# Lectures on Random Matrices (Spring 2025)
# Lecture 8: Cutting corners and loop equations

Leonid Petrov

Wednesday, February 26, 2025*

## Contents



## 1 Cutting corners: polynomial equation and distribution

### 1.1 Recap: polynomial equation

Recall the polynomial equation we proved in the last Lecture 7. Fix $\lambda = (\lambda_1 \ge \ldots \ge \lambda_n)$. Let $H \in \operatorname{Orbit}(\lambda)$ be a random Hermitian matrix defined as

$$H = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n) U^\dagger,$$

*`Course webpage` • `Live simulations` • `TeX Source` • Updated at 02:25, Friday 28$^{\text{th}}$ February, 2025

where $U$ is Haar-distributed unitary matrix from $U(n)$. This is the case $\beta = 2$, but the statement holds for the cases $\beta = 1, 4$ with appropriate modifications. Let $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of the $(n-1) \times (n-1)$ corner $H^{(n-1)}$.

**Lemma 1.1.** *The distribution of $\mu_1, \ldots, \mu_{n-1}$ is the same as the distribution of the roots of the polynomial equation*

$$\sum_{i=1}^{n} \frac{\xi_i}{z - \lambda_i} = 0, \tag{1.1}$$

*where $\xi_i$ are i.i.d. random variables with the distribution $\chi^2_\beta$.*

Recall also that this passage from $\lambda$ to $\mu$ works inductively, and the distribution of the next level eigenvalues $\nu = (\nu_1 \geq \ldots \geq \nu_{n-2})$ is given by the same polynomial equation, but with $\lambda$ replaced by $\mu$. In this way, we can define a *Markov map* from $\lambda$ to $\mu$, which is then iterated to construct the full array of eigenvalues of the corners of $H$.

For $\beta = \infty$, this map is deterministic, and is equivalent to successive differentiating the characteristic polynomial of $H$.

### 1.2 Extension to general $\beta$

We extend the polynomial equation to general $\beta$, by *declaring* (defining) that the general $\beta$ corners distribution is powered by the passage from $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ to $\mu = (\mu_1 \geq \ldots \geq \mu_{n-1})$, where $\mu$ solves (1.1) with $\xi_i$ i.i.d. $\chi^2_\beta$. In this way, $\mu$ interlaces with $\lambda$. For $\beta = 1, 2, 4$, this definition reduces to the one with invariant ensembles with fixed eigenvalues $\lambda$.

### 1.3 Distribution of the eigenvalues of the corners

Let $\mu$ be obtained from $\lambda$ by the general $\beta$ corners operation.

**Theorem 1.2.** *The density of $\mu$ with respect to the Lebesgue measure is given by*

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} \prod_{1 \leq i < j \leq n-1} (\mu_i - \mu_j) \prod_{i=1}^{n-1} \prod_{j=1}^{n} |\mu_i - \lambda_j|^{\beta/2-1} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^{1-\beta}.$$

*Proof.* Let $\varphi_i = \xi_i / \sum_{j=1}^n \xi_j$. It is well-known[1] the joint density of $(\varphi_1, \ldots, \varphi_n)$ is the (symmetric) Dirichlet density

$$\frac{\Gamma(N\beta/2)}{\Gamma(\beta/2)^n} w_1^{\beta/2-1} \ldots w_n^{\beta/2-1} dw_1 \ldots dw_{n-1}$$

(note that the density is $(n-1)$-dimensional).

We need to compute the Jacobian of the transformation from $\varphi$ to $\mu$, if we write

$$\sum_{i=1}^{n} \frac{\varphi_i}{z - \lambda_i} = \frac{\prod_{i=1}^{n-1}(z - \mu_i)}{\prod_{i=1}^{n}(z - \lambda_i)},$$

[1]See Problem H.3.

and compute (as a decomposition into partial fractions):

$$\varphi_a = \frac{\prod_{i=1}^{n-1}(\lambda_a - \mu_i)}{\prod_{i\neq a}(\lambda_a - \lambda_i)}.$$

Therefore,

$$\frac{\partial \varphi_a}{\partial \mu_b} = \frac{\prod_{i=1}^{n-1}(\lambda_a - \mu_i)}{\prod_{i\neq a}(\lambda_a - \lambda_i)} \frac{1}{\mu_b - \lambda_a}, \qquad a = 1, \ldots, n, \quad b = 1, \ldots, n-1. \tag{1.2}$$

The Jacobian is essentially the determinant of the matrix $1/(\mu_b - \lambda_a)$, which is the Cauchy determinant (Problems H.1 and **??**). The final density is obtained from the symmetric Dirichlet density, but we plug in $w = \varphi$, and also multiply by the Jacobian. This completes the proof. □

