Lectures on Random Matrices (Spring 2025)
Lecture 9: Loop equations and asymptotics to Gaussian Free Field

Leonid Petrov

Wednesday, March 5, 2025*

1 Recap

1.1 (Dynamical) loop equations

Theorem 1.1. *We fix $n = 1, 2, \ldots$ and $n+1$ real numbers $\lambda_1 \geq \cdots \geq \lambda_{n+1}$. For $\beta > 0$, consider $n+1$ i.i.d. χ^2_β random variables ξ_i and set*

$$w_i = \frac{\xi_i}{\sum_{j=1}^{n+1} \xi_j}, \qquad 1 \leq i \leq n+1.$$

We define n random points $\{\mu_1, \ldots, \mu_n\}$ as n solutions to the equation

$$\sum_{i=1}^{n+1} \frac{w_i}{z - \lambda_i} = 0. \tag{1.1}$$

Take any polynomial $W(z)$ *and consider the complex function:*

$$f_W(z) = \mathbb{E}\left[\prod_{j=1}^{n} \exp\bigl(W(\mu_j)\bigr) \frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} \left(W'(z) + \sum_{i=1}^{n+1} \frac{\beta/2 - 1}{z - \lambda_i} + \sum_{j=1}^{n} \frac{1}{z - \mu_j}\right)\right]. \tag{1.2}$$

Then $f_W(z)$ is an entire function *of z, in the following sense:*

- *For $z \in \mathbb{C} \setminus [\lambda_{n+1}, \lambda_1]$, the expectation in (1.2) defines a holomorphic function of z.*
- *This function has an analytic continuation to $\mathbb{C}$, which has no singularities.*

We proved this statement for $\beta > 2$, but it is valid for all $\beta > 0$.

*`Course webpage` • `Live simulations` • `TeX Source` • Updated at 10:04, Wednesday 5th March, 2025

1.2 Loop equations for $W = 0$

When $W = 0$, the loop equation (1.2) becomes

$$f_0(z) = \frac{(n+1)\beta}{2} - 1,$$

so

$$\mathbb{E}\left[\frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)}\left(\sum_{i=1}^{n+1}\frac{\beta/2-1}{z-\lambda_i}+\sum_{j=1}^{n}\frac{1}{z-\mu_j}\right)\right] = \frac{(n+1)\beta}{2} - 1.$$

Recall that we defined

$$G_\lambda(z) = \frac{1}{n}\sum_{i=1}^{n+1}\frac{1}{z-\lambda_i}, \qquad G_\mu(z) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{z-\mu_j}.$$

We also define the "logarithmic potentials" (indefinite integrals of the Stieltjes transforms):

$$\int G_\lambda(z)dz = \frac{1}{n}\sum_{i=1}^{n+1}\ln(z-\lambda_i), \qquad \int G_\mu(z)dz = \frac{1}{n}\sum_{j=1}^{n}\ln(z-\mu_j).$$

We understand the integrals up to the same integration constant (and branch), so the exponent of the difference yields the original product:

$$\frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} = \exp\left(n\left(\int G_\lambda(z) - \int G_\mu(z)\right)\right)$$

We can rewrite the loop equation as:

$$\mathbb{E}\left[\exp\left(n\left(\int G_\lambda(z)\,dz - \int G_\mu(z)\,dz\right)\right)\left(\left(\frac{\beta}{2}-1\right)G_\lambda(z)+G_\mu(z)\right)\right] = \frac{\beta}{2}+\frac{1}{n}\left(\frac{\beta}{2}-1\right). \tag{1.3}$$

1.3 The full corners process

Assume n is going to infinity, and we fix a sequence of top-level eigenvalues $\lambda_j^{(n)}$, $1 \le j \le n$, growing in some way. This sequence can be random (like GβE rescaled to have eigenvalues in a bounded interval) or deterministic (for example, $\lambda^{(n)}$ has $n/10$ points at 0, $n/10$ points at 1, and $8n/10$ points at 2, see Figure 1).

