Lectures on Random Matrices (Spring 2025)
Lecture 2: Wigner semicircle law

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Contents

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1 Recap

We are working on the Wigner semicircle law.

1. Wigner matrices W: real symmetric random matrices with iid entries X_{ij}, $i > j$ (mean 0, variance σ^2); and iid diagonal entries X_{ii} (mean 0, some other variance and distribution).

2. Empirical spectral distribution (ESD)

$$\nu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i/\sqrt{n}},$$

which is a random probability measure on $\mathbb{R}$.

3. Semicircle distribution μ_{sc}:

$$\mu_{\mathrm{sc}}(dx) = \frac{1}{2\pi} \sqrt{4 - x^2}\, dx, \qquad x \in [-2, 2].$$

4. Computation of expected traces of powers of W (with variance 1). We showed that

$$\int_{\mathbb{R}} x^k\, \nu_n(dx) \to \#\left\{\text{rooted planar trees with } k/2 \text{ edges}\right\}.$$

Remark 1.1. If the off-diagonal elements of the matrix have variance σ^2, then the semicircle distribution should be scaled to be supported on $[-2\sigma, 2\sigma]$. We assume that the variance of the off-diagonal elements is 1 in most arguments throughout the lecture.

2 Two computations

First, we finish the combinatorial part, and match the limiting expected traces of powers of W to moments of the semicircle law.

2.1 Moments of the semicircle law

We also need to match the Catalan numbers to the moments of the semicircle law. Let $k = 2m$, and we need to compute the integral

$$\int_{-2}^{2} x^{2m} \frac{1}{2\pi} \sqrt{4 - x^2}\, dx.$$

By symmetry, we write:

$$\int_{-2}^{2} x^{2m} \rho(x)\, dx = \frac{2}{\pi} \int_{0}^{2} x^{2m} \sqrt{4 - x^2}\, dx.$$

Using the substitution $x = 2\sin\theta$, we have $dx = 2\cos\theta\, d\theta$. The integral becomes:

$$\frac{2}{\pi} \int_{0}^{\pi/2} (2\sin\theta)^{2m} (2\cos\theta)(2\cos\theta\, d\theta) = \frac{2^{2m+2}}{\pi} \int_{0}^{\pi/2} \sin^{2m}\theta \cos^2\theta\, d\theta.$$

Using $\cos^2\theta = 1 - \sin^2\theta$, we split the integral:

$$\frac{2^{2m+2}}{\pi}\left(\int_0^{\pi/2} \sin^{2m}\theta\, d\theta - \int_0^{\pi/2} \sin^{2m+2}\theta\, d\theta\right).$$

Using the standard formula (cf. Problem B.1)

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2}\frac{(2n)!}{2^{2n}(n!)^2}, \tag{2.1}$$

we compute each term:

$$\frac{2^{2m+2}}{\pi}\left(\frac{\pi}{2}\frac{(2m)!}{2^{2m}(m!)^2} - \frac{\pi}{2}\frac{(2m+2)!}{2^{2m+2}((m+1)!)^2}\right).$$

After simplification, this becomes C_m, the m-th Catalan number.

2.2 Counting trees and Catalan numbers

Throughout this section, for a random matrix trace moment of order k, we use $m = k/2$ as our main parameter. Note that m can be arbitrary (not necessarily even).

Definition 2.1 (Dyck Path)**.** A *Dyck path* of semilength m is a sequence of $2m$ steps in the plane, each step being either $(1, 1)$ (up step) or $(1, -1)$ (down step), starting at $(0, 0)$ and ending at $(2m, 0)$, such that the path never goes below the x-axis. We denote an up step by U and a down step by D.

Definition 2.2 (Rooted Plane Tree)**.** A *rooted plane tree* is a tree with a designated root vertex where the children of each vertex have a fixed left-to-right ordering. The size of such a tree is measured by its number of edges, which we denote by m.

Definition 2.3 (Catalan Numbers)**.** The sequence of *Catalan numbers* $\{C_m\}_{m\geq 0}$ is defined recursively by:

$$C_0 = 1, \quad C_{m+1} = \sum_{j=0}^{m} C_j C_{m-j} \quad \text{for } m \geq 0. \tag{2.2}$$

Alternatively, they have the closed form[1]

$$C_m = \frac{1}{m+1}\binom{2m}{m} = \binom{2m}{m} - \binom{2m}{m+1}. \tag{2.3}$$

These numbers appear naturally in the moments of random matrices, where $m = k/2$ for trace moments of order k.

