

Lectures on Random Matrices (Spring 2025)

Lecture 14: Matching Random Matrices to Random Growth II

Leonid Petrov

Wednesday, April 16, 2025*

Contents

1	Recap	1
1.1	Main goal	1
1.2	Spiked Wishart ensembles and the largest eigenvalue process	2
1.3	Inhomogeneous last-passage percolation	3
1.4	RSK via toggles: definitions and weight preservation	3
2	Distributions of last-passage times in geometric LPP	4
2.1	Matching RSK to last-passage percolation	4
2.2	Distributions in RSK	5
N	Problems (due 2025-04-29)	6
N.1	Non-Markovianity	6
N.2	Schur polynomials — equivalence of definitions	6
N.3	Schur polynomials — stability property	6
N.4	Cauchy identity for Schur polynomials	6

1 Recap

1.1 Main goal

In the previous [Lecture 13](#), we began establishing a remarkable correspondence between two a priori different objects:

- The *spiked Wishart ensemble*: an $n \times n$ Hermitian random-matrix process $\{M(t)\}_{t \geq 0}$ whose entries come from columns of independent Gaussian random vectors of suitably chosen covariance.
- An *inhomogeneous last-passage percolation (LPP)* model: an array $\{W_{i,j}\}$ of exponential random weights on a portion of the two-dimensional lattice, whose last-passage times $L(t, n)$ match the largest eigenvalues of $M(t)$, jointly for all $t \in \mathbb{Z}_{\geq 0}$.

*[Course webpage](#) • [Live simulations](#) • [TeX Source](#) • Updated at 13:00, Tuesday 15th April, 2025

This equivalence, originally due to [DW08] (following [Def10], [FR06]; see also [Bar01], [Joh00] for earlier results of this kind), can be fully understood by passing to a *discrete* version of LPP with geometric site-weights and then applying the *Robinson–Schensted–Knuth* (RSK) correspondence.

1.2 Spiked Wishart ensembles and the largest eigenvalue process

We defined the *generalized* (or *spiked*) Wishart matrix $M(t)$ of size $n \times n$ by setting

$$M(t) = \sum_{m=1}^t A^{(m)} (A^{(m)})^*$$

where $\{A^{(m)}\}_{m=1}^\infty$ are i.i.d. complex Gaussian column vectors of length n , with

$$\text{Var}(A_i^{(m)}) = \frac{1}{\pi_i + \hat{\pi}_m}.$$

Here, $\pi = (\pi_1, \dots, \pi_n)$ and $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \dots)$ are positive and nonnegative parameters, respectively. Writing $\lambda_1(t) \geq \dots \geq \lambda_n(t) \geq 0$ for the eigenvalues of $M(t)$, we then saw:

1. The vectors $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))$ form a Markov chain in the *Weyl chamber* $\mathbb{W}^n = \{x_1 \geq \dots \geq x_n \geq 0\}$.
2. There is an *interlacing* property: each update $M(t-1) \mapsto M(t)$ via the rank-one matrix $A^{(t)}(A^{(t)})^*$ forces $\lambda(t)$ to interlace with $\lambda(t-1)$:

$$\lambda_1(t) \geq \lambda_1(t-1) \geq \lambda_2(t) \geq \dots \geq \lambda_n(t-1) \geq \lambda_n(t).$$

In [Lecture 13](#), we wrote down the transition kernel from $\lambda(t-1)$ to $\lambda(t)$:

Theorem 1.1 ([DW08]). *Fix an integer $n \geq 1$. Let $\pi = (\pi_1, \dots, \pi_n)$ be a strictly positive n -vector, and let $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \dots)$ be any sequence of nonnegative real parameters. Under the probability measure $P^{\pi, \hat{\pi}}$, the eigenvalues of the $n \times n$ generalized Wishart matrices $\{M(t)\}_{t \geq 0}$ form a time-inhomogeneous Markov chain $\{\text{sp}(M(t))\}_{t \geq 0}$ in the Weyl chamber*

$$\mathbb{W}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n : x_1 \geq x_2 \geq \dots \geq x_n\}.$$