**Corollary 1.3** (Joint density of the corners)**.** *The eigenvalues* $\lambda^(k)_j$, $1 \le j \le k \le n$, *of a random matrix from* Orbit$(\lambda)$ *form an interlacing array, with the joint density*

$$\propto \prod_{k=1}^{n} \prod_{1\le i<j\le k} \left(\lambda_j^{(k)} - \lambda_i^{(k)}\right)^{2-\beta} \prod_{a=1}^{k+1}\prod_{b=1}^{k} \left|\lambda_a^{(k+1)} - \lambda_b^{(k)}\right|^{\beta/2-1}.$$

For $\beta = 2$, all factors disappear, and we get the uniform distribution on the interlacing array. This is the *uniform Gibbs property* which is important for other models, including discrete ensembles.

# 2 Loop equations

Let us write down the *loop equations* for the passage from the eigenvalues $\lambda$ to the eigenvalues $\mu$. These loop equations are due to [GH24] by a limit from a discrete system (related to Jack symmetric polynomials). Note that despite the name, these are not **equations**, but rather a statement that some expectations are holomorphic. We stick to the random matrix setting, and present a formulation and a proof given by [Gor25].

## 2.1 Formulation

**Theorem 2.1.** *We fix* $n = 1, 2, \ldots$ *and* $n+1$ *real numbers* $\lambda_1 \ge \cdots \ge \lambda_{n+1}$. *For* $\beta > 0$, *consider* $n+1$ *i.i.d.* $\chi^2_\beta$ *random variables* $\xi_i$ *and set*

$$w_i = \frac{\xi_i}{\sum_{j=1}^{n+1} \xi_j}, \qquad 1 \le i \le n+1.$$

*We define* $n$ *random points* $\{\mu_1, \ldots, \mu_n\}$ *as* $n$ *solutions to the equation*

$$\sum_{i=1}^{n+1} \frac{w_i}{z - \lambda_i} = 0. \tag{2.1}$$

*Take any* polynomial $W(z)$ *and consider the complex function:*

$$f_W(z) = \mathbb{E}\left[\prod_{j=1}^{n} \exp\big(W(\mu_j)\big) \frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} \left(W'(z) + \sum_{i=1}^{n+1} \frac{\beta/2-1}{z-\lambda_i} + \sum_{j=1}^{n} \frac{1}{z-\mu_j}\right)\right]. \qquad (2.2)$$

*Then* $f_W(z)$ *is an* entire function *of* $z$*, in the following sense:*

- *For* $z \in \mathbb{C} \setminus [\lambda_{n+1}, \lambda_1]$*, the expectation in* (2.2) *defines a holomorphic function of* $z$*.*
- *This function has an analytic continuation to* $\mathbb{C}$*, which has no singularities.*

**Remark 2.2.** Note that for $z$ in $[\lambda_{n+1}, \lambda_1]$, the integral determining (2.2) might be divergent, and, therefore, analytic continuation is the proper way to define $f_W(z)$, $z \in [\lambda_{n+1}, \lambda_1]$.

**Corollary 2.3.** *We have*

$$f_0(z) = \frac{(n+1)\beta}{2} - 1.$$

*Here* $f_0$ *means* $f_W$ *with* $W \equiv 0$*.*

*Proof.* This is obtained by sending $z \to \infty$ in (2.2). □

## 2.2 Proof of Theorem 2.1 for $\beta > 2$

Theorem 2.1 remains valid for $\beta > 0$, but we only prove it for $\beta > 2$ here. We also assume that $\lambda_1 > \ldots > \lambda_n$.

We begin by observing that for $z \in \mathbb{C} \setminus [\lambda_{n+1}, \lambda_1]$, the expectation in (2.2) is well-defined and holomorphic in $z$. This follows since for such $z$, the denominators $z - \lambda_i$ and $z - \mu_j$ are bounded away from zero with probability 1. The key challenge is to show that $f_W(z)$ can be analytically continued to an entire function. Potential singularities of $f_W(z)$ are inside the intervals $(\lambda_{i+1}, \lambda_1)$. We will show that these singularities do not actually occur.

