Incremental Nonlinear Dynamic Inversion Control for a Missile

Zhidong Lu
Technical University of Munich, Garching, Germany
Started on 25.04.2025, finished on 27.04.2025

I. Introduction

This report present the derivation and application of an Incremental Nonlinear Dynamic Inversion (INDI) control law for normal acceleration tracking of a missile. The objective is to design a robust control system that can accurately track normal acceleration commands under varying flight conditions. Its performance is compared to a gain-scheduled cascaded controller. Due to limited preparation time, only preliminary results and comparison have been performed in this report.

II. Missile Dynamics Model

The missile dynamics used in this analysis are based on a hypothetical tail-controlled missile, previously adopted as a baseline in nonlinear control research [1, 2]. The model is representative of a missile traveling at Mach 3 at an altitude of 20,000 ft; however, it does not correspond to any specific operational airframe. The missile model assumes a constant mass (post-burnout), zero roll rate, zero roll angle, no side-slip, and no yaw rate. Under these assumptions, the longitudinal nonlinear equations of motion for a rigid airframe reduce to two force equations, one moment equation, and three kinematic equations. Using body-axis components, the six equations are given as:

$$\dot{u} + qw = \sum F_{Bx}/m \tag{1}$$

$$\dot{w} - qu = \sum F_{Bz}/m \tag{2}$$

$$\dot{q} = \sum M_Y/I_Y \tag{3}$$

$$\dot{\theta} = q \tag{4}$$

Assuming a flat Earth model with the positive z-axis directed downward, the forces and moment about the center of gravity are defined as:

$$\sum F_{Bx} = F_A - mg\sin\theta \tag{5}$$

$$\sum F_{Bz} = F_N + mg\cos\theta \tag{6}$$

$$\sum M_Y = M \tag{7}$$

The axial force, normal force and pitching moment are expressed as:

$$F_A = \frac{1}{2}\rho V^2 S_{\mathrm{ref}} C_A, \quad F_N = \frac{1}{2}\rho V^2 S_{\mathrm{ref}} C_N, \quad M = \frac{1}{2}\rho V^2 S_{\mathrm{ref}} d_{\mathrm{ref}} C_m \tag{8}$$

where the aerodynamic coefficients are modeled by:

$$C_A = C_A 0(\alpha, Ma) + C_{A\delta}\delta \tag{9}$$

$$C_N = C_N 0(\alpha, Ma) + C_{N\delta}\delta \tag{10}$$

$$C_m = C_m 0(\alpha, Ma) + C_{m\delta}\delta + C_{mq} q \tag{11}$$

The reference area and reference distance are

$$S_{\mathrm{ref}} = 0.04087\mathrm{m}^2, \quad d_{\mathrm{ref}} = 0.2286\mathrm{m}$$

Some coefficients are regraded constant and their numeric values are

$$C_A 0(\alpha, Ma) = -0.3, \quad C_{A\delta} = 0, \quad C_{N\delta} = -1.9481, \quad C_{m\delta} = -11.803, \quad C_{mq} = -1.719$$

The angle of attack α, airspeed V_a, and Mach number Ma are computed as:

$$\alpha = \text{atan2}(w, u), \quad V_a = \sqrt{u^2 + w^2}, \quad Ma = \frac{V_a}{a_s(h)} \tag{12}$$

where $a_s(h)$ is the speed of sound as a function of altitude h.

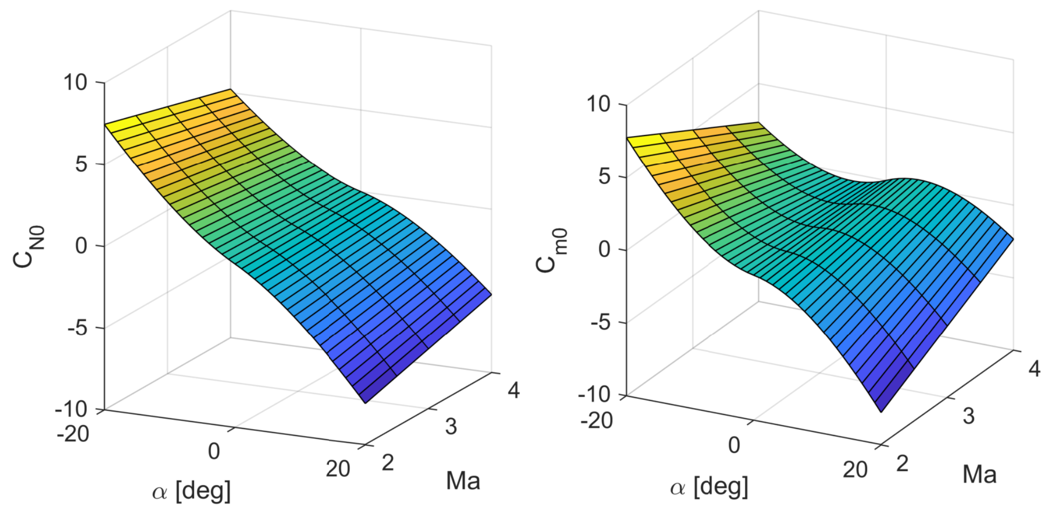

Fig. 1 Basic aerodynamic coefficients varying with angle of attack and Mach number.

