1. PCA, SVD

1) Find the SVD of X, $U\Sigma V^T$, where $X = \begin{pmatrix} 4 & 2 \\ \sqrt{2} & 2\sqrt{2} \\ -\sqrt{2} & -2\sqrt{2} \end{pmatrix}$

$$X^T X = \begin{pmatrix} 4 & \sqrt{2} & -\sqrt{2} \\ 2 & 2\sqrt{2} & -2\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 2 \\ \sqrt{2} & 2\sqrt{2} \\ \sqrt{2} & -2\sqrt{2} \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 18 \end{pmatrix}$$

Eigenvalues:

$\lambda_1 = 10$, $\lambda_2 = 18$

Eigenvectors:

$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, for $\lambda_1 = 10$

$V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, for $\lambda_2 = 18$

Compute SV's:

$$\Sigma = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{18} \end{pmatrix}$$

Compute Singular Vectors U and V:

$$U = XV\Sigma^{-1} = \begin{pmatrix} 4 & 2 \\ \sqrt{2} & 2\sqrt{2} \\ -\sqrt{2} & -2\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{10}} & 0 \\ 0 & \frac{1}{\sqrt{18}} \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{10}} & \frac{2}{\sqrt{18}} \\ \frac{\sqrt{2}}{\sqrt{10}} & \frac{2\sqrt{2}}{\sqrt{18}} \\ -\frac{\sqrt{2}}{\sqrt{10}} & -\frac{2\sqrt{2}}{\sqrt{18}} \end{pmatrix}$$

This gives us the following SVD:

$$X = U\Sigma V^T = \begin{pmatrix} \frac{4}{\sqrt{10}} & \frac{2}{\sqrt{18}} \\ \frac{\sqrt{2}}{\sqrt{10}} & \frac{2\sqrt{2}}{\sqrt{18}} \\ -\frac{\sqrt{2}}{\sqrt{10}} & -\frac{2\sqrt{2}}{\sqrt{18}} \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{18} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T$$

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2) Perform PCA on X:

Given $X = U\Sigma V^T$, we can use the PC's to perform PCA:

$$PC_1 = \begin{pmatrix} \frac{4}{\sqrt{10}} \\ \frac{\sqrt{2}}{\sqrt{10}} \\ -\frac{\sqrt{2}}{\sqrt{10}} \end{pmatrix}, \quad PC_2 = \begin{pmatrix} \frac{2}{\sqrt{18}} \\ \frac{2\sqrt{2}}{\sqrt{18}} \\ \frac{-2\sqrt{2}}{\sqrt{18}} \end{pmatrix}$$

To project the three 2-dimensional points in X into a 1-dimensional space, we can choose the first PC:

$$Z = X \cdot PC_1 = X \cdot \begin{pmatrix} \frac{4}{\sqrt{10}} \\ \frac{\sqrt{2}}{\sqrt{10}} \\ -\frac{\sqrt{2}}{\sqrt{10}} \end{pmatrix}$$

Z will be a 1-dimensional representation of the points in X.

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