2025 FRM®
Exam Prep

SchweserNotes™
Valuation and Risk Models

Part I Book 4

KAPLAN SCHWESER

Book 4: Valuation and Risk Models

SchweserNotes™ 2025

FRM Part I

KAPLAN SCHWESER

SCHWESERNOTES™ 2025 FRM® PART I BOOK 4: VALUATION AND RISK MODELS

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47. Measures of Financial Risk

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 1.

After completing this reading, you should be able to:

a. describe the mean-variance framework and the efficient frontier.
b. compare the normal distribution with the typical distribution of returns of risky financial assets such as equities.
c. define the VaR measure of risk, describe assumptions about return distributions and holding periods, and explain the limitations of VaR.
d. explain and calculate ES and compare and contrast VaR and ES.
e. define the properties of a coherent risk measure and explain the meaning of each property.
f. explain why VaR is not a coherent risk measure.

48. Calculating and Applying VaR

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 2.

After completing this reading, you should be able to:

a. explain and provide examples of linear and non-linear portfolios.
b. describe and explain the historical simulation approach for computing VaR and ES.
c. describe the delta-normal approach and calculate VaR for non-linear derivatives using delta-normal approach.
d. describe and calculate VaR for linear derivatives.
e. describe the limitations of the delta-normal method.
f. explain the Monte Carlo simulation method for calculating VaR and ES and identify its strengths and weaknesses.
g. describe the implications of correlation breakdown for a VaR or ES analysis.
h. describe worst-case scenario analysis and compare it to VaR.

49. Measuring and Monitoring Volatility

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 3.

After completing this reading, you should be able to:

a. explain how asset return distributions tend to deviate from the normal distribution.
b. explain reasons for fat tails in a return distribution and describe their implications.
c. differentiate between conditional and unconditional distributions and describe regime switching.
d. compare and contrast different approaches for estimating conditional volatility.
e. apply the exponentially weighted moving average (EWMA) approach to estimate volatility, and describe alternative approaches to weighting historical return data.
f. apply the GARCH (1,1) model to estimate volatility.
g. explain and apply approaches to estimate long horizon volatility/VaR and describe the process of mean reversion according to a GARCH (1,1) model.
h. evaluate implied volatility as a predictor of future volatility and its shortcomings.
i. describe an example of updating correlation estimates.

STUDY SESSION 13

50. External and Internal Credit Ratings

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 4.

After completing this reading, you should be able to:

a. describe external rating scales, the rating process, and the link between ratings and default.

b. define conditional and unconditional default probabilities and explain the distinction between the two.
c. define hazard rate and calculate the unconditional default probability of a credit asset using hazard rate.
d. define recovery rate and calculate the expected loss from a loan.
e. explain and compare the through-the-cycle and point-in-time ratings approaches.
f. describe alternative methods to credit ratings produced by rating agencies.
g. compare external and internal ratings approaches.
h. describe, calculate, and interpret a rating transition matrix and explain its uses.
i. describe the relationships between changes in credit ratings and changes in stock prices, bond prices, and credit default swap spreads.
j. explain historical failures and potential challenges to the use of credit ratings in making investment decisions.

51. Country Risk: Determinants, Measures, and Implications

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 5.

After completing this reading, you should be able to:

a. explain how a country's economic growth rates, political risk, legal risk, and economic structure relate to its risk exposure.
b. evaluate composite measures of risk that incorporate multiple components of country risk.
c. compare instances of sovereign default in both foreign currency debt and local currency debt and explain common causes of sovereign defaults.
d. describe the consequences of sovereign default.
e. describe factors that influence the level of sovereign default risk; explain and assess how rating agencies measure sovereign default risks.
f. describe the characteristics of sovereign credit spreads and sovereign credit default swaps (CDS) and compare the use of sovereign spreads to credit ratings.

52. Measuring Credit Risk

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 6.

After completing this reading, you should be able to:

a. explain the distinctions between economic capital and regulatory capital and describe how economic capital is derived.
b. describe the degree of dependence typically observed among the loan defaults in a bank's loan portfolio, and explain the implications for the portfolio's default rate.
c. define and calculate expected loss (EL).
d. define and explain unexpected loss (UL).
e. estimate the mean and standard deviation of credit losses assuming a binomial distribution.
f. describe the Gaussian copula model and its application.
g. describe and apply the Vasicek model to estimate default rate and credit risk capital for a bank.
h. describe the CreditMetrics model and explain how it is applied in estimating economic capital.
i. describe and apply Euler's theorem to determine the contribution of a loan to the overall risk of a portfolio.
j. explain why it is more difficult to calculate credit risk capital for derivatives than for loans.
k. describe challenges to quantifying credit risk.

53. Operational Risk

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 7.

After completing this reading, you should be able to:

a. describe the different categories of operational risk and explain how each type of risk can arise.
b. compare the basic indicator approach, the standardized approach, and the advanced measurement approach for calculating operational risk regulatory capital.
c. describe the standardized measurement approach and explain the reasons for its introduction by the Basel Committee.
d. explain how a loss distribution is derived from an appropriate loss frequency distribution and loss severity distribution using Monte Carlo simulation.
e. describe the common data issues that can introduce inaccuracies and biases in the estimation of loss frequency and severity distributions.

f. describe how to use scenario analysis in instances when data are scarce.
g. describe how to identify causal relationships and how to use Risk and Control Self-Assessment (RCSA), Key Risk Indicators (KRIs), and education to understand and manage operational risks.
h. describe the allocation of operational risk capital to business units.
i. explain how to use the power law to measure operational risk.
j. explain how the moral hazard and adverse selection problems faced by insurance companies relate to insurance against operational risk.

54. Stress Testing

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 8.

After completing this reading, you should be able to:
a. describe the rationale for the use of stress testing as a risk management tool.
b. describe the relationship between stress testing and other risk measures, particularly in enterprise-wide stress testing.
c. describe stressed VaR and stressed ES, including their advantages and disadvantages, and compare the process of determining stressed VaR and ES to that of traditional VaR and ES.
d. explain key considerations and challenges related to developing stress testing scenarios and building stress testing models.
e. describe reverse stress testing and describe an example of regulatory stress testing.
f. describe the responsibilities of the board of directors, senior management, and the internal audit function in stress testing governance.
g. describe the role of policies and procedures, validation, and independent review in stress testing governance.
h. describe the Basel stress testing principles for banks regarding the implementation of stress testing.

STUDY SESSION 14

55. Pricing Conventions, Discounting, and Arbitrage

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 9.

After completing this reading, you should be able to:
a. define discount factor and calculate present and future values using a discount function.
b. define the “law of one price,” explain it using an arbitrage argument, and describe how it can be applied to bond pricing.
c. identify arbitrage opportunities for fixed-income securities with certain cash flows.
d. identify the components of a U.S. Treasury coupon bond and compare the structure to Treasury STRIPS, including the difference between P-STRIPS and C-STRIPS.
e. construct a replicating portfolio using multiple fixed-income securities to match the cash flows of a given fixed-income security.
f. differentiate between “clean” and “dirty” bond pricing and explain the implications of accrued interest with respect to bond pricing.
g. describe the common day-count conventions used to calculate interest on a fixed-income security.

56. Interest Rates

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 10.

After completing this reading, you should be able to:
a. calculate and interpret the impact of different compounding frequencies on a bond’s value.
b. define spot rate and calculate discount factors given spot rates.
c. interpret the forward rate and calculate forward rates given spot rates.
d. define par rate and describe how to determine the par rate of a bond.
e. interpret the relationship between spot, forward, and par rates.
f. assess the impact of a change in time to maturity on the price of a bond.
g. define the “flattening” and “steepening” of rate curves and describe a trade to reflect expectations that a curve will flatten or steepen.
h. describe a swap transaction and explain how a swap market defines par rates.

57. Bond Yields and Return Calculations

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 11.

After completing this reading, you should be able to:

a. differentiate between gross and net realized returns and calculate the realized return for a bond over a holding period including reinvestments.
b. define and interpret the spread of a bond and explain how a spread is derived from a bond price and a term structure of rates.
c. define, interpret, and apply a bond's yield to maturity (YTM) to bond pricing.
d. explain how to calculate a bond's YTM given its structure and price.
e. calculate the price of an annuity and a perpetuity.
f. explain the relationship between spot rates and YTM.
g. define the coupon effect and explain the relationship between coupon rate, YTM, and bond prices.
h. explain the decomposition of the profit and loss (P&L) for a bond position or portfolio into separate factors including carry roll-down, rate change, and spread change effects.
i. describe the common assumptions made about interest rates when calculating carry roll-down, and calculate carry roll-down under these assumptions.

58. Applying Duration, Convexity, and DV01

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 12.

After completing this reading, you should be able to:

a. describe a one-factor interest rate model and identify common examples of interest rate factors.
b. define and calculate the DV01 of a fixed-income security given a change in rates and the resulting change in price.
c. calculate the face amount of bonds required to hedge an interest rate-sensitive position given the DV01 of each.
d. define, calculate, and interpret the effective duration of a fixed-income security given a change in rates and the resulting change in price.
e. compare and contrast DV01 and effective duration as measures of price sensitivity.
f. define, calculate, and interpret the convexity of a fixed-income security given a change in rates and the resulting change in price.
g. calculate the DV01, duration, and convexity of a portfolio of fixed-income securities.
h. explain the hedging of a position based on effective duration and convexity.
i. construct a barbell portfolio to match the cost and duration of a given bullet investment and explain the advantages and disadvantages of bullet and barbell portfolios.

59. Modeling Non-Parallel Term Structure Shifts and Hedging

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 13.

After completing this reading, you should be able to:

a. describe principal components analysis and identify the factors that are the most important drivers of term structure movements.
b. describe key rate shift analysis and define key rate 01 (KR01).
c. calculate the KR01s of a portfolio given a set of key rates.
d. calculate the positions in hedging instruments necessary to hedge the key rate risks of a portfolio.
e. apply key rate analysis and principal components analysis to estimating portfolio volatility.
f. describe an interest rate bucketing approach, define forward bucket 01, and compare forward bucket 01s to KR01s.
g. calculate the corresponding duration measure given a KR01 or forward bucket 01.

STUDY SESSION 15

60. Binomial Trees

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 14.

After completing this reading, you should be able to:

a. calculate the value of an American and a European call or put option using a one-step and two-step binomial model.

b. describe how volatility is captured in the binomial model.
c. describe how the value calculated using a binomial model converges as time periods are added.
d. define and calculate delta of a stock option.
e. explain how the binomial model can be altered to price options on stocks with dividends, stock indices, currencies, and futures.

61. The Black-Scholes-Merton Model

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 15.

After completing this reading, you should be able to:

a. explain the lognormal property of stock prices, the distribution of rates of return, and the calculation of expected return.
b. calculate the realized return and historical volatility of a stock.
c. describe the assumptions underlying the Black-Scholes-Merton option pricing model.
d. calculate the value of a European option on a non-dividend-paying stock using the Black-Scholes-Merton model.
e. define implied volatilities and describe how to calculate implied volatilities from market prices of options using the Black-Scholes-Merton model.
f. explain how dividends affect the decision to exercise early for American call and put options.
g. calculate the value of a European option on a dividend-paying stock, futures, or foreign currency using the Black-Scholes-Merton model.
h. describe warrants, calculate the value of a warrant, and calculate the dilution cost of the warrant to existing shareholders.

62. Option Sensitivity Measures: The "Greeks"

Global Association of Risk Professionals. *Valuation and Risk Models*. New York, NY: Pearson, 2022. Chapter 16.

After completing this reading, you should be able to:

a. describe and assess the risks associated with naked and covered option positions.
b. describe the use of a stop-loss hedging strategy, including its advantages and disadvantages, and explain how this strategy can generate naked and covered option positions.
c. calculate the delta of an option.
d. explain delta hedging for an option position, including its dynamic aspects.
e. define and describe vega, gamma, theta, and rho for option positions and calculate the gamma and vega of an option.
f. explain how to implement and maintain a delta-neutral and gamma-neutral position.
g. describe the relationship between delta, theta, gamma, and vega.
h. calculate the delta, gamma, and vega of a portfolio.
i. describe how to implement portfolio insurance and how this strategy compares with delta hedging.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 1.

READING 47

MEASURES OF FINANCIAL RISK

Study Session 12

EXAM FOCUS

The assumption regarding the shape of the underlying return distribution is critical in determining an appropriate risk measure. The mean-variance framework can only be applied under the assumption of an elliptical distribution, such as the normal distribution. The value at risk (VaR) measure can calculate risk measures when the return distribution is nonelliptical, but the measurement is unreliable and no estimate of the amount of loss is provided. Expected shortfall is a more robust risk measure that satisfies all the properties of a coherent risk measure with less restrictive assumptions. For the exam, focus your attention on the calculation of VaR, properties of coherent risk measures, and the expected shortfall methodology.

MODULE 47.1: PORTFOLIO THEORY AND VALUE AT RISK

Mean-Variance Framework

LO 47.a: Describe the mean-variance framework and the efficient frontier.

The traditional mean-variance model estimates the amount of financial risk for portfolios in terms of the portfolio's expected return (i.e., mean) and risk (i.e., standard deviation or variance). Under the **mean-variance framework**, it is necessary to assume that return distributions for portfolios are elliptical distributions. The most commonly known elliptical probability distribution function is the normal distribution.

The **normal distribution** is a continuous distribution that illustrates all possible outcomes for random variables. Recall that the standard normal distribution has a mean of zero and a standard deviation of one. If returns are normally distributed, approximately 66.7% of returns will occur within plus or minus one standard deviation of the mean, and approximately 95% of the observations will occur within plus or minus two standard deviations of the mean. Thus, given this type of distribution, returns are more likely to occur closer to the mean return.

For a portfolio, the following equations are used to calculate the mean and standard deviation. A two-asset portfolio will have the following equations, with *w* reflecting weights in a portfolio, μ equaling expected returns, σ equaling standard deviations, and ρ representing the correlation coefficient.

$$\text{Mean: } \mu_P = w_1\mu_1 + w_2\mu_2$$

$$\text{Standard deviation: } \sigma_P = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho}$$

For a portfolio with *n* investments, the mean and standard deviation of the portfolio are as follows:

$$\text{Portfolio mean: } \mu_P = \sum_{i=1}^{n} w_i\mu_i$$

$$\text{Portfolio standard deviation: } \sigma_P = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_iw_j\sigma_i\sigma_j\rho_{ij}}$$

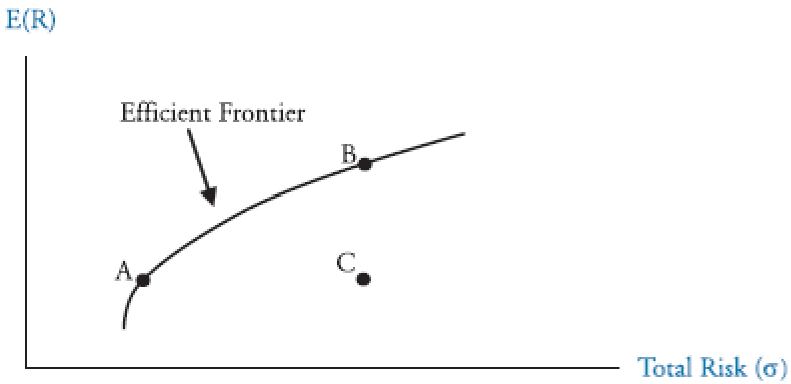
Portfolio managers are concerned with measuring downside risk and, therefore, are particularly interested in measuring the possibility of outcomes to the left or below the expected return. If the return distribution is symmetrical (like the normal distribution), then the standard deviation is an appropriate measure of risk when determining the probability that an undesirable outcome will occur.

If we assume that return distributions for all risky securities are normally distributed, then we can choose portfolios based on the expected returns and standard deviations of all possible combinations of risky securities. Figure 47.1 illustrates the concept of the **efficient frontier**.

In theory, all investors prefer securities or portfolios that lie on the efficient frontier. Consider Portfolios A, B, and C in Figure 47.1. If you had to choose between Portfolios A and C, which one would you prefer and why? Since Portfolios A and C have the same expected return, a risk-averse investor would choose the portfolio with the least amount of risk (which would be Portfolio A). Now if you had to choose between Portfolios B and C, which one would you choose and why? Because Portfolios B and C have the same amount of risk, a risk-averse investor would choose the portfolio with the higher expected return (which would be Portfolio B). We say that Portfolio B dominates Portfolio C with respect to expected return, and that Portfolio A dominates Portfolio C with respect to risk. Likewise, all portfolios on the efficient frontier dominate all other portfolios in the investment universe of risky assets with respect to either risk, return, or both.

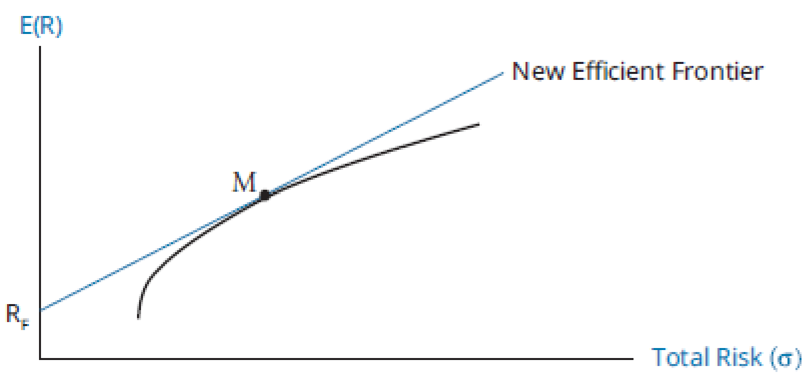
There are an almost unlimited number of combinations of risky assets to the right and below the efficient frontier. However, in the absence of a risk-free security, portfolios to the left and above the efficient frontier are not possible. Therefore, all investors will choose some portfolio on the efficient frontier. If an investor is more risk-averse, she may choose a portfolio on the efficient frontier closer to Portfolio A. If an investor is less risk-averse, she will choose a portfolio on the efficient frontier closer to Portfolio B.

Figure 47.1: The Efficient Frontier



If we now assume that there is a risk-free security, then the mean-variance framework is extended beyond the efficient frontier. Figure 47.2 illustrates that the optimal set of portfolios now lie on a straight line (the new efficient frontier) that runs from the risk-free security through the **market portfolio**, *M*. The market portfolio consists (in theory) of all investments available in the market in their respective proportions. All investors will now seek investments by holding some portion of the risk-free security and the market portfolio. To achieve points on the line to the right of the market portfolio, an investor who is very aggressive will borrow money (at the risk-free rate) and invest in more of the market portfolio. More risk-averse investors will hold some combination of the risk-free security and the market portfolio to achieve portfolios on the line segment between the risk-free security and the market portfolio.

Figure 47.2: The Efficient Frontier With the Risk-Free Security



Mean-Variance Framework Limitations

LO 47.b: Compare the normal distribution with the typical distribution of returns of risky financial assets such as equities.

In order to apply the mean-variance framework, certain assumptions must be made. First, the means, standard deviations, and correlations between investment returns are assumed to be consistent from the perspective of all investors. Second, there is an assumption that the mean and the standard deviation are all that matters in regard to portfolios. Finally, all investors are assumed to be able to borrow at the risk-free rate of interest. Each of these assumptions are challenging from a practical perspective.

Another limitation is that the use of the standard deviation as a risk measure is not appropriate for nonnormal distributions. If the shape of the underlying return density function is not symmetrical, then the standard deviation does not capture the appropriate probability of obtaining undesirable return outcomes.

As noted earlier, the standard normal distribution has a mean of zero and a standard deviation of one. The cumulative distribution for the standard normal distribution is estimated by calculating a *z*-score. The cumulative probability of the area under the standard normal distribution up to a specific value (*x*) can be found using the following formula:

$$z = \frac{x - \mu}{\sigma}$$

For example, if the mean is three and the standard deviation is six, to determine the cumulative probability that the value is less than two, the *z*-score is calculated as:

$$z = \frac{2 - 3}{6} = -0.1667$$

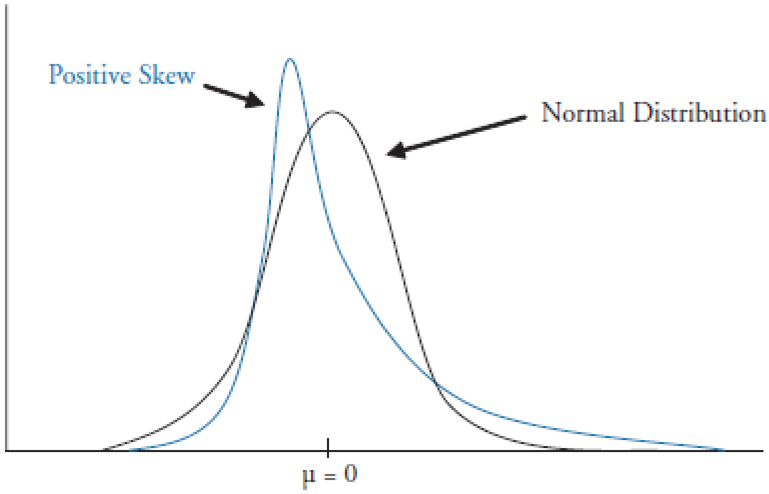
A *z*-score of –0.1667 equates to a probability of 0.4338, or 43.4%. This means that the cumulative probability that the value will be two or below is equal to 43.4%.

The assumption with financial variables is that they follow a normal distribution. However, for risky financial assets like equities, the distribution tends to have fatter tails and be more peaked than the normal distribution. This implies that extreme events are more likely in reality than what is predicted based on the normal distribution.

For example, in reviewing the returns of the S&P 500 over the 20-year period from 1998–2017, actual returns and standard deviations showed that small and large changes happen much more frequently than would be expected if returns followed a normal distribution.

Figure 47.3 illustrates two probability distribution functions. One probability distribution function is the normal distribution with a mean of zero. The other probability distribution is positively skewed. This positively skewed distribution has the same mean and standard deviation as the normal distribution. The degree of skewness alters the entire distribution. For the positively skewed distribution, outcomes below the mean are more likely to occur closer to the mean. Clearly normality is an important assumption when using the mean-variance framework. Thus, the mean-variance framework is unreliable when the assumption of normality is not met.

Figure 47.3: Normal Distribution vs. Positively Skewed Distribution



Value at Risk

LO 47.c: Define the VaR measure of risk, describe assumptions about return distributions and holding periods, and explain the limitations of VaR.

Value at risk (**VaR**) is interpreted as the worst possible loss under normal conditions over a specified period. Another way to define VaR is as an estimate of the maximum loss that can occur with a given confidence level. If an analyst says, “for a given month, the VaR is $1 million at a 95% level of confidence,” then this translates to mean “under normal conditions, in 95% of the months (19 out of 20 months), we expect the fund to either earn a profit or lose no more than $1 million.” Analysts may also use other standard confidence levels (e.g., 90% and 99%). **Delta-normal VaR** can be computed using the following expression: $[\mu - (z)(\sigma)] \times$ asset value.

EXAMPLE: **Calculating value at risk**

For a $100,000,000 portfolio, the expected 1-week portfolio return and standard deviation are 0.00188 and 0.0125, respectively. **Calculate** the 1-week VaR with a 95% confidence level.

Answer:

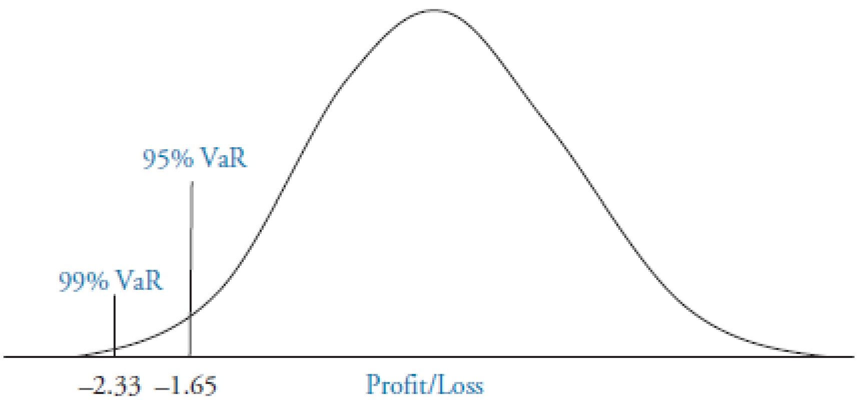
$$\begin{aligned}\text{VaR} &= (\mu - z\sigma) \times \text{portfolio value} \\ &= [0.00188 - 1.65(0.0125)] \times \$100{,}000{,}000 \\ &= -0.018745 \times \$100{,}000{,}000 \\ &= -\$1{,}874{,}500\end{aligned}$$

The manager can be 95% confident that the maximum 1-week loss will not exceed $1,874,500.

A major limitation of the VaR measure for risk is that two arbitrary parameters are used in the calculation—the confidence level and the holding period. The confidence level indicates the likelihood or probability that we will obtain a value greater than or equal to VaR. The holding period can be any predetermined time period measured in days, weeks, months, or years.

Figure 47.4 illustrates VaR parameters for a confidence level of 95% and 99%. As you can see, the level of risk is dependent on the degree of confidence chosen. VaR increases when the confidence level increases. In addition, VaR will increase at an increasing rate as the confidence level increases.

Figure 47.4: VaR Measurements for a Normal Distribution



The second arbitrary parameter is the holding period. VaR will increase with increases in the holding period. The rate at which VaR increases is determined in part by the mean of the distribution. If the return distribution has a mean, μ, equal to 0, then VaR rises with the square root of the holding period (i.e., the square root of time). If the return distribution has a $\mu > 0$, then VaR rises at a lower rate and will eventually decrease. Thus, the mean of the distribution is an important determinant for estimating how VaR changes with changes in the holding period.

VaR estimates are also subject to both model risk and implementation risk. Model risk is the risk of errors resulting from incorrect assumptions used in the model. Implementation risk is the risk of errors resulting from the implementation of the model.

Another major limitation of the VaR measure is that it does not tell the investor the amount or magnitude of the actual loss. VaR only provides the maximum value we can lose for a given confidence level. Two different return distributions may have the same VaR, but very different risk exposures. A practical example of how this can be a serious problem is when a portfolio manager is selling out-of-the-money options. For a majority of the time, the options will have a positive return and, therefore, the expected return is positive. However, in the unfavorable event that the options expire in the money, the resulting loss can be very large. Thus, different strategies focusing on lowering VaR can be very misleading since the magnitude of the loss is not calculated.

To summarize, VaR measurements work well with elliptical return distributions, such as the normal distribution. VaR is also able to calculate the risk for nonnormal distributions; however, VaR estimates may be unreliable in this case. Limitations in implementing the VaR model for determining risk result from the underlying return distribution, arbitrary confidence level, arbitrary holding period, and the inability to calculate the magnitude of losses. The measure of VaR also violates the coherent risk measure property of subadditivity when the return distribution is not elliptical. This property is further explained later in this reading.

MODULE QUIZ 47.1

1. The mean-variance framework is inappropriate for measuring risk when the underlying return distribution:
   A. is normal.
   B. is elliptical.
   C. has a kurtosis equal to three.
   D. is positively skewed.

2. Assume an investor is very risk-averse and is creating a portfolio based on the mean-variance model and the risk-free asset. The investor will most likely choose an investment on the:
   A. left-hand side of the efficient frontier.
   B. right-hand side of the efficient frontier.
   C. line segment connecting the risk-free rate to the market portfolio.
   D. line segment extending to the right of the market portfolio.

3. An investor has purchased an equity security which is part of the S&P 500 index. Relative to the normal distribution, she can reasonably expect the security's returns to:
   A. be less peaked.
   B. exhibit zero skewness.
   C. have no excess kurtosis.
   D. be more extreme in both directions.

MODULE 47.2: COHERENT RISK MEASURES AND EXPECTED SHORTFALL

Coherent Risk Measures

LO 47.e: Define the properties of a coherent risk measure and explain the meaning of each property.

In order to properly measure risk, one must first clearly define what is meant by a measure of risk. If we allow *R* to be a set of random events and ρ(R) to be the risk measure for the random events, then **coherent risk measures** should exhibit the following properties:

1. **Monotonicity:** a portfolio with greater future returns will likely have less risk: $R_1 \geq R_2$, then $\rho(R_1) \leq \rho(R_2)$
2. **Subadditivity:** the risk of a portfolio is at most equal to the risk of the assets within the portfolio: $\rho(R_1 + R_2) \leq \rho(R_1) + \rho(R_2)$
3. **Positive homogeneity:** the size of a portfolio, β, will impact the size of its risk: for all $\beta > 0$, $\rho(\beta R) = \beta\rho(R)$
4. **Translation invariance:** the risk of a portfolio is dependent on the assets within the portfolio: for all constants *c* (representing cash), $\rho(c + R) = \rho(R) - c$

The first, third, and fourth properties are more straightforward properties of well-behaved distributions. Monotonicity infers that if a random future value R_1 is always greater than a random future value R_2, then the risk of the return distribution for R_1 is less than the risk of the return distribution for R_2. Positive homogeneity suggests that the risk of a position is proportional to its size. Positive homogeneity should hold as long as the security is in a liquid market. Translation invariance implies that the addition of a sure amount reduces the risk at the same rate as the cash needed to make the position acceptable.

Subadditivity is the most important property for a coherent risk measure. The property of subadditivity states that a portfolio made up of subportfolios will have equal or less risk than the sum of the risks of each individual subportfolio. This assumes that when individual risks are combined, there may be some diversification benefits or, in the worst case, no diversification benefits and no greater risk. This implies grouping or adding risks does not increase the overall aggregate risk amount.

Expected Shortfall

LO 47.d: Explain and calculate ES and compare and contrast VaR and ES.

LO 47.f: Explain why VaR is not a coherent risk measure.

Value at risk is the minimum percent loss, equal to a pre-specified worst-case quantile return (typically the 5th percentile return). **Expected shortfall (ES)** is the expected loss given that the portfolio return already lies below the pre-specified worst-case quantile return (i.e., below the 5th percentile return). In other words, expected shortfall is the mean percent loss among the returns falling below the q-quantile. Expected shortfall is also known as **conditional VaR** or **expected tail loss (ETL)**, and by definition, must exceed VaR.

For example, assume an investor is interested in knowing the 5% VaR (the 5% VaR is equivalent to the 5th percentile return) for a fund. Further, assume the 5th percentile return for the fund equals –20%. Therefore, 5% of the time, the fund earns a return less than –20%. The value at risk is –20%. However, VaR does not provide good information regarding the expected size of the loss if the fund performs in the lower 5% of the possible outcomes. That question is answered by the expected shortfall amount, which is the expected value of all returns falling below the fifth percentile return (i.e., below –20%). Therefore, expected shortfall will equal a larger loss than the VaR.

For a normal distribution with a mean equal to μ and a standard deviation equal to σ, the following equation can be used to calculate the expected shortfall:

$$\text{ES} = \mu + \sigma \frac{e^{-(z^2/2)}}{(1-x)\sqrt{2\pi}}$$

In this equation, x equals the confidence level and z equals the point in the distribution that has a probability of being exceeded of x%.

EXAMPLE: Calculating expected shortfall

For a $100,000,000 portfolio the expected 1-week portfolio return is zero and the standard deviation is 0.0125. **Calculate** the 1-week expected shortfall in dollar terms with a 95% confidence level.

Answer:

$$
\begin{aligned}
\%\text{ES} &= \mu + \sigma \frac{e^{-(z^2/2)}}{(1-x)\sqrt{2\pi}} \\
&= 0.0125 \frac{e^{-(1.65^2/2)}}{(1-0.95)\sqrt{2 \times 3.1416}} \\
&= 0.0125 \frac{0.2563}{0.12533} = 0.02556 \\
\$\text{ES} &= 0.02556 \times \$100{,}000{,}000 = \$2{,}556{,}000
\end{aligned}
$$

Unlike VaR, ES has the ability to satisfy the property of subadditivity. Under this property, the risk of a portfolio should never exceed the combined risk of the assets within the portfolio. Assuming most assets have less than perfect positive correlation with each other, overall risk will decline when assets within a portfolio are combined. With VaR, the combined VaR may exceed the summation of the individual assets' VaRs; with expected shortfall, this is not the case.

The ES method provides an estimate of how large of a loss is expected if an unfavorable event occurs. VaR does not provide any estimate of the magnitude of losses, only the probability that they might occur. The property of subadditivity under the ES framework is also beneficial in eliminating another problem for VaR. When adjusting both the holding period and confidence level at the same time, an ES surface curve showing the interactions of both adjustments is convex. This implies that the ES method is more appropriate than the VaR method in solving portfolio optimization problems.

ES is similar to VaR in that both provide a common consistent risk measure across different positions. ES can be implemented in determining the probability of losses the same way that VaR is implemented as a risk measure, and they both appropriately account for correlations.

However, ES is a more appropriate risk measure than VaR for the following reasons:

- ES satisfies all of the properties of coherent risk measurements including subadditivity. VaR only satisfies these properties for normal distributions.
- The portfolio risk surface for ES is convex because the property of subadditivity is met. Thus, ES is more appropriate for solving portfolio optimization problems than the VaR method.
- ES gives an estimate of the magnitude of a loss for unfavorable events. VaR provides no estimate of how large a loss may be.
- ES has less restrictive assumptions regarding risk/return decision rules.

MODULE QUIZ 47.2

1. $\rho(X + Y) \leq \rho(X) + \rho(Y)$ is the mathematical equation for which property of a coherent risk measure?
   A. Monotonicity.
   B. Subadditivity.
   C. Positive homogeneity.
   D. Translation invariance.

2. Which of the following is not a reason that expected shortfall (ES) is a more appropriate risk measure than value at risk (VaR)?
   A. For normal distributions, only ES satisfies all the properties of coherent risk measurements.
   B. For nonelliptical distributions, the portfolio risk surface formed by holding period and confidence level is more convex for ES.
   C. ES gives an estimate of the magnitude of a loss.
   D. ES has less restrictive assumptions regarding risk/return decision rules than VaR.

KEY CONCEPTS

LO 47.a

The traditional mean-variance model estimates the amount of financial risk for portfolios in terms of the portfolio's expected return (mean) and risk (standard deviation or variance). A necessary assumption for this model is that return distributions for the portfolios are elliptical distributions.

The efficient frontier is the set of portfolios that dominate all other portfolios in the investment universe of risky assets with respect to risk and return. When a risk-free security is introduced, the optimal set of portfolios consists of a line from the risk-free security that is tangent to the efficient frontier at the market portfolio.

Assumptions needed in order to apply the mean-variance framework are (1) all investors assume the same means, standard deviations, and correlation coefficients for investments, (2) mean and standard deviation are all that matter, and (3) all investors can borrow at the risk-free rate. The mean-variance framework is unreliable when the underlying return distribution is not normal or elliptical. The standard deviation is not an accurate measure of risk and does not capture the probability of obtaining undesirable return outcomes when the underlying return density function is not symmetrical.

LO 47.b

The standard normal distribution has a mean of zero and a standard deviation of one. The *z*-score can be used to determine the cumulative probability of a return landing at or below a specific value. Returns on risky financial assets tend to exhibit a more peaked distribution with fatter tails, implying that the probability of more extreme events is higher than would be expected with the normal distribution.

LO 47.c

Value at risk (VaR) is a risk measurement that determines the probability of an occurrence in the left-hand tail of a return distribution at a given confidence level. VaR can be defined as $[\mu - (z)(\sigma)]$. The underlying return distribution, arbitrary choice of confidence levels and holding periods, and the inability to calculate the magnitude of losses result in limitations in implementing the VaR model when determining risk.

LO 47.d

Expected shortfall (ES) is a more accurate risk measure than VaR for the following reasons:

- ES satisfies all the properties of coherent risk measurements including subadditivity.
- The portfolio risk surface for ES is convex since the property of subadditivity is met. Thus, ES is more appropriate for solving portfolio optimization problems than the VaR method.
- ES gives an estimate of the magnitude of a loss for unfavorable events. VaR provides no estimate of how large a loss may be.
- ES has less restrictive assumptions regarding risk/return decision rules.

LO 47.e

The properties of a coherent risk measure are the following:

- Monotonicity: $Y \geq X \Rightarrow \rho(Y) \leq \rho(X)$; greater future returns imply less risk.
- Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$; portfolio risk ≤ sum of individual asset risks.
- Positive homogeneity: $\rho(\beta X) = \beta\rho(X)$ for $h > 0$; portfolio size impacts risk.
- Translation invariance: $\rho(c + X) = \rho(X) - c$; adding cash reduces risk by the amount of cash.

LO 47.f

Subadditivity, the most important property for a coherent risk measure, states that a portfolio made up of subportfolios will have equal or less risk than the sum of the risks of each individual subportfolio. VaR violates the property of subadditivity.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 47.1

1. **D** The mean-variance framework is only appropriate when the underlying distribution is elliptical. The normal distribution is a special case of elliptical distributions where skewness is equal to zero and kurtosis is equal to three. If there is any skewness, the distribution function will not be symmetrical, and standard deviation will not be an appropriate risk measure. (LO 47.a)

2. **C** Under the mean-variance framework, when a risk-free security is included in the analysis, the optimal set of portfolios lies on a straight line that runs from the risk-free security to the market portfolio. All investors will hold some portion of the risk-free security and the market portfolio. More risk-averse investors will hold some combination of the risk-free security and the market portfolio to achieve portfolios on the line segment between the risk-free security and the market portfolio. (LO 47.a)

3. **D** Returns on risky assets such as equities do not tend to follow the normal distribution. They tend to follow a distribution that is more peaked, has fatter tails, and will likely not align with the normal distribution's properties of zero skewness and zero excess kurtosis. The fatter tails indicate that more extreme returns (in both directions) are likely. (LO 47.b)

Module Quiz 47.2

1. **B** The property of subadditivity states that a portfolio made up of subportfolios will have equal or less risk than the sum of the risks of each individual subportfolio. (LO 47.e)
2. **A** VaR and ES both satisfy all the properties of coherent risk measures for normal distributions. However, only ES satisfies all the properties of coherent risk measures when the assumption of normality is not met. (LO 47.f)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 2.

READING 48

CALCULATING AND APPLYING VaR

Study Session 12

EXAM FOCUS

In this reading, risk measurement approaches are discussed for linear and nonlinear derivatives. Methods for calculating value at risk (VaR) and expected shortfall (ES) under the historical simulation approach, the delta-normal approach, and full revaluation approach are then discussed, including the advantages and disadvantages and underlying assumptions of the various approaches. Finally, structured Monte Carlo, stress testing, and worst-case scenario (WSC) analysis are presented as useful methods in extending VaR techniques to more appropriately measure risk for complex derivatives and scenarios.

MODULE 48.1: LINEAR AND NON-LINEAR DERIVATIVES

LO 48.a: Explain and provide examples of linear and non-linear portfolios.

A **linear derivative** reflects a relationship between an underlying factor and the derivative that is linear in nature. For example, an equity index futures contract is a linear derivative, while an option on the same index is nonlinear. The delta (rate of change) for a linear derivative must be constant for all levels of the underlying factor, but not necessarily equal to one.

A forward contract on an asset is a linear derivative because the forward contract is a linear function of the asset. The value of the forward contract at any time prior to expiry can be expressed as:

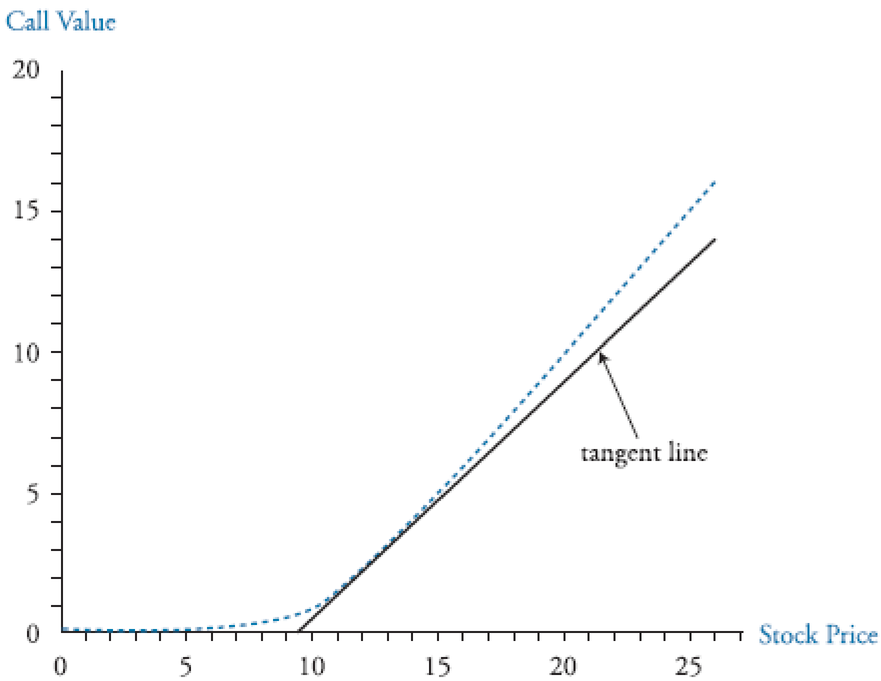
forward value = S – PV(K)

where *S* is the current asset price and *PV(K)* is the present value of the asset price at a future time *T*. This reflects a linear relationship.

The value of a **nonlinear derivative** is a function of the change in the value of the underlying asset and is dependent on the state of the underlying asset, and reflects a relationship between an underlying factor and the derivative that is not linear in nature. A call option is a good example of a nonlinear derivative. At expiry, the call option payoff is zero if the asset price is below the strike price, but if the asset price is above the strike price the payoff is *S* – *X*. Prior to expiry, the value of the call option does not increase (decrease) at a constant rate when the underlying asset increases (decreases) in value.

The change in the value of the call option is dependent in part on how far away the market value of the asset is from the exercise price. Figure 48.1 illustrates how the value of the call option on a stock does not change at a constant rate with the change in the value of the underlying stock. The curved line represents the actual change in value of the call option. The tangent line at any point on the curve illustrates how this is not a linear change in value.

Figure 48.1: Call Option Value Given Underlying Stock Price



LO 48.d: Describe and calculate VaR for linear derivatives.

In general, the **value at risk (VaR)** of a long position in a linear derivative is $VaR_p = \delta VaR_f$, where VaR_f is the VaR of the underlying factor and the derivative's delta, δ, is the sensitivity of the derivative's price to changes in the underlying factor. Delta is assumed to be positive here because we're modeling a long position. Delta can also be represented as:

$$\delta = \frac{\Delta P}{\Delta S}$$

where:

ΔP = change in portfolio

ΔS = change in risk factor

The local delta is defined as the percentage change in the derivative's price for a 1% change in the underlying asset. For small changes in the underlying price of the asset the change in price of the derivative can be extrapolated based on the local delta.

EXAMPLE: Futures contract VaR

Determine how a risk manager could estimate the VaR of an equity index futures contract. Assume a one-point increase in the index increases the value of a long position in the contract by $500.

Answer:

This relationship is shown mathematically as: $F_t = \$500S_t$, where F_t is the futures contract and S_t is the underlying index. The VaR of the futures contract is calculated as the amount of the index point movement in the underlying index, S_t, times the multiple, $500 as follows: $\text{VaR}(F_t) = \$500\text{VaR}(S_t)$.

Historical Simulation

LO 48.b: Describe and explain the historical simulation approach for computing VaR and ES.

The **historical simulation method** can be used to calculate both VaR and expected shortfall. It is a simple and straightforward method. Historical simulation starts with identifying risk factors that could include interest rates, exchange rates, stock prices, volatilities, credit spreads, or other factors. Typically, daily data is then collected on these factors and scenario analysis is used to simulate daily changes. These scenarios can then be applied to calculate daily gains or losses of a portfolio. Under the historical simulation approach, these returns would then be ordered from smallest (largest losses) to largest (largest gains). Suppose that simulated data for 300 days are ranked by loss amount as shown in Figure 48.2.

Figure 48.2: Simulated Loss Data

Scenario Number	Loss (In Canadian Dollar Millions)
53	12.8
225	11.4
101	11.0
5	9.5
189	7.8
…	…

To calculate the VaR over a one-day time horizon with a 99% confidence (1% significance), we would take the third worst outcome (3/300 = 0.01). The VaR is CAD 11.0 million, and the **expected shortfall (ES)** is the average of losses greater than the VaR level:

$$\text{ES} = \frac{12.8 + 11.4}{2} = 12.1, \text{ or CAD } 12.1 \text{ million}$$

The *advantage* of this approach is that it may identify a crisis event that was previously overlooked for a specific asset class. The focus is on identifying extreme changes in valuation. The *disadvantage* of the historical simulation approach is that it is limited to actual historical data.

Delta-Normal Approach

LO 48.c: Describe the delta-normal approach and calculate VaR for non-linear derivatives using delta-normal approach.

In contrast to the historical simulation method, the **delta-normal method** explicitly assumes a distribution for the underlying observations. The delta-normal method can be used for portfolios that are linearly dependent on the underlying market variables. Assuming the returns on these variables is multivariate normal, then portfolio value changes will be normally distributed. This makes VaR and ES calculations more intuitive.

Note that there are two categories of risk factors: (1) those where percentage changes are more practical (e.g., stock and commodity prices), and (2) those where actual changes are more practical (e.g., interest rates and credit spreads). With these risk factor types, we can generate an equation for portfolio value changes as follows:

$$\Delta P = \sum a_i x_i$$

where:

$x_i = \Delta S_i / S_i$ and $a_i = \delta_i S_i$ for risk factors where percentage changes are more practical

or

$x_i = \Delta S_i$ and $a_i = \delta_i$ for risk factors where actual changes are more practical

Under these assumptions, the portfolio risk (i.e., variance) can be rewritten as a variation of the well-known portfolio risk formula:

$$\sigma_P^2 = a_i^2 \sigma_i^2 + 2\sum_{i>j} a_i a_j \rho_{ij} \sigma_i \sigma_j$$

where σ_i is the mean and standard deviation of x_i, respectively, and ρ_{ij} is the coefficient of correlation between x_i and x_j.

Assuming the change in portfolio value is normal, the portfolio variance can be used to calculate VaR and ES:

$$\text{VaR} = \mu_P - z\sigma_P$$

$$\text{ES} = \mu_P + \sigma_P \frac{e^{-(z^2/2)}}{(1-x)\sqrt{2\pi}}$$

where *x* is the confidence level and *z* is the point on the normal distribution that has an *x* probability of being exceeded.

If each of the x_i is normal, the change in the value of a linear portfolio is normally distributed. Therefore, for risk factors like equity price, the percentage changes are normal, whereas for factors like interest rates, the actual changes are normal.

For portfolios with many nonnormal variables, the central limit theorem states that the portfolio can be approximately normal. It is also often assumed that the mean change in each risk factor is zero, in which case the VaR and ES calculations simplify by omitting μ.

Risk factors like equity or commodity prices are easy to monitor because they can be described by a single number. Risk factors such as credit spreads or interest rates are more difficult to monitor because they are usually described by a set of points (i.e., a term structure). If certain maturity spreads or interest rates are not observable in the market, **linear interpolation** can be used to determine the missing rates, by looking at the linear distance between two observable interest rates in the term structure. For example, if the six-month interest rate is 2.2% and the one-year rate is 2.8%, we can assume that the nine-month rate would be 2.5%.

LO 48.e: Describe the limitations of the delta-normal method.

While the delta-normal method works well for linear portfolios, it is only an approximation for nonlinear products (e.g., call option) or portfolios. Delta works well for small price changes but does not work well for large price changes. This is because delta is a linear measure. The **gamma** parameter helps adjust for the delta's linearity through curvature (i.e., nonlinearity). Therefore, delta works much better with in-the-money options than at-the-money options, because the gamma of *in-the-money* options is much lower.

The approximate portfolio value change with both delta and gamma can be estimated by the following equation:

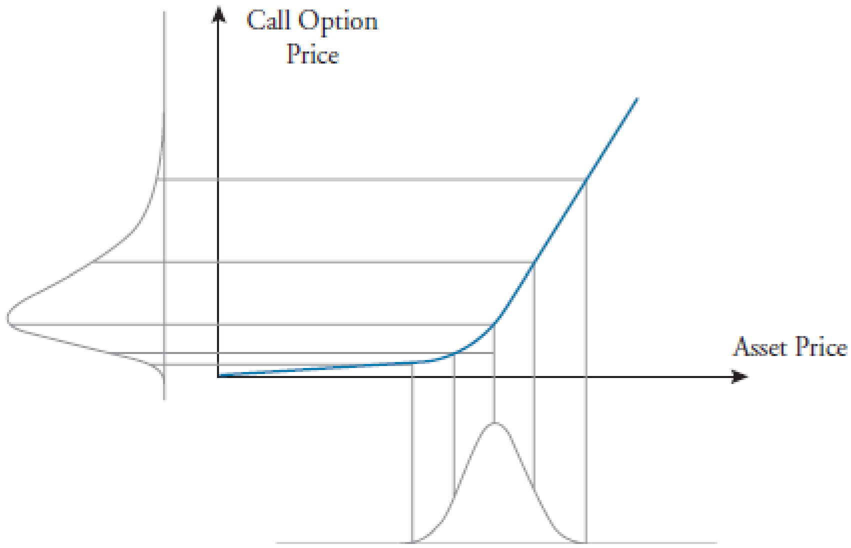
$$\Delta P = \delta \Delta S + \frac{1}{2}\gamma(\Delta S)^2$$

where:
γ = gamma of the risk factor

In addition to call options, nonlinear products include Asian options, barrier options and mortgage-backed securities (MBSs). Butterfly spreads, which involve three positions, are examples of highly nonlinear derivatives.

Contrary to linear products, a normal distribution of the underlying asset price translates into a skewed (nonnormal) distribution for nonlinear products. This is illustrated in Figure 48.3.

Figure 48.3: Skewed Distribution for Call Option Price



Therefore, the delta-normal method (which translates a normal distribution for the asset price into a normal distribution of the derivative) would understate the probability of high option values and would overstate the probability of low option values. This can create significant problems for portfolio managers because the delta-normal method does not consider the curvature (gamma) of a nonlinear portfolio.

MODULE QUIZ 48.1

1. A call option and a mortgage-backed security are good examples of:
   A. a linear and nonlinear derivative, respectively.
   B. a nonlinear and linear derivative, respectively.
   C. linear derivatives.
   D. nonlinear derivatives.

2. Which of the following statements regarding linear and nonlinear derivatives is true?
   A. The delta of a linear derivative is equal to one.
   B. A forward contract is an example of a nonlinear derivative.
   C. A linear derivative's delta must be constant for all levels of value for the underlying factor.
   D. The value of the call option changes at a constant rate with the change in the value of the underlying stock.

MODULE 48.2: MONTE CARLO, STRESS TESTING, AND SCENARIO ANALYSIS

Monte Carlo Simulation Method

LO 48.f: Explain the Monte Carlo simulation method for calculating VaR and ES and identify its strengths and weaknesses.

LO 48.g: Describe the implications of correlation breakdown for a VaR or ES analysis.

The **Monte Carlo approach** generates scenarios using random samples and simulates thousands of valuation outcomes for the underlying assets. The VaR and ES for the portfolio of derivatives is then calculated from the simulated outcomes.

The Monte Carlo approach involves six steps:

Step 1: Using current values of risk factors, value the portfolio today.

Step 2: Apply sampling techniques from the multivariate normal probability distribution for the change in *x* (Δx_i).

Step 3: Using the sampled values of Δx_i, determine the values of the risk factors at the end of the period.

Step 4: Revalue the portfolio using the updated risk factor values.

Step 5: Subtract the revalued portfolio value from the current value. This will determine the amount of loss.

Step 6: Repeat Steps 2 through 5 to create a loss distribution.

Once this process is complete, we can calculate daily VaR and expected loss, using a similar approach as historical simulation. For example, if Monte Carlo produces 500 trials, the daily VaR with a 99% confidence level will be the fifth worst loss (= 1% × 500), and the expected loss will be the average of the four worst losses. Remember, to calculate longer time period VaR and expected losses, the daily values will be multiplied by the square root of time:

$$\text{VaR}(T, X) = \text{VaR}(1, X) \times \sqrt{T}$$
$$\text{ES}(T, X) = \text{ES}(1, X) \times \sqrt{T}$$

where:
T = *T*-day time horizon

An *advantage* of the Monte Carlo approach is that it is able to address multiple risk factors by assuming an underlying distribution and modeling the correlations among the risk factors. For example, a risk manager can simulate 10,000 outcomes and then determine the probability of a specific event occurring. In order to run the simulations, the risk manager just needs to provide parameters for the mean and standard deviation and assume all possible outcomes are normally distributed. Another significant advantage is that Monte Carlo simulation can assume any distribution type as long as correlations between the risk factors can be determined.

The primary *disadvantage* of Monte Carlo simulation are that the process is slow and computationally intensive. The Monte Carlo approach is typically used for large portfolios which is time consuming.

Stress Testing Methods

In times of crisis, correlations increase (some substantially) and strategies that rely on low correlations fall apart and underestimate risk. Certain economic or crisis events can cause diversification benefits to deteriorate in times when the benefits are most needed. A **contagion effect** often occurs when volatility and correlations both increase, thus mitigating any diversification benefits. Stressing correlation, VaR, and ES is a method used to model the contagion effect that could occur in a crisis event.

One approach for stress testing is to examine historical crisis events, for example stress testing a bank for the 2007–2009 financial crisis. After the crisis is identified, the impact on the current portfolio is determined. The *advantage* of this approach is that no assumptions of underlying asset returns or normality are needed. The biggest *disadvantage* of using historical events for stress testing is that it is limited to only evaluating events that have actually occurred.

An alternative approach is to analyze different predetermined **stressed scenarios**. For example, a financial institution could evaluate a 3% increase in short-term rates or a 4% depreciation of the domestic currency relative to a foreign currency. As in the previous method, the next step is then to evaluate the effect of the stress scenarios on the current portfolio. The main advantage of this approach is that it is not limited to the evaluation of risks that have occurred historically. It can be used to address any possible scenarios.

Worst-Case Scenario Analysis

LO 48.h: Describe worst-case scenario analysis and compare it to VaR.

The **worst-case scenario (WCS)** focuses on the distribution of worst possible outcomes given an unfavorable event. An expected loss is then determined from this worst-case distribution analysis. Thus, the WCS information extends the VaR analysis by estimating the extent of the loss given an unfavorable event occurs. For example, an investor may be concerned about the worst possible daily outcome over a six-month period, and would look at the distribution of returns during the six-month period. While useful, WCS should not be viewed as an alternative to VaR and ES calculations.

MODULE QUIZ 48.2

1. Which of the following statements regarding stress scenarios is incorrect? A contagion effect:
   A. results from a crisis event.
   B. increases diversification benefits.
   C. occurs with a rise in both volatility and correlation.
   D. causes a different return generating process in the underlying asset.

KEY CONCEPTS

LO 48.a

A derivative is described as linear when the relationship between an underlying factor and the derivative's value is linear in nature (e.g., a forward currency contract). A nonlinear derivative's value is a function of the change in the value of the underlying asset and is dependent on the state of the underlying asset (e.g., a call option).

LO 48.b

Historical simulation involves identifying risk factors, collecting data on these factors, and using scenario analysis to simulate daily portfolio changes to calculate daily gains or losses of a portfolio. The returns are then ranked to calculate the VaR. The expected shortfall is the average of the losses greater than the VaR level.

LO 48.c

The delta-normal method can be used for portfolios that are linearly dependent on the underlying market variables. Assuming the returns on these variables is multivariate normal, then portfolio value changes will be normally distributed. Assuming the change in portfolio value is normal, VaR and ES can be calculated as:

$$\text{VaR} = \mu_P - z\,\sigma_P$$

$$\text{ES} = \mu_P + \sigma_P \frac{e^{-(z^2/2)}}{(1-x)\sqrt{2\pi}}$$

The central limit theorem states that the portfolio can be approximately normal with many nonnormal variables.

LO 48.d

In general, the value at risk (VaR) of a linear derivative is $\text{VaR}_p = \delta \text{VaR}_f$, where the derivative's local delta is the sensitivity of the derivative's price to a 1% change in the underlying asset's value.

LO 48.e

The delta-normal method is only an approximation for nonlinear products, because delta is a linear parameter that works well only for small price changes but does not work well for large price changes. The gamma parameter helps adjust for the delta's linearity through nonlinearity.

