Lectures on Random Matrices (Spring 2025)
Lecture 14: Matching Random Matrices to Random Growth II

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Contents

1 Recap

1.1 Main goal

In the previous Lecture 13, we began establishing a remarkable correspondence between two a priori different objects:

- The *spiked Wishart ensemble*: an $n \times n$ Hermitian random-matrix process $\{M(t)\}_{t\geq 0}$ whose entries come from columns of independent Gaussian random vectors of suitably chosen covariance.
- An *inhomogeneous last-passage percolation (LPP)* model: an array $\{W_{i,j}\}$ of exponential random weights on a portion of the two-dimensional lattice, whose last-passage times $L(t,n)$ match the largest eigenvalues of $M(t)$, jointly for all $t \in \mathbb{Z}_{\geq 0}$.

This equivalence, originally due to [DW08] (following [Def10], [FR06]; see also [Bar01], [Joh00] for earlier results of this kind), can be fully understood by passing to a *discrete* version of LPP with geometric site-weights and then applying the *Robinson–Schensted–Knuth* (RSK) correspondence.

*Course webpage • Live simulations • TeX Source • Updated at 11:39, Tuesday 15th April, 2025

1.2 Spiked Wishart ensembles and the largest eigenvalue process

We defined the *generalized* (or spiked) Wishart matrix $M(t)$ of size $n \times n$ by setting

$$M(t) \;=\; \sum_{m=1}^{t} A^{(m)} \big(A^{(m)}\big)^{*}$$

where $\{A^{(m)}\}_{m=1}^{\infty}$ are i.i.d. complex Gaussian column vectors of length n, with

$$\mathrm{Var}\big(A_i^{(m)}\big) \;=\; \frac{1}{\pi_i + \hat{\pi}_m}\,.$$

Here, $\pi = (\pi_1, \ldots, \pi_n)$ and $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \ldots)$ are positive and nonnegative parameters, respectively. Writing $\lambda_1(t) \geq \cdots \geq \lambda_n(t) \geq 0$ for the eigenvalues of $M(t)$, we then saw:

1. The vectors $\lambda(t) = \big(\lambda_1(t), \ldots, \lambda_n(t)\big)$ form a Markov chain in the *Weyl chamber* $\mathbb{W}^n = \{x_1 \geq \cdots \geq x_n \geq 0\}$.
2. There is an *interlacing* property: each update $M(t-1) \mapsto M(t)$ via the rank-one matrix $A^{(t)}\big(A^{(t)}\big)^*$ forces $\lambda(t)$ to interlace with $\lambda(t-1)$:
$$\lambda_1(t) \;\geq\; \lambda_1(t-1) \;\geq\; \lambda_2(t) \;\geq\; \cdots \;\geq\; \lambda_n(t-1) \;\geq\; \lambda_n(t).$$

In Lecture 13, we wrote down the transition kernel from $\lambda(t-1)$ to $\lambda(t)$:

Theorem 1.1 ([DW08])**.** *Fix an integer $n \geq 1$. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a strictly positive n-vector, and let $\widehat{\pi} = (\widehat{\pi}_1, \widehat{\pi}_2, \ldots)$ be any sequence of nonnegative real parameters. Under the probability measure $P^{\pi,\widehat{\pi}}$, the eigenvalues of the $n \times n$ generalized Wishart matrices $\{M(t)\}_{t\geq 0}$ form a time-inhomogeneous Markov chain $\{\mathrm{sp}(M(t))\}_{t\geq 0}$ in the Weyl chamber*

$$\mathbb{W}^n \;=\; \big\{\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} : x_1 \geq x_2 \geq \cdots \geq x_n \big\}.$$

More precisely, writing $x = \mathrm{sp}(M(t-1))$ and $y = \mathrm{sp}(M(t))$, the one-step transition law from time $(t-1)$ to t is absolutely continuous on the interior of $\mathbb{W}^n$ and can be factored as

$$Q^{\pi,\widehat{\pi}}_{t-1,t}(x,\, dy) \;=\; \Big[\prod_{i=1}^{n}(\pi_i + \widehat{\pi}_t)\Big] \cdot \frac{h_\pi(y)}{h_\pi(x)}\; \exp\Big(-(\widehat{\pi}_t - 1)\sum_{i=1}^{n}(y_i - x_i)\Big) \;\times\; Q^{(0)}(x,\, dy), \tag{1.1}$$

where

- *$Q^{(0)}\big(x,\, dy\big)$ is the* standard *(null-spike) Wishart transition kernel, given explicitly by*
$$Q^{(0)}(x,\, dy) \;=\; \frac{\Delta(y)}{\Delta(x)}\; \exp\Big(-\sum_{i=1}^{n}(y_i - x_i)\Big)\, \mathbf{1}_{\{x \prec y\}}\; dy, \tag{1.2}$$
with $\Delta(z) = \prod_{1\leq i<j\leq n}(z_i - z_j)$ the Vandermonde determinant.
- *The function h_π is the (continuous) Harish-Chandra orbit integral factor*
$$h_\pi(z) \;=\; \frac{(-1)^{\binom{n}{2}}}{0!1!\cdots(n-1)!}\frac{\det\big(e^{-\pi_i\, z_j}\big)_{i,j=1}^{n}}{\Delta(\pi)\,\Delta(z)}.$$
Note that $h_\pi(0) = 1$.