Lemma 2.4. *Formulas (2.2) and (2.3) are equivalent.*

[1] See Problem B.4 for a combinatorial proof of the second inequality.

Proof. One can check that the closed form satisfies the recurrence relation by direct substitution. The other direction involves generating functions. Namely, (2.2) can be rewritten for the generating function

$$C(z) = \sum_{m=0}^{\infty} C_m z^m$$

as

$$C(z) = 1 + zC(z)^2.$$

Solving for $C(z)$, we get

$$C(z) = \frac{1 \pm \sqrt{1-4z}}{2z}. \tag{2.4}$$

We need to pick the solution which is nonsingular at $z = 0$, and it corresponds to the minus sign. Taylor expansion of the right-hand side of (2.4) at $z = 0$ gives the closed form. $\square$

Remark 2.5. Catalan numbers enumerate many (too many!) combinatorial objects. For a comprehensive treatment, see [Sta15].

Proposition 2.6 (Dyck Path–Rooted Tree Correspondence)**.** *For any m, there exists a bijection between the set of Dyck paths of semilength m and the set of rooted plane trees with m edges.*

Proof. Given a Dyck path of semilength m, we build the corresponding rooted plane tree as follows (see Figure 1 for an illustration):

1. Start with a single root vertex
2. Read the Dyck path from left to right:
   - For each up step (U), add a new child to the current vertex
   - For each down step (D), move back to the parent of the current vertex
3. The order of children is determined by the order of up steps

This is clearly a bijection, and we are done. $\square$

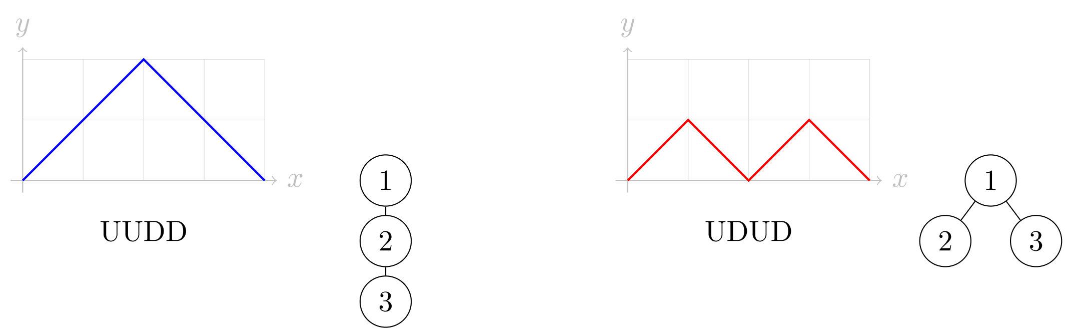

Figure 1: The two possible Dyck paths of semilength $m = 2$ and their corresponding rooted plane trees.

It remains to show that the Dyck paths or rooted plane trees are counted by the Catalan numbers, by verifying the recursion (2.2) for them. By Proposition 2.6, it suffices to consider only Dyck paths.

Proposition 2.7. *The number of Dyck paths of semilength m satisfies the Catalan recurrence* (2.2).

Proof. We need to show that the number of Dyck paths of semilength $m+1$ is given by the sum in the right-hand side of (2.2). Consider a Dyck path of semilength $m+1$, and let the *first* time it returns to zero be at semilength $j+1$, where $j = 0, \ldots, m$. Then the first and the $(2j+1)$-st steps are, respectively, U and D. From 0 to $2j+2$, the path does not return to the x-axis, so we can remove the first and the $(2j+1)$-st steps, and get a proper Dyck path of semilength j. The remainder of the Dyck path is a Dyck path of semilength $m-j$. This yields the desired recurrence. $\square$