Denote the eigenvalues of the $k \times k$ beta corner (that is, obtained by successively solving the polynomial equation (1.1) $n-k$ times) by $\lambda_j^{(k)}$, $1 \le j \le k$. As $n \to \infty$, we postulate that

> The empirical distribution of $\lambda_j^{(k)}$ converges to some deterministic probability measure $\mathfrak{m}_t$, where $k/n \to t \in [0,1]$. Consequently, the Stieltjes transform $G_{\lambda^{(k)}}(z)$ converges to $G_t(z)$, for z in a complex domain outside of the support of $\mathfrak{m}_t$.

Note that we do not assume the scaling of the $\lambda_j^{(k)}$'s, for convenience.

Denote by $G_t(z) = \int_{\mathbb{R}} \frac{\mathfrak{m}_t(dx)}{z-x}$ the Stieltjes transform of the measure $\mathfrak{m}_t$.

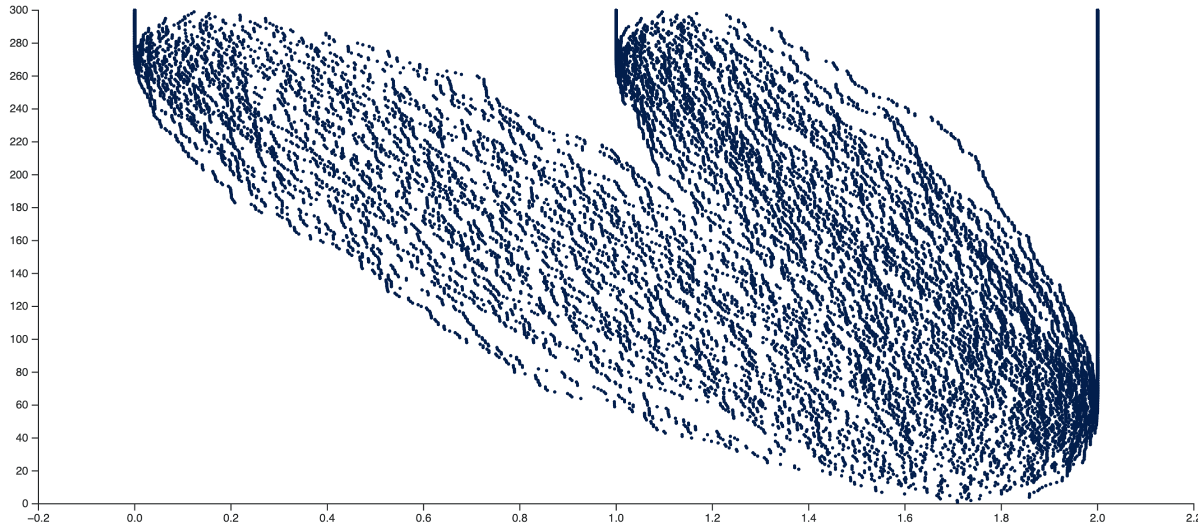

Figure 1: Corners process for $n = 300$, $\beta = 1$, with $n/10$ points at 0, $n/10$ points at 1, and $8n/10$ points at 2 on the top level.

Proposition 1.2. *The functions $G_t(z)$ satisfy the complex Burgers equation*

$$\frac{\partial}{\partial t} G_t(z) + \frac{1}{G_t(z)} \frac{\partial}{\partial z} G_t(z) = 0.$$

Proof. We have in (1.3), if λ and μ live on levels t and $t - \frac{1}{n}$, respectively:

$$G_\lambda(z) - G_\mu(z) \approx \frac{1}{n} \frac{\partial}{\partial t} G_t(z), \qquad \left(\frac{\beta}{2} - 1\right) G_\lambda(z) + G_\mu(z) \approx \frac{\beta}{2} G_t(z) - \frac{1}{n} \frac{\partial}{\partial t} G_t(z) \approx \frac{\beta}{2} G_t(z).$$