Actuator dynamics are also included in the model. The actuator is represented by a second-order system:

$$\begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_a^2 & -2\zeta\omega_a \end{bmatrix} \begin{bmatrix} \delta \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_a^2 \end{bmatrix} \delta_c \tag{13}$$

where $\zeta = 0.7$ is the damping ratio and $\omega_a = 150$ is the natural frequency (in rad/s). Saturation limits on actuator deflection and rate are also considered:

- Maximum deflection: $+30°$
- Minimum deflection: $-30°$
- Maximum deflection rate: $\pm 500°/\text{s}$

III. Gain Scheduled Controller

This section presents the gain-scheduled cascaded controller, which serves as the baseline control law. Across the entire flight envelope, a cascaded control structure is employed. This structure is illustrated in Fig. 2. The objective is to control the missile's normal acceleration a_z by adjusting the tail fin deflection δ (Fin demand).

The cascaded controller consists of three loops:

- **Pitch Rate Feedback Loop**: The innermost loop stabilizes the system by adding damping. It anticipates pitch motion and corrects deviations rapidly.
- **Pitch Rate Tracking Loop**: The second loop uses a pure integral gain KI_q to drive the pitch rate error to zero:

$$\delta_I(t) = KI_q \cdot \int (q_{\text{cmd}} - q_{\text{mea}})\, dt \tag{14}$$

- **Normal Acceleration Tracking Loop**: The outermost loop tracks the desired normal acceleration based on the stabilized pitch rate inner loop. It consists of a proportional gain KP_a on the normal acceleration error, and a feed-forward gain KF_a. The feedforward gain predicts the necessary fin deflection to achieve the desired acceleration under nominal conditions, improving transient response and reducing the burden on the feedback loops.

Since missile dynamics vary significantly with angle of attack α and Mach number Ma, fixed-gain controllers may become unstable at high α or underperform during transonic flight. Therefore, all controller gains (KI_q, KP_a, and KF_a) are scheduled as functions of (α, Ma) through lookup tables. Following the principle of "scheduling based on slow dynamics variables", two slow-varying parameters, namely Mach number and angle of attack, are selected as the scheduling variables. The resulting gain schedules are shown in Fig. 3. It can be observed that these gains vary significantly with Mach number.

To validate the controller, a simulation is conducted using a doublet command on normal acceleration. The missile is initialized at Mach 3 in horizontal flight, with zero pitch angle and zero angle of attack. Over the course of eight seconds, the missile accelerates to approximately Mach 3.5. The simulation results are presented in Fig. 4. Significant overshoot is observed in the normal acceleration tracking, as well as oscillations in the angle of attack, pitch rate, and fin deflection.

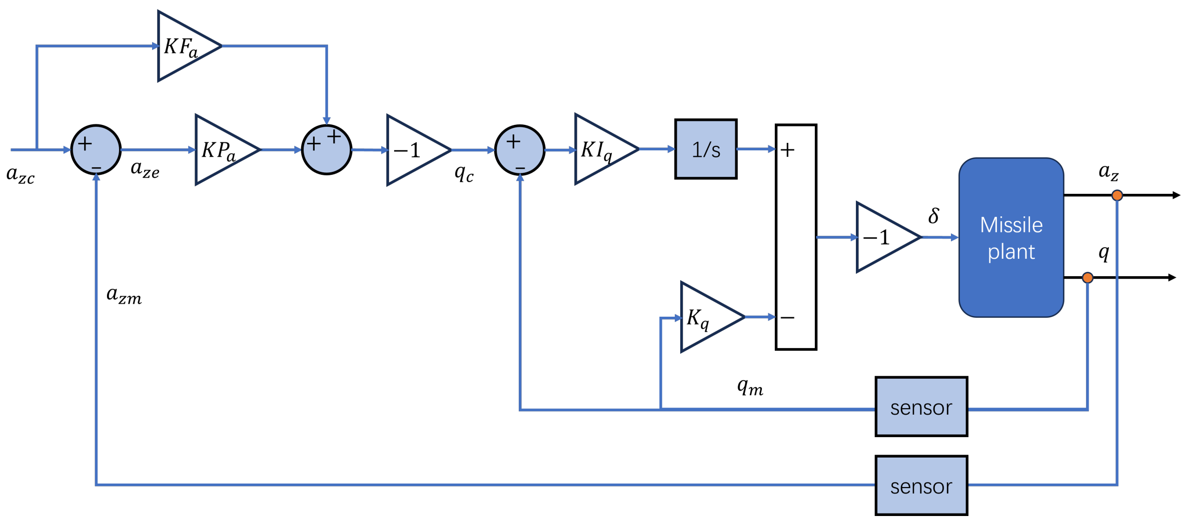

Fig. 2 Cascaded PID controller structure.

