

# Lectures on Random Matrices (Spring 2025)

## Lecture 6: Double contour integral kernel. Steepest descent and local statistics

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### Notes for the lecturer

1. GUE det structure
2. Formulate Cauchy–Binet and Andreief
3. Recall that  $\rho_n = P_n$  and it is  $(\det[\psi_i(x_j)]_{n \times n})^2$ , then reproduce the proofs here.

## 1 Recap: Determinantal structure of the GUE

Last time, we proved the following result:

**Theorem 1.1.** *The GUE correlation functions are given by*

$$\rho_k(x_1, \dots, x_k) = \det \left[ K_n(x_i, x_j) \right]_{i,j=1}^k,$$

with the correlation kernel

$$K_n(x, y) = \sum_{j=0}^{n-1} \psi_j(x) \psi_j(y).$$

Here

$$\psi_j(x) = \frac{1}{\sqrt{h_j}} p_j(x) e^{-x^2/4},$$

where  $p_j(x)$  are the monic Hermite polynomials, and  $h_j$  are the normalization constants so that  $\psi_j(x)$  are orthonormal in  $L^2(\mathbb{R})$ .

For this theorem, we need Cauchy–Binet summation formula and Andreief identity (which is essentially the same as Cauchy–Binet, but when summation is replaced by integration). Having these, we can write

$$\rho_k(x_1, \dots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p(x_1, \dots, x_n) dx_{k+1} \cdots dx_n$$

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$$\begin{aligned}
&= \frac{1}{(n-k)! \widehat{Z}_{n,2}} \sum_{\substack{\sigma, \tau \in S_n \\ \sigma(k+1)=\tau(k+1), \dots, \sigma(n)=\tau(n)}} \text{sgn}(\sigma) \text{sgn}(\tau) \prod_{i=1}^k \psi_{\sigma(i)-1}(x_i) \psi_{\tau(i)-1}(x_i) \\
&= \text{const}_n \sum_{I \subseteq [n], |I|=k} \sum_{\sigma', \tau' \in S(I)} \text{sgn}(\sigma') \text{sgn}(\tau') \prod_{i=1}^k \psi_{\sigma'(i)-1}(x_i) \psi_{\tau'(i)-1}(x_i) \\
&= \text{const}_n \sum_{I \subseteq [n], |I|=k} \det [\psi_{i_\alpha}(x_j)]_{\alpha, j=1}^k \det [\psi_{i_\alpha}(x_j)]_{\alpha, j=1}^k,
\end{aligned}$$

where  $I = \{i_1, \dots, i_k\}$  is a subset of  $[n]$  of size  $k$ , and  $S(I)$  is the set of permutations of  $I$ . The last sum of products of two determinants is written by the Cauchy–Binet formula as

$$\text{const}_n \cdot \det \left[ \sum_{j=0}^{n-1} \psi_j(x_\alpha) \psi_j(x_\beta) \right]_{\alpha, \beta=1}^k,$$

and finally the constant is equal to 1 by Andreief identity.

## 2 Double Contour Integral Representation for the GUE Kernel

### 2.1 One contour integral representation for Hermite polynomials

Recall that the GUE kernel is defined by

$$K_N(x, y) = \sum_{n=0}^{N-1} \psi_n(x) \psi_n(y),$$

with the orthonormal functions

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} p_n(x) e^{-x^2/4},$$

where the (monic, probabilists') Hermite polynomials are given by

$$p_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}. \quad (2.1)$$

Note that the monic Hermite polynomials are uniquely defined by the orthogonality property. We are not proving (2.1) here, it is an exercise.

**Lemma 2.1** (Generator function for Hermite polynomials). *We have*

$$\exp\left(xt - \frac{t^2}{2}\right) = \sum_{n \geq 0} p_n(x) \frac{t^n}{n!}.$$

*The series converges for all  $t$  since the left-hand side is an entire function of  $t$ .*

*Proof.* Write the generating function as

$$\sum_{n \geq 0} p_n(x) \frac{t^n}{n!} = \sum_{n \geq 0} \frac{(-1)^n t^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

Since the factor  $e^{x^2/2}$  does not depend on  $n$ , we can factor it out:

$$\sum_{n \geq 0} p_n(x) \frac{t^n}{n!} = e^{x^2/2} \sum_{n \geq 0} \frac{(-t)^n}{n!} \frac{d^n}{dx^n} e^{-x^2/2}.$$

