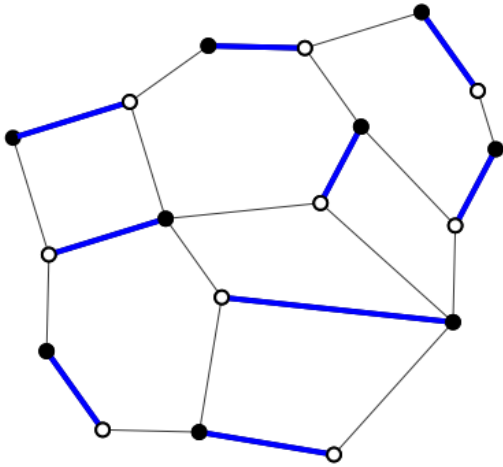


Lecture 1

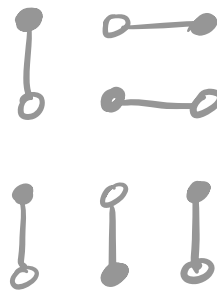
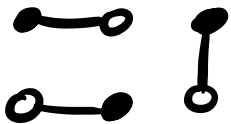
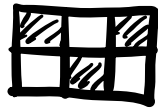
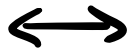
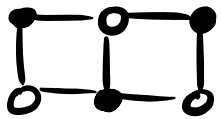
Kasteleyn theorem

G-planar, bipartite graph



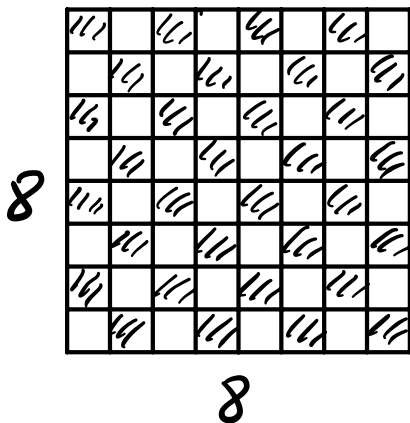
dimer configurations /
perfect matchings

- Today: Dimer model on \mathbb{Z}^2
(domino tilings)



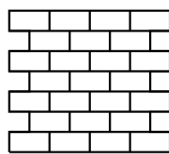
$$\#[\text{domino}]^2 = 3$$

Q: How many domino tilings?

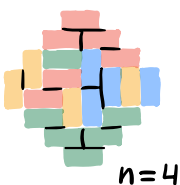


$$\#(\text{tilings of } \square_8^8) = 12\,988\,816$$

1



$$2^{n(n+1)/2}$$



n=4

Thm: $(\# \text{ dimer configurations})^2 = \text{Perm } \tilde{A}$,

where \tilde{A} is an adjacency matrix of G

$$\tilde{A} = \begin{matrix} & \begin{matrix} w_1, \dots, w_n & b_1, \dots, b_n \end{matrix} \\ \begin{matrix} w_1 \\ \vdots \\ w_n \\ b_1 \\ \vdots \\ b_n \end{matrix} & \begin{pmatrix} \textcircled{1} & \tilde{K} \\ \tilde{K}^T & \textcircled{1} \end{pmatrix} \end{matrix} \quad a_{ij} = \begin{cases} 1, & \text{if } i \sim j \\ 0, & \text{otherwise} \end{cases}$$

Reminder: $\text{Perm } \tilde{A} = \sum_{\sigma \in S_{2n}} a_{1\sigma(1)} \dots a_{2n\sigma(2n)}$

Proof: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 5 & 8 & 6 & 7 \end{pmatrix} = (1324)(5)(687)$

$$\# \left(\begin{array}{cc} \text{[Diagram 1: 8 vertices, 4 red edges, 4 green edges]} & \text{[Diagram 2: 8 vertices, 4 red edges, 4 green edges]} \end{array} \right) = (\# \text{ dimer configurations})^2$$

(□)

Rmk: $\text{Perm } \tilde{A} = (\text{Perm } \tilde{K})^2$

WANTED: $\text{det } A (= (\text{det } K)^2)$

signs $\tau_{w_i, b_j} \in \mathbb{C}, |\tau_{w_i, b_j}| = 1$

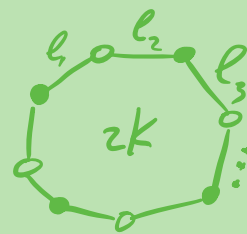
$$K = \begin{matrix} & \begin{matrix} b_1, \dots, b_n \end{matrix} \\ \begin{matrix} w_1 \\ \vdots \\ w_n \end{matrix} & \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{matrix} \quad k_{w_i, b_j} = \begin{cases} \tau(w_i, b_j) & \text{if } w_i \sim b_j \\ 0 & \text{otherwise} \end{cases}$$

$$\text{det } K = \sum_{\sigma \in S_n} \text{sign}(\sigma) k_{w_1, b_{\sigma(1)}} \dots k_{w_n, b_{\sigma(n)}}$$

