

Lectures on Random Matrices (Spring 2025)

Lecture 14: Matching Random Matrices to Random Growth II

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1 Recap

1.1 Main goal

In the previous [Lecture 13](#), we began establishing a remarkable correspondence between two a priori different objects:

- The *spiked Wishart ensemble*: an $n \times n$ Hermitian random-matrix process $\{M(t)\}_{t \geq 0}$ whose entries come from columns of independent Gaussian random vectors of suitably chosen covariance.
- An *inhomogeneous last-passage percolation (LPP)* model: an array $\{W_{i,j}\}$ of exponential random weights on a portion of the two-dimensional lattice, whose last-passage times $L(t, n)$ match the largest eigenvalues of $M(t)$, jointly for all $t \in \mathbb{Z}_{\geq 0}$.

This equivalence, originally due to [\[DW08\]](#) (following [\[Def10\]](#), [\[FR06\]](#); see also [\[Bar01\]](#), [\[Joh00\]](#) for earlier results of this kind), can be fully understood by passing to a *discrete* version of LPP with geometric site-weights and then applying the *Robinson–Schensted–Knuth* (RSK) correspondence.

*[Course webpage](#) • [Live simulations](#) • [TeX Source](#) • Updated at 11:12, Tuesday 15th April, 2025

1.2 Spiked Wishart ensembles and the largest eigenvalue process

We defined the *generalized* (or spiked) Wishart matrix $M(t)$ of size $n \times n$ by setting

$$M(t) = \sum_{m=1}^t A^{(m)} (A^{(m)})^*$$

where $\{A^{(m)}\}_{m=1}^\infty$ are i.i.d. complex Gaussian column vectors of length n , with

$$\text{Var}(A_i^{(m)}) = \frac{1}{\pi_i + \hat{\pi}_m}.$$

Here, $\pi = (\pi_1, \dots, \pi_n)$ and $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \dots)$ are positive and nonnegative parameters, respectively. Writing $\lambda_1(t) \geq \dots \geq \lambda_n(t) \geq 0$ for the eigenvalues of $M(t)$, we then saw:

1. The vectors $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))$ form a Markov chain in the *Weyl chamber* $\mathbb{W}^n = \{x_1 \geq \dots \geq x_n \geq 0\}$.
2. There is an *interlacing* property: each update $M(t-1) \mapsto M(t)$ via the rank-one matrix $A^{(t)}(A^{(t)})^*$ forces $\lambda(t)$ to interlace with $\lambda(t-1)$:

$$\lambda_1(t) \geq \lambda_1(t-1) \geq \lambda_2(t) \geq \dots \geq \lambda_n(t-1) \geq \lambda_n(t).$$

In [Lecture 13](#), we wrote down the transition kernel from $\lambda(t-1)$ to $\lambda(t)$:

Theorem 1.1 ([DW08]). *Fix an integer $n \geq 1$. Let $\pi = (\pi_1, \dots, \pi_n)$ be a strictly positive n -vector, and let $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \dots)$ be any sequence of nonnegative real parameters. Under the probability measure $P^{\pi, \hat{\pi}}$, the eigenvalues of the $n \times n$ generalized Wishart matrices $\{M(t)\}_{t \geq 0}$ form a time-inhomogeneous Markov chain $\{\text{sp}(M(t))\}_{t \geq 0}$ in the Weyl chamber*

$$\mathbb{W}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}_{\geq 0}^n : x_1 \geq x_2 \geq \dots \geq x_n\}.$$

More precisely, writing $x = \text{sp}(M(t-1))$ and $y = \text{sp}(M(t))$, the one-step transition law from time $(t-1)$ to t is absolutely continuous on the interior of \mathbb{W}^n and can be factored as

$$Q_{t-1,t}^{\pi, \hat{\pi}}(x, dy) = \left[\prod_{i=1}^n (\pi_i + \hat{\pi}_t) \right] \cdot \frac{h_\pi(y)}{h_\pi(x)} \exp\left(-(\hat{\pi}_t - 1) \sum_{i=1}^n (y_i - x_i)\right) \times Q^{(0)}(x, dy), \quad (1.1)$$

where

- $Q^{(0)}(x, dy)$ is the standard (null-spike) Wishart transition kernel, given explicitly by

$$Q^{(0)}(x, dy) = \frac{\Delta(y)}{\Delta(x)} \exp\left(-\sum_{i=1}^n (y_i - x_i)\right) \mathbf{1}_{\{x \prec y\}} dy, \quad (1.2)$$

with $\Delta(z) = \prod_{1 \leq i < j \leq n} (z_i - z_j)$ the Vandermonde determinant.

