UVA Summer School Invariant Measures PS 2

July 16, 2024

The number of stars next to an exercise indicates the expected time needed to fully solve the exercise.

Exercise 1 [*]

Give a proof of Lemma 4.2, i.e. for $N = 1, 2, \dots,$

$$\mathcal{J}^N = \frac{\mathcal{Z}_{N-1}}{\mathcal{Z}_N},$$

where we recall

$$\mathcal{Z}_N = \Big\langle W \Big| (D+E)^N \Big| V \Big\rangle,$$

is the partition function, and

$$\mathcal{J}^N = \mu(\eta(i) = 1 \text{ and } \eta(i+1) = 0) - q\mu(\eta(i) = 0 \text{ and } \eta(i+1) = 1) \quad \text{for all } i \in [N-1],$$

is the current of the open ASEP.

Exercise 2 [*]

Let $\gamma = \delta = 0$. Set $a = (1-\alpha)\alpha^{-1}$ and $b = (1-\beta)\beta^{-1}$. Use the matrices D and E given by

$$D = \begin{pmatrix} 1+b & \sqrt{1-ab} & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1+a & 0 & 0 & 0 & \cdots \\ \sqrt{1-ab} & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}$$

with respect to the vectors

$$V = (1\ 0\ 0\ 0\ \cdots)^{\mathrm{T}} \quad \text{and} \quad W = (1\ 0\ 0\ 0\ \dots)$$

in order to prove Theorem 4.4 for the special case $q = 0$ and $\alpha, \beta \in (0,1)$, i.e. that a basic weight function can be used to express the stationary distribution of the open ASEP.
Bonus Exercise: Find a representation for general $q \in (0,1)$, and prove the equivalence to a basic weight function.

Exercise 3 [**]

When $ACq^{\ell} = 1$ for some $\ell \in \mathbb{N}_0$, find an $\ell + 1$ dimensional representation for the matrix product ansatz.
Hint: One possible solution may use that D is a diagonal matrix and E is tridiagonal.

Exercise 4 [**]

Let $\alpha = \beta = 1$ and $\gamma = \delta = 0$. We denote by

$$[\ell]_q := \frac{1-q^\ell}{1-q} = 1 + q + q^2 + \cdots + q^{\ell-1} \tag{1}$$

the q-analogue of a natural number $\ell \in \mathbb{N}$. Use the matrices D and E given by

$$D = \begin{pmatrix} [1]_q & [2]_q & 0 & 0 & \cdots \\ 0 & [2]_q & [3]_q & 0 & \\ 0 & 0 & [3]_q & [4]_q & \\ 0 & 0 & 0 & [4]_q & \\ \vdots & & & & \ddots \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} [1]_q & 0 & 0 & 0 & \cdots \\ [1]_q & [2]_q & 0 & 0 & \\ 0 & [2]_q & [3]_q & 0 & \\ 0 & 0 & [3]_q & [4]_q & \\ \vdots & & & & \ddots \end{pmatrix}$$

with respect to the vectors

$$V = (1\ 0\ 0\ 0\ \cdots)^{\mathrm{T}} \quad \text{and} \quad W = (1\ 0\ 0\ 0\ \ldots)$$

in order to verify that Theorem 4.5 holds, i.e. that the bi-colored Motzkin paths can be used to express the stationary distribution of the open ASEP.
Bonus Exercise: Show that this Exercise holds true without the assumption $\alpha = \beta = 1$.