Due to the concentration assumption, we can ignore the expectation. Then, taking the logarithm of (1.3), and differentiating with respect to z, we get the Burgers equation. $\square$

1.4 Example: GβE and the semicircle law

The Stieltjes transform of the semicircular law is given by:

$$G(z) = \int_{-2}^{2} \frac{1}{z - x} \frac{\sqrt{4 - x^2}}{2\pi} dx = \frac{1}{2} \left(z - \sqrt{z^2 - 4} \right).$$

We take this as the function $G_t(z)$ for $t = 1$. Then, for each $0 \le t \le 1$, the GβE solution should be

$$\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{z - \lambda_i^{(\lfloor nt \rfloor)}} \to t G^{(\sqrt{t})}(z),$$

where

$$G^{(c)}(z) := \frac{z - \sqrt{z^2 - 4c^2}}{2c^2},$$

is the Stieltjes transform of the semicircular law on $[-2c, 2c]$.

Lemma 1.3. *The function $G_t(z) \coloneqq t G^{(\sqrt{t})}(z)$ satisfies the Burgers equation.*

Proof. Straightforward verification. $\square$

2 Gaussian Free Field

The *Gaussian Free Field* (GFF) is a fundamental object in probability theory and mathematical physics. Roughly speaking, it can be viewed as a multi-dimensional analog of Brownian motion: instead of one-dimensional "time," the underlying parameter space is a multi-dimensional domain (often two-dimensional). In one dimension, the GFF reduces to an ordinary Brownian bridge (or motion). In higher dimensions, it becomes a random generalized function (a "distribution") whose covariance structure is governed by the appropriate Green's function of the Laplacian. Below we provide an introduction, starting from finite-dimensional Gaussian vectors and culminating in the GFF as a random distribution.

2.1 Gaussian correlated vectors and random fields

Recall that an n-dimensional real-valued random vector $X = (X_1, \ldots, X_n)$ is called *Gaussian* if every linear combination

$$\alpha_1 X_1 + \cdots + \alpha_n X_n$$

of its components is a univariate Gaussian random variable. The law of such a vector is completely determined by its mean vector $m \in \mathbb{R}^n$ and its covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$. The density function, for invertible Σ, is

$$f_X(x) \;=\; \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\Bigl(-\tfrac{1}{2}\,(x-m)^\top \Sigma^{-1}(x-m)\Bigr).$$

For simplicity, we will assume that $m = 0$ (the centered case).

2.2 Gaussian fields as random generalized functions

A natural extension from finite-dimensional Gaussian vectors to infinite-dimensional settings leads us to Gaussian fields. Informally, a Gaussian field is a collection of Gaussian random variables indexed by points in some space.

For a domain $D \subset \mathbb{R}^d$, we might wish to define a random function $\Phi : D \to \mathbb{R}$ such that for any finite collection of points $x_1, \ldots, x_n \in q$, the vector $(\Phi(x_1), \ldots, \Phi(x_n))$ is a Gaussian vector. However, such a random function may not exist as a proper function in the usual sense. The reason is that we would like to consider analogues of linear combinations of the form

$$\Phi(f) = \int_D \Phi(x) f(x)\, dx, \tag{2.1}$$

For example, if we wish the vector $(\Phi(x_1), \ldots, \Phi(x_n))$ to have independent components, we would need to assign a value to each point in D. This means that the hypothetical function Φ would be too irregular, and even non-measurable, and the integral (2.1) would not be well-defined.

Instead, for the field with independent values at all points, we would like $\Phi(f)$ to be normal with mean zero and variance (paralleling the finite-dimensional story)

$$\mathrm{Var}\,(\Phi(f)) = \|f\|^2_{L^2(D)} \;=\; \int_D f(x)^2\,dx.$$

So, Gaussian fields (in particular, our topic, the *Gaussian Free Field*)a are defined as random distributions, not as functions. That is, rather than assigning a value to each point, we assign a random value to each test function f in some appropriate space via (2.1).

The covariance structure of the mean zero Gaussian random variables $\Phi(f_1), \ldots, \Phi(f_n)$ is given by a certain bilinear form determined by the domain D.

