (Adding a list of numbers). Consider an i.i.d. sequence of random variables $\{\xi_j\}$ (each taking values in $\{0, 1\}$), and define the partial sums $S_n = \sum_{j=1}^n \xi_j$. The occupancy process, marking site S_n as “occupied,” forms a 1-dependent sequence. By Theorem 5.3, it is thus determinantal.

D Problems (due 2025-02-28)

D.1 Eigenvalue density of $G\beta E$

Read and understand the main principles of the proof of Theorem 2.5 in [DE02].

D.2 Chi-square mean and variance

Let X be a random variable with χ_ν^2 distribution. Compute the mean and variance of X . (If ν is an integer, you can use the fact that χ_ν^2 is a sum of ν independent squares of standard normal random variables. How to extend this to non-integer ν ?)

D.3 Edge contributions in the tridiagonal moment computation

Show that the cases when the i_ℓ 's are close to the edge ($\theta = 0$ or 1) in (3.2) do not contribute to the limit of the moments.

D.4 Hermite polynomials and three-term recurrence

Show that the monic Hermite polynomials $H_k(x)$ (2.4) satisfy the three-term recurrence relation

$$H_k(x) = xH_{k-1}(x) - (k-1)H_{k-2}(x).$$

References

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