IV. Incremental Nonlinear Dynamic Inversion

A. Pitch Rate Inner Loop

The longitudinal pitch rate dynamics of the missile can be described as:

$$\dot{q} = \frac{M}{I_y} = \frac{1}{I_y}\left(\frac{1}{2}\rho V^2 S_{\text{ref}} d_{\text{ref}} C_m\right) \tag{15}$$

where the pitching moment coefficient C_m is modeled as:

$$C_m = C_{m_0}(\alpha, M) + C_{m_\delta}\delta + C_{m_q} q \tag{16}$$

Here, δ denotes the tail fin deflection angle, and C_{m_0}, C_{m_δ}, and C_{m_q} are aerodynamic coefficients that vary with the angle of attack α and Mach number Ma.

It can be observed that $\dot{q}$ is an explicit function of δ, and the system possesses a relative degree of one with respect to the control input. The pitch rate dynamics can therefore be rewritten in control-affine form as:

$$\dot{q} = g(\alpha, M, q, V) + B_\delta \delta \tag{17}$$

where $g(\cdot)$ aggregates all terms independent of δ.

The control effectiveness is defined as:

$$B_\delta = \frac{\partial \dot{q}}{\partial \delta} \tag{18}$$

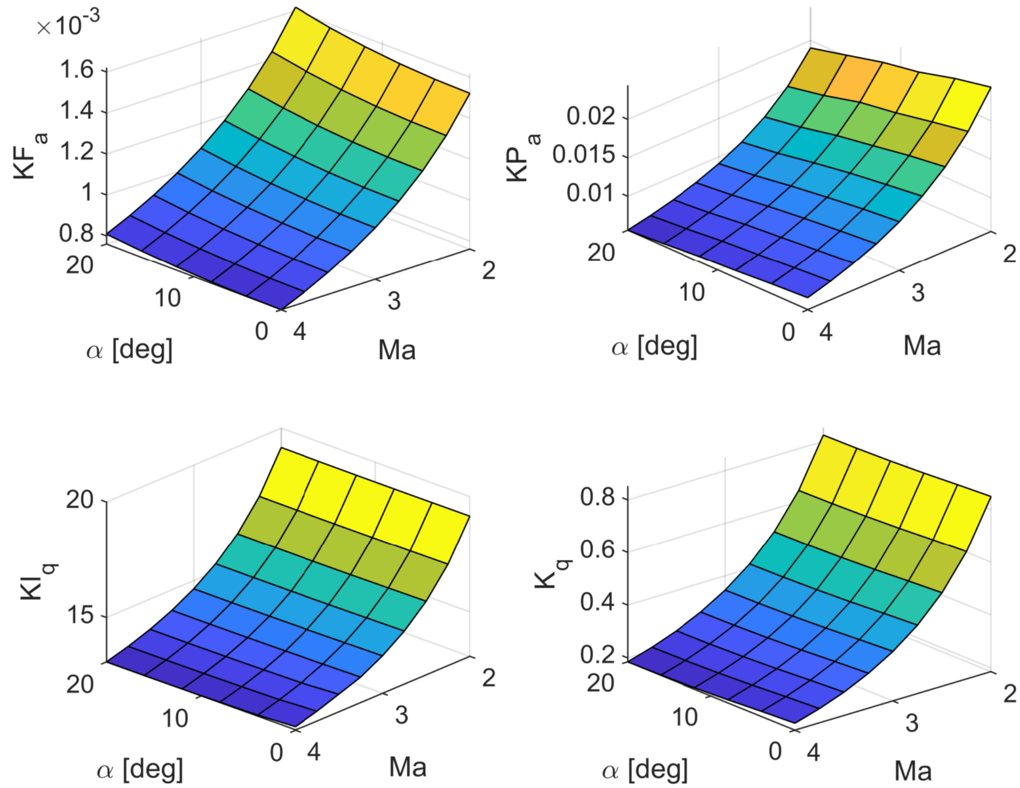

Fig. 3 Gain schedule with angle of attack and Mach number.

Expanding the derivatives:

$$\frac{\partial \dot{q}}{\partial C_m} = \frac{1}{2I_y}\left(\rho V^2 S_{\text{ref}} d_{\text{ref}}\right) \quad \text{and} \quad \frac{\partial C_m}{\partial \delta} = C_{m_\delta} \tag{19}$$

Thus, the control effectiveness becomes:

$$B_\delta = \frac{1}{2I_y}\left(\rho V^2 S_{\text{ref}} d_{\text{ref}}\right) C_{m_\delta} \tag{20}$$

Since the system is control-affine, the incremental form over a small time interval Δt can be approximated by:

$$\Delta \dot{q} \approx B_\delta \Delta \delta \tag{21}$$

where higher-order terms are neglected.