2.3 Concrete treatment via orthogonal functions

Let us now construct the Gaussian Free Field more concretely. Consider a bounded domain $D \subset \mathbb{R}^d$ with smooth boundary. Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L^2(D)$ consisting of eigenfunctions of the Laplacian with Dirichlet boundary conditions:

$$\begin{cases} -\Delta f_n = \lambda_n f_n & \text{in } D, \\ f_n = 0 & \text{on } \partial D, \end{cases} \tag{2.2}$$

where $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ are the corresponding eigenvalues.

We can now define the Gaussian Free Field on D as:

$$\Phi = \sum_{n=1}^{\infty} \frac{\alpha_n}{\sqrt{\lambda_n}} f_n, \tag{2.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ are independent standard Gaussian random variables. This series does not converge pointwise, but it does converge in the space of distributions almost surely.

For any test function $g \in C_0^{\infty}(D)$, we have:

$$\Phi(g) = \int_D \Phi(x) g(x)\,dx = \sum_{n=1}^{\infty} \frac{\alpha_n}{\sqrt{\lambda_n}} \int_D f_n(x) g(x)\,dx, \tag{2.4}$$

which is a well-defined Gaussian random variable.

2.4 Connection to Brownian bridge

The Gaussian Free Field in one dimension is closely related to the Brownian bridge. Consider the interval $[0, 1]$ with the Dirichlet Laplacian. The eigenfunctions are $f_n(x) = \sqrt{2}\sin(n\pi x)$ with eigenvalues $\lambda_n = n^2\pi^2$. The Gaussian Free Field on $[0, 1]$ can be expressed as:

$$\Phi(x) = \sqrt{2} \sum_{n=1}^{\infty} \frac{\alpha_n}{n\pi} \sin(n\pi x). \tag{2.5}$$

This series representation converges to a continuous function, which is precisely the Brownian bridge on $[0, 1]$. The Brownian bridge is a Gaussian process B_t with mean zero and covariance function:

$$\mathbb{E}[B_s B_t] = \min(s, t) - st. \tag{2.6}$$

The key difference between the one-dimensional and higher-dimensional cases is that in one dimension, the Gaussian Free Field is a continuous function, whereas in dimensions two and higher, it is a genuine distribution (not a function). This reflects the fact that Brownian motion is a continuous path in one dimension but becomes increasingly irregular in higher dimensions.

2.5 Covariance structure and Green's function

The covariance structure of the Gaussian Free Field is intimately connected to the Green's function of the Laplacian. For test functions $f, g \in C_0^\infty(D)$, we have:

$$\mathbb{E}[\Phi(f)\Phi(g)] = \mathbb{E}\left[\sum_{n,m=1}^{\infty} \frac{\alpha_n \alpha_m}{\sqrt{\lambda_n \lambda_m}} \int_D f_n(x) f(x)\, dx \int_D f_m(y) g(y)\, dy\right] \tag{2.7}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_D f_n(x) f(x)\, dx \int_D f_n(y) g(y)\, dy. \tag{2.8}$$

Define the Green's function $G_D(x,y)$ for the Dirichlet Laplacian on D as the solution to:

$$\begin{cases} -\Delta_x G_D(x,y) = \delta(x-y) & \text{for } x, y \in D, \\ G_D(x,y) = 0 & \text{for } x \in \partial D \text{ or } y \in \partial D. \end{cases} \tag{2.9}$$

The Green's function has the eigenfunction expansion:

$$G_D(x,y) = \sum_{n=1}^{\infty} \frac{f_n(x) f_n(y)}{\lambda_n}. \tag{2.10}$$

Using this, we can rewrite the covariance as:

$$\mathbb{E}[\Phi(f)\Phi(g)] = \int_D \int_D G_D(x,y) f(x) g(y)\, dx\, dy. \tag{2.11}$$

This relationship between the covariance of the GFF and the Green's function is fundamental. It shows that the GFF can be viewed as a random solution to the equation $-\Delta\Phi = W$, where W is white noise. Here the white noise is the Gaussian field with covariance $\delta(x-y)$ — the object which is the correct way of constructing a Gaussian field with i.i.d. values at all points.