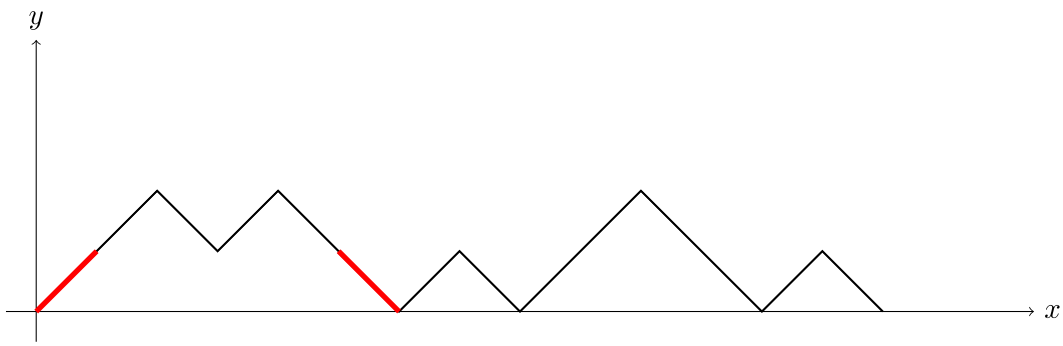

Figure 2: Illustration of a Dyck path decomposition for the proof of Proposition 2.7.

3 Analysis steps in the proof

We are done with combinatorics, and it remains to justify that the computations lead to the desired semicircle law from Lecture 1.

Let us remember that so far, we showed that

$$\lim_{n\to\infty} \frac{1}{n^{k/2+1}} \mathbb{E}\left[\operatorname{Tr} W^k\right] = \begin{cases} \sigma^{2m} C_m & \text{if } k = 2m \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Here, W is real Wigner (unnormalized) with mean 0, where its off-diagonal entries are iid with variance σ^2.

3.1 The semicircle distribution is determined by its moments

We use (without proof) the known Carleman's criterion for the uniqueness of a distribution by its moments.

Proposition 3.1 (Carleman's criterion [ST43, Theorem 1.10], [Akh65])**.** *Let X be a real-valued random variable with moments $m_k = \mathbb{E}[X^k]$ of all orders. If*

$$\sum_{k=1}^{\infty} (m_{2k})^{-1/(2k)} = \infty, \tag{3.1}$$

then the distribution of X is uniquely determined by its moments $(m_k)_{k \geq 1}$.

Remark 3.2. Note that we do not assume that the measure is symmetric, but use only even moments for the Carleman criterion. Indeed, in determining uniqueness, the decisive aspect is how the distribution mass "escapes" to $\pm\infty$. Since $\int |x|^n d\mu(x)$ can be bounded by twice $\int x^{2\lfloor n/2 \rfloor} d\mu(x)$ (roughly speaking), controlling $\int x^{2n} d\mu(x)$ also controls $\int |x|^n d\mu(x)$. Thus, one does not need to worry about positive or negative signs in x; the even powers handle both sides of the real line at once.

Moreover, the convergence of (3.1), as for any infinite series, is only determined by arbitrarily large moments, for the same reason.

Remark 3.3. By the Stone-Wierstrass theorem, the semicircle distribution on $[-2, 2]$ is unique among distributions with an arbitrary, but fixed compact support with the moments $\sigma^{2k} C_k$. However, we need to guarantee that there are no distributions on $\mathbb{R}$ with the same moments.

Now, the moments satisfy the asymptotics

$$m_{2k} = C_k \sigma^{2k} \sim \frac{4^k}{k^{3/2}\sqrt{\pi}} \sigma^{2k},$$

so

$$\sum_{k=1}^{\infty} (m_{2k})^{-1/(2k)} \sim \sum_{k=1}^{\infty} \left(\frac{k^{3/2}\sqrt{\pi}}{4^k} \right)^{1/2k} \sigma^{-1}.$$

The k-th summands converges to $1/(2\sigma)$, so the series diverges.

Remark 3.4. See also Problem A.4 from Lecture 1 on an example of a distribution not determined by its moments.

3.2 Convergence to the semicircle law

Recall [Bil95, Theorem 30.2] that convergence of random variables in moments plus the fact that the limiting distribution is uniquely determined by its moments implies convergence in distribution. However, we need weak convergence in probability or almost surely (see the previous Lecture 1). which deals with random variables

$$\int_{\mathbb{R}} f(x)\, \nu_n(dx), \qquad f \in C_b(\mathbb{R}),$$

and we did not compute the moments of these random variables.