Kasteleyn sign condition:

For each face of degree $2k$

$$\frac{\tau(e_1)}{\tau(e_2)} \cdot \frac{\tau(e_3)}{\tau(e_4)} \cdots \frac{\tau(e_{2k-1})}{\tau(e_{2k})} = (-1)^{k-1}$$

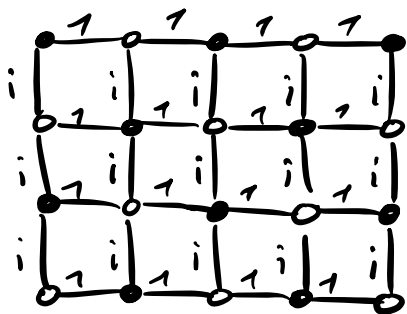


Thm: (Percus, Kasteleyn)

$$\# \text{ dimer configurations} = |\det K|$$

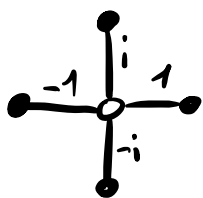
Two choices of Kasteleyn signs on \mathbb{Z}^2

I



\leadsto formula for the
tilings of $m \times n$ rectangle

II



\leadsto

$$K^{-1} \cdot K = \text{Id}$$

\Downarrow

$F_w(b) := K^{-1}(b, w_0)$ is
discrete holomorphic

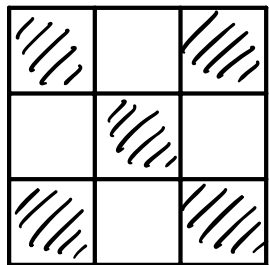
Next time

Proposition:

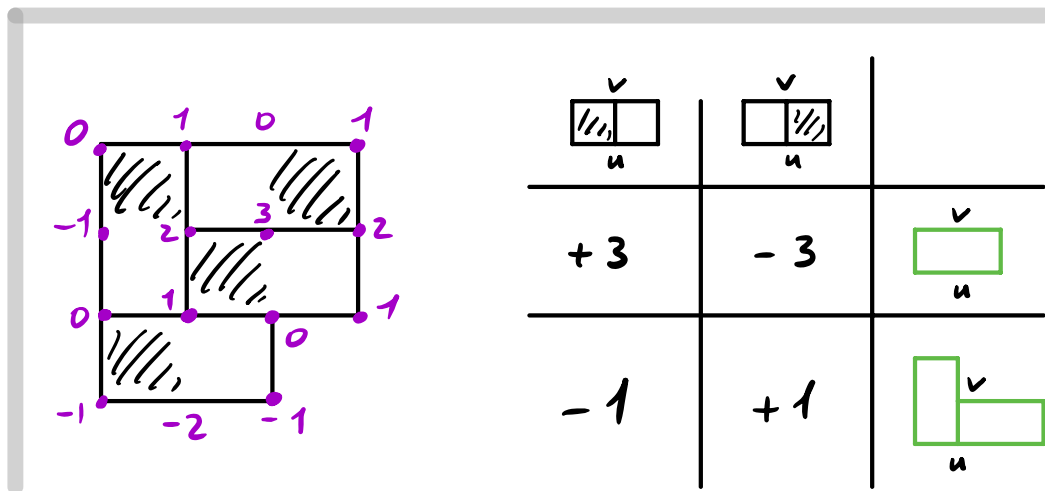
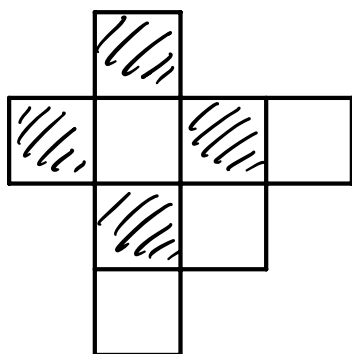
$$\left(\# \text{ tilings } m \times n \right)^4 = \prod_{p=1}^m \prod_{q=1}^n 4 \left(\cos^2 \frac{\pi p}{m+1} + \cos^2 \frac{\pi q}{n+1} \right)$$

Thurston theorem

Q: When discrete domain is tileable?



black faces
= # white Faces
NOT enough:



Def: [Thurston height Function]

Given a tiling of a simply connected domain Ω with $\partial\Omega$ on \mathbb{Z}^2 , the height function h is a \mathbb{Z} -valued fct on the set of vertices of Ω defined by local rules: $\forall u \sim v$

$$h(v) - h(u) = \begin{cases} \pm 1 & \text{if } uv \text{ does not cross domino} \\ \mp 3 & \text{if } uv \text{ crosses domino} \end{cases}$$

Rmk: h is defined up to a global additive constant by local rules.