More precisely, writing $x = \text{sp}(M(t-1))$ and $y = \text{sp}(M(t))$, the one-step transition law from time $(t-1)$ to t is absolutely continuous on the interior of \mathbb{W}^n and can be factored as

$$Q_{t-1,t}^{\pi, \hat{\pi}}(x, dy) = \left[\prod_{i=1}^n (\pi_i + \hat{\pi}_t) \right] \cdot \frac{h_\pi(y)}{h_\pi(x)} \exp\left(-(\hat{\pi}_t - 1) \sum_{i=1}^n (y_i - x_i)\right) \times Q^{(0)}(x, dy), \quad (1.1)$$

where

- $Q^{(0)}(x, dy)$ is the standard (null-spike) Wishart transition kernel, given explicitly by

$$Q^{(0)}(x, dy) = \frac{\Delta(y)}{\Delta(x)} \exp\left(-\sum_{i=1}^n (y_i - x_i)\right) \mathbf{1}_{\{x \prec y\}} dy, \quad (1.2)$$

with $\Delta(z) = \prod_{1 \leq i < j \leq n} (z_i - z_j)$ the Vandermonde determinant.

- The function h_π is the (continuous) Harish-Chandra orbit integral factor

$$h_\pi(z) = \frac{(-1)^{\binom{n}{2}}}{0!1!\cdots(n-1)!} \frac{\det(e^{-\pi_i z_j})_{i,j=1}^n}{\Delta(\pi) \Delta(z)}.$$

Note that $h_\pi(0) = 1$.

In particular, the chain starts from $\text{sp}(M(0)) = 0$ (the zero matrix).

1.3 Inhomogeneous last-passage percolation

On the random growth side, we considered an array of site-weights $\{W_{i,j}\}_{i,j \geq 1}$ such that each $W_{i,j}$ is exponentially distributed with rate $\pi_i + \hat{\pi}_j$. For every integer $t \geq 1$, we define $L(t, n)$ to be the maximum total weight of all up-right paths from $(1, 1)$ to (t, n) :

$$L(t, n) = \max_{\Gamma: (1,1) \rightarrow (t,n)} \sum_{(i,j) \in \Gamma} W_{i,j}.$$

One checks that $L(\cdot, n)$ satisfies a simple additive recursion:

$$L(i, j) = W_{i,j} + \max\{L(i-1, j), L(i, j-1)\},$$

The main claim which we show in today's lecture is the equality in distribution:

$$(L(1, n), L(2, n), \dots, L(t, n)) \stackrel{d}{=} (\lambda_1(1), \lambda_1(2), \dots, \lambda_1(t)). \quad (1.3)$$

1.4 RSK via toggles: definitions and weight preservation

The *Robinson–Schensted–Knuth* correspondence (RSK) was the main new mechanism in [Lecture 13](#). In our setup, we adopt a *toggle-based* viewpoint: we encode arrays by diagonals and successively *toggle* the diagonals to achieve a fully *ordered* array R . The key to how RSK links LPP and random matrices is its *weight preservation* property.

We work with arrays $W = \{W_{ij}\}_{1 \leq i \leq t, 1 \leq j \leq n}$ and $R = \{R_{ij}\}_{1 \leq i \leq t, 1 \leq j \leq n}$, where W is a nonnegative integer array and R is an ordered array, that is, $R_{i,j} \leq R_{i,j+1}$ and $R_{i,j} \leq R_{i+1,j}$ for all i, j . Using RSK, we showed in [Lecture 13](#) that there is a bijection which maps W to R .

We also started to prove the following result, which we now complete:

Theorem 1.2 (Weight preservation). *Let $W = \{W_{i,j}\}$ be a nonnegative integer array, and $R = \text{RSK}(W)$. Denote*

$$\text{row}_i = \sum_{j=1}^n W_{i,j}, \quad \text{col}_j = \sum_{i=1}^t W_{i,j}$$

(which are essentially the cdf's of the array W), and for R define the diagonal sums starting at each (i, j) and going diagonally down and to the right:

$$\text{diag}_{i,j} = \sum_{k=0}^{\min(i,j)-1} R_{i-k, j-k}.$$

Then for each $1 \leq j \leq n$ and $1 \leq i \leq t$, we have

$$\text{diag}_{t,j} = \sum_{m=1}^j \text{col}_m, \quad \text{diag}_{i,n} = \sum_{m=1}^i \text{row}_m. \quad (1.4)$$

In particular, the total sum of W over all cells equals the total sum of R over all cells.