Therefore, the delta-normal method understates the probability of high option values and overstates the probability of low option values, creating problems for portfolio managers.

LO 48.f

The Monte Carlo approach simulates thousands of possible movements in the underlying asset and then uses those outcomes to estimate the VaR for a portfolio of derivatives.

An advantage of the Monte Carlo approach is that it can assume any distribution type, and is able to address multiple risk factors by generating correlated scenarios based on a statistical distribution. The main disadvantages are that the process is slow and computationally intensive.

LO 48.g

Crisis events cause diversification benefits to deteriorate due to a contagion effect that occurs when a rise in volatility and correlation result in a different return generating process for the underlying asset. This creates problems when using simulations for traditional scenario.

LO 48.h

The worst-case scenario (WCS) extends VaR risk measurement by estimating the extent of the loss given the worst result within a specific timeframe.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 48.1

1. **D** A nonlinear derivative's value is a function of the change in the value of the underlying asset and is dependent on the state of the underlying asset. (LO 48.a)

2. **C** The delta, or slope, of a linear derivative must be constant (the delta of a nonlinear derivative changes for different levels of the underlying factor). The delta does not necessarily equal to one. A forward contract is an example of a linear derivative. The value of the call option does *not* change at a constant rate with the change in the value of the underlying stock. (LO 48.a)

Module Quiz 48.2

1. **B** A contagion effect occurs with a rise in volatility and correlation causing a different return generating process. A specific example of events leading to the breakdown of historical correlation matrices causing a contagion effect is the 2007–2009 global financial crisis. A contagion effect often occurs when volatility and correlations both increase, thus mitigating any diversification benefits. (LO 48.h)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 3.

READING 49

MEASURING AND MONITORING VOLATILITY

Study Session 12

EXAM FOCUS

The accurate estimation of volatility is crucial to understanding potential risk exposure. Asset value can be evaluating using a normal distribution; however, deviations from normality will create challenges for the risk manager in measuring both volatility and value at risk (VaR). In this reading, we will discuss issues with volatility estimation and different weighting methods that can be used to determine VaR. The advantages, disadvantages, and underlying assumptions of the various methodologies will also be discussed. For the exam, understand why deviations from normality occur and have a general understanding of the approaches to measuring VaR (parametric and nonparametric). Also, be able to estimate volatility using both the exponentially weighted moving average (EWMA) and the generalized autoregressive conditional heteroskedasticity [GARCH (1,1)] models, and discuss the mean-reverting characteristic of volatility.

MODULE 49.1: RETURN DISTRIBUTIONS AND MARKET REGIMES

Value at Risk

Value at risk (VaR) is a probabilistic method of measuring the potential loss in portfolio value over a given time period and for a given distribution of historical returns. VaR is the dollar or percentage loss in portfolio (asset) value that will be equaled or exceeded only *x*% of the time. In other words, there is an *x*% probability that the loss in portfolio value will be equal to or greater than the VaR measure. VaR can be calculated for any percentage probability of loss and over any time period. A 1%, 5%, and 10% VaR would be denoted as VaR(1%), VaR(5%), and VaR(10%), respectively. The risk manager selects the *x*% probability of interest and the time period over which VaR will be measured. Generally, the time period selected (and the one we will use) is one day.

A brief example will help solidify the VaR concept. Assume a risk manager calculates the daily 5% VaR as $10,000. The VaR(5%) of $10,000 indicates that there is a 5% chance that on any given day, the portfolio will experience a loss of $10,000 or more.

We could also say that there is a 95% chance that on any given day the portfolio will experience either a loss less than $10,000 or a gain. If we further assume that the $10,000 loss represents 8% of the portfolio value, then on any given day there is a 5% chance that the portfolio will experience a loss of 8% or greater, but there is a 95% chance that the loss will be less than 8% or a percentage gain greater than zero.

Deviations From the Normal Distribution

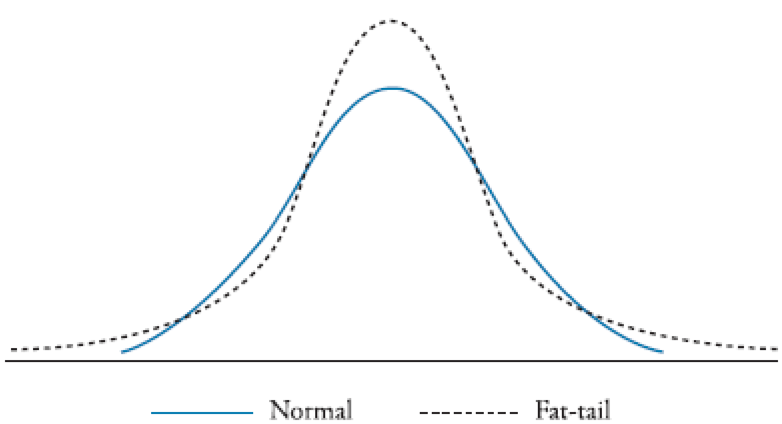
LO 49.a: Explain how asset return distributions tend to deviate from the normal distribution.

LO 49.b: Explain reasons for fat tails in a return distribution and describe their implications.

Three common deviations from normality that are problematic in modeling risk result from asset return distributions that are (1) fat-tailed, (2) skewed (nonsymmetrical), or (3) unstable.

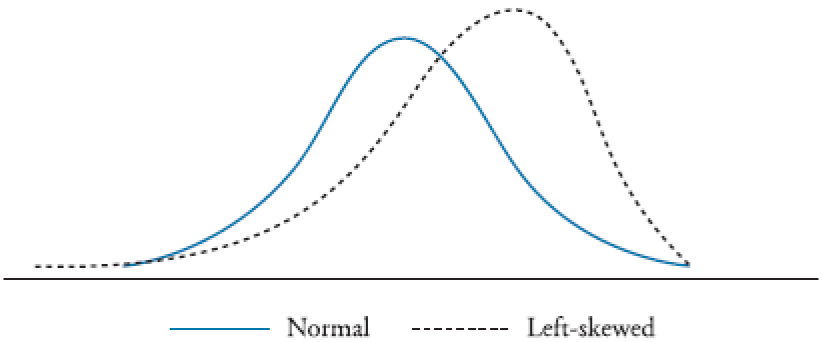
Fat-tailed return distributions refers to a distribution with a higher probability of observations occurring in the tails relative to the normal distribution. As illustrated in Figure 49.1, there is a larger probability of an observation occurring further away from the mean of the distribution. The first two moments (mean and variance) of the distributions are similar for the fat-tailed and normal distribution. However, in addition to the greater mass in the tails, in order to keep the standard deviation equal to that of the normal distribution, there must also be greater probability mass around the center for the fat-tailed distribution. Furthermore, there is less probability mass in the intermediate range (around +/– one standard deviation) for the fat-tailed distribution compared to the normal distribution.

Figure 49.1: Illustration of Fat-Tailed and Normal Distributions



A **nonsymmetrical distribution** is skewed and there is a higher probability of either large negative or large positive returns. Combining two distributions with different means and standard deviations can produce skewed distributions. If the resulting distribution has more probability in the left tail than right tail, it is referred to as left skewed. This is illustrated in Figure 49.2.

Figure 49.2: Left-Skewed and Normal Distributions



The return distributions are *unstable* if the parameters are not constant but vary over time. In modeling this risk, a number of assumptions are necessary. For example, if expected equity returns, interest rates, and inflation are changing over time, this will affect the volatility of the returns through time.

Conditional and Unconditional Distributions

LO 49.c: Differentiate between conditional and unconditional distributions and describe regime switching.

The phenomenon of "fat tails" is most likely the result of the volatility and/or the mean of the distribution changing over time. If the volatility changes in an unpredictable way, we refer to it as **stochastic volatility** where we need to distinguish between unconditionally normal and conditionally normal distributions.

The distribution of returns is an **unconditional distribution** if the mean and standard deviation are the same for asset returns for any given day. The distribution of returns is a **conditional distribution** if different market or economic conditions may cause the mean and variance of the return distribution to change over time.

Assume we separate the full data sample into two normally distributed subsets based on market environment with *conditional* means and variances. Pulling a data sample at different points in time from the full sample could generate fat tails in the unconditional distribution even if the conditional distributions are normally distributed with similar means but different volatilities. Consider that we observe historical data with a standard deviation that averaged 2% over time, but we currently estimate volatility at 3%. It would produce more accurate VaR and expected shortfall results if we assumed a normal distribution with a 3% standard deviation, than if we assumed fat-tailed distributions from historical data or if we assumed a normal distribution but with a 2% standard deviation.

Regime Switching

In modeling data, it is often assumed that volatilities change slowly. This means that high volatility is often followed by high volatility periods and low volatility by low volatility periods. However, volatilities may also change quickly and abruptly. When this happens, it is referred to as **regime switching**. For example, an unexpected central bank or government announcement may cause market volatility to spike immediately, then sharply decline once markets absorb the news.

MODULE QUIZ 49.1

1. Fat-tailed asset return distributions are most likely the result of time-varying:
   A. volatility for the unconditional distribution.
   B. means for the unconditional distribution.
   C. volatility for the conditional distribution.
   D. means for the conditional distribution.

2. The problem of fat tails when measuring volatility is least likely in a(n):
   A. unstable distribution.
   B. skewed distribution.
   C. regime-switching model.
   D. unconditional distribution.

MODULE 49.2: ESTIMATING VALUE AT RISK

LO 49.d: Compare and contrast different approaches for estimating conditional volatility.

A value at risk (VaR) method for estimating risk is typically either a historical-based approach or an implied-volatility-based approach. Under the historical-based approach, the shape of the conditional distribution is estimated based on historical time series data.

Parametric vs. Nonparametric VaR Methods

Historical-based approaches typically fall into two subcategories: parametric and nonparametric.

1. The **parametric approach** requires specific assumptions regarding the asset returns distribution. A parametric model typically assumes asset returns are normally or lognormally distributed with time-varying volatility. The most common example of the parametric method in estimating future volatility is based on calculating historical variance or standard deviation using "mean squared deviation." For example, the following equation is used to estimate future variance based on a window of the *K* most recent returns data.[1]

   $$\sigma_t^2 = (r_{t-K,t-K+1}^2 + \ldots + r_{t-3,t-2}^2 + r_{t-2,t-1}^2 + r_{t-1,t}^2)/K$$

   If we assume asset returns follow a random walk, the mean return is zero. Alternatively, an analyst may assume a conditional mean different from zero and a volatility for a specific period of time.

PROFESSOR'S NOTE

The delta-normal method (shown in previous reading) is an example of a parametric approach.

EXAMPLE: Estimating a conditional mean

Assuming K = 100 (an estimation window using the most recent 100 asset returns), **estimate** a conditional mean assuming the market is known to decline 15%.

Answer:

The daily conditional mean asset return, μ_t, is estimated to be –15 bps/day:

μ_t = –1500 bps/100 days = –15 bps/day

2. The **nonparametric approach** is less restrictive in that there are no underlying assumptions of the asset returns distribution. Common nonparametric approaches model volatility using (1) the historical simulation method and (2) the multivariate density estimation method.

Historical Simulation Method

The six lowest returns for an estimation window of 100 days (K = 100) are listed in Figure 49.3. Under the **historical simulation method**, all returns are weighted equally based on the number of observations in the estimation window (1/K). Thus, in this example, each return has a weight of 1/100, or 0.01.

EXAMPLE: Calculating VaR using historical simulation

Estimate VaR of the fifth percentile using historical simulation and the data provided in Figure 49.3.

Figure 49.3: Historical Simulation Example

Six Lowest Returns	Historical Simulation Weight	Historical Simulation Cumulative Weight
–4.70%	0.01	0.0100
–4.10%	0.01	0.0200
–3.70%	0.01	0.0300
–3.60%	0.01	0.0400
–3.40%	0.01	0.0500
–3.20%	0.01	0.0600

Answer:

Estimating VaR of 5% requires identifying the 5th percentile. With 100 observations, the 5th percentile would be the 5th lowest return. In this case, the 5th lowest return corresponds to –3.40%.

Nonparametric vs. Parametric VaR Methods

Advantages of nonparametric methods compared to parametric methods:

- Nonparametric models do not require assumptions regarding the entire distribution of returns to estimate VaR.
- Fat tails, skewness, and other deviations from some assumed distribution are no longer a concern in the estimation process for nonparametric methods.

Disadvantages of nonparametric methods compared to parametric methods:

- Data is used more efficiently with parametric methods than nonparametric methods. Therefore, large sample sizes are required to precisely estimate volatility using historical simulation.
- Separating the full sample of data into different market regimes reduces the amount of usable data for historical simulations.

MODULE QUIZ 49.2

1. Which of the following statements is an advantage of nonparametric methods compared to parametric methods?
   A. Data is used more efficiently with nonparametric methods.
   B. Nonparametric methods are not concerned with fat tails or skewness during the estimation process.
   C. Nonparametric models require assumptions regarding the entire distribution of returns to estimate VaR.
   D. Separating data into different market regimes increases the amount of usable data for nonparametric methods.

MODULE 49.3: ESTIMATING VOLATILITY AND MEAN REVERSION

LO 49.e: Apply the exponentially weighted moving average (EWMA) approach to estimate volatility, and describe alternative approaches to weighting historical return data.

The volatility of a variable, σ, is represented as the standard deviation of that variable's continuously compounded return. With option pricing, volatility is typically expressed as the standard deviation of return over a one-year period. This differs from risk management, where volatility is typically expressed as the standard deviation of return over a one-day period.

The traditional measure of volatility first requires a measure of change in asset value from period to period, which is calculated as follows:

$$r_i = \frac{S_i - S_{i-1}}{S_{i-1}}$$

From a risk management perspective, the daily volatility of an asset usually refers to the standard deviation of the daily proportional change in asset value.

By collecting continuously compounded return data, r_i, over a number of days, we can compute the mean return of the individual returns as follows:

$$\bar{r} = \frac{1}{m}\sum_{i=1}^{m} r_{n-i}$$

where:
m = number of observations leading up to the present period

If we assume that the mean return is zero, which would be true when the mean is small compared to the variability, we obtain the maximum likelihood estimator of variance:

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} r_{n-i}^2$$

In simplest terms, historical data is used to generate returns in an asset-pricing series. This historical return information is then used to generate a volatility parameter, which can be used to infer expected realizations of risk. However, these straightforward approaches weight each observation equally in that more distant past returns have the same influence on estimated volatility as observations that are more recent. If the goal is to estimate the current level of volatility, we may want to weight recent data more heavily. There are various weighting schemes, which can all essentially be represented as:

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i r_{n-i}^2$$

where:
α_i = weight on the return *i* days ago

The weights (the αs) must sum to one, and if the objective is to generate a greater influence on recent observations, then the αs will decline in value for older observations.

An alternative approach for estimating volatility is to use average historical returns, rather than squared returns. The advantage of this approach is that it provides a better forecast for nonnormal distribution with fat tails.

EWMA Model

The **exponentially weighted moving average (EWMA) model** is a specific case of the general weighting model. The main difference is that the weights are assumed to decline exponentially back through time. The estimated volatility on day *n* is derived by applying the weights to past squared returns, where data from longer ago carries lower weight. Using the EWMA approach, conditional variance is estimated using the following formula:

$$\sigma_n^2 = w_0 r_{n-1}^2 + w_0 \lambda r_{n-2}^2 + w_0 \lambda^2 r_{n-3}^3 + \ldots$$

where:
n = number of observations used to estimate volatility

This model is then simplified to just two periods of data (*n*−*1* and *n*−*2*). In addition, we can substitute w_0 = (1 − λ) because the two weights must sum to one. This assumption results in a specific relationship for variance:

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)r_{n-1}^2$$

where:

λ = weight on previous volatility estimate (λ is a positive constant between zero and one)

This equation is also known as an **adaptive volatility estimation** because it updates prior beliefs about volatility with new data. This model of EWMA overcomes two problems: (1) if *n* is very large, but volatility is cyclical, the model may over- or underestimate volatility; (2) very large negative or positive returns from longer ago could unduly impact the model.

EXAMPLE: **EWMA model**

The decay factor in an exponentially weighted moving average model is estimated to be 0.94 for daily data. Daily volatility is estimated to be 1%, and today's stock market return is 2%. **Calculate** the new estimate of volatility using the EWMA model.

Answer:

$$\sigma_n^2 = 0.94 \times 0.01^2 + (1-0.94) \times 0.02^2 = 0.000118$$

$$\sigma_n = \sqrt{0.000118} = 1.086\%$$

The EWMA model was used by **RiskMetrics**, formerly a division of JPMorgan. The RiskMetrics approach is just an EWMA model that uses a prespecified decay factor for daily data (0.94) and monthly data (0.97). The simplest interpretation of the EWMA model is that the day-*n* volatility estimate is calculated as a function of the volatility calculated as of day *n*–*1* and the most recent squared return. Depending on the weighting term λ, which ranges between zero and one, the previous volatility and most recent squared returns will have differential impacts. High values of λ will minimize the effect of daily percentage returns, whereas low values of λ will tend to increase the effect of daily percentage returns on the current volatility estimate.

One benefit of the EWMA is that it requires few data points. Specifically, all we need for calculating the variance is the current estimate of the variance and the most recent squared return on day *n*–*1*. The current estimate of variance will then feed into the next period's estimate, as will this period's squared return. Technically, the only "new" piece of information for the volatility calculation will be that attributed to the squared return.

GARCH (1,1) Model

LO 49.f: Apply the GARCH (1,1) model to estimate volatility.

One of the most popular methods of estimating volatility is the **GARCH (1,1) model** where GARCH stands for generalized autoregressive conditional heteroskedasticity. This is a time-series model used by analysts to predict time-varying volatility. A GARCH (1,1) model not only incorporates the most recent estimates of variance and squared return, but also a variable that accounts for a long-run average level of variance. The (1,1) in the GARCH (1,1) model refers to the weight given to one squared return (the most recently observed) and one variance rate (most recent estimate). The best way to describe a GARCH (1,1) model is to take a look at the formula representing its determination of variance, which can be shown as:

$$\sigma_n^2 = \omega + \alpha r_{n-1}^2 + \beta \sigma_{n-1}^2$$

where:

α = weighting on the previous period's return

β = weighting on the previous variance estimate

ω = weighted long-run variance = γV_L

V_L = long-run average variance = $\dfrac{\omega}{1 - \alpha - \beta}$

$\alpha + \beta + \gamma = 1$

$\alpha + \beta < 1$ for stability so that γ is not negative

The EWMA is nothing other than a special case of a GARCH (1,1) volatility process, with $\omega = 0$, $\alpha = 1 - \lambda$, and $\beta = \lambda$. Similar to the EWMA model, β represents the exponential decay rate of information. The GARCH (1,1) model adds to the information generated by the EWMA model in that it also assigns a weighting to the average long-run variance estimate. An additional characteristic of a GARCH (1,1) estimate is the implicit assumption that variance tends to revert to a long-term average level. Recognition of a mean-reverting characteristic in volatility is an important feature when pricing derivatives securities such as options.

EXAMPLE: GARCH (1,1) model

The parameters of a generalized autoregressive conditional heteroskedastic [GARCH (1,1)] model are $\omega = 0.000003$, $\alpha = 0.04$, and $\beta = 0.92$. If daily volatility is estimated to be 1%, and today's stock market return is 2%, **calculate** the new estimate of volatility using the GARCH (1,1) model, and the implied long-run volatility level.

Answer:

$$\sigma_n^2 = 0.000003 + 0.04 \times 0.02^2 + 0.92 \times 0.01^2 = 0.000111$$

$$\sigma_n = \sqrt{0.000111} = 1.054\%$$

$$\text{long-run average variance} = \frac{\omega}{(1 - \alpha - \beta)} = \frac{0.000003}{(1 - 0.04 - 0.92)} = 0.000075$$

$$\bar{\sigma} = \sqrt{0.000075} = 0.866\% = \text{long-run volatility}$$

One of the useful features of GARCH models is that they do a very good job at modeling volatility clustering when periods of high volatility tend to be followed by other periods of high volatility, and periods of low volatility tend to be followed by subsequent periods of low volatility. Thus, there is autocorrelation in r_i^2. If GARCH models do a good job of explaining volatility changes, there should be very little autocorrelation in r_i^2 / σ_i^2. GARCH models appear to do a very good job of explaining volatility.

Alternative Weighting Approaches

Under **multivariate density estimation (MDE)**, an analysis is first done to see which past periods correspond to the current period, and then a weight is assigned to the historical data depending on how similar it is to the current data. For example, the volatility in interest rates tends to vary with the level of interest rates. Under the MDE method, more weight is given to the level of interest rates more similar to the current level of interest rates.

Advantages of the MDE method include:

- MDE allows for weights to vary based on how relevant the data is to the current market environment, regardless of the timing of the most relevant data.
- MDE is very flexible in introducing dependence on economic variables (called *state variables* or *conditioning variables*).

Disadvantages of the MDE method include:

- MDE may lead to data snooping or overfitting in identifying required assumptions regarding the weighting scheme identification of relevant conditioning variables and the number of observations used to estimate volatility.
- MDE requires a large amount of data that is directly related to the number of conditioning variables used in the model.

Mean Reversion and Long Time Horizons

LO 49.g: Explain and apply approaches to estimate long horizon volatility/VaR and describe the process of mean reversion according to a GARCH (1,1) model.

By assuming daily returns are independent with the same level of variation, daily volatility can be extended over a longer period of time, *T*, by multiplying the standard deviation of return by the square root of *T* to calculate long horizon volatility. Similarly, VaR can be extended to a longer-term basis by multiplying VaR by the square root of the number of days (i.e., the square root rule). For example, to convert daily VaR to weekly VaR, multiply the daily VaR by the square root of five. For example, if the daily volatility is 1.5%, the standard deviation of the return (compounded continuously) over a 10-day period would be computed as $1.5\% \times \sqrt{10} = 4.74\%$. When converting daily volatility to annual volatility the usual practice is to use the square root of 252 days, which is the number of business days in a year, as opposed to the number of calendar days in a year.

However, empirical data indicates that volatility exhibits a mean-reverting characteristic. This means that if current volatility is high, we expect it to decline; if it is low, we expect it to increase. This indicates that if we expect volatility to decline, we will overestimate volatility if we multiply the standard deviation of the return by the square root time. The term V_L in GARCH (1,1) provides a pull back toward the long-term average mean. EWMA does not provide this pull.

For example, assume the calculated current daily variance is 0.000225, which corresponds to a current daily standard deviation of 1.5%. Using the square root of time rule, the 30-day volatility would be:

$$\sqrt{30} \times 0.015 = 0.0822, \text{ or } 8.22\%$$

However, assume that as a result of mean reversion we expect daily average variance over the next 30 days to be higher at 0.000324 (standard deviation of 1.8%). In this case, we should base VaR and expected shortfall estimates for volatility over a 30-day time horizon using the higher volatility estimate. This would result in a volatility of 9.9%:

$$\sqrt{30} \times 0.018 = 0.0986, \text{ or } 9.9\%$$

In this case, by applying mean reversion, the long horizon conditional volatility estimate is higher.

Implied Volatility

LO 49.h: Evaluate implied volatility as a predictor of future volatility and its shortcomings.

Estimating future volatility using historical data requires time to adjust to current changes in the market. An alternative method for estimating future volatility is **implied volatility**. Whereas volatilities calculated from historical data (including EWMA or GARCH) are backward looking, implied volatility calculated from options prices is forward looking. Option prices are dependent on volatilities, and as volatilities increase so do options prices. As a result, volatilities are implied from options prices.

Implied volatilities of options are usually expressed as annual volatilities. Less than annual volatilities would have to be adjusted by the square root of time. For example, if annual volatility is 18%, the daily volatility would be $18\% / \sqrt{252} = 1.13\%$ (assuming 252 trading days in a year).

The most widely used index for publishing implied volatility is the Chicago Board Options Exchange (CBOE) Volatility Index (ticker symbol: VIX). The VIX demonstrates implied volatility on a wide variety of 30-day calls and puts on the S&P 500 Index. Note that trading in futures and options on the VIX is a bet on volatility only. Since its inception, the VIX has mainly traded between 10 and 20 (which corresponds to volatility of 10%–20% on the S&P 500 Index options), but it reached a peak of close to 80 in October 2008, after the collapse of Lehman Brothers. The VIX is often referred to as the fear index by market participants because it reflects current market uncertainties.

Updating Correlation Estimates

LO 49.i: Describe an example of updating correlation estimates.

Managers need to monitor correlations in addition to volatilities. Similar to estimating volatility from variances, we can estimate correlations from covariances. We can establish a general covariance formula using the EWMA model between return X and return Y:

$$\text{cov}_n = \lambda\text{cov}_{n-1} + (1-\lambda)X_{n-1}Y_{n-1}$$

where:

cov_n = covariance estimate for day n

X_n = value of X on day n

Y_n = value of Y on day n

Recall that correlation between two assets is their covariance divided by the product of their standard deviations. Alternatively, covariance between two assets is their correlation multiplied by the product of their standard deviations:

$$\text{cov}_{X,Y} = (\text{corr}_{X,Y})(\sigma_X)(\sigma_Y)$$

EXAMPLE: Calculating correlation

Suppose an analyst is looking to estimate the updated correlation between two asset returns. The analyst observes on day *n–1* that return X is 2% and Y is 4%, and the correlation between X and Y is 0.3. The volatility of return X and Y is 1% and 2%, respectively. The analyst estimates a value for λ of 0.92. **Calculate** the new coefficient of correlation assuming the updated volatilities of X and Y will be 1.11% and 2.23%, respectively.

Answer:

The covariance on day *n–1* can be calculated as:

$$\text{cov}_{n-1} = 0.3 \times 0.01 \times 0.02 = 0.00006$$

For day *n*, the covariance is updated as follows:

$$\begin{aligned}\text{cov}_n &= \lambda\text{cov}_{n-1} + (1-\lambda)X_{n-1}Y_{n-1}\\ &= 0.92 \times 0.00006 + 0.08 \times 0.02 \times 0.04\\ &= 0.0001192\end{aligned}$$

Assuming the same λ of 0.92, the volatilities of X and Y are updated to 1.11% and 2.23%, respectively. We can now calculate the new coefficient of correlation:

$$\text{corr}_{x,y} = \frac{0.0001192}{0.0111 \times 0.0223} = 0.48$$

MODULE QUIZ 49.3

1. The parameters of a GARCH (1,1) model are ω = 0.00003, α = 0.04, and β = 0.92. If daily volatility is estimated to be 1.5%, and today's stock market return is 0.8%, what is the new estimate of the standard deviation?
   A. 1.68%.
   B. 1.55%.
   C. 1.45%.
   D. 2.74%.

2. The λ of an exponentially weighted moving average (EWMA) model is estimated to be 0.9. Daily standard deviation is estimated to be 1.5%, and today's stock market return is 0.8%. What is the new estimate of the standard deviation?
   A. 1.68%.
   B. 1.55%.
   C. 1.45%.

D. 2.74%.

3. The parameters of a GARCH (1,1) model are $\omega = 0.00003$, $\alpha = 0.04$, and $\beta = 0.92$. These figures imply a long-run daily standard deviation of:
   A. 1.68%.
   B. 1.55%.
   C. 1.45%.
   D. 2.74%.

4. GARCH (1,1) models can only be used to estimate volatility in the case where:
   A. $\alpha + \beta > 0$.
   B. $\alpha + \beta < 1$.
   C. $\alpha > \beta$.
   D. $\alpha < \beta$.

KEY CONCEPTS

LO 49.a

Three common deviations from normality that are problematic in modeling risk result from asset returns that are fat tailed, skewed, or unstable. Fat tailed refers to a distribution with a higher probability of observations occurring in the tails relative to the normal distribution. A distribution is skewed when the distribution is not symmetrical and there is a higher probability of outliers. Parameters of the model that vary over time are said to be unstable.

LO 49.b

The phenomenon of fat tails is most likely the result of the volatility and/or the mean of the distribution changing over time.

LO 49.c

If the mean and standard deviation are the same for asset returns for any given day, the distribution of returns is referred to as an unconditional distribution of asset returns. However, different market or economic conditions may cause the mean and variance of the return distribution to change over time. In such cases, the return distribution is referred to as a conditional distribution.

Contrary to when volatilities change slowly, volatilities may also change quickly and abruptly. The abrupt change in volatilities is referred to as regime switching.

LO 49.d

A value at risk (VaR) method for estimating risk can be based either on historical-based approach or an implied-volatility-based approach. Historical-based approaches of measuring VaR typically fall into two subcategories: parametric and nonparametric.

The parametric approach typically assumes asset returns are normally or lognormally distributed with time-varying volatility (i.e., historical standard deviation or exponential smoothing).

The nonparametric approach is less restrictive in that there are no underlying assumptions of the asset returns distribution (i.e., historical simulation).

LO 49.e

The EWMA model generates volatility estimates based on weightings of the last estimate of volatility and the latest current price change information. The objective is to account for previous volatility estimates, as well as to account for the latest return information.

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) r_{n-1}^2$$

where:

λ = weight on previous volatility estimate (λ between zero and one)

LO 49.f

GARCH (1,1) models not only incorporate the most recent estimates of volatility and return, but also incorporate a long-run average level of variance.

$$\sigma_n^2 = \omega + \alpha r_{n-1}^2 + \beta \sigma_{n-1}^2$$

where:

α = weighting on the previous period's return

β = weighting on the previous variance estimate

ω = weighted long-run variance = γV_L

V_L = long-run average variance = $\dfrac{\omega}{1 - \alpha - \beta}$

$\alpha + \beta + \gamma = 1$

$\alpha + \beta < 1$ for stability so that γ is not negative

GARCH (1,1) estimates of volatility have a better theoretical justification than the EWMA model. In the event that model parameter estimates indicate instability, however, EWMA volatility estimates may be used.

Under multivariate density estimation (MDE), a weight is assigned to the historical data depending on how similar it is to the current data.

LO 49.g

Volatility tends to exhibit a mean-reverting characteristic, meaning that if current volatility is high, it is expected to decline; if it is low, it is expected to increase.

If volatility is forecast to decline, we will overestimate volatility if we multiply the standard deviation of the return by the square root time. GARCH (1,1) provides a pull back toward the long-term average mean, EWMA does not.

If mean reversion exists, the long horizon risk (and resulting VaR calculation) will be smaller than the square root volatility.

LO 49.h

Implied volatility is implied from options prices, which are dependent on volatilities. As volatilities increase so do options prices. Implied volatilities of options are usually expressed as annually; less than annual volatilities are adjusted by the square root of time. The most widely used index for publishing implied volatility is the Chicago Board Options Exchange (CBOE) Volatility Index (VIX).

LO 49.i

Correlations can be estimated from covariances. Correlation between two assets is their covariance divided by the product of their standard deviations.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 49.1

1. **A** The most likely explanation for "fat tails" is that the volatility is time-varying for the unconditional distribution. For example, this explanation is much more likely given observed changes in volatility in interest rates prior to a much-anticipated Federal Reserve announcement. Examining a data sample at different points of time from the full sample could generate fat tails in the unconditional distribution, even if the conditional distributions are normally distributed. (LO 49.c)

2. **C** The regime-switching model captures the conditional normality and may resolve the fat-tailed problem and other deviations from normality. A regime-switching model allows for conditional means and volatility. Thus, the conditional distribution can be normally distributed even if the unconditional distribution is not. (LO 49.c)

Module Quiz 49.2

1. **B** Fat tails, skewness, and other deviations from an assumed distribution are not a concern in the estimation process for nonparametric methods. (LO 49.d)

Module Quiz 49.3

1. **B** $\sigma_n^2 = 0.00003 + (0.008)^2 \times 0.04 + (0.015)^2 \times 0.92 = 0.00023956$

   $\sigma_n = \sqrt{0.00023956} = 0.0155 = 1.55\%$

   (LO 49.f)

2. **C** $\sigma_n^2 = 0.9 \times (0.015)^2 + (1 - 0.9) \times (0.008)^2 = 0.0002089$

   $\sigma_n^2 = \sqrt{0.0002089} = 0.0145 = 1.45\%$

   (LO 49.e)

3. **D** The long-run variance rate can be estimated by dividing the ω of a GARCH (1,1) model by 1 – α – β. This yields 0.00003 / (1 – 0.04 – 0.92) = 0.00075; long-run standard deviation $= \sqrt{0.00075} = 0.0274 = 2.74\%$. (LO 49.f)

4. **B** Stable GARCH (1,1) models require $\alpha + \beta < 1$; otherwise the model is unstable. (LO 49.f)

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[1] To adjust for one degree of freedom related to the conditional mean, the denominator in the formula is K – 1. In practice, adjusting for the degrees of freedom makes little difference when large sample sizes are used.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 4.

READING 50

EXTERNAL AND INTERNAL CREDIT RATINGS

Study Session 13

EXAM FOCUS

Credit ratings can apply to whole companies or individual investments. Agencies determine external ratings with both qualitative and quantitative methods, and the historical relationship between ratings and subsequent defaults is quite strong. For the exam, know how to interpret a default probability table and a ratings transition matrix. Also, understand the value of the hazard rate and the recovery rate as they pertain to expected losses. Banks generate their own internal ratings, and like agencies they may use a point-in-time approach or a through-the-cycle approach. Have a general understanding on how external and internal credit ratings are established.

MODULE 50.1: EXTERNAL CREDIT RATINGS

Rating Agencies

LO 50.a: Describe external rating scales, the rating process, and the link between ratings and default.

The three primary rating agencies in the United States (Moody's, Standard and Poor's [S&P], and Fitch) serve as *external sources* of credit risk data, providing independent opinions on credit risk. The focus of these agencies is on money market instruments and bonds issued by governments and corporations, although their ratings in other areas like structured products played a significant role in the financial crisis of 2007–2009. Because rating agencies usually only focus on firms with publicly traded instruments, banks who lend money to borrowers without external ratings may rely on their own internal rating systems.

External credit ratings are typically issued for an instrument issued by an entity, as opposed to a rating on the entity itself. The rating is established for the purpose of assessing how likely it is that an entity will default on its obligations. Because agencies tend to give all offerings from a single entity the same rating, it would seem that the rating applies to the overall entity as well.

Bond ratings are considered long-term ratings due to the fact that bonds offer periodic interest payments and principal payback at maturity to their debtholders. Money market instruments, which typically last one year or less and simply return one final payment, have ratings which are considered short term.

For Moody's, the high-level ratings scale reads as follows:

> Aaa, Aa, A, Baa, Ba, B, Caa, Ca, C, and D

The probability of default increases down the scale, as Aaa rated bonds are highly unlikely to default while D rated bonds are already in default. Further delineations of ratings occur at each level except Aaa and the lowest rating categories, as Aa ratings for example are divided into Aa1, Aa2, and so on.

S&P ratings are similar to Moody's, with ratings of

> AAA, AA, A, BBB, BB, B, CCC, CC, C, and D

S&P also further delineates their ratings other than AAA and their lowest categories, with AA divided into AA+, AA, and AA–. Fitch ratings align with S&P ratings.

A key dividing line for both Moody's and S&P are the Baa3 and BBB– ratings, respectively. Any instruments with ratings at or above this line are considered **investment grade**, whereas any instruments with ratings below this line are considered **noninvestment grade** (also known as speculative grade or junk bonds). Within the investment grade categories, firms have the capacity to meet their obligations and they have adequate protection in place to withstand adverse economic conditions or other changes in circumstances. Increased vulnerability is reflected in lower ratings, but investment grade is still considered relatively safe. Noninvestment grade ratings indicate considerable capacity and uncertainty issues. Investors would expect to be compensated with higher returns for taking on additional risk as ratings decline.

For short-term ratings on money market instruments, Moody's prime rating categories (essentially, investment grade) are P-1, P-2, and P-3 (with P-1 being the strongest rating). A rating of NP (non-prime) is considered noninvestment grade. S&P ratings are similar at the top to Moody's, with their A-1+ and A-1 ratings aligning with Moody's P-1. S&P's A-2 and A-3 align with P-2 and P-3. S&P uses three lower rating categories (B, C, and D). Fitch ratings are similar to S&P, with a scale of F1+, F1, F2, F3, B, C, and D.

Conditional and Unconditional Default Probabilities

LO 50.b: Define conditional and unconditional default probabilities and explain the distinction between the two.

Default probability data can also be conveyed with conditional and unconditional default probabilities. The **conditional default probability** estimates the probability of default in a given year conditional on no default from previous years. In this case, it is computed as the **unconditional default probability** of the stated year divided by the probability of survival from the previous years.

The following three figures provide subsets of data on defaults from 1981 to 2016. This information can be used to assess the conditional and unconditional default probabilities based on ratings.

Figure 50.1: Cumulative Percentage Default Probabilities (Based on S&P Ratings)

	Years Until Default				
Rating	1	2	3	4	5
AAA	0.00	0.03	0.13	0.24	0.35
AA	0.02	0.06	0.13	0.23	0.33
A	0.06	0.15	0.25	0.38	0.53
BBB	0.18	0.51	0.88	1.33	1.78
BB	0.72	2.24	4.02	5.80	7.45
B	3.76	8.56	12.66	15.87	18.32
CCC/C	26.78	35.88	40.96	44.06	46.42

Figure 50.2: Unconditional Percentage Default Probabilities (Based on S&P Ratings)

	Years Until Default				
Rating	1	2	3	4	5
AAA	0.00	0.03	0.10	0.11	0.11
AA	0.02	0.04	0.07	0.10	0.10
A	0.06	0.09	0.10	0.13	0.15
BBB	0.18	0.33	0.37	0.45	0.45
BB	0.72	1.52	1.78	1.78	1.65
B	3.76	4.80	4.10	3.21	2.45
CCC/C	26.78	9.10	5.08	3.10	2.36

Figure 50.3: Probability of Default Conditional on No Default Earlier (Based on S&P Ratings)

	Years Until Default				
Rating	1	2	3	4	5
AAA	0.00	0.03	0.10	0.11	0.11
AA	0.02	0.04	0.07	0.10	0.10
A	0.06	0.09	0.10	0.13	0.15
BBB	0.18	0.33	0.37	0.45	0.46
BB	0.72	1.53	1.82	1.85	1.75
B	3.76	4.99	4.48	3.68	2.91
CCC/C	26.78	12.43	7.92	5.25	4.22

To illustrate how the interpret this data, Figure 50.1 shows a B-rated bond had an 8.56% chance of defaulting within 2 years and a 12.66% chance of defaulting within 3 years. Figure 50.2 shows that the unconditional probability of the B-rated bond defaulting during the third year is 4.10% (which is also equal to the difference between 8.56% and 12.66%). The probability that the bond does not default through the first two years is

91.44% (= 100% – 8.56%). If the unconditional probability that the B-rated bond defaults in the third year is 4.10%, then the conditional probability of it defaulting in that year (conditional on the fact that it did not default earlier) is 4.48% as shown in Figure 50.3 (= 4.10% / 91.44% = 4.48%).

During the first several years, the probability of default increases as a function of time for investment grade bonds, because over time, there is a greater chance that the financial health of the issuing entity declines. For the lowest rated bonds, the opposite situation exists; surviving the first year or two represents an increased likelihood of an improvement in financial health.

Additional Impacts on Ratings

Ratings agencies determine the external rating of a firm or bond using current information with the goal of indicating the probability of future events such as default and/or loss. The probability of default given any rating at the beginning of a cycle *increases with the horizon*. The increase in the default rates, or cumulative default rate, is much more dramatic for noninvestment grade bonds. In addition to the condition of the firm, forecasted events in the horizon will affect the probabilities. The most notable events are the *economic and industrial cycles*. Since the rating should apply to a long horizon, in many cases, ratings agencies try to give *a rating that incorporates the effect of an average cycle*. This practice leads to the ratings being relatively stable over an economic or industrial cycle. Unfortunately, this averaging practice may lead to an over- or underestimate during periods when the economic conditions deviate too far from an average cycle. Also, the default rate of lower-grade bonds is correlated with the economic cycle, while the default rate of high-grade bonds is fairly stable.

A goal of rating agencies is to issue ratings that reflect the same thing across different countries and different industries, such that a BB rating in one country is equivalent to a BB rating in another. Note that the three rating agencies covered in this reading are all U.S. based, whereas other rating agencies in other countries have much shorter histories and, therefore, may not be entirely comparable. Figure 50.4 from S&P (2016) shows five-year cumulative default probabilities for U.S. firms, emerging market firms, and European firms.

Figure 50.4: Five-Year Cumulative Default Probabilities (Reported by S&P)

Initial Rating	U.S. Firms	Emerging Market Firms	European Firms
AAA	0.42	N.A.	0.00
AA	0.45	0.00	0.21
A	0.73	0.05	0.29
BBB	2.05	2.59	0.65
BB	8.38	6.26	4.20
B	19.57	12.59	13.86
CCC/C	51.31	25.88	48.01
All rated	7.57	6.53	2.54
Investment grade	1.17	1.69	0.38
Speculative grade	17.00	10.19	10.84

From an industry perspective, ratings consistency is more difficult to capture. Financial corporations have had higher default rates than nonfinancial corporations with the same rating, although past experiences are certainly not indicative of the future.

Hazard Rates

LO 50.c: Define hazard rate and calculate the unconditional default probability of a credit asset using hazard rate.

A **hazard rate** (also known as default intensity) is the rate at which defaults happen at time *t*. This rate can be used to calculate unconditional default probabilities. Assuming that $\bar{h}$ is the average hazard rate between two time periods, $\bar{h}_1$ and that $\bar{h}_2$ and are the average hazard rates between times t_1 and t_2, the following equations can be applied:

unconditional default probability between t_0 and t: $1 - e^{(-\bar{h}t)}$

probability of survival (also known as the survival rate): $e^{(-\bar{h}t)}$

unconditional probability of default between times t_1 and t_2: $e^{(-\bar{h}_1 t_1)} - e^{(-\bar{h}_2 t_2)}$

If the hazard rate were to be constant at 2% per year, the probability that a bond would default by the end of the fourth year would be equal to 7.69% [= $1 - e^{(-0.02\times4)}$]. The unconditional probability of default happening during the fifth year would be 1.83% [= $e^{(-0.02\times4)} - e^{(-0.02\times5)}$].

The conditional probability of a default during the fifth year, given that the bond has survived through four years, is equal to 1.98% [= 1.83% / (1 − 7.69%)]. For the B-rated bond described earlier, Figure 50.1 shows a cumulative default probability of 12.66% over three years. The average hazard rate can then be calculated using $1 - e^{(-\bar{h}t)}$. In this case, $0.1266 = 1 - e^{(-\bar{h}\times3)}$; $\bar{h} = 4.51\%$.

Recovery Rates and Expected Losses

LO 50.d: Define recovery rate and calculate the expected loss from a loan.

The two questions that lenders are interested in when it comes to lending money are (1) what is the probability of default (PD) and (2) if the borrower were to default, how much could the lender expect to recover?

The **recovery rate (RR)** (for a bond) is equal to its value just after default, expressed as a percentage of face value. The **loss given default (LGD)** is equal to 100 – the percentage RR. So over a period of time, the **expected loss (EL)** is equal to:

PD × LGD

or

PD × (1 – RR)

For senior secured bonds, the recovery rate is approximately 50%; for junior bonds, it is around 25%. Default rates and recovery rates are negatively correlated, as recovery rates tend to be low when default rates are high.

Assume the B-rated bond has a 15.87% chance of defaulting in the next four years. If the recovery rate is estimated at 20%, the expected loss will be 15.87% × 0.80, or 12.70% of bond principal. To compensate themselves for the risk, a bondholder may charge interest of 12.70% / 4 years, or 318 basis points above the risk-free rate. This excess 318 basis points is considered the **credit spread** (actuarial compensation), and in reality bondholders will require a risk premium on top of the credit spread as protection from the systematic risk of bonds defaulting in unison during weak economic times.

Through-the-Cycle and Point-in-Time Ratings

LO 50.e: Explain and compare the through-the-cycle and point-in-time ratings approaches.

As noted earlier, rating agencies will provide ratings for money market instruments and publicly traded bonds. Key factors that go into ratings include financial information (both historical and projected), economic data, industry/peer data, and qualitative elements such as the company's organizational structure, meetings with management, governance, et cetera.

Ratings, which are given when instruments are first issued and then subsequently reviewed on an annual or more frequent basis, are paid for by the firm receiving the rating. This arrangement is similar to how firms pay for their auditors. Rating agencies also offer **outlooks**, which are expectations as to where the rating may go in the medium term. Outlooks can be positive (rating may be raised), negative (rating may be lowered), stable (rating is unlikely to change), or developing (rating may change, but the direction is undetermined). A **watchlist** is designed to reflect a short-term (three months or less) anticipated change, which can be positive or negative.

Because ratings are used in financial contracts and by bond traders whose holdings are in large part driven by ratings, **ratings stability** is a key goal for agencies that should only be adjusting ratings if there has been a long-term change in the issuing firm's overall creditworthiness. Due to the nature of the economy and its business cycles, a rating agency has two options for how they will rate issuances; they can either rate **through-the-cycle** or at a **point-in-time**. Through-the-cycle will evaluate creditworthiness over several years, thereby insulating it from the inevitable ups and downs of business cycles. Point-in-time is the best estimate of the probability of future default at the present time. Given the goal of ratings stability, the through-the-cycle approach tends to be the one used by agencies. Adjustments can be made to convert these ratings to a point-in-time measure in order to better reflect changes in the economy as they happen.

MODULE QUIZ 50.1

1. Which of the following bonds is likely to have the highest yield?
   A. A bond rated AA by S&P.
   B. A bond rated BB by S&P.
   C. A bond rated B by Moody's.
   D. A bond rated Baa3 by Moody's.

2. If the hazard rate is constant at 4% per year, the probability of default for a bond by the end of the fifth year is closest to:
   A. 1.87%.
   B. 3.92%.
   C. 18.13%.
   D. 20.00%.

3. A BBB rated bond has a 1.78% chance of defaulting within five years. If the expected loss on bond principal is 0.445%, the recovery rate will be closest to:
   A. 5%.
   B. 15%.
   C. 25%.
   D. 75%.

4. A bond rating is most likely to fall during a contractionary business cycle using which of the following rating methodologies?
   A. Forecast.
   B. Point-in-time.
   C. Historical driven.
   D. Through-the-cycle.

MODULE 50.2: RATING CHANGES AND ADDITIONAL RATING METHODS

Rating Transition Matrix

LO 50.h: Describe, calculate, and interpret a rating transition matrix and explain its uses.

A **rating transition matrix** is designed to show the probability that a bond issuer may go from one rating to another during a one-year time period. Based on data from 1981 to 2016, Figure 50.5 represents one-year rating transitions per S&P.

Figure 50.5: Transition Matrix (From S&P)

	AAA	AA	A	BBB	BB	B	CCC/C	D	NR
AAA	87.05	9.03	0.53	0.05	0.08	0.03	0.05	0	3.17
AA	0.52	86.82	8.00	0.51	0.05	0.07	0.02	0.02	3.99
A	0.03	1.77	87.79	5.33	0.32	0.13	0.02	0.06	4.55
BBB	0.01	0.10	3.51	85.56	3.79	0.51	0.12	0.18	6.23
BB	0.01	0.03	0.12	4.97	76.98	6.92	0.61	0.72	9.63
B	0.00	0.03	0.09	0.19	5.15	74.26	4.46	3.76	12.06
CCC/C	0.00	0.00	0.13	0.19	0.63	12.91	43.97	26.78	15.39

Based on this data, it is clear that investment grade bonds have a significantly higher chance of maintaining their ratings over the course of the year. The highest rated bonds (AAA) had an 87.05% chance of keeping that rating. Contrast that with a CCC/C rated bond, which had only a 43.97% chance of keeping the rating (with a 26.78% chance of actually being downgraded to a default [D] rating and a 15.39% chance of not being rated [NR]).

The ratings matrix phenomenon reflects the fact that a downgrade (upgrade) in one year has a higher likelihood of being followed by another downgrade (upgrade) in the next year. Also, transition matrices like Figure 50.5 tend to be driven by the economic cycle in spite of the fact that they are through-the-cycle ratings.

EXAMPLE: Ratings migration

Given the following one-year transition matrix, **compute** the probability that a B-rated firm will default over a two-year period.

Rating From	Rating To			
	A	B	C	Default
A	90%	5%	5%	0%
B	5%	85%	5%	5%
C	0%	5%	80%	15%

Answer:

At the end of year one, there is a 5% chance of default and an 85% chance that the firm will maintain a B rating.

In year two, there is a 5% chance of default if the firm was rated B after one year (85% × 5% = 4.25%). There is a 0% chance of default if the firm was rated A after one year (5% × 0% = 0%). Also, there is a 15% chance of default if the firm was rated C after one year (5% × 15% = 0.75%).

The probability of default is 5% from year one plus 5% chance of default from year two (i.e., 4.25% + 0% + 0.75%) for a total probability of default over a two-year period of 10%.

Anticipating Rating Changes

LO 50.i: Describe the relationships between changes in credit ratings and changes in stock prices, bond prices, and credit default swap spreads.

When ratings are downgraded, the bond and stock market tend to have significant reactions (especially to movements from investment grade to noninvestment grade). Upgrades, on the other hand, do not garner the same reaction. As such, the information content of ratings has been something that is up for debate. One study found that watchlist reviews for downgrades have significant information, while negative outlooks and actual downgrades do not. While credit default swap spread changes do seem to forecast ratings changes, stocks and bonds do not consistently follow suit.

Rating Structured Products

LO 50.j: Explain historical failures and potential challenges to the use of credit ratings in making investment decisions.

As noted in the beginning of the reading, rating agencies involvement in rating structured products a key role in the economic crisis of 2007–2009. Much of this had to do with structured product ratings' reliance on modeling, and modeling inputs such as correlations between mortgage defaults were overly optimistic. Once the models were understood, structured product designs were tailored in order to achieve desired ratings. Also, the independence of ratings agencies when it came to modeling these products was questioned due to the extremely profitable nature of the business. As a result, rating agencies now have far more oversight than they had in the past.

Alternative Methods to Credit Ratings

LO 50.f: Describe alternative methods to credit ratings produced by rating agencies.

Kamakura and Moody's are organizations which use models that incorporate factors such as firm equity market value, equity volatility, and debt in the firm's capital structure to estimate default probabilities. If there is a point in time where assets are less than debt repayments required, the firm is in default and equity is either negative or at most, zero. The estimates are point-in-time, and because the models are more responsive to changing circumstances, the goal of stability is not a factor.

External vs. Internal Ratings

LO 50.g: Compare external and internal ratings approaches.

Internal credit ratings used to evaluate potential borrowers are developed by financial institutions for three primary reasons: (1) there may not be external ratings available, (2) accounting standards require banks to account for default probabilities when they value loans on their balance sheet, and (3) probabilities of default drive regulatory credit risk capital. While internal ratings can also be either through-the-cycle or point-in-time, they tend to be the point-in-time (which drives reductions in lending when the economy is in a downturn).

It is critical for banks to test their internal rating procedures in order to ensure that their methodologies are correct. The early years of lending decision algorithms were determined by Altman's Z-score (1968), which used the following five financial ratios in a multiple regression: (1) working capital to total assets, (2) retained earnings to total assets, (3) earnings before interest and taxes (EBIT) to total assets, (4) equity market value to total liability book value, and (5) sales to total assets. Current algorithms are much more sophisticated now and use myriad input variables and data to estimate default probabilities.

MODULE QUIZ 50.2

1. Which of the following statements is most accurate in regard to a ratings matrix?
   A. A bond has a greater likelihood of following a ratings upgrade with a downgrade.
   B. A bond with a AAA rating in one year is likely to keep that rating in the next year.
   C. A bond with a CCC rating in one year is likely to keep that rating in the next year.
   D. A bond has a greater likelihood of following a ratings downgrade with an upgrade.

2. With respect to the effect on the price of a bond, the effect of a bond upgrade will:
   A. be positive and stronger than the downward effect of a bond downgrade.
   B. be positive and weaker than the downward effect of a bond downgrade.
   C. have about the same negative effect, in absolute value terms, as a bond downgrade.
   D. be negative and about equal to that of a bond downgrade.

3. The role that rating agency evaluations of structured products played in the economic crisis of 2007–2009 was largely a result of:
   A. underestimated correlations between defaults.
   B. the high degree of regulatory oversight in place.
   C. the low impact of potential profits to the agencies.
   D. the minimal impact of structured product performance modeling.

4. The models developed by organizations such as Moody's (KMV) are:
   A. driven by a goal of ratings stability.
   B. based on a through-the-cycle approach.
   C. nonresponsive to equity and debt volatility.
   D. highly responsive to changing circumstances.

5. All of the following represent rationales for developing internal rating systems except:
   A. they do not require testing like external ratings.
   B. situations where external ratings are not available.
   C. default probabilities driving regulatory credit risk capital.

D. accounting requirements tied to recognizing default probabilities on loan valuations.

KEY CONCEPTS

LO 50.a

The three primary rating agencies in the United States (Moody's, Standard and Poor's [S&P], and Fitch) serve as external sources of credit risk data, providing independent opinions on credit risk. The focus of these agencies is on money market instruments and bonds issued by governments and corporations. Because rating agencies usually only focus on firms with publicly traded instruments, banks who lend money to borrowers without external ratings may rely on their own internal rating systems.

An external credit rating is typically issued for an instrument issued by an entity, as opposed to a rating on the entity itself. The rating is established for the purpose of assessing how likely it is that an entity will default on its obligations. For all three agencies, the probability of default increases down the scale. A key dividing line is where investments are considered investment grade or noninvestment grade (also known as speculative grade or junk bonds).

LO 50.b

Conditional default probabilities estimate the probability of default in a given year conditional on no default from previous years (e.g., default in Year 4 assuming no default in Years 1–3). Unconditional default probabilities estimate the probability of default in a specific year (e.g., default in Year 4) regardless of the outcomes from other years.

LO 50.c

A hazard rate (also known as default intensity) is the rate at which defaults happen at a specific time and can be used to calculate unconditional default probabilities. Assuming that $\bar{h}$ is the average hazard rate between two time periods, and that $\bar{h}_1$ and $\bar{h}_2$ are the average hazard rates between times t_1 and t_2, the following equations can be applied:

unconditional default probability between t_0 and t: $1 - e^{(-\bar{h}t)}$

probability of survival (also known as the survival rate): $e^{(-\bar{h}t)}$

unconditional probability of default between times t_1 and t_2: $e^{(-\bar{h}_1 t_1)} - e^{(-\bar{h}_2 t_2)}$

LO 50.d

The recovery rate (for a bond) is equal to its value just after default, expressed as a percentage of face value. The loss given default is equal to 100 – the percentage recovery rate. So over a period of time, the expected loss is equal to either the probability of default × the loss given default, or the probability of default × (1 – recovery rate).

LO 50.e

A rating agency has two options for how they will rate issuances; they can either rate through-the-cycle or at a point-in-time. Through-the-cycle will evaluate creditworthiness over several years, thereby insulating it from the inevitable ups and downs of business cycles. Point-in-time is the current best estimate of the probability of future default. Given the goal of ratings stability, the through-the-cycle approach tends to be the one used by agencies.

LO 50.f

In contrast to external ratings, there are organizations which use models that incorporate factors such as firm equity market value, equity volatility, and debt in the firm's capital structure to estimate default probabilities. These estimates are point-in-time, and because the models are more responsive to changing circumstances, the goal of stability is not a factor. Internal ratings may also be developed and used to evaluate the creditworthiness of potential borrowers.

LO 50.g

Internal rating systems used to evaluate potential borrowers are developed by financial institutions for three primary reasons: (1) there may not be external ratings available, (2) accounting standards require banks to account for default probabilities when they value loans on their balance sheet, and (3) probabilities of default drive regulatory credit risk capital. While internal ratings can also be either through-the-cycle or point-in-time, they tend to be the point-in-time.

LO 50.h

A rating transition matrix is designed to show the probability that a bond issuer may go from one rating to another during a one-year time period. Based on historical data, investment grade bonds had a significantly higher chance of maintaining their ratings over the course of the year than noninvestment grade bonds.

The ratings matrix phenomenon reflects the fact that a downgrade (upgrade) in one year has a higher likelihood of being followed by another downgrade (upgrade) in the next year.

LO 50.i

Ratings downgrades tend to have a significant impact on bond and stock prices, whereas upgrades have had much less of an impact. Also, bonds and stocks do not seem to contain any information regarding the future outlook for ratings changes.