To specify the pitch rate command response, a first-order reference model is adopted, as shown in Fig. 6:

$$\dot{q}_{\text{ref}} = KR_q \left(q_{\text{cmd}} - q_{\text{ref}}\right) \tag{22}$$

Accordingly, the desired pitch rate acceleration command is formulated as:

$$\dot{q}_{\text{cmd}} = \dot{q}_{\text{ref}} + K_q \left(q_{\text{ref}} - q_{\text{est}}\right) \tag{23}$$

Due to physical limitations of the actuator, such as position and rate saturation, a reaction deficit in the virtual control may occur. For this pitch rate control loop, the reaction deficit is quantified by:

$$\dot{q}_{\text{h}} = \Delta \dot{q}_{\text{cmd}} - B_\delta \cdot \Delta \delta_{\text{cmd}} \tag{24}$$

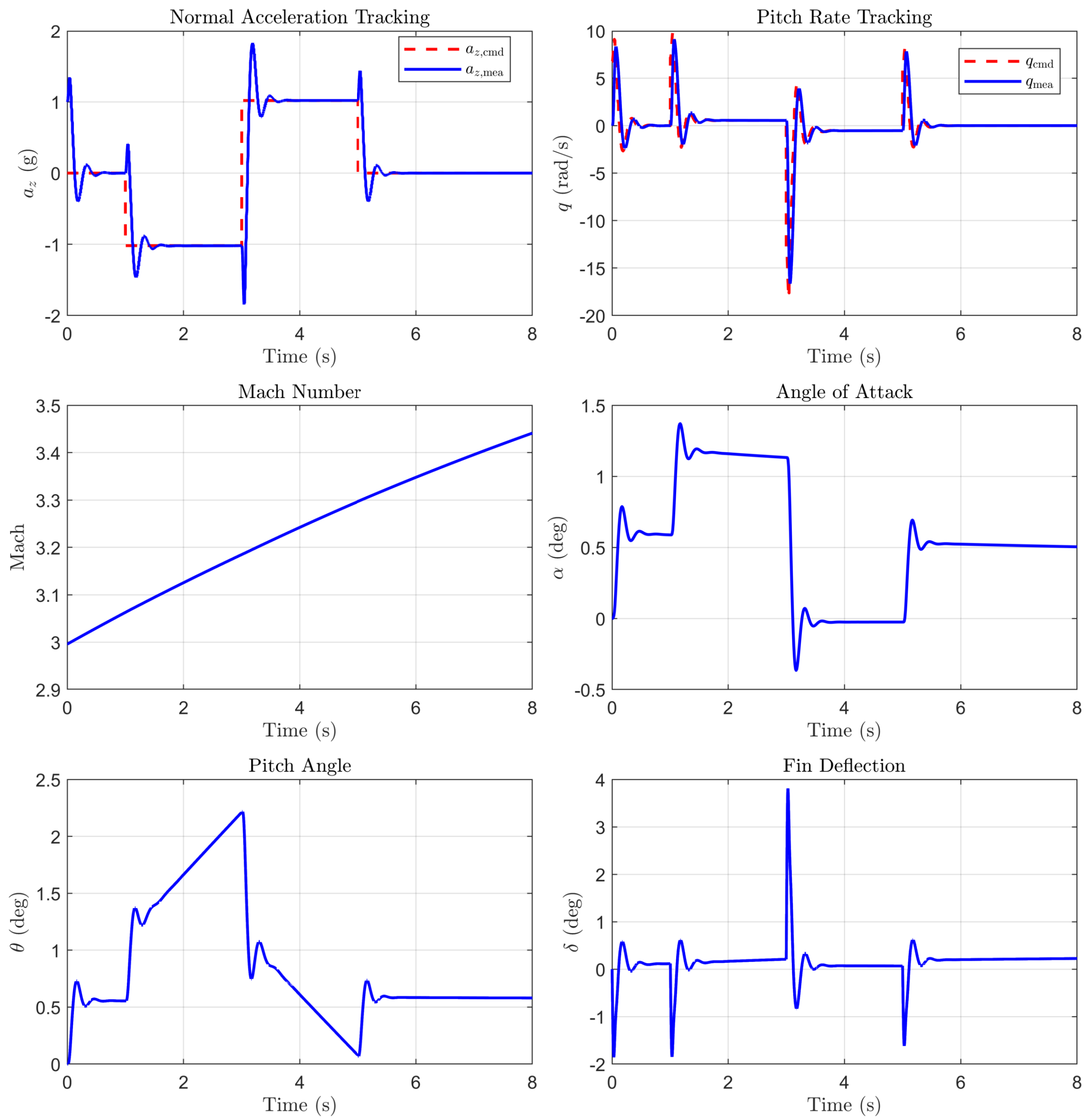

Fig. 4 Doublet a_z command tracking results using the gain scheduled PID controller.

This deficit is subtracted from the $\dot{q}$ reference signal, as shown in Fig. 6, to slow down the reference dynamics. This technique is called the "pseudo control hedging" [3, 4].