In particular, the chain starts from $\mathrm{sp}(M(0)) = 0$ (the zero matrix).

1.3 Inhomogeneous last-passage percolation

On the random growth side, we considered an array of site-weights $\{W_{i,j}\}_{i,j\geq 1}$ such that each $W_{i,j}$ is exponentially distributed with rate $\pi_i + \hat{\pi}_j$. For every integer $t \geq 1$, we define $L(t,n)$ to be the maximum total weight of all up-right paths from $(1,1)$ to (t,n):

$$L(t,n) \;=\; \max_{\Gamma\colon(1,1)\to(t,n)} \sum_{(i,j)\in\Gamma} W_{i,j}.$$

One checks that $L(\,\cdot\,,n)$ satisfies a simple additive recursion:

$$L(i,j) \;=\; W_{i,j} \;+\; \max\bigl\{\, L(i-1,j),\; L(i,j-1)\bigr\},$$

The main claim which we show in today's lecture is the equality in distribution:

$$\bigl(L(1,n),\, L(2,n),\, \ldots,\, L(t,n)\bigr) \;\stackrel{d}{=}\; \bigl(\lambda_1(1),\, \lambda_1(2),\, \ldots,\, \lambda_1(t)\bigr). \tag{1.3}$$

1.4 RSK via toggles: definitions and weight preservation

The *Robinson–Schensted–Knuth* correspondence (RSK) was the main new mechanism in Lecture 13. In our setup, we adopt a *toggle-based* viewpoint: we encode arrays by diagonals and successively *toggle* the diagonals to achieve a fully *ordered* array R. The key to how RSK links LPP and random matrices is its *weight preservation* property.

We work with arrays $W = \{W_{ij}\}_{1\leq i\leq t,\ 1\leq j\leq n}$ and $R = \{R_{ij}\}_{1\leq i\leq t,\ 1\leq j\leq n}$, where W is a nonnegative integer array and R is an ordered array, that is, $R_{i,j} \leq R_{i,j+1}$ and $R_{i,j} \leq R_{i+1,j}$ for all i,j. Using RSK, we showed in Lecture 13 that there is a bijection which maps W to R.

We also started to prove the following result, which we now complete:

Theorem 1.2 (Weight preservation)**.** *Let $W = \{W_{i,j}\}$ be a nonnegative integer array, and $R = \mathrm{RSK}(W)$. Denote*

$$\mathrm{row}_i \;=\; \sum_{j=1}^{n} W_{i,j}, \quad \mathrm{col}_j \;=\; \sum_{i=1}^{t} W_{i,j}$$

(which are essentially the cdf's of the array W), and for R define the diagonal sums starting at each (i,j) and going diagonally down and to the right:

$$\mathrm{diag}_{i,j} \;=\; \sum_{k=0}^{\min(i,j)-1} R_{i-k,\,j-k}.$$

Then for each $1 \leq j \leq n$ and $1 \leq i \leq t$, we have

$$\mathrm{diag}_{t,j} \;=\; \sum_{m=1}^{j} \mathrm{col}_m, \qquad \mathrm{diag}_{i,n} \;=\; \sum_{m=1}^{i} \mathrm{row}_m. \tag{1.4}$$

In particular, the total sum of W over all cells equals the total sum of R over all cells.

Proof (sketch). One inductively builds R by adding the sites (i,j) one at a time. Each toggle modifies exactly one diagonal. After adding a box (i,j), the diagonal-sum identity

$$\mathrm{diag}_{i,j} \;=\; \mathrm{diag}_{i-1,j} + \mathrm{diag}_{i,j-1} \;-\; \mathrm{diag}_{i-1,j-1} \;+\; W_{i,j}$$

holds, expressing that W captures the discrete "mixed second differences" of the diagonal sums in R. Thus, the cdf's of W must coincide with the diagonal sums of R, as desired. ☐

2 Distributions of last-passage times in geometric LPP

2.1 Conditional distribution

Recall that we are working with the independent geometric random variables

$$\mathrm{Prob}\,(W_{ij} = k) = (a_i b_j)^k (1 - a_i b_j), \qquad k = 0, 1, \ldots.$$

The parameters $a_1, \ldots, a_t$ and $b_1, \ldots, b_n$ are positive real numbers, and we assume that $a_i b_j < 1$ for all i, j, so that the random variables W_{ij} are well-defined.

Recall the notation

$$Z(t) = (L(t,1), \ldots, L(t,n))\,, \qquad t \in \mathbb{Z}_{\geq 0}.$$

Using the weight-preservation property (Theorem 1.2), we can now compute the conditional distribution of $Z(t)$ given $Z(t-1)$, and, in particular, show that this is a Markov chain.

Theorem 2.1.

N Problems (due 2025-04-29)

N.1 Non-Markovianity

Show that the sequence of random variables defined in the exponential LPP model,

$$L(1,n), L(2,n), \ldots, L(t,n),$$

is **not** a Markov chain. By virtue of the equivalence with the spiked Wishart ensemble (1.3), you may alternatively show that the sequence of maximal eigenvalues

$$\lambda_1(1), \lambda_1(2), \ldots, \lambda_1(t)$$

of successive Wishart matrices $M(1), M(2), \ldots, M(t)$ is **not** a Markov chain either.

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