LO 50.j

Rating agencies have been fairly successful in the area of money market instruments and bonds. A critical failure was in the structured products market, which was one of the drivers of the economic crisis in 2007–2009. Ratings for these products were based on modeling, which tended to be overly optimistic and easily solvable by issuers looking to create products that would garner higher ratings. Rating agency independence has also been questioned, although more significant regulations are in place now to prevent some of the issues identified from the past.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 50.1

1. **C** The bond with the highest yield will be the one with the lowest rating (because it will have the highest likelihood of default, investors will require a higher return to compensate for the risk). Of the choices given, the bond with the lowest rating is the Moody's bond rated B. (LO 50.a)

2. **C** If the hazard rate were to be constant at 4% per year, the probability that a bond would default by the end of the fifth year is equal to 18.13% [$= 1 - e^{(-0.04 \times 5)}$]. (LO 50.c)

3. **D** With a 1.78% chance of defaulting within five years and an expected loss of 0.445% of bond principal, the recovery rate must be 75% [= 1 − (0.00445 / 0.0178)]. (LO 50.d)

4. **B** The point-in-time approach is likely to result in a bond rating decline during a contractionary cycle, while the through-the-cycle approach is used to achieve ratings consistency by not having rating changes as the economy progresses through business cycles. The other two options (forecast and historical driven) are not actual methodologies. (LO 50.e)

Module Quiz 50.2

1. **B** A bond that is AAA rated in one year is highly likely (over an 87% chance historically) to maintain that rating in the next year. For CCC rated bonds, there is less than a 50% chance that the rating will be maintained one year later. The ratings matrix phenomenon has shown that a ratings upgrade (downgrade) is likely to be followed by another upgrade (downgrade). (LO 50.h)

2. **B** A bond's upgrade will have a positive effect on the bond's price, but the negative effect of a bond downgrade is generally stronger. (LO 50.i)

3. **A** Underestimated correlations between defaults caused significant problems, as defaults often snowball and follow each other. The degree of regulatory oversight was minimal at the time relative to present day. Potential profits were significant, which called into question the independence of rating agencies. Modeling was a significant driver in what ultimately led to the economic crisis. (LO 50.j)

4. **D** KMV models used as an alternative to external ratings are highly responsive to changing circumstances because they are more point-in-time versus external rating agencies which tend to strive for ratings stability and use a through-the-cycle methodology. The KMV models are highly sensitive to equity and debt volatility. (LO 50.f)

5. **A** Internal ratings methodologies absolutely require testing in order to determine that they are functioning as expected. All of the other statements represent legitimate rationales for creating internal ratings. (LO 50.g)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 5.

READING 51

COUNTRY RISK: DETERMINANTS, MEASURES, AND IMPLICATIONS

Study Session 13

EXAM FOCUS

Sovereign risks vary across countries. Factors such as the country's political risk, legal risk, position in the economic growth cycle, and economic diversity affect the investor's overall risk. Rating agencies rate sovereign risks and also assess rating transitions. Country risk can also be analyzed using sovereign default risk spreads. For the exam, be able to compare and contrast the advantages of sovereign debt ratings and sovereign default risk spreads and be able to identify sources of sovereign risk and describe the consequences that result from both local currency and foreign currency sovereign defaults.

MODULE 51.1: COUNTRY RISK

LO 51.a: Explain how a country's economic growth rates, political risk, legal risk, and economic structure relate to its risk exposure.

Investors are increasingly exposed to **country risk**, both directly and indirectly. Country risk refers to the risk of investing or operating in a foreign country and includes political risk, legal risk, and corruption. Innovations in financial markets have made investing in nondomestic companies easier for investors. Investors now have a range of investment options that make it easier and cheaper to invest in foreign markets, including through exchange-traded funds (ETFs). Risk exposures vary across countries and there are different sources of risk that contribute to these variations.

Key sources of country risk include (1) the level of a country's economic growth, (2) political risk, (3) legal risk, including risks that arise from both the structure and the efficiency of legal systems, and (4) economic structure including the level of diversification.

Economic Growth Life Cycle

A country's level and growth in its gross domestic product (GDP) and especially GDP net of inflation (real GDP growth) is important in evaluating economic prospects and risks in that country. For example, high GDP growth is positive, but if it is coupled with high inflation then real growth rates will be low. How a country will react to economic cycles is also an important determinant of country risk. Developing countries often experience larger declines in real GDP growth rates than developed countries, often because of a higher reliance on commodities which are more sensitive to global downturns. For example, during the global recession of 2007–2009, Mexico experienced a real GDP decline of 4.7% while the United States saw a 2.8% real GDP decline.

Political Risk

Political risk is the risk that government actions or changes, or corruption and violence could impact a firm's or investor's profitability. Political risk is broad and includes everything from whether a country is a democracy or a dictatorship to the smoothness with which a country transfers political power (e.g., military coups vs. democratic elections). There are four broad components of political risk:

- *Continuous vs. discontinuous policy*. Risks in democracies are continuous but generally low. In contrast, risks in dictatorships or authoritarian governments are discontinuous. Policies change much less frequently, but changes are often severe and difficult to protect against (i.e., discontinuity in policy). Studies are mixed regarding which system, authoritarian or democratic, results in higher economic growth.
- *Corruption*. There are costs associated with government corruption. Corruption costs of bribes can be likened to an implicit tax, directly reducing company profits and returns and indirectly reducing investor returns. Because the tax is not explicit and may also result in legal sanctions against the firm operating in the corrupt system (e.g., if a firm is caught bribing an official), it increases risk. Using survey data, Transparency International ranks countries based on the level of corruption that is present. According to its 2016 survey, Denmark, New Zealand, and Finland were recognized as the three least corrupt countries, while Somalia, South Sudan, and North Korea were listed as the most corrupt in the survey.
- *Physical violence*. There are economic costs (e.g., insurance and security costs) and physical costs (e.g., possible physical harm to employees or investors) associated with countries in conflict. The Institute for Peace and Economics publishes a Global Peace Index which indicates similar rankings as the rankings on the corruption index.
- *Nationalization or expropriation risk*. Firms that perform well may see their profits expropriated via arbitrary taxation by governments. This risk is heightened for firms working natural resources. A firm may be nationalized, in which case the owners will receive much less than the true value of the company. Risks are greater in countries where nationalization and/or expropriation is possible.

Legal Risk

Legal risk is the risk of losses to an investor or firm due to risks in the legal system. The protection of property rights (i.e., the structure of the legal system) and the efficiency of dispute resolution affect risk. If disputes cannot be settled in a timely fashion, the risks to profitability rise. A well-functioning legal system protects property rights and should allow either the firm or its management to be sued. Several nongovernment organizations have created international property rights indices, including the International Property Rights Index published by the Property Rights Alliance. North America, Australia, New Zealand, and Western Europe tend to be ranked the highest, while Latin America and Africa tend to be ranked the lowest in terms of protection of property rights. Not surprisingly, many of the countries that are ranked the least corrupt by Transparency International are also ranked high on the property rights index.

Economic Structure

In addition to GDP growth, **economic structure** including the level of economic diversification and competitive advantage are important factors in assessing country risk. A disproportionate reliance on a single commodity or service in an economy increases a country's risk exposure. A downturn in the demand (and/or price) for the good or service on which the country is dependent can create significant losses in the economy, increasing risks for businesses and investors. The trade-off between short-term and long-term growth is also an important consideration. A reliance on specific commodities is what determines short-term growth, but developing other industries may be considered in the long term. Economic diversification tends to be greater in large countries, including China, India, and Brazil. Regardless of size, however, countries should be able to hedge their economic risks through long-term trade contracts with other countries.

The competitive advantage of a country is dependent on four factors according to Michael Porter:[1]

- *Factor conditions*: relative availability of production factors, including labor and capital
- *Demand conditions*: level of demand for a country's products and services
- *Related industries*: availability of other competitive industries
- *Firm strategy/structure/rivalry*: level of domestic conditions affecting a country's firms

Investors must assess all of the sources of country risk not only in isolation, but also in conjunction with each other, when analyzing nondomestic investment opportunities.

Evaluating Country Risk

LO 51.b: Evaluate composite measures of risk that incorporate multiple components of country risk.

There are numerous services that attempt to evaluate country risk in its entirety (in other words, **total risk**), including:

- *Political Risk Services (PRS).* PRS evaluates countries on the key areas of country risk. Political, economic, and financial risk dimensions are evaluated using 22 variables to measure risks. The firm provides a composite score as well as a score on each of the three dimensions.

- *Media outlets*. Media outlets that measure country risk include *Euromoney* and *The Economist*. *Euromoney* uses a survey of 400 economists who assess country risk factors, while *The Economist* assesses currency risk, sovereign debt risk, and banking risks to develop country risk scores.
- *The World Bank.* The World Bank uses risk measures that assess six areas, including the level of corruption, government effectiveness, political stability, rule of law, accountability, and regulatory quality.

There are many limitations associated with these risk services that may diminish their value to businesses and investors. First, not every component is relevant to all investors. Second, there is no standardization across the information providers. This makes it difficult to compare country risk assessments across providers. Third, the scores are better used as rankings than as true scores. For example, a country with a ranking of 70 in *The Economist* rankings can be interpreted as riskier than a country with a ranking of 35. However, one cannot conclude, using the rankings, that a country with a score of 70 is twice as risky as a country with a score of 35.

MODULE QUIZ 51.1

1. Ravi Chowdhury, a portfolio manager for a hospital foundation, is considering the inclusion of sovereign bonds in the fixed income portion of the foundation's portfolio. Chowdhury, much to the surprise of his colleagues, plans to purchase the bonds of a country that has long been under authoritarian rule. He cites "lower political risk" when asked about his investment decision. Which of the following statements best supports his assertion of lower risk?
   A. Authoritarian regimes are more likely to control corruption in government agencies.
   B. Government policies that may affect debt repayment are often more stable under an authoritarian regime.
   C. Relative to a democracy, risks are greater on a day-to-day basis, but the effects are less detrimental overall.
   D. In most authoritarian countries, property rights are protected and property disputes are settled quickly.

2. In an attempt to understand country risk, an analyst at Global Funds examines multiple sources of information to determine the truest measure of risk. She considers sovereign risk ratings, default risk spreads, and composite measures of risk. Which of the following sources relies on surveys of several hundred economists to measures sovereign risk?
   A. Political Risk Services.
   B. *The Economist*.
   C. World Bank.
   D. *Euromoney*.

MODULE 51.2: SOVEREIGN DEFAULT RISK

LO 51.c: Compare instances of sovereign default in both foreign currency debt and local currency debt and explain common causes of sovereign defaults.

Sovereign default risk refers to the risk that holders of government-issued debt fail to receive the full amount of promised interest and principal payments during the specified time period. Sovereign default risk can be used as a proxy for country risk.

Sovereign default categories include foreign currency defaults and local currency defaults.

Foreign Currency Defaults

Throughout history, governments have often relied on debt borrowed from other countries or banks in those countries denominated in foreign currency. However, countries may have insufficient foreign currency to meet their obligations. Given that they are unable to print money in a foreign currency to repay the debt, in extreme cases they may default on the foreign currency debt. A large proportion of sovereign defaults are foreign currency defaults. Between 2010 and 2016, there have been eight foreign currency defaults. The largest sovereign default was by Greece in March 2012, defaulting on more than USD 260 billion (Greece also subsequently defaulted in December 2012 on USD 42 billion), resulting in investor losses of up to 70%.

Over the last 200 years there have been many instances of default. Many of the defaults during this long time period were by South American countries, including Argentina (6 times), Brazil (5 times), and Paraguay (6 times).

Local Currency Defaults

Many of the countries that defaulted on foreign currency debt over the last several decades were simultaneously defaulting on local (i.e., domestic) country debt. Defaulting countries included Brazil in 1990 and Russia in 1998.

A study by Moody's finds that countries are increasingly defaulting on both foreign and local currency debts concurrently. Reasons why countries may default on local currency debt are as follows:

1. *Gold reserves*. Prior to 1971, some countries followed the gold standard. This means the country was required to have gold reserves to back currency. The gold standard thus limited the amount of currency a country could print, reducing its flexibility in terms of printing currency to repay debt.
2. *Currency union*. Shared currency in a currency union limits the ability of individual countries to print money. For example, printing money in the European Union is the responsibility of the European Central Bank. During the recent Greek debt crisis, Greece was not able to print currency to pay off debt.
3. *Currency debasement.* Printing money may debase a local currency and lead to higher inflation. There are costs associated with default and costs associated with printing money.

Rating agencies provide ratings for both local and foreign currency debt. These ratings represent the agencies' opinion of the potential for default on a currency. Generally, the local currency debt rating is a few notches higher than the ratings on foreign currency debt.

Consequences of Sovereign Default

LO 51.d: Describe the consequences of sovereign default.

Historically, defaults were often followed by military actions. While this typically does not happen in the modern era, defaults are often preceded by difficult economic conditions. For example, Argentina's 2001 default was preceded by an economic depression starting in 1998. Countries that default usually suffer a loss of reputation, making it more difficult and more expensive to borrow in the future. Countries also experience reduced investment in stock and bond markets, face an economic downturn, or see political instability as a result of a default. The International Monetary Fund (IMF) is often involved in restructuring these defaulted obligations but typically impose significant austerity conditions.

Factors Influencing Sovereign Default Risk

LO 51.e: Describe factors that influence the level of sovereign default risk; explain and assess how rating agencies measure sovereign default risks.

When credit rating agencies assess corporate ratings, they often look at the debt to equity ratio. Because countries do not have equivalent equity measures, the debt to equity ratio is not a meaningful measure, and ratings are often assessed by looking at the debt to GDP ratio (see Figure 51.1). However, the overall country debt is also influenced by the level of debt of subnational governments, including states and cities.

Figure 51.1: Government Debt as a Percentage of GDP, 2017[2]

Country	Government Debt (% of GDP)
Japan	240.30
United States	108.14
France	96.84
United Kingdom	89.48
Brazil	83.36
India	68.69
Germany	65.01
China	47.61
Russia	17.35

Several factors influence a country's sovereign default risk:

1. *Social security commitments.* Governments make significant commitments to their citizens, including health care and pension payments. Also, as these commitments increase, a country's availability of liquidity (cash) declines.
2. *Tax base.* Governments must pay debt obligations in both good and bad economic times. This means the revenue stream must be stable to meet these fixed

obligations. Countries with more diversified economies are more likely to have stable tax receipts.

3. *Political risk.* Autocracies may be more likely to default than democracies because, as noted previously, defaults put pressure on, and may cause a change in, the leadership of the country. There may be less pressure on the leaders of dictatorships if the country defaults. Also, the more independent the central bank, the more difficult it may be for a country to print money.
4. *Implicit guarantees.* Rating agencies and other market participants also look at implicit guarantees by other entities. For example, countries in the European Union that are in financial distress are assumed to receive support from stronger countries like Germany and France to avoid a default. However, there is no explicit guarantee.

In summary, sovereign default risk is multifaceted and must be analyzed from many perspectives. The country's level of indebtedness, obligations to its citizens for things like pensions and health care, and its tax systems are all relevant to assessing the risk of default. In addition, the trustworthiness of the government and the nature of the economy must also be considered when evaluating sovereign default risk.

Generally, the local currency rating is at least as high as the foreign currency rating, because, as noted, countries can print money in local currency to repay debt. It is, however, possible for the local currency rating to be lower. Sovereign ratings may change across time, but change much less frequently than corporate bond ratings.

Looking at the 10-year sovereign and corporate default rates in 2016,[3] countries with AAA and AA ratings (or equivalent) performed well historically with near zero default rates, better than the corporate debt default rates during the same period. Countries with an A rating did not perform as well as corporate debt. Countries with BBB, BB, and B ratings performed similar to corporate debt defaults for foreign currency debt but performed significantly better (lower default rates) for local currency debt. Countries with CCC or C ratings had much higher foreign currency default rates than corporate debt, but lower local currency debt default rates.

Rating agencies also assess rating transitions. For example, a one-year rating transition for sovereign ratings looks at the probability that a country would maintain its current rating or transition to a higher (unless it's already AAA) or lower rating category.

Sovereign Default Spread

LO 51.f: Describe the characteristics of sovereign credit spreads and sovereign credit default swaps (CDS) and compare the use of sovereign spreads to credit ratings.

A measure of sovereign default risk is the **sovereign credit spread**. This measure is generated by the market and is continuously updated as sovereign bonds are traded. It reflects the sovereign bond yield compared to a riskless investment in the same currency and maturity. For example, consider a 10-year, U.S. dollar-denominated developing market bond yielding 8% while the 10-year U.S. Treasury bond yields 2%.

Because the U.S. Treasury bond is assumed to be risk free, the 6% spread between the two bond yields reflects the market's assessment of the default risk of the developing market bond. The 6% difference is therefore the sovereign credit spread.

Market-based spreads are more dynamic than ratings. As bonds trade and bond yields rise and fall, credit risk spreads change, revealing information about the market's perception of risk. Sovereign credit spreads also adjust more quickly to new information regarding the sovereign relative to bond ratings. This means investors are signaled earlier of impending threats and can adjust portfolios accordingly.

Credit default swaps (CDSs) provide important information about sovereign credit spreads. They are similar to insurance contracts, because credit protection is offered in exchange for premium payments. CDSs ensure that bondholders are made whole when there is a bond default (in other words, they would be in the same position as if there was no bond default). The CDS market is not without criticism, as speculator activity has been blamed for the very high Greek CDS spreads in 2010. In response, the European Union introduced legislation designed to curb speculator activity by banning the purchases of uncovered sovereign CDS contracts. In addition, CDS contracts can be illiquid and buyers may be exposed to the default of the protection seller.

MODULE QUIZ 51.2

1. Which of the following statements regarding foreign currency defaults is most accurate?
   A. In recent years, defaults have often been followed by military actions.
   B. Greater central bank independence means less difficulty for a country to print money.
   C. Prior to the 20th century, no country had ever defaulted on funds borrowed in a foreign currency.
   D. Countries are more likely to default on funds borrowed from foreign banks than on sovereign bond issues.

2. Which of the following reasons least likely explains local currency defaults?
   A. Countries may decide that the costs of higher inflation are less than the costs of default.
   B. Countries may decide that the costs of currency debasement are higher than the costs of default.
   C. Shared currencies like the euro make it difficult for countries to control their own monetary policy.
   D. Prior to 1971, the use of the gold standard prior made it more difficult for some countries to print money.

KEY CONCEPTS

LO 51.a

Key sources of country risk include (1) the level of a country's economic growth; (2) political risk; (3) legal risk, including risks that arise from both the structure and the efficiency of legal systems; and (4) economic structure including the level of diversification.

Regarding economic growth life cycle, more mature markets and companies within those markets are less risky than those firms and countries in the early stages of growth.

Regarding political risk, there are at least four components of political risk, including the level of corruption in the country, the occurrences of physical violence due to wars or civil unrest, the possibility of nationalization and expropriations, and the continuity and severity of risks versus discontinuous risks.

Regarding legal risks, the protection of property rights and the speed with which disputes are settled affect default risk.

Regarding economic structure, the level of economic diversification and competitive advantage are important factors in assessing country risk. A disproportionate reliance on a single commodity or service in an economy increases a country's risk exposure.

LO 51.b

Companies such as Political Risk Services (PRS) and media outlets such as *The Economist* and *Euromoney* evaluate more than 100 countries on key areas of country risk. Some are critical of these composite risk measures because not all measures are equally important to all investors, and they are not readily comparable with each other due to a lack of standardization across the information providers. Also, the scores are better used as rankings than as a way to interpret the relative risk of countries.

LO 51.c

Sovereign credit risk refers to the risk that holders of government-issued debt fail to receive the full amount of promised interest and principal payments. Countries are often without the foreign currency to meet the debt obligation and are unable to print money to repay the debt. This makes up a large proportion of sovereign defaults.

Many of the countries that defaulted on foreign currency debt over the last several decades were simultaneously defaulting on local currency debt. Three reasons may explain local currency defaults: (1) the use of the gold standard prior to 1971 made it more difficult for some countries to print money, (2) shared currencies, such as the euro, make it impossible for countries to control their own monetary policy, and (3) some counties must conclude that the costs of currency debasement and potentially higher inflation are greater than the costs of default.

LO 51.d

Sovereign defaults may be preceded by difficult economic conditions. Defaulting countries typically suffer reputational loss, increasing future borrowing costs and reduced investment in stock and bond markets. Countries may also be faced with an economic downturn and political instability as a result of a default.

LO 51.e

Several factors determine a country's sovereign default risk. The country's level of indebtedness, its social security commitments, the level and stability of tax receipts, political risks, and implicit guarantees from other countries all impact a country's likelihood of defaulting on sovereign debt.

Rating agencies consider several factors when evaluating default risk. These factors are related to the economic, political, and institutional characteristics of a country with respect to its ability to repay debt. Rating agencies also assess rating transitions, looking at the probability that a country would maintain its current rating or transition to a different rating category.

LO 51.f

The sovereign credit spread reflects the sovereign bond yield compared to a riskless investment in the same currency and maturity. Advantages of the sovereign risk spreads relative to sovereign bond ratings are that changes occur in real time and risk premiums adjust to new information more quickly. Credit default swaps (CDSs), which are similar to insurance contracts, provide important information about sovereign credit spreads.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 51.1

1. **B** Some investors prefer the stability of investing in countries with autocratic governments because government policies are locked in and generally more predictable compared to democratic countries where an election can significantly change government policies. Risks in a democracy are continuous, but usually low. In contrast, risks in a dictatorship are discontinuous. Policies change much less frequently, but changes are often severe and difficult to protect against. Chowdhury is willing to accept the bigger, discontinuous risk as a trade-off for the more frequent, but less damaging, continuous risk. (LO 51.a)

2. **D** Numerous services attempt to evaluate country risk in its entirety. They include Political Risk Services (PRS), *The Economist*, *Euromoney*, and the World Bank. *Euromoney* surveys 400 economists who assess country risk factors and rank countries from 0 to 100, with higher numbers indicating lower risk. (LO 51.b)

Module Quiz 51.2

1. **D** Historically, countries have been more likely to default on foreign bank debt than on sovereign bonds. Defaults were often followed by military actions, but this is not true in recent times. Greater central bank independence means it is more difficult for a country to print money. Over the last 200 years, there are many instances of default. (LO 51.e)

2. **A** Countries may decide that the costs of *higher* inflation are higher (not lower) than the costs of default, so a default is comparatively less expensive. In general, the following factors explain local currency defaults: (1) prior to 1971, the use of the gold standard made it more difficult for some countries to print money, (2) shared currencies including the euro make it impossible for countries to control their own monetary policy, and (3) some counties may conclude that the costs of currency debasement and potentially higher inflation are *greater* than the costs of default. (LO 51.c)

[1] M. E. Porter, “The Competitive Advantage of Nations,” Harvard Business Review, March–April, 1990. https://hbr.org/1990/03/the-competitive-advantage-of-nations.

[2] International Monetary Fund, www.imf.org.

[3] Source: S&P

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 6.

READING 52

MEASURING CREDIT RISK

Study Session 13

EXAM FOCUS

Since financial institutions hold many assets, we need to examine the expected and unexpected losses in a portfolio setting. For the exam, be familiar with the calculations of expected loss, unexpected loss, and the risk contribution of each asset in a portfolio. In addition, know the two approaches for calculating credit risk capital under Basel II. This reading also looks at different ways of measuring credit losses and modeling credit risk, including the Gaussian copula model, the Vasicek model, the CreditMetrics model, and Euler's theorem. While calculations help absorb difficult concepts, your goal should be to understand the primary rationale and benefits (and limitations) of using these models.

MODULE 52.1: EXPECTED AND UNEXPECTED LOSS

Economic Capital vs. Regulatory Capital

LO 52.a: Explain the distinctions between economic capital and regulatory capital and describe how economic capital is derived.

The best estimate of the devaluation of a risky asset is expected loss. However, the unexpected loss can exceed the expected loss by a wide margin. The risk of losses from counterparty nonpayment or default is measured by credit risk. If the bank holds inadequate reserves relative to losses, there is a possibility that the bank will become impaired. Therefore, it is vital that the bank hold capital reserves to buffer itself from unexpected losses so it can absorb large losses and continue to operate. When there are losses, they are absorbed from equity capital.

A bank's own estimate of capital is called **economic capital**. The question of how much capital a bank needs to hold depends on a bank's estimate of possible losses, but it also depends on its capital structure, including its level of debt relative to equity.

By contrast, **regulatory capital** is the capital that regulators require banks to keep. Globally, banks' regulatory capital is determined by the Basel Committee on Banking Supervision. The Basel Committee's first main regulatory framework known as the

Basel I Accord came into effect in 1996 and determined the required level of credit risk capital for banks. In 2004, the Basel II Accord was introduced which changed the capital requirements for credit risk and also introduced capital requirements for market and operational risk. These frameworks will be superseded by the Basel III Accord.

Basel II features two approaches for calculating credit risk capital: (1) the **standardized approach**, and the (2) **internal ratings-based (IRB) approach**. The standardized approach involves the use of credit ratings. The IRB approach involves the use of the Vasicek model, which we will cover later in this reading.

Default Data

LO 52.b: Describe the degree of dependence typically observed among the loan defaults in a bank's loan portfolio, and explain the implications for the portfolio's default rate.

Evaluating default data is important when assessing credit risk. Defaults tend to cluster, meaning that they tend not to default independently of each other. Some years will have higher (corporate) default rates, while others will have lower. For example, there was a significant increase in default rates in 1990, 2001, and 2009. One of the important reasons for this is due to systemic risk in the economy. Good economic conditions tend to result in lower probability of default, while weaker conditions increase defaults and correlations. The increase in correlations during bad economic times is called *credit contagion*, which is especially important for financial companies because it can create systemic risk in the banking system. Economic conditions represent risk that cannot be diversified away.

Credit Risk Factors

The **probability of default (PD)** is the likelihood that a borrower will default; however, this measure is not necessarily the creditor's greatest concern. A borrower may briefly default and then quickly recover by making a payment or paying interest charges or penalties for missed payments. Creditors must rely on other measures of risk in addition to PD.

The **exposure**, also referred to as **exposure at default (EAD)**, is the loss exposure stated as a dollar amount (e.g., the loan balance outstanding). EAD can also be stated as a percentage of the nominal amount of the loan or the maximum amount available on a credit line.

The **loss rate**, also referred to as **loss given default (LGD)**, represents the likely percentage loss if the borrower defaults. The severity of a default is equally as important to the creditor as the likelihood that the default would occur in the first place. If the default is brief and the creditor suffers no loss as a result, it is less of a concern than if the default is permanent and the creditor suffers significant losses. Both PD and LGD are expressed as percentages. Note that, by definition, LGD = 1 – recovery rate (RR). Therefore, the factors that affect the loss rate will also impact the recovery rate.

Expected and Unexpected Loss

LO 52.c: Define and calculate expected loss (EL).

LO 52.d: Define and explain unexpected loss (UL).

Expected loss (EL) is the expected decline in the value of a risky asset that the bank has taken onto its balance sheet. EL is calculated as the product of the expected default rate (i.e., probability of default [PD]) and loss in the event of default (i.e., loss given default [LGD]):

EL = PD × LGD

EXAMPLE: Computing expected loss

Calculate expected loss if a bank expects a 1.8% default rate on its loans assuming the recovery rate in the event of default is 60%.

Answer:

EL = 1.8% × (1 − 0.60) = 0.72%

Note that the expected loss is an approximation. The 0.72% in the above example represents an average loss expectation for the bank's portfolio in percentage terms. In dollar terms, EL would be multiplied by EAD, meaning that EL increases with increases in exposure:

EL = EAD × PD × LGD

Note that the actual loss in the event of default on its assets may be higher or lower than the expected loss. The difference between the actual loss and expected loss is called the **unexpected loss (UL)**.

The unanticipated loss on the risky asset can arise from the incidence of default or credit migration. Credit migration denotes the possible deterioration in creditworthiness of the borrower. While a shift in migration may not result in immediate default, the probability of such an event increases. It is also possible for the reverse to occur, that is, the credit quality of the obligor improving over time.

Linking this to our discussion on value at risk (VaR) in previous readings, required capital is equal to VaR minus EL, over a one-year time horizon at an x% confidence level. The Basel Committee sets x at 99.9% for regulatory capital under the IRB approach. A 99.9% probability would correspond to events that are expected to occur once every 1,000 years only. Interestingly, banks tend to be even more conservative than this probability in order to maintain their credit ratings. For example, a AA-rated bank would have a default probability of no more than 0.02% in one year to keep its AA rating, corresponding to a 99.98% probability of no default (i.e., x = 99.98%).

MODULE QUIZ 52.1

1. XYZ Bank is trying to forecast the expected loss on a loan to a mid-size corporate borrower. It determines that there will be a 75% loss if the borrower does not perform the financial obligation.

This risk measure is the:
A. probability of default.
B. loss rate.
C. unexpected loss.
D. exposure at default.

2. Which of the following statements about expected loss (EL) and unexpected loss (UL) is true?
A. Expected loss always exceeds unexpected loss.
B. Unexpected loss always exceeds expected loss.
C. Expected loss requires quantifying the actual loss.
D. Expected loss is directly related to the exposure amount.

3. The relationship between expected loss (EL), unexpected loss (UL), and actual loss can be best described as:
A. actual loss = EL + UL.
B. actual loss = EL – UL.
C. actual loss = EL × UL.
D. actual loss = UL – EL.

MODULE 52.2: MEASURING CREDIT LOSSES AND MODELING CREDIT RISK

LO 52.e: Estimate the mean and standard deviation of credit losses assuming a binomial distribution.

For a portfolio consisting of *n* loans, the loss on the *i*th loan default with a face value of *L* will be $L_i(1 - RR_i)$, where RR_i is the recovery rate in the event of default on the *i*th loan. The probability distribution of loan losses in the portfolio takes on a binomial distribution, meaning that there will be a probability of losses equal to (PD_i) and a probability of no losses equal to $(1 - PD_i)$.

Therefore, the mean loss can be calculated as:

$$PD_i \times L_i(1 - RR_i) + (1 - PD_i) \times 0 = PD_i \times L_i(1 - RR_i)$$

The standard deviation of loss from the *i*th loan can be calculated as:

$$\sigma_i^2 = E(Loss^2) - [E(Loss)]^2$$

Here, $E(Loss) = PD_i \times L_i(1 - R_i)$. Therefore, we can rewrite the standard deviation of loss from the *i*th loan as:

$$\sigma_i = \sqrt{PD_i - PD_i^2} \times \left[L_i(1 - RR_i)\right]$$

From the individual standard deviation of loan losses, we can then calculate the standard deviation of the loss on a portfolio of loans, by introducing the correlation between losses, ρ_{ij}:

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\sigma_i\sigma_j$$

If we now make the simplifying assumptions that all loans have the same principal *L* and the standard deviation of the loss from loan *i* is the same for all *i*, then the standard deviation simplifies to:

$$\sigma = \sqrt{PD - PD^2} \times [L(1 - RR)]$$

In addition, we can also calculate the standard deviation of the loss from the loan portfolio as a percentage of its size:

$$\alpha = \frac{\sigma_P}{nL} = \frac{\sigma\sqrt{1 + (n - 1)\rho}}{\sqrt{n} \times L}$$

EXAMPLE: Computing standard deviation of loss

Suppose that a bank has a portfolio with 10,000 loans, and each loan is EUR 1 million and has a 0.5% PD in a year. Also assume that the recovery rate is 30% and correlation between losses is 0.2. **Calculate** the standard deviation of the loss from each individual loan as well as the standard deviation of the loss from the loan portfolio as a percentage of its size. Assume that L = 1.

Answer:

$$\sigma = \sqrt{0.005 - 0.000025} \times (1 \times 0.7) = 0.04937$$

$$\alpha = \frac{0.04937\sqrt{1 + (9{,}999 \times 0.2)}}{\sqrt{10{,}000} \times 1} = 0.02208$$

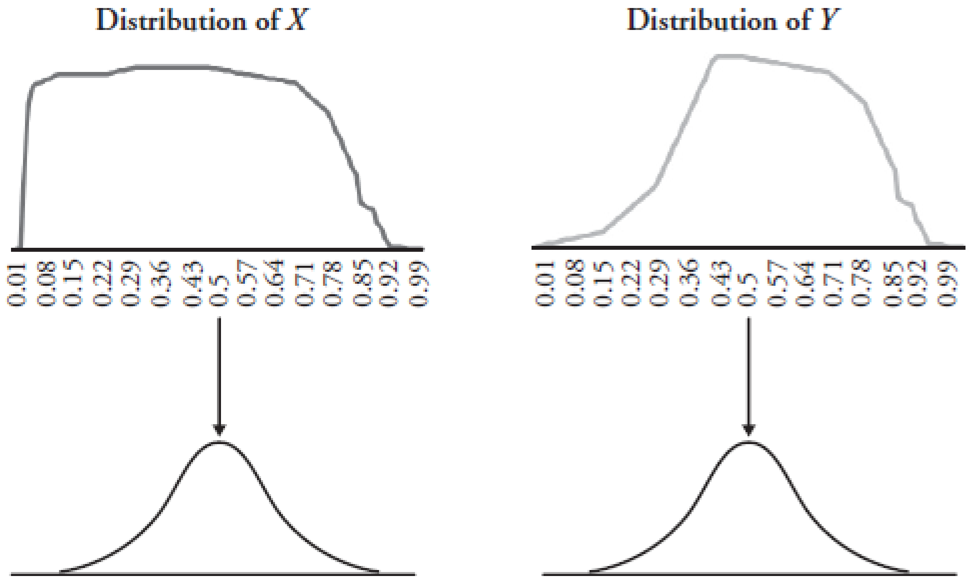
Gaussian Copula Model

LO 52.f: Describe the Gaussian copula model and its application.

A **copula** creates a joint probability distribution between two or more variables while maintaining their individual marginal distributions. The **Gaussian copula model** maps the probability distributions of two variables that may have unique shapes to the standard normal distribution. The normal distribution has a mean of zero and a standard deviation of one. The mapping of each variable to the new standard normal distribution is done on a percentile-to-percentile basis. The Gaussian copula is used by both bank and regulator models. Note that it is just one of several copula models that can be used to define joint distributions.

Figure 52.1 illustrates how the observations of the nonnormal distributions are mapped to the standard normal distribution with the Gaussian copula.

Figure 52.1: Mapping Nonnormal Distributions to the Standard Normal Distribution



For example, the one-percentile point in the nonnormal distributions V_1 and V_2 are mapped to the one-percentile points on the standard normal distributions U_1 and U_2, which has a value of −2.326. Similarly, the five-percentile points are mapped to the five-percentile points on the standard normal distribution which has a value of −1.645. This is repeated for each observation on a percentile-to-percentile basis.

Now a correlation structure can be defined between the distributions U_1 and U_2 which are bivariate normal. However, defining correlations may be challenging, and a large number of distributions requires a large number of correlation parameters. One way to solve for this problem is to use a one-factor model. A typical one-factor model can be illustrated as:

$$U_i = a_i F + \sqrt{1 - a_i^2}\, Z_i$$

In this equation, F is a factor common to all the U_i and Z_i is the component of U_i unrelated to F. The Z_i factors are uncorrelated with each other. Also, the a_i parameters have values between −1 and +1.

When the correlation between the two standard normal distributions is considered, the coefficient of correlation becomes:

$$E\left[\left(a_i F + \sqrt{1 - a_i^2}\, Z_i\right)\left(a_j F + \sqrt{1 - a_j^2}\, Z_j\right)\right]$$

Since F is uncorrelated with all the Z_i parameters, and Z_i is uncorrelated with Z_j:

$$(FZ_i) = (FZ_j) = (Z_i Z_j) = 0$$

Therefore, the coefficient of correlation between U_i and U_j is $E(a_i a_j F^2)$, which is simply $a_i a_j$, since $E(F^2) = 1$.

The best-known one-factor model is the capital asset pricing model (CAPM), where the single factor is the return from the market index. In the next section, we will discuss a one-factor model that applies to probabilities of default.

Vasicek Model

LO 52.g: Describe and apply the Vasicek model to estimate default rate and credit risk capital for a bank.

The **Vasicek model** is a mathematical model that is used to determine regulatory capital. The model helps assess counterparty risk by defining both EL and UL. The model is concerned with the left tail of a distribution, that is, credit losses and defaults.

The left tail risk can be measured through the standard normal distribution U_i, where a company defaults if:

$$U_i \le N^{-1}(PD)$$

where N^{-1} is the inverse cumulative normal distribution

For example, if the PD = 2%, a company defaults if $U_i \le N^{-1}(0.02) = -2.054$. Any value of U_i less than −2.054 corresponds to a default, while any value above it (that is, between −2.054 and positive infinity) corresponds to no default.

Assuming a 99.9% probability of no default (i.e., a 0.1% PD), we can set up the 99.9 percentile for default rate as:

$$N\left(\frac{N^{-1}(PD) + \sqrt{\rho}\,N^{-1}(0.999)}{\sqrt{1-\rho}}\right)$$

This model allows us to convert the average PD into a portfolio default with a 0.1% probability (something that occurs only 1 in every 1,000 years). The default rate increases with both the PD and the correlation coefficient.

We can also extend the Vasicek model to calculate the UL for an individual loan or a loan portfolio. Under the IRB approach of Basel II capital requirements, the unexpected loss will be:

$$UL = (WCDR - PD) \times LGD \times EAD$$

where:

WCDR = worst-case default rate, or the 99.9 percentile of the default rate distribution

PD = probability of default

LGD = loss given default (or 1 – recovery rate)

EAD = exposure at default, or the sum of all loan exposures at default

Note that in this equation, all loans have the same PD, same correlation, and same LGD. Also, for a portfolio of loans, the UL would be the sum of the individual loan ULs.

CreditMetrics Model

LO 52.h: Describe the CreditMetrics model and explain how it is applied in estimating economic capital.

The **CreditMetrics model** is mainly used by financial institutions to determine economic capital. Under the model, borrowers are assigned both an internal and external credit rating, and the probability that the borrower's rating will change within a one-year period is assessed through a credit loss distribution. The assessments are typically done through Monte Carlo simulations, which use the Gaussian copula model. The Monte Carlo simulation transforms each borrower's rating transition into a normal distribution incorporating the correlations between the normal distributions. Correlations are often estimated from the correlations between exchange-traded equity returns.

The CreditMetrics model differs from the Vasicek model because it not only considers the impact of defaults, but also the impact of rating changes. Under the CreditMetrics model, a financial institution may incur a credit loss even if there was no default but there was a rating transition.

Euler's Theorem

LO 52.i: Describe and apply Euler's theorem to determine the contribution of a loan to the overall risk of a portfolio.

Euler's theorem allows a portfolio's homogeneous risk functions to be decomposed into their component contributions. This can be particularly useful when, for example, we are interested in determining the contribution of each underlying loan to the overall loan portfolio risk.

The homogeneous function can be written as a set of risk variables for functions F and a constant λ:

$$(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = (x_1, x_2, \ldots, x_n)$$

The relationship between the functions and variables can then be written as:

$$Q_i = x_i \frac{\Delta F_i}{\Delta x_i} = \frac{\Delta F_i}{\frac{\Delta x_i}{x_i}}$$

where:
Δx_i = small change in variable i
ΔF_i = resultant small change in F

Euler showed that as x_i gets smaller, the risk function F simplifies to the sum of the individual Q_i components:

$$F = \sum_{i=1}^{n} Q_i$$

Consider a portfolio with three loans. Suppose that the standard deviation of losses for loan(1) = 1.2, loan(2) = 0.8, and loan(3) = 0.8. The correlation matrix for the three loans is given in the following table.

	Loan(1)	Loan(2)	Loan(3)
Loan(1)	1	0	0
Loan(2)	0	1	0.6
Loan(3)	0	0.6	1

We can determine the portfolio's standard deviation of total losses as:

$$\sqrt{(1.2^2 + 0.8^2 + 0.8^2 + 2 \times 0.6 \times 0.8 \times 0.8)} = 1.87$$

We can then decompose this total risk into the individual contributions of the three loans. In order to do that, suppose that the size of loan(1) increased by 1%. The standard deviation of the loss from loan(1) would then increase to 1.2 × 1.01 = 1.212. We can now calculate the increase in the standard deviation of the loan portfolio:

$$\sqrt{(1.212^2 + 0.8^2 + 0.8^2 + 2 \times 0.6 \times 0.8 \times 0.8)} - \sqrt{(1.2^2 + 0.8^2 + 0.8^2 + 2 \times 0.6 \times 0.8 \times 0.8)} = 0.007733$$

From our earlier description of Q_i, Q_1 is then 0.007733 / 0.01 = 0.7733.

Similarly, Q_2 = 0.5492 and Q_3 = 0.5492. Thus:

$$Q_1 + Q_2 + Q_3 = 0.7733 + 0.5492 + 0.5492 = 1.87$$

We have therefore decomposed the portfolio's standard deviation of total losses into the individual contributions of the three loans.

MODULE QUIZ 52.2

Use the following information to answer Questions 1 and 2.

Suppose that a bank has a portfolio with 50,000 loans, and each loan is USD 1 million with a 1.1% PD in a year. The recovery rate is 40% and the correlation between loans is 0.2. Assume that L = 1.

1. The standard deviation of the loss from each individual loan is closest to:
   A. 0.01100.
   B. 0.01088.
   C. 0.04172.
   D. 0.06258.

2. The standard deviation of the loss from the loan portfolio as a percentage of its size is closest to:
   A. 0.015288.
   B. 0.027988.
   C. 0.041622.
   D. 0.055975.

3. A risk manager is looking to determine the contribution of each underlying loan to the overall loan portfolio risk. The risk manager should use:
   A. CreditMetrics.
   B. Euler's theorem.
   C. the Gaussian copula.

D. the Vasicek model.

MODULE 52.3: CHALLENGES IN ASSESSING CREDIT RISK

LO 52.j: Explain why it is more difficult to calculate credit risk capital for derivatives than for loans.

Calculating required capital for banks includes calculating EAD. However, while calculating EAD is easier for loans since it is simply the amount advanced, it is more challenging for derivatives. This is because derivatives exposures vary daily as the value of the derivative changes. Basel rules set EAD for derivatives as current exposure plus an add-on amount. Here, *current exposure* is the amount that the bank could lose if its counterparty defaulted. The *add-on amount* serves as a buffer if the exposure became worse by the time of the default.

In addition, unlike loans, derivatives are subject to netting agreements. As a result, if netting exists, all derivatives with a single counterparty are considered a single derivative if the counterparty defaulted, and therefore calculating required capital must be done on a counterparty basis (rather than on a transaction basis).

LO 52.k: Describe challenges to quantifying credit risk.

Assessing credit risk has several limitations:

1. Regulators require that banks calculate *through-the-cycle* PD for regulatory capital because it removes the volatility of business cycles. However, banks may prefer a *point-in-time* PD for their internal purposes. The PD over an economic cycle may not be the best representation of current PD, in which case the point-in-time PD would be a better predictor of credit risk.
2. An economic downturn has a dual negative effect on banks because default rates increase during downturns, while the recovery rate declines (and therefore LGD increases).
3. Calculating EAD is more challenging for derivatives than for loans. In addition, **wrong-way risk** captures the increase in credit exposure of an entity to a counterparty at the same time that the credit risk of the counterparty increases. In other words, an increase in the exposure value to a counterparty coincides with a higher default risk of the counterparty. This is especially important in derivatives transactions.
4. Correlations can be difficult to estimate.
5. In addition to credit risk, entities face many other risks, including market, operational, liquidity and strategic risk that could all impact a bank's required capital.

MODULE QUIZ 52.3

1. Which of the following statements regarding challenges in calculating credit risk capital for derivatives is least accurate?
   A. It is easier to calculate EAD for loans than for derivatives.
   B. Netting arrangements simplify calculating credit risk capital for derivatives.

C. Basel rules set EAD for derivatives as current exposure plus an add-on amount.
D. For derivatives with netting arrangements, calculating required capital must be done on a counterparty basis.

KEY CONCEPTS

LO 52.a

A bank's own estimate of the capital it requires is called economic capital. A bank must hold enough economic capital to buffer itself from unexpected losses in order to continue to operate.

While economic capital measures a bank's own estimate of capital, regulatory capital is the capital regulators require banks to keep. Basel II requires that banks calculate credit risk capital either under the standardized approach or the internal ratings-based (IRB) approach.

LO 52.b

Weak economic conditions increase defaults and correlations. The increase in correlations during bad economic times is known as credit contagion.

LO 52.c

Expected loss (EL) is the expected decline in the value of a risky asset that the bank has taken onto its balance sheet. EL is the product of the expected default rate (i.e., probability of default [PD]) and loss in the event of default (i.e., loss given default [LGD]):

$$EL = PD \times LGD$$

LO 52.d

The difference between the actual loss and expected loss is called the unexpected loss (UL).

LO 52.e

For a portfolio consisting of n loans, the loss on the ith loan default with a face value of L will be $L_i(1 - RR_i)$. If all loans have the same principal L and the standard deviation of the loss from loan i is the same for all i, then the standard deviation simplifies to:

$$\sigma = \sqrt{PD - PD^2} \times [L(1 - RR)]$$

The standard deviation of the loss from the loan portfolio as a percentage of its size is:

$$\alpha = \frac{\sigma_P}{nL} = \frac{\sigma\sqrt{1 + (n-1)\rho}}{\sqrt{n} \times L}$$

LO 52.f

The Gaussian copula is used by both banks and regulators to map the probability distributions of two variables that may have unique shapes to the standard normal distribution, where the normal distribution has a mean of zero and a standard deviation of one. Mapping of variables is done on a percentile-to-percentile basis.

LO 52.g

The Vasicek model can be used to determine regulatory capital and assess counterparty risk by defining both EL and UL. The model looks at the left tail distribution of defaults. UL is defined as (WCDR − PD) × LGD × EAD.

LO 52.h

The CreditMetrics model is used to determine economic capital. Borrowers are assigned credit ratings and Monte Carlo simulation is used to assess the probability that the borrower's rating could change during a one-year period. The CreditMetrics model differs from the Vasicek model because it not only considers the impact of defaults, but also the impact of rating changes.

LO 52.i

Euler's theorem lets a portfolio's risk functions to be decomposed into their individual contributions and determines what the contribution of each underlying loan to the overall loan portfolio risk is.

LO 52.j

Calculating EAD is more challenging for derivatives than for loans because derivatives exposure varies daily. In addition, unlike loans, derivatives are subject to netting agreements, under which derivatives with a single counterparty are considered a single derivative if the counterparty defaulted.

LO 52.k

Limitations of assessing credit risk include (1) banks' need to calculate both through-the-cycle PD and point-in-time PD, (2) economic downturns lead to higher default rates while at the same time increasing LGD, (3) challenges in calculating EAD for derivatives, which may also be subject to wrong-way risk, (4) difficulty in estimating correlations, and (5) other risks including market, operational, liquidity, and strategic risk all impact a bank's required capital.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 52.1

1. **B** Current measures used to evaluate credit risk include the firm's probability of default, which is the likelihood that a borrower will default, the loss rate (loss given default), which represents the likely percentage loss if the borrower defaults, the exposure amount (exposure at default), and the expected loss. The stated 75% loss if the borrower defaults is the loss rate. (LO 52.b)

2. **D** EL increases with increases in the exposure amount. UL typically exceeds EL, but they are both equal to zero when probability of default is zero. UL, not EL, requires quantifying actual loss. (LO 52.c)

3. **A** Since UL can be defined as the difference between the actual loss and expected loss, actual loss = EL + UL. (LO 52.d)

Module Quiz 52.2

1. **D** The standard deviation of the loss on each individual loan can be calculated as:

   $$\sigma = \sqrt{PD - PD^2} \times [L(1 - RR)]$$

   Therefore,

   $$\sigma = \sqrt{(0.011 - 0.000121)} \times 1 \times (1 - 0.4) = 0.062581$$

   (LO 52.e)

2. **B** The standard deviation of the loss from the loan portfolio as a percentage of its size can be calculated as:

   $$\alpha = \frac{\sigma_P}{nL} = \frac{\sigma\sqrt{1 + (n - 1)\rho}}{\sqrt{n} \times L}$$

   Therefore,

   $$\alpha = \frac{0.062581\sqrt{1 + (49{,}999 \times 0.2)}}{\sqrt{50{,}000} \times 1} = 0.027988$$

   (LO 52.e)

3. **B** Euler's theorem allows a portfolio's risk functions to be decomposed into their component contributions; for example by determining the contribution of each underlying loan to the overall loan portfolio risk. (LO 52.i)

Module Quiz 52.3

1. **B** Netting arrangements make calculating risk capital for derivatives more challenging because all derivatives with a single counterparty are considered a single derivative if the counterparty defaulted, and therefore calculating required capital must be done on a counterparty basis rather than on a transaction basis. (LO 52.j)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 7.

READING 53

OPERATIONAL RISK

Study Session 13

EXAM FOCUS

This reading introduces operational risk by defining operational risk, which includes both internal failures and external events, and discussing the types of operational risk and bank business lines that must be considered when calculating operational risk capital requirements. Collecting data for loss frequency and loss severity distributions is an important component of allocating operational risk capital among various business lines. Methods for finding the necessary operational loss data points are based on both internal and external data and historical and forward-looking approaches.

MODULE 53.1: DEFINING AND MEASURING OPERATIONAL RISK

Some firms define **operational risk** as all risk that is not credit or market risk. However, most people agree that this definition is far too broad. Other definitions of operational risk include the following:

- Risks arising from operational mistakes, including incorrect processing of bank transactions but excluding fraud, cyberattacks, and damage to assets
- Financial risk that is not caused by market risk (e.g., unexpected asset price movements) or credit risk (e.g., the failure of a counterparty to meet financial obligations)
- Any risk developing from a breakdown in normal operations (e.g., system failures or processing mistakes)
- Adverse change from operations events, including internal failures and external events

In 2011, the Basel Committee on Banking Supervision defined operational risk as "the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events."[1]

Traditional definitions of operational risk look at internal functions or processes, human factors, systems, firm infrastructure, and outside events. These operational risk definitions explicitly include legal risk, but do not address reputational risk or strategic risk.

Operational Risk Categories

LO 53.a: Describe the different categories of operational risk and explain how each type of risk can arise.

The Basel Committee disaggregates operational risk into seven types. A majority of the operational risk losses result from clients, products, and business practices.

1. *Internal fraud*. Disobeying the law, regulations, and/or company policy, or misuse of company property. Examples include misreporting data, employee theft, and insider trading.
2. *External fraud*. Actions by a third party that disobey the law or misuse property. Examples include check kiting, robbery, and computer hacking.
3. *Employment practices and workplace safety*. Actions that do not follow laws related to employment or health and safety. Examples include workers' compensation, organized labor activities, discrimination disputes, and disobeying health and safety rules.
4. *Clients, products, and business practices*. Failure (either intentional [negligent] or unintentional) to perform obligations for clients. Examples include mishandling of confidential information, breaches in fiduciary duty, and money laundering.
5. *Damage to physical assets*. Damage occurring from events, such as natural disasters. Examples include vandalism, terrorist attacks, earthquakes, and fires.
6. *Business disruption and system failures*. Examples include computer failures, both hardware- and software-related, and utility outages.
7. *Execution, delivery, and process management*. Failure to correctly process transactions and the inability to uphold relations with counterparties. Examples include data entry errors, collateral management problems, and incomplete legal documents.

Large Operational Risks

Operational risk looks at the severity of outcomes, and large risks can cause the largest severity losses. There are three broad types of large risks: cyber risks, compliance risks, and rogue trader risk.

Cyber risks are risks of losses from external attacks on an institution's systems. They can originate from many sources, including organized crime, hackers, and insiders. They generally lead to fraud, embezzlement, loss/theft of personal or company data, and theft of intellectual property. Companies can protect against cyber risks through conducting regular phishing exercises to educate employees, user account controls, firewalls, and various intruder detection software.

Some of the best-known examples of cyber risks include the 2011 cyberattack on Yahoo, which resulted in the data breach of 3 billion user accounts; the 2016 hacking of the Central Bank of Bangladesh's networks; and the 2017 cyberattack on Equifax, which affected the data of 143 million users.

Compliance risks are risks that an organization will incur fines and penalties as a result of intentional or unintentional failures to follow laws and regulations. Regulatory infractions are especially important because they can result from a small part of an organization's global activities but can lead to hefty fines. Designing adequate software and instituting internal training can mitigate these risks.

Examples include the USD 1.9 billion fine levied on HSBC in 2012 due to lack of adequate anti-money laundering programs; the USD 8.9 billion payment by BNP Paribas to the U.S. government as a result of transacting with sanctioned countries; and the USD 2.8 billion fine levied on Volkswagen for cheating on emissions tests.

Rogue trader risk is the risk that a single employee's (trader's) activities, if not properly detected and supervised, could lead to significant losses for an institution. However, institutions may be tempted to let rogue activities, even if detected, continue unsanctioned if they lead to a profit. One of the most effective ways to protect against rogue trader risk is to separate the activities of the front and back offices: the front office should conduct trading activities, while an independent back office should be responsible for verifying transactions and recordkeeping.

Two of the best-known examples of rogue trading are Nick Leeson at Barings Bank, which resulted in a nearly USD 1 billion loss for the bank that led to its collapse in 1995, and the EUR 4.9 billion loss suffered by Société Générale in 2008 due to the activities of Jérôme Kerviel. Other rogue trading examples involve UBS in 2011 and Allied Irish Bank in 2002.

Operational Risk Regulatory Capital

LO 53.b: Compare the basic indicator approach, the standardized approach, and the advanced measurement approach for calculating operational risk regulatory capital.

The Basel Committee updated its regulations under what became known as Basel II by proposing three approaches for determining the operational risk capital requirement (i.e., the amount of capital needed to protect against the possibility of operational risk losses): (1) the **basic indicator approach**, (2) the **standardized approach**, and (3) the **advanced measurement approach (AMA)**. These regulations explicitly recognized that many large bank losses were operational in nature.

The basic indicator approach and the standardized approach require relatively simple calculations and determine capital requirements as a multiple of gross income at either the business line or the institutional level. The AMA is significantly more complex and offers institutions the possibility to lower capital requirements in exchange for investing in risk assessment and management technologies.

Basic Indicator Approach

With the basic indicator approach, operational risk capital is based on 15% of the bank's annual gross income over a 3-year period, where gross income includes both net interest income and noninterest income:

gross income = interest earned + noninterest income − interest paid

Standardized Approach

For the standardized approach, the bank similarly calculates gross income, but uses eight business lines to calculate a percentage capital charge. With this approach, the capital factor of each business line is multiplied by the annual gross income amount over a 3-year period. The results are then summed to arrive at the total operational risk capital charge. The percentages used in the standardized approach are shown in Figure 53.1.

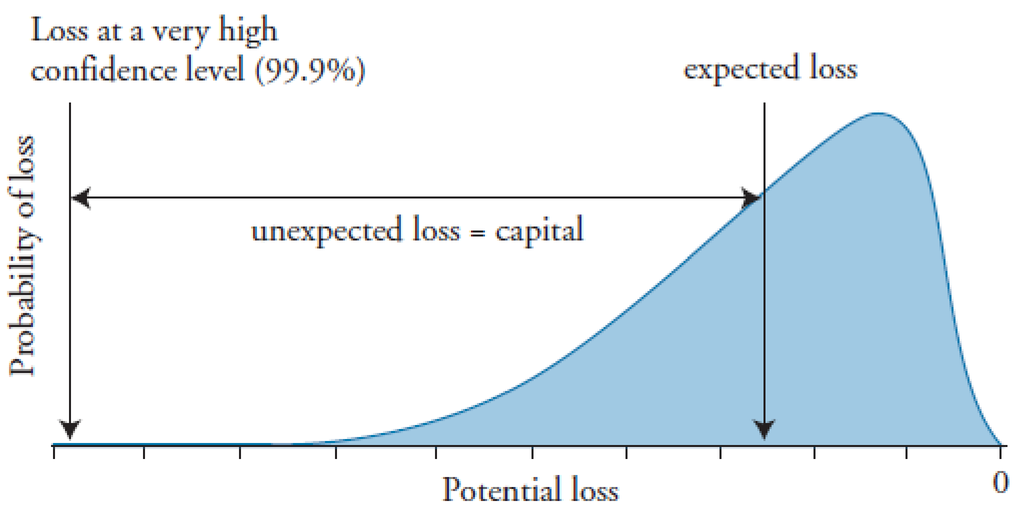
Figure 53.1: Standardized Operational Risk Capital

Business Line	Capital (% of Gross Income)
Corporate finance	18%
Trading and sales	18%
Retail banking	12%
Commercial banking	15%
Payment and settlement	18%
Agency services	15%
Asset management	12%
Retail brokerage	12%

Advanced Measurement Approach (AMA)

Banks that want to take advantage of the possible lower capital requirements available by using the AMA will be required to determine the operational risk capital charge based on internal criteria that are both qualitative and quantitative in nature. In order to use the AMA, banks must satisfy certain requirements in addition to being able to approximate unexpected losses. The operational risk capital requirement is equal to the 99.9 percentile of the loss distribution minus the expected operational loss. This concept is illustrated in Figure 53.2.

