Lectures on Random Matrices (Spring 2025)
Lecture 14: Matching Random Matrices to Random Growth II

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Contents

1 Recap

In the previous Lecture 13, we established a remarkable correspondence between two a priori different objects:

- The *spiked Wishart ensemble*: an $n \times n$ Hermitian random-matrix process $\{M(t)\}_{t \geq 0}$ whose entries come from columns of independent Gaussian random vectors of suitably chosen covariance.
- An *inhomogeneous last-passage percolation (LPP)* model: an array $\{W_{i,j}\}$ of exponential (or geometric) random weights on a portion of the two-dimensional lattice, whose last-passage times $L(t, n)$ relate closely to the largest eigenvalues of $M(t)$.

Our goal was to demonstrate that under certain parameter choices $(\pi, \hat{\pi})$ determining both the Wishart covariances and the LPP rates, the *joint* distributions of $\{\lambda_1(1), \ldots, \lambda_1(t)\}$ (the largest eigenvalue of each $M(s)$ for $s = 1, \ldots, t$) coincide exactly with those of $\{L(1, n), \ldots, L(t, n)\}$ (the last-passage times along the top row of the LPP model). This equivalence, originally demonstrated by Dieker–Warren (2009), can be fully understood by passing to a *discrete* version of LPP with geometric site-weights and then applying the *Robinson–Schensted–Knuth* (RSK) correspondence.

1. Spiked Wishart ensembles and the largest eigenvalue process.

We defined the *generalized* (or spiked) Wishart matrix $M(t)$ of size $n \times n$ by setting

$$M(t) \;=\; \sum_{m=1}^{t} A^{(m)} \big(A^{(m)}\big)^*$$

*Course webpage • Live simulations • TeX Source • Updated at 10:21, Tuesday 15th April, 2025 where $\{A^{(m)}\}_{m=1}^{\infty}$ are i.i.d. complex Gaussian column vectors of length n, with

$$\operatorname{Var}\big(A_i^{(m)}\big) \;=\; \frac{1}{\pi_i + \hat{\pi}_m}\,.$$

Here, $\pi = (\pi_1, \dots, \pi_n)$ and $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \dots)$ are positive and nonnegative parameters, respectively. Writing $\lambda_1(t) \geq \cdots \geq \lambda_n(t) \geq 0$ for the eigenvalues of $M(t)$, we then saw:

1. The vectors $\lambda(t) = \big(\lambda_1(t), \dots, \lambda_n(t)\big)$ form a Markov chain in the *Weyl chamber* $\mathbb{W}^n = \{x_1 \geq \cdots \geq x_n \geq 0\}$.
2. There is an *interlacing* property: each update $M(t-1) \mapsto M(t)$ via the rank-one matrix $A^{(t)}\big(A^{(t)}\big)^*$ forces $\lambda(t)$ to interlace with $\lambda(t-1)$:

$$\lambda_1(t) \;\geq\; \lambda_1(t-1) \;\geq\; \lambda_2(t) \;\geq\; \cdots \;\geq\; \lambda_n(t-1) \;\geq\; \lambda_n(t).$$

By writing down the transition kernels and normalizing appropriately (see Theorem 13.6 in Lecture 13), one obtains explicit factorized forms for $Q^{\pi;\hat{\pi}}_{t-1,t}(\,\cdot\,,\,\cdot)$, the one-step law from $\lambda(t-1)$ to $\lambda(t)$. Notably, for $\pi_i = 1$ and $\hat{\pi}_j = 0$ (the *null* Wishart case), one recovers the classical Laguerre orthogonal polynomial ensemble at fixed times.

2. Inhomogeneous last-passage percolation with exponential (or geometric) weights.

On the combinatorial side, we considered an array of site-weights $\{W_{i,j}\}_{i,j\geq 1}$ such that each $W_{i,j}$ is exponentially distributed with rate $\pi_i + \hat{\pi}_j$. For every integer $t \geq 1$, we define $L(t,n)$ to be the maximum total weight of all up-right paths from $(1,1)$ to (t,n). Formally,

$$L(t,n) \;=\; \max_{\Gamma\colon (1,1)\to(t,n)} \sum_{(i,j)\in\Gamma} W_{i,j}.$$

One checks that $L(\,\cdot\,,n)$ satisfies a simple additive recursion, plus boundary conditions:

$$L(i,j) \;=\; W_{i,j} \;+\; \max\big\{\, L(i-1,j),\; L(i,j-1)\big\},$$

with $L(1,j) = \sum_{k=1}^{j} W_{1,k}$ and $L(i,1) = \sum_{k=1}^{i} W_{k,1}$. As t grows, $\{L(t,n)\}_{t\geq 1}$ also forms a process in $\mathbb{R}$. The main claim, to be completed in subsequent arguments, is that

$$\big(L(1,n),\, L(2,n),\, \dots,\, L(t,n)\big) \;\stackrel{d}{=}\; \big(\lambda_1(1),\, \lambda_1(2),\, \dots,\, \lambda_1(t)\big),$$

i.e. they share the same joint law across all times. The proof route: discretize the exponential random variables to geometric ones, and then use the RSK insertion *bijection* to see how LPP times map into eigenvalue-like data.