The desired increment in the control input is:

$$\Delta\delta_{\text{cmd}} = \text{sat}(\frac{1}{B_\delta}\Delta\dot{q}_{\text{cmd}}) = \max\left(\Delta\delta_{\min}, \min(\Delta\delta, \Delta\delta_{\max})\right) \tag{25}$$

where $\Delta\dot{q}_{\text{cmd}}$ is the desired change in pitch rate, $\Delta\delta_{\min}$ and $\Delta\delta_{\max}$ are the lower and upper bound of the incremental input, which is determined by the position and rate limits of the fin actuator.

Equation (25) serves as the control allocation module, determining the required incremental control input $\Delta\delta_{\text{cmd}}$ based on the estimated control effectiveness B_δ and the desired virtual command increment $\Delta\dot{q}_{\text{cmd}}$. Since only one control input and one virtual control command are considered in this case, the control allocation reduces to a simple scalar multiplication and saturation operation. For multi-input multi-output (MIMO) systems, more complex control allocation methods, such as Moore–Penrose pseudo-inverse techniques, would be necessary.

Finally, the total control law is expressed as:

$$\delta_{\mathrm{cmd}} = \delta_0 + \Delta\delta_{\mathrm{cmd}} = \delta_0 + \frac{1}{B_\delta}\Delta\dot{q}_{\mathrm{cmd}} \tag{26}$$

where δ_0 denotes the previously applied fin deflection input.

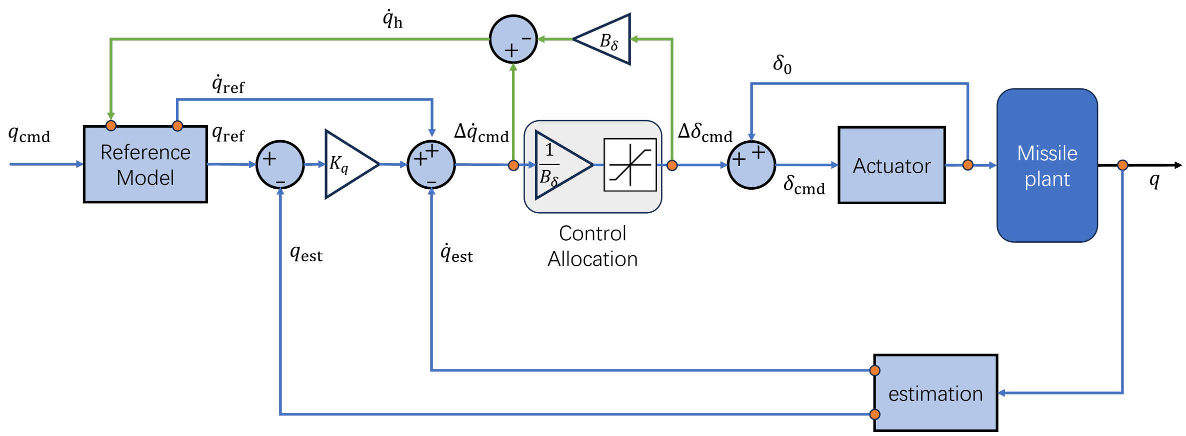

Fig. 5 Pitch rate INDI controller structure.

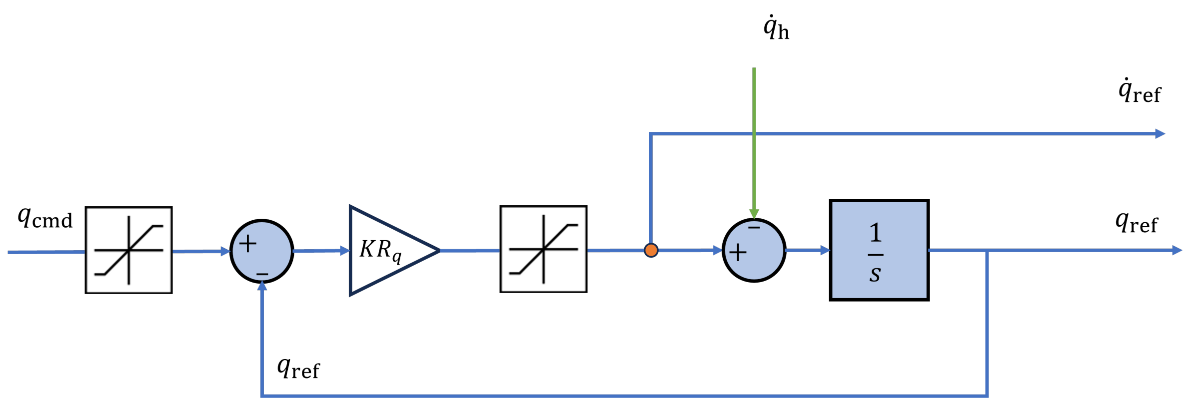

Fig. 6 Pitch rate reference model.