To complete the argument, let us show that for each fixed integer $k \geq 1$, we have almost sure convergence of the moments (of a random distribution, so that the $Y_{n,k}$'s are random variables):

$$Y_{n,k} := \int_{\mathbb{R}} x^k\, \nu_n(dx) \xrightarrow[n\to\infty]{\text{a.s.}} m_k, \qquad n \to \infty,$$

where m_k are the moments of the semicircle distribution, and ν_n is the ESD corresponding to the scaling of the eigenvalues as $\lambda_i/\sqrt{n}$.

As typical in asymptotic probability, we not only need the expectation of $Y_{n,k}$, but also their variances, to control the almost sure convergence. Recall that we showed $\mathbb{E}(Y_{n,k}) \to m_k$. Let us assume the following:

Proposition 3.5 (Variance bound)**.** *For each fixed integer $k \geq 1$ and large enough n, we have*

$$\mathrm{Var}(Y_{n,k}) \leq \frac{m_k}{n^2}.$$

We will prove Proposition 3.5 in Section 4 below. Let us finish the proof of convergence to the semicircle law modulo Proposition 3.5.

3.2.1 A concentration bound and the Borel–Cantelli lemma

From Chebyshev's inequality,

$$\mathbb{P}\Big(\big|Y_{n,k} - \mathbb{E}[Y_{n,k}]\big| \geq n^{-\frac{1}{4}}\Big) \leq \mathrm{Var}[Y_{n,k}]\sqrt{n} = O(n^{-\frac{3}{2}}),$$

where in the last step we used Proposition 3.5.

Hence the probability that $|Y_{n,k} - \mathbb{E}[Y_{n,k}]| > n^{-\frac{1}{4}}$ is summable in n. By the Borel–Cantelli lemma, with probability 1 only finitely many of these events occur. Since $\mathbb{E}[Y_{n,k}] \to m_k$, we conclude

$$\big|Y_{n,k} - m_k\big| \;\leq\; \big|Y_{n,k} - \mathbb{E}[Y_{n,k}]\big| + \big|\mathbb{E}[Y_{n,k}] - m_k\big| \;\xrightarrow[n\to\infty]{}\; 0 \quad \text{almost surely.}$$

3.2.2 Tightness of $\{\nu_n\}$ and subsequential limits

Since $|Y_{n,k}| = \big|\int x^k\,\nu_n(dx)\big|$ stays almost surely bounded for each k, one readily checks (Problem B.5) that almost surely, for each fixed k,

$$\nu_n\big(\{x : |x| > M\}\big) \;\leq\; \frac{C}{M^k}. \tag{3.2}$$

By choosing k large, we see that ν_n puts arbitrarily little mass outside any large interval $[-m, m]$. Thus, the sequence of probability measures $\{\nu_n\}$ is *tight*. By Prokhorov's theorem [Bil95, Theorem 25.10], there exists a subsequence ν_{n_j} converging weakly to some probability measure ν^*. We will now characterize all subsequential limits ν^* of ν_n.

3.2.3 Characterizing the limit measure

We claim that $\nu^* = \mu_{\mathrm{sc}}$, the semicircle distribution (and in particular, this measure is not random). Indeed, fix k. Since x^k is a bounded function on a sufficiently large interval, and $\nu_{n_j} \to \nu^*$ weakly, we have

$$\int_{\mathbb{R}} x^k\,\nu_{n_j}(dx) \;\to\; \int_{\mathbb{R}} x^k\,\nu^*(dx).$$

On the other hand, we have already shown

$$\int_{\mathbb{R}} x^k \, \nu_{n_j}(dx) \;=\; Y_{n_j,k} \;\xrightarrow[j\to\infty]{\text{a.s.}}\; m_k \;=\; \int_{\mathbb{R}} x^k \, \mu_{\mathrm{sc}}(dx).$$

Thus

$$\int_{\mathbb{R}} x^k \, \nu^*(dx) \;=\; m_k \;=\; \int_{\mathbb{R}} x^k \, \mu_{\mathrm{sc}}(dx) \qquad \text{for all } k \ge 1.$$

By Proposition 3.1, the measure ν^* is uniquely determined by its moments. Hence ν^* must coincide with μ_{sc}.