Figure 53.2: AMA Capital Requirement



Assuming that a bank was active in each of the eight business lines listed in Figure 53.1, it would have 56 different combinations to estimate 1-year losses (56 = 8 business lines × 7 risk categories).

Similar regulations to Basel II were introduced in 2016 by the European Union: the Solvency II regulatory framework for insurance companies, which specifically incorporate capital requirements for operational risk.[2]

MODULE QUIZ 53.1

1. In constructing the operational risk capital requirement for a bank under the advanced measurement approach (AMA), risks are aggregated for:
   A. commercial and retail banking.
   B. investment banking and asset management.
   C. each of the seven risk types and eight business lines that are relevant.
   D. only those business lines that generate at least 20% of the gross revenue of the bank.

2. Under Basel II regulations, banks using the advanced measurement approach must calculate the operational risk capital charge at a:
   A. 99 percentile confidence level and a 1-year time horizon.
   B. 99 percentile confidence level and a 5-year time horizon.
   C. 99.9 percentile confidence level and a 1-year time horizon.
   D. 99.9 percentile confidence level and a 5-year time horizon.

3. Which of the following is not one of the seven types of operational risk identified by the Basel Committee?
   A. Failed business strategies.
   B. Clients, products, and business practices.
   C. Employment practices and workplace safety.
   D. Execution, delivery, and process management.

4. Which of the following measurement approaches for assessing operational risk would be most appropriate for small banks?
   A. The standardized approach.
   B. The basic indicator approach.
   C. The advanced measurement approach (AMA).
   D. Either the standardized approach or the AMA.

MODULE 53.2: STANDARDIZED MEASUREMENT APPROACH AND LOSS DISTRIBUTION APPROACH

Standardized Measurement Approach

LO 53.c: Describe the standardized measurement approach and explain the reasons for its introduction by the Basel Committee.

While the AMA methodology helped enhance awareness among banks of operational risk, it proved challenging because of the large variability among banks in calculating risk capital. In response, the Basel Committee in 2016 replaced the previous approaches with the **standardized measurement approach (SMA)**.

Under the SMA, a bank is required to first calculate **business indicator (BI)**, which is similar to gross income but adjusted for the bank's size, including trading losses and operating expenses. Based on the range of the BI, a **BI component (BIC)** is computed as a percentage of the BI. In addition, operational loss exposure is determined by computing the **loss component (LC)**:

$$7X + 7Y + 5Z$$

In this equation, X, Y, and Z are estimates of 10-year average annual operational risk losses. X includes all losses, Y includes losses larger than EUR 10 million, and Z includes losses larger than EUR 100 million. For an average bank, the calculations should ensure that the BI component and the loss component are equal.

The loss component allows the SMA to become more sensitive to risk than simply using the BI component alone. It is used to determine the bank's **internal loss multiplier (ILM)**. Depending on the BI range, operational risk capital will equal: BI component × ILM. For an average bank, the internal loss multiplier will equal 1, so operational risk capital is the same as the BI component.

Loss Distribution Approach

LO 53.d: Explain how a loss distribution is derived from an appropriate loss frequency distribution and loss severity distribution using Monte Carlo simulation.

Operational risk losses can be classified along two dimensions: loss frequency and loss severity. **Loss frequency** is defined as the average number of losses in one year, and **loss severity** is defined as the probability distribution of losses. It can be reasonably assumed that these two dimensions are independent.

Loss frequency is most often modeled with a **Poisson distribution** (a distribution that models random events). Using the expected number of losses as λ, the probability of n losses during a year is:

$$\frac{e^{-\lambda}\lambda^{n}}{n!}$$

The parameter λ is equal to the average number of losses over a given time horizon. So if 10 losses occurred over a 5-year period, λ would equal 2 per year (10 / 5 = 2).

Loss severity is often modeled with a **lognormal distribution**. This distribution is asymmetrical (the frequency of high-impact, low-frequency losses is not equal to the frequency of low-impact, high-frequency losses) and fat-tailed (rare events occur more often than would be indicated by a normal distribution). The mean (μ) and standard deviation (σ) are derived from the logarithm of losses. The mean is equal to:

$$\ln\left(\frac{\mu\sigma}{\sqrt{1+w}}\right)$$

The variance of the logarithm of the size of loss is equal to:

$$\ln(1 + w)$$

where:

$w = (\sigma / \mu)^2$

For example, if the mean of the loss size is EUR 60 million and the standard deviation of the loss size is EUR 20 million, then $w = (20 / 60)^2 = 0.11$. Therefore, the mean can be calculated as:

$$\ln\left(\frac{60 \times 20}{\sqrt{1.11}}\right) = \ln\left(\frac{1{,}200}{1.0536}\right) = 7.04$$

The variance is then calculated as:

$\ln(1.11) = 0.104$

Loss frequency and loss severity are combined in an effort to simulate a probability distribution of the loss, typically by using a **Monte Carlo simulation** process. There are four steps in this process:

Step 1: Sample from the Poisson distribution to determine the number of loss events (n).

Step 2: Sample n times from the lognormal distribution of the loss size for each of the n loss events.

Step 3: Sum these n loss sizes for the total loss.

Step 4: This process is continued several thousand times to create a Poisson loss distribution (between 0 and 1).

Having created the loss distribution, the desired percentile value can be measured directly. For example, the 99th percentile would correspond with the loss amount that is greater than 99% of the distribution's data. The difference between the losses at the selected percentile and the mean loss of the distribution equals the unexpected losses at the corresponding confidence level.

Data Limitations

LO 53.e: Describe the common data issues that can introduce inaccuracies and biases in the estimation of loss frequency and severity distributions.

Estimation Procedures

The historical record of operational risk loss data is currently inadequate. This creates challenges when trying to accurately estimate frequency and severity. Given the extreme risk that operational problems create, firms are beginning to build a database of potential loss events. Compared to credit risk losses, the data available for operational risk losses is lacking. Financial institutions need to combine objective data with subjective judgment. It is recommended that banks use internal data when estimating the frequency of losses and utilize both internal and external data when estimating the severity of losses. Institutions can also use sharing agreements to estimate losses experienced by other financial institutions (this may be obtained through vendors like Factiva or Lexis-Nexis).

Potential Biases

Vendor data is less reliable given the inherent reporting biases, since loss data likely only contains relatively large losses from firms that have weak internal controls. As a result, vendor data is more appropriate when used to determine relative loss severity. For example, a bank may use vendor data to determine loss Type 1, and the data indicates it is three times as severe as loss Type 2. If the bank has no information on loss Type 1 but has its own data for loss Type 2, it can conclude that the severity of loss Type 1 would be three times the loss Type 2 it calculated.

Loss severity estimates should be adjusted for inflation. In addition, when viewing external data from other banks, it is necessary to use a scale adjustment that applies the loss event to a bank's own circumstances in order to account for a potential bias in loss size. For instance, if Bank X has a $5 million operational risk loss, how would this loss apply to Bank Y? A simple mathematical proportion will likely over- or underestimate the actual loss. As a result, the accepted scale adjustment for loss size is as follows:

$$\text{estimated loss for Bank Y} = \text{observed loss for Bank X} \times \left(\frac{\text{Bank Y revenue}}{\text{Bank X revenue}}\right)^{\beta}$$

It is generally estimated that a β of 0.23 provides a good fit.

EXAMPLE: Loss size scale adjustment

If the observed loss for Bank Z is $5 million and it has $1 billion in revenue, what will be the estimated loss size adjustment for Bank Y, which has revenue of $2 billion? Assume a β of 0.23.

Answer:

$$\text{estimated loss}_{\text{Bank Y}} = \$5 \text{ million} \times \left(\frac{\$2 \text{ billion}}{\$1 \text{ billion}}\right)^{0.23} = \$5{,}864{,}175$$

Notice that this loss is much less than the proportional estimate of a $10 million loss given that Bank Y has twice the revenue.

Applying Scenario Analysis

LO 53.f: Describe how to use scenario analysis in instances when data are scarce.

Another method for obtaining additional operational risk data points is to use **scenario analysis**. Scenario analysis is particularly useful for estimating losses for high-severity, low-frequency events. Regulators encourage the use of scenarios because this approach allows management to incorporate events that have not yet occurred. This has a positive effect on the firm, since management is actively seeking ways to immunize against potential operational risk losses. The drawback is the amount of time spent by management developing scenarios and contingency plans.

Scenario analysis can incorporate scenarios from a bank's own previous experience or hypothetical scenarios modeled through risk teams. Losses can also be categorized based on estimated frequency of occurrence, including categories for losses that could happen once every 1,000 years, every 100 years, and so on. The key point of this exercise is to consider losses that may not have previously happened (or may not happen frequently), but may happen in the future.

MODULE QUIZ 53.2

1. In modeling risk frequency, it is common to:
   A. use a Poisson distribution.
   B. assume that risks are highly correlated.
   C. assume risk frequency and severity are the same.
   D. use straight-line projection from the most recent loss data.

MODULE 53.3: OPERATIONAL RISK MANAGEMENT

Forward-Looking Approaches

LO 53.g: Describe how to identify causal relationships and how to use Risk and Control Self-Assessment (RCSA), Key Risk Indicators (KRIs), and education to understand and manage operational risks.

It is important for management to use forward-looking approaches in an attempt to prepare for future operational risk losses. One way to accomplish this objective is to learn from the mistakes of other companies, including financial disasters and losses suffered due to rogue traders.

Analyzing causes of losses is a convenient method of identifying potential operational risks. Relationships are analyzed to check for a correlation between firm actions and operational risk losses. For example, if employee turnover or the use of a new computer system demonstrates a strong correlation with losses, the firm should investigate the matter. It is necessary to conduct a cost-benefit analysis if significant relationships are discovered.

One of the most frequently used tools in operational risk identification and measurement is the **risk and control self-assessment (RCSA)** program. The basic approach of an RCSA is to survey those managers directly responsible for the operations of the various business lines, including interviewing managers and staff, conducting questionnaires, and reviewing internal and external risk reports (such as those by auditors and regulators). It is presumed that the managers are the closest to the operations and are, therefore, in the best position to evaluate the risks. The issue with this assumption is that you cannot reasonably expect managers to disclose risks that are out of control. A sound risk management program should quantify the frequency and severity of losses and should update assessments periodically (typically annually).

The identification of appropriate **key risk indicators (KRIs)** may also be very helpful when attempting to identify operational risks. Examples of KRIs include employee turnover, number of temporary positions, unfilled positions, and number of transactions that ultimately fail. In order to be valuable as risk indicators, the factors must (1) have a predictive relationship to losses and (2) be accessible and measurable in a timely fashion. The idea of utilizing KRIs is to provide the firm with a system that warns of possible losses before they happen.

Educating employees about good business practices is another important step in mitigating operational risk and avoiding legal disputes and adverse publicity. Education should include training on email and phone communications.

Allocating Operational Risk Capital

LO 53.h: Describe the allocation of operational risk capital to business units.

Allocating operational risk capital to each business unit encourages managers to improve their management of operational risks. Less capital will be allocated to those business units that are able to reduce the frequency and severity of risks. The reduction in capital will increase the unit's return on invested capital measure; however, reducing capital may not be ideal if the costs of reducing certain risks outweigh the potential benefits. It is, therefore, necessary to conduct a cost-benefit analysis and for each business line manager to be allocated the optimal amount of operational risk capital.

Power Law

LO 53.i: Explain how to use the power law to measure operational risk.

The **power law** is useful when evaluating the nature of the tail of a given distribution. The use of this law is appropriate because operational risk losses are likely to occur in the tails. The law states that for a range of variables:

$$P(v > x) \approx Kx^{-\alpha}$$

where:

v = random variable

x = high value of v

K and α = model parameters

The probability that v is greater than x depends on a scale parameter K and on α, where α indicates the thickness of the tail where extreme losses occur (the smaller α is, the thicker the tail).

In general, it has been shown that this equation is true for x values in the top 5% of the distribution. Applicability of the power law has been wide, including incomes of individuals, sizes of corporations, stock trading volumes, and even earthquakes. More recently, applicability of the power law has been extended to operational risk losses.

Insurance

LO 53.j: Explain how the moral hazard and adverse selection problems faced by insurance companies relate to insurance against operational risk.

Managers have the option to insure against the occurrence of operational risks. The important considerations are how much insurance to buy and which operational risks to insure. Insurance companies offer polices on everything from losses related to fire to losses related to a rogue trader. A bank using the AMA for calculating operational risk capital requirements can use insurance to reduce its capital charge. Two issues facing insurance companies and risk managers are moral hazard and adverse selection.

A **moral hazard** occurs when an insurance policy causes an insured company to act differently with the presence of insurance protection. For example, if a firm is insured against a fire, it may be less motivated to take the necessary fire safety precautions. To help protect against the moral hazard issue, insurance companies use deductibles, policy limits, and coinsurance provisions. With coinsurance provisions, the insured firm pays a percentage of the losses in addition to the deductible.

An interesting dilemma exists for rogue trader insurance. A firm with a rogue trader has the potential for profits that are far greater than potential losses, given the protection of insurance. As a result, insurance companies that offer these policies are careful to specify trading limits, and some may even require the insured firm not to reveal the presence of the policy to traders. These insurance companies are also banking on the fact that the discovery of a rogue trader would greatly increase the firm's insurance premiums and greatly harm the firm's reputation.

Adverse selection occurs when an insurance company cannot discern between good insurance risk (low risk) and bad insurance risk (high risk). Since the insurance company offers the same polices to all firms, it will attract more bad risks, because those firms with poor internal controls are more likely to desire insurance. For example, companies with weak cyber security will more likely purchase insurance than companies with strong cyber risk protection, which may find it too expensive. To combat adverse selection, insurance companies must take an active role in understanding each firm's internal controls. Like auto insurance, premiums can be adjusted to adapt to different situations with varying levels of risk.

MODULE QUIZ 53.3

1. A shortcoming of the risk and control self-assessment (RCSA) program is that it does not consider the:
   A. expert opinion of managers.
   B. identification of expected losses.
   C. independent verification of risk identification and measurement.
   D. ongoing assessment of the effectiveness of risk management activities.

KEY CONCEPTS

LO 53.a

The Basel definition of operational risk is "the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events."

Operational risk can be divided into seven types: (1) internal fraud, (2) external fraud, (3) employment practices and workplace safety, (4) clients, products, and business practices, (5) damage to physical assets, (6) business disruption and system failures, and (7) execution, delivery, and process management.

Large risks can cause the largest severity losses, and include cyber risks, compliance risks, and rogue trader risk.

LO 53.b

The three methods for calculating operational risk capital requirements are (1) the basic indicator approach, (2) the standardized approach, and (3) the advanced measurement approach (AMA). The basic indicator and standardized approaches are simpler, while the AMA is more complex. Large banks are encouraged to move from the standardized approach to the AMA in an effort to reduce capital requirements.

LO 53.c

Under the standardized measurement approach (SMA) a bank is required to calculate a business indicator (BI), a BI component (BIC), and a loss component (LC). For an average bank, the BI component and the loss component should be equal.

LO 53.d

Operational risk losses can be classified along two dimensions: loss frequency and loss severity. Loss frequency is defined as the number of losses over a specific time period, and loss severity is defined as the size of a loss, should a loss occur.

Loss frequency is typically modeled with a Poisson distribution. Loss severity is often modeled with a lognormal distribution. A Monte Carlo simulation combines the loss frequency and loss severity to generate a probability distribution of the loss.

LO 53.e

Banks should use internal data to combine objective data with subjective judgment when estimating the frequency of losses, and utilize both internal and external data when estimating the severity of losses. Regarding external data, banks can use sharing agreements to estimate losses experienced by other financial institutions.

Vendor data may contain reporting biases, and as a result vendor data is more appropriate when used to determine relative loss severity.

Loss severity estimates should be adjusted for inflation and also adjusted for loss size using a scale adjustment that applies the loss event to the financial institution's own circumstances.

LO 53.f

Scenario analysis is a method for obtaining additional operational risk data points. It is used to estimate losses for high-severity, low-frequency events using a bank's own previous experience or hypothetical scenarios. Regulators encourage the use of scenarios because they allow management to incorporate events that have not yet occurred.

LO 53.g

Forward-looking approaches are also used to discover potential operational risk loss events. Forward-looking methods include (1) causes of losses, (2) risk and control self-assessment (RCSA), and (3) key risk indicators. Educating employees about good business practices is another important step in mitigating operational risk.

LO 53.h

Allocating operational risk capital to each business unit encourages managers to improve their management of operational risks. However, this should be combined with a cost-benefit analysis.

LO 53.i

The power law is useful in evaluating the nature of the loss tail of a given distribution. The use of this law is appropriate because operational risk losses are likely to occur in the tails.

LO 53.j

Two issues facing insurance companies that provide insurance for operational risks are moral hazard and adverse selection. A moral hazard occurs when an insurance policy causes a company to act differently (that is, riskier behavior) with insurance protection. Adverse selection occurs when an insurance company cannot differentiate between good (low risk) and bad (high risk) insurance risks.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 53.1

1. **C** The construction of the operational risk capital for a bank requires that risks be aggregated over each of the seven types of risk and each of the eight business lines that are relevant for the particular bank. (LO 53.b)

2. **C** Basel II regulations require that for banks using the advanced measurement approach, operational risk capital be calculated at the 99.9th percentile level over a one-year horizon. (LO 53.b)

3. **A** Failed business strategies are not included in the definition of operational risk, which includes (1) internal fraud, (2) external fraud, (3) employment practices and workplace safety, (4) clients, products, and business practices, (5) damage to physical assets, (6) business disruption and system failures, and (7) execution, delivery, and process management. (LO 53.a)

4. **B** The basic indicator approach is more common for less-sophisticated, typically smaller banks. (LO 53.b)

Module Quiz 53.2

1. **A** It is common to use a Poisson distribution to model loss frequency. A Poisson distribution has a single parameter that can be varied to accurately describe loss data. (LO 53.d)

Module Quiz 53.3

1. **C** An RCSA provides no independent verification of risk measurement and identification. (LO 53.g)

[1] Basel Committee on Banking Supervision, *Principles for the Sound Management of Operational Risk* (Bank for International Settlements, June 2011), https://www.bis.org/publ/bcbs195.pdf.

[2] The European Commission, "Risk Management and Supervision of Insurance Companies (Solvency 2)."

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 8.

READING 54

STRESS TESTING

Study Session 13

EXAM FOCUS

Stress testing focuses on extreme events, which have a low probability of occurrence but are high impact if they do occur. The goal is to ensure that an institution has enough liquid assets and capital to manage through these events. For the exam, understand how scenarios are chosen, models are created, and coverage is tested. The relationship between value at risk (VaR) and expected shortfall (ES) to stress testing is important as well, with stressed risk metrics having advantages and disadvantages relative to the more traditional risk measures. Governance over the stress-testing process and the roles of the board of directors, senior management, and internal audit functions are critical to a well-functioning stress-testing framework. Policies and procedures, documentation, validation, and the review of the stress-testing process are all key components of stress-testing governance. Finally, be familiar with the Basel stress-testing principles for banks.

MODULE 54.1: STRESS-TESTING INTEGRATION AND METHODS

Stress Testing vs. Other Risk Measures

LO 54.a: Describe the rationale for the use of stress testing as a risk management tool.

When answering the question of whether an entity has enough liquid assets and capital to manage through various adverse scenarios, **stress testing** can play a critical role in this evaluation. Stress tests themselves are focused on extreme events, which tend to have a low probability of occurrence but high impact if they do occur. While normal market conditions tend to occur with greater frequency, institutions may be unprepared to manage through nonnormal situations. Stress tests can be used alongside value at risk (VaR) and expected shortfall (ES) to present a detailed and enterprise-wide view of financial institution risk.

LO 54.b: Describe the relationship between stress testing and other risk

measures, particularly in enterprise-wide stress testing.

VaR and ES allow a financial institution to determine the percentage certainty that its losses will not exceed the VaR level during a specific time and, if they do exceed this level, the average or expected tail loss (the ES amount). VaR and ES analyses are backward-looking, which allows for the estimation of a loss distribution across a wide range of scenarios over a relatively short time horizon.

Stressed Risk Metrics

LO 54.c: Describe stressed VaR and stressed ES, including their advantages and disadvantages, and compare the process of determining stressed VaR and ES to that of traditional VaR and ES.

While VaR and ES measures typically use comprehensive data over the previous several years, **stressed VaR** and **stressed ES** use data from periods that were specifically stressed, and as such the data is conditional on a recurrence of a specific stressed period. The question to answer is: If we had a repeat of a specific stressful period, would we be able to survive that period? Stressed VaR and ES account for extreme movements during a single stressed period, but short-horizon stress tests are able to account for large movements from many historical stressed periods.

A key advantage of using stressed risk metrics is that they are conservative relative to the typical scenarios incorporated into VaR and ES analyses. In other words, the nature of the calculations should allow more than sufficient capital to be set aside for unexpected losses for future stressed events. Also, stress testing is forward-looking and facilitates questions such as: If this happens, how will it impact our portfolio? Stress testing also focuses on longer time periods than traditional VaR and ES approaches.

On the other hand, a key disadvantage is that risk metrics are stressed and will not necessarily respond to current market conditions; instead, they will be impacted mainly by portfolio assets. Also, stress testing does not provide a probability distribution for losses, as it cannot reasonably estimate all possible outcomes. Backtesting data is also not feasible with stress testing due to the extreme outcomes underlying its modeling.

Key Aspects of Stress Testing

LO 54.d: Explain key considerations and challenges related to developing stress testing scenarios and building stress testing models.

Selecting a time horizon is the first step in creating a stress-test scenario, with the most common periods ranging from three months to two years. Scenarios can be chosen internally or set by outside regulators, and, if based on historical data, are predicated on the assumption that variables will behave in the future how they have in the past. For some variables, proportional changes are assumed; for others, actual changes from the stressed period are modeled. Although linear relationships between risk factors are often assumed, the reality is that correlations tend to increase during stressed economic times.

Various significant historical events and times can be used as a basis for stress-test modeling, including October 1987 (stock market crash), April 1992 (significant movements in 10-year bond yields), August 1998 (Russia defaulting on its bonds), September 2001 (terrorist attacks in the United States), 2007–2009 (U.S. housing-driven recession), September 2008 (Lehman Brothers declares bankruptcy), and 2020 (COVID-19 pandemic).

As an entity creates scenarios for stress testing, it may choose to focus on large changes in critical variables, including equity prices, interest rates, GDP, unemployment rates, overall volatility, commodity prices, default rates, and exchange rates.

These stress tests are typically going to occur on a routine basis, likely as frequent as monthly. **Ad hoc stress tests** are nonroutine scenarios that are also used to assess future negative events and the exposure that an institution has to these events. As an example, a stress test could be performed to estimate how an institution would be impacted by an election having a particular result. Government regulations, policy changes, and other compliance requirements could also be modeled.

The most basic models typically only stress a few key variables. However, a more comprehensive approach to modeling captures the potential behavior of a range of variables. The most critical variables will be **core variables**, while the other, less important inputs are considered **peripheral variables**. Using historical stressed periods is going to be more valuable than periods with normal market conditions. While the immediate impact of adverse scenarios must be captured, models should also reflect **knock-on effects**, which reflect the consequences of other firms' reactions to adverse events. When the U.S. housing market reached its peak in the mid-2000s, a decline in prices as well as an increase in mortgage portfolio losses would have been modeled. What was not adequately modeled was the impact of upside-down mortgages on mortgage-backed securities, increased inventories from homeowners walking away from their mortgages, declines in bond and equity prices due to perceived riskiness, and a decline in interbank lending.

Reverse Stress Testing

LO 54.e: Describe reverse stress testing and describe an example of regulatory stress testing.

While stress testing involves creating scenarios and evaluating their impacts, **reverse stress testing** looks at the combination of situations that could result in the demise of a financial institution. One option for reverse stress testing involves taking a time in the past and accentuating the scenarios that unfolded in order to determine what it would have taken for the institution to fail in that situation. Another approach is to use changes in key factors and model combinations of them in order to create scenarios that would drive firm failure.

Regulatory Stress Testing

In addition to stress tests performed by financial institutions, regulators also design stress tests for banks and insurance companies. For example, the Federal Reserve conducts stress testing on all U.S. banks with over $50 billion in assets. This process is known as the **Comprehensive Capital Analysis and Review (CCAR)**.

Bank regulators will choose from the following scenarios: (1) baseline, (2) adverse, (3) severely adverse, and (4) internal. By selecting the scenarios, regulators can evaluate a bank's ability to survive challenging conditions in a consistent manner. An example of a severely adverse hypothetical scenario could be a global recession where unemployment rises to 10% and credit markets become extremely stressed.

Another regulatory stress test is the **Dodd-Frank Act Stress Test (DFAST)**, which applies to banks with between $10 billion and $50 billion in assets. DFAST scenarios are similar to CCAR. However, unlike CCAR, DFAST does not mandate banks to submit a capital plan that justifies their stress testing results. Note that if a bank fails a regulatory stress test, they will most likely be required to restrict dividends and raise additional capital.

MODULE QUIZ 54.1

1. Each of the following statements accurately reflects why stress testing is an appropriate risk management tool except:
   A. normal market conditions can present a false sense of security.
   B. the extreme scenarios that are modeled are unlikely but still possible.
   C. extreme events tend to have a high probability of occurrence with a moderate impact.
   D. an institution must have sufficient liquid assets and capital to survive an extreme event.

2. Which of the following reasons best explains why institutions use reverse stress tests?
   A. To identify liquidity risk.
   B. To identify risk concentrations.
   C. To assess where multiple risks occur simultaneously.
   D. To test events that threaten the viability of the institution.

3. Relative to other measures of risk, stress testing is more likely to:
   A. use relatively short time horizons.
   B. capture both positive and negative events.
   C. capture a large number of extreme scenarios.
   D. be forward-looking without providing probabilities for loss distributions.

4. Which of the following key variable inputs is least likely to be incorporated into stress-test models?
   A. A 5% decrease in the stock market.
   B. A decline in GDP of 300 basis points.
   C. An increase in interest rates of 300 basis points.
   D. A 5% increase in the national unemployment rate.

5. Which of the following statements most likely describes an advantage of using stressed risk metrics?
   A. The risk metric will be more realistic.
   B. The risk metric will be more conservative.
   C. The risk metric will mirror the portfolio returns.

D. The risk metric will respond to current market conditions.

MODULE 54.2: STRESS-TESTING GOVERNANCE

LO 54.f: Describe the responsibilities of the board of directors, senior management, and the internal audit function in stress testing governance.

To ensure there is adequate oversight, institutions should have separation of duties between the board of directors and senior management. This separation of duties also applies to stress testing. Although the board and management share several common responsibilities, they each have distinct responsibilities within an institution.

Responsibilities of the Board of Directors

The board of directors has oversight for an organization's key strategies and decisions, sets the organization's risk culture and risk appetite, and is responsible and accountable for the entire organization. It is important that the board discuss and evaluate information received from senior management and review it with a critical eye. The board should be sufficiently knowledgeable about the organization's stress-testing activities to ask informed questions, even if it is not directly involved with and does not possess expert knowledge in stress-testing activities or their technical details. Board members should be critical of stress tests by actively challenging assumptions.

Stress-test results are important because they are used to inform the board of the institution's risk appetite and risk profile, as well as its operating and strategic decisions. Stress-test results provide forward-looking assessments and are especially important in decisions regarding capital and liquidity adequacy and capital funding plans. However, boards should view results with some degree of skepticism and should not rely on a single stress-test exercise, but rather supplement it with other tests and quantitative and qualitative information. Stress testing can serve as an early warning sign, especially in nonstress times, allowing the board to take actions that include adjusting capital levels, increasing liquidity, adjusting risks, or engaging in or withdrawing from certain activities.

Responsibilities of Senior Management

Senior management is accountable to the board and is responsible for the satisfactory implementation of the stress-testing activities authorized by the board. This entails establishing robust policies and procedures to ensure compliance with these activities, reviewing and coordinating stress-testing activities, and remedying any issues. To avoid inconsistencies, gaps, or problems, management must ensure that stress-test assumptions remain transparent and are used in a clear manner.

It is senior management's responsibility to ensure that stress-testing activities do not simply rely on a single test, but instead rely on a series of stress tests to evaluate risks. Stress tests should aid management's decision-making relating to business strategies, risk limits, and the institution's capital, liquidity, and risk profile. It is prudent practice to benchmark results against an adequate benchmark to aid comparison and allow management to properly evaluate the stress-test results. Senior management should actively challenge these results; therefore, it is critical that management remain knowledgeable of the details of the stress-testing activities.

Stress tests should be appropriately aggregated, remedial actions should be carried out appropriately, and results should be adequately documented. Senior management should also consider the effectiveness of mitigation techniques and the possibility that these remedial actions could break down during stressed times. Stress tests can be used to supplement other risks as well as capital and adequacy measures.

Senior management should regularly report back to the board on stress-test results and developments and on the adequacy of compliance procedures. Reports should be clear and concise, and they should explain the main elements of the stress-testing activities, their assumptions, and any limitations. Senior management should also report on the governance, validation, and independent review of stress tests and stress-test results. In addition, senior management should ensure that stress-testing activities are reviewed by an independent, unbiased party (e.g., through an internal audit).

It is also senior management's responsibility to regularly update stress-testing activities given changing risks, data sources, and internal or external operating environments. Stress-testing activities and their underlying models should be reviewed, adjusted, and refined on an ongoing basis; senior management should confirm that the models and activities remain appropriate for the institution. Finally, management is also responsible for conducting stress tests to ensure that the institution is sufficiently flexible to withstand new risks and vulnerabilities.

Role of Internal Audits

The internal audit forms a crucial component of an institution's governance and controls. It is intended to assess the integrity and reliability of an institution's policies and procedures, including those pertaining to stress tests. Auditors should be independent and have sufficient knowledge and technical expertise to conduct their reviews.

The internal audit should verify that stress tests are conducted thoroughly, consistently, and as intended, and that the individuals in charge of these activities possess the necessary expertise and adhere to the appropriate policies and procedures. An internal audit should also review the procedures pertaining to the documentation, review, and approval of stress tests. Any deficiencies or potential improvements in stress tests should also be identified and communicated to senior management.

Policies, Procedures, and Documentation

LO 54.g: Describe the role of policies and procedures, validation, and independent review in stress testing governance.

Clear and comprehensive policies, procedures, and documentation are critical in codifying an institution's risk practices, including its stress-testing activities. Policies and procedures should be clear and concise in order to provide assurance that all parts of an organization apply stress tests consistently.

Policies, procedures, and documentation should address the implementation of stress-testing activities, including the following:

- Describing the overall purpose of stress tests
- Describing roles and responsibilities
- Indicating the use of stress tests and who uses the tests
- Outlining the stress-testing process, including scenario design and selection
- Determining frequency and priority of stress-testing activities
- Tracking for how stress-test results change over time
- Establishing consistent and adequate stress-testing practices, including how independent reviews will function
- Providing documentation for how third-party models and software acquisitions function
- Providing transparency to third parties regarding the stress-testing process, which allows third parties to evaluate the tests and their components
- Updating and reviewing policies and procedures to remain consistent with the institution's risk appetite, risk exposures, and changing market conditions

Validation and Independent Review

Prudent governance should also incorporate ongoing validation and independent review of stress-testing activities. These should be done in an unbiased manner using a critical review to ensure stress tests were conducted appropriately. Validation and independent review of stress tests should be incorporated into an institution's overall validation and review processes.

The individuals who review stress-testing procedures should be independent of the individuals who conduct the actual stress tests. A thorough review will ensure that there is a sound theory underlying the tests, ensure the acknowledgment of any uncertainties or limitations, cover any judgmental or qualitative components of the tests, and monitor the results on a routine and ongoing basis.

The challenge with stress-test validations is the reality that stress tests represent rare events. Variables and their correlations during normal markets are unlikely to mirror behaviors in stressed market conditions, as the latter tend to drive increased correlations and lower recovery rates; this reality must be incorporated into stress-test models. Results from multiple models (as opposed to results from a single model) are often used to provide a range of losses versus a single estimated loss.

Basel II Stress-Testing Principles

LO 54.h: Describe the Basel stress testing principles for banks regarding the implementation of stress testing.

The Basel Committee requires that stress testing be "rigorous and comprehensive." In 2009, the Committee published the principles of stress testing for banking institutions with a goal of understanding how much capital is needed to manage losses from large adverse events. Underlying the principles is the role that stress testing plays in assessing risk in the future, driving risk tolerance levels, facilitating risk mitigation strategies, supporting communications, overcoming modeling limitations, and helping to drive liquidity and capital planning procedures.

In addition to acknowledging that stress testing is especially critical after lengthy periods of normal market conditions, the Basel Committee noted that the role of the board and senior management is critical (as described earlier), stress testing needs to be done at an enterprise-wide level incorporating exposures across different areas of an institution, the scenarios need to incorporate realistic correlations between variables without such a heavy reliance on historical scenarios, new products should be considered, the scenarios should not underestimate the likely impacts of adverse events, and durations should be lengthy.

The most recent set of revised principles published by the Basel Committee[1] are as follows:

- Stress-testing frameworks should have clearly articulated and formally adopted objectives.
- Stress-testing frameworks should include an effective governance structure.
- Stress testing should be used as a risk management tool and to inform business decisions.
- Stress-testing frameworks should capture material and relevant risks and apply stresses that are sufficiently severe.
- Resources and organizational structures should be adequate to meet the objectives of the stress-testing framework.
- Stress tests should be supported by accurate and sufficiently granular data and by robust IT systems.
- Models and methodologies to assess the impacts of scenarios and sensitivities should be fit for purpose.
- Stress-testing models, results, and frameworks should be subject to challenge and regular review.
- Stress-testing practices and findings should be communicated within and across jurisdictions.

MODULE QUIZ 54.2

1. Which of the following statements about governance structure is accurate?
   A. Senior management has ultimate oversight responsibility and accountability for an entire institution.
   B. The board of directors has responsibility for implementing authorized stress-testing activities.
   C. The board of directors can change an institution's capital levels and exposures following a review of stress-test results.
   D. Senior management should use scenario analysis, not stress testing, to evaluate an institution's risk decisions.

2. Which of the following statements about documentation of stress tests is most appropriate?
   A. Institutions are not concerned if their vendors document stress-testing activities.
   B. Institutions should incentivize documenting stress tests to increase efficiency.

C. Documentation is not useful for stress-test developers, but it is important to senior management.
D. Documentation should not include a description of the types of stress tests and methodologies, but it should include a description of the key assumptions and limitations.

3. Which of the following actions is least likely a component of the validation and independent review of stress tests?
   A. Using expert-based judgment.
   B. Testing data during nonstress periods.
   C. Communicating stress-test results to all stress-test users.
   D. Reviewing the qualitative but not the judgmental aspects of stress tests.

4. Which of the following statements best reflects the responsibilities of an internal audit?
   A. An internal audit should not assess the staff involved in stress-testing activities.
   B. An internal audit must independently assess each stress test used.
   C. An internal audit should review the manner in which stress-testing efficiencies are identified and tracked.
   D. The internal audit function needs to be impartial but does not need to be independent.

5. Which of the following statements accurately reflects a Basel Committee stress-testing principle?
   A. Stress-testing models should be reviewed at least twice per year.
   B. Stress-test results should not be communicated beyond senior management and the board.
   C. The risk captured in a stress-testing framework should be comprehensive, ranging from mild to extreme.
   D. Stress-testing framework objectives should be aligned with the overall risk management framework.

KEY CONCEPTS

LO 54.a

Stress testing is an important tool that enables a bank to identify, assess, monitor, and manage risk. Stress testing also helps a financial institution answer the question of whether it has enough liquid assets and capital to survive adverse events.

LO 54.b

Backward-looking measures like value at risk (VaR) and expected shortfall (ES) allow an institution to determine the likelihood that its losses will exceed a specific level over a period of time and, if that should occur, the average expected tail loss. Stress tests can be used in conjunction with VaR and ES measures in order to focus on how an institution would manage more extreme periods.

LO 54.c

Stressed VaR and stressed ES use data from periods that are particularly stressed, which allows an institution to understand how it might perform if that period were to occur again.

Key advantages of stressed risk metrics are that they are conservative, are forward-looking, lend themselves to hypothetical (what-if) scenarios, and focus on relatively long time horizons. Key disadvantages are that risk metrics will not necessarily respond to current market conditions, do not produce a probability distribution for losses, and cannot feasibly be backtested.

LO 54.d

Selecting a time horizon is the first step in creating a stress-test scenario, with the most common periods ranging from three months to two years. Scenarios can be chosen internally or set by outside regulators and, if based on historical data, are predicated on the assumption that variables will behave in the future how they have in the past.

As an entity creates scenarios for stress testing, it may choose to focus on large changes in critical variables, including equity prices, interest rates, GDP, unemployment rates, overall volatility, commodity prices, default rates, and exchange rates.

Stress tests may occur on a routine basis or they may be ad hoc, which are nonroutine scenarios that are used to assess future negative events and the exposure that an institution has to these events.

The most basic models typically only stress a few key variables. However, a more comprehensive approach to modeling captures the potential behavior of a range of variables. While the immediate impact of adverse scenarios must be captured, models should also reflect knock-on effects, which reflect the consequences of other firms' reactions to adverse events.

LO 54.e

While stress testing involves creating scenarios and evaluating their impacts, reverse stress testing looks at the combination of situations that could result in the demise of a financial institution.

Bank regulators perform stress testing based on the following scenarios: (1) baseline, (2) adverse, (3) severely adverse, and (4) internal. Two types of regulatory stress tests are the Comprehensive Capital Analysis and Review (CCAR) and the Dodd-Frank Act Stress Test (DFAST).

LO 54.f

The board of directors is accountable for the entire organization and must be sufficiently knowledgeable about the organization's stress-testing activities.

Stress-test results inform the board of the institution's risk appetite, risk profile, and operating and strategic decisions. The results may serve as early warning signs of upcoming pressures.

Boards should actively challenge the results of stress tests and supplement them with other tests as well as both quantitative and qualitative information.

Senior management, with oversight from the board, is responsible for establishing robust policies for stress tests, which could supplement other risk, capital, and adequacy measures. Management is also responsible for reviewing and coordinating stress-testing activities, assigning competent staff, challenging results and assumptions, and incorporating remedies to potential problems.

Senior management should ensure there is a sufficient range of stress-testing activities to evaluate risks. Results should be benchmarked and regularly updated given that risks, data sources, and the operating environment can change. An independent auditor should verify test results.

Senior management should regularly report to the board of directors on stress-test results in a clear and concise way, highlighting the key elements of the stress-testing activities and any limitations.

The internal audit assesses the integrity, consistency, and reliability of policies and procedures. It should verify that stress tests are conducted thoroughly and as intended, by staff with the relevant expertise.

The internal audit should also review the procedures relating to the documentation, review, and approval of stress tests. Any deficiencies and areas for potential improvement should be identified and communicated to senior management.

LO 54.g

Stress-testing activities should be governed by clear and comprehensive policies and procedures that are kept up to date and documented in an appropriate manner. Appropriate documentation allows senior management to track and analyze results over time.

Stress-testing policies, procedures, and documentation should address a range of issues, including describing the purpose and process of stress-testing activities, establishing consistent and adequate practices, defining roles and responsibilities, determining the frequency and priority of stress-testing activities, tracking changes in results over time, and updating policies and procedures to adapt to changing risks and market conditions.

Ongoing validation and independent review of stress-testing activities is an important component of an institution's governance. Validation and independent review should be unbiased and critical, and it should form part of the institution's overall validation and review processes.

Reviews should be based on sound theory, acknowledge limitations, cover subjective elements, and monitor results on a routine basis. Reviews should also account for the reality of increased variable correlations during stressed market conditions, with multiple models used together typically providing a more accurate means of assessing potential losses.

LO 54.h

Underlying the principles is the role that stress testing plays in assessing risk in the future, driving risk tolerance levels, facilitating risk mitigation strategies, supporting communications, overcoming modeling limitations, and helping to drive liquidity and capital planning procedures. Specifically, stress testing should have clear and formal objectives, include an effective governance structure, be used as a risk management tool, capture material and relevant risks, be adequately resourced, be supported by granular data and robust IT systems, include appropriate models and methodologies, be subject to review and challenge, and be communicated within and across jurisdictions.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 54.1

1. **C** Extreme events tend to have a low probability of occurrence with a high impact. Stress testing is designed to show how an institution will respond to these types of events and ensure that they have enough capital and liquid assets to manage during these times. (LO 54.a)

2. **D** Institutions use reverse stress tests to assess the events that are outside of normal business expectations and could threaten the institution’s viability. (LO 54.e)

3. **D** Stress tests are forward-looking and do not provide probabilities for loss distributions. The time horizons are typically long, only negative events are captured, and the number of extreme scenarios tends to be relatively small. (LO 54.b)

4. **A** A 5% decrease in the stock market could easily occur over the course of a few trading sessions and is not as likely to be included in stress-test models as a 300 basis point decline in GDP, a 300 basis point increase in interest rates, or a 5% increase in the unemployment rate. (LO 54.c)

5. **B** A key advantage of using stressed risk metrics is that they are conservative. In examining capital adequacy for unexpected losses and considering stressed metrics, the amount of capital is likely to be more than sufficient. In other words, a risk metric that is stressed is likely to be more conservative. A more conservative risk metric does not necessarily mean it is more realistic. One of the disadvantages of using stressed inputs is that the risk metric becomes unresponsive to current market conditions and is more dependent on the investments within the portfolio. (LO 54.c)

Module Quiz 54.2

1. **C** Stress testing can serve as an early warning sign of upcoming pressures and risks. The board of directors can take actions that include adjusting capital levels, increasing liquidity, adjusting risks, or engaging in or withdrawing from certain activities.

   The board of directors has ultimate oversight responsibility and accountability for an entire institution. Senior management is responsible for implementing authorized stress-testing activities. Senior management should use stress testing, complemented with scenario analysis, to evaluate an institution’s risk decisions. (LO 54.f)

2. **B** Institutions should offer incentives for documenting stress tests to ensure that documentation is effective and complete.

   Institutions should ensure that other market participants, including management, vendors, and reviewers, adequately document their stress-testing activities. Documentation is useful for both stress-test developers and senior management. Documentation should include a description of the types of stress tests and methodologies, as well as a description of the key assumptions and limitations. (LO 54.g)

3. **D** Validation and independent review of stress tests includes a review of both the qualitative and judgmental aspects of stress tests.

   Validation and independent review of stress tests should also use expert-based judgment, test data during nonstress periods, and involve communication of stress-test results to all stress-test users. (LO 54.g)

4. **C** An internal audit should review the manner in which stress-testing efficiencies are identified, tracked, and remedied.

   An internal audit should assess not only the stress-testing activities, but also the staff involved in stress-testing activities. An internal audit does not need to independently assess each stress test used. The internal audit function needs to be independent and objective. (LO 54.f)

5. **D** Stress-testing objectives should align with the overall risk management framework for an institution. There is no requirement that the models should be reviewed twice per year. Results should be communicated beyond just senior management and the board. Mild risks are not what stress testing is intended to capture; risks should be extreme. (LO 54.h)

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[1] Basel Committee on Banking Supervision, *Stress Testing Principles* (Bank for International Settlements, September 2018), https://www.bis.org/bcbs/publ/d450.pdf.

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 9.

READING 55

PRICING CONVENTIONS, DISCOUNTING, AND ARBITRAGE

Study Session 14

EXAM FOCUS

This reading provides an overview of the fundamentals of bond valuation. The value of a bond is simply the present value of its cash flows discounted at the appropriate periodic required return. Discount factors are used for pricing coupon bonds and for determining whether bonds are trading cheap or rich. If a mispricing exists among securities, a riskless arbitrage profit can be made from the violation of the law of one price, which states that securities with identical future cash flows should sell for the same price.

FUNDAMENTALS OF BOND VALUATION

The general procedure for valuing fixed-income securities (or any security) is to take the present values of all the expected cash flows and add them up to get the value of the security.

There are three steps in the bond valuation process:

Step 1: Estimate the cash flows over the life of the security. For a bond, there are two types of cash flows: (1) the coupon payments and (2) the return of principal.

Step 2: Determine the appropriate discount rate based on the risk of (uncertainty about) the receipt of the estimated cash flows.

Step 3: Calculate the present value of the estimated cash flows by multiplying the bond's expected cash flows by the appropriate discount factors.

For a Treasury bond, the appropriate rate used to value the promised cash flows is the risk-free rate. This may be a single rate, used to discount all of the cash flows, or a series of discount rates that correspond to the times until each cash flow arrives.

For non-Treasury securities, we must add a risk premium to the risk-free (Treasury) rate to determine the appropriate discount rate. This risk premium is the added yield to compensate for greater risk (credit risk, liquidity risk, call risk, prepayment risk, and so on). When using a single discount rate to value bonds, the risk premium is added to the risk-free rate to get the appropriate discount rate for all of the expected cash flows.

Calculating the Value of a Coupon Bond

Valuation With a Single Yield (Discount Rate)

For an option-free coupon bond, the coupon payments can be valued as an annuity. In order to take into account the payment of the par value at maturity, we will enter this final payment as the future value. This is the basic difference between valuing a coupon bond and valuing an annuity.

For simplicity, consider a security that will pay $100 per year for 10 years and make a single $1,000 payment at maturity (in 10 years). If the appropriate discount rate is 8% for all the cash flows, the value is:

$$\frac{100}{1.08}+\frac{100}{1.08^2}+\frac{100}{1.08^3}+\frac{100}{1.08^4}+\ldots+\frac{100}{1.08^{10}}+\frac{1,000}{1.08^{10}}$$

$$= \$1,134.20 \text{ (present value of expected cash flows)}$$

This is simply the sum of the present values of the future cash flows, $100 per year for 10 years and $1,000 (the principal repayment) to be received at the end of the 10th year, at the same time as the final coupon payment.

The calculator solution is:

N = 10; PMT = 100; FV = 1,000; I/Y = 8; CPT → PV = –$1,134.20
where:
N = number of years
PMT = *annual* coupon payment
I/Y = *annual* discount rate
FV = par value or selling price at the end of an assumed holding period

PROFESSOR'S NOTE

Take note of a couple of points here. The discount rate is entered as a whole number in percent, 8, not 0.08. The 10 coupon payments of $100 each are taken care of in the N = 10 entry; the principal repayment is in the FV = 1,000 entry. Lastly, note that the PV is negative—it will be the opposite sign to the sign of the PMT and FV. The calculator is just "thinking" that if you receive the payments and future value (you own the bond), you must pay the present value of the bond today (you must buy the bond). That's why the PV amount is negative—it is a cash outflow to a bond buyer. Just make sure that you give the payments and future value the same sign, and then you can ignore the sign on the answer (PV).

Valuation With a Single Yield and Semiannual Cash Flows

Let's calculate the value of the same bond with semiannual payments.

Rather than $100 per year, the security will pay $50 every six months. Adjust the discount rate of 8% per year to 4% per six months. The par value remains $1,000.

The calculator solution is:

N = 20; PMT = 50; FV = 1,000; I/Y = 4; CPT → PV = –1,135.90

where:

N = number of semiannual periods

PMT = *semiannual* coupon payment

I/Y = *semiannual* discount rate

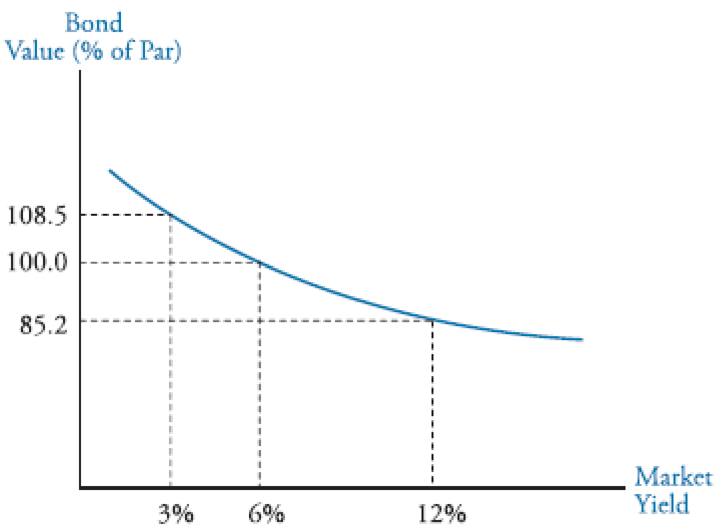
FV = par value

Price-Yield Curve

Bond values and bond yields are inversely related. An *increase* in the discount rate will *decrease* the present value of a bond's expected cash flows; a *decrease* in the discount rate will *increase* the present value of a bond's expected cash flows. The change in bond value in response to a change in the discount rate can be calculated as the difference between the present values of the cash flows at the two different discount rates.

If you plot a bond's yield to its corresponding value, you'll get a graph like the one shown in Figure 55.1. Here we see that higher prices are associated with lower yields. This graph is called the **price-yield curve**. Note that it is not a straight line, but curved. For option-free bonds, the price-yield curve is convex toward the origin, meaning it looks like half of a smile.

Figure 55.1: The Price-Yield Profile



MODULE 55.1: DISCOUNT FACTORS AND ARBITRAGE

LO 55.a: Define discount factor and calculate present and future values using a discount function.

Discount factors are used to determine present values. The discount function is expressed as *d(n)*, where *n* denotes the number of coupon periods until the next payment. Discount factors are often used to value Treasury bills and Treasury bonds.

Discount Factors for Treasury Bills

Treasury bills are securities that mature within one year and are issued by governments to finance their short-term funding needs. The cash price paid for a Treasury bill is a function of the maturity price (e.g., 100), quoted price (Q), and number of calendar days until maturity (n). This can be expressed as:

$$\text{cash price} = 100 - \frac{Qn}{360}$$

If the Treasury bill matures in one year and n = 360, then the price paid by a buyer would be 100 – Q. In other words, the buyer would pay 100 – Q today and receive 100 in 360 days. Note that the quoted price, Q, is essentially the annualized discount of the Treasury bill. The quoted price is referred to as the clean price and does not include accrued interest. The cash price is referred to as the dirty price and includes accrued interest.

We could calculate a cash price based on both the bid and ask quotes for the bill. The midpoint of these two values is the discount factor. For example, if the cash price based on the bid for a security that matures in 80 days is calculated as 99.60 and the cash price based on the ask is 99.65, the midpoint is 99.625, or 0.99625. A security worth $100,000 in 80 days would be priced at $99,625 today. Therefore, the 0.99625 is the discount factor for this maturity date. If a security was priced at $100,000 today, it would be worth $100,000 / 0.99625 = $100,376.41 in 80 days based on the midmarket discount factor.

EXAMPLE: **Calculating a bond value using discount factors**

Suppose that the discount factor for a Treasury bill maturing in 180 days is d(0.5) = 0.92432. **Calculate** the price of a Treasury bill that matures in 180 days from today.

Answer:

Since $1 to be received in six months is worth $0.92432 today, $100 received in six months is worth 0.92432 × $100 = $92.432 today.

Discount Factors for Treasury Bonds

Treasury bonds are securities issued by governments to finance their mid- or long-term needs. They promise a stream of future cash flows, and are therefore defined by their face value, cash flow (coupons), and maturity. A series of Treasury bond (T-bond) prices can be used to generate the discount function.

EXAMPLE: **Calculating discount factors given bond prices**

Figure 55.2 shows selected T-bond prices for semiannual coupon $100 face value bonds. Settlement is T + 1.

Figure 55.2: Selected T-Bond Prices

Bond	Coupon	Maturity	Price
1	4.25%	November 15, 2021	101.50
2	7.25%	May 15, 2022	105.98
3	2.00%	November 15, 2022	101.22
4	12.00%	May 15, 2023	120.94
5	5.75%	November 15, 2023	110.42

Generate discount factors for the dates indicated.

Answer:

Bond 1:

When this bond matures on November 15, 2021, it will repay its principal of 100 and will make its last interest payment of:

$$\left(\frac{0.0425}{2} \times \$100\right) = 2.125$$

The current cash price of the bond is 101.50. Therefore:

$$d(1) = \frac{101.50}{102.125} = 0.9939$$

Moving farther out on the curve, the function becomes slightly more complex, as each point of the curve must be included. For example, to solve for Bond 2, we must include both d(1) and d(2).

Bond 2:

The coupon payment at Time 1 is 7.25 / 2 = 3.625. The final cash flow at Time 2 is 100 + 3.625 = 103.625. These two cash flows discounted back to present value using the discount function should equal the price of the bond:

[3.625 × d(1)] + [103.625 × d(2)] = 105.98

Since it's already known that d(1) = 0.9939:

(3.625 × 0.9939) + [103.625 × d(2)] = 105.98

d(2) = 0.9880

Using the same methodology for Bonds 3, 4, and 5:

Bond 3:

[1.0 × d(1)] + [1.0 × d(2)] + [101 × d(3)] = 101.22

Thus:

d(3) = 0.9825

Bonds 4 and 5:

d(4) = 0.9731

d(5) = 0.9633

Figure 55.3: Summary of the Discount Factors

Time to Maturity	Discount Factor
0.5	0.9939
1.0	0.9880
1.5	0.9825
2.0	0.9731
2.5	0.9633

Determining Value Using Discount Functions

LO 55.b: Define the "law of one price," explain it using an arbitrage argument, and describe how it can be applied to bond pricing.

LO 55.c: Identify arbitrage opportunities for fixed-income securities with certain cash flows.

The discount functions previously mentioned can be used to estimate the value of a bond. Because investors do not care about the origin of a cash flow, all else equal, a cash flow from one bond is just as good as a cash flow from another bond. Instruments with identical future cash flows should sell for the same price. This is known as the **law of one price**.

If investors are able to exploit a mispricing because of the law of one price, it is referred to as an arbitrage opportunity. To take advantage of an arbitrage opportunity, investors should short sell the more expensive instrument/portfolio and buy the cheaper instrument/portfolio. Since both provide identical future cash flows, the investor can then generate a profit.

Short positions are important considerations for arbitrage. Short positions involve selling securities the investor does not own, with the expectation that the security price declines and the investor can repurchase the security at a lower price. If the security pays income (coupon for bonds or dividends for equities), the investor must pay this income to the lender of the security. The risk of short positions is that the security price moves up, or that the security can no longer be borrowed and the investor needs to buy back the security (potentially at a loss).

EXAMPLE: **Identifying arbitrage opportunities**

Suppose you observe the annual coupon bonds shown in Figure 55.4.

Figure 55.4: Observed Bond Yields and Prices

Maturity	YTM	Coupon (Annual Payments)	Price (% of Par)
1 year	4%	0%	96.154
2 years	8%	0%	85.734
2 years	8%	8%	100.000

The 2-year spot rate is 8.167%. Is there an arbitrage opportunity? If so, **describe** the trades necessary to exploit the arbitrage opportunity.

Answer:

The answer is yes. An arbitrage profit may be realized because the YTM on the 2-year zero-coupon is too low (8% vs. 8.167%), which means the bond is trading *rich* (the bond price is too high). To exploit this violation of the law of one price, buy the 2-year, 8% coupon bond, strip the coupons, and short sell them separately. The discount factors are derived from the prices of the zero-coupon bonds.

Figure 55.5: Discount Factors

Time to Maturity	Discount Factor
1.0	0.96154
2.0	0.85734

To demonstrate the process of exploiting the arbitrage opportunity for a $1 million position, consider the following 3-step process:

Step 1: Buy $1 million of the 2-year, 8% coupon bonds.

Step 2: Short sell $80,000 of the 1-year, zero-coupon bonds at 96.154.

Step 3: Short sell $1.08 million of the 2-year, zero-coupon bonds at 85.734.