Proof (sketch). One inductively builds R by adding the sites (i, j) one at a time. Each toggle modifies exactly one diagonal. After adding a box (i, j) , the diagonal-sum identity

$$\text{diag}_{i,j} = \text{diag}_{i-1,j} + \text{diag}_{i,j-1} - \text{diag}_{i-1,j-1} + W_{i,j}$$

holds, expressing that W captures the discrete “mixed second differences” of the diagonal sums in R . Thus, the cdf’s of W must coincide with the diagonal sums of R , as desired. \square

2 Distributions of last-passage times in geometric LPP

2.1 Matching RSK to last-passage percolation

Recall that we are working with the independent geometric random variables

$$\text{Prob}(W_{ij} = k) = (a_i b_j)^k (1 - a_i b_j), \quad k = 0, 1, \dots$$

The parameters a_1, \dots, a_t and b_1, \dots, b_n are positive real numbers, and we assume that $a_i b_j < 1$ for all i, j , so that the random variables W_{ij} are well-defined. Let $R = \text{RSK}(W)$.

Lemma 2.1. *The distribution of the top row of the array R , $R_{t,1}, \dots, R_{t,n}$, is the same as the distribution of the last-passage times $L(t, 1), \dots, L(t, n)$, defined in the same environment $W = \{W_{ij}\}$.*

Note that this statement does not rely on the exact distribution of W , and holds for any fixed or random nonnegative integer array W .

Proof of Lemma 2.1. The values in R update according to the toggle rule. Denote by $R^{(i)}$ the array obtained after toggling the i -th row (and all previous rows) of W . Then, the top row of $R^{(i)}$ updates as

$$R_{i,j}^{(i)} = W_{i,j} + \max\{R_{i-1,j}^{(i-1)}, R_{i,j-1}^{(i)}\}.$$

By the induction hypothesis, we have

$$R_{i-1,j}^{(i-1)} = L(i-1, j), \quad R_{i,j-1}^{(i)} = L(i, j-1).$$

This implies that $L(i, j) = R_{i,j}^{(i)}$, and we may proceed by induction on j and then on i . \square

Remark 2.2. The correspondence between $R_{t,j}$ and $L(t, j)$ holds only for the top row of the final array $R = R^{(t)}$. For rows below the top row (i.e., for $R_{k,j}$ with $k < t$), there is no such direct correspondence with one-path last-passage times. On the other hand, the whole array R can be defined through multipath last-passage times. This is known as *Greene’s theorem* [Sag01] for RSK, and falls outside the scope of this course.

2.2 Distributions in RSK

Fix t, n , and consider the following quantities in a diagonal of the array $R = \text{RSK}(W)$:

$$\lambda_1 := R_{t,n}, \lambda_2 := R_{t-1,n-1}, \dots, \lambda_n := R_{t-n+1,1}.$$

Clearly, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ (we pad diag's by zeroes if necessary), and these are integers. We regard $\lambda = (\lambda_1, \dots, \lambda_n)$ as an integer partition, or a Young diagram. Denote by $T(\lambda)$ the space of all *semistandard Young tableaux* (SSYT) of shape λ , that is, all collections of numbers r_{ij} which interlace as

$$r_{i,j} \leq r_{i,j+1}, \quad r_{i,j} \leq r_{i+1,j}, \quad i = 1, \dots, t, \quad j = 1, \dots, n; \quad r_{t-k+1,n-k+1} = \lambda_k, \quad k = 1, \dots, n.$$

We are after the distribution of the random Young diagram λ .

Definition 2.3 (Schur polynomial). For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$, the Schur polynomial $s_\lambda(x_1, \dots, x_n)$ in n variables is defined as:

$$s_\lambda(x_1, \dots, x_n) = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\det(x_i^{n-j})_{i,j=1}^n} = \frac{\det(x_i^{\lambda_j+n-j})_{i,j=1}^n}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}. \quad (2.1)$$

Alternatively, the Schur polynomial has a combinatorial interpretation as a sum over semistandard Young tableaux:

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in T(\lambda)} \prod_{i=1}^n x_i^{m_i(T)} \quad (2.2)$$

where $T(\lambda)$ is the set of all semistandard Young tableaux of shape λ with entries from $\{1, 2, \dots, n\}$, and $m_i(T)$ is the number of occurrences of i in the tableau T .

From (2.1), it is evident that $s_\lambda(x_1, \dots, x_n)$ is a symmetric polynomial in x_1, \dots, x_n . This is highly non-obvious from the combinatorial definition (2.2). See Problem N.2 for a proof of the equivalence of the two definitions.