Remark 3.6. In Sections 3.2.2 and 3.2.3 we tacitly assumed that we choose an elementary outcome ω, and view ν_n as measures depending on ω. Then, since the convergence of moments is almost sure, ω belongs to a set of full probability. The limiting measure ν^* must coincide with μ_{sc} for this ω, and thus, ν^* is almost surely nonrandom.

Any subsequence of $\{\nu_n\}$ has a further sub-subsequence convergent to ν. By a standard diagonal argument, this forces $\nu_n \to \nu$ in the weak topology (almost surely). This completes the proof that the ESD of our Wigner matrix (rescaled by $\sqrt{n}$) converges to the semicircle distribution weakly almost surely, modulo Proposition 3.5. (See also Problem B.6 for the weakly in probability convergence.)

4 Proof of Proposition 3.5: bounding the variance

There is one more "combinatorial" step in the proof of the semicircle law: we need to show that the variance of the moments of the ESD is bounded by m_k/n^2.

Recall that

$$Y_{n,k} = \int_{\mathbb{R}} x^k \, \nu_n(dx) = \frac{1}{n^{1+\frac{k}{2}}} \sum_{i_1,\ldots,i_k=1}^{n} X_I, \quad \text{where} \quad X_I = X_{i_1 i_2} X_{i_2 i_3} \cdots X_{i_k i_1}.$$

Here we use the notation I for the multi-index $(i_1, \ldots, i_k)$, and throughout the computation below, we use the notation $I \in [n]^k$, where $[n] = \{1, \ldots, n\}$. We have

$$\mathrm{Var}\big(Y_{n,k}\big) = \frac{1}{n^{2+k}} \, \mathrm{Var}\Big(\sum_{I \in [n]^k} X_I \Big) = \frac{1}{n^{2+k}} \sum_{I,J \in [n]^k} \mathrm{Cov}\big(X_I, X_J\big).$$

We claim that the sum of all covariances is bounded by a constant times n^k, which then implies $\mathrm{Var}\big(Y_{n,k}\big) \le \mathrm{const} \cdot n^k / n^{2+k} = O\big(\frac{1}{n^2}\big)$.

Step 1. Identifying when $\mathrm{Cov}\big(X_I, X_J\big)$ can be nonzero. For each k-tuple $I = (i_1, i_2, \ldots, i_k) \in [n]^k$, the product

$$X_I = X_{i_1 i_2} \, X_{i_2 i_3} \, \ldots \, X_{i_k i_1}$$

is the product of the entries of our Wigner matrix corresponding to the directed "edges" $(i_1 \to i_2), (i_2 \to i_3), \ldots, (i_k \to i_1)$. Similarly, X_J is determined by the edges of another closed directed walk J.

1. If I and J use disjoint collections of matrix entries, then X_I and X_J are independent, and hence $\operatorname{Cov}(X_I, X_J) = 0$.

2. If there is an edge (say, $X_{i_1 i_2}$) which appears *only once* in exactly one of I or J but not both, then that edge factor is independent and forces $\operatorname{Cov}(X_I, X_J) = 0$ since $\mathbb{E}[X_{i_1 i_2}] = 0$. Indeed, for example if $X_{i_1 i_2}$ appears only in X_I, then

$$\mathbb{E}\left[X_I\right] = \mathbb{E}\left[X_{i_1 i_2}\right] \cdot \mathbb{E}\left[\text{other factors}\right] = 0, \qquad \mathbb{E}\big[X_I X_J\big] = \mathbb{E}[X_{i_1 i_2}] \cdot \mathbb{E}\big[\text{other factors}\big] = 0.$$

Thus, the only way we could get a nonzero covariance is if *every* edge that appears in $I \cup J$ appears at least twice overall. Graphically, let us represent each k-tuple I by a directed closed walk in the complete graph on $[n]$. The union $I \cup J$ must be a connected subgraph in which every directed edge has total multiplicity ≥ 2.

Step 2. Counting the contributions to the sum. Denote by $q = |V(I \cup J)|$ the number of distinct vertices involved in the union $I \cup J$. In principle, there are $O(n^q)$ ways to choose q vertices from $[n]$. Then we need to specify how the edges form two closed walks of length k.