Now, recall Taylor's theorem: for any analytic function  $f$  we have

$$f(x-t) = \sum_{n \geq 0} \frac{(-t)^n}{n!} f^{(n)}(x).$$

Applying this with  $f(x) = e^{-x^2/2}$ , we deduce that

$$\sum_{n \geq 0} \frac{(-t)^n}{n!} \frac{d^n}{dx^n} e^{-x^2/2} = e^{-(x-t)^2/2}.$$

Thus, our generating function becomes

$$\sum_{n \geq 0} p_n(x) \frac{t^n}{n!} = e^{x^2/2} e^{-(x-t)^2/2},$$

as desired. □

By Cauchy's integral formula we can write using Lemma 2.1:

$$p_n(x) = \frac{n!}{2\pi i} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{n+1}} dt, \quad (2.2)$$

where the contour  $C$  is a simple closed curve encircling the origin. Indeed, here we use the complex analysis property

$$\frac{1}{2\pi i} \oint_C \frac{1}{z^{k+1}} dz = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0, \end{cases}$$

so (2.2) is simply a complex analysis version of the operation of extracting the coefficient of  $t^n$  in the Taylor expansion.

Therefore,

$$\psi_n(x) = \frac{1}{\sqrt{h_n}} p_n(x) e^{-x^2/4} = \frac{e^{-x^2/4}}{\sqrt{h_n}} \frac{n!}{2\pi i} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{n+1}} dt.$$

## 2.2 Another contour integral representation for Hermite polynomials

We start with the Fourier transform identity

$$\int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2} + i t x\right) dt = \sqrt{2\pi} e^{-x^2/2}.$$

Differentiating both sides  $n$  times with respect to  $x$  yields

$$\frac{d^n}{dx^n} \left( e^{-x^2/2} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i t)^n e^{-t^2/2 + i t x} dt.$$

Recalling the definition

$$p_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left( e^{-x^2/2} \right),$$

we obtain

$$p_n(x) = \frac{(-1)^n e^{x^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (i t)^n e^{-t^2/2 + i t x} dt.$$

Next, perform the change of variable

$$s = i t, \quad \text{so that} \quad t = -i s, \quad dt = -i ds.$$

Under this substitution the factors transform as follows:

$$(i t)^n = s^n,$$

and the exponent becomes

$$-\frac{t^2}{2} + i t x = -\frac{(-i s)^2}{2} + i (-i s) x = \frac{s^2}{2} + s x.$$

Thus, the integral transforms into

$$\int_{-\infty}^{\infty} (i t)^n e^{-t^2/2 + i t x} dt = -i \int_{-i\infty}^{i\infty} s^n e^{s^2/2 + s x} ds.$$

Substituting back we have

$$p_n(x) = \frac{(-1)^n e^{x^2/2}}{\sqrt{2\pi}} (-i) \int_{-i\infty}^{i\infty} s^n e^{s^2/2 + s x} ds.$$

That is,

$$p_n(x) = \frac{i (-1)^{n+1} e^{x^2/2}}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} s^n e^{s^2/2 + s x} ds.$$

Finally, change the sign of  $s$ , and we get:

$$p_n(x) = \frac{i e^{x^2/2}}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} s^n e^{s^2/2 - s x} ds.$$

Therefore,

$$\psi_n(x) = \frac{i e^{x^2/4}}{\sqrt{2\pi} h_n} \int_{-i\infty}^{i\infty} s^n e^{s^2/2 - s x} ds.$$

### 2.3 Double contour integral representation for the GUE kernel

**Lemma 2.2.** *We have*

$$h_n = \int_{-\infty}^{\infty} p_n(x)^2 e^{-x^2/2} dx = n! \sqrt{2\pi}.$$

*Proof.* Multiply the generating function

$$\exp\left(xt - \frac{t^2}{2}\right) = \sum_{n \geq 0} p_n(x) \frac{t^n}{n!}$$

with a second copy (with parameter  $s$ ):

$$\exp\left(xs - \frac{s^2}{2}\right) = \sum_{m \geq 0} p_m(x) \frac{s^m}{m!}.$$

Then,

$$\exp\left(xt - \frac{t^2}{2}\right) \exp\left(xs - \frac{s^2}{2}\right) = \sum_{n, m \geq 0} p_n(x) p_m(x) \frac{t^n s^m}{n! m!}.$$