Consider a specific interval $(\lambda_2, \lambda_1)$. We need to show that $f_W(z)$ has no singularities in this interval. From Theorem 1.2, the probability distribution of $\mu = (\mu_1, \ldots, \mu_n)$ has density proportional to:

$$\prod_{i<j}(\mu_i - \mu_j) \prod_{i=1}^{n} \prod_{j=1}^{n+1} |\mu_i - \lambda_j|^{\beta/2-1}.$$

Let us analyze the function in (2.2). For $z \in (\lambda_2, \lambda_1)$, we need to demonstrate that the expectation

$$\mathbb{E}\left[\prod_{j=1}^{n} \exp\big(W(\mu_j)\big) \frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} \left(W'(z) + \sum_{i=1}^{n+1} \frac{\beta/2-1}{z-\lambda_i} + \sum_{j=1}^{n} \frac{1}{z-\mu_j}\right)\right]$$

is holomorphic. We need to show that the integral

$$\int_{\mu_i \in [\lambda_{i+1}, \lambda_i]} \prod_{i<j}(\mu_i - \mu_j) \prod \prod (\mu_j - \lambda_i)^{\beta/2-1} \prod e^{W(\mu_j)} \frac{\prod(z-\lambda_i)}{\prod(z-\mu_j)}$$

$$\times \left( W'(z) + \sum \frac{\beta/2-1}{z-\lambda_i} + \sum \frac{1}{z-\mu_j} \right) d\mu_1 \dots d\mu_n$$

is holomorphic for $z \in (\lambda_2, \lambda_1)$. Note that (here we are using the fact that $\beta > 2$)

$$0 = \int\limits_{\mu_i \in [\lambda_{i+1}, \lambda_i]} \frac{\partial}{\partial \mu_1} \left( \underbrace{\prod_{i<j} (\mu_i - \mu_j) \prod \prod (\mu_j - \lambda_i)^{\beta/2-1} \prod e^{W(\mu_j)} \frac{\prod (z-\lambda_i)}{\prod (z-\mu_j)}}_{(*)} \right) d\mu_1 \dots d\mu_n$$

$$= \int\limits_{\mu_i \in [\lambda_{i+1}, \lambda_i]} (*) \cdot \left[ \sum_{j=2}^{n} \frac{1}{\mu_1 - \mu_j} + \sum_{i=1}^{n+1} \frac{\beta/2-1}{\mu_1 - \lambda_i} + W'(\mu_1) + \frac{1}{z-\mu_1} \right] d\mu_1 \dots d\mu_n$$

Subtracting this expression from our original integral and noting that

$$\left( W'(z) + \sum \frac{\beta/2-1}{z-\lambda_i} + \sum \frac{1}{z-\mu_j} \right) - \left( \sum_{j=2}^{n} \frac{1}{\mu_1 - \mu_j} + \sum_{i=1}^{n+1} \frac{\beta/2-1}{\mu_1 - \lambda_i} + W'(\mu_1) + \frac{1}{z-\mu_1} \right)$$

has zero at $z = \mu_1$, we conclude that our integral has no singularity at $\mu_1$, and therefore no singularities in the $[\lambda_2, \lambda_1]$ interval. This completes the proof of Theorem 2.1 for $\beta > 2$.

# 3 Applications of loop equations

The loop equations provide a powerful tool for analyzing the spectral properties of random matrices through their eigenvalue distributions. Let us derive an equation for the Stieltjes transform of the empirical measures.

## 3.1 Stieltjes transform equations

Starting from Theorem 2.1 with $W = 0$, we have:

$$\mathbb{E} \left[ \frac{\prod_{i=1}^{n+1} (z-\lambda_i)}{\prod_{j=1}^{n} (z-\mu_j)} \left( \sum_{i=1}^{n+1} \frac{\beta/2-1}{z-\lambda_i} + \sum_{j=1}^{n} \frac{1}{z-\mu_j} \right) \right] = \frac{(n+1)\beta}{2} - 1. \tag{3.1}$$

Let us introduce the empirical Stieltjes transforms:

$$G_\lambda(z) = \frac{1}{n} \sum_{i=1}^{n+1} \frac{1}{z-\lambda_i},$$

$$G_\mu(z) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{z-\mu_j}.$$

We also define the logarithmic potentials (indefinite integrals of the Stieltjes transforms):

$$\int G_\lambda(z)dz = \frac{1}{n}\sum_{i=1}^{n+1} \ln(z-\lambda_i),$$

$$\int G_\mu(z)dz = \frac{1}{n}\sum_{j=1}^{n} \ln(z-\mu_j).$$

Noting that

$$\frac{\prod_{i=1}^{n+1}(z-\lambda_i)}{\prod_{j=1}^{n}(z-\mu_j)} = \exp\left(n\left(\int G_\lambda(z) - \int G_\mu(z)\right)\right),$$

we can rewrite equation (3.1) as:

$$\mathbb{E}\left[\exp\left(n\left(\int G_\lambda(z)\,dz - \int G_\mu(z)\,dz\right)\right)\left(\left(\frac{\beta}{2}-1\right)G_\lambda(z) + G_\mu(z)\right)\right] = \frac{\beta}{2} + \frac{1}{n}\left(\frac{\beta}{2}-1\right). \tag{3.2}$$

### 3.2 Asymptotic behavior

Equation (3.2) can be reinterpreted in terms of a time evolution of eigenvalue distributions. This perspective offers significant insights into the asymptotic behavior of the corners process.