We split into two cases:

1. $\boldsymbol{q \leq k}$**.** Then the n-power in the sum over I, J is at most n^k, which yields the overall contribution $O(n^{-2})$, as desired.

2. $\boldsymbol{q \geq k+1}$**.** Ignoring directions and multiplicities, we see that the subgraph corresponding to $I \cup J$ contains at most k edges. Since $q \geq k+1$, we must have $q = k+1$ (by connectedness). Thus, $I \cup J$ is a double tree. Since I and J are subsets of this double tree and $q = k+1$, they also must be double trees. Thus, there exists an edge which appears in both I and J, and at least twice in I and twice in J, so four times in $I \cup J$. This contradicts the assumption that $I \cup J$ is a double tree.

   This implies that there are no leading contributions to the sum when $q \geq k+1$.

Combining these two cases, we conclude that the total number of pairs (I, J) with nonzero covariance is of order at most n^k, This yields the desired bound on the variance, and completes the proof of Proposition 3.5.

With that, we are done with the Wigner semicircle law proof for real Wigner matrices (with weakly almost sure convergence; see Lecture 1 for the definitions).

Also, see Problem B.7 for the complex case of the Wigner semicircle law.

5 Remark: Variants of the semicircle law

Let us briefly outline a few examples of the semicircle law for real/complex Wigner matrices which relax the iid conditions and the conditions that all moments of the entries must be finite. This list is not comprehensive, it is presented as an illustration of the universality / robustness of the semicircle law.

Theorem 5.1 (Gaussian β-Ensembles [Joh98], [For10])**.** *Let $\beta > 0$, and consider an $n \times n$ random matrix ensemble with joint eigenvalue density:*

$$p_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} \exp\left(-\frac{\beta}{4}\sum_{i=1}^{n} \lambda_i^2\right) \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^\beta \tag{5.1}$$

where $Z_{n,\beta}$ is the normalization constant.[2] *Then the ESD of the normalized eigenvalues $\lambda_i/\sqrt{n}$ converges weakly almost surely to the semicircle law.*

Theorem 5.2 (Correlated entries [SSB05]). *Let $W_n = \left(\frac{1}{\sqrt{n}} X_{pq}\right)_{1\le p,q\le n}$ be a sequence of $n \times n$ Hermitian random matrices where:*

1. *The entries X_{pq} are complex random variables that are:*
   - *Centered: $\mathbb{E}[X_{pq}] = 0$,*
   - *Unit variance: $\mathbb{E}[|X_{pq}|^2] = 1$,*
   - *Moment bound: $\sup_n \max_{p,q=1,\ldots,n} \mathbb{E}\left[|X_{pq}|^k\right] < \infty$ for all $k \in \mathbb{N}$.*
2. *There exists an equivalence relation $\sim_n$ on pairs of indices (p,q) in $\{1,\ldots,n\}^2$ such that:*
   - *Entries $X_{p_1q_1},\ldots,X_{p_jq_j}$ are independent when $(p_1,q_1),\ldots,(p_j,q_j)$ belong to distinct equivalence classes.*
   - *The relation satisfies the following bounds:*

     (a) $\max_p \#\big\{(q,p',q') \in \{1,\ldots,n\}^3 \mid (p,q) \sim_n (p',q')\big\} = o(n^2)$,

     (b) $\max_{p,q,p'} \#\big\{q' \in \{1,\ldots,n\} \mid (p,q) \sim_n (p',q')\big\} \le B$ *for some constant B,*

     (c) $\#\big\{(p,q,p') \in \{1,\ldots,n\}^3 \mid (p,q) \sim_n (q,p')$ *and* $p \ne p'\big\} = o(n^2)$.
3. *The matrices are Hermitian: $X_{pq} = \overline{X_{qp}}$. In particular, $(p,q) \sim_n (q,p)$, and this is consistent with the conditions on the equivalence relation.*

Then, as $n \to \infty$, the ESD of W_n converges to the semicircle law.

There are variants of this theorem without the assumption that all moments of the entries are finite.