B. Normal Acceleration Loop

Classical nonlinear dynamic inversion (NDI) approaches require the system's input/output characteristics to be minimum phase. In general, tail-controlled missiles tend to exhibit non-minimum phase behavior when the IMU-measured normal acceleration is used as the output. In this section, a nonlinear dynamic inversion (NDI) control law for normal acceleration a_z tracking is derived.

The body-frame normal acceleration can be expressed as:

$$a_z = \frac{F_N}{m} + g\cos\theta = \frac{1}{m}\left(\bar{q}S_{\mathrm{ref}}C_N\right) + g\cos\theta \tag{27}$$

where $\bar{q} = \frac{1}{2}\rho V^2$ is the dynamic pressure.

Taking the time derivative yields:

$$\dot{a}_z = \frac{\bar{q}S_{\mathrm{ref}}}{m}\frac{\partial C_N}{\partial\alpha}\dot{\alpha} + g(-\sin\theta)\dot{\theta} \tag{28}$$

The angle of attack dynamics are given by:

$$\dot{\alpha} = \frac{a_z \cos\alpha - a_x \sin\alpha}{V} + q \tag{29}$$

where a_x is the body-frame longitudinal acceleration and a_z is the body frame normal acceleration.

Substituting $\dot{\alpha}$ and $\dot{\theta} = q$ into the derivative of a_z leads to:

$$\dot{a}_z = \frac{\bar{q}S_{\text{ref}}}{m}\frac{\partial C_N}{\partial \alpha}\left(\frac{a_z \cos\alpha - a_x \sin\alpha}{V}\right) + \left(\frac{\bar{q}S_{\text{ref}}}{m}\frac{\partial C_N}{\partial \alpha} - g\sin\theta\right) q \tag{30}$$

Grouping terms, the normal acceleration dynamics can be written in the control-affine form with respect to the pitch rate:

$$\dot{a}_z = f_{a_z}(x) + g_{a_z}(x) q \tag{31}$$

where

$$f_{a_z}(x) = \frac{\bar{q}S_{\text{ref}}}{m}\frac{\partial C_N}{\partial \alpha}\left(\frac{a_z \cos\alpha - a_x \sin\alpha}{V}\right)$$

and

$$g_{a_z}(x) = \frac{\bar{q}S_{\text{ref}}}{m}\frac{\partial C_N}{\partial \alpha} - g\sin\theta$$

Applying nonlinear dynamic inversion, the commanded pitch rate q_{cmd} to achieve a desired $\dot{a}_{z,\text{cmd}}$ is obtained as:

$$q_{\text{cmd}} = g_{a_z}(x)^{-1}\left(\dot{a}_{z,\text{cmd}} - f_{a_z}(x)\right)$$

that is,

$$q_{\text{cmd}} = \left(\frac{\bar{q}S_{\text{ref}}}{m}\frac{\partial C_N}{\partial \alpha} - g\sin\theta\right)^{-1}\left[\dot{a}_{z,\text{cmd}} - \frac{\bar{q}S_{\text{ref}}}{m}\frac{\partial C_N}{\partial \alpha}\frac{a_z \cos\alpha - a_x \sin\alpha}{V}\right]$$

To implement this dynamic inversion, estimates of $g_{a_z}(x)$ and $f_{a_z}(x)$ are required. It is assumed that the angle of attack α, velocity V, and accelerations a_x and a_z can be estimated or measured. The partial derivative $\frac{\partial C_N}{\partial \alpha}$ is estimated from the aerodynamic model. Fig. 7 shows the surface of $\frac{\partial C_N}{\partial \alpha}$, which is observed to be symmetric about zero angle of attack. A polynomial approximation of this partial derivative is fitted as:

$$\frac{\partial C_N}{\partial \alpha} = -19.5264 + 61.0047\|\alpha\| + 3.2391Ma + 55.6785\|\alpha\|^2 \tag{32}$$

The overall NDI control structure for normal acceleration is illustrated in Fig. 8.

Combined with the INDI pitch rate controller, a complete normal acceleration controller is designed. An advantage of the INDI controller is that it avoids tedious gain scheduling across different flight conditions. All error gains and reference model gains are kept constant throughout the flight envelope. The controller gains are set as:

$$KP_q = 30, \quad KR_q = 50, \quad KP_a = 3, \quad KR_a = 10 \tag{33}$$

A doublet normal acceleration command tracking simulation, similar to the one shown in Fig. 4, is conducted using the INDI controller. The simulation results are presented in Fig. 9. Compared to the gain-scheduled PID controller, the INDI controller exhibits:

- Smaller overshoot (0% compared to 40%)
- Similar settling time
- Lower pitch rate and smoother response without abrupt changes
- Reduced usage of fin deflection and smoother fin command behavior

