Lectures on Random Matrices (Spring 2025)
Lecture 8: Cutting corners and loop equations

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1 Cutting corners: polynomial equations and distribution

1.1 Recap: polynomial equations

Recall the polynomial equation we proved in the last Lecture 7. Fix $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$. Let $H \in \operatorname{Orbit}(\lambda)$ be a random Hermitian matrix defined as

$$H = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n) U^{\dagger},$$

where U is Haar-distributed unitary matrix from $U(n)$. This is the case $\beta = 2$, but the statement holds for the cases $\beta = 1, 4$ with appropriate modifications. Let $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of the $(n-1) \times (n-1)$ corner $H^{(n-1)}$.

Lemma 1.1. *The distribution of $\mu_1, \ldots, \mu_{n-1}$ is the same as the distribution of the roots of the polynomial equation*

$$\sum_{i=1}^{n} \frac{\xi_i}{z - \lambda_i} = 0, \tag{1.1}$$

where ξ_i are i.i.d. random variables with the distribution χ^2_β.

Recall also that this passage from λ to μ works inductively, and the distribution of the next level eigenvalues $\nu = (\nu_1 \geq \ldots \geq \nu_{n-2})$ is given by the same polynomial equation, but with λ

*Course webpage • Live simulations • TeX Source • Updated at 11:24, Tuesday 25th February, 2025 replaced by μ. In this way, we can define a *Markov map* from λ to μ, which is then iterated to construct the full array of eigenvalues of the corners of H.

For $\beta = \infty$, this map is deterministic, and is equivalent to successive differentiating the characteristic polynomial of H.

1.2 Distribution of the eigenvalues of the corners

Theorem 1.2. *The density of μ with respect to the Lebesgue measure is given by*

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} \prod_{1\le i<j\le n-1} (\mu_i - \mu_j) \prod_{i=1}^{n-1} \prod_{j=1}^{n} |\mu_i - \lambda_j|^{\beta/2-1} \prod_{1\le i<j\le n} (\lambda_i - \lambda_j)^{1-\beta}.$$

Proof. Let $\varphi_i = \xi_i / \sum_{j=1}^n \xi_j$. The joint density of $(\varphi_1, \ldots, \varphi_n)$ is the (symmetric) Dirichlet density

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} w_1^{\beta/2-1} \ldots w_n^{\beta/2-1} dw_1 \ldots dw_{n-1}$$

(note that the density is $(n-1)$-dimensional).

We need to compute the Jacobian of the transformation from φ to μ, if we write

$$\sum_{i=1}^{n} \frac{\varphi_i}{z - \lambda_i} = \frac{\prod_{i=1}^{n-1}(z-\mu_i)}{\prod_{i=1}^{n}(z-\lambda_i)},$$

and compute (as a decomposition into partial fractions):

$$\varphi_a = \frac{\prod_{i=1}^{n-1}(\lambda_a-\mu_i)}{\prod_{i\ne a}(\lambda_a-\lambda_i)}.$$

Therefore,

$$\frac{\partial \varphi_a}{\partial \mu_b} = \frac{\prod_{i=1}^{n-1}(\lambda_a-\mu_i)}{\prod_{i\ne a}(\lambda_a-\lambda_i)} \frac{1}{\mu_b - \lambda_a}.$$

The Jacobian is essentially the determinant of the matrix $1/(\mu_b - \lambda_a)$, which is the Cauchy determinant (Problem H.1). The final density is obtained from the symmetric Dirichlet density, but we plug in $w = \varphi$, and also multiply by the Jacobian. This completes the proof. □

Corollary 1.3 (Joint density of the corners)**.** *The eigenvalues $\lambda^(k)_j$, $1 \le j \le k \le n$, of a random matrix from* Orbit(λ) *form an interlacing array, with the joint density*

$$\propto \prod_{k=1}^{n} \prod_{1\le i<j\le k} \left(\lambda_j^{(k)} - \lambda_i^{(k)}\right)^{2-\beta} \prod_{a=1}^{k+1} \prod_{b=1}^{k} \left|\lambda_a^{(k+1)} - \lambda_b^{(k)}\right|^{\beta/2-1}.$$

For $\beta = 2$, all factors disappear, and we get the uniform distribution on the interlacing array. This is the *uniform Gibbs property* which is important for other models, including discrete ensembles.

H Problems (due 2025-03-25)

H.1 Cauchy determinant

Prove the Cauchy determinant formula:

$$\det \left(\frac{1}{x_i - y_j} \right)_{1 \le i,j \le n} = \frac{\prod_{i<j} (x_i - x_j)(y_i - y_j)}{\prod_{i,j} (x_i - y_j)}.$$

References

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