UVA Summer School Invariant Measures PS 3

July 17, 2024

The number of stars next to an exercise indicates the expected time needed to fully solve the exercise.

Exercise 1 [*]

Assume $AC < 1$ and $A > \max(1, C)$ in open ASEP. Use Theorem 5.5 to show that

$$\langle W|(E+D)^N|V\rangle \sim \frac{(1+A)^{2n}}{A^n(1-q)^n} \frac{(A^{-2}, BC, BD, CD)_\infty}{(B/A, C/A, D/A, ABCD)_\infty}.$$

Exercise 2 [**]

Under the additional assumption

$$a^2, b^2, c^2, d^2, ab, ac, ad, bc, bd, cd \notin \{q^{-l} : l \in \mathbb{N}_0\}, \tag{1}$$

verify that the Askey–Wilson polynomials (w_m) are orthogonal with respect to the Askey–Wilson signed measure ν, i.e., show that

$$\int_{\mathbb{R}} \nu(\mathrm{d}x; a, b, c, d) w_j(x) w_k(x) = \delta_{jk} \frac{(1 - q^{j-1}abcd)(q, ab, ac, ad, bc, bd, cd)_j}{(1 - q^{2j-1}abcd)(abcd)_j} \tag{2}$$

holds for all $j, k \in \mathbb{N}_0$.
Hint: You may use the complex contour integral version of the orthogonality stated as Theorem 5.1.

Exercise 3 [***]

Show that the orthogonality (2) in **Exercise 2** holds true without the additional assumption (1).
Hint: You may use continuity arguments.

Exercise 4 [****]

Use the orthogonality described in **Exercise 3** above, together with the projection formula

$$\int_{\mathbb{R}} p_j(y; t) P_{s,t}(x, \mathrm{d}y) = p_j(x; s) \quad \text{for } x \in U_s \text{ and } j = 0, 1, \ldots \tag{3}$$

where we recall

$$p_j(x; t) := t^{j/2}(ABt)_j^{-1} w_j\left(x; A\sqrt{t}, B\sqrt{t}, C/\sqrt{t}, D/\sqrt{t}\right) \quad \text{for } j \in \mathbb{N}_0$$

and

$$P_{s,t}(x, \mathrm{d}y) = \nu(\mathrm{d}y; A\sqrt{t}, B\sqrt{t}, \sqrt{s/t}(x + \sqrt{x^2-1}), \sqrt{s/t}(x - \sqrt{x^2-1})), \quad \forall s < t,\ s, t \in \mathbb{R}_+,\ x \in U_s,$$

to demonstrate the so-called **time-reversal property** of the Askey–Wilson signed measures:

$$\pi^{(A,B,C,D)}_{t_1,\ldots,t_m}(\mathrm{d}x_1,\ldots,\mathrm{d}x_m) = \pi^{(C,D,A,B)}_{1/t_m,\ldots,1/t_1}(\mathrm{d}x_m,\ldots,\mathrm{d}x_1)\,, \tag{4}$$

for any $0 < t_1 \leq \cdots \leq t_m$ such that both sides above are well-defined.
Hint: First prove the cases $m = 1$ and $m = 2$, then proceed by induction over m.