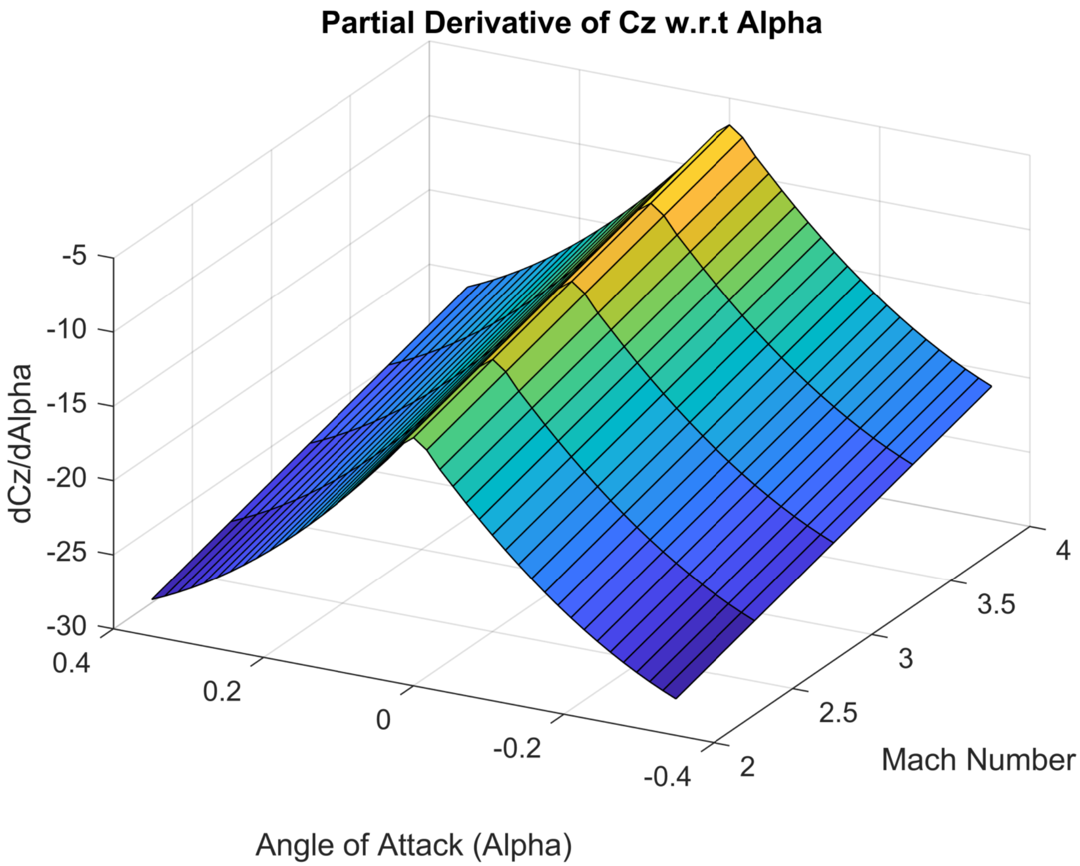

Fig. 7 Partial derivative $\frac{\partial C_N}{\partial \alpha}$.

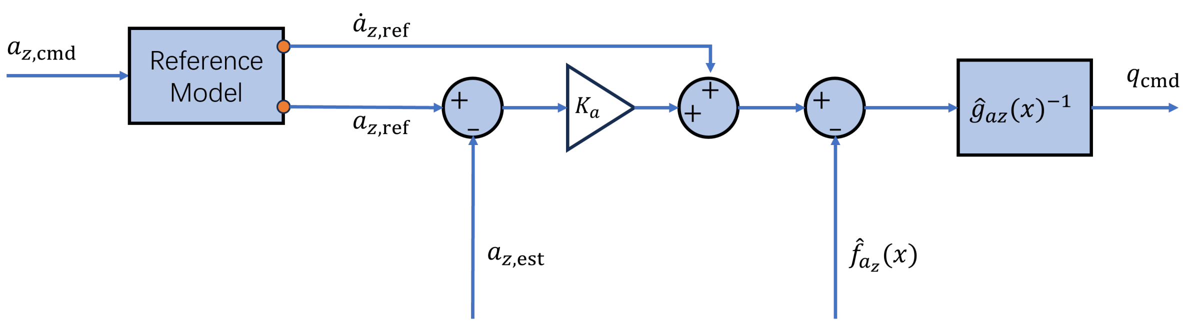

Fig. 8 Nonlinear dynamic inversion control for the normal acceleration.

V. Discussion

This report has introduced two control designs for a longitudinal missile model: a gain-scheduled controller and an incremental nonlinear dynamic inversion (INDI) controller. In the same doublet normal acceleration tracking simulation task, the INDI controller exhibits smoother responses, fewer oscillations or abrupt changes in pitch rate and angle of attack, and reduced usage of fin deflection.

Most importantly, the INDI controller eliminates the need for gain scheduling across the angle of attack and Mach number, as constant proportional gains and reference model gains are sufficient.