Theorem 5.3 ([BGK16]). *Let $M_n = [X_{ij}]_{i,j=1}^n$ be a symmetric $n \times n$ matrix with random entries such that:*

- *The off-diagonal elements X_{ij}, for $i < j$, are i.i.d. random variables with $\mathbb{E}[X_{ij}] = 0$ and $\mathbb{E}[X_{ij}^2] = 1$.*
- *The diagonal elements X_{ii} are i.i.d. random variables with $\mathbb{E}[X_{ii}] = 0$ and a finite second moment, $\mathbb{E}[X_{ii}^2] < \infty$, for $1 \le i \le n$.*

Then the ESD of M_n, normalized by $\sqrt{n}$, converges to the semicircle law.

Theorem 5.4. *For each $n \in \mathbb{Z}_+$, let $M_n = [X_{ij}]_{i,j=1}^n$ be a symmetric $n \times n$ matrix with real random entries satisfying the following conditions:*

- *The entries X_{ij} are independent (but not necessarily identically distributed) random variables with $\mathbb{E}[X_{ij}] = 0$ and $\mathbb{E}[X_{ij}^2] = 1$.*

[2] For $\beta = 1, 2, 4$, this is the joint eigenvalue density of the Gaussian Orthogonal, Unitary, and Symplectic Ensembles, respectively. For general β, there is no invariant random matrix distribution (while the eigenvalue density (5.1) makes sense), and we can still treat all the β cases in a unified manner.

- *There exists a constant C such that $\sup_{i,j,n} \mathbb{E}\left[|X_{ij}|^4\right] < C$.*

Then the ESD of M_n, normalized by $\sqrt{n}$, converges to the semicircle law almost surely. The second condition can also be replaced by a uniform integrability condition on the variances.

Theorem 5.5 (For example, see [SB95])**.** *Let $M_n = [X_{ij}]_{i,j=1}^n$ be a symmetric $n \times n$ matrix with random entries. Assume that the expected matrix $\mathbb{E}[M_n]$ has rank $r(n)$, where*

$$\lim_{n\to\infty} \frac{r(n)}{n} = 0.$$

Additionally, suppose $\mathbb{E}[X_{ij}] = 0$, $\mathrm{Var}(X_{ij}) = 1$, and

$$\sup_{i,j,n} \mathbb{E}\left[|X_{ij} - \mathbb{E}[X_{ij}]|^4\right] < \infty.$$

Then the ESD of M_n, normalized by $\sqrt{n}$, converges to the semicircle law almost surely.

B Problems (due 2025-02-15)

B.1 Standard formula

Prove formula (2.1):

$$\int_0^{\pi/2} \sin^{2n}\theta \, d\theta = \frac{\pi}{2} \frac{(2n)!}{2^{2n}(n!)^2}.$$

B.2 Tree profiles

Show that the expected height of a uniformly random Dyck path of semilength m is of order $\sqrt{m}$.

B.3 Ballot problem

Suppose candidate A receives p votes and candidate B receives q votes, where $p > q \geq 0$. In how many ways can these votes be counted such that A is always strictly ahead of B in partial tallies?

B.4 Reflection principle

Show the equality

$$C_m = \binom{2m}{m} - \binom{2m}{m-1},$$

where C_m counts the number of lattice paths from $(0,0)$ to $(2m,0)$ with steps $(1,1)$ and $(1,-1)$ that never go below the x-axis, and binomial coefficients count arbitrary lattice paths from $(0,0)$ to $(2m,0)$ or to $(2m,2)$ with steps $(1,1)$ and $(1,-1)$. In other words, show that the difference between the number of paths to $(2m,0)$ and to $(2m,2)$ is C_m, the number of paths that never go below the x-axis.

B.5 Bounding probability in the proof

Show inequality (3.2).

B.6 Almost sure convergence and convergence in probability

Show that in Wigner's semicircle law, the weakly almost sure convergence of random measures ν_n to μ_{sc} implies weak convergence in probability.

B.7 Wigner's semicircle law for complex Wigner matrices

Complex Wigner matrices are Hermitian symmetric, with iid complex off-diagonal entries, and real iid diagonal entries (all mean zero). Each complex random variable has independent real and imaginary parts.

1. Compute the expected trace of powers of a complex Wigner matrix.
2. Outline the remaining steps in the proof of Wigner's semicircle law for complex Wigner matrices.

B.8 Semicircle law without the moment condition

Prove Theorem 5.3.

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