2.6 The GFF on the upper half-plane

In the complex upper half-plane $\{\operatorname{Im} z > 0\}$ with $\mathbb{R}$ as the boundary, the Green function has the form

$$G(z,w) = -\frac{1}{\pi} \ln|z-w| + \frac{1}{\pi} \ln|z-\overline{w}|.$$

The covariance is

$$\mathbb{E}\left[\Phi(f)\,\Phi(g)\right] = \int\int |dz|^2\, |dw|^2 f(z) g(w) G(z,w).$$

3 Fluctuations

3.1 Height function and related definitions

Let us define the *height function* using the corners process $\{\lambda_j^{(k)} : 1 \le j \le k \le n\}$:

$$h(t, x) \coloneqq \#\{\text{eigenvalues } \lambda_j^{(\lfloor nt \rfloor)} \text{ which are } \le x\}.$$

Recall that in our regime, we do not scale x. Throughout the following, we will interchangeably use the parameters n and $\varepsilon \coloneqq 1/n$.

Our goal is to understand the asymptotic behavior of the centered height function

$$h(\varepsilon^{-1}t, x) - \mathbb{E}[h(\varepsilon^{-1}t, x)],$$

defined inside the region of the (t, x) plane. Note that in contrast with the usual Central Limit Theorem, the fluctuations are not scaled by $\varepsilon^{1/2}$, but rather are unscaled. Note that the law of large numbers is going to be

$$\varepsilon h(\varepsilon^{-1}t, x) \to \mathfrak{h}(t, x),$$

where $\mathfrak{h}(t, x)$ is the limiting height function (for a fixed t, this is the cumulative distribution function of the measure $\mathfrak{m}_t$). We will see that these unscaled fluctuations are converging to a Gaussian Free Field. Thus, the unscaled fluctuations are "just barely" going to infinity, while retaining nontrivial and bounded correlations.

Define

$$\rho(t, x) \coloneqq h(t, x-\varepsilon) - h(t, x) = \sum_{i=1}^{\lfloor nt \rfloor} \mathbf{1}_{\lambda_i^{(\lfloor nt \rfloor)} \le x \le \lambda_i^{(\lfloor nt \rfloor)} + \varepsilon}, \qquad \text{where } \mathbf{1}_A \text{ is the indicator of the event } A. \tag{3.1}$$

This is a discrete analogue of the x-derivative of $h(t, x)$.

3.2 Deformed ensemble

The rest of this section recreates the argument analogous to [GH24, Theorem 4.5], but in the random matrix setting. In the interest of time, we are following the main steps in a non-rigorous manner, as outlined in [GH24, Section 4.2] before the actual proof.

This theorem is an asymptotic expansion of the Stieltjes transform of the one-step transition from λ to μ. We assume that the support of λ is in $[l, r]$. Denote

$$\Pi_\lambda(z) \coloneqq \prod_{i=1}^{n+1}(z - \lambda_i), \qquad \Pi_\mu(z) \coloneqq \prod_{j=1}^{n}(z - \mu_j).$$

Also assume that $W(z)$ is fixed and nice, and that μ_j are distributed according to a modified density, which includes $W(z)$:

$$\frac{1}{Z} \prod_{1 \le i < j \le n} (\mu_i - \mu_j) \prod_{i=1}^{n} \prod_{j=1}^{n+1} |\mu_i - \lambda_j|^{\beta/2-1} \prod_{1 \le i < j \le n+1} (\lambda_i - \lambda_j)^{1-\beta} \prod_{j=1}^{n} e^{W(\mu_j)}.$$

From now on, all expectations will be over the W-modified density.