- The function h_π is the (continuous) Harish-Chandra orbit integral factor

$$h_\pi(z) = \frac{(-1)^{\binom{n}{2}}}{0!1!\dots(n-1)!} \frac{\det(e^{-\pi_i z_j})_{i,j=1}^n}{\Delta(\pi) \Delta(z)}.$$

Note that $h_\pi(0) = 1$.

In particular, the chain starts from $\text{sp}(M(0)) = 0$ (the zero matrix).

1.3 Inhomogeneous last-passage percolation

On the random growth side, we considered an array of site-weights $\{W_{i,j}\}_{i,j \geq 1}$ such that each $W_{i,j}$ is exponentially distributed with rate $\pi_i + \hat{\pi}_j$. For every integer $t \geq 1$, we define $L(t, n)$ to be the maximum total weight of all up-right paths from $(1, 1)$ to (t, n) :

$$L(t, n) = \max_{\Gamma: (1,1) \rightarrow (t,n)} \sum_{(i,j) \in \Gamma} W_{i,j}.$$

One checks that $L(\cdot, n)$ satisfies a simple additive recursion:

$$L(i, j) = W_{i,j} + \max\{L(i-1, j), L(i, j-1)\},$$

The main claim which we show in today's lecture is the equality in distribution:

$$(L(1, n), L(2, n), \dots, L(t, n)) \stackrel{d}{=} (\lambda_1(1), \lambda_1(2), \dots, \lambda_1(t)). \quad (1.3)$$

1.4 RSK via toggles: definitions and weight preservation

The *Robinson–Schensted–Knuth* correspondence (RSK) was the main new mechanism in [Lecture 13](#). In our setup, we adopt a *toggle-based* viewpoint: we encode arrays by diagonals and successively *toggle* the diagonals to achieve a fully *ordered* array R . The key to how RSK links LPP and random matrices is its *weight preservation* property.

We work with arrays $W = \{W_{ij}\}_{1 \leq i \leq t, 1 \leq j \leq n}$ and $R = \{R_{ij}\}_{1 \leq i \leq t, 1 \leq j \leq n}$, where W is a non-negative integer array and R is an ordered array, that is, $R_{i,j} \leq R_{i,j+1}$ and $R_{i,j} \leq R_{i+1,j}$ for all i, j . Using RSK, we showed in [Lecture 13](#) that there is a bijection which maps W to R .

We also started to prove the following result, which we now complete:

Theorem 1.2 (Weight preservation). *Let $W = \{W_{i,j}\}$ be a nonnegative integer array, and $R = \text{RSK}(W)$. Denote*

$$\text{row}_i = \sum_{j=1}^n W_{i,j}, \quad \text{col}_j = \sum_{i=1}^t W_{i,j}$$

(which are essentially the cdf's of the array W), and for R define the diagonal sums starting at each (i, j) and going diagonally down and to the right:

$$\text{diag}_{i,j} = \sum_{k=0}^{\min(i,j)-1} R_{i-k, j-k}.$$

Then for each $1 \leq j \leq n$ and $1 \leq i \leq t$, we have

$$\text{diag}_{t,j} = \sum_{m=1}^j \text{col}_m, \quad \text{diag}_{i,n} = \sum_{m=1}^i \text{row}_m. \quad (1.4)$$

In particular, the total sum of W over all cells equals the total sum of R over all cells.

Proof (sketch). One inductively builds R by adding the sites (i, j) one at a time. Each toggle modifies exactly one diagonal. After adding a box (i, j) , the diagonal-sum identity

$$\text{diag}_{i,j} = \text{diag}_{i-1,j} + \text{diag}_{i,j-1} - \text{diag}_{i-1,j-1} + W_{i,j}$$

holds, expressing that W captures the discrete “mixed second differences” of the diagonal sums in R . Thus, the cdf's of W must coincide with the diagonal sums of R , as desired. \square

2 Distributions of last-passage times in geometric LPP

2.1

N Problems (due 2025-04-29)

N.1 Non-Markovianity

Show that the sequence of random variables defined in the exponential LPP model,

$$L(1, n), L(2, n), \dots, L(t, n),$$

is **not** a Markov chain. By virtue of the equivalence with the spiked Wishart ensemble (1.3), you may alternatively show that the sequence of maximal eigenvalues

$$\lambda_1(1), \lambda_1(2), \dots, \lambda_1(t)$$

of successive Wishart matrices $M(1), M(2), \dots, M(t)$ is **not** a Markov chain either.

References

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