If we think of $\lambda$ as configuration at time $t = 1$ and $\mu$ as configuration at time $t = 1 - \frac{1}{n}$, then denoting the general time parameter as $t$ and setting $G_\lambda = G_1$, $G_\mu = G_{1-\frac{1}{n}}$, we obtain a continuous time evolution of Stieltjes transforms. (And similarly for all $t$, of course.)

As $n \to \infty$, equation (3.2) transforms into:

$$\frac{\beta}{2}\exp\left(\frac{\partial}{\partial t}\int G_t(z)\,dz\right)\cdot G_t(z) = \frac{\beta}{2}.$$

This implies

$$\frac{\partial}{\partial t}\int G_t(z)\,dz + \ln G_t(z) = 0.$$

Taking the derivative with respect to $z$, we get:

$$\frac{\partial}{\partial t}G_t(z) + \frac{1}{G_t(z)}\frac{\partial}{\partial z}G_t(z) = 0. \tag{3.3}$$

This is precisely the inviscid Burgers equation, a fundamental nonlinear PDE in fluid dynamics. The appearance of this equation indicates that the eigenvalue distributions evolve according to a hydrodynamic flow as we move through the corners of the random matrix from full size down to zero.

**Remark 3.1.** We see that the Burgers equation (3.3) does not depend on $\beta$, which is expected. Indeed, for example, G$\beta$E eigenvalues have the same Wigner semicircle law as $\beta = 2$, up to an overall rescaling.

### 3.3 Example with semicircle law

The Stieltjes transform of the semicircular law is given by:

$$G(z) = \int_{-2}^{2} \frac{1}{z-x} \frac{\sqrt{4-x^2}}{2\pi} dx = \frac{1}{2}\left(z - \sqrt{z^2-4}\right).$$

Let us define

$$G^{(c)}(z) := \frac{z - \sqrt{z^2 - 4c^2}}{2c^2},$$

this is the Stieltjes transform of the semicircular law on $[-2c, 2c]$.

## H Problems (due 2025-03-25)

### H.1 Cauchy determinant

Prove the Cauchy determinant formula:

$$\det\left(\frac{1}{x_i - y_j}\right)_{1\le i,j\le n} = \frac{\prod_{i<j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(x_i - y_j)}.$$

### H.2 Jacobian from $n-1$ to $n$ dependent variables

Explain how the factor $\prod_{i=1}^{n-1}\prod_{j=1}^{n}|\mu_i - \lambda_j|$ appears from the Jacobian of the transformation from $\varphi$ to $\mu$ (1.2), even though $\partial\varphi_a/\partial\mu_b$ is defined for $a = 1, \ldots, n$, $b = 1, \ldots, n-1$, but the $\varphi_i$'s are not independent.

### H.3 Dirichlet density

Find in the literature or prove on your own the first statement in the proof of Theorem 1.2 about the symmetric Dirichlet density arising from normalizing the $\xi_i$'s to $\varphi_i$'s.

### H.4 General $\beta$ Corners Process Simulation

This problem explores computational aspects of the general $\beta$ corners process.

(a) Write code for generating a sample from the distribution of $\mu = (\mu_1, \ldots, \mu_{n-1})$ given $\lambda = (\lambda_1, \ldots, \lambda_n)$ for arbitrary $\beta > 0$, using the polynomial equation characterization.

(b) Let $\lambda = (n, n-1, \ldots, 2, 1)$. For $n = 7$, compute (numerically) the expected values $\mathbb{E}[\mu_i]$ for each $i$, when $\beta = 1, 2, 4$, and 10. Describe the behavior as $\beta$ increases.

## References


[GH24] V. Gorin and J. Huang, *Dynamical loop equation*, Ann. Probab. **52** (2024), no. 5, 1758–1863. arXiv:2205.15785 [math.PR]. ↑3

[Gor25] V. Gorin, *Private communication*, 2025. ↑3



L. Petrov, University of Virginia, Department of Mathematics, 141 Cabell Drive, Kerchof Hall, P.O. Box 400137, Charlottesville, VA 22904, USA

E-mail: `lenia.petrov@gmail.com`