3. RSK via toggles: definitions and weight preservation.

The *Robinson–Schensted–Knuth* correspondence (RSK) was the main new mechanism in Lecture 13. In our setup, we adopt a *toggle-based* viewpoint: we encode arrays by diagonals and successively *toggle* the diagonals to achieve a fully *ordered* array R. Concretely:

Definition 1.1 (Nonnegative and ordered arrays)**.** For integers $t, n \geq 1$:

- A *nonnegative array* W is a collection of integers $W_{i,j} \in \mathbb{Z}_{\geq 0}$, $1 \leq i \leq t$, $1 \leq j \leq n$.
- An *ordered array* R (of the same size) satisfies $R_{i,j} \leq R_{i,j+1}$ and $R_{i,j} \leq R_{i+1,j}$ for all valid i, j.

Definition 1.2 (Toggle operation)**.** Let $w \in \mathbb{Z}_{\geq 0}$ and let (λ, κ, μ) be three interlacing sequences of nonnegative integers, symbolically written as $\lambda \succ \kappa \prec \mu$. Then

$$\mathrm{T}\big(w;\, \lambda, \kappa, \mu\big) \;=\; (\lambda, \nu, \mu)$$

is defined by leaving λ, μ unchanged and setting

$$\nu_1 \;=\; w + \max(\lambda_1,\, \mu_1), \qquad \nu_i \;=\; \max(\lambda_i,\, \mu_i) \;+\; \min(\lambda_{i-1},\, \mu_{i-1}) \;-\; \kappa_{i-1}, \quad i \geq 2.$$

From a straightforward check (Problem 13.7), toggling preserves total weights in a precise sense, and we always end up with $\lambda \prec \nu \succ \mu$.

Theorem 1.3 (RSK is a bijection, cf. Lecture 13)**.** *Given a nonnegative array W of size $t \times n$, the* RSK map *outputs an ordered array $R = \mathrm{RSK}(W)$ by the following procedure:*

- *Process the cells (i, j) of W in an arbitrary order (e.g. row by row from bottom to top).*
- *For each cell (i, j),* toggle *the diagonal containing that cell in the partial R, inserting weight $w = W_{i,j}$.*

All toggles commute on different diagonals, so the final ordered array R does not depend on the insertion order. Moreover, $W \mapsto R$ is a bijection between nonnegative arrays and ordered arrays.

The key to how RSK links LPP and random matrices is its *weight preservation* property, which we restate in a concise form here:

Theorem 1.4 (Weight preservation, cf. Proposition 13.25)**.** *Let $W = \{W_{i,j}\}$ be a nonnegative integer array, and $R = \mathrm{RSK}(W)$. Denote*

$$\mathrm{row}_i \;=\; \sum_{j=1}^{n} W_{i,j}, \quad \mathrm{col}_j \;=\; \sum_{i=1}^{t} W_{i,j},$$

and for R define the diagonal sums

$$\mathrm{diag}_{i,j} \;=\; \sum_{k=0}^{\min(i,j)-1} R_{i-k,\, j-k}.$$

Then for each $1 \leq j \leq n$ and $1 \leq i \leq t$, we have

$$\mathrm{diag}_{t,j} \;=\; \sum_{m=1}^{j} \mathrm{col}_m, \qquad \mathrm{diag}_{i,n} \;=\; \sum_{m=1}^{i} \mathrm{row}_m, \tag{1.1}$$

ensuring that the total sum of W over all cells equals the total sum of R over all cells.

Proof (sketch). One inductively builds R by adding the sites (i, j) one at a time. Each toggle modifies exactly one diagonal and preserves an inclusion–exclusion count on neighboring diagonals. Concretely, after adding a box (i, j), the diagonal-sum identity

$$\mathrm{diag}_{i,j} \;=\; \mathrm{diag}_{i-1,j} + \mathrm{diag}_{i,j-1} \;-\; \mathrm{diag}_{i-1,j-1} \;+\; W_{i,j}$$

holds, expressing that R captures the discrete "second difference" of W. Since toggles commute on disjoint diagonals, the partial sums assemble to match the row and column sums of W regardless of the order of addition. □

Thus, applying RSK to random arrays W (in particular, to a geometric LPP environment) yields an ordered array R whose interlacing diagonals reflect precisely the combinatorial structure of the LPP. By interpreting each diagonal as encoding eigenvalue increments, one connects R to the same interlacing patterns arising in Hermitian random matrices of $\beta = 2$ type. This observation is what ultimately shows the distributional identity between $\big(L(1, n), \ldots, L(t, n)\big)$ and $\big(\lambda_1(1), \ldots, \lambda_1(t)\big)$ under appropriate limiting and deformation parameters.

Outline of Next Steps

In the upcoming lecture, we will:

- Translate the RSK interlacing arrays directly into a form resembling eigenvalue distributions.
- Show how the parameter choices $(\pi, \hat{\pi})$ in the geometric version correspond to the spiked Wishart setup.
- Conclude the proof of Theorem 13.10 (the exact matching of the largest eigenvalues of spiked Wishart and the last-passage times in the exponential LPP).

These steps will complete our new perspective on why matrix spectra in the Wishart class align so precisely with the maximum-weight growth in an LPP model.

N Problems (due 2025-04-29)

References

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