Figure 55.6: Cash Flow Diagram

Time = 0		1 Year		2 Years	
–1,000,000.00	(Cost of 2-year, 8% coupon bonds)	+80,000	(Coupon, interest)	+1,080,000	(Coupon, interest)
+76,923.20*	(Proceeds from 1-year, 0% bond)	–80,000	(Maturity)		
+925,927.20**	(Proceeds from 2-year, 0% bond)			–1,080,000	(Maturity)
+2,850.40	Net	0		0	

*76,923.20 = 0.96154 × 80,000

**925,927.20 = 0.85734 × 1,080,000

The result is receiving positive income today in return for no future obligation, which is an *arbitrage opportunity*.

MODULE QUIZ 55.1

Use the following information to answer Questions 1 and 2.

Maturity	Coupon	Price
6 months	5.5%	101.3423
1 year	14.0%	102.1013
2 years	8.5%	99.8740

1. Which of the following is the closest to the discount factor for the 6-month discount factor, d(0.5)?
   A. 0.8923.
   B. 0.9304.
   C. 0.9525.
   D. 0.9863.

2. Which of the following is the closest to the discount factor for the 1-year discount factor, d(1)?
   A. 0.8897.
   B. 0.9394.
   C. 0.9525.
   D. 0.9746.

MODULE 55.2: BOND COMPONENTS AND PRICING

LO 55.d: Identify the components of a U.S. Treasury coupon bond and compare the structure to Treasury STRIPS, including the difference between P-STRIPS and C-STRIPS.

Zero-coupon bonds issued by the Treasury are called **STRIPS** (which stands for Separate Trading of Registered Interest and Principal Securities). STRIPS are created by request when a coupon bond is presented to the Treasury. The bond is “stripped” into two components: principal (**P-STRIPS**, TPs, or Ps) and coupons (**C-STRIPS**, TINTs, or INTs).

Arbitrage opportunities exist when the price of C-STRIPS relative to P-STRIPS diverges. However, transaction costs may negate the benefit of these arbitrage opportunities, which are then left unexploited.

Constructing a Replicating Portfolio

LO 55.e: Construct a replicating portfolio using multiple fixed-income securities to match the cash flows of a given fixed-income security.

Suppose a 2-year fixed-income security exists with \$100 face value and a 10% coupon rate. The coupons are paid on a semiannual basis, and the security’s YTM is assumed to be 4.5%. The present value of this bond, Bond 1, and its cash flows are calculated as follows:

$$PV_{B1} = \frac{5}{1.0225^1} + \frac{5}{1.0225^2} + \frac{5}{1.0225^3} + \frac{105}{1.0225^4} = \$110.41$$

or

N = 2 × 2 = 4; I/Y = 4.5 / 2 = 2.25%; FV = 100; PMT = 10 / 2 = 5;
CPT → PV = $110.41

If this bond is determined to be trading cheap, then a trader can conduct an arbitrage trade by purchasing the undervalued bond and shorting a replicating portfolio that mimics the bond's cash flows. To demonstrate the creation of a replicating portfolio, assume the following four fixed-income securities exist in addition to the bond we are trying to replicate.

Figure 55.7: Bonds for Replicating Portfolio

Bond	Coupon	PV	FV	Time Horizon
2	7%	$101.22	$100	6 months (0.5 years)
3	12%	$107.25	$100	12 months (1 year)
4	5%	$100.72	$100	18 months (1.5 years)
5	6%	$102.84	$100	24 months (2 years)

To create a replicating portfolio using multiple fixed-income securities, we must determine the position of each bond to purchase that would match Bond 1's cash flows in each semiannual period.

$$\text{Bond 1 } CF_t = F_2 \times \frac{7\%}{2} + F_3 \times \frac{12\%}{2} + F_4 \times \frac{5\%}{2} + F_5 \times \frac{6\%}{2}$$

When doing this calculation by hand, it is easiest to start from the end—with the bond that matches Bond 1's time horizon. In this case, that security is Bond 5:

$$FV_5 \times \frac{100 + \frac{6}{2}}{100} = \$105$$

Solving this equation for FV_5 yields the face amount percentage we need to purchase of Bond 5 (FV_5 = 101.94). Because the coupon rate on Bond 5 is lower than that of Bond 1, it makes sense that we'll need to purchase more of Bond 5 (101.94%) than the $100 face value of Bond 1. We can now use the value of FV_5 to solve for FV_4:

$$FV_4 \times \frac{100 + \frac{5}{2}}{100} + 101.94 \times \frac{3}{100} = \$5$$

The remaining unknowns (FV_2 and FV_3) are solved in a similar fashion. The replicating portfolio can now be purchased (or sold for the arbitrage trade) using the face amount percentages listed in the following table. Notice, in the last two rows, how the total cash flows from these four bonds exactly matches the cash flows from Bond 1.

The cash flows from the replicating portfolio are computed by multiplying each bond's initial cash flows by face amount percentage. For example, regarding Bond 5, the 2-year cash flow is computed as \$103 × 1.0194 = \$105, and the 1-year cash flow is computed as \$3 × 1.0194 = \$3.0582.

Figure 55.8: Bond Cash Flow Summary

	Coupon	Face Amount	CF (t = 0.5)	CF (t = 1)	CF (t = 1.5)	CF (t = 2)
Bond 2	7%	1.73%	1.7871			
Bond 3	12%	1.79%	0.1074	1.8945		
Bond 4	5%	1.89%	0.0473	0.0473	1.9418	
Bond 5	6%	101.94%	3.0582	3.0582	3.0582	105
Bond 1 CFs			5	5	5	105

Pricing Conventions Between Coupon Dates

LO 55.f: Differentiate between "clean" and "dirty" bond pricing and explain the implications of accrued interest with respect to bond pricing.

As noted, bonds are frequently purchased between coupon dates. We must account for three items in this situation: accrued interest, fractional period compounding, and the day count convention of the bond.

Clean and Dirty Bond Pricing

The **dirty price** is the price that the seller of the bond must be paid to give up ownership. It includes the present value of the bond plus the accrued interest. The **clean price** is the dirty price less accrued interest:

clean price = dirty price − accrued interest

The dirty price of the bond is sometimes referred to as the full price or invoice price. The clean price of the bond is sometimes referred to as the flat price or quoted price.

Note that the dirty price includes the discounted value of the next coupon so that the method of calculating accrued interest does not matter. As long as the clean price is calculated as dirty price minus accrued interest, the sum of the clean price and accrued interest will equal the dirty price.

Accrued Interest

Accrued interest is the interest earned between coupon dates; it is what is owed by the buyer to the seller. It is calculated as the coupon interest times the ratio of time passed since the last coupon and time between coupon dates. Consider a \$100 par value bond that pays 3% coupon semiannually. This means a coupon of \$1.50 is paid every six months. If the bond is sold (and settles) 41 days after the last coupon, the buyer will need to pay the seller $\$1.50 \times \frac{41}{182} = \$0.3379$ for every \$100 purchased. When calculating the discount factor, the amount of the accrued interest needs to be added to both the bid and ask quotes before calculating the midpoint.

EXAMPLE: Computing accrued interest

A $1,000 par value U.S. corporate bond pays a semiannual 10% coupon. Assume the last coupon was paid 90 days ago and there are 30 days in each month. **Compute** the accrued interest.

Answer:

Accrued interest is computed as follows:

$$AI = \$50\left(\frac{90}{180}\right) = \$25$$

Day Count Conventions

LO 55.g: Describe the common day-count conventions used to calculate interest on a fixed-income security.

Several **day count conventions** are used in practice in the bond markets. The following are the most common:

- Actual/actual (in period)
- Actual/360
- 30/360

The day count used will depend on the type of security. For example, U.S. government bonds pay coupons semiannually and have an actual/actual day count. U.S. corporate and municipal bonds pay semiannual interest with a 30/360 day count. An actual/365 day count is typically used for money market securities in Canada, New Zealand, and Australia.

We need to modify the bond pricing formula to incorporate the appropriate day count convention. Specifically, the bond pricing equation becomes:

$$P = \frac{C}{(1+y)^{w}} + \frac{C}{(1+y)^{1+w}} + \frac{C}{(1+y)^{2+w}} + \ldots + \frac{C}{(1+y)^{n-1+w}} + \frac{M}{(1+y)^{n-1+w}}$$

where:
P = bond price
C = semiannual coupon
y = periodic required yield
n = number of periods remaining
M = par value of the bond
w = number of days until the next coupon payment divided by the number of days in the coupon period

When expressing *w* in the preceding equation, the number of days to use for the coupon period is determined by the appropriate day count convention. For example, the denominator is 180 for semiannual bonds that use the 30/360 convention. This equation computes the dirty price the bond because it includes the discounted value of the first full coupon payment even though the accrued interest belongs to the seller of the bond.

MODULE QUIZ 55.2

1. Which of the following day count conventions would most likely be used in pricing an Australian money market security?
   A. 30/360.
   B. Actual/360.
   C. Actual/365.
   D. Actual/actual.

2. A EUR 100,000 par value French corporate bond pays 3.5% coupon with a semiannual frequency. Assume the last coupon was paid 75 days ago and there are 30 days in each month. The accrued interest is closest to:
   A. 729.
   B. 1,094.
   C. 1,458.
   D. 2,917.

KEY CONCEPTS

LO 55.a

Discount factors are used to determine present values, and they are often used to value Treasury bills and Treasury bonds. The discount function is expressed as d(n), where *n* denotes the number of coupon periods until the next payment.

LO 55.b

Instruments with identical future cash flows should sell for the same price, which is known as the law of one price.

LO 55.c

Arbitrage opportunities allow investors to exploit mispricings when two securities have identical future cash flows but different current prices. By short selling a rich security and using the proceeds to purchase a similar cheap security, an investor can take advantage of an arbitrage opportunity and make a riskless profit with no investment.

Short positions are important considerations for arbitrage, but if the security pays income (coupon for bonds or dividends for equities), the investor must pay this income to the lender of the security.

LO 55.d

STRIPS are zero-coupon bonds issued from Treasury securities, which create principal strips (P-STRIPS) and coupon strips (C-STRIPS).

Arbitrage opportunities exist when the price of C-STRIPS relative to P-STRIPS diverges.

LO 55.e

To create a replicating portfolio using multiple fixed-income securities, we must determine the face amounts of each fixed-income security to purchase, which match the cash flows from the bond we are trying to replicate.

LO 55.f

Accrued interest is an important consideration when trading fixed-income securities. It is the interest earned between coupon dates, calculated as the coupon interest times the ratio of time passed since the last coupon and time between coupon dates.

Valuation of bonds between coupon payment dates requires the calculation of accrued interest and a modification to the bond pricing formula. Values derived between coupon dates will include accrued interest. This is also known as the dirty price. Subtracting the accrued interest from the dirty price gives the clean price of the bond.

LO 55.g

Accrued interest calculations vary across classes of bonds because of differing day count conventions. The most common day count conventions are actual/actual and 30/360.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 55.1

1. **D** 101.3423 = 102.75 × d(0.5)

   d(0.5) = 0.9863

   (LO 55.a)

2. **A** 102.1013 = [7 × d(0.5)] + [107 × d(1)]

   102.1013 = 7(0.9863) + 107 × d(1)

   95.1972 = 107 × d(1)

   d(1) = 0.8897

   (LO 55.a)

Module Quiz 55.2

1. **C** The actual/365 day count convention is typically used for money market securities in Canada, New Zealand, and Australia. (LO 55.g)

2. **A** 100,000 × 3.5% / 2 = 1,750

   $$1,750 \times \frac{75}{180} = \text{EUR } 729.17$$

   (LO 55.f)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 10.

READING 56

INTEREST RATES

Study Session 14

EXAM FOCUS

Interest rates can be compounded at various frequencies, which will impact the value of an investment in the future and the present value of an investment today. For the exam, be familiar with the various types of rates applicable to debt instruments, including the spot rate, forward rate, par rate, and swap rate. The relationships between these rates, as well as the impact maturity has on bond valuations, is extremely critical to understand. Also, understand yield curves in terms of their slopes, what drives the flattening and steepening of curves, and the strategies to deploy in these situations.

MODULE 56.1: COMPOUNDING

LO 56.a: Calculate and interpret the impact of different compounding frequencies on a bond's value.

Most financial institutions pay and charge interest over much shorter periods than annually. For instance, if an account pays interest every six months, we say interest is compounded semiannually. Every three months represents quarterly compounding, and every month is monthly compounding. The compounding frequency describes how often interest is applied within a single year and defines how an interest rate is measured.

Imagine a $1,000 investment with an interest rate of 5% per year. With annual compounding, the formula to determine how much the investment will be worth after *n* years is $1{,}000 \times 1.05^{n}$. Semiannual compounding implies that 2.5% will be applied every six months, such that the value of the investment after *n* years is $1{,}000 \times 1.025^{2n}$.

The highest level frequency is continuous compounding. A 5% annual rate on a $1,000 investment, compounded continuously and to be received at time *t*, will have a value of $1{,}000e^{0.05 \times t}$.

Figure 56.1 shows the effect of compounding on a $1,000 investment earning an annual rate of 5%. Note the increase in value as the compounding frequency increases.

Figure 56.1: Future Values Based on Compounding Frequencies

Compounding Frequency	Rate Compounding Times per Year	Value of $1,000 Investment After One Year
Annual	1	$1,050.00
Semiannual	2	$1,050.63
Quarterly	4	$1,050.95
Monthly	12	$1,051.16
Daily	365	$1,051.27
Continuously	Continuous	$1,051.27

A similar exercise involves determining the present value of an amount to be received in the future, which will vary depending on the compounding frequency. Figure 56.2 illustrates the present value of a $1,000 investment to be received in three years based on various compounding frequencies at a 5% annual interest rate.

Figure 56.2: Present Values Based on Compounding Frequencies

Compounding Frequency	Rate Compounding Times per Year	Present Value of $1,000 Investment Received in Three Years
Annual	1	$863.84
Semiannual	2	$862.30
Quarterly	4	$861.51
Monthly	12	$860.98
Daily	365	$860.72
Continuously	Continuous	$860.71

A financial calculator is the easiest way to derive the present values shown in the preceding figure, but they can also be calculated manually. As an example, for quarterly compounding of a three-year investment, the calculation would be:

$$\frac{1{,}000}{(1+0.0125)^{12}} = 861.51$$

Continuous compounding uses the following equation:

$$1{,}000e^{-0.05 \times t}$$

Another useful exercise is comparing interest rates compounded at different frequencies in order to determine equivalent rates. To determine equivalent rates, use the following formula:

$$R_2 = \left[\left(1+\frac{R_1}{m_1}\right)^{m_1/m_2} - 1\right] \times m_2$$

where:
R = rate
m = number of times per year the rate is applied

For example, an interest rate of 6% compounded on a semiannual basis is equal to 5.93% on a monthly basis:

$$R_2 = \left[\left(1+\frac{0.06}{2}\right)^{2/12} - 1\right] \times 12 = 5.93\%$$

For calculating equivalent rates when continuous compounding is involved, assume again that the interest rate is 6% compounded semiannually. The equivalent continuously compounded rate is 5.91%. This is computed using the following equation:

$$2\ln\left(1+\frac{0.06}{2}\right) = 0.0591, \text{ or } 5.91\%$$

A 5% rate expressed with continuous compounding would be equal to 5.06% on a semiannual basis, computed as follows:

$$2(e^{0.05/2} - 1) = 0.0506, \text{ or } 5.06\%$$

MODULE QUIZ 56.1

1. The present value of a 2-year, $1,000 bond with an annualized interest rate of 3% compounded monthly is closest to:
   A. $940.00.
   B. $941.84.
   C. $941.98.
   D. $942.60.

MODULE 56.2: SPOT, FORWARD, PAR, AND SWAP RATES

Defining and Deriving Spot Rates

LO 56.b: Define spot rate and calculate discount factors given spot rates.

The **spot rate** (also known as the zero-coupon interest rate or the zero) is the rate earned on an investment when it is received at a single point in the future. In a situation where a single dollar is invested today and repaid as a lump-sum amount in the future, the spot rate will equate that future amount with the single dollar today. For example, $50 is invested today and $58 is returned to the investor two years from today. The spot rate is derived using the following equation, where *R* is determined to be 7.7033% and is applied annually:

$$50(1 + R)^2 = 58; R = 7.7033\%$$

The spot rate and the **discount factor**, d(t), provide the same information, such that the applicable discount factor for a spot rate of 7.7033% on a two-year investment is 0.86207. In other words, a $58 future amount, multiplied by 0.86207, is equal to $50. On a financial calculator, the discount factor and spot rate can be derived as long as one of the values is known. If the discount factor is known, by setting the present value equal to the discount factor, setting the future value to $1, and applying the number of years, the spot rate can be calculated. If the spot rate is known, that will be the interest rate, the future value will be \$1, and the number of years is applied in order to determine the present value (the discount factor). The formula to calculate the discount rate given the spot rate at time r(t), assuming semiannual compounding, is:

$$d(t) = \left(1 + \frac{r(t)}{2}\right)^{-2t}$$

For a continuous compounding spot rate, the formula is:

$$d(t) = e^{-r(t)t}$$

Forward Rates

LO 56.c: Interpret the forward rate and calculate forward rates given spot rates.

Forward rates are future spot rates that are based on current spot rates. In theory, an investor should be indifferent and earn the same return for an investment that spans two years versus one that lasts one year and then requires reinvestment in the second year. For example, an investor has \$1,000 and can earn 2% for the first year. Alternatively, the investor is offered a two-year investment that pays 3.5%. The forward rate is the rate earned on the second year that should make the investor indifferent between the two options.

The following formula is used to derive the forward rate, *F*:

$$1{,}000 \times 1.02 \times (1 + F) = 1{,}000 \times 1.035^2$$

$$F = 1.035^2 / 1.02 - 1 = 5.02\%$$

In a situation where rates are semiannual (spot rates R_1 and R_2), the 6-month forward rate can be expressed in the following equation. Note that this rate can be annualized by multiplying by two.

$$\frac{\left(1 + \frac{R_2}{2}\right)^{T+0.5}}{\left(1 + \frac{R_1}{2}\right)^{T}} - 1$$

For continuously compounded rates, the forward rate for a period that lies between times T_1 and T_2 is:

$$F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

EXAMPLE: Calculating forward rates

Assuming continuously compounded spot rates of 4.25 for 3 years and 4.40% for 3.5 years, **calculate** the forward rate for the period between Year 3 and Year 3.5.

Answer:

$$\frac{0.0440 \times 3.5 - 0.0425 \times 3.0}{3.5 - 3.0} = 0.530, \text{ or } 5.30\%$$

By lending or borrowing at spot rates, a financial institution can lock in the forward rate. Spot rates are equal to forward rates compounded for successive periods. If rates are expressed on a semiannual basis, the spot rate is derived from forward rates as follows:

$$\left(1+\frac{R}{2}\right)^{n} = \left(1+\frac{F_1}{2}\right)\left(1+\frac{F_2}{2}\right)\ldots\left(1+\frac{F_n}{2}\right)$$

A **forward rate agreement (FRA)** is a financial instrument that guarantees a specific rate to be paid or earned during a future period. The FRA is worth zero when the current forward rate (*F*) is equal to the guaranteed rate (*R*). Assuming there is a difference, the present value of that difference between *R* and *F*, applied to the principal amount, equals the value of the FRA. When *R* is greater than *F*, the value of the FRA is positive; when *R* is less than *F*, the value of the FRA is negative.

Par Rates

LO 56.d: Define par rate and describe how to determine the par rate of a bond.

The **par rate** at maturity is the rate at which the present value of a bond equals its par value. Assuming a 2-year bond pays semiannual coupons and has a par value of $100, the 2-year par rate can be computed by incorporating bond discount factors from each semiannual period as follows:

$$\frac{\text{par rate}}{2}[d(0.5) + d(1.0) + d(1.5) + d(2.0)] + 100 \times d(2.0) = 100$$

For example, assume the maturities, swap rates, and discount factors shown in Figure 56.3.

Figure 56.3: Swap Rates and Discount Factors

Maturity (Years)	Swap Rate	Discount Factor
0.5	0.65%	0.9968
1.0	0.80%	0.9920
1.5	1.02%	0.9848
2.0	1.16%	0.9771

From this data, we can compute the 2-year par rate as:

$$\frac{\text{par rate}}{2}(0.9968 + 0.9920 + 0.9848 + 0.9771) + 100 \times 0.9771 = 100$$

$$\frac{\text{par rate}}{2}(3.9507) + 97.71 = 100$$

$$\text{par rate} = 1.16\%$$

From Figure 56.3, notice how the par rate of 1.16% is exactly equal to the Year 2 swap rate of 1.16%. This equality occurs because swap rates are, in fact, par rates. Therefore, because we used swap rates to represent bond coupon payments when deriving discount factors, we can also say that par rates represent bond coupon payments when a bond's price is equal to its par value.

We can generalize the par rate equation stated previously to compute the par rate for any maturity, P_T, assuming a par value of $1 and the sum of discount factors as the annuity factor, A_T, as follows:

$$P_T = \frac{2[1 - d(T)]}{A_T}$$

EXAMPLE: Calculating par rates

The spot rates for each semiannual period over two years, along with their respective discount factors, are shown below.

Maturity (Years)	Spot Rate	Discount Factor
0.5	3.25%	0.9840
1.0	3.50%	0.9662
1.5	4.00%	0.9427
2.0	4.25%	0.9201

Calculate the 2-year par rate.

Answer:

When T = 2, A_T = 0.9840 + 0.9662 + 0.9427 + 0.9201 = 3.813 and d(T) = 0.9201.

$$P_T = \frac{2(1 - 0.9201)}{3.813} = 0.419, \text{ or } 4.19\%$$

The output can be interpreted as follows: a 2-year bond paying a coupon every six months at a rate of 4.19% per year will be worth exactly par.

The par rate and annuity factors can also be used to value bonds with other coupons using the following formula (assuming par is $1):

$$V = 1 + \frac{c - P}{2} A_T$$

where:
c = coupon rate
V = value of the bond

With a par rate of 4.19% and a coupon of 3.25%, and based on the discount factors given previously, the value of the bond is equal to:

$$V = 1 + \frac{0.0325 - 0.0419}{2}(3.813) = 0.9821, \text{ or } 98.21\% \text{ of par}$$

Relationship Between Spot, Forward, and Par Rates

LO 56.e: Interpret the relationship between spot, forward, and par rates.

The relationship between spot, forward, and par rates (for a given term structure) is shown in Figure 56.4.

Figure 56.4: Relationship Between Spot Rates, Par Rates, and Forward Rates

Term Structure	Spot Rates	Par Rates	Forward Rates
Flat	All equal	Same as spot rates	Same as spot rates
Upward sloping	Middle	Lowest	Highest
Downward sloping	Middle	Highest	Lowest

Effect of Maturity on Bond Prices and Returns

LO 56.f: Assess the impact of a change in time to maturity on the price of a bond.

As noted earlier, the forward rate agreement (FRA) is positive when the guaranteed rate (R) exceeds the forward rate (F). In line with this relationship, the value of a bond will fall if its coupon rate exceeds the forward rate for the final payment period. An FRA is negative when the forward rate is greater than the guaranteed rate. The value of a bond will rise when the forward rate for the last period is greater than the coupon rate, which tends to happen in an upward-sloping term structure.

Imagine a situation where the 2-year, continuously compounded rate is 3% and the 3-year, continuously compounded rate is 3.5%. The forward rate for the third year will be equal to 4.5%. The strategy to deploy if an investor feels that the third-year rate will be less than 4.5% would be to borrow for two years at 3% and invest for three years at 3.5%. If the third-year rate does come in under 4.5%, she will make a profit because the overall borrowing rate will be less than 3.5%. If the same investor feels that the third-year rate will be greater than 4.5%, she will invest for two years and borrow for three years. This will be a profitable strategy if the third-year rate turns out to be higher than 4.5%.

Swaps and Swap Rates

LO 56.h: Describe a swap transaction and explain how a swap market defines par rates.

A swap is a derivatives transaction where two parties agree to exchange payments based on the movement of an underlying asset. A fixed rate for floating rate swap involves one party making payments based on a fixed rate and receiving payments based on a floating rate, while the counterparty has the opposite position. Each party believes that interest rates are moving in opposite directions and is looking to synthetically produce a more favorable outcome based on rate movements. By definition, a derivatives transaction is a zero-sum game: one party wins and one party loses. Based on the frequency of payment, a net payment will be made each period from the party that is losing to the party that is winning.

EXAMPLE: Pay fixed, receive floating interest rate swap

Regist Bank has issued floating rate debt of $10 million and will be paying its debtholders on a quarterly basis. The bank is concerned about rising interest rates and would like to lock in a fixed rate. They enter into a pay fixed, receive floating interest rate swap with Counterparty X. In exchange for fixed payments at a rate of 3% annually (0.75% per quarter), Regist will receive payments based on the market reference rate (e.g., SOFR). **Describe** what will happen if the reference rate falls below 3%.

Answer:

The bank has entered into this swap, which will entitle them to pay a fixed rate and receive a floating rate, in order to protect themselves from rising rates. They have their own floating rate debt issuance and will receive payments from the counterparty at a floating rate, which will allow them to meet their interest payment requirements. However, if reference rates drop, the bank will be the party that has to make the net payment every quarter to the counterparty because the fixed rate they have locked in will exceed the reference rate.

The **notional principal** (which is never exchanged between the parties in an interest rate swap) is the amount ($10 million in the preceding example) that the interest rates are applied to in order to determine the net payment each period.

The swap market is used to define par rates, in that a 2-year swap rate defines a 2-year bond selling for par. Par rates, as described earlier, can be used to determine spot rates and discount factors.

MODULE QUIZ 56.2

1. If the 3-year, continuously compounded spot rate is 6.25%, the discount factor will be closest to:
   A. 0.8125.
   B. 0.8290.
   C. 0.9394.

D. 1.2062.

2. Assume you are given the following bonds and forward rates:

Maturity	YTM	Coupon	Price
1 year	4.5%	0%	95.694
2 years	7%	0%	87.344
3 years	9%	0%	77.218

- 1-year forward rate 1 year from today = 9.56%
- 1-year forward rate 2 years from today = 10.77%
- 2-year forward rate 1 year from today = 11.32%

Which of the following statements about the forward rates, based on the bond prices, is true?
A. The 1-year forward rate 1 year from today is too low.
B. The 2-year forward rate 1 year from today is too high.
C. The 1-year forward rate 2 years from today is too low.
D. The forward rates and bond prices provide no opportunities for arbitrage.

3. The current yield curve shows a 1-year spot rate of 2.5%, a 2-year spot rate of 3.25%, and a 10-year spot rate of 4.75%. Which of the following par and forward rate combinations for the 10-year bond is most likely to be found on this curve?
A. Forward rate of 4.25%; par rate of 5.00%.
B. Forward rate of 5.25%; par rate of 4.50%.
C. Forward rate of 4.75%; par rate of 4.75%.
D. Forward rate of 4.50%; par rate of 4.50%.

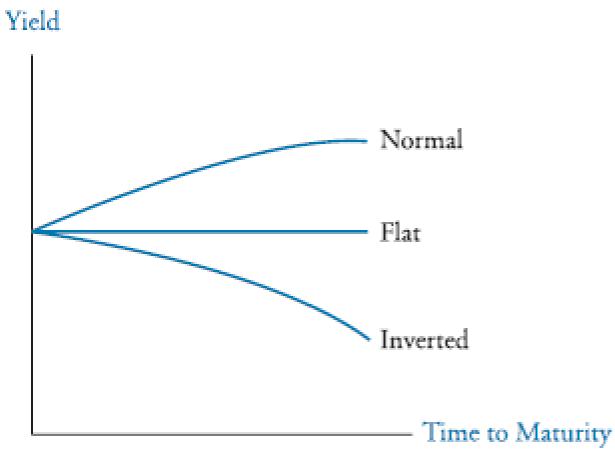
4. The 3-year continuously compounded rate is 2.25%, and the 4-year continuously compounded rate is 2.375%. An investor will borrow for three years, invest for four, and make a profit if the forward rate for the fourth year is:
A. equal to 2.75%.
B. equal to 2.25%.
C. less than 2.75%.
D. greater than 2.75%.

MODULE 56.3: YIELD CURVE SHAPES

LO 56.g: Define the "flattening" and "steepening" of rate curves and describe a trade to reflect expectations that a curve will flatten or steepen.

Historically, the yield curve has taken on three fundamental shapes, as shown in Figure 56.5.

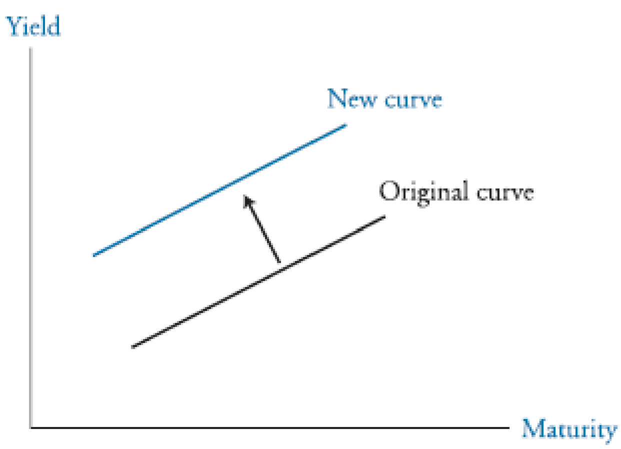
Figure 56.5: Yield Curve Shapes



A *normal* yield curve is one in which long-term rates are greater than short-term rates, so the curve has a positive slope. A *flat* yield curve represents the situation where the yield on all maturities is essentially the same. An *inverted* yield curve reflects the condition where long-term rates are less than short-term rates, giving the yield curve a negative slope.

When the yield curve undergoes a **parallel shift**, the yields on all maturities change in the same direction and by the same amount. As indicated in Figure 56.6, the slope of the yield curve remains unchanged following a parallel shift.

Figure 56.6: Parallel Yield Curve Shift

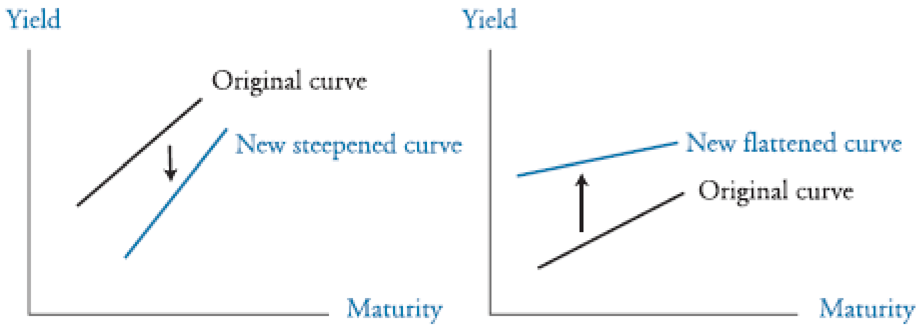


When the yield curve undergoes a **nonparallel shift**, the yields for the various maturities change by differing amounts. The slope of the yield curve after a nonparallel shift is not the same as it was prior to the shift. Nonparallel shifts fall into two general categories: twists and butterfly shifts.

Yield curve twists are yield curve changes when the slope becomes either flatter or steeper. With an upward-sloping yield curve, a *flattening* of the yield curve means that the spread between short- and long-term rates has narrowed. Conversely, a *steepening* of the yield curve occurs when spreads widen.

As shown in Figure 56.7, the most common shifts tend to be either a downward shift and a steepened curve or an upward shift and a flattened curve.

Figure 56.7: Nonparallel Yield Curve Shifts—Twists



The curve will flatten when long-term rates fall by more than short-term rates (a bull flattener) or when short-term rates rise by more than long-term rates (a bear flattener). The curve will steepen when long-term rates rise by more than short-term rates (a bear steepener) or when short-term rates fall by more than long-term rates (a bull steepener).

Trading strategies are based on expectations of steepening and flattening yield curves.

EXAMPLE: Strategies for yield curve shifts

Investor A expects an upward-sloping term structure to flatten in the coming months, with long-term rates falling and short-term rates rising. Investor B expects the same term structure to go in the opposite direction. **Describe** the appropriate strategies for each investor.

Answer:

Investor A will take a long position in longer-term bonds and a short position in shorter-term bonds. A flattening curve means longer-term bonds will see lower rates, which will increase their value and produce a profit on the long position. Short-term bonds will see relatively higher rates, lowering their value and benefitting the short position.

Investor B will take a short position in longer-term bonds and a long position in short-term bonds. A steepening curve means the longer-term bonds will see higher rates, which will lower the price of the bonds and allow the investor to cover his short position at a lower price. The shorter-term bonds will have lower rates, which will increase the price of these bonds and improve the investor's long position.

Yield curve butterfly shifts are changes in the degree of curvature. A positive butterfly means the yield curve has become less curved. For example, if rates increase, the short and long maturity yields increase by more than the intermediate maturity yields. A negative butterfly means there is more curvature to the yield curve.

MODULE QUIZ 56.3

1. A bond investor projects a decline in 2-year rates from 3% to 2.5% and a decline in 20-year rates from 6% to 4.5%. This situation is best described as a:
   A. bull flattener.
   B. bear flattener.
   C. bull steepener.

D. bear steepener.

KEY CONCEPTS

LO 56.a

The compounding frequency describes how often interest is applied within a single year and defines how an interest rate is measured. Compounding can be annually, semiannually, quarterly, monthly, daily, or continuous. The value of an investment increases as the compounding frequency increases. The present value of a fixed amount to be received or paid in the future decreases as the compounding frequency increases. Interest rates compounded at different frequencies can be compared in order to determine and express equivalent rates.

LO 56.b

The spot rate is the rate earned on an investment when it is received at a single point in the future. The future value received is linked to the current value today via the spot rate. The spot rate and the discount factor are related in that either can be used to equate the present and future value of an investment over a period of time.

LO 56.c

Forward rates are future spot rates based on current spot rates. The forward rate is the rate earned on a future-year investment that should make an investor indifferent between shorter and longer investment options for the same instrument. Spot rates are equal to forward rates compounded for successive periods.

A forward rate agreement (FRA) guarantees a specific rate to be paid or earned during a future period. The value of the FRA is equal to the present value of the difference between the guaranteed rate and the current forward rate. When the guaranteed rate is higher than the current forward rate, the value of the FRA is positive; when the guaranteed rate is lower than the current forward rate, the value of the FRA is negative.

LO 56.d

The par rate at maturity is the rate at which the present value of a bond equals its par value. Par rates represent bond coupon payments when a bond’s price is equal to its par value. Annuity factors, which represent the sum of each period’s discount factor, can be used to calculate the par rate. The par rate and annuity factors can also be used to value bonds with other coupon rates.

LO 56.e

When the term structure of rates is flat, all spot rates are equal and par and forward rates will both equal spot rates. When the term structure is upward sloping, forward rates will be the highest, with spot rates in the middle and par rates as the lowest rates. When the term structure is downward sloping, par rates will be the highest, with spot rates in the middle and forward rates as the lowest rates.

LO 56.f

The value of a bond will rise if the forward rate for the final period is greater than its coupon rate, which is what tends to happen in an upward-sloping term structure. If the forward rate for the final period is less than the coupon rate, the value of the bond will fall.

Investors make borrowing and investing decisions based on their expectations of forward rates relative to the actual forward rates implied by looking at investment alternatives that span different maturities.

LO 56.g

A normal yield curve is upward sloping and reflects long-term rates that are greater than short-term rates. An inverted yield curve is the opposite: it is downward sloping and reflects short-term rates that are greater than long-term rates. A flat yield curve has the same rates across all maturities.

When the yield curve undergoes a parallel shift, the yields on all maturities change in the same direction and by the same amount. The slope of the yield curve remains unchanged following a parallel shift.

When the yield curve undergoes a nonparallel shift, the yields for the various maturities do not necessarily change in the same direction or by the same amount. The slope of the yield curve after a nonparallel shift is not the same as it was before the shift.

- Twists are yield curve changes when the slope becomes either flatter or steeper. A flattening of the yield curve means the spread between short- and long-term rates has narrowed. When both long- and short-term rates decrease, if the long maturity rate decreases more, it is referred to as a bull flattener; if the short maturity rate decreases more, it is a bull steepener. When both rates increase but the short-term rate increases more than the long-term rate, it is referred to as a bear flattener; if the long-term rate increases more, it is a bear steepener.
- Butterfly shifts are changes in curvature of the yield curve. A positive butterfly means the yield curve has become less curved. A negative butterfly means there is more curvature to the yield curve.

LO 56.h

A swap is a derivatives transaction where two parties agree to exchange payments based on the movement of an underlying asset. A fixed rate for floating rate swap involves one party making payments based on a fixed rate and receiving payments based on a floating rate, while the counterparty has the opposite position.

The notional principal is the amount that the interest rates are applied to in order to determine the net payment each period.

The swap market is used to define par rates, in that an *n*-year swap rate defines an *n*-year bond selling for par.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 56.1

1. **B** The annual rate of 3.00%, divided by 12 months, is 0.25% (or 0.0025), and the number of periods is 24, based on monthly compounding for two years.

   The formula for calculating the present value of a future investment, applying the numbers from the question, is:

   $$\frac{1{,}000}{(1 + 0.0025)^{24}} = 941.84$$

   (LO 56.a)

Module Quiz 56.2

1. **B** For a continuously compounded spot rate, the formula to use is:

   $$d(t) = e^{-r(t)t}$$

   Plugging in 6.25% as the rate and 3 as the number of years (*t*), the discount rate is equal to:

   $$d(t) = e^{-0.0625(3)} = 0.8290$$

   (LO 56.b)

2. **C** Given the bond spot rates on the zero-coupon bonds, the appropriate forward rates should be as follows:
   - 1-year forward rate 1 year from today =

     $[(1 + 0.07)^2 / (1 + 0.045)] - 1 = 0.0956$, or 9.56%
   - 1-year forward rate 2 years from today =

     $[(1 + 0.09)^3 / (1 + 0.07)^2] - 1 = 0.1311$, or 13.11%
   - 2-year forward rate 1 year from today =

     $[(1 + 0.09)^3 / (1 + 0.045)] = 1.2393$

     $1.2393^{0.5} - 1 = 0.1132$, or 11.32%

   Therefore, the 1-year forward rate 2 years from today is too low. (LO 56.c)

3. **B** With an upward-sloping yield curve (which occurs when long-term rates exceed short-term rates), the forward rate will be the highest rate, followed by the spot rate in the middle and the par rate as the lowest rate. The only combination for the 10-year rates that will meet this criterion are the forward rate of 5.25%, the spot rate of 4.75%, and the par rate of 4.50%. (LO 56.e)

4. **C** The forward rate for the fourth year is calculated by comparing the 4-year continuously compounded rate to the third-year rate using the following formula:

   $$F = \frac{R_2T_2 - R_1T_1}{T_2 - T_1}$$

   $$F = \frac{0.02375(4) - 0.0225(3)}{4 - 3} = 0.0275, \text{ or } 2.75\%$$

If the anticipated forward rate for the fourth year is less than 2.75%, the investor would borrow for three years at 2.25% and invest for four years at 2.375%. The cost of borrowing in the fourth year, which would be less than 2.75%, would make the overall cost of borrowing lower than the 2.375% earned on the 4-year investment. (LO 56.f)

Module Quiz 56.3

1. **A** A decline in long-term rates which that is greater than the decline in short-term rates is considered a bull flattener. A bear flattener occurs when short-term rates rise more than long-term rates. A bull steepener occurs when short-term rates fall faster than long-term rates, while a. A bear steepener occurs when long-term rates increase at a faster rate than short-term rates. (LO 56.g)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 11.

READING 57

BOND YIELDS AND RETURN CALCULATIONS

Study Session 14

EXAM FOCUS

In this reading, we look at bond yields and spreads, and how reinvestment of a coupon is important in determining overall return. For coupon bonds, yield to maturity (YTM) may not be a good measure of actual returns to maturity. An investor receiving coupon payments bears the risk that these cash flows will be reinvested at a rate of return that is lower than the original promised yield on the bond if market yields have declined. For the exam, know how to calculate YTM given different compounding frequencies and how to calculate and interpret the various components of bond returns.

MODULE 57.1: REALIZED RETURNS AND BOND SPREADS

LO 57.a: Differentiate between gross and net realized returns and calculate the realized return for a bond over a holding period including reinvestments.

A bond's **realized return** compares its ending investment value with its beginning value, while factoring in any coupon payment or coupon reinvestment. The **gross realized return** (or simply **gross return**) of a bond is its end-of-period total value minus its beginning-of-period value divided by its beginning-of-period value, but it does not factor in any financing cost. The end-of-period total value would include both ending bond price and any coupons paid during the period. If we denote the current bond price at time t as BV_t, coupons received during time period t as C_t, and the initial bond price as BV_{t-1}, then the realized return for a bond from time period $t - 1$ to t is computed as follows:

$$R_{t-1,t} = \frac{BV_t + C_t - BV_{t-1}}{BV_{t-1}}$$

EXAMPLE: Calculating gross realized return

What is the gross realized return for a bond that is currently selling for $112 if it was purchased exactly six months ago for $105 and paid a $2 coupon today?

Answer:

Substituting the appropriate values into the realized return equation, we get:

$$R_{t-1,t} = \frac{\$112 + \$2 - \$105}{\$105}$$
$$= 8.57\%$$

To compute the realized return for a bond over multiple periods, we must keep track of the rates at which coupons received are reinvested. When a bondholder receives coupon payments, the investor runs the risk that these cash flows will be reinvested at a rate that is lower than the expected rate. For example, if interest rates go down across the board, the reinvestment rate will also be lower. This is known as **reinvestment risk**.

EXAMPLE: **Calculating realized return with reinvested coupons**

What is the realized return for a bond that is currently selling for \$112 if it was purchased exactly one year ago for \$105, paid a \$2 coupon today, and paid a \$2 coupon six months ago? Assume the coupon received six months ago was reinvested at an annual rate of 1%.

Answer:

$$R_{t-1,t} = \frac{\$112 + \$2 + \left[\$2 \times \left(1 + \frac{1\%}{2}\right)\right] - \$105}{\$105}$$
$$= \frac{\$112 + \$2 + 2.01 - \$105}{\$105}$$
$$= 10.49\%$$

The **net realized return** (or simply **net return**) of a bond is its gross realized return minus per period financing costs. Cost of financing would arise from borrowing cash to purchase the bond. When the bond is fully financed, the initial cash outlay would be zero; however, convention is to use the initial bond price as the beginning-of-period value.

EXAMPLE: **Calculating net realized return**

What is the net realized return for a bond that is currently selling for \$112 and paid a \$2 coupon today if its purchase price of \$105 was entirely financed at an annual rate of 0.6% exactly six months ago?

Answer:

Substituting the appropriate values into the realized return equation and then subtracting per period financing costs, we get:

$$R_{t-1,t} = \frac{\$112 + \$2 - \$105}{\$105} - \frac{0.6\%}{2}$$
$$= 8.57\% - 0.3\%$$
$$= 8.27\%$$

So far, we assumed bonds are valued at coupon dates. What happens if we buy or sell a bond between coupon dates? In this case, we need to calculate the bond's dirty price, which is the quoted price plus accrued interest. For example, consider a semiannual coupon bond with maturity at time *T*. In this case, the bond valuation can be expressed through this formula:

$$P^* = \frac{c/2}{(1+y/2)^{2t_1}} + \frac{c/2}{(1+y/2)^{2t_2}} + \ldots + \frac{c/2}{(1+y/2)^{2t_{n-1}}} + \frac{100 + c/2}{(1+y/2)^{2T}}$$

LO 57.b: Define and interpret the spread of a bond and explain how a spread is derived from a bond price and a term structure of rates.

The market price of a bond may differ from the computed price of a bond using spot rates or forward rates. Any difference between bond market price and bond price according to the term structure of interest rates is known as the **bond spread**. An investor may be concerned with a particular Treasury security's excess return relative to other Treasury securities, and in particular would want to know what spread is necessary to be added to Treasury forward rates to ensure that the present value of bond cash flows equals bond price.

Assume a 2-year bond pays annual coupon payments and a principal payment at the end of Year 2. The value of this bond can be computed by discounting all cash flows by the corresponding 1-year forward rates as follows:

$$\text{bond value} = \frac{\text{coupon}}{(1 + \text{0–12-month forward rate})} + \frac{(\text{coupon} + \text{principal})}{[(1 + \text{0–12-month forward rate}) \times (1 + \text{12–24-month forward rate})]}$$

We can simplify this equation by using discount factors. The 1-year discount factor would be 1 / (1 + 0–12-month forward rate) and the 2-year discount rate would be 1 / [(1 + 0–12-month forward rate) × (1 + 12–24-month forward rate)]. For example, if the 0–12-month forward rate is 2% and the 12–24-month forward rate is 3%, the 1-year discount factor would be 1 / 1.02 = 0.980392 and the 2-year discount factor would be 1 / (1.02 × 1.03) = 0.951837.

If the market price of this bond is different than this computed value, an investor would need to determine the discount rates that would equate the discounted cash flows to this market price. We can determine this by adding a spread (*s*) to the forward rates (or to the discount factors):

$$\text{bond value} = \frac{\text{coupon}}{(1 + \text{0–12-month forward rate} + s)} + \frac{(\text{coupon} + \text{principal})}{[(1 + \text{0–12-month forward rate} + s) \times (1 + \text{12–24-month forward rate} + s)]}$$

By deriving this spread, we can identify how much the bond is trading cheap or rich in terms of the bond's return. Spreads will generally increase with maturity.

MODULE QUIZ 57.1

1. Reinvestment risk would not occur if:
   A. interest rates shifted over the time period the bond is held.
   B. the bonds were callable.
   C. bonds are issued at par.
   D. only zero-coupon bonds are purchased.

MODULE 57.2: YIELD TO MATURITY

LO 57.c: Define, interpret, and apply a bond's yield to maturity (YTM) to bond pricing.

LO 57.d: Explain how to calculate a bond's YTM given its structure and price.

The **yield to maturity** (**YTM**) of a fixed-income security is equivalent to its internal rate of return. The YTM is the single discount rate that equates the present value of all cash flows associated with the instrument to its price.

When the YTM is less than the coupon rate, the bond will trade at a *premium*. When the YTM is greater than the coupon rate, the bond will trade at a *discount*. When the YTM equals the coupon rate, the bond trades at *par*.

For a security that pays a series of known annual cash flows, the computation of yield uses the following relationship:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \ldots + \frac{C_N}{(1+y)^N}$$

where:
P = price of the security (i.e., present value [PV])
C_k = annual cash flow in year k, which for a bond would include the principal at maturity
N = term to maturity in years
y = annual yield or YTM on the security

EXAMPLE: Yield to maturity

Suppose a fixed-income instrument offers annual payments in the amount of $100 for 10 years. The current value for this instrument is $700. **Compute** the YTM on this security.

Answer:

The YTM is the *y* that solves the following equation:

$$\$700 = \frac{\$100}{(1+y)^1} + \frac{\$100}{(1+y)^2} + \frac{\$100}{(1+y)^3} + \ldots + \frac{\$100}{(1+y)^{10}}$$

We can solve for YTM using a financial calculator:

N = 10; PMT = 100; PV = –700; CPT → I/Y = 7.07%

If cash flows occur more frequently than annually, the previous equation can be repurposed as:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \ldots + \frac{C_n}{(1+y)^n}$$

where:

n = N × m = number of periods (years multiplied by payments per year)

C_k = periodic cash flow in time period k

y = periodic yield or periodic interest rate

EXAMPLE: Periodic yield and YTM

Suppose now that the security in the previous example pays the $100 semiannually for five years. **Compute** the periodic yield and the YTM on this security.

Answer:

The periodic yield is the *y* that solves the following equation:

$$\$700 = \frac{\$100}{(1+y)^1} + \frac{\$100}{(1+y)^2} + \frac{\$100}{(1+y)^3} + \ldots + \frac{\$100}{(1+y)^{10}}$$

Using a financial calculator:

N = 10; PMT = 100; PV = –700; CPT → I/Y = 7.07%. To compute the annual YTM, we must multiply the periodic yield by the number of periods per year, *m*, which in this case is equal to 2. This produces a YTM of 14.14%.

PROFESSOR'S NOTE

Reinvestment risk is a major threat to the bond's computed YTM, as it is assumed in such calculations that the coupon cash flows can be reinvested at a rate of return that's equal to the computed yield (e.g., if the computed yield is 8%, it is assumed the investor will be able to reinvest all coupons at 8%). If the average reinvestment rate is below the YTM, the realized yield will be below the YTM. For this reason, it is often stated that the yield to maturity assumes cash flows will be reinvested at the YTM and assumes that the bond will be held until maturity.

Annuity and Perpetuity

LO 57.e: Calculate the price of an annuity and a perpetuity.

We can easily calculate the price of cash flows (annuities) if given the YTM and cash flows.

EXAMPLE: Present value of an annuity

Suppose a fixed-income instrument offers annual payments in the amount of $100 for 10 years. The YTM for this instrument is 10%. **Compute** the price (PV) of this security.

Answer:

The price is the present value that solves the following equation:

$$PV = \frac{\$100}{(1+0.10)^1} + \frac{\$100}{(1+0.10)^2} + \frac{\$100}{(1+0.10)^3} + \ldots + \frac{\$100}{(1+0.10)^{10}}$$

Using a financial calculator, the price equals $614.46:

N = 10; PMT = 100; I/Y = 10; CPT PV = $614.46

The perpetuity formula is straightforward and does not require an iterative process:

$$\text{PV of a perpetuity} = \frac{C}{y}$$

where:
C = cash flow that will occur every period into perpetuity
y = yield to maturity

EXAMPLE: Price of a perpetuity

Suppose we have a security paying $1,000 annually into perpetuity. The interest rate is 10%. **Calculate** the price of the perpetuity.

Answer:

We don't need a financial calculator to do this calculation. The price of the perpetuity is simply $10,000:

$$PV = \frac{\$1,000}{0.10} = \$10,000$$

Spot Rates and YTM

LO 57.f: Explain the relationship between spot rates and YTM.

The relationship between spot rates and YTM is complex, but we can make some important observations. With larger coupons, the earlier spot rates will be more important in determining the YTM. When the term structure of interest rates is upward sloping, the earlier spot rates are lower than the spot rate for the final payment date. This is because the largest cash flow occurs as the bond matures. Therefore, the YTM declines as the coupon rate increases. If the spot curve is flat, the spot rate for early payment is the same as for the final maturity. If the term structure of interest rates is downward sloping, the YTM increases as the coupon rate increases.

Relationship Between YTM, Coupon Rate, and Price

LO 57.g: Define the coupon effect and explain the relationship between coupon rate, YTM, and bond prices.

The **coupon effect** describes a scenario where two bonds with identical maturities but different coupons will have different yields to maturity. If two bonds are identical in all respects except their coupon, the bond with the smaller coupon will be more sensitive to interest rate changes. That is, for any given change in yield, the smaller-coupon bond will experience a bigger percentage change in price than the larger-coupon bond. All else being equal:

- the lower the coupon rate, the greater the interest rate risk; and
- the higher the coupon rate, the lower the interest rate risk.

Figure 57.1 summarizes the relationship between bond price sensitivity and coupon size. The bonds have equal maturities but different coupons. Assume semiannual coupons for both bonds.

Figure 57.1: Bond Price Reactions to Changes in Yield

	Price Change From Par ($1,000)	
Change in Interest Rates	20-Year, 8%	20-Year, 12%
–2%	+231.15	+171.59
–1%	+106.77	+80.23
0%	0	0
+1%	–92.01	–70.73
+2%	–171.59	–133.32

For the same change in interest rates, the 20-year, 8% bond experiences a greater change in price than the 20-year, 12% bond. This suggests that bonds with similar maturities, but different coupon rates, can have different yields to maturity.

Japanese Yields

The yield convention in Japan differs from the U.S. yield convention. Japanese bond yields are typically quoted on a simple yield basis without factoring in compounding. Simple yields only require the coupon (c), principal (p), and maturity (T):

$$\text{yield} = \frac{c}{p} + \frac{(100 - p)}{pT}$$

For example, consider a Japanese bond with a 3% coupon and 6 years to maturity, with a price of 98:

$$\text{yield} = \frac{3}{98} + \frac{(100 - 98)}{98 \times 6} = 0.034, \text{ or } 3.4\%$$

MODULE QUIZ 57.2

1. An annuity pays $10 every year for 100 years and currently costs $100. The YTM is closest to:
   A. 5%.
   B. 7%.
   C. 9%.
   D. 10%.

2. A $1,000 par bond carries a 7.75% semiannual coupon rate. Prevailing market rates are 8.25%. What is the price of the bond?
   A. Less than $1,000.
   B. $1,000.
   C. Greater than $1,000.
   D. Not enough information to determine.

3. A $1,000 par bond carries a coupon rate of 10%, pays coupons semiannually, and has 13 years remaining to maturity. Market rates are currently 9.25%. The price of the bond is closest to:
   A. $586.60.
   B. $1,036.03.
   C. $1,055.41.
   D. $1,056.05.

4. An investment pays $50 annually into perpetuity and yields 6%. Which of the following is closest to the price?
   A. $120.
   B. $300.
   C. $530.
   D. $830.

MODULE 57.3: BOND RETURN DECOMPOSITION

LO 57.h: Explain the decomposition of the profit and loss (P&L) for a bond position or portfolio into separate factors including carry roll-down, rate change, and spread change effects.

Return decomposition for a bond breaks down bond **profit and loss (P&L)** into component parts. This decomposition of P&L helps bond investors understand how their investments are making or losing money. A bond's profitability or loss is generated through price appreciation and explicit cash flows (e.g., cash-carry), such as coupons and financing costs. The change in the bond's price can be broken down into three component parts for price effect analysis: carry roll-down, rate change, and spread change effects. Dividing each component return by the bond's price will give us components of the gross return.

The **carry roll-down** is the estimated return from bond price movements and coupon payment assuming no change to interest rate expectations. In other words, the expected forward rates are realized and become the spot rates. Different carry roll-down scenarios for this expected term structure will be discussed in the next learning objective. This component does not account for spread changes.

The **rate change** is the realized return when this realized return is different from what was assumed under the carry roll-down. Similar to carry roll-down, this component does not account for spread changes.

The **spread change** component accounts for price changes due to changes in the bond's spread relative to other bonds. Expected changes in the spread are frequently the subject of investments for traders who are betting that a security is trading either cheap or rich.

For example, consider the information for a bond with a 4% coupon paid semiannually:

Bond initial price:	102.65
Carry roll-down:	0.85
Rate changes:	0.30
Spread changes:	0.08
Bond final value:	101.88
Cash-and-carry:	2.00

The gain on the bond is 101.88 + 2.00 – 102.65 = 1.23.

The total gain can be broken down into its component yields: 1.23 = 0.85 (carry roll-down) + 0.30 (impact of rate changes) + 0.08 (impact of spread change).

We can also show these figures as a component of gross return, where the gross return is 1.23 / 102.65 = 1.198%.

There are two extensions to P&L analysis:

1. *Consider the impact of financing*. If financing is considered, the *cost of financing* should be added as a fourth component of the P&L, and both gross and net returns would be calculated.
2. *Consider accrued interest on both the initial and final valuation dates*. So far, we looked at returns simply between two coupon dates. However, both the initial and final bond valuations could be between coupon dates. In this case, it is necessary to add a fourth component for the impact of *accrued interest*.

Carry Roll-Down Scenarios

LO 57.i: Describe the common assumptions made about interest rates when calculating carry roll-down, and calculate carry roll-down under these assumptions.

As mentioned, the carry roll-down is the estimated return from bond price movements and coupon payment assuming no change to interest rate expectations. Traders make investment return calculations based on their expectations, and many traders will consider scenarios where rates do not change. Given this expectation, term structure choices for no-change scenarios include realized forward, unchanged term structure, and unchanged yields.

The *realized forward* scenario assumes that forward rates for future periods remain unchanged as time passes. This means that as forward rates are realized, they will be equal to the expected future spot rates. As a result, when the beginning of a forward period is reached, the forward rate becomes the spot rate.

For example, consider a Treasury bond with a 2-year maturity and 2% coupon, currently valued at $100.785. In addition, assume the forward rates shown in Figure 57.2.

Figure 57.2: Forward Rates for Realized Forward Scenario

Period (Years)	Forward Rate (%) Semiannual Compounding	Realized Forward Rate After 6 Months
0–0.5	0.8	
0.5–1.0	1.0	1.0
1.0–1.5	1.2	1.2
1.5–2.0	1.4	1.4

PROFESSOR'S NOTE

Recall that forward rates are always quoted on an annual basis.