Integrate both sides against  $e^{-x^2/2} dx$ . Using the orthogonality

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) e^{-x^2/2} dx = h_n \delta_{nm},$$

the right-hand side becomes

$$\sum_{n \geq 0} \frac{h_n}{(n!)^2} (ts)^n.$$

On the left-hand side, we have

$$\int_{-\infty}^{\infty} e^{-x^2/2} \exp\left(x(t+s) - \frac{t^2+s^2}{2}\right) dx.$$

Completing the square in  $x$  or recalling the standard Gaussian integral yields

$$\sqrt{2\pi} \exp\left(\frac{(t+s)^2 - (t^2 + s^2)}{2}\right) = \sqrt{2\pi} \exp(ts).$$

Thus, we obtain

$$\sqrt{2\pi} \exp(ts) = \sum_{n \geq 0} \frac{h_n}{(n!)^2} (ts)^n.$$

Expanding the left side as

$$\sqrt{2\pi} \sum_{n \geq 0} \frac{(ts)^n}{n!},$$

and comparing coefficients, we conclude that

$$\frac{h_n}{(n!)^2} = \frac{\sqrt{2\pi}}{n!} \implies h_n = n! \sqrt{2\pi}.$$

This completes the proof. □

Therefore, we can sum up the kernel (another proof of the Christoffel–Darboux formula):

$$\begin{aligned}
K_n(x, y) &= \sum_{k=0}^{n-1} \psi_k(x) \psi_k(y) \\
&= \sum_{k=0}^{n-1} \frac{e^{-x^2/4}}{\sqrt{h_k}} \frac{k!}{2\pi i} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{k+1}} dt \frac{i e^{y^2/4}}{\sqrt{2\pi h_k}} \int_{-i\infty}^{i\infty} s^k e^{s^2/2 - sy} ds \\
&= e^{(y^2 - x^2)/4} \sum_{k=0}^{n-1} \frac{1}{4\pi^2} \oint_C \frac{\exp\left(xt - \frac{t^2}{2}\right)}{t^{k+1}} dt \int_{-i\infty}^{i\infty} s^k e^{s^2/2 - sy} ds.
\end{aligned}$$

We can now extend the sum to  $k = -\infty$ , and get a formula for the GUE kernel as a double contour integral:

$$K_n(x, y) = \frac{e^{(y^2 - x^2)/4}}{4\pi^2} \oint_C \int_{-i\infty}^{i\infty} \frac{\exp\left\{\frac{s^2}{2} - sy - \frac{t^2}{2} + tx\right\}}{s - t} \left(\frac{s}{t}\right)^n ds dt.$$

Details will be in the next [Lecture 6](#).

**Remark 2.3.** Many other versions of the GUE / unitary invariant ensembles admit determinantal structure:

1. The GUE corners process [[JN06](#)]
2. The Dyson Brownian motion (e.g., add a GUE to a diagonal matrix) [[NF98](#)]
3. GUE corners plus a fixed matrix [[FF14](#)]
4. Corners invariant ensembles with fixed eigenvalues  $UDU^\dagger$ , where  $D$  is a fixed diagonal matrix and  $U$  is Haar distributed on the unitary group [[Met13](#)]

## F Problems (due DATE)

### References

- [FF14] P. Ferrari and R. Frings, *Perturbed GUE minor process and Warren’s process with drifts*, J. Stat. Phys **154** (2014), no. 1-2, 356–377. [arXiv:1212.5534 \[math-ph\]](#). [↑6](#)
- [JN06] K. Johansson and E. Nordenstam, *Eigenvalues of GUE minors*, Electron. J. Probab. **11** (2006), no. 50, 1342–1371. [arXiv:math/0606760 \[math.PR\]](#). [↑6](#)
- [Met13] A. Metcalfe, *Universality properties of Gelfand-Tsetlin patterns*, Probab. Theory Relat. Fields **155** (2013), no. 1-2, 303–346. [arXiv:1105.1272 \[math.PR\]](#). [↑6](#)
- [NF98] T. Nagao and P.J. Forrester, *Multilevel dynamical correlation functions for Dyson’s Brownian motion model of random matrices*, Physics Letters A **247** (1998), no. 1-2, 42–46. [↑6](#)

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