The Schur polynomials satisfy the stability property:

$$s_\lambda(x_1, \dots, x_{n-1}, x_n) \Big|_{x_n=0} = \begin{cases} s_\lambda(x_1, \dots, x_{n-1}) & \text{if } \lambda_n = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (2.3)$$

Theorem 2.4. Let $\mu = (\mu_1, \dots, \mu_n)$ be a fixed Young diagram. Then, for $R = \text{RSK}(W)$, where W is the array of independent geometric random variables, we have

$$\text{Prob}(R_{t,n} = \mu_1, \dots, R_{t-n+1,1} = \mu_n) = \prod_{i=1}^t \prod_{j=1}^n (1 - a_i b_j) \cdot s_\mu(a_1, \dots, a_t) s_\mu(b_1, \dots, b_n). \quad (2.4)$$

Note that if $t < n$, then $\mu_{t+1} = \dots = \mu_n = 0$, as it should be. Note also that the statement of the theorem implies that the expressions in the right-hand side of (2.4) sum to one over all $\mu_1 \geq \dots \geq \mu_n \geq 0$, which is the celebrated *Cauchy identity* for Schur polynomials. One can alternatively establish the Cauchy identity from the Cauchy-Binet formula, using the determinantal formulas (2.1). See Problem N.4.

Proof of Theorem 2.4.

□

N Problems (due 2025-04-29)

N.1 Non-Markovianity

Show that the sequence of random variables defined in the exponential LPP model,

$$L(1, n), L(2, n), \dots, L(t, n),$$

is **not** a Markov chain. By virtue of the equivalence with the spiked Wishart ensemble (1.3), you may alternatively show that the sequence of maximal eigenvalues

$$\lambda_1(1), \lambda_1(2), \dots, \lambda_1(t)$$

of successive Wishart matrices $M(1), M(2), \dots, M(t)$ is **not** a Markov chain either.

N.2 Schur polynomials — equivalence of definitions

Show the equivalence of the two definitions of Schur polynomials (2.1) and (2.2).

Hint: Substitute $x_n = 1$ and consider how both formulas expand as linear combinations of Schur polynomials $s_\mu(x_1, \dots, x_{n-1})$ in $n-1$ variables. This induction (together with the fact that Schur polynomials are a linear basis in the ring of symmetric polynomials in a given fixed number of variables) will show that the two definitions are equivalent.

N.3 Schur polynomials — stability property

Show the stability property of Schur polynomials (2.3).

N.4 Cauchy identity for Schur polynomials

Let a_1, \dots, a_t and b_1, \dots, b_n be positive parameters satisfying $a_i b_j < 1$ for all pairs (i, j) . Prove the Cauchy identity for Schur polynomials:

$$\sum_{\mu: \mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0} s_\mu(a_1, \dots, a_t) s_\mu(b_1, \dots, b_n) = \prod_{i=1}^t \prod_{j=1}^n \frac{1}{1 - a_i b_j}.$$

References

- [Bar01] Yu. Baryshnikov, *GUEs and queues*, Probab. Theory Relat. Fields **119** (2001), 256–274. [↑2](#)
- [Def10] M. Defosseux, *Orbit measures, random matrix theory and interlaced determinantal processes*, Ann. Inst. H. Poincaré Probab. Statist. **46** (2010), no. 1, 209–249. arXiv:0810.1011 [math.PR]. [↑2](#)
- [DW08] A. B. Dieker and J. Warren, *On the largest-eigenvalue process for generalized Wishart random matrices*, arXiv preprint (2008). arXiv:0812.1504 [math.PR]. [↑2](#)
- [FR06] P. J. Forrester and E. M. Rains, *Jacobians and rank 1 perturbations relating to unitary Hessenberg matrices*, Int. Math. Res. Not. **2006** (2006), Art. ID 48306. arXiv:math/0505552 [math.PR]. [↑2](#)
- [Joh00] K. Johansson, *Shape fluctuations and random matrices*, Commun. Math. Phys. **209** (2000), no. 2, 437–476. arXiv:math/9903134 [math.CO]. [↑2](#)

[Sag01] B.E. Sagan, *The symmetric group: representations, combinatorial algorithms, and symmetric functions*, Springer Verlag, 2001. [↑4](#)

L. PETROV, UNIVERSITY OF VIRGINIA, DEPARTMENT OF MATHEMATICS, 141 CABELL
DRIVE, KERCHOF HALL, P.O. BOX 400137, CHARLOTTESVILLE, VA 22904, USA
E-mail: lenia.petrov@gmail.com