Under the realized forward scenario, in six months the realized forward rates will be 1.0% for 0–0.5 years, 1.2% for 0.5–1.0 years, and 1.4% for 1.0–1.5 years. We can therefore value the bond, which now has 1.5 years left to maturity, compounded on a semiannual basis:

$$\frac{1.0}{1.005} + \frac{1.0}{(1.005 \times 1.006)} + \frac{101}{(1.005 \times 1.006 \times 1.007)} = \$101.188$$

Therefore, if forward rates are realized in six months, the bond's price is expected to change to $101.188.

An alternative way of calculating the bond's price using the carry roll-down is to assume that the return earned is equal to the currently prevailing 1-period forward rate. In our example, the 6-month forward rate, semiannually compounded, is 0.8%. Therefore, the carry roll-down can be calculated as:

$100.785 × 0.004 = $0.403

And therefore:

$100.785 + $0.403 = $101.188

The *unchanged term structure* scenario assumes that the term structure will remain unchanged over the investment horizon. This means that the gross realized return will depend greatly on the relationship between the bond's coupon rate and the last forward rate before the bond matures. This scenario implies that there is a risk premium built into forward rates. For example, if the term structure is upward sloping and remains unchanged, the term structure shape must reflect an investor risk premium that increases over the investment horizon.

Continuing with our previous example of a Treasury bond with a 2-year maturity and 2% coupon, currently valued at $100.785, the unchanged term structure assumption means that the assumed forward rates will materialize. This is summarized in Figure 57.3.

Figure 57.3: Forward Rates for Unchanged Term Structure Scenario

Period (Years)	Forward Rate (%) Semiannual Compounding	Realized Forward Rate After 6 Months
0–0.5	0.8	
0.5–1.0	1.0	0.8
1.0–1.5	1.2	1.0
1.5–2.0	1.4	1.2

We can therefore value the bond, which now has 1.5 years left to maturity, compounded on a semiannual basis:

$$\frac{1.0}{1.004} + \frac{1.0}{(1.004 \times 1.005)} + \frac{101}{(1.004 \times 1.005 \times 1.006)} = \$101.487$$

As the name suggests, the *unchanged yields* scenario assumes that bond yields remain unchanged over the investment horizon. This means that the 1-period gross realized return will equal a bond's yield (i.e., its yield to maturity). Thus, this scenario assumes that bond coupon payments are reinvested at the YTM. As stated earlier, there are limitations to this reinvestment assumption because the term structure is unlikely to be flat and remain unchanged.

MODULE QUIZ 57.3

1. Assume the 1-year spot rate is 4%, the 1-year forward rate starting in 1 year is 5%, and the 1-year forward rate starting in 2 years is 6%. Under the realized forward scenario, the realized 1-year rate in 1 year would be:
   A. 4%.
   B. 4.5%.
   C. 5%.
   D. 5.5%.

KEY CONCEPTS

LO 57.a

The gross realized return for a bond is the difference between end-of-period total value (including end-of-period price and coupons) and starting period value all divided by starting period value:

$$R_{t-1,t} = \frac{BV_t + C_t - BV_{t-1}}{BV_{t-1}}$$

The net realized return for a bond is its gross realized return minus per period financing costs.

LO 57.b

The difference between bond market price and bond model price (according to the term structure) is known as the spread. By deriving the spread, we can identify how much the bond is trading cheap or rich in terms of the bond's return. Spreads will generally increase with maturity.

LO 57.c

The yield to maturity (YTM) is the single discount rate that equates the present value of all cash flows associated with the instrument to its price. An iterative process is used for the actual computation of yield. On a financial calculator, it can be found by inputting all other variables and solving for YTM.

When the YTM is less than the coupon rate, the bond will trade at a premium. When the YTM is greater than the coupon rate, the bond will trade at a discount. When the YTM equals the coupon rate, the bond trades at par.

LO 57.d

For a security that pays a series of known annual cash flows, the computation of yield uses the following relationship:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \ldots + \frac{C_N}{(1+y)^N}$$

LO 57.e

The present value of an annuity (equal future cash flows) can be found by using the YTM and cash flows. The present value, or price, of a perpetuity can be found by dividing the coupon payment by the YTM.

LO 57.f

When pricing a bond, YTM or spot rates can be used. The YTM will be a blend of the spot rates for the bond. The YTM declines as the coupon rate increases. If the term structure of interest rates is downward sloping, the YTM increases as the coupon rate increases.

LO 57.g

The coupon effect describes a scenario where two bonds with identical maturities but different coupons will have different yields to maturity. The bond with the smaller coupon will be more sensitive to interest rate changes.

When the bond is trading at par, the coupon rate is equal to the YTM. When the bond is trading below par, the coupon rate is less than the YTM, and is said to trade at a discount. When a bond is trading above par, the coupon rate is greater than the YTM, and the bond then trades at a premium.

LO 57.h

A bond's P&L is generated through price appreciation and explicit cash flows. Total price appreciation can be broken down into three component parts for price effect analysis: carry roll-down, rate changes, and spread changes.

LO 57.i

For an expected term structure, no-change scenarios include realized forward, unchanged term structure, and unchanged yields. The realized forward scenario assumes that forward rates are equal to expected future spot rates. The unchanged term structure scenario assumes that the term structure will remain unchanged. The unchanged yields scenario assumes that bond yields remain unchanged.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 57.1

1. **D** Callable bonds have reinvestment risk because the principal can be prematurely retired. The higher the coupon, the higher the reinvestment risk, holding all else constant. A bond being issued at par has nothing to do with reinvestment risk. (LO 57.a)

Module Quiz 57.2

1. **D** N = 100; PMT = 10; PV = –100; CPT → I/Y = 10% (LO 57.e)
2. **A** Because the coupon rate is less than the market interest rate, the bond is a discount bond and trades less than par. (LO 57.g)
3. **D** N = 26; PMT = 50; I/Y = 4.625; FV = 1,000; CPT → PV = $1,056.05 (LO 57.c)
4. **D** PV = C/I = $50 / 0.06 = $833.33 (LO 57.e)

Module Quiz 57.3

1. **C** Under the realized forward scenario, as forward rates are realized, they will be equal to the expected future spot rates. As a result, the realized 1-year rate in 1 year would be 5%. (LO 57.i)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 12.

READING 58

APPLYING DURATION, CONVEXITY, AND DV01

Study Session 14

EXAM FOCUS

This reading looks at ways to measure and hedge risk for fixed-income securities. The three main concepts covered are DV01, duration, and convexity. DV01 is an acronym for the dollar value of a basis point, which measures how much the price of a bond changes from a one basis point change in yield. Duration, specifically effective duration, measures the percentage change in a bond's value for small, parallel changes in rates. Using both DV01 and duration can measure price volatility, but they do not capture the curvature in the relationship between bond yield and price. To capture the curvature effects of the price-yield relationship, we use convexity to complement these measures. For the exam, be able to compare, contrast, and calculate DV01, duration, and convexity.

MODULE 58.1: DOLLAR VALUE OF A BASIS POINT

LO 58.a: Describe a one-factor interest rate model and identify common examples of interest rate factors.

Measures of interest rate sensitivity allow investors to evaluate bond price changes as a result of interest rate changes. Price changes are based on **interest rate factors**, which are variables that influence individual interest rates along the yield curve. Under a simple one-factor approach (i.e., **single-factor approach**), a change in one rate can be used to determine the movement in all other interest rates. For example, if the two-year spot rate increases by three basis points, all other spot rates are assumed to increase by three basis points.

Under another one-factor approach, interest rates could move in the same direction, but not in a parallel way. For example, short-term interest rates could increase more than long-term rates. This is still considered a one-factor model because the change in only one rate is necessary to calculate all rate changes. Alternatively, the shape of the term structure of interest rates could also change completely (e.g., changing from upward sloping to downward sloping).

LO 58.b: Define and calculate the DV01 of a fixed-income security given a change in rates and the resulting change in price.

The **dollar value of a basis point (DV01)** is the dollar value change in a fixed-income security's price for a one basis point change in interest rates. The *01* refers to one basis point (i.e., 0.0001).

DV01 is computed using the following formula:

$$\text{DV01} = -\frac{\Delta P}{\Delta y}$$

where:
ΔP = change in the value of the portfolio
Δy = size of a parallel shift in the interest rate term structure

The DV01 formula is preceded by a negative sign, so when rates decline and prices increase, DV01 will be positive.

EXAMPLE: **Computing DV01**

Suppose that a bond with a face value of $100,000 and a coupon of 6% (compounded semiannually) matures in five years. The bond is currently priced at $100,750.00. Also, suppose there is a parallel shift in the interest rate term structure by 10 basis points, and that the bond's price increases to $101,181.44. **Compute** the DV01.

Answer:

We can calculate the new bond price using a financial calculator. The bond's price will increase to $101,181.44. The DV01 will therefore be the change in bond price divided by the change in interest rates:

$$\text{DV01} = -\frac{\Delta P}{\Delta y} = -\frac{\$101{,}181.44 - \$100{,}750.00}{-10} = \$43.14$$

Since bond prices exhibit convexity to changes in interest rates (discussed in LO 58.f), the bond price increase will be larger when rates decline compared to the bond price decline when rates increase by the same percentage. For instance, in our previous example, the bond's price increase was $431.44 for a 10 basis point decline in interest rates. However, for a 10 basis point increase, the bond's price would decline by $429.24 (to $100,320.76). This is summarized in Figure 58.1.

Figure 58.1: Bond Price Change for Same Change in Rates

Rate Change	New Bond Price	Price Change	DV01
+10 bps	$100,320.76	–$429.24	$42.92
–10 bps	$101,181.44	+$431.44	$43.14

We can also calculate DV01 using yields. In this case, we would take the yield to maturity (YTM) and recalculate the bond prices by increasing and decreasing the YTM by the same basis point change. We would then average the two prices to calculate the DV01. This is best accomplished by using a financial calculator to compute bond price changes for one basis point changes in yield.

Analysts may also distinguish between the following definitions of DV01:

- *Yield-based DV01*. The change in bond price from a one basis point change in yield.
- *DVDZ* or *DPDZ*. The change in bond price from a one basis point change in spot (zero) rates.
- *DVDF* or *DPDF*. The change in bond price from a one basis point change in forward rates.

DV01 Application to Hedging

LO 58.c: Calculate the face amount of bonds required to hedge an interest rate-sensitive position given the DV01 of each.

Sensitivity measures like DV01 are commonly used to assess hedges between the position to be hedged and the instrument used to hedge the position. For example, a DV01 for an investor of –$250 implies that investor's position would increase (decrease) by $250 for a one basis point decrease (increase) in all rates.

The investor can hedge the risk of its position change through hedging with a bond. A hedge ratio of 1 means that the hedging instrument and the position have the same interest rate sensitivity. The goal of a hedge is to produce a combined position (the initial position combined with the hedge position) that will not change in value for a small change in yield. The **hedge ratio (HR)** is expressed as:

$$\text{HR} = \frac{\text{DV01 (of position to be hedged)}}{\text{DV01 (of hedging instrument)}}$$

The face amount to be hedged can then be computed as:

face value of hedging instrument = face value of initial position × HR

EXAMPLE: Computing the amount of bonds needed to hedge

An investor's $1 million portfolio has a DV01_P of $340. This portfolio can be hedged with a 3% coupon, 5-year bond with a DV01_B of $285. **Calculate** the face amount of the bond required to hedge the investor's initial position.

Answer:

We can determine the face amount of the bond required to hedge the investor's position as:

$$\text{face value of hedging instrument} = \$1 \text{ million} \times \frac{\$340}{\$285} = \$1{,}192{,}982.46$$

Therefore, the investor's $1 million portfolio with a DV01_P of $340 can be hedged with a $1,192,982 face value 3% coupon, 5-year bond with a DV01_B of $285.

MODULE QUIZ 58.1

Use the following information to answer Questions 1 and 2.

An investor has a short position valued at $100 in a 10-year, 5% coupon bond with a 7% yield to maturity (YTM). Assume discounting occurs on a semiannual basis.

1. Which of the following is closest to the dollar value of a basis point (DV01)?
   A. 0.033.
   B. 0.047.
   C. 0.056.
   D. 0.065.

2. Using a 20-year T-bond with a DV01 of 0.085 to hedge the interest rate risk in the 10-year bond with a DV01 of 0.065, which of the following actions should the investor take?
   A. Buy $76.50 of the hedging instrument.
   B. Sell $76.50 of the hedging instrument.
   C. Buy $130.75 of the hedging instrument.
   D. Sell $130.75 of the hedging instrument.

MODULE 58.2: DURATION AND CONVEXITY

Duration

LO 58.d: Define, calculate, and interpret the effective duration of a fixed-income security given a change in rates and the resulting change in price.

Duration is the most widely used measure of bond price volatility. A bond's price volatility is a function of its coupon, maturity, and initial yield. Duration captures the impact of all three of these variables in a single measure.

Effective duration measures the percentage change in a bond's price, *P*, for a given small change in rates (Δy). It is commonly interpreted as the percentage change in a bond's price for a 100 basis point change in rates.

$$D = -\frac{\Delta P / P}{\Delta y} = -\frac{\Delta P}{P \Delta y}$$

This equation can be rewritten to show the change in the bond's price:

$$\Delta P = -D \times P \times \Delta y$$

where:
D = duration
P = price
Δy = change in rates

EXAMPLE: Computing bond price change

Suppose a 15-year bond with an annual coupon of 7% is currently trading at $98,550. The bond has a duration of 3.28. **Compute** and **interpret** the bond's price change for a 25 basis point decrease in all rates.

Answer:

We can compute the change in the bond's price as:

$$\Delta P = -D \times P \times \Delta y = -3.28 \times \$98{,}550 \times -0.0025 = \$808.11$$

Therefore, for a given 25 basis point *decrease* in rates, the bond's price is expected to *increase* by $808.11.

It is common to calculate a bond's price change for a 100 basis point change in rates by first calculating the price change for a one basis point change and then scale it up by multiplying by a factor of 100.

Callable and Putable Bonds

A **callable bond** is a bond that gives its issuer the right to buy back the bond from investors at a predetermined price in the future. The issuer will typically want to buy back the bond when interest rates have declined and it can refinance at lower, more beneficial rates. As a result, the upside price appreciation in response to decreasing rates is limited and capped at the call price.

When considering the effective duration of a callable bond, a common approach is to use the bond's duration calculated as if the bond would not be called in the future. However, this is incorrect, as it ignores the possibility that the bond will be called. An alternative approach is to calculate the effective duration as the weighted duration of the bond, with the weights as the probability that the bond would not be called and the probability that it would be called.

For example, consider a 7-year bond that can be called in four years, with a 43% probability that it will be called. The effective duration would then be calculated as:

$$D = 0.43(D_{\text{called}}) + 0.57(D_{\text{not called}})$$

While this method is better than the first approach, it ignores the reduction in probability that the bond is called when rates increase. The correct approach to calculate duration requires valuing the bond's price today, revaluing the price today for an assumed one basis point parallel change in interest rates, and calculating duration from the percentage change in the bond's price.

A **putable bond** gives the bondholder the right to sell the bond back to the issuer at a predetermined price in the future. The effective duration of a putable bond is calculated in a similar fashion as callable bonds.

DV01 vs. Duration

LO 58.e: Compare and contrast DV01 and effective duration as measures of price sensitivity.

While DV01 measures the change in *dollar value* of a security for every basis point change in rates, duration measures the *percentage* change in a security's value for a unit change in rates.

Duration is more convenient than DV01 in an investing context, given that a high duration number can easily alert a bond investor of a large percentage change in value.

However, when analyzing trading or derivatives hedging situations, percentage changes are not that useful because dollar amounts of the two sides of the transaction are different. In this case, DV01 would be more useful.

Convexity

LO 58.f: Define, calculate, and interpret the convexity of a fixed-income security given a change in rates and the resulting change in price.

Duration is a good approximation of price changes for relatively small, parallel changes in interest rates. Like DV01, duration is a linear estimate since it assumes that the price change will be the same regardless of whether interest rates go up or down. However, the price calculated using duration will always underestimate the actual price change in the bond, which will always be above the duration-calculated price.

Convexity measures the sensitivity of duration to changes in interest rates. It is a measure of the curvature in the relationship between changes in interest rates and the change in bond price. Because duration will always result in a price below the bond's true price change, convexity will help adjust (lift) the price calculated by duration. It is always a positive adjustment. Generally, the higher the convexity number, the higher the price volatility.

While a precise calculation of convexity involves the use of calculus, an approximate measure of convexity can be generated as:

$$C = \frac{1}{P}\left[\frac{P^{+} + P^{-} - 2P}{(\Delta y)^2}\right] = \left[\frac{P^{+} + P^{-} - 2P}{P(\Delta y)^2}\right]$$

where:
C = convexity
P = initial bond price
P^{+} = new (lower) bond price when rates increase
P^{-} = new (higher) bond price when rates decline

EXAMPLE: Computing convexity

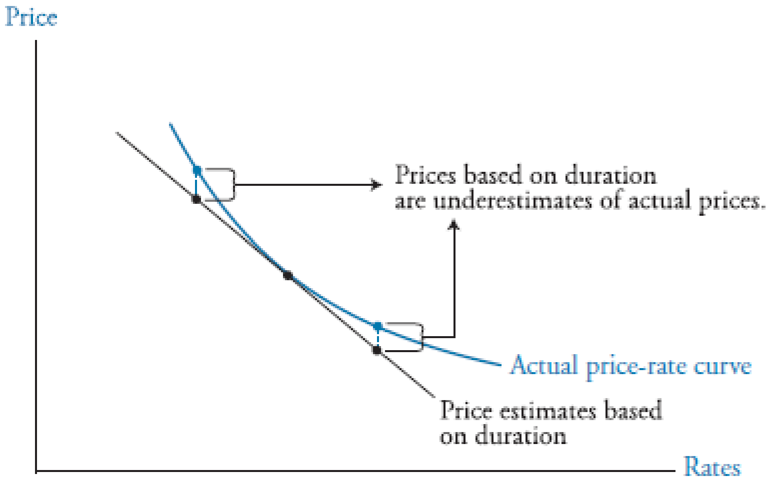
Suppose there is a 15-year option-free noncallable bond with an annual coupon of 7% trading at par. If interest rates rise by 50 basis points (0.50%), the estimated price of the bond is 95.586. If interest rates fall by 50 basis points, the estimated price of the bond is 104.701. **Calculate** the convexity of this bond.

Answer:

$$\text{convexity} = \frac{104.701 + 95.586 - 2(100)}{(100)(0.005)^2} = 114.8$$

Unlike duration, a convexity of 114.8 cannot be conveniently converted into some measure of potential price volatility. Indeed, the convexity value means nothing in isolation, although a higher number does mean more price volatility than a lower number. This value can become very useful, however, because it can be combined with a bond's duration to provide a more accurate estimate of potential price change. Figure 58.2 illustrates why convexity is important and why estimates of price changes based solely on duration alone are inaccurate.

Figure 58.2: Duration-Based Price Estimates vs. Actual Bond Prices



Price Change Using Both Duration and Convexity

Now, by combining duration and convexity, a far more accurate estimate of the change in the price of a bond can be obtained, especially for larger changes in yield. The price change of the bond can be calculated by adding the convexity effect to the duration effect:

$$\text{price change} \approx \text{duration effect} + \text{convexity effect}$$

$$\Delta P = -D \times P \times \Delta y + \frac{1}{2} \times C \times P \times \Delta y^2$$

EXAMPLE: Estimating bond price changes using both duration and convexity

Using the duration/convexity approach, **estimate** the effect of a 150 basis point increase and decrease on a 15-year, 7%, option-free bond currently trading at par. The bond has a duration of 9.115 and a convexity of 114.8.

Answer:

Using the duration/convexity approach, we see the following:

$$\Delta P = -9.115 \times 100 \times 0.015 + \frac{1}{2} \times 114.8 \times 100 \times 0.015^2$$
$$= -12.381, \text{ or a decline of } \$12.381$$

Using duration alone would have implied a price change of −9.115 × 100 × 0.015 = −13.6725; that is, a bond price decline of $13.6725. In other words, duration would have overestimated the decline.

Portfolio Duration and Portfolio Convexity

LO 58.g: Calculate the DV01, duration, and convexity of a portfolio of fixed-income securities.

We will now look at calculating the DV01, duration, and convexity of a portfolio of fixed-income securities.

The DV01 of a portfolio of bonds is simply the sum of the DV01s, without the use of weights. For example, a portfolio that consists of four bonds with DV01s of 15, 25, 28, and 35 would be 103 (= 15 + 25 + 28 + 35).

The duration of a portfolio of individual securities equals the weighted sum of the individual durations. Each security's weight is its value taken as a percentage of the overall portfolio value.

$$\text{duration of portfolio} = \sum_{j=1}^{K} w_j \times D_j$$

where:
D_j = duration of bond *j*
w_j = market value of bond *j* divided by market value of the total bond portfolio
K = number of bonds in portfolio

Like portfolio duration, portfolio convexity is calculated as the value-weighted average of the individual bond convexities within a portfolio.

EXAMPLE: **Computing portfolio duration and convexity**

Assume there are three bonds in a portfolio. Portfolio weightings and individual durations and convexities are shown in the following table.

Coupon	Maturity (Years)	YTM	Price (% of Par)	Value (In $ Millions)	Duration	Convexity
1%	5	0.75%	102.92	2.0	4.02	14.05
2%	15	1.25%	109.58	3.5	9.63	71.80
3%	30	2.13%	118.30	4.5	13.75	170.29

Calculate portfolio duration and convexity.

Answer:

The total portfolio value is $10 million (= $2M + $3.5M + $4.5M). The duration and convexity of the portfolio are therefore:

$$\text{duration} = \left(\frac{2}{10} \times 4.02\right) + \left(\frac{3.5}{10} \times 9.63\right) + \left(\frac{4.5}{10} \times 13.75\right) = 10.36$$

$$\text{convexity} = \left(\frac{2}{10} \times 14.05\right) + \left(\frac{3.5}{10} \times 71.80\right) + \left(\frac{4.5}{10} \times 170.29\right)$$
$$= 104.57$$

Hedging Example

LO 58.h: Explain the hedging of a position based on effective duration and convexity.

In the section on DV01, we introduced the concept of hedging a portfolio with bonds using DV01. We can now extend those calculations to hedging a portfolio with bonds using duration and convexity.

Hedging an investment with a value V and a duration D_V can be done with a bond with value P and duration of D_P:

$$\Delta V = -D_V \times V \times \Delta y$$
$$\Delta P = -D_P \times P \times \Delta y$$

For the investment to be fully hedged, it must be that ΔV = ΔP. We can then rearrange these equations to state:

$$P = -\frac{V \times D_V}{D_P}$$

EXAMPLE: **Computing the amount of bonds needed to hedge an investment**

An investor has a $1.7 million investment with a duration of 6.5. The investor is looking to hedge the investment using a bond with a duration of 7.9. **Calculate** the face amount of the bond required to hedge the investor's position, and **demonstrate** why this will result in a fully hedged position.

Answer:

We can determine the face amount of the bond required to hedge the investor's position as:

$$\text{bond value} = -\$1.7 \text{ million } \frac{6.5}{7.9} = -\$1.4 \text{ million}$$

The investor should therefore short $1.4 million bonds. The portfolio is fully hedged against parallel rate changes:

$$\Delta V = -6.5 \times 1.7 \times \Delta y = -11.1\Delta y$$
$$\Delta P = -7.9 \times (-1.4) \times \Delta y = +11.1\Delta y$$

An investor could also hedge an investment using both duration and convexity. This could be done by using two bonds with prices P_1 and P_2, durations D_1 and D_2, and convexities C_1 and C_2:

$$\Delta V = -V \times D_V \times \Delta y + \frac{1}{2} \times C_V \times V \times \Delta y^2$$

$$\Delta P_1 = -P_1 \times D_1 \times \Delta y + \frac{1}{2} \times C_1 \times P_1 \times \Delta y^2$$

$$\Delta P_2 = -P_2 \times D_2 \times \Delta y + \frac{1}{2} \times C_2 \times P_2 \times \Delta y^2$$

For the investment to be fully hedged, both durations and convexities must be zero. We can then rearrange the equation to state:

$$-V \times D_V - P_1 \times D_1 - P_2 \times D_2 = 0$$
$$+V \times C_V + P_1 \times C_1 + P_2 \times C_2 = 0$$

EXAMPLE: Computing the amount of bonds needed to hedge an investment

An investor has a $3 million investment with a duration of 6 and convexity of 25. The investor is looking to hedge the investment using two bonds: Bond 1 has a duration of 7 and convexity 20, and Bond 2 has a duration of 5 and convexity of 19. **Calculate** the face amount of the bond required to hedge the investor's position.

Answer:

We can determine the face amount of the bond required to hedge the investor's position as:

$$-V \times D_V - P_1 \times D_1 - P_2 \times D_2 = 0 = -\$3(6) - P_1(7) - P_2(5)$$
$$+V \times C_V + P_1 \times C_1 + P_2 \times C_2 = 0 = \$3(25) + P_1(20) + P_2(19)$$

Solving for P_1 and P_2, we get $P_1 = 1$ and $P_2 = -5$.

The investor can therefore hedge the $3 million investment with a $1 million long position in Bond 1 and a $5 million short position in Bond 2. This ensures that the investment will be hedged against even larger parallel changes in rates.

Constructing a Barbell Portfolio

LO 58.i: Construct a barbell portfolio to match the cost and duration of a given bullet investment and explain the advantages and disadvantages of bullet and barbell portfolios.

A **barbell investment** is typically used when an investor invests in bonds with short and long maturities, thus forgoing any intermediate-term bonds. A **bullet investment** is used when an investor buys a single bond, typically concentrated in the intermediate maturity range.

The advantages and disadvantages of a barbell versus a bullet portfolio depend on the investor's view on interest rates. If the manager believes that rates will be especially volatile, the barbell portfolio would be preferred over the bullet portfolio.

Consider details for the following three bonds in Figure 58.3.

Figure 58.3: Bonds for Barbell and Bullet Investments

Bond	Coupon	Maturity	Value	Effective Duration	Effective Convexity
1	0.75%	5 years	100.02	4.12	21.9
2	1.70%	10 years	99.04	7.65	59.8
3	2.75%	30 years	97.46	14.93	310.5

Assume that an investor is looking to construct a $100,000 portfolio with a duration of 7.65. The investor could buy $100,000 of the 10-year bond (Bond 2) with a duration of 7.65. Because only one bond is purchased that matches the desired duration, this is a bullet investment. This portfolio would have a convexity of 59.8, which is the convexity of Bond 2.

Instead of a bullet investment, the investor could construct a portfolio using a shorter and longer maturity bond (using the 5-year and 30-year bonds) with a weighted duration of 7.65. This is a barbell investment, which would have the same duration as the individual bullet investment:

$4.12\,w_1 + 14.93\,(1 - w_1) = 7.65$, where w_1 is the weight in Bond 1

$w_1 = 0.6735$

Therefore, the investor could construct a portfolio with a duration of 7.65 by investing 67.35% of funds in the 5-year Bond 1 and investing (1 – 67.35%) = 32.65% in the 30-year Bond 3.

The barbell portfolio will have a combined convexity of:

$0.6735 \times 21.9 + 0.3265 \times 310.5 = 116.1$

Note that although both the bullet and barbell portfolios have the same amount of duration risk, the barbell portfolio's convexity of 116.1 is greater than the bullet portfolio's convexity of 59.8. Higher convexity is beneficial because it improves the investor's position for parallel changes in interest rates.

Arbitrageurs could anticipate changes in interest rates to decide whether an arbitrage opportunity exists using both bullet and barbell strategies. If interest rates are expected to increase in a parallel way, the barbell strategy would typically outperform because of the higher convexity. If this is the case, an investor could profit by buying the barbell portfolio and short selling the same amount of the bullet portfolio. However, for nonparallel changes in interest rates, the bullet strategy often outperforms. For example, in a typical upward sloping yield curve scenario, the yield of the bullet investment would be greater than the yield of the barbell strategy. Researchers typically strive to create models of interest rate movements that result in no-arbitrage opportunities to investors.

MODULE QUIZ 58.2

1. The duration of a portfolio can be computed as the sum of the value-weighted durations of the bonds in the portfolio. Which of the following is the most limiting assumption of this approach?
   A. All weights must be different.
   B. The portfolio must be equally weighted.

C. The rate changes are assumed to be parallel.
D. All the bonds in the portfolio must be in the same risk class or along the same yield curve.

2. What is the estimate for the percentage price change in bond price from a 25 basis point increase in rates on a bond with a duration of 7 and a convexity of 243?
   A. 1.67% decrease.
   B. 1.67% increase.
   C. 1.75% increase.
   D. 1.75% decrease.

KEY CONCEPTS

LO 58.a

Interest rate factors are random variables that influence individual interest rates along the yield curve.

LO 58.b

The DV01 is the dollar value change in a fixed-income security's price for a one basis point change in interest rates. DV01 can be calculated by (1) assuming a parallel shift in the interest rate term structure, or (2) using yields by taking the YTM and recalculate the bond prices by increasing and decreasing the YTM by the same basis point change, and averaging the two prices.

Analysts can also distinguish between yield-based DV01, and changes in bond prices from changes in spot rates (DVDZ or DPDZ) or forward rates (DVDF or DPDF).

LO 58.c

The goal of a hedge is to produce a combined investment + hedge position that will not change in value for a small change in yield. The HR when hedging a bond with another bond is calculated as:

$$HR = \frac{\text{DV01 (initial position)}}{\text{DV01 (hedging instrument)}}$$

The HR is then multiplied by the face value of the initial position to arrive at the face value of the hedging instrument.

LO 58.d

Effective duration measures the percentage change in a security's value for a particular unit's change in rates. It can be interpreted as the percentage change in a bond's price for a 100 basis point change in rates. The common duration equation is:

$$\Delta P = -D \times P \times \Delta y$$

For callable bonds, the bond's duration can be calculated as if the bond would not be called in the future. An alternative approach is to calculate duration as the weighted probability that the bond would not be called and the probability that it would be called. A third, more precise, approach calculates duration valuing the bond's price today, revaluing the price for an assumed 1 basis point parallel change in interest rates, and calculating duration from the percentage change in the bond's price. The effective duration of a putable bond is calculated similar to callable bonds.

LO 58.e

DV01 works better for hedgers, while duration is more convenient for traditional investors.

LO 58.f

Convexity is the sensitivity of duration to changes in interest rates. Convexity is positive and will always increase the price calculated by duration only. Convexity can be calculated as:

$$C = \frac{1}{P}\left[\frac{P^{+} + P^{-} - 2P}{(\Delta y)^2}\right] = \left[\frac{P^{+} + P^{-} - 2P}{P(\Delta y)^2}\right]$$

Combining both duration and convexity gives a more accurate estimate of bond price changes when rates change. The combined effect can be calculated as:

$$\Delta P = -D \times P \times \Delta y + \frac{1}{2} \times C \times P \times \Delta y^2$$

LO 58.g

The duration of a portfolio of individual securities equals the weighted sum of the individual durations:

$$\text{duration of portfolio} = \sum_{j=1}^{K} w_j \times D_j$$

where:
D_j = duration of bond *j*
w_j = market value of bond *j* divided by market value of portfolio
K = number of bonds in portfolio

Like portfolio duration, convexity for the entire portfolio is simply the value-weighted average of each individual security's convexity within the portfolio.

LO 58.h

Hedging an investment with a value V and a duration D_V can be done with a bond with value P and duration of D_P:

$$\Delta V = -D_V \times V \times \Delta y$$
$$\Delta P = -D_P \times P \times \Delta y$$

For the investment to be fully hedged, it must be that ΔV = ΔP. We can then rearrange the equation to state:

$$P = -\frac{V \times D_V}{D_P}$$

An investor could also hedge an investment using both duration and convexity. This could be done by using two bonds with prices P_1 and P_2, durations D_1 and D_2, and convexities C_1 and C_2:

$$-V \times D_V - P_1 \times D_1 - P_2 \times D_2 = 0$$
$$+V \times C_V + P_1 \times C_1 + P_2 \times C_2 = 0$$

LO 58.i

A barbell strategy is typically used when an investment manager uses bonds with short and long maturities, thus forgoing any intermediate-term bonds. A bullet strategy is used when an investment manager buys bonds concentrated in the intermediate maturity range.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 58.1

1. **D** For a 7.01% bond, N = 20; I/Y = 7.01 / 2 = 3.505%; PMT = 2.5; FV = 100; CPT → PV = −85.723

   For a 6.99% bond, N = 20; I/Y = 6.99 / 2 = 3.495%; PMT = 2.5; FV = 100; CPT → PV = −85.852

   DV01 = |85.852 − 85.723| / 2 = 0.0645

   (LO 58.b)

2. **A** The HR is 0.065 / 0.085 = 0.765. Since the investor has a short position in the bond, this means the investor needs to buy $0.765 of par value of the hedging instrument for every $1 of par value for the 10-year bond. (LO 58.c)

Module Quiz 58.2

1. **C** Duration measures assume a parallel shift in the yield curve. Duration is not a good measure of nonparallel shifts. (LO 58.g)

2. **A** $\Delta P = [-7 \times 0.0025 \times P] + \left[\frac{1}{2} \times 243 \times (0.0025)^2\right] \times P$

   $= -1.67\% \times P$, or a decrease of 1.67%

   (LO 58.f)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 13.

READING 59

MODELING NON-PARALLEL TERM STRUCTURE SHIFTS AND HEDGING

Study Session 14

EXAM FOCUS

This reading divides the term structure of interest rates into several regions, and makes assumptions regarding how rates change for each region. Key rate analysis measures a portfolio's exposure to changes in a few key rates—for instance, 2-year, 5-year, 10-year, and 30-year rates. The key rate method is straightforward and assumes that rates change in the region of the key rate chosen. The forward-bucket method is similar to the key rate approach, but instead uses information from a greater array of rates, specifically those built into the forward rate curve. For the exam, understand how to apply key rate shift analysis and be familiar with key rate '01, key rate duration, and the calculations associated with hedging positions given a specific key rate exposure profile.

MODULE 59.1: FACTORS, PRINCIPAL COMPONENTS, AND KEY RATES

A single-factor approach to measuring and hedging risk in fixed-income markets is quite limiting because it assumes that within the term structure of interest rates (typically referred to as the yield curve), all rate changes are driven by a single factor (i.e., the term structure shifts in a parallel fashion).

The simplifying assumption that rates of all terms move up or down by the same amount based on one factor, such as the 10-year par rate (i.e., the 10-year swap rate), is restrictive; thus, practitioners apply multifactor approaches allowing for nonparallel shifts. These multifactor approaches, such as key rate and bucket approaches, assume that rate changes are a function of two or more factors.

Principal Components Analysis

LO 59.a: Describe principal components analysis and identify the factors that are the most important drivers of term structure movements.

Principal components analysis (PCA) is a technique used to analyze term structure movements in historical data. Based on daily movements in rates across maturities, the technique identifies uncorrelated factors which, combined linearly, are used to create the daily term structure of rate movements. Based on this analysis, multiple factors impact the movements in the term structure (with the first two to three often having the biggest impact).

Figure 59.1 reflects a hypothetical subset of factors applicable to Treasury rates for a five-year period. The interpretation of each factor loading is that the interest rate moves by the amount shown in the table for each single-factor unit. The factors are ordered such that Factor 1 is the most important, Factor 2 is next, and so on.

Figure 59.1: Factor Loadings for Treasury Rates

	Factor			
Rate Maturity	**1**	**2**	**3**	**4**
1-year	–0.165	0.326	0.698	–0.465
2-year	–0.292	0.415	0.124	0.525
3-year	–0.350	0.472	0.103	0.219
5-year	–0.434	0.169	–0.324	–0.185

As an example of how to interpret the table, when there is +1 unit of Factor 3, the 2-year rate changes by 0.124 basis points and the 3-year rate changes by 0.103 basis points. Positive units of Factor 1 drive all associated rates lower, while negative units of Factor 1 would drive all associated rates higher (i.e., a term structure shift where all rates move in one direction). The change on any given day is a by-product of the linear combination of the factors.

Factor scores are variable values relating to a specific data point covering daily changes, with their standard deviations aligned with the relative importance of each factor. The variance of all of the factor scores, when summed, equal the total variance of all rate movements. Figure 59.2 provides standard deviations of the factor scores for the four factors shown in Figure 59.1.

Figure 59.2: Standard Deviations of the Factor Scores

Factors			
1	2	3	4
12.96	5.82	2.14	1.79

The total variance is equal to the following: $(12.96)^2 + (5.82)^2 + (2.14)^2 + (1.79)^2 =$ 209.62. Each factor's relative importance can be ascertained by comparing its variance to the total variance. For instance, Factor 1's variance of 167.96 (which is 12.96

squared) represents 80.13% of the total. The factors themselves may reflect situations where all rates move in the same direction (but the amounts are different), where short-term and long-term rates move in different directions, and situations where intermediate rates move in the opposite direction to short- and long-term rates.

Key Rate Shift Analysis

LO 59.b: Describe key rate shift analysis and define key rate 01 (KR01).

Key rate exposures help describe how the risk of a bond portfolio is distributed along the term structure, and they assist in setting up a proper hedge for a bond portfolio. Key rate exposures are used for measuring and hedging risk in bond portfolios using rates from the most liquid bonds available, which are generally government bonds that have been issued recently and are selling at or near par.

Similar to key rate exposures, **partial '01s** are used for measuring and hedging risk in swap portfolios (or a portfolio with a combination of bonds and swaps). These partial '01s are derived from the most liquid money market and swap instruments for which a swap curve is usually constructed. **Forward-bucket '01s** are also used in swap and combination bond/swap contexts, but instead of measuring risk based on other securities, they measure risk based on changes in the shape of the yield curve. Thus, forward-bucket '01s enable us to understand a portfolio's yield curve risk. Partial '01s and forward-bucket '01s are similar to key rate approaches but use more rates, which divide the term structure into many more regions.

Key rate shift analysis makes the simplifying assumption that all rates can be determined as a function of a few key rates. To cover risk across the entire term structure, a small number of key rates are used, pertaining only to the most liquid government securities.

The most common key rates used for the U.S. Treasury and related markets are par yield bonds—2-, 5-,10-, and 30-year par yields. If one of these key rates shifts by one basis point, it is called a **key rate shift**. Note that par yields are also referred to as par rates. The key rate shift technique is an approach to nonparallel shifts in the yield curve, which allows for changes in all rates to be determined by changes from selected key rates.

LO 59.c: Calculate the KR01s of a portfolio given a set of key rates.

A **key rate '01** is the effect of a dollar change of a one basis point shift around each key rate on the value of the security. They are expressed as $KR01_1$, $KR01_2$, ..., $KR01_N$, with each key rate representing the reduction in portfolio value for a one basis point increase in that particular spot rate. Assume DV01 is equal to the impact of a one basis point shift in all spot rates on the value of a portfolio. DV01 is equal to the sum of all of the individual KR01s. So, while DV01 can be used to hedge against parallel shifts in the interest rate term structure, using a similar process to hedge against KR01s, the hedge covers a wider range of structure movements.

Key rate duration is a calculation that measures the sensitivity of a portfolio's value to a 100 basis point change in yield for a specific maturity. In effect, it is the percentage change in the value of the portfolio resulting from this interest rate change.

The following example demonstrates the calculation of key rate '01 and key rate duration, using a 30-year zero-coupon bond. Zero-coupon securities are also referred to as STRIPS (separate trading of registered interest and principal securities). Investors of zero-coupon bonds receive payment from STRIPS at maturity.

Figure 59.3: Key Rate '01s and Durations of a C-STRIP

	(1) Value	(2) Key Rate '01	(3) Key Rate Duration
Initial value	25.11584		
2-year shift	25.11681		
5-year shift	25.11984	–0.0040	–1.59
10-year shift	25.13984		
30-year shift	25.01254		

Column (1) in Figure 59.3 provides the initial price of a C-STRIP and its present value after application of key rate one basis point shifts. Column (2) in Figure 59.3 shows the key rate '01s. For example, the key rate '01 with respect to the five-year shift is calculated as follows:

$$-\frac{1}{10{,}000}\frac{25.11984 - 25.11584}{0.01\%} = -0.0040$$

This implies that the C-STRIP increases in price by 0.0040 per $100 face value for a positive one basis point five-year shift. Like DV01, the key rate '01 is negative when value, after a given shift, increases relative to the initial value. The key rate '01 would be positive if value, after a given shift, declines relative to the initial value.

Continuing with the same five-year shift, its key rate duration is calculated as follows:

$$-\frac{1}{25.11584}\frac{25.11984 - 25.11584}{0.01\%} = -1.59$$

Completing Column (3) in Figure 59.3 and summing all key rate durations would give us the effective duration of the 30-year C-STRIP. Note that key rate duration can also be computed using the corresponding key rate '01 and initial value as follows:

$$\frac{-0.0040}{25.115840} \times 10{,}000 = -1.59$$

MODULE QUIZ 59.1

1. Assume the standard deviations for factor scores are 10.25, 7.16, 4.12, and 3.08. How much of an impact do the first two factors together have relative to the total variance?
   A. 41.65%.
   B. 57.48%.
   C. 70.74%.
   D. 85.52%.
2. Which of the following statements regarding partial '01s is most accurate?

A. They reflect a 100 basis point change in rates.
B. They are derived from highly liquid instruments.
C. They cannot be used to hedge risk in swap portfolios.
D. They differ significantly from key rate exposures in terms of functionality.

3. Which of the following maturities is least likely associated with a key rate for U.S. Treasuries?
   A. 2-year.
   B. 10-year.
   C. 15-year.
   D. 30-year.

4. Key rate duration is most accurately described as the:
   A. dollar change in portfolio value associated with a 1 basis point change in yield.
   B. dollar change in portfolio value associated with a 100 basis point change in yield.
   C. percentage change in portfolio value associated with a 1 basis point change in yield.
   D. percentage change in portfolio value associated with a 100 basis point change in yield.

MODULE 59.2: KEY RATE EXPOSURES AND HEDGING

LO 59.d: Calculate the positions in hedging instruments necessary to hedge the key rate risks of a portfolio.

Assume DV01 is equal to the impact of a one basis point shift in all spot rates on the value of a portfolio. Further assume three spot rates: the 1-year, 3-year, and 10-year rates. The impact of shifts in these three spot rates are key rates '01s (KR01s) or partial '01s. $KR01_1$ will represent a decrease in portfolio value from a one basis point increase in the 1-year spot rate, $KR01_2$ will represent the same for the 3-year spot rate, and $KR01_3$ will represent the same for the 10-year spot rate. DV01 will therefore equal the sum of $KR01_1$, $KR01_2$, and $KR01_3$. So, while DV01 can be used to hedge against parallel shifts in the interest rate term structure, using a similar process to hedge against KR01s, the hedge covers a wider range of structure movements.

Key rate shifts allow for better hedging of a bond position, and when summed across all key rates, they assume a parallel shift across all maturities in the maturity spectrum. Although key rate exposure analysis is a useful tool for measuring bond price sensitivity, it makes very strong assumptions about how the term structure behaves. It assumes that the rate of a given term is affected only by the key rates that surround it, when in reality, shifts are not always perfectly linear.

Assume an investment portfolio contains equal investments in four zero-coupon bonds maturing in 1, 5, 10, and 15 years. Figure 59.4 shows the portfolio value decrease associated with one basis point increases in the relevant spot rates, while Figure 59.5 shows the shift in the key rates associated with each spot rate maturity.

Figure 59.4: Portfolio Value Decreases for Spot Rate Increases

Spot Rate Maturity (Years)	1	5	10	15
Decrease in portfolio value	48.75	302.65	595.10	856.45

Figure 59.5: Key Rate Shifts

Shift	Spot Rate Maturity (Years)			
	1	5	10	15
1-year spot	1	0.50	0	0
3-year spot	0	0.50	0.40	0.15
10-year spot	0	0	0.60	0.85

For the portfolio just listed, key rates are calculated in Figure 59.6.

Figure 59.6: Key Rate Calculations

Partial '01s	Calculations
$KR01_1$	$(48.75 \times 1) + (302.65 \times 0.50) = 200.08$
$KR01_2$	$(302.65 \times 0.50) + (595.10 \times 0.40) + (856.45 \times 0.15) = 517.83$
$KR01_3$	$(595.10 \times 0.60) + (856.45 \times 0.85) = 1{,}085.04$

To hedge a portfolio using KR01s, assume Figure 59.7 presents KR01s for a portfolio along with three available hedging instruments.

Figure 59.7: KR01s for Hedging Instruments

Key Rates	Portfolio	Hedge 1	Hedge 2	Hedge 3
$KR01_1$	68	14	2	3
$KR01_2$	106	3	12	4
$KR01_3$	169	7	1	15

Hedging involves setting KR01s equal to zero. Based on Figure 59.7, algebraic equations with three unknown variables are set up and solved as follows:

$$68 + 14x_1 + 2x_2 + 3x_3 = 0$$
$$106 + 3x_1 + 12x_2 + 4x_3 = 0$$
$$169 + 7x_1 + x_2 + 15x_3 = 0$$

Solving for the three variables, short positions of 2, 5, and 10 for x_1, x_2, and x_3, respectively, will hedge the portfolio.

MODULE QUIZ 59.2

1. A key rate ($KR01_1$) for a 2-year spot rate will represent the increase in portfolio value from a:
   A. one basis point increase in the 2-year spot rate.
   B. one basis point decrease in the 2-year spot rate.
   C. one basis point increase in the 1- and 2-year spot rates.
   D. one basis point decrease in the 1- and 2-year spot rates.

2. Assume the following KR01s along with two hedging instruments.

Key Rates	Portfolio	Hedge 1	Hedge 2
$KR01_1$	41	3	2
$KR01_2$	48	–4	1

To properly hedge this portfolio, which of the following positions should an investor take?

A. A short position of 1 and long position of 3 for x_1 and x_2.

B. A long position of 6 and short position of 4 for x_1 and x_2.

C. A long position of 5 and short position of 28 for x_1 and x_2.

D. A long position of 46 and long position of 45 for x_1 and x_2.

MODULE 59.3: FORWARD BUCKETS AND VOLATILITY

LO 59.f: Describe an interest rate bucketing approach, define forward bucket 01, and compare forward bucket 01s to KR01s.

Instead of using individual key rates, buckets can be used as segments of interest rates covering the term structure. The dollar impact on the portfolio value is then derived from changing every spot rate in each bucket by one basis point. If B_1 represents a decrease in portfolio value from a one-basis-point increase in spot rates associated with the first bucket, and B_2 and B_3 are defined similarly, DV01 will therefore equal the sum of B_1, B_2, and B_3.

Forward rates can also be used in buckets. Assume three buckets are covering a 30-year term, and are divided as follows: 0–3 years, 3–15 years, and 15–30 years. If forward rates are calculated in six-month periods, each six-month rate will be increased by one basis point. A **forward-bucket '01** represents the decrease in portfolio value for a one basis point increase in all forward rates within a bucket. DV01 is then calculated by summing the forward-bucket '01s. Spot rates can be derived from forward rates using the following formula if N is the specific year's spot rate (R):

$$(1+0.5R)^{2N} = (1+0.5f_0)(1+0.5f_{0.5})(1+0.5f_1)\ldots(1+0.5f_{N-0.5})$$

The impact of a one basis point change in forward rates on spot rates and the resulting portfolio value can then be calculated.

As an example, a portfolio contains a 3-year bond (face value of $100) and a coupon rate of 4.5% per year. Compounding is semiannual, and the term structure is a flat 3.5%. The value of the bond is equal to:

$$\frac{2.25}{1.0175}+\frac{2.25}{1.0175^2}+\frac{2.25}{1.0175^3}+\frac{2.25}{1.0175^4}+\frac{2.25}{1.0175^5}+\frac{102.25}{1.0175^6} = 102.8245$$

Assuming three equal buckets covering 0–1 year, 1–2 years, and 2–3 years, the value of the bond when each bucket increases by one basis point is shown as follows:

- Bucket (0–1 year):

$$\frac{2.25}{1.01755}+\frac{2.25}{1.01755^2}+\frac{2.25}{1.01755^2\times 1.0175}+\frac{2.25}{1.01755^2\times 1.0175^2}$$
$$+\frac{2.25}{1.01755^2\times 1.0175^3}+\frac{102.25}{1.01755^2\times 1.0175^4}=102.8145$$

Forward-bucket '01 = 102.8245 – 102.8145 = 0.0100

- Bucket (1–2 years):

$$\frac{2.25}{1.0175}+\frac{2.25}{1.0175^2}+\frac{2.25}{1.0175^2\times 1.01755}+\frac{2.25}{1.0175^2\times 1.01755^2}$$
$$+\frac{2.25}{1.0175^2\times 1.01755^2\times 1.0175}+\frac{102.25}{1.0175^2\times 1.01755^2\times 1.0175^2}=102.8148$$

Forward-bucket '01 = 102.8245 – 102.8148 = 0.0097

- Bucket (2–3 years):

$$\frac{2.25}{1.0175}+\frac{2.25}{1.0175^2}+\frac{2.25}{1.0175^3}+\frac{2.25}{1.0175^4}+\frac{2.25}{1.0175^4\times 1.01755}$$
$$+\frac{102.25}{1.0175^4\times 1.01755^2}=102.8153$$

Forward-bucket '01 = 102.8245 – 102.8153 = 0.0092

Figure 59.8 summarizes the results just listed.

Figure 59.8: Forward-Bucket '01s (3-Year Bond)

Forward Bucket	Bond Price After Rate Shift	Forward-Bucket '01
0–1 year	102.8145	0.0100
1–2 years	102.8148	0.0097
2–3 years	102.8153	0.0092
Total		**0.0289**

Now, assume all forward rates are increased by one basis point:

$$\frac{2.25}{1.01755}+\frac{2.25}{1.01755^2}+\frac{2.25}{1.01755^3}+\frac{2.25}{1.01755^4}+\frac{2.25}{1.01755^5}+\frac{102.25}{1.01755^6}=102.7957$$

The price difference between 102.8245 and 102.7957 with rounding is approximately 0.0289.

LO 59.e: Apply key rate analysis and principal components analysis to estimating portfolio volatility.

Banks must calculate the impact of one basis-point shifts in 10 KR01s, including spot rates at three and six months, along with rates at 1, 2, 3, 5, 10, 15, 20, and 30 years. Banks must also calculate expected shortfall and value at risk (VaR) measures using

KR01 exposures as they relate to the standard deviations and correlations between the 10 rates noted here.

In Figure 59.2, factor score standard deviations were shown. If the standard deviation of the factor score of the *i*th factor is σ_i and f_i is the change in portfolio value, then given the significant relative impact of the first three factors, the standard deviation of the first three factors is calculated using the following formula:

$$\sigma_P = \sqrt{\sigma_1^2 f_1^2 + \sigma_2^2 f_2^2 + \sigma_3^2 f_3^2}$$

This equation can be used to determine the standard deviation of daily changes in portfolio value once the portfolio value changes are known. Assume for the first three factors that the portfolio value changes are +10, −20, and −30. Using the standard deviations from Figure 59.2, the standard deviation of daily changes is calculated as:

$$\sqrt{12.96^2 \times 10^2 + 5.82^2 \times -20^2 + 2.14^2 \times -30^2} = 185.65$$

Although spot rates have been used to define key rate shifts up to this point, another option is to use par yields. Because market prices for actively traded instruments are typically used to derive the term structure of spot rates, par yields ultimately reflect spot rates.

LO 59.g: Calculate the corresponding duration measure given a KR01 or forward bucket 01.

Similar to how DV01 can be segmented into spot rates, par rates, or forward rates, the same concept can be applied to duration calculations. In place of DV01, an ‘01 measure can be substituted such that the duration calculation is expressed as:

$$\text{duration} = \frac{10{,}000 \times (\text{‘01 measure})}{\text{portfolio value}}$$

Using forward-bucket ‘01s derived and shown in Figure 59.8 and the initial portfolio value of 102.8245, forward-bucket durations calculated with the formula just listed are shown in Figure 59.9.

Figure 59.9: Forward-Bucket Durations

Forward Bucket	Forward-Bucket ‘01	Forward-Bucket Durations
0–1 year	0.0100	0.9725
1–2 years	0.0097	0.9434
2–3 years	0.0092	0.8947
Total	**0.0289**	**2.8106**

MODULE QUIZ 59.3

1. The value of a 3-year bond is 103.960. If forward rates are increased by one basis point, the value falls to 103.925. Which combination of three forward-bucket ‘01s is feasible for this bond?
   A. 0.010, 0.009, 0.008.
   B. 0.014, 0.012, 0.009.

C. 0.017, 0.017, 0.017.
D. 0.020, 0.011, 0.006.

2. With an initial portfolio value of 100.565 and face value of 100, the forward-bucket duration for a forward-bucket ‘01 of 0.0095 is closest to:
   A. 0.9447.
   B. 0.9500.
   C. 94.47.
   D. 95.00.

KEY CONCEPTS

LO 59.a

The single-factor approach to hedging is limiting because it assumes that all future rate changes are driven by a single factor and that interest rates move in a parallel fashion.

Principal components analysis is a technique used to analyze term structure movements in historical data. The technique identifies multiple factors that impact the movements in the term structure, with the first two to three typically having the biggest impact.

Factor scores are variable values relating to a specific data point, with their standard deviations aligned with the relative importance of each factor. The sum of the variance of all of the factor scores equals the total variance of all rate movements.

The factors themselves may reflect situations where all rates move in the same direction (similar to a parallel shift), situations where short-term and long-term rates move in different directions, and situations where intermediate rates move in the opposite direction to short- and long-term rates.

LO 59.b

Key rate exposures help describe how the risk of a bond portfolio is distributed along the term structure, and they assist in setting up a proper hedge for a bond portfolio. Partial ‘01s are used for measuring and hedging risk in swap portfolios (or a portfolio with a combination of bonds and swaps). These partial ‘01s are derived from the most liquid money market and swap instruments for which a swap curve is usually constructed. Forward-bucket ‘01s are also used in swap and combination bond/swap contexts and measure risk based on changes in the shape of the yield curve. Partial ‘01s and forward-bucket ‘01s are similar to key rate approaches but use more rates, which divide the term structure into many more regions.

Key rate shift analysis makes the simplifying assumption that all rates can be determined as a function of a few key rates. If a key rate shifts by one basis point, it is called a key rate shift. The key rate shift technique is an approach to nonparallel shifts in the yield curve, which allows for changes in all rates to be determined by changes from selected key rates.

LO 59.c

A key rate ‘01 is the effect of a dollar change of a one basis point shift around each key rate on the value of the security. Key rate duration is a percentage change calculation that measures the sensitivity of a portfolio's value to a 100 basis point change in yield for a specific maturity.

Key rate '01s are calculated as follows:

$$DV01^k = \frac{1}{10{,}000}\frac{\Delta BV}{\Delta y^k}$$

Key rate durations are calculated as follows:

$$D^k = \frac{1}{BV}\frac{\Delta BV}{\Delta y^k}$$

LO 59.d

DV01, which is equal to the impact of a one basis point shift in all spot rates on the value of a portfolio, can be divided into several key rate exposures. DV01 can be used to hedge against parallel shifts in the interest rate term structure, while key rate shifts allow for better hedging of a bond position given nonparallel shifts in the curve. Key rate exposure analysis is very useful for measuring bond price sensitivity, but it assumes that the rate of a given term is affected only by the key rates that surround it.

Key rate exposures can be calculated using changes in portfolio values for given spot rate maturities, assessing the shifts of the various spot rates, and calculating partial '01s. Hedging involves setting KR01s to zero and algebraically solving for the combination of hedging investments that will produce perfect hedges.

LO 59.e

Banks must calculate the impact of one basis point shifts in 10 KR01s, including spot rates at three and six months, along with rates at 1, 2, 3, 5, 10, 15, 20, and 30 years. Banks must also calculate expected shortfall and VaR measures using KR01 exposures as they relate to the standard deviations and correlations between the rates.

Key rate shifts can be defined using spot or par rates. Because market prices for actively traded instruments are typically used to derive the term structure of spot rates, par yields ultimately reflect spot rates.