Rmk: Ω -simply connected \Rightarrow

$h|_{\partial\Omega}$ does not depend on a tiling

Thm: Simply-connected Ω is tileable iff both conditions hold:

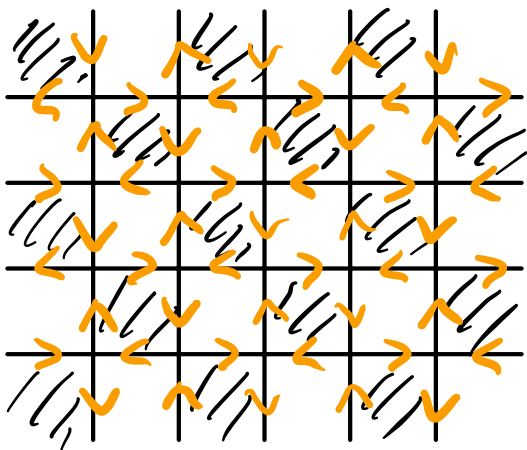
1) h is well-defined on $\partial\Omega$

2) $\forall u, v \in \partial\Omega$

$$h(v) - h(u) \leq d(u, v)$$

Def: Oriented graph distance:

" \square "-orientation

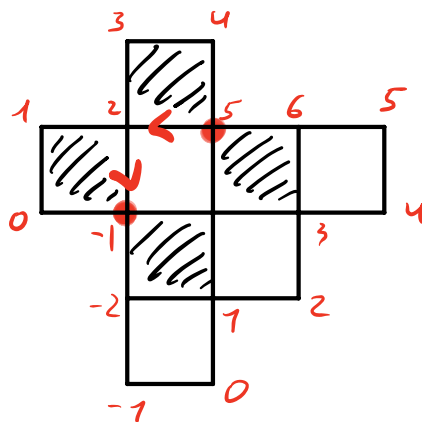
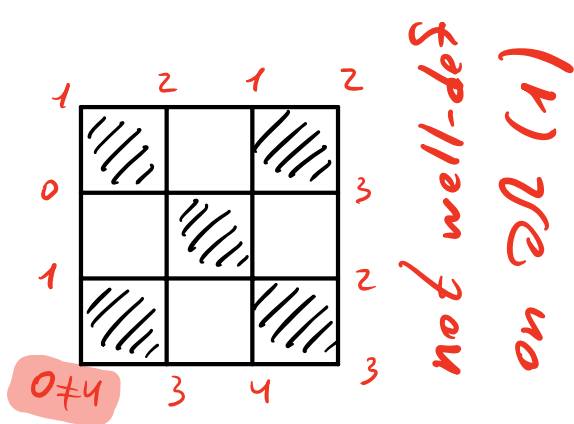


$d(u, v)$ = edge length
of shortest positive
oriented path
from u to v
(within $\bar{\Omega} = \Omega \cup \partial\Omega$)

Rmk: $d(u, v)$ NOT symmetric

$$d(u, v) = 1 \quad \& \quad d(v, u) = 3$$





$$h(v) - h(u) = 6$$

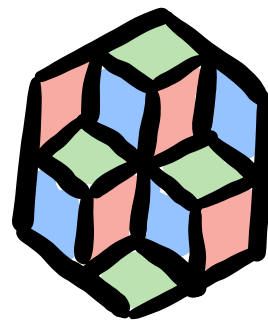
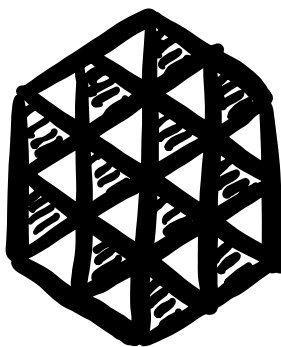
$$> d(u, v) = 2$$

(2)

Height Function of lozenge tilings:

$$h(v) - h(u) = \begin{cases} \pm 1 \\ \mp 2 \end{cases}$$

\updownarrow
distance from surface
to $(1,1,1)$ -plane



"3D picture"