We aim to analyze the quantity

$$\mathcal{A}(z) := \mathbb{E}\left[\frac{\Pi_\lambda(z)}{\Pi_\mu(z)}\right],$$

which enters the loop equation. Moreover, the loop equation states the holomorphicity of

$$\mathcal{C}(z) = \mathcal{A}(z)\left[W'(z) + \frac{\beta}{2}\frac{1}{n}\sum_{i=1}^{n+1}\frac{1}{z-\lambda_i}\right] + \mathbb{E}\left[\frac{\Pi_\lambda(z)}{\Pi_\mu(z)}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{z-\mu_j} - \frac{1}{n}\sum_{i=1}^{n+1}\frac{1}{z-\lambda_i}\right)\right].$$

The first summand is the leading term, and the second summand will be negligible. Indeed, it contains the difference of $G_\mu(z)$ and $G_\lambda(z)$, and these Stieltjes transforms are close to each other, so the difference is $O(\varepsilon)$.

3.3 Wiener-Hopf like factorization

Denote

$$\mathcal{B}(z) = W'(z) + \frac{\beta}{2}G_\lambda(z).$$

Decompose $\mathcal{B}(z)$ using the Cauchy residue formula:

$$\ln\mathcal{B}(z) = \frac{1}{2\pi i}\oint_{\omega_+}\frac{\ln\mathcal{B}(w)}{w-z}dw - \frac{1}{2\pi i}\oint_{\omega_-}\frac{\ln\mathcal{B}(w)}{w-z}dw,$$

where ω_+ is positively oriented and encloses $[l,r]$ and z, while ω_- is also positively oriented and encloses $[l,r]$ but not z. Then define

$$h_+(u)\colon \frac{1}{2\pi i}\oint_{\omega_+}\frac{\ln\mathcal{B}(w)}{w-u}dw, \qquad h_-(u)\colon \frac{1}{2\pi i}\oint_{\omega_-}\frac{\ln\mathcal{B}(w)}{w-u}dw.$$

Thus, we get the Wiener-Hopf like factorization

$$\mathcal{B}(z) = e^{h_+(z)}e^{-h_-(z)},$$

where h_+ is holomorphic in a neighborhood of $[l,r]$, and h_- is a holomorphic in a neighborhood of ∞, with behavior $O(1/u)$ at infinity. The factorization is valid in an annulus between the two contours ω_+ and ω_-.

3.4 First order asymptotics of $\mathcal{A}(z)$

The next step is to understand the asymptotics of $\mathcal{A}(z)$. Recall that

$$\mathcal{A}(z) = \mathbb{E}\left[\frac{\Pi_\lambda(z)}{\Pi_\mu(z)}\right]. \tag{3.2}$$

From the loop equation, we know that $\mathcal{C}(z)$ is entire, and the leading term involves $\mathcal{A}(z)\mathcal{B}(z)$. That is,

$$\mathcal{A}(z)\mathcal{B}(z) = \text{entire function} + O(\varepsilon). \tag{3.3}$$

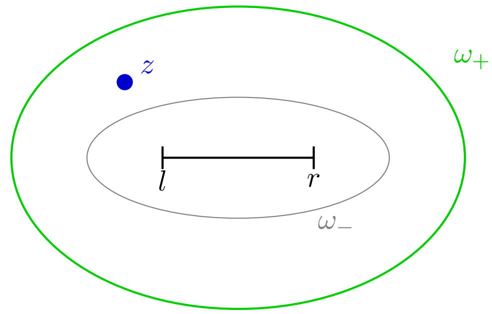

Figure 2: Positively oriented contours ω_+ and ω_- in the complex plane.

Using the Wiener-Hopf factorization of $\mathcal{B}(z)$, let us multiply (3.3) by $e^{-h_+(z)}$. The entire function remains entire in a complex neighborhood of $[l, r]$. Therefore, we can integrate over ω_-, and get

$$\begin{aligned}0 = \frac{1}{2\pi i}\oint_{\omega_-}\frac{\mathcal{C}(w)e^{-h_+(w)}dw}{w-z} &= \frac{1}{2\pi i}\oint_{\omega_-}\frac{\mathcal{A}(w)e^{-h_-(w)}dw}{w-z} + O(\varepsilon)\\ &= -\mathcal{A}(z)e^{-h_-(z)} + \frac{1}{2\pi i}\oint_{\omega_+}\frac{\mathcal{A}(w)e^{-h_-(w)}}{w-z}dw + O(\varepsilon).\end{aligned}$$

In the last equality, we took a residue at $w = z$, and replaced the integral by an integral over ω_+.