LO 59.f

Instead of using individual key rates, buckets can be used as segments of interest rates covering the term structure. The dollar impact on the portfolio value is then derived from changing every spot rate in each bucket by one basis point. Forward rates can also be used in buckets. A forward-bucket '01 represents the decrease in portfolio value for a one basis point increase in all forward rates within a bucket.

LO 59.g

Similar to how DV01 can be segmented into spot rates, par rates, or forward rates, the same concept can be applied to duration calculations. In place of DV01, an '01 measure can be substituted such that the duration calculation is expressed as:

$$\text{duration} = \frac{10,000 \times (\text{‘01 measure})}{\text{portfolio value}}$$

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 59.1

1. **D** The total variance is equal to the following: $(10.25)^2 + (7.16)^2 + (4.12)^2 + (3.08)^2 =$ 182.79. The first two variances are 105.06 and 51.27 respectively, totaling 156.33 (which is 85.52% of the total). (LO 59.a)

2. **B** Partial ‘01s are derived from highly liquid instruments. They reflect a one basis point change in rates, they are very often used to hedge swap portfolio risk, and they are very similar to key rate exposures in terms of functionality. (LO 59.b)

3. **C** Key rates for U.S. Treasuries are typically 2-, 5-, 10-, and 30-year maturities. The 15-year maturity is not usually a key rate. (LO 59.b)

4. **D** Key rate duration is a measure of percentage change, not dollar change. The relevant change in yield is 100 basis points, not 1 basis point. (LO 59.c)

Module Quiz 59.2

1. **B** Interest rates and portfolio values move in opposite directions. For a 2-year spot rate, a key rate $KR01_1$ will represent the increase in portfolio value from a one basis point decrease in the 2-year spot rate. (LO 59.d)

2. **C** The correct algebraic equations to set up based on the table are:

   $41 + 3x_1 + 2x_2 = 0$

   $48 - 4x_1 + x_2 = 0$

   Solving for the variables, we have 5 for x_1 and −28 for x_2. This represents a long position of 5 and a short position of 28. (LO 59.d)

Module Quiz 59.3

1. **B** The difference in price is equal to 0.035 (103.960 − 103.925). The combination of three forward-bucket ‘01s must equal 0.035. The only choice that sums to 0.035 is 0.014, 0.012, and 0.009. (LO 59.f)

2. **A** The formula to calculate the duration is equal to:

   $$\text{duration} = \frac{10,000 \times (0.0095)}{100.565} = 0.9447$$

   (LO 59.g)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 14.

READING 60

BINOMIAL TREES

Study Session 15

EXAM FOCUS

This reading introduces the binomial model for valuing options on stock and serves as an introduction to the Black-Scholes-Merton model you'll encounter in the next reading. For the exam, you should be able to calculate the value of a European or American option using a one-step or two-step binomial model. This will require you to know the formulas for the sizes of upward and downward movements, as well as the risk-neutral probabilities in both up and down states. Also, be familiar with the concept of delta and how it is applied to hedging. Finally, understand how the binomial model can be adjusted to expand beyond just modeling for individual, non-dividend-paying stocks.

MODULE 60.1: ONE-STEP BINOMIAL MODEL

LO 60.a: Calculate the value of an American and a European call or put option using a one-step and two-step binomial model.

LO 60.b: Describe how volatility is captured in the binomial model.

A **one-step binomial model** is best described within a two-state world where the price of a stock will either go up once or down once and the change will occur one step ahead at the end of the holding period.

To see how this works, let's first define some terms. Then, we'll work through a calculation:

- P = stock's current price
- X = call option's exercise price
- t = time to option expiration
- i = risk-free interest rate
- S_U = stock value in "up" state
- S_D = stock value in "down" state
- c = value of the call option today

EXAMPLE: One-step binomial model

Calculate the value of a call option, assuming the following values:

P = \$100

X = \$125

t = 1 year

i = 8% (continuously compounded)

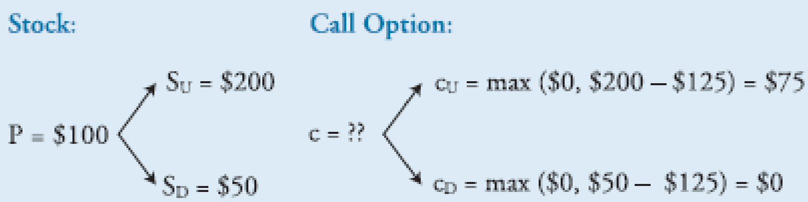
S_U = \$200

S_D = \$50

Answer:

First, set up the binomial trees as follows:

Figure 60.1: One-Period Binomial Trees



Next, assume a portfolio that consists of a long position in Δ shares and a short position in 1 call option. If the stock price goes up to 200, the portfolio value is equal to 200Δ – 75. If the stock price goes down to 50, the portfolio value is equal to 50Δ. Setting both values equal to each other, Δ is equal to 0.5 and the portfolio value in one year is certain to equal 25. A value of 25 one year from today, with rates continuously compounded at 8%, is equal to 23.08 today ($25e^{-0.08\times1}$).

If the value of the call is *c*, and the value of the long position today is 50 (= 100 × 0.5), the value of the portfolio today is equal to 50 – c. Setting 50 – c = 23.08, the value of the call is equal to 26.92.

Risk-Neutral Valuation

The **law of one price** dictates that if two investments provide the same cash flows at the same times in the future, they both should sell for exactly the same price. This presents a no-arbitrage situation, which is illustrated for derivatives using binomial trees. Binomial trees also reflect **risk-neutral valuation**, a critical concept in derivatives pricing that values derivatives, assuming the expected return for market participants is equal to the risk-free rate.

The one-step binomial model can also be expressed in terms of probabilities and call prices. The sizes of the upward and downward movements are defined as functions of the volatility and the length of the steps in the binomial model:

$$U = \text{size of the up-move factor} = e^{\sigma\sqrt{t}}$$

$$D = \text{size of the down-move factor} = e^{-\sigma\sqrt{t}} = \frac{1}{e^{\sigma\sqrt{t}}} = \frac{1}{U}$$

where:

σ = annual volatility of the underlying asset's returns

t = the length of the step in the binomial model

PROFESSOR'S NOTE

On the exam, you may be provided with approximate sizes of the up and down moves (e.g., up 15% and down 15%). If that is not the case, you would use the previous formulas.

The risk-neutral probabilities of upward and downward movements are then calculated as:

$$\pi_u = \text{probability of an up move} = \frac{e^{rt} - D}{U - D}$$

$$\pi_d = \text{probability of a down move} = 1 - \pi_u$$

where:

r = continuously compounded annual risk-free rate

PROFESSOR'S NOTE

These two probabilities are not the actual probability of an up or down move. They are risk-neutral probabilities that would exist if investors were risk neutral.

We can calculate the value of an option on the stock by

- calculating the payoff of the option at maturity in both the up-move and down-move states,
- calculating the expected value of the option in one year as the probability-weighted average of the payoffs in each state, and
- discounting the expected value back to today at the risk-free rate.

EXAMPLE: Risk-neutral approach to option valuation

The current price of Downhill Ski Equipment, Inc., is $20. The annual standard deviation is 14%. The continuously compounded risk-free rate is 4% per year. Assume Downhill pays no dividends. **Compute** the value of a 1-year European call option with a strike price of $20 using a one-period binomial model.

Answer:

The up-move and down-move factors are:

$$U = e^{0.14 \times \sqrt{1}} = 1.15$$

$$D = \frac{1}{1.15} = 0.87$$

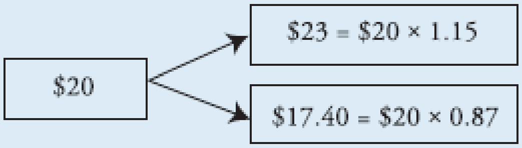
The risk-neutral probabilities of an up move and down move are:

$$\pi_u = \frac{e^{0.04\times1} - D}{U - D} = \frac{1.0408 - 0.87}{1.15 - 0.87} = 0.61$$

$$\pi_d = 1 - 0.61 = 0.39$$

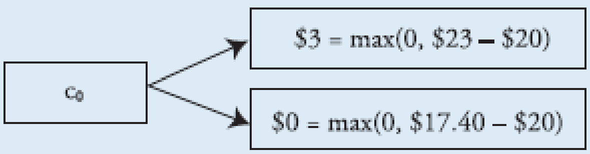
The binomial tree for the stock is shown in Figure 60.2:

Figure 60.2: Binomial Tree—Stock



The binomial tree for the option is shown in Figure 60.3:

Figure 60.3: Binomial Tree—Option



Notice that the call option is in the money in the up state, so its value is $3. It is out of the money in the down state, so its value is zero.

The expected value of the option in one year is:

$$c_U \times \pi_U + c_D \times \pi_D \text{ or } (\$3 \times 0.61) + (\$0 \times 0.39) = \$1.83$$

The present value of the option's expected value is:

$$c_0 = \frac{\$1.83}{e^{0.04\times1}} = \frac{\$1.83}{1.0408} = \$1.76$$

EXAMPLE: Put option valuation using put-call parity

The current price of Downhill Ski Equipment, Inc., is $20, the risk-free rate is 4% per year, and the price of a 1-year call option with a strike price of $20 is $1.76. **Compute** the value of a 1-year European put option on Downhill Ski Equipment with a strike price of $20.

Answer:

Put-call parity formula:

$$\text{put} = \text{call} - \text{stock} + Xe^{-rt}$$

$$\text{put} = \$1.76 - \$20 + \$20e^{-0.04(1)} = \$0.98$$

Notice from the previous examples that a high standard deviation will result in a large difference between the stock price in an up state, S_U, and the stock price in a down state, S_D. If the standard deviation were zero, the binomial tree would collapse into a straight line and S_U would equal S_D. Obviously, the higher the standard deviation, the greater the dispersion between stock prices in up and down states. Therefore, volatility (as measured here by standard deviation) can be captured by evaluating stock prices at each time period considered in the tree.

Using Delta to Develop a Replicating Portfolio

LO 60.d: Define and calculate delta of a stock option.

The value of the option can also be solved by creating a **perfect hedge**:

- **Hedging** is the elimination of price variation through the short sale of an asset exhibiting the same price volatility as the asset to be hedged. A perfect hedge creates a riskless position.
- The **hedge ratio** indicates the number of asset units needed to completely eliminate the price volatility of one call option. The hedge ratio is also known as option **delta**, which is a key measure of option sensitivity. Constructing a riskless position is sometimes known as **delta hedging**.

In the first example for this reading, we could have created a perfect hedge had we sold one share of stock short at $100 and purchased two call options. No matter which way the stock price moves, the hedged portfolio will be worth $50:

- If the stock price *falls* to $50, the two options will have zero value, so the net asset position is the gain on the short sale: $100 – $50 = $50.
- If the stock price *rises* to $200, the two calls will have a combined value of $150, leaving a net asset value of $50 after considering a $100 loss ($100 – $200) from the short position in the stock.

Since the terminal value of this strategy (short one share and long two calls) always nets $50, the present value of the strategy is $50e^{-0.08\times1}$ = $46.16. Therefore, the value of one call option must be $26.92 ($100 – 2c = $46.16, c = $26.92).

The delta of a stock option, Δ, tells us how many units of the stock to hold per call option to be shorted to make the hedge work (i.e., create a riskless position). In the single-period model, the delta is calculated as:

$$\Delta = \frac{c_U - c_D}{S_U - S_D} = \frac{\$75 - \$0}{\$200 - \$50} = 0.5$$

A delta of 0.5 says that one option contract is needed for each half-share of stock. The reciprocal of the delta is equivalent to the number of option contracts to buy per share of stock that was sold short. Note that delta does not remain constant and will change over time.

MODULE QUIZ 60.1

1. An investor is analyzing a 1-year European call option with an exercise price of $18. The stock value in the up state is $30, while the value in the down state is $10. The delta for this option is closest to:
   A. 0.40.
   B. 0.60.
   C. 0.67.

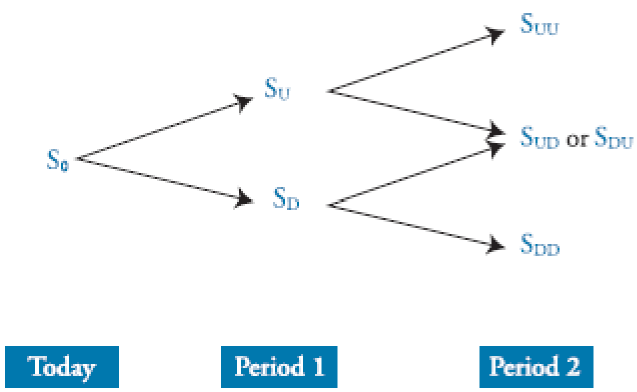
D. 0.90.

2. Suppose a 1-year European call option exists on XYZ stock. The current continuously compounded risk-free rate is 3% and XYZ does not pay a dividend. Assume an annual standard deviation of 8%. The risk-neutral probability of an up move for the XYZ call option is:
   A. 0.31.
   B. 0.69.
   C. 0.92.
   D. 1.08.

MODULE 60.2: TWO-STEP BINOMIAL MODEL AND MODEL MODIFICATIONS

In the two-period and multiperiod models, the *tree* is expanded to provide for a greater number of potential outcomes. The stock price tree for the two-period model is shown in Figure 60.4.

Figure 60.4: Two-Step Binomial Model Stock Price Tree



For multistep trees, the length of the tree step is defined as Δt. The value of the option is now expressed as:

$$c = e^{-r\Delta t}\left[\pi_u c_u + \pi_d c_d\right]$$

Additional relevant equations applicable to multistep trees include:

$$\pi_u = \frac{e^{r\Delta t} - D}{U - D}$$

$$U = e^{\sigma\sqrt{\Delta t}}$$

$$D = e^{-\sigma\sqrt{\Delta t}}$$

EXAMPLE: Option valuation with a two-step binomial model

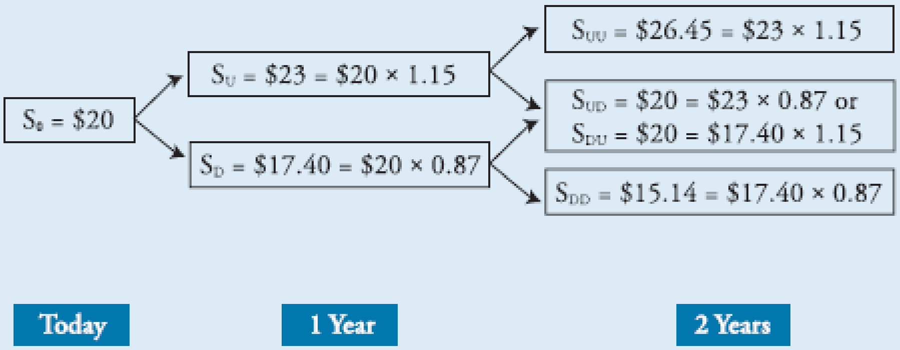
Let's continue with the Downhill Ski Equipment example. The current price of Downhill Ski Equipment, Inc., is $20. The annual standard deviation is 14%. The risk-free rate is 4% per year.

Assume Downhill pays no dividends. Using information from the previous example, **compute** the values of a 2-year European call and a 2-year European put option with strike prices of $20.

Answer:

First compute the theoretical value of the stock in each period using the up- and down-stock price movements from the preceding examples, as shown in Figure 60.5:

Figure 60.5: Theoretical Stock Value



From this information, the values of the call option in each of the possible outcomes can be determined. Notice that the only time the option is in the money is when two upward price movements lead to an ending price of \$26.45 and a call value of \$6.45. The expected value of the option at the end of Year 2 is the value of the option in each state multiplied by the probability of that state occurring:

$$\text{expected call value in two years} = (0.61 \times 0.61 \times \$6.45) + (0.61 \times 0.39 \times \$0) + (0.39 \times 0.61 \times \$0) + (0.39 \times 0.39 \times \$0) = (0.3721 \times \$6.45) = \$2.40$$

The value of the call option today is the expected value in two years discounted at the risk-free rate of 4%:

$$\text{call} = \frac{\$2.40}{e^{0.04(2)}} = \$2.21$$

The value of the put option, using put-call parity, is then:

$$\text{put} = \$2.21 - \$20 + \$20e^{-0.04(2)} = \$0.67$$

Modifying the Binomial Model

LO 60.e: Explain how the binomial model can be altered to price options on stocks with dividends, stock indices, currencies, and futures.

The binomial option pricing model can be altered to value a stock that pays a continuous **dividend yield**, *q*. Since the total return in a risk-neutral setting is the risk-free rate, *r*, and dividends provide a positive yield, capital gains must be equal to r – q. The risk-neutral probabilities of upward and downward movements incorporate a dividend yield as follows:

$$\pi_u = \frac{e^{(r-q)t} - D}{U - D}$$

$$\pi_d = 1 - \pi_u$$

The equations for the size of the up-move and down-move factors will be the same. Options on stock indices are valued in a similar fashion to stocks with dividends, because it is assumed that stocks underlying the index pay a dividend yield equal to q.

For options on currencies, it is assumed that a foreign currency asset provides a return equal to the foreign risk-free rate of interest, r_{FC}. As a result, the upward probability in the binomial model is altered by replacing e^{rt} with $e^{(rDC - rFC)t}$:

$$\pi_u = \frac{e^{(r_{DC} - r_{FC})t} - D}{U - D}$$

where:
r_{DC} = domestic currency risk-free rate of interest
r_{FC} = foreign currency risk-free rate of interest

The binomial model can also incorporate the unique characteristics of options on futures. Since futures contracts are costless to enter into, they are considered in a risk-neutral setting to be zero-growth instruments. To account for this characteristic, e^{rt} is simply replaced with a 1:

$$\pi_u = \frac{1 - D}{U - D}$$

American Options

Valuing American options with a binomial model requires the consideration of the ability of the holder to exercise early. In the case of a two-step model, that means determining whether early exercise is optimal at the end of the first period. If the payoff from early exercise (the intrinsic value of the option) is greater than the option's value (the present value of the expected payoff at the end of the second period), then it is optimal to exercise early.

EXAMPLE: American put option valuation

The current price of Uphill Mountaineering is \$10. The up-move factor is 1.20, and the down-move factor is 0.833. The probability of an up move is 0.51, and the probability of a down move is 0.49. The risk-free rate is 2%. **Compute** the value of a 2-year American put option with strike price of \$12.

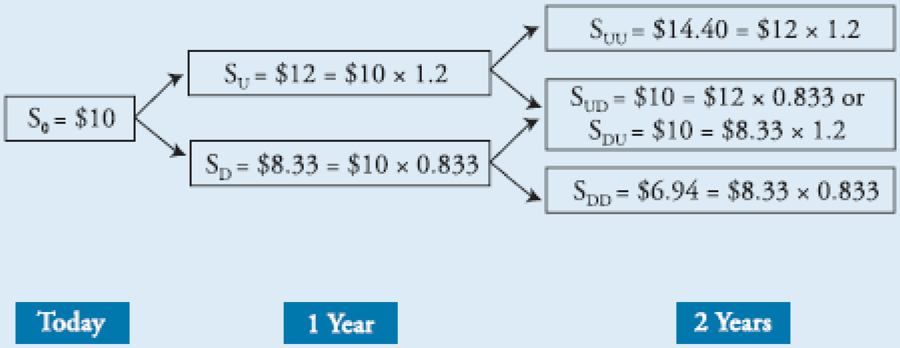
PROFESSOR'S NOTE

The calculation of the risk-neutral probabilities depends on the length of the time step. So, for a 2-year option with two time steps, the change in t is 1 year. For example, the probability of an up move in the information just listed is calculated as: $(e^{0.02 \times 1} - 0.833) / (1.2 - 0.833) = 0.51$.

Answer:

The stock price tree is shown in Figure 60.6:

Figure 60.6: Stock Price Tree



The \$12 put option is in the money when the stock price finishes at \$10 or at \$6.94; the option is worth \$2.00 (= \$12 – \$10) or \$5.06 (= \$12 – \$6.94). It is out of the money at \$14.40. The Year 1 value of the expected payoff on the option in Year 2, given that the Year 1 move is an up move, is:

$$\frac{(\$0.00 \times 0.51) + (\$2.00 \times 0.49)}{e^{(0.02)(1)}} = \$0.96$$

The payoff from early exercise at the Year 1 up node is:

max (\$12 – \$12, 0), since the option is at the money

Early exercise is not optimal in this case because the option is worth more unexercised (\$0.96) than if exercised (\$0).

At the down node at the end of Year 1, the value of the expected option payoff in Year 2 is:

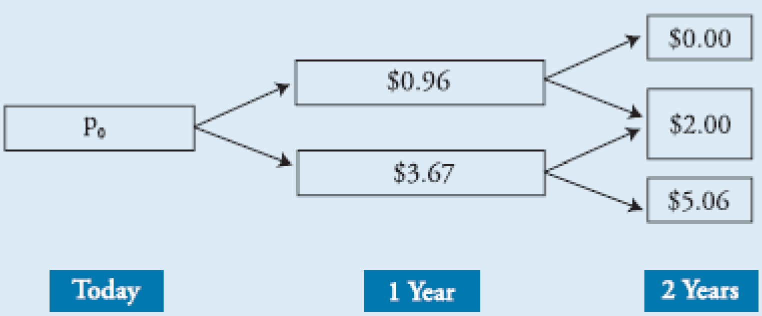
$$\frac{(\$2.00 \times 0.51) + (\$5.06 \times 0.49)}{e^{(0.02)(1)}} = \$3.43$$

The payoff from early exercise at the down node at the end of the first year is:

max (\$12 – \$8.33) = \$3.67

In this case, early exercise is optimal because the option is worth more if exercised (\$3.67) than if not exercised (\$3.43). The option tree is shown in Figure 60.7:

Figure 60.7: Option Tree



The value of the option today is calculated as:

$$\frac{(\$0.96 \times 0.51) + (\$3.67 \times 0.49)}{e^{(0.02)(1)}} = \$2.24$$

Note that \$3.67 appears in the bottom node of Year 1 since the early exercise value (\$3.67) exceeds the unexercised value (\$3.43).

PROFESSOR'S NOTE

When evaluating American options, you need to assess early exercise at each node in the tree. This includes the initial node (Node 0). If the option price today (calculated via the binomial model) is less than the value of early exercise today, then the option should be exercised early. In this example, if the value of the option today was worth less than \$2, the option would be exercised today since the put option is currently equal to \$2.

Increasing the Number of Time Periods

LO 60.c: Describe how the value calculated using a binomial model converges as time periods are added.

If we shorten the length of the intervals in a binomial model, there are more intervals over the same time period, more branches to consider, and more possible ending values. If we continue to shrink the length of intervals in the model until they are what mathematicians call *arbitrarily small*, we approach continuous time as the limiting case of the binomial model. The model for option valuation in the next reading (i.e., the Black-Scholes-Merton model) is a continuous-time model. The binomial model converges to this continuous-time model as we make the time periods arbitrarily small.

MODULE QUIZ 60.2

1. A 1-year American put option with an exercise price of \$50 will be worth either \$8 at maturity with a probability of 0.45 or \$0 with a probability of 0.55. The current stock price is \$45. The risk-free rate is 3%. The optimal strategy is to:
   A. exercise the option because the payoff from exercise exceeds the present value of the expected future payoff.
   B. not exercise the option because the payoff from exercise is less than the discounted present value of the future payoff.
   C. exercise the option because it is currently in the money.
   D. not exercise the option because it is currently out of the money.

2. Assume the stock price is currently \$80, the stock price annual up-move factor is 1.15, and the risk-free rate is 3.9%. The value of a 2-year European call option with an exercise price of \$62 using a two-step binomial model is closest to:
   A. \$0.00.
   B. \$18.00.
   C. \$23.07.
   D. \$24.92.

3. Assume the stock price is currently \$80, the stock price annual up-move factor is 1.15, and the risk-free rate is 3.9%. The value of a 2-year European put option with an exercise price of \$62 using a two-step binomial model is closest to:
   A. \$0.42.

B. $16.89.
C. $18.65.
D. $21.05.

4. The annual standard deviation for Baker stock is 11%. The continuously compounded risk-free rate is 3.5% per year, and Baker pays dividends at a yield of 2%. The risk-neutral probability of a downward move π_D is closest to:
   A. 0.366.
   B. 0.459.
   C. 0.541.
   D. 0.634.

5. Using a binomial model, the price of a call option is equal to $3.46. For the same option, the Black-Scholes-Merton model produces a price of $3.38. If the time intervals used in the binomial model are shortened, the expectation is that:
   A. $3.38 will get closer to $3.46.
   B. $3.46 will get closer to $3.38.
   C. the price for both models will be approximately $3.42.
   D. there will be no change in the gap between prices.

KEY CONCEPTS

LO 60.a

The value of a European option can be calculated using a binomial tree as the probability-weighted expected value of the option at maturity discounted at the risk-free rate. The value of an American option reflects early exercise features. An American option will be exercised at the end of the first period if the intrinsic value is greater than the discounted value of the expected option payoff at the end of the second period.

Given the volatility of the underlying stock and the length of the steps in the binomial tree, the size of the up- and down-move factors are calculated as:

$$U = \text{size of the up-move factor} = e^{\sigma\sqrt{t}}$$

$$D = \text{size of the down} - \text{move factor} = \frac{1}{U}$$

The risk-neutral probabilities of up and down moves are calculated as:

$$\pi_u = \text{probability of an up move} = \frac{e^{rt} - D}{U - D}$$

$$\pi_d = \text{probability of a down move} = 1 - \pi_u$$

The value of the comparable European put option can be calculated using put-call parity, which is written as: put = call – stock + Xe^{-rT}.

LO 60.b

The higher the standard deviation, the greater the dispersion between stock prices in up and down states. Volatility, as measured by standard deviation, can be captured by evaluating stock prices at each time period considered in the tree. The sizes of the upward and downward movements are defined as functions of the volatility and the length of the steps in the binomial model.

LO 60.c

As the period covered by a binomial model is divided into arbitrarily small, discrete time periods, the model results converge to those of the continuous-time model (the Black-Scholes-Merton model).

LO 60.d

The delta of a stock option, Δ, tells us how many units of the stock to hold per call option to be shorted to make a hedge work. Delta is computed as the ratio of the change in the stock option price to the change in the underlying stock price.

LO 60.e

The binomial option pricing model can be altered to value a stock that pays a continuous dividend yield, q.

$$\pi_u = \frac{e^{(r-q)t} - D}{U - D}$$

$$\pi_d = 1 - \pi_u$$

Options on stock indices are valued in a similar fashion to stocks with dividends.

For options on currencies, the upward probability in the binomial model is altered by replacing e^{rt} with $e^{(rDC - rFC)t}$ such that:

$$\pi_u = \frac{e^{(r_{DC} - r_{FC})t} - D}{U - D}$$

The binomial model can also incorporate the unique characteristics of options on futures by replacing the numerator of the upward probability equation with 1 – D, such that:

$$\pi_u = \frac{1 - D}{U - D}$$

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 60.1

1. **B** Delta provides the number of units of stock to hold per call option to be shorted to implement a hedge. If the stock price goes up to \$30, the call option with an exercise price of \$18 will be worth \$12. If the stock price goes down to \$0, the call option will be worth \$0.

$$\Delta = \frac{c_U - c_D}{S_U - S_D} = \frac{\$12 - \$0}{\$30 - \$10} = 0.60$$

(LO 60.d)

2. **B** First, calculate the size of the up- and down-move factors:

$$U = e^{\sigma\sqrt{t}} = e^{0.08\sqrt{1}} = 1.08$$

$$D = \frac{1}{U} = 0.92$$

The risk-neutral probability of an up move is then calculated as:

$$\pi_u = \frac{e^{rt} - D}{U - D} = \frac{e^{0.03} - 0.92}{1.08 - 0.92} = 0.69$$

(LO 60.b)

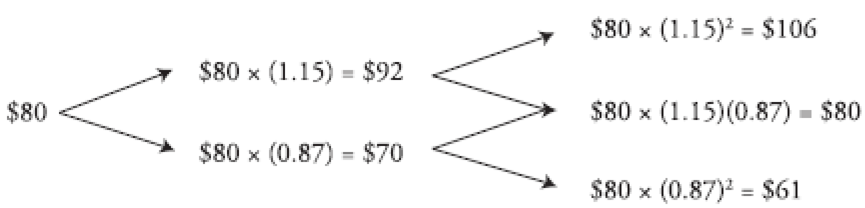
Module Quiz 60.2

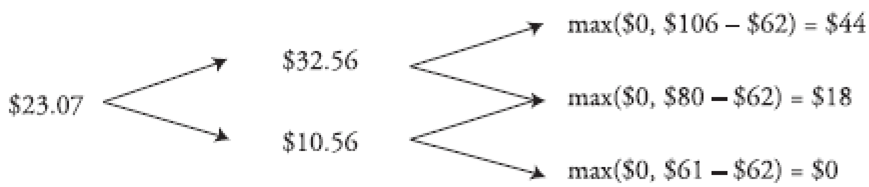
1. **A** The payoff from exercising the option is the exercise price minus the current stock price: \$50 – \$45 = \$5. The discounted value of the expected future payoff is:

$$\frac{(0 \times 0.55) + (8 \times 0.45)}{e^{(0.03)\times 1}} = \$3.49$$

It is optimal to exercise the option early because it is worth more exercised (\$5.00) than if not exercised (\$3.49). (LO 60.a)

2. **C**





$$U = 1.15$$

$$D = \frac{1}{1.15} = 0.8696$$

$$\pi_U = \frac{(e^{0.039}) - (0.87)}{1.15 - 0.87} = 0.61$$

$$\pi_D = 1 - 0.61 = 0.39$$

$$\pi_{UU} = 0.61^2 = 0.372$$

$$\pi_{UD} = \pi_{DU} = 0.61 \times 0.39 = 0.238$$

$$\pi_{DD} = 0.39^2 = 0.152$$

$$C_{UU} = \$44$$

$$C_{UD} = \$18$$

$$C_{DU} = \$18$$

$$C_{DD} = \$0$$

PROFESSOR'S NOTE

You may have a slightly different result due to rounding. Focus on the mechanics of the calculation.

$$c_t = \frac{(0.372 \times \$44) + (0.238 \times \$18) + (0.238 \times \$18) + (0.152 \times \$0)}{e^{(0.039) \times 2}}$$

$$= \$23.07$$

(LO 60.a)

3. **A** put = call – stock + (exercise price × e^{-rT})

$$= \$23.07 - \$80 + [\$62 \times e^{-(0.039)(2)}] = \$0.42$$

(LO 60.a)

4. **B** There are several steps needed for this calculation. The first is to calculate *U* (the size of the up-move factor), which is equal to $e^{0.11\sqrt{1}} = 1.116278$.

D (the size of the down-move factor) is therefore 1 / 1.116278, or 0.895834.

Next, the risk-neutral probability of an upward move is calculated as:

$$\pi_u = \frac{e^{(0.035-0.02)} - 0.895834}{1.116278 - 0.895834} = 0.541$$

Finally, the risk-neutral probability of a downward move is calculated as:

$$\pi_d = 1 - 0.541 = 0.459$$

(LO 60.e)

5. **B** As time intervals are shortened, the price produced by the binomial model will converge toward the Black-Scholes-Merton model price. In this case, the $3.46 price will get lower until it eventually lands at $3.38. (LO 60.c)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 15.

READING 61

THE BLACK-SCHOLES-MERTON MODEL

Study Session 15

EXAM FOCUS

The Black-Scholes-Merton (BSM) option pricing model (often referred to as the Black-Scholes model) is based on the assumption that stock prices are lognormally distributed. In this reading, we examine the calculation of call and put options using the BSM option pricing model. Also, we discuss how volatility can be estimated using a combination of the BSM model and current option prices. For the exam, know how to calculate the value of a call and put option using the BSM model and be able to incorporate dividends, currencies, and futures into the model if necessary. Put-call parity can be applied to calculate call or put values since the BSM model requires the use of European options.

MODULE 61.1: STOCK PRICE AND RETURN DISTRIBUTIONS

LO 61.a: Explain the lognormal property of stock prices, the distribution of rates of return, and the calculation of expected return.

LO 61.b: Calculate the realized return and historical volatility of a stock.

The model used to develop the Black-Scholes-Merton (BSM) model assumes stock prices are lognormally distributed:

$$\ln S_T \sim N\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

where:

S_T = stock price at time T

S_0 = stock price at time 0

μ = expected return on stock per year

σ = volatility of the stock price per year

$N[m, s]$ = normal distribution with mean m and standard deviation s

Since $\ln S_T$ is normally distributed, S_T has a lognormal distribution.

EXAMPLE: Calculating mean and standard deviation

Assume a stock has an initial price S_0 = $25, an expected annual return of 12%, and an annual volatility of 20%. **Calculate** the mean and standard deviation of the distribution of the stock price in three months.

Answer:

The probability distribution of the stock price, S_T , in three months would be:

$$\ln S_T \sim N\left\{\ln 25 + \left[\left(0.12 - \frac{0.2^2}{2}\right) \times 0.25\right], 0.2 \times \sqrt{0.25}\right\}$$

$$\ln S_T \sim N(3.244, 0.10)$$

Since $\ln S_T$ is normally distributed, 95% of the values will fall within 1.96 standard deviations of the mean. Therefore, $\ln S_T$ will lie between 3.244 ± (1.96 × 0.1). Stated another way:

$$e^{(3.244-1.96\times0.1)} < S_T < e^{(3.244+1.96\times0.1)}$$

$$21.073 < S_T < 31.187$$

Dividing the mean and standard deviation by T results in the continuously compounded annual return of a stock price. Specifically, the continuously compounded annual returns are *normally distributed* with a mean of:

$$\left(\mu - \frac{\sigma^2}{2}\right)$$

and a standard deviation of:

$$\frac{\sigma}{\sqrt{T}}$$

PROFESSOR'S NOTE

Notice that the BSM model assumes stock prices are lognormally distributed, but stock returns are normally distributed. Also, notice in the standard deviation formula that volatility will be lower for longer time periods.

EXAMPLE: Return distribution

Assume a stock has an expected annual return of 12% and an annual volatility of 20%. **Calculate** the mean and standard deviation of the probability distribution for the continuously compounded average rate of return over a four-year period.

Answer:

$$\text{mean} = 0.12 - \frac{0.2^2}{2} = 0.10$$

$$\text{standard deviation} = \frac{0.2}{\sqrt{4}} = 0.10$$

Expected Value

Using the properties of a *lognormal distribution*, we can show that the expected value of S_T, $E(S_T)$, is:

$$E(S_T) = S_0 e^{\mu T}$$

where:

μ = expected rate of return

EXAMPLE: Expected stock price

Assume a stock is currently priced at $25 with an expected annual return of 20%. **Calculate** the expected value of the stock in six months.

Answer:

$$E(S_T) = \$25 \times e^{0.2 \times 0.5} = \$27.63$$

When computing the **realized return** for a portfolio, we want to chain-link the returns, as shown in the following example.

EXAMPLE: Realized return

Consider a portfolio that has the following asset returns: 5%, –4%, 9%, 6%. **Calculate** the return realized by this portfolio.

Answer:

$$\text{realized portfolio return} = (1.05 \times 0.96 \times 1.09 \times 1.06)^{1/4} - 1 = 3.9\%$$

Estimating Historical Volatility

The volatility for short time periods can be scaled to longer time periods. For example, if the weekly standard deviation is 5%, and we want the annual standard deviation, we simply scale it by the square root of the number of periods in a year, or $\sqrt{52}$. So, the annual standard deviation in this case is 36.06%. Conversely, if we knew that the annual standard deviation was 36.06%, then the weekly standard deviation can be found using this formula: $36.06\% / \sqrt{52}$, which is 5%.

The volatility estimation process is a matter of collecting daily price data and then computing the standard deviation of the series of corresponding continuously compounded returns. Continuously compounded returns can be calculated as: $\ln(S_i / S_{i-}$

$_{1}$). The annualized volatility is simply the estimated volatility multiplied by the square root of the number of trading days in a year. Typically, 90–180 trading days of data is sufficient for this estimation technique, but a common rule of thumb is to use data covering a period equal to the length of the projection period. In other words, to estimate the volatility for the next year, we should use a year's worth of historical data.

An additional element to consider when estimating historical volatility is the impact of dividends. On the ex-dividend date, a stock price will naturally decline. When using historical data to calculate volatility, the best approach is to remove stock price changes on ex-dividend dates from the dataset.

PROFESSOR'S NOTE

We will examine the calculation for historical volatility later in this reading.

MODULE QUIZ 61.1

1. XYZ stock has a current price of \$30 and an expected value in nine months of \$34. The expected annual return is closest to:
   A. 11.76%.
   B. 12.52%.
   C. 13.33%.
   D. 16.69%.

2. Assuming a portfolio has the following asset returns: 6%, 2%, 8%, –3%, what is the realized portfolio return?
   A. 3.16%.
   B. 3.25%.
   C. 4.72%.
   D. 4.75%.

MODULE 61.2: BLACK-SCHOLES-MERTON MODEL

LO 61.c: Describe the assumptions underlying the Black-Scholes-Merton option pricing model.

The BSM model values options in continuous time and is derived from the same no-arbitrage assumption used to value options with the binomial model. In the binomial model, the hedge portfolio is riskless over the next period, and the no-arbitrage option price is the one that ensures that the hedge portfolio will yield the risk-free rate. To derive the BSM model, an instantaneously riskless portfolio (one that is riskless over the next instant) is used to solve for the option price based on the same logic.

In addition to the no-arbitrage condition, the assumptions underlying the BSM model are the following:

- The price of the underlying asset follows a lognormal distribution. A variable that is lognormally distributed is one where the logs of the values (in this case, the continuous returns) are normally distributed. It has a minimum of zero and conforms to prices better than the normal distribution (which would produce negative prices).
- The (continuous) risk-free rate is constant, known, and always available for borrowing or lending.

- Trading is continuous.
- The volatility of the underlying asset is constant and known. Option values depend on the volatility of the price of the underlying asset or interest rate.
- Markets are "frictionless." There are no taxes, no transactions costs, and no restrictions on short sales or the use of short-sale proceeds.
- The underlying asset has no cash flow, such as dividends or coupon payments.
- The options valued are European options, which can only be exercised at maturity. The model does not correctly price American options.

Black-Scholes-Merton Formulas

LO 61.d: Calculate the value of a European option on a non-dividend-paying stock using the Black-Scholes-Merton model.

The call and put formulas for the BSM model are:

$$c_0 = [S_0 \times N(d_1)] - [X \times e^{-R_f^c \times T} \times N(d_2)]$$

$$p_0 = \{X \times e^{-R_f^c \times T} \times [1 - N(d_2)]\} - \{S_0 \times [1 - N(d_1)]\}$$

where:

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + [R_f^c + (0.5 \times \sigma^2)] \times T}{\sigma \times \sqrt{T}}$$

$$d_2 = d_1 - (\sigma \times \sqrt{T})$$

T = time to maturity (as % of a 365-day year)
S_0 = asset price
X = exercise price
R_f^c = continuously compounded risk-free rate
σ = volatility of continuously compounded returns on the stock
$N(\bullet)$ = cumulative normal probability

Both call and put formulas are provided here, but remember that if you're given one of those prices, you can always use **put-call parity** (with continuously compounded interest rates) to calculate the other one:

$$c_0 = p_0 + S_0 - (X \times e^{-R_f^c \times T})$$

or

$$p_0 = c_0 - S_0 + (X \times e^{-R_f^c \times T})$$

$N(d_1)$ and $N(d_2)$ are found in a table of probability values (i.e., the z-table), so any question about the value of an option will provide those values. The rest is a straightforward calculation.

EXAMPLE: Using the BSM model to value a European call option

Suppose that the stock of Vola, Inc., is trading at $50, and there is a call option on Vola available with an exercise price of $45 that expires in three months. The continuously compounded risk-free rate is 5%, and the annualized standard deviation of returns is 12%. Using the BSM model, **calculate** the value of the call option.

Answer:

First, we must compute d_1 and d_2 as follows:

$$d_1 = \frac{\ln\left(\frac{50}{45}\right) + [0.05 + (0.5 \times 0.12^2)] \times 0.25}{0.12 \times \sqrt{0.25}} = 1.99$$

$$d_2 = 1.99 - \left(0.12 \times \sqrt{0.25}\right) = 1.93$$

Now, look up these values in the normal probability tables in Figure 61.1.

Figure 61.1: Partial Cumulative Normal Distribution Table*

	0.03	0.04	0.05	0.06	0.07	0.08	0.09
1.8	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	**0.9732**	0.9738	0.9744	0.9750	0.9756	0.9761	**0.9767**
2.0	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

*Note: This table is incomplete. To view an example of a complete cumulative normal table, see the table included at the back of this book.

From the table, we determine that $N(d_1)$ is 0.9767 and $N(d_2)$ is 0.9732. Now that we have everything we need to apply the main call option formula, the value of the call is:

$$c_0 = (\$50 \times 0.9767) - (\$45 \times e^{-(0.05 \times 0.25)} \times 0.9732) = \$48.84 - \$43.25$$
$$= \$5.59$$

To price the corresponding put option using the data in our example, we simply solve put-call parity for the put option price.

EXAMPLE: Calculating put option value

Calculate the value of a Vola 3-month put option with an exercise price of $45, given the information in the previous example.

Answer:

We can use put-call parity to find the value of the comparable put:

$$P_0 = \$5.59 - \$50.00 + [\$45.00 \times e^{-(0.05 \times 0.25)}] = \$0.03$$

We can also use the BSM put formula:

$$P_0 = [\$45 \times e^{-0.05 \times 0.25} \times (1 - 0.9732)] - [\$50 \times (1 - 0.9767)] = \$0.03$$

PROFESSOR'S NOTE

You should know how to look up $N(d_1)$ and $N(d_2)$ in the normal probability table given d_1 and d_2. It's possible, however unlikely, that you will have to calculate d_1 and d_2 without the formulas. To value the put option, memorize put-call parity and use it to solve for the put value given the call value.

MODULE QUIZ 61.2

1. Which of the following is not an assumption underlying the BSM options pricing model?
   A. The underlying asset does not generate cash flows.
   B. Continuously compounded returns are lognormally distributed.
   C. The option can only be exercised at maturity.
   D. The risk-free rate is constant.

2. A European put option has the following characteristics: S_0 = \$50; X = \$45; r = 5%; T = 1 year; and σ = 25%. Which of the following is closest to the value of the put?
   A. \$1.88.
   B. \$3.28.
   C. \$9.06.
   D. \$10.39.

3. A European call option has the following characteristics: S_0 = \$50; X = \$45; r = 5%; T = 1 year; and σ = 25%. Which of the following is closest to the value of the call?
   A. \$1.88.
   B. \$3.28.
   C. \$9.06.
   D. \$10.39.

4. A security sells for \$40. A 3-month call with a strike of \$42 has a premium of \$2.49. The risk-free rate is 3%. What is the value of the put according to put-call parity?
   A. \$1.89.
   B. \$3.45.
   C. \$4.18.
   D. \$6.03.

5. Stock ABC trades for \$60 and has 1-year call and put options written on it with an exercise price of \$60. The annual standard deviation estimate is 10%, and the continuously compounded risk-free rate is 5%. The value of both the call and put using the BSM option pricing model are closest to which of the following?

	Call	Put
A.	$6.21	$1.16
B.	$4.09	$3.28
C.	$4.09	$1.16
D.	$6.21	$3.28

MODULE 61.3: DIVIDENDS, WARRANTS, AND IMPLIED VOLATILITY

Valuation of European Options

LO 61.g: Calculate the value of a European option on a dividend-paying stock, futures, or foreign currency using the Black-Scholes-Merton model.

Just as we subtracted the present value of expected cash flows from the asset price when valuing forwards and futures, we can subtract it from the asset price in the BSM model. To do this, we need to relax the BSM assumption that the underlying asset does not have cash flows. Since the BSM model is in continuous time, in practice, $S_0 \times e^{-qT}$ is substituted for S_0 in the BSM formula, where q is equal to the continuously compounded rate of the dividend payment. Over time, the asset price is discounted by a greater amount to account for the greater amount of cash flows. Cash flows will increase put values and decrease call values.

EXAMPLE: **Valuing a call option on a stock with a continuous dividend yield**

Let's revisit Vola, Inc., and this time we'll assume the stock pays a continuous dividend yield of 2%. Here's the basic information again. Suppose the stock of Vola is trading at $50, and there is a call option available with an exercise price of $45 that expires in three months. The continuously compounded risk-free rate is 5%, and the annualized standard deviation of returns is 12%. Using the BSM model, **calculate** the value of the call option.

Answer:

The adjusted price of the stock is:

$$e^{-0.02\times0.25} \times \$50.00 = \$49.75$$

Recalculate d_1 and d_2 using the adjusted price:

$$d_1 = \frac{\ln\left(\frac{49.75}{45}\right) + \{0.05 + [0.5 \times (0.12)^2]\} \times 0.25}{0.12 \times \sqrt{0.25}} = 1.91$$

$$d_2 = 1.91 - (0.12 \times \sqrt{0.25}) = 1.85$$

Now, look up these values in the normal probability tables in Figure 61.2.

Figure 61.2: Partial Cumulative Normal Distribution Table*

	0.00	0.01	0.02	0.03	0.04	0.05	0.06
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	**0.9678**	0.9686
1.9	0.9713	**0.9719**	0.9726	0.9732	0.9738	0.9744	0.9750
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803

*Note: This table is incomplete. To view an example of a complete cumulative normal table, see the table included at the back of this book.

From the table, we can determine that $N(d_1)$ is 0.9719 and $N(d_2)$ is 0.9678. The adjusted price is \$49.75. Now that we have everything we need to apply the BSM model, the value of the call is:

$$c_0 = (\$49.75 \times 0.9719) - (\$45 \times e^{-(0.05\times0.25)} \times 0.9678) = \$48.35 - \$43.01$$
$$= \$5.34$$

The value of the Vola call with no dividend yield was \$5.59 from our earlier example. The 2% dividend yield reduced the call value by \$0.25, from \$5.59 to \$5.34.

For options on foreign currencies, the foreign risk-free rate (r_f) is substituted in place of the continuous dividend yield (q) in the $S_0 \times e^{qT}$ component of the BSM model. The values are calculated in the model in the same way as they were for continuous dividends. For options on futures, the stock price S_0 in the $S_0 \times e^{qT}$ component is replaced with the futures price (F) and the dividend yield (q) is replaced with the domestic risk-free rate (r). The resulting BSM equations, when used for options on futures, are called **Black's model**, and the volatility of the futures price is used in place of the volatility of the stock price under the traditional model. Black's model can be used to value an option on an asset's spot price in terms of its forward price.

On the exam, it may be the case that you are provided with the dollar amount of the dividend rather than the dividend yield. The process for computing option value is similar, but instead of discounting the stock price with a continuously compounded dividend rate, you would compute the present value of the dividend(s) and then subtract that amount from the stock price. The following example demonstrates this technique.

EXAMPLE: Pricing options on a dividend-paying stock

Assume we have a non-dividend-paying stock with a current price of \$100 and volatility of 20%. If the risk-free rate is 7%, the price of a 6-month at-the-money call option, according to the BSM model, will be \$7.43. The corresponding put option price will be \$3.99. Now, assume that the same stock instead pays a \$1 dividend in two months and a \$1 dividend in five months. **Compute** the value of a 6-month call option on the dividend-paying stock.

Answer:

The present value of the first dividend is $1e^{-0.07(0.1667)} = 0.9884$, and the present value of the second dividend is $1e^{-0.07(0.4167)} = 0.9713$. The stock price then becomes:

$$S_0 = 100 - 0.9884 - 0.9713 = \$98.04$$

We now know the following: S_0 is \$98.04; X is \$100; σ is 20%; r is 7%; and T is 0.5.

So, d_1 and d_2 are computed as follows:

$$d_1 = \frac{\ln\left(\frac{98.04}{100}\right) + \left(0.07 + \frac{0.2^2}{2}\right) \times 0.5}{0.2 \times \sqrt{0.5}} = 0.1783$$

$$d_2 = 0.1783 - 0.2\sqrt{0.5} = 0.03688$$

From the cumulative standard normal tables, we find:

$$N(d_1) = 0.5708 \text{ and } N(d_2) = 0.5147$$

Substituting back into the call option price formula yields:

$$c = 98.04 \times 0.5708 - 100e^{-0.07 \times 0.5} \times 0.5147 = \$6.26$$

Using put-call parity, the corresponding put option price is:

$$p = \$6.26 + 100e^{-0.07 \times 0.5} - 98.04 = \$4.78$$

Since the dividend reduces the value of the stock, the call value decreased, and the put value increased compared to the non-dividend-paying stock.

Impact of Dividends on American Options

LO 61.f: Explain how dividends affect the decision to exercise early for American call and put options.

Recall that when no dividends are paid, there is no difference between European and American call options. This is because the unexercised value of a call option, $S_0 - Xe^{-rT}$, was always more valuable than the exercised value of the option, $S_0 - X$. When a stock pays a dividend, D, at time n, the exercise decision becomes more complicated.

At the last dividend date before expiration, t_n, the exercised value of the option is:

$$S(t_n) - X$$

If the call option is unexercised and the dividend is paid, its unexercised value is:

$$S(t_n) - D_n - Xe^{-r(T - t_n)}$$

An investor will only exercise when:

$$S(t_n) - X > S(t_n) - D_n - Xe^{-r(T - t_n)}$$

or

$$D_n > X(1 - e^{-r(T - t_n)})$$

So, the closer the option is to expiration and the larger the dividend, the more optimal early exercise will become. The previous result can be generalized to show that early exercise is not optimal if:

$$D_i \leq X(1 - e^{-r(t_{i+1} - t_n)}) \text{ for } i < n$$

For American put options, early exercise becomes less likely with larger dividends. The value of the put option increases as the dividend is paid. Early exercise is, therefore, not optimal as long as:

$$D_n \geq X(1 - e^{-r(T - t_n)})$$

Valuation of Warrants

LO 61.h: Describe warrants, calculate the value of a warrant, and calculate the dilution cost of the warrant to existing shareholders.

Warrants are attachments to a bond issue that give the holder the right to purchase shares of a security at a stated price. After purchasing the bond, warrants can be exercised separately or stripped from the bond and sold to other investors. Hence, warrants can be valued as a separate call option on the firm's shares.

One distinction is necessary, though. With call options, the shares are already outstanding, and the exercise of a call option triggers the trading of shares among investors at the strike price of the call options. When an investor exercises warrants, the investor purchases shares directly from the firm. The distinction is that the value of all outstanding shares can be affected by the exercise of warrants, as the amount paid for the shares will (in all likelihood) be less than their pro rata market value, so the value of equity per share will fall with exercise (i.e., dilution can occur).

Assuming there is no benefit to the company from issuing warrants, the value of each warrant is computed as:

$$\frac{N}{N + M} \times \text{value of regular call option}$$

where:

N = number of shares outstanding

M = number of new warrants issued

Thus, the total cost of issuing warrants is *M* times the value of each warrant. With no perceived market benefit, the company's stock price will decline by: M / (N + M) × value of regular call option.

For example, suppose a company with 1 million shares outstanding worth $50 each is contemplating issuing 500,000 warrants. Each warrant would grant the holder the right to purchase one share with a strike price of $65 in two years. Assuming the value of a corresponding two-year European call option is worth $6.00, the value of each warrant would be computed as:

$$\frac{1{,}000{,}000}{1{,}000{,}000 + 500{,}000} \times 6.00 = \$4.00$$

Thus, the total cost of the warrant issue would be 500,000 × $4 = $2 million. In the event that there is no perceived benefit in the marketplace from issuing the warrants, we would expect the initial stock price of $50 to decline by $2 to $48 per share:

$$\frac{500{,}000}{1{,}000{,}000 + 500{,}000} \times 6.00 = \$2.00$$

Volatility Estimation

LO 61.e: Define implied volatilities and describe how to calculate implied volatilities from market prices of options using the Black-Scholes-Merton model.

Notice in call and put equations that volatility is unobservable. Historical data can serve as a basis for what volatility might be going forward, but it is not always representative of the current market. Consequently, practitioners will use the BSM option pricing model along with market prices for options and solve for volatility. The result is what is known as **implied volatility**. Before we discuss implied volatility further, let's first examine the calculation of historical volatility.

The steps in computing **historical volatility** for use as an input in the BSM continuous-time options pricing model are as follows:

- Convert a time series of *N* prices to returns:

$$R_i = \frac{P_i - P_{i-1}}{P_{i-1}},\ i = 1 \text{ to } N$$

- Convert the returns to continuously compounded returns:

$$R_i^c = \ln(1 + R_i),\ i = 1 \text{ to } N$$

- Calculate the variance and standard deviation of the continuously compounded returns:

$$\sigma^2 = \frac{\sum_{i=1}^{N}\left(R_i^c - \overline{R}_i^c\right)^2}{N - 1}$$

$$\sigma = \sqrt{\sigma^2}$$

Continuously compounded returns can be calculated using a set of price data with the following equation:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

Arriving at the continuously compounded return value is no different than taking the holding period return and then taking the natural log of (1 + holding period return). For example, if we assume that a stock price is currently valued at $50 and was $47 yesterday, the continuously compounded return can be computed as either:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) = \ln\left(\frac{50}{47}\right) = 6.19\%$$

or

$$R_i = \frac{P_i - P_{i-1}}{P_{i-1}} = \frac{50 - 47}{47} = 6.38\%$$

$$R_i^c = \ln(1 + 0.0638) = 6.19\%$$

Implied volatility is the value for standard deviation of continuously compounded rates of return that is implied by the market price of the option. Of the five inputs into the BSM model, four are observable: (1) stock price, (2) exercise price, (3) risk-free rate, and (4) time to maturity. If we use these four inputs in the formula and set the BSM formula equal to market price, we can solve for the volatility that satisfies the equality.

Volatility enters into the equation in a complex way, and there is no closed-form solution for the volatility that will satisfy the equation. Rather, it must be found by iteration (trial and error). If a value for volatility makes the value of a call calculated from the BSM model lower than the market price, it needs to be increased (and vice versa) until the model value equals market price (remember, option value and volatility are positively related).

The reality with volatility is that volatility varies with strike prices such that graphically, plotting implied volatility against strike prices produces a **volatility smile**. Assessing the **volatility surface** involves looking at implied volatilities relative to strike price and time to maturity.

MODULE QUIZ 61.3

1. Which of the following statements is most accurate regarding implied volatility in the BSM model?
   A. Volatility is constant across strike prices.
   B. Volatility is most accurately applied using historical data.
   C. The process for estimating volatility involves two steps at most.
   D. Volatility is often derived using the BSM market price and the other inputs.

2. Compared to the value of a call option on a stock with no dividends, a call option on an identical stock expected to pay a dividend during the term of the option will have a:
   A. lower value in all cases.
   B. higher value in all cases.
   C. lower value only if it is an American-style option.
   D. higher value only if it is an American-style option.

3. Assume a current stock price of $35 with a continuously compounded dividend yield of 2.5%. There is a 6-month call option on the stock with an exercise price of $33. What is the adjusted stock price to use for the BSM model?
   A. $30.12.
   B. $32.59.
   C. $34.57.
   D. $35.44.

4. There are 3 million outstanding shares of ABC stock currently selling at $42 each. ABC is considering issuing 1 million warrants with a strike price of $45 exercisable in one year. If the current value of a 1-year European call option is $2.12, the expected stock price after announcing the warrant (assuming no perceived benefit to issuance) will be closest to:
   A. $40.41.

B. $41.47.
C. $42.53.
D. $43.59.

KEY CONCEPTS

LO 61.a

The BSM model suggests that stock prices are lognormal over longer time periods, but suggests that stock returns are normally distributed.

LO 61.b

The realized return for a portfolio is computed using a geometric return (i.e., chain-linking returns).

The volatility estimation process involves collecting daily price data and then computing the standard deviation of the series of corresponding continuously compounded returns. The annualized volatility is the estimated volatility multiplied by the square root of the number of trading days in a year. In using historical data to calculate volatility, the best approach is to remove stock price changes on ex-dividend dates from the dataset.

LO 61.c

There are assumptions underlying the BSM model:

- The price of the underlying asset follows a lognormal distribution.
- The (continuous) risk-free rate is constant and known.
- Trading is continuous.
- The volatility of the underlying asset is constant, known, and always available.
- Markets are frictionless.
- The underlying asset generates no cash flows.
- The options are European.