Further work is needed to provide a more comprehensive assessment of these two controllers, including:

- Evaluation of stability margins across the flight envelope
- Analysis of robustness under varying flight conditions and model uncertainties

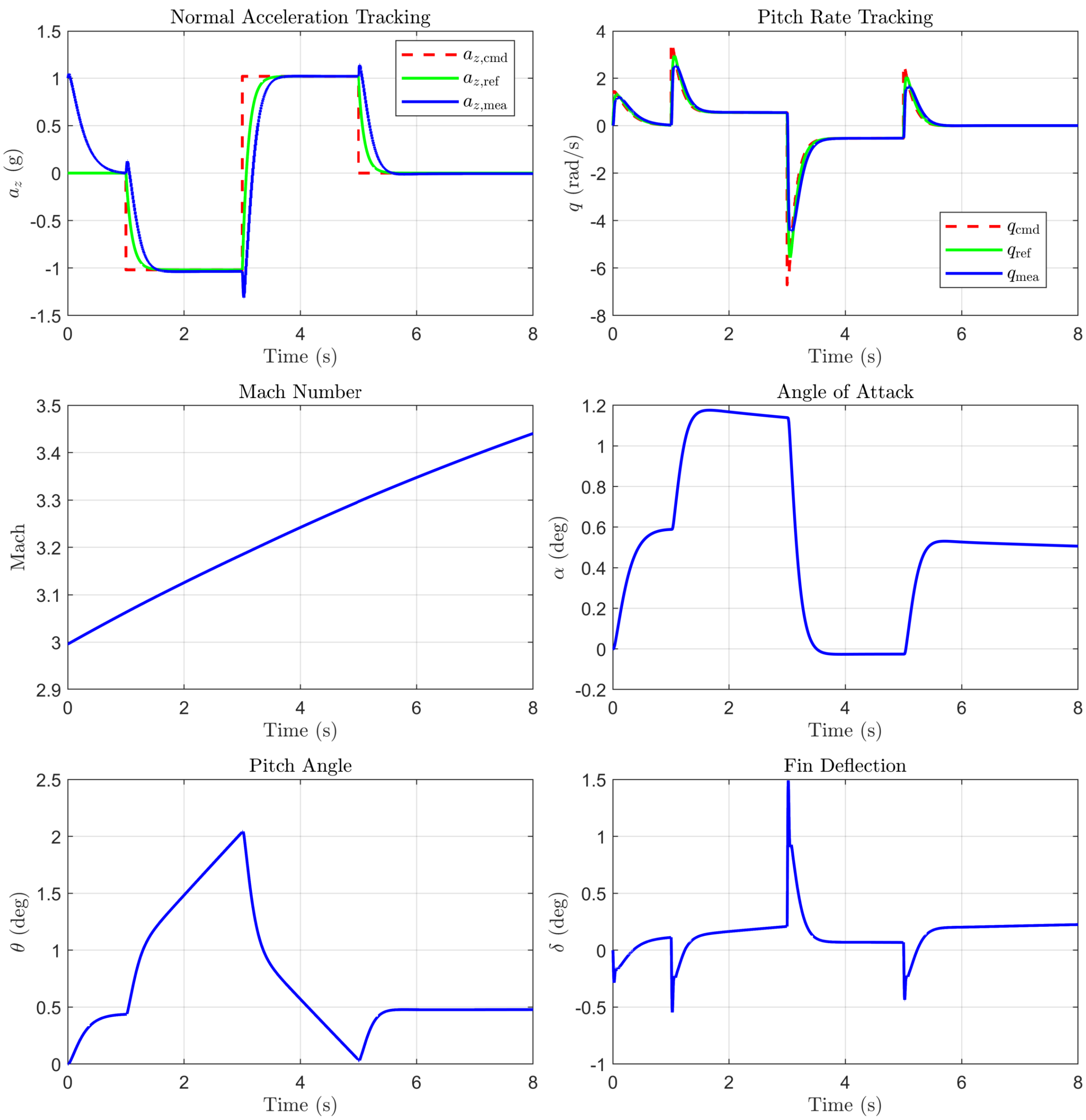

Fig. 9 Doublet a_z command tracking results using the INDI controller.

- Investigation of disturbance rejection capabilities, such as under wind gusts and atmospheric turbulence

References

[1] Shamma, J. S., and Cloutier, J. R., "Gain-scheduled missile autopilot design using linear parameter varying transformations," *Journal of guidance, Control, and dynamics*, Vol. 16, No. 2, 1993, pp. 256–263.

[2] Bennani, S., Willemsen, D., Scherer, C., Scherer, C., Bennani, S., and Willemsen, D., "Robust LPV control with bounded parameter rates," *Guidance, Navigation, and Control Conference*, 1997, p. 3641.

[3] Zhang, J., and Holzapfel, F., "Saturation Protection with Pseudo Control Hedging: A Control Allocation Perspective," *AIAA Scitech 2021 Forum*, 2021, p. 0370.

[4] Lu, Z., and Holzapfel, F., "Stability and performance analysis for SISO incremental flight control," *arXiv preprint arXiv:2012.00129*, 2020.