The integrand has no singularities outside ω_+, and thus is just the residue at infinity. Using the fact that $e^{-h_-(u)} = e^{1+O(1/u)} = 1 + O(1/u)$, $u \to \infty$ and the fact that the expectation $\mathcal{A}(u)$ is balanced in u (hence it is $1 + O(1/u)$), we see that the residue at infinity is simply equal to 1. Therefore,

$$0 = -\mathcal{A}(z)e^{-h_-(z)} + 1 + O(\varepsilon), \qquad \ln\mathcal{A}(z) = h_-(z) + O(\varepsilon).$$

rewrite

Let $\rho_\mu(x)$ stand for (3.1) with μ instead of $\lambda^{(\lfloor nt \rfloor)}$. Define

$$\mathcal{G}_\mu(z) = \exp\left[\int_l^r \frac{\rho_\mu(x)}{z-x}\,dx\right], \qquad \mathcal{B}_\mu(z) = 1+\mathcal{G}_\mu(z)$$ loop eq

We have

$$\int_l^r \frac{\rho_\mu(x)}{z-x}dx = \sum_{i=1}^n \sum_{i=1}^n \int_{\mu_i}^{\mu_i+\varepsilon} \frac{dx}{z-x} = \sum_{i=1}^n \left(\ln(z-\mu_i+\varepsilon) - \ln(z-\mu_i)\right),$$

so

$$\mathcal{G}_\mu(z) = \prod_{i=1}^n \frac{z-\mu_i+\varepsilon}{z-\mu_i} = 1+\varepsilon\sum_{i=1}^n \frac{1}{z-\mu_i} + O(\varepsilon^2)..$$

Theorem 3.1. *We have as $\varepsilon \to 0$:*

$$\frac{1}{\varepsilon}\int_l^r \frac{\rho_\mu(x)-\rho_\lambda(x)}{z-x}\,dx = \frac{1}{\pi i \beta}\oint_{\omega_-} \frac{\ln \mathcal{B}_\mu(z)}{(w-z)^2}\,dw + \varepsilon\cdot(\text{explicit expression}) + \Delta M(z) + O(\varepsilon^2),$$

where the contour ω_- encloses $[l,r]$ but not z, and $\Delta M(z)$ are mean 0 random variables such that $\varepsilon^{-1/2}\Delta M(z)$, for $z\mathbb{C}\setminus[l,r]$, are asymptotically Gaussian with the limiting covariance

$$\frac{\mathbb{E}\left[\Delta M(z_1)\Delta M(z_2)\right]}{\varepsilon} = \frac{1}{\pi i\beta}\oint_{\omega_-} \frac{\mathcal{G}(w)\varphi^+(w)}{\mathcal{B}(w)}\frac{dw}{(w-z_1)^2(w-z_2)^2} + o(1).$$

The higher order joint moments of $\varepsilon^{-1/2}\Delta M(z)$ also converge as $\varepsilon \to 0$ to Gaussian moments.

I Problems (due 2025-04-29)

I.1 Brownian bridge

Derive the covariance structure of the Brownian bridge (2.6) from the series representation (2.5).

References

[GH24] V. Gorin and J. Huang, *Dynamical loop equation*, Ann. Probab. **52** (2024), no. 5, 1758–1863. arXiv:2205.15785 [math.PR]. ↑7

L. Petrov, University of Virginia, Department of Mathematics, 141 Cabell Drive, Kerchof Hall, P.O. Box 400137, Charlottesville, VA 22904, USA

E-mail: `lenia.petrov@gmail.com`