LO 61.d

The formulas for the BSM model are:

$$c_0 = \left[S_0 \times N(d_1)\right] - \left[Xe^{-R_f^c \times T} \times N(d_2)\right]$$
$$p_0 = \left\{Xe^{-R_f^c \times T} \times \left[1 - N(d_2)\right]\right\} - \left\{S_0 \times \left[1 - N(d_1)\right]\right\}$$

Put-call parity (with continuously compounded interest rates) can be used to calculate the value of a call or put if you have the value of the other.

$$c_0 = p_0 + S_0 - (X \times e^{-R_f^c \times T})$$

or

$$p_0 = c_0 - S_0 + (X \times e^{-R_f^c \times T})$$

$N(d_1)$ and $N(d_2)$ are found in a table of probability values (i.e., the z-table), so any question about the value of an option will provide those values.

Cash flows on the underlying asset decrease call prices and increase put prices.

LO 61.e

Historical volatility is the standard deviation of a past series of continuously compounded returns for the underlying asset. Implied volatility is the volatility that, when used in the BSM formula, produces the current market price of the option.

There is no closed-form solution for calculating implied volatility, as it must be found by iteration (trial and error). If a value for volatility makes the value of a call calculated from the BSM model lower than the market price, it needs to be increased (and vice versa) until the model value equals market price (remember, option value and volatility are positively related).

Volatility varies with strike prices such that graphically it produces a volatility smile. Assessing the volatility surface involves looking at implied volatilities relative to strike price and time to maturity.

LO 61.f

When no dividends are paid, there is no difference between European and American call options. Dividends complicate the early exercise decision for American-style options because a dividend payment effectively decreases the price of the stock. The closer the option is to expiration and the larger the dividend, the more optimal early exercise will become.

For American put options, early exercise becomes less likely with larger dividends. The value of the put option increases as the dividend is paid.

LO 61.g

To adjust the BSM model for assets with a continuously compounded rate of dividend payment equal to q, S_0e^{-qT} is substituted for S_0 in the formula. Options on foreign currencies use the same formulas as when there are continuously compounded dividends, except that the foreign risk-free rate (r_f) is substituted in place of the continuous dividend yield (q). Options on futures use the same formulas as when there are continuously compounded dividends, except that the stock price S_0 is replaced with the futures price (F), and the dividend yield (q) is replaced with the domestic risk-free rate (r).

LO 61.h

Warrants are attachments to a bond issue that give the holder the right to purchase shares of a security at a stated price. Warrants can be valued as a separate call option on the firm's shares. Assuming there is no benefit to the company from issuing warrants, the value of each warrant is computed as:

$$\frac{N}{N + M} \times \text{value of regular call option}$$

where N = number of shares outstanding and M = number of new warrants issued.

The total cost of issuing warrants is M times the value of each warrant, and assuming no perceived market benefit, the company's stock price will decline by: M / (N + M) × value of regular call option.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 61.1

1. **D** With a current price of \$30, a future price of \$34, and a time period of nine months, the formula setup is as follows:

   $\$30 \times e^{\mu 0.75} = \$34$

   By dividing \$34 by \$30 and then taking the natural log of both sides, we can solve for μ (the expected rate of return):

   $\mu = 0.1669$, or 16.69%

   (LO 61.a)

2. **A** Realized portfolio return = $(1.06 \times 1.02 \times 1.08 \times 0.97)^{1/4} - 1 = 3.16\%$ (LO 61.b)

Module Quiz 61.2

1. **B** Assumptions underlying the BSM options pricing model include the following:
   - Asset price (not returns) follows a lognormal distribution.
   - The (continuous) risk-free rate is constant.
   - Trading is continuous.
   - The volatility of the underlying asset is constant.
   - Markets are frictionless.
   - The asset has no cash flows.
   - The options are European (i.e., they can only be exercised at maturity).

   (LO 61.c)

2. **A** $S_0 = \$50$; $X = \$45$; $r = 5\%$; $T = 1$ year; and $\sigma = 25\%$.

   $$d_1 = \frac{\ln\left(\frac{50}{45}\right) + \left(0.05 + \frac{0.0625}{2}\right)1}{0.25(1)} = \frac{0.18661}{0.25} = 0.74644$$

   $d_2 = 0.74644 - 0.25 = 0.49644$

   from the cumulative normal table:

   $N(-d_1) = 0.2266$

   $N(-d_2) = 0.3085$*

   $p = 45e^{-0.05(1)}(0.3085) - 50(0.2266) = 1.88$

(*note rounding)

(LO 61.d)

3. **C** S_0 = \$50; X = \$45; r = 5%; T = 1 year; and σ = 25%.

$$d_1 = \frac{\ln\left(\frac{50}{45}\right) + \left(0.05 + \frac{0.0625}{2}\right)1}{0.25(1)} = \frac{0.18661}{0.25} = 0.74644$$

$d_2 = 0.74644 - 0.25 = 0.49644$

from the cumulative normal table:

$N(d_1) = 0.7731$

$N(d_2) = 0.6915$*

$c = 50(0.7731) - 45e^{-0.05}(0.6915) = 9.055$

(*note rounding)

(LO 61.d)

4. **C** $p = c + Xe^{-rT} - S = 2.49 + 42e^{-0.03\times 0.25} - 40 = \$4.18$

(LO 61.d)

5. **C** First, let's compute d_1 and d_2 as follows:

$$d_1 = \frac{\ln\left(\frac{60}{60}\right) + [0.05 + (0.5 \times 0.10^2)] \times 1.0}{0.1 \times \sqrt{1.0}} = 0.55$$

$$d_2 = 0.55 - (0.1 \times \sqrt{1.0}) = 0.45$$

Now, look up these values in the normal table at the back of this book. These values are $N(d_1) = 0.7088$ and $N(d_2) = 0.6736$. Hence, the value of the call is:

$$\begin{aligned} c_0 &= \$60(0.7088) - [\$60 \times e^{-(0.05\times 1.0)} \times (0.6736)] \\ &= \$42.53 - \$38.44 \\ &= \$4.09 \end{aligned}$$

According to put-call parity, the put's value is:

$$\begin{aligned} P_0 &= c_0 - S_0 + (X \times e^{-R_f^c \times T}) \\ &= \$4.09 - \$60.00 + [\$60.00 \times e^{-(0.05\times 1.0)}] \\ &= \$1.16 \end{aligned}$$

(LO 61.d)

Module Quiz 61.3

1. **D** Volatility is not directly observable, and so to estimate it, the price of the option using the BSM model and the other observable inputs (stock price, exercise price, risk-free rate, and time to maturity) are put into the model to derive volatility. Volatility is not constant across strike prices. Using historical data to estimate volatility is helpful, but it does not predict current or future volatility. The process for estimating volatility requires many steps, as it is a trial-and-error process. (LO 61.e)

2. **A** An expected dividend during the term of an option will decrease the value of a call option. (LO 61.f)

3. **C** The adjusted stock price is calculated as:

   adjusted stock price = $\$35 \times e^{-0.025 \times 0.5} = \$34.57$

   (LO 61.g)

4. **B** The value of each warrant is equal to:

   $$\frac{3{,}000{,}000}{3{,}000{,}000 + 1{,}000{,}000} \times \$2.12 = \$1.59$$

   The total warrant cost is 1,000,000 × \$1.59 = \$1.59 million.

   The initial stock price will therefore decline by:

   $$\frac{1{,}000{,}000}{3{,}000{,}000 + 1{,}000{,}000} \times \$2.12 = \$0.53$$

   So, the stock price = \$42.00 – \$0.53 = \$41.47. (LO 61.h)

The following is a review of the Valuation and Risk Models principles designed to address the learning objectives set forth by GARP®. Cross-reference to GARP FRM Part I Valuation and Risk Models, Chapter 16.

READING 62

OPTION SENSITIVITY MEASURES: THE "GREEKS"

Study Session 15

EXAM FOCUS

The level of risk associated with an option position is dependent in large part on the following factors: relationship between the value of a position involving options and the value of the underlying assets, time until expiration, asset value volatility, and the risk-free rate. Measures that capture the effects of these factors are referred to as "the Greeks" due to their names: delta, theta, gamma, vega, and rho. Thus, a large part of this reading covers the evaluation of option Greeks. Once option participants are aware of their Greek exposures, they can more effectively hedge their positions to mitigate risk. This reading also introduces the common hedging concepts of delta-neutral portfolios and portfolio insurance.

MODULE 62.1: NAKED AND COVERED POSITIONS

LO 62.a: Describe and assess the risks associated with naked and covered option positions.

A **naked position** occurs when one party sells a call option without owning the underlying asset. A **covered position** occurs when the party selling a call option owns the underlying asset.

Suppose a firm can sell 10,000 call options on a stock that is currently trading at $20. The strike price of the option is $23, and the option premium is $4. A naked position would generate $40,000 in revenue, and as long as the stock price is below $23 at expiration, the firm can retain the income without cost. However, the initial income will be reduced by $10,000 for every dollar above $23 that the stock reaches at expiration. For example, if the stock is at $30 per share when the option expires, the naked position results in a negative payoff of $70,000 and a net loss of $30,000. The potential loss from a naked written position is unlimited, assuming the stock's price can rise without bound. The maximum potential gain is capped at the level of the premium received. If the stock price at expiration is $23 or less, the writer makes a profit equal to the premium of $40,000.

With a covered call, the firm owns 10,000 shares of the underlying stock, so if the stock price rises above the $23 strike price and the option is exercised, the firm will sell shares that it already owns. This minimizes the cost of the short options by locking in the revenue from the option sale. However, if the stock falls to $10 per share, the long stock position decreases in value by $100,000, which is substantially larger than the premium received from the option sale.

Stop-Loss Strategy

LO 62.b: Describe the use of a stop-loss hedging strategy, including its advantages and disadvantages, and explain how this strategy can generate naked and covered option positions.

Stop-loss strategies with call options are designed to limit the losses associated with short option positions (i.e., those taken by call writers). The strategy requires purchasing the underlying asset for a naked call position when the asset rises above the option's strike price. The asset is then sold as soon as it goes below the strike price. The objective here is to hold a naked position when the option is out of the money and a covered position when the option is in the money.

The strategy tends to work well when an option is initially in the money. The main drawbacks to this simplistic approach are transaction costs and price uncertainty. The costs of buying and selling the asset can become high as the frequency of stock price fluctuations about the exercise price increases. In addition, there is great uncertainty as to whether the asset will be above (or below) the strike price at expiration.

MODULE QUIZ 62.1

1. An investor takes a short position in a call option on ABC stock with a current price of $15 and a strike price of $18. If the investor does not own the underlying stock, the biggest risk to the investor is:
   A. a loss on the premium paid.
   B. the stock price rising above $18.
   C. the stock price falling below $15.
   D. a decline in the overall stock market.

2. Stop-loss strategies with call options require purchasing the underlying asset for a:
   A. naked call position when the asset falls below the option's strike price.
   B. naked call position when the asset rises above the option's strike price.
   C. covered call position when the asset falls below the option's strike price.
   D. covered call position when the asset rises above the option's strike price.

MODULE 62.2: DELTA AND DELTA HEDGING

LO 62.c: Calculate the delta of an option.

The **delta** of an option, Δ, is the ratio of the change in price of an option, (c for call and p for put), to the change in price of the underlying asset, s, for small changes in s. Mathematically (for a call option):

$$\text{delta} = \Delta = \frac{\partial c}{\partial s}$$

where:
∂c = change in the call option price
∂s = change in the stock price

As illustrated in Figure 62.1, delta is the slope of the call option pricing function at the current stock price. As shown in Figure 62.2, call option deltas range from zero to positive one, while put option deltas range from negative one to zero.

Figure 62.1: Delta of a Call Option

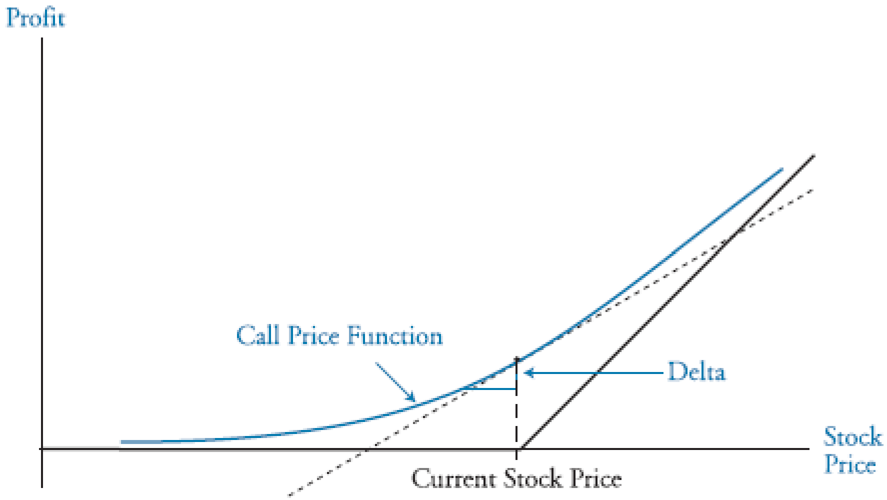
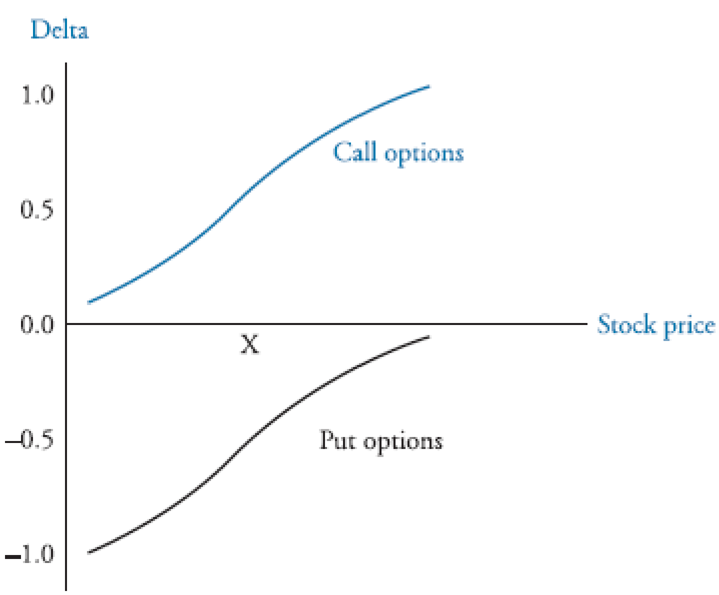


Figure 62.2: Call and Put Option Deltas



Option Delta

A call delta equal to 0.6 means that the price of a call option on a stock will change by approximately $0.60 for a $1.00 change in the value of the stock. To completely hedge a long stock or short call position, an investor must purchase the number of shares of stock equal to delta times the number of options sold. Another term for being completely hedged is **delta neutral**. For example, if an investor is short 1,000 call options, he will need to be long 600 (0.6 × 1,000) shares of the underlying. When the value of the underlying asset increases by $1.00, the underlying position increases by $600, while the value of his option position decreases by $600. When the value of the underlying asset decreases by $1.00, there is an offsetting increase in value in the option position.

Delta can also be calculated as the $N(d_1)$ in the Black-Scholes-Merton option pricing model. Recall from the previous reading that d_1 is equal to:

$$d_1 = \frac{\ln(S_0/X) + (R_F + \sigma^2/2) \times T}{\sigma \times \sqrt{T}}$$

EXAMPLE: Computing delta

Suppose that Stock XYZ is trading at $50, and there is a call option that trades on XYZ with an exercise price of $45, which expires in three months. The risk-free rate is 5%, and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's delta.

Answer:

$$d_1 = \frac{\ln(50/45) + (0.05 + 0.12^2/2) \times 0.25}{0.12 \times \sqrt{0.25}} = 1.99$$

Next, look up this value in the normal probability tables, which can be found in appendix at the end of this book. From the normal probability tables $N(1.99)$, and, in turn, delta is 0.9767. This means that when the stock price changes by $1, the option price will change by 0.9767.

Forward Delta

The delta of a forward position is equal to one, implying a one-to-one relationship between the value of the forward contract and its underlying asset. A forward contract position can easily be hedged with an offsetting underlying asset position with the same number of securities.

PROFESSOR'S NOTE

When the underlying asset pays a dividend, q, the delta of an option or forward must be adjusted. If a dividend yield exists, the delta for a call option equals $e^{-qT} \times N(d_1)$, the delta of a put option equals $e^{-qT} \times [N(d_1) - 1]$, and the delta of a forward contract equals e^{-qT}.

Futures Delta

Unlike forward contracts, the delta of a futures position is not ordinarily one because of the spot-futures parity relationship. For example, the delta of a futures position is e^{rT} on a stock or stock index that pays no dividends, where r is the risk-free rate and T is the time to maturity. Assets that pay a dividend yield, q, would generate a delta equal to $e^{(r}$

$-q)T$. An investor would hedge short futures positions by going long the amount of the deliverable asset.

Dynamic Aspects of Delta Hedging

LO 62.d: Explain delta hedging for an option position, including its dynamic aspects.

As we saw in Figure 62.1, the delta of an option is a function of the underlying stock price. This means that when the stock price changes, so does the delta. When the delta changes, the portfolio will no longer be hedged (i.e., the number of options and underlying stocks will no longer be in balance), and the investor will need to either purchase or sell the underlying asset. This rebalancing must be done on a continual basis to maintain the delta-neutral hedged position.

The goal of a **delta-neutral portfolio** (or delta-neutral hedge) is to combine a position in an asset with a position in an option *so that the value of the portfolio does not change with changes in the value of the asset*. In referring to a stock position, a delta-neutral portfolio can be made up of a risk-free combination of a long stock position and a short call position where the number of calls to short is given by $1 / \Delta_c$.

$$\text{number of options needed to delta hedge} = \frac{\text{number of shares hedged}}{\text{delta of call option}}$$

EXAMPLE: **Delta-neutral portfolio—Part 1**

An investor owns 60,000 shares of ABC stock that is currently selling for \$50. A call option on ABC with a strike price of \$50 is selling at \$4 and has a delta of 0.60. **Determine** the number of call options necessary to create a delta-neutral hedge.

Answer:

To determine the number of call options necessary to hedge against instantaneous movements in ABC's stock price, calculate:

$$\begin{aligned}\text{number of options needed to delta hedge} &= \frac{60{,}000}{0.6} = 100{,}000 \text{ options} \\ &= 1{,}000 \text{ call option contracts}\end{aligned}$$

Because he is long the stock, he needs to short the options.

EXAMPLE: **Delta-neutral portfolio—Part 2**

Calculate the effect on portfolio value of a \$1 increase in the price of ABC stock.

Answer:

Assuming the price of ABC stock increased instantly by \$1, then the value of the call option position would decrease by \$0.60 because the investor is *short* (or has sold) the call option contracts. Therefore, the net impact of the price change would be zero, as illustrated here:

$$\text{total value of increase in stock position} = (60{,}000) \times (\$1) = \$60{,}000$$

$$\text{total value of decrease in option position} = (100{,}000) \times (-\$0.60) = -\$60{,}000$$

$$\text{total change in portfolio value} = \$60{,}000 - \$60{,}000 = \$0$$

Recall that when short a call (or other asset), as the price of the underlying rises, the position loses value, and when the price of the underlying declines, the value of the position increases.

Maintaining the Hedge

A key consideration in delta-neutral hedging is that the *delta-neutral position only holds for very small changes in the value of the underlying stock*. Hence, the delta-neutral portfolio must be frequently (continuously) *rebalanced to maintain the hedge*. As the underlying stock price changes, so does the delta of the call option. The delta of the option is an approximation of a nonlinear function: the change in value of the option that corresponds with a change in the value of the underlying asset. As the delta changes, the number of calls that need to be sold to maintain a risk-free position also changes. Hence, continuously maintaining a delta-neutral position can be very costly in terms of transaction costs associated with either closing out options or selling additional contracts.

Adjusting the hedge on a frequent basis is known as **dynamic hedging**. If, instead, the hedge is initially set up but never adjusted, it is referred to as **static hedging**. This type of hedge is also known as a *hedge-and-forget strategy*.

EXAMPLE: Delta-neutral portfolio—Part 3

Continuing with the previous example, assume now that the price of the underlying stock has moved to $51, and consequently, the delta of the call option with a strike price of $50 has increased from 0.60 to 0.62. How would the investor's portfolio of stock and options have to be adjusted to maintain the delta-neutral position?

Answer:

To determine the number of call options necessary to maintain the hedge against instantaneous movements in ABC's stock price, recalculate the number of short call options needed:

$$\text{number of options needed} = \frac{\text{number of shares hedged}}{\text{delta of call option}}$$

$$= \frac{60{,}000}{0.62} = 96{,}744$$

She will need 96,774 call options, or approximately 968 option contracts. In other words, 32 option contracts would need to be purchased to maintain the delta-neutral position. If the hedge were not modified, then another price change would result in a greater movement in the value of the options than in the underlying stock. With the rebalanced hedge, the change in value of her stock position will again be offset by the change in value of her short position. Assume the price of ABC stock increased (decreased) instantly by $1.00, then the value of the short call option position would decrease (increase) by $0.62. Therefore, the net impact of the price change would be zero:

$$\text{increase in stock position} = (60{,}000) \times (\$1) = \$60{,}000$$

$$\text{decrease in short position} = (96{,}774) \times (-\$0.62) = -\$60{,}000$$

Other Portfolio Hedging Approaches

It's also possible to develop a delta-neutral hedge by buying put options in sufficient numbers so that the current gain or loss on the underlying asset is offset by the current gain or loss on the puts. Hence, similar to the discussion of delta-neutral portfolios using call options, a delta-neutral position can be created by *purchasing* the correct number of put options so that:

$$\Delta \text{ value of puts} = -\Delta \text{ value of long stock position}$$

When using puts in constructing a delta-neutral portfolio, *purchase* [1 / (call delta – 1)] put options to protect a share of stock held long. When using calls, you would *sell* (1 / call delta) call options for each long share of stock. Rebalancing is just as important with puts as it is with calls.

EXAMPLE: Delta-neutral portfolio—Part 4

Using our earlier example, assume the investor owns 60,000 shares of ABC stock that is currently selling for $50. A call option on ABC with a strike price of $50 is selling at $4 and has a delta of 0.60. **Determine** the number of put options necessary to create a delta-neutral hedge.

Answer:

First, compute the delta of the put option. The investor knows that the delta of a call option is 0.60. The delta of the put option is then equal to (call delta – 1), or (0.60 – 1) = –0.40. To determine the number of put options necessary to hedge against instantaneous movements in ABC's stock price, calculate:

$$\begin{aligned}\text{number of options needed to delta hedge} &= \frac{-60{,}000}{-0.4} = 150{,}000 \text{ options}\\ &= 1{,}500 \text{ put option contracts}\end{aligned}$$

Because he is long the stock, he needs to purchase the put options.

MODULE QUIZ 62.2

1. If the risk-free rate is 3% and the time to maturity is nine months, the delta of a forward position is closest to:
   A. 0.98.
   B. 1.00.
   C. 1.02.

D. 2.25.

2. Which of the following choices will effectively hedge a short call option position that exhibits a delta of 0.5?
   A. Sell two shares of the underlying for each option sold.
   B. Buy two shares of the underlying for each option sold.
   C. Sell the number of shares of the underlying equal to half the options sold.
   D. Buy the number of shares of the underlying equal to half the options sold.

3. A static hedging strategy will be least effective when the underlying stock price:
   A. increases from $4 to $6.
   B. increases from $20 to $21.
   C. decreases from $26 to $25.
   D. decreases from $35 to $34.

MODULE 62.3: THETA, GAMMA, VEGA, AND RHO

LO 62.e: Define and describe vega, gamma, theta, and rho for option positions and calculate the gamma and vega of an option.

LO 62.f: Explain how to implement and maintain a delta-neutral and gamma-neutral position.

LO 62.g: Describe the relationship between delta, theta, gamma, and vega.

Theta

Theta, Θ, measures the option's sensitivity to a decrease in time to expiration. Theta is also termed the *time decay* of an option. Theta varies as a function of both time and the price of the underlying asset. Figure 62.3 illustrates theta as a function of stock price and days until expiration.

Theta for a call option is calculated using the following equation:

$$\Theta = \frac{\partial c}{\partial t}$$

where:
∂c = change in the call price
∂t = change in time

For European call options on non-dividend-paying stocks, theta can be calculated using the Black-Scholes-Merton formula as follows:

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rXe^{-rT}N(d_2)$$

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rXe^{-rT}N(-d_2)$$

where:

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-(x^2/2)}$$

Note that theta in these equations is measured in years. It can be converted to a daily basis by dividing by 365. To find the theta for each trading day, you would divide by 252.

EXAMPLE: Computing theta

Suppose that Stock XYZ is trading at $50, and there is a call option that trades on XYZ with an exercise price of $45, which expires in three months. The risk-free rate is 5%, and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's theta per trading day. Assume d_1 is 1.99 and d_2 is 1.93. From the normal probability tables, $N(d_1)$ is 0.9767 and $N(d_2)$ is 0.9732.

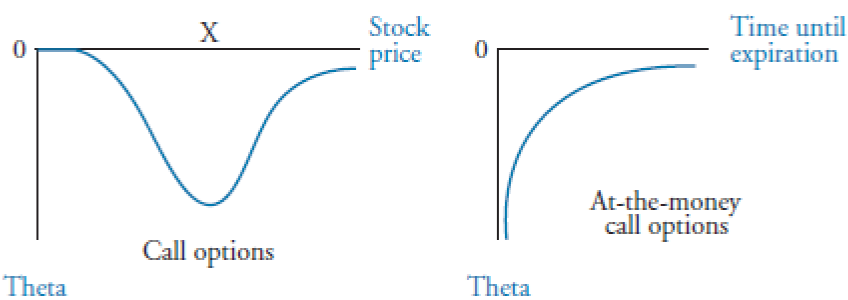
Answer:

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-(1.99^2/2)} = 0.055$$

$$\begin{aligned}\Theta(\text{call}) &= -\frac{50 \times 0.055 \times 0.12}{2\sqrt{0.25}} - 0.05 \times 45\, e^{-0.05 \times 0.25} \times 0.9732 \\ &= -0.33 - 2.16 \\ &= -2.49\end{aligned}$$

Theta per trading day is: –2.49 / 252 = –0.00988.

Figure 62.3: Theta as a Function of Stock Price and Time to Expiration



The specific characteristics of theta are as follows:

- Theta affects the value of put and call options in a similar way (e.g., as time passes, most call and put options decrease in value, all else equal).
- Theta varies with changes in stock prices and as time passes.
- Theta is most pronounced when the option is at the money, especially nearer to expiration. The left side of Figure 62.3 illustrates this relationship.
- Theta values are usually negative, which means the value of the option decreases as it gets closer to expiration.
- Theta usually increases in absolute value as expiration approaches. The right side of Figure 62.3 illustrates this relationship.
- It is possible for a European put option that is in the money to have a positive theta value.

Gamma

Gamma, Γ, represents the expected change in the delta of an option. It measures the curvature of the option price function not captured by delta (see Figure 62.1). The specific mathematical relationship for gamma is:

$$\Gamma = \frac{\partial^2 c}{\partial s^2}$$

where:

$\partial^2 c$ and ∂s^2 = second partial derivatives of the call and stock prices, respectively

The calculation of gamma for European call or put options on non-dividend-paying stocks can also be found using the following formula, where N'(x) is calculated in the same fashion as it is for theta.

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

EXAMPLE: Computing gamma

Suppose that Stock XYZ is trading at $50, and there is a call option that trades on XYZ with an exercise price of $45, which expires in three months. The risk-free rate is 5%, and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's gamma. Assume d_1 is 1.99 and $N(d_1)$ is 0.9767.

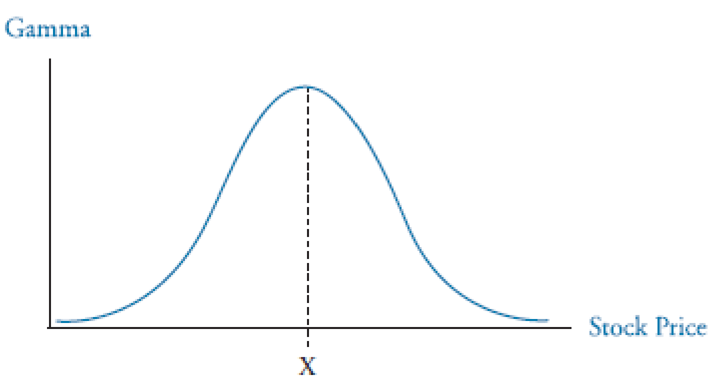
Answer:

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} = \frac{0.055}{50 \times 0.12 \times \sqrt{0.25}} = \frac{0.055}{3} = 0.0183$$

Gamma measures the rate of change in the option's delta, so for a $1 change in the price of the stock, the delta will change by 0.0183.

Figure 62.4 illustrates the relationship between gamma and the stock price for a stock option. As indicated in Figure 62.4, gamma is largest when an option is at the money (at stock price = X). When an option is deep in the money or out of the money, changes in stock price have little effect on gamma.

Figure 62.4: Gamma vs. Stock Price



When gamma is large, delta will be changing rapidly. On the other hand, when gamma is small, delta will be changing slowly. Since gamma represents the curvature component of the call price function not accounted for by delta, it can be used to minimize the *hedging error* associated with a linear relationship (delta) to represent the curvature of the call price function.

Delta-neutral positions can hedge the portfolio against small changes in stock price, while gamma can help hedge against relatively large changes in stock price. Therefore, it is not only desirable to create a delta-neutral position, but also to create one that is **gamma neutral**. In that way, neither small nor large stock price changes adversely affect the portfolio's value.

Since underlying assets and forward instruments generate linear payoffs, they have zero gamma and, hence, cannot be employed to create gamma-neutral positions. Gamma-neutral positions have to be created using instruments that are not linearly related to the underlying instrument, such as options. The specific relationship that determines the number of options that must be added to an existing portfolio to generate a gamma-neutral position is $-(\Gamma_p/\Gamma_T)$, where Γ_p is the gamma of the existing portfolio position, and Γ_T is the gamma of a traded option that can be added. Let's take a look at an example.

EXAMPLE: Creating a gamma-neutral position

Suppose an existing short option position is delta neutral but has a gamma of –6,000. Here, gamma is negative because we are short the options. Also, assume that there exists a traded call option with a delta of 0.6 and a gamma of 1.25. **Create** a gamma-neutral position.

Answer:

To gamma-hedge, we must buy 4,800 options (6,000 / 1.25). Now, the position is gamma neutral, but the added options have changed the delta position of the portfolio from 0 to 2,880 (= 4,800 × 0.6). This means that 2,880 shares of the underlying position will have to be sold to maintain not only a gamma-neutral position, but also the original delta-neutral position.

Relationship Among Delta, Theta, and Gamma

Stock option prices are affected by delta, theta, and gamma as indicated in the following relationship:

$$r\Pi = \Theta + rS\Delta + 0.5\sigma^2 S^2 \Gamma$$

where:

r = risk-neutral, risk-free rate of interest
Π = price of the option
Θ = option theta
S = price of the underlying stock
Δ = option delta
σ^2 = variance of the underlying stock
Γ = option gamma

This equation shows that the change in the value of an option position is directly affected by its sensitivities to the Greeks.

For a delta-neutral portfolio, $\Delta = 0$, so the preceding equation reduces to:

$$r\Pi = \Theta + 0.5\sigma^2 S^2 \Gamma$$

The left side of the equation is the dollar risk-free return on the option (risk-free rate times option value). Assuming the risk-free rate is small, this demonstrates that for large positive values of theta, gamma tends to be large and negative, and vice versa, which explains the common practice of using theta as a proxy for gamma.

Vega

PROFESSOR'S NOTE

Vega is not actually a letter of the Greek alphabet, but we still call vega one of the Greeks in option pricing.

Vega measures the sensitivity of the option's price to changes in the volatility of the underlying stock. For example, a vega of 8 indicates that for a 1% increase in volatility, the option's price will increase by 0.08. For a given maturity, exercise price, and risk-free rate, the vega of a call is equal to the vega of a put.

Vega for a call option is calculated using the following equation:

$$\text{vega} = \frac{\partial c}{\partial \sigma}$$

where:

∂c = change in the call price

$\partial \sigma$ = change in volatility

Vega for European calls and puts on non-dividend-paying stocks is calculated as:

$$\text{vega} = S_0 N'(d_1)\sqrt{T}$$

EXAMPLE: Computing vega

Suppose that Stock XYZ is trading at $50, and there is a call option that trades on XYZ with an exercise price of $45, which expires in three months. The risk-free rate is 5%, and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's vega. Assume d_1 is 1.99 and $N(d_1)$ is 0.9767.

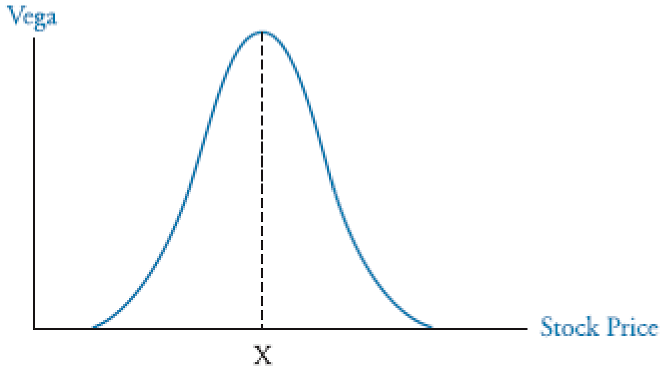
Answer:

$$\text{vega} = S_0 N'(d_1)\sqrt{T} = 50 \times 0.055 \times \sqrt{0.25} = 1.375$$

The interpretation for this value is that for a 1% increase in the volatility of the option (in this example, 12% to 13%), the value of the option will increase by approximately 0.01 × 1.375 = 0.01375.

Options are most sensitive to changes in volatility when they are at the money. Deep out-of-the-money or deep in-the-money options have little sensitivity to changes in volatility (i.e., vega is close to zero). The diagram in Figure 62.5 illustrates this behavior.

Figure 62.5: Vega of a Stock Option



Rho

Rho, ρ, measures an option's sensitivity to changes in the risk-free rate. Keep in mind, however, that equity options are not as sensitive to changes in interest rates as they are to changes in the other variables (e.g., volatility and stock price). Large changes in rates have only small effects on equity option prices. Rho is a much more important risk factor for fixed-income derivatives.

Rho for a call option is calculated using the following equation:

$$\text{rho} = \frac{\partial c}{\partial r}$$

where:
∂c = change in the call price
∂r = change in interest rate

In-the-money calls and puts are more sensitive to changes in rates than out-of-the-money options. Increases in rates cause larger *increases* for in-the-money call prices (vs. out-of-the-money calls) and larger *decreases* for in-the-money puts (vs. out-of-the-money puts).

For European calls on a non-dividend-paying stock, rho is measured as:

$$\text{rho(call)} = XTe^{-rT}\,N(d_2)$$

For European puts, rho is:

$$\text{rho(put)} = -XTe^{-rT}\,N(-d_2)$$

EXAMPLE: Computing rho

Suppose that Stock XYZ is trading at $50, and there is a call option that trades on XYZ with an exercise price of $45, which expires in three months. The risk-free rate is 5%, and the standard deviation of returns is 12% annualized. **Determine** the value of the call option's rho. Assume d_2 is 1.93 and $N(d_2)$ is 0.9732.

Answer:

$$\text{rho(call)} = 45 \times 0.25 \times e^{-0.05\times 0.25} \times 0.9732 = 10.81$$

Similar to the interpretation of vega, a 1% increase in the risk-free rate (from 5% to 6%) will increase the value of the call option by approximately 0.01 × 10.81 = 0.1081.

Greek Letters for a Portfolio

LO 62.h: Calculate the delta, gamma, and vega of a portfolio.

The delta of a portfolio of options on a single underlying asset can be calculated as the weighted average delta of each option position in the portfolio:

$$\text{portfolio delta} = \Delta_p = \sum_{i=1}^{n} w_i \Delta_i$$

where:
w_i = portfolio weight of each option position
Δ_i = delta of each option position

Therefore, portfolio delta represents the expected change of the overall option portfolio value given a small change in the price of the underlying asset. Portfolio vega and portfolio gamma are also computed as the weighted sum of the corresponding vega and gamma positions.

Hedging Activities in Practice

One of the main problems facing options traders is the expense associated with trying to maintain positions that are neutral to the Greeks. Although delta-neutral positions can be created, it is not as easy to find securities at reasonable prices that can mitigate the negative effects associated with gamma and vega. Traders must also contend with trading limits on the Greeks. For example, a delta limit of $200,000 on an underlying asset price of $50 implies a limit of $4,000. A gamma limit of 300 implies that a $1 change in asset price cannot result in a delta change of more than 300. A vega limit of $50,000 per 1% implies a 1% change in volatility, which cannot lead to a change of more than $50,000 in the portfolio.

To make things somewhat more manageable, large financial institutions usually adjust to a delta-neutral position and then monitor exposure to the other Greeks. Two offsetting situations assist in this monitoring activity. First, institutions that have sold options to their clients are exposed to negative gamma and vega, which tend to become more negative as time passes (if the option stays close to the money). In contrast, when the options are initially sold at the money, the level of sensitivity to gamma and vega is highest, but as time passes, the options tend to go either in the money or out of the money. The farther in the money or out of the money an option becomes, the less the impact of gamma and vega on the delta-neutral position.

Additional Greeks calculated by traders include: charm (delta sensitivity over time), vanna (delta sensitivity to volatility), and vomma (vega sensitivity to an implied volatility change).

Scenario analysis involves calculating expected portfolio gains or losses over desired periods using different inputs for underlying asset price and volatility. In this way, traders can assess the impact of changing various factors individually, or simultaneously, on their overall position.

Portfolio Insurance

LO 62.i: Describe how to implement portfolio insurance and how this strategy compares with delta hedging.

Recall that delta hedging involves combining an initial investment with an opposite position. For example, if an investor has a short call option, they can hedge it with a long position in the underlying stock. Hedging a long position or portfolio with long put options (or synthetic put options) is known as **portfolio insurance**.

Portfolio insurance is the combination of (1) an underlying instrument and (2) either cash or a derivative that generates a floor value for the portfolio in the event that market values decline, while still allowing for upside potential in the event that market values rise.

The simplest way to create portfolio insurance is to buy put options on an underlying portfolio. In this case, any loss on the portfolio may be offset with gains on the long put position.

Simply buying puts on the underlying portfolio may not be feasible because the put options needed to generate the desired level of portfolio insurance may not be available. As an alternative to buying the puts, a synthetic put position can be created with index futures contracts. This is accomplished by selling index futures contracts in an amount equivalent to the proportion of the portfolio dictated by the delta of the required put option. The main reasons traders may prefer synthetically creating the portfolio insurance position with index futures include substantially lower trading costs and relatively higher levels of liquidity.

MODULE QUIZ 62.3

Use the following information to answer Questions 1 and 2.

A delta-neutral position exhibits a gamma of –3,200. An existing option with a delta equal to 0.5 exhibits a gamma of 1.5.

1. Which of the following will generate a gamma-neutral position for the existing portfolio?
   A. Buy 2,133 of the available options.
   B. Sell 2,133 of the available options.
   C. Buy 4,800 of the available options.
   D. Sell 4,800 of the available options.

2. Which of the following actions would have to be taken to restore a delta-neutral hedge to the gamma-neutral position?
   A. Buy 1,067 shares of the underlying stock.
   B. Sell 1,067 shares of the underlying stock.
   C. Buy 4,266 shares of the underlying stock.
   D. Sell 4,266 shares of the underlying stock.

3. Which of the following statements about the Greeks is true?
   A. Rho for fixed-income options is small.
   B. Call option deltas range from –1 to +1.

C. A vega of 10 suggests that for a 1% increase in volatility, the option price will increase by 0.10.
D. Theta is the most negative for out-of-the-money options.

4. An option with a strike price of $12 and a current stock price of $12 that has one week until expiration is likely to have a gamma to an option seller that is:
   A. positive and large.
   B. positive and small.
   C. negative and large.
   D. negative and small.

5. A portfolio consists of three options. Option 1 has a weighting of 20% and a delta of 0.75, Option 2 has a weighting of 35% and a delta of 0.45, and Option 3 has a weighting of 45% and a delta of 0.60. The portfolio delta is closest to:
   A. 0.27.
   B. 0.58.
   C. 0.60.
   D. 1.80.

6. Portfolio insurance payoffs would not involve which of the following?
   A. Selling call options in the proportion 1/delta.
   B. Buying put options one to one relative to the underlying.
   C. Buying and selling the underlying in the proportion of delta of a put.
   D. Buying and selling futures in the proportion of delta of a put.

KEY CONCEPTS

LO 62.a

A naked call option is written without owning the underlying asset, whereas a covered call is a short call option where the writer owns the underlying asset. Neither of these positions is a hedged position.

LO 62.b

Stop-loss trading strategies are designed to minimize losses in the event the price of the underlying exceeds the strike price of a short call option position.

LO 62.c

To completely hedge a long stock or short call position, an investor must purchase the number of shares of stock equal to delta times the number of options sold. Another term for being completely hedged is delta neutral.

A forward/futures contract position can easily be hedged with an offsetting underlying asset position with the same number of securities.

The delta of an option, Δ, is the ratio of the change in price of the call option, c, to the change in price of the underlying asset, s, for small changes in s.

LO 62.d

Delta-neutral hedges are sophisticated hedging methods that minimize changes in a portfolio's position due to changes in the underlying security.

Delta-neutral hedges are only appropriate for small changes in the underlying asset and need to be rebalanced when large changes in the asset's value occur.

LO 62.e

Theta, also referred to as the time decay of an option, measures the sensitivity of an option's price to decreases in time to expiration.

Gamma measures the sensitivity of an option's price to changes in the option's delta.

Vega measures the sensitivity of an option's price to changes in the underlying asset's volatility.

Rho measures the sensitivity of an option's price to changes in the level of interest rates.

LO 62.f

Gamma is used to correct the hedging error associated with delta-neutral positions by providing added protection against large movements in the underlying asset's price.

Gamma-neutral positions are created by matching the gamma of the portfolio with an offsetting option position.

LO 62.g

Theta, delta, and gamma directly affect the rate of return of an option portfolio.

Stock option prices are affected by delta, theta, and gamma as indicated in the following relationship:

$$r\Pi = \Theta + rS\Delta + 0.5\sigma^2S^2\Gamma$$

where:

r = risk-neutral risk-free rate of interest
Π = price of the option
Θ = option theta
S = price of the underlying stock
Δ = option delta
σ^2 = variance of the underlying stock
Γ = option gamma

LO 62.h

The delta, vega, and gamma of a portfolio are a weighted average of the respective deltas, vegas, and gammas of each portfolio position.

LO 62.i

Portfolio insurance is the combination of (1) an underlying instrument and (2) either cash or a derivative that generates a floor value of the portfolio in the event that market valuations decline, while allowing for upside potential in the event that market valuations rise.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 62.1

1. **B** Selling a call without the underlying stock to support it is called a naked call position, and the investor who sells the call is therefore vulnerable to the

underlying stock increasing above the strike price. The higher the price goes above the strike price, the more likely it is that the call will be exercised and the investor will then have to go out into the market and buy the stock (at now higher prices) to cover the call. The seller of the option receives the premium, so there will not be a loss on the premium paid for the seller. (LO 62.a)

2. **B** Stop-loss strategies with call options are designed to limit the losses associated with short option positions. The strategy requires purchasing the underlying asset for a naked call position when the asset rises above the option's strike price. (LO 62.b)

Module Quiz 62.2

1. **B** This question does not require any calculations, as the relationship between a forward and the underlying asset is one to one, making the delta equal to exactly 1.00. (LO 62.c)

2. **D** To hedge a short call option position, a manager would have to buy enough of the underlying to equal the delta times the number of options sold. In this case, delta = 0.5, so for every two options sold, the manager would have to buy a share of the underlying security. (LO 62.c)

3. **A** An increase in the underlying stock price from $4 to $6 is not only the largest dollar change of the choices given, but also it is the largest percentage change. Static hedging (the hedge-and-forget strategy) is only effective when there are small changes in the stock price. To protect against larger changes, dynamic hedging needs to be deployed. (LO 62.d)

Module Quiz 62.3

1. **A** To create a gamma-neutral position, a manager must add the appropriate number of options that equals the existing portfolio gamma position. In this case, the existing gamma position is –3,200, and an available option exhibits a gamma of 1.5, which translates into buying approximately 2,133 options (= 3,200 / 1.5). (LO 62.f)

2. **B** The gamma-neutral hedge requires the purchase of 2,133 options, which will then increase the delta of the portfolio to 1,067 (= 2,133 × 0.5). Therefore, this would require selling approximately 1,067 shares to maintain a delta-neutral position. (LO 62.f)

3. **C** Theta is the most negative for at-the-money options. Call option deltas range from 0 to 1. A vega of 10 suggests that for a 1% increase in volatility, the option price will increase by 0.10. Rho for equity options is small. (LO 62.e)

4. **C** Gamma is the most negative for at-the-money options near expiration for an option *seller*. An option with a strike price and current price of $12 will be at the money. (LO 62.g)

5. **B** The portfolio delta is a weighted average of the individual option deltas, calculated as follows: (0.20)(0.75) + (0.35)(0.45) + (0.45)(0.60) = 0.58. (LO 62.h)

6. **A** Portfolio insurance can be created by all of the statements except selling call options in the proportion 1/delta. This action generates a delta-neutral hedge, not portfolio insurance. (LO 62.i)

FORMULAS

Reading 47

mean: $\mu_P = w_1\mu_1 + w_2\mu_2$

standard deviation: $\sigma_P = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho}$

delta-normal VaR: $\text{VaR} = (\mu - z\sigma) \times \text{portfolio value}$

expected shortfall: $ES = \mu + \sigma\dfrac{e^{-(z^2/2)}}{(1-x)\sqrt{2\pi}}$

Reading 48

delta:

$$\delta = \frac{\Delta P}{\Delta S}$$

where:
ΔP = change in portfolio
ΔS = change in risk factor

$\text{VaR}(T, X) = \text{VaR}(1, X) \times \sqrt{T}$
$\text{ES}(T, X) = \text{ES}(1, X) \times \sqrt{T}$

where:
T = *T*-day time horizon

Reading 49

exponentially weighted moving average (EWMA) model:

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)r_{n-1}^2$$

where:
λ = weight on previous volatility estimate (λ is a positive constant between zero and one)

GARCH (1,1) model:

$$\sigma_n^2 = \omega + \alpha r_{n-1}^2 + \beta\sigma_{n-1}^2$$

where:
α = weighting on the previous period's return
β = weighting on the previous variance estimate
ω = weighted long-run variance = γV_L
V_L = long-run average variance = $\dfrac{\omega}{1-\alpha-\beta}$
$\alpha + \beta + \gamma = 1$
$\alpha + \beta < 1$ for stability so that γ is not negative

Reading 52

expected loss: $EL = EAD \times PD \times LGD$

standard deviation of credit loss: $\sigma = \sqrt{PD - PD^2} \times [L(1 - RR)]$

standard deviation of credit loss as percentage of size: $\alpha = \dfrac{\sigma_P}{nL} = \dfrac{\sigma\sqrt{1 + (n-1)\rho}}{\sqrt{n} \times L}$

unexpected loss:

$UL = (WCDR - PD) \times LGD \times EAD$

where:

WCDR = worst-case default rate, or the 99.9 percentile of the default rate distribution

PD = probability of default

LGD = loss given default (or 1 – recovery rate)

EAD = exposure at default, or the sum of all loan exposures at default

Reading 55

clean price = dirty price – accrued interest

bond pricing:

$$P = \frac{C}{(1+y)^w} + \frac{C}{(1+y)^{1+w}} + \frac{C}{(1+y)^{2+w}} + \ldots + \frac{C}{(1+y)^{n-1+w}} + \frac{M}{(1+y)^{n-1+w}}$$

where:

P = bond price

C = semiannual coupon

y = periodic required yield

n = number of periods remaining

M = par value of the bond

w = number of days until the next coupon payment divided by the number of days in the coupon period

Reading 56

compounding frequencies:

$$R_2 = \left[\left(1 + \frac{R_1}{m_1}\right)^{m_1/m_2} - 1\right] \times m_2$$

where:

R = rate

m = number of times per year the rate is applied

discount factor: $d(t) = \left(1 + \dfrac{r(t)}{2}\right)^{-2t}$

forward rate: $F = \dfrac{R_2T_2 - R_1T_1}{T_2 - T_1}$

Reading 57

realized return: $R_{t-1,t} = \frac{BV_t + C_t - BV_{t-1}}{BV_{t-1}}$

bond price:

$$P = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \ldots + \frac{C_N}{(1+y)^N}$$

where:
P = price of the security (i.e., present value [PV])
C_k = annual cash flow in year k, which for a bond would include the principal at maturity
N = term to maturity in years
y = annual yield or YTM on the security

perpetuity price:

$$\text{PV of a perpetuity} = \frac{C}{y}$$

where:
C = cash flow that will occur every period into perpetuity
y = yield to maturity

Reading 58

DV01:

$$DV01 = -\frac{\Delta P}{\Delta y}$$

where:
ΔP = change in the value of the portfolio
Δy = size of a parallel shift in the interest rate term structure

hedge ratio: $HR = \frac{\text{DV01 (of position to be hedged)}}{\text{DV01 (of hedging instrument)}}$

duration: $D = -\frac{\Delta P / P}{\Delta y} = -\frac{\Delta P}{P \Delta y}$

convexity:

$$C = \frac{1}{P}\left[\frac{P^+ + P^- - 2P}{(\Delta y)^2}\right] = \left[\frac{P^+ + P^- - 2P}{P(\Delta y)^2}\right]$$

where:
C = convexity
P = initial bond price
P^+ = new (lower) bond price when rates increase
P^- = new (higher) bond price when rates decline

percentage price change ≈ duration effect + convexity effect

$$\Delta P = -D \times P \times \Delta y + \frac{1}{2} \times C \times P \times \Delta y^2$$

$$\text{duration of portfolio} = \sum_{j=1}^{K} w_j \times D_j$$

where:
D_j = duration of bond j
w_j = market value of bond j divided by market value of the total bond portfolio
K = number of bonds in portfolio

Reading 59

$$\text{key rate '01: } DV01^k = \frac{1}{10{,}000} \frac{\Delta BV}{\Delta y^k}$$

$$\text{key rate duration: } D^k = \frac{1}{BV} \frac{\Delta BV}{\Delta y^k}$$

Reading 60

$$U = \text{size of the up-move factor} = e^{\sigma\sqrt{t}}$$

$$D = \text{size of the down-move factor} = e^{-\sigma\sqrt{t}} = \frac{1}{e^{\sigma\sqrt{t}}} = \frac{1}{U}$$

where:
σ = annual volatility of the underlying asset's returns
t = length of the step in the binomial model

$$\pi_u = \text{probability of an up move} = \frac{e^{rt} - D}{U - D}$$

$$\pi_d = \text{probability of a down move} = 1 - \pi_u$$

where:
r = continuously compounded annual risk-free rate

Reading 61

Black-Scholes-Merton option pricing model:

$$c_0 = \left[S_0 \times N(d_1)\right] - \left[X \times e^{-R_f^c \times T} \times N(d_2)\right]$$

$$p_0 = \left\{X \times e^{-R_f^c \times T} \times \left[1 - N(d_2)\right]\right\} - \left\{S_0 \times \left[1 - N(d_1)\right]\right\}$$

where:

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left[R_f^c + (0.5 \times \sigma^2)\right] \times T}{\sigma \times \sqrt{T}}$$

$$d_2 = d_1 - (\sigma \times \sqrt{T})$$

T = time to maturity (as % of a 365-day year)
S_0 = asset price
X = exercise price
R_f^c = continuously compounded risk-free rate
σ = volatility of continuously compounded returns on the stock
$N(\bullet)$ = cumulative normal probability

put-call parity:

$$c_0 = p_0 + S_0 - (X \times e^{-R_f^c \times T})$$

or

$$p_0 = c_0 - S_0 + (X \times e^{-R_f^c \times T})$$

valuation of warrants:

$$\frac{N}{N + M} \times \text{value of regular call option}$$

where:
N = number of shares outstanding
M = number of new warrants issued

continuously compounded returns: $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$

Reading 62

delta:

$$\text{delta} = \Delta = \frac{\partial c}{\partial s}$$

where:
∂c = change in the call option price
∂s = change in the stock price

theta:

$$\Theta = \frac{\partial c}{\partial t}$$

where:
∂c = change in the call price
∂t = change in time

gamma:

$$\Gamma = \frac{\partial^2 c}{\partial s^2}$$

where:
$\partial^2 c$ and ∂s^2 = second partial derivatives of the call and stock prices, respectively

relationship among delta, theta, and gamma:

$$r\prod = \Theta + rS\Delta + 0.5\sigma^2 S^2 \Gamma$$

where:
r = risk-neutral, risk-free rate of interest
$\prod$ = price of the option
Θ = option theta
S = price of the underlying stock
Δ = option delta
σ^2 = variance of the underlying stock
Γ = option gamma

vega:

$$\text{vega} = \frac{\partial c}{\partial \sigma}$$

where:
∂c = change in the call price
$\partial \sigma$ = change in volatility

rho:

$$\text{rho} = \frac{\partial c}{\partial r}$$

where:
∂c = change in the call price
∂r = change in interest rate

portfolio delta:

$$\text{portfolio delta} = \Delta_p = \sum_{i=1}^{n} w_i \Delta_i$$

where:
w_i = portfolio weight of each option position
Δ_i = delta of each option position

APPENDIX

USING THE CUMULATIVE *Z*-TABLE

Probability Example

Assume that the annual earnings per share (EPS) for a large sample of firms is normally distributed with a mean of $5.00 and a standard deviation of $1.50. What is the approximate probability of an observed EPS value falling between $3.00 and $7.25?

If EPS = x = $7.25, then $z = (x - \mu) / \sigma = (\$7.25 - \$5.00) / \$1.50 = +1.50$.

If EPS = x = $3.00, then $z = (x - \mu) / \sigma = (\$3.00 - \$5.00) / \$1.50 = -1.33$.

For z-value of 1.50: Use the row headed 1.5 and the column headed 0 to find the value 0.9332. This represents the area under the curve to the left of the critical value 1.50.

For z-value of –1.33: Use the row headed 1.3 and the column headed 3 to find the value 0.9082. This represents the area under the curve to the left of the critical value +1.33. The area to the left of –1.33 is 1 – 0.9082 = 0.0918.

The area between these critical values is 0.9332 – 0.0918 = 0.8414, or 84.14%.

Hypothesis Testing—One-Tailed Test Example

A sample of a stock's returns on 36 non-consecutive days results in a mean return of 2.0%. Assume the population standard deviation is 20.0%. Can we say with 95% confidence that the mean return is greater than 0%?

H_0: $\mu \leq 0.0\%$, H_A: $\mu > 0.0\%$. The test statistic = *z*-statistic = $\dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$
= (2.0 – 0.0) / (20.0 / 6) = 0.60.

The significance level = 1.0 – 0.95 = 0.05, or 5%.

Since this is a one-tailed test with an alpha of 0.05, we need to find the value 0.95 in the cumulative *z*-table. The closest value is 0.9505, with a corresponding critical z-value of 1.65. Since the test statistic is less than the critical value, we fail to reject H_0.

Hypothesis Testing—Two-Tailed Test Example

Using the previous assumptions, suppose that the analyst now wants to determine with 99% confidence that the stock's return is not equal to 0.0%.

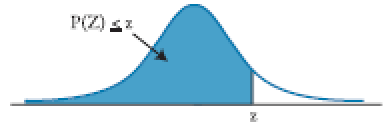
H_0: $\mu = 0.0\%$, H_A: $\mu \neq 0.0\%$. The test statistic (*z*-value) = (2.0 – 0.0) / (20.0 / 6) = 0.60. The significance level = 1.0 – 0.99 = 0.01, or 1%.

Since this is a two-tailed test with an alpha of 0.01, there is a 0.005 rejection region in both tails. Thus, we need to find the value 0.995 (1.0 – 0.005) in the table. The closest value is 0.9951, which corresponds to a critical *z*-value of 2.58. Since the test statistic is less than the critical value, we fail to reject H_0 and conclude that the stock's return equals 0.0%.

Cumulative *Z*-Table

$P(Z \le z) = N(z)$ for $z \ge 0$

$P(Z \le -z) = 1 - N(z)$



z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

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