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Logarithmic Fluctuations for Internal DLA and Grid Brownian Motions

Diplomarbeit

vorgelegt von

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Betreuer

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Abstract

In my thesis I consider internal diffusion limited aggregation—a random growth model—on the two-dimensional lattice. With each step, a new particle starts in the origin and performs a random walk until it hits an unoccupied lattice point, where it stops. I am interested in the asymptotic growth of the occupied cluster. Lawler, Bramson, and Griffeath [GFLG92] proved that the asymptotic shape is a Euclidean ball. This statement was remarkably improved by Jerison, Levine, and Sheffield [JLS12], who showed that the fluctuations from circularity are of logarithmic order. This result is subject of my thesis. I give a thorough overview of the proof by filling in the omitted details. One of the most technical steps that was left out in the paper and is filled in in this thesis is an extension of the classical result that a harmonic function of Brownian motion is a martingale to grid-harmonic functions and grid Brownian motions.

Contents

1 Introduction

1.1 Model and Result

Internal diffusion limited aggregation (IDLA) is a growth model. In this thesis we consider IDLA on $\mathbb{Z}^2$, which inductively can be defined as follows: at the beginning the IDLA cluster $A(n)$ only contains the origin. In each step a simple random walk starting in the origin is run until it reaches an unoccupied lattice point; this point is then added to the cluster. To be more precise, for independent simple random walks S^1, S^2, S^3, ... with start in 0, let $A(0) = \{0\}$; for larger n, let

$$A(n) = A(n-1) \cup \{S^n(\tau^n)\},$$

where

$$\tau^n = \inf\{j \geq 0 \,|\, S^n(j) \notin A(n-1)\}.$$

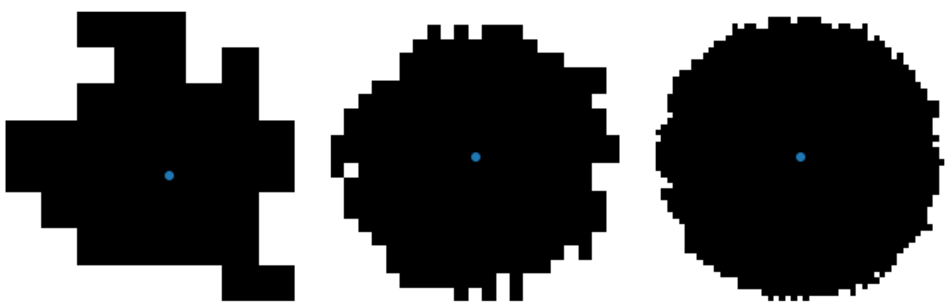

Figure 1: IDLA clusters for n = 40, 300, and 1800. [1]

By time πr^2 we expect the IDLA cluster to have the approximate shape of the ball $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$, where $|\cdot|$ denotes the Euclidean norm. The main result of this theses proves this asymptote and furthermore shows that fluctuations away from the ball are logarithmic. For real numbers $t \geq 0$ let $A(t) := A(\lceil t \rceil)$ and let $\mathbb{B}_r = B_r \cap \mathbb{Z}^2$. With these definitions we can already state the main theorem.

[1] The code for all simulations and visualizations can be found in the following repository: `https://github.com/lennartCln/IDLA`

Theorem 1 (Logarithmic Fluctuations). *For each $\gamma \geq 1$ there is an $a = a(\gamma) < \infty$ and $r_0 = r_0(\gamma)$ such that*

$$\mathbb{P}\big(\{\mathbb{B}_{r-a\ln r} \subset A(\pi r^2) \subset \mathbb{B}_{r+a\ln r}\}^c\big) \leq r^{-\gamma},$$

for all $r > r_0$.

Since $\sum_{r\in\mathbb{N}} r^{-\gamma}$ converges for $\gamma > 1$, Theorem 1 implies (using Borel-Cantelli) that almost surely there is just a finite number of $r \in \mathbb{N}$ such that $\mathbb{B}_{r-a\ln r} \not\subset A(\pi r^2)$ or $A(\pi r^2) \not\subset \mathbb{B}_{r+a\ln r}$.

For the proof of Theorem 1, I will follow Jerison, Levine, and Sheffield [JLS12] and fill out the steps omitted in the paper.

1.2 A Brief History of IDLA Shape Results

The model was first proposed in chemical physics by Meakin and Deutch [MD86] who already questioned the smoothness of the boundary of the cluster. Lawler, Bramson, and Griffeath [GFLG92] were the first who identified the asymptotic shape (in every dimension) as the trace of Euclidean balls. More precisely, they proved that for $\epsilon > 0$ it is $\mathbb{B}_{r-\epsilon r} \subset A(\pi r^2) \subset \mathbb{B}_{r+\epsilon r}$, almost surely for r sufficiently large. Later, this result was better quantified by Lawler [Law95], showing that the fluctuation from circularity is (up to logarithmic factors) at most $O(r^{1/3})$. The latter result had not been improved until Asselah-Gaudillière [AG13a] and Jerison-Levine-Sheffield [JLS12] independently obtained that the fluctuation is bounded by $O(\log r)$. Even though simulations [FL11] indicate that the fluctuation is of logarithmic order, proving that they are of no smaller order is still an open problem. The best lower bound on the fluctuation for IDLA so far is $O(\sqrt{\log r})$, see [AG11].

IDLA has been used to understand anodic polishing [MD86]; two dimensional IDLA has been studied as a model for viscous fluid displacement in porous media [Pat84, Tan77] and for the diffusion of oil and water particles [CGHL17].

This thesis is restricted to the two dimensional case. When the dimension d is larger than or equal to three, Jerison-Levin-Sheffield [JLS12] and Asselah-Gaudillière [AG13b] showed even smaller fluctuations, namely

$\mathbb{B}_{r-C\sqrt{\log r}} \subset A(\omega_d r^d) \subset \mathbb{B}_{r+\sqrt{\log^2 r}}$, where ω_d is the volume of the d-dimensional Euclidean ball of radius 1. Recent work analysed the shape of more complex cases than clusters based on the simple random walk on $\mathbb{Z}^d$, such as IDLA on the cylinder lattice [JLS14], comb lattices [HS12], and supercritical percolation clusters [She10].

1.3 Early and Late Points

We reformulate Theorem 1 in terms of early and late points. We would expect a lattice point z to join the IDLA cluster at time $\pi|z|^2$. With that in mind, we call z to be m-early if z joins the cluster at the time where we would expect the shape of the cluster being $\mathbb{B}_{|z|-m}$; more precisely, $z \in \mathbb{Z}^2$ is m-**early** if $z \in A(\pi(|z|-m)^2)$. Let $E_z^m = \{z \in A(\pi(|z|-m)^2)\}$ denote the event that z is m-early, then let

$$\mathcal{E}^m(T) = \bigcup_{z \in A(T)} E_z^m$$

be the event that up to time T there was an m-early point in the cluster.

Similarly, we call $z \in \mathbb{Z}^2$ to be l-**late** if $z \notin A(\pi(|z|+l)^2)$. By L_z^l denote the event of z being l-late. Let

$$\mathcal{L}^l(T) = \bigcup_{z \in \mathbb{B}_{\sqrt{T/\pi}-l}} L_z^l \tag{1.1}$$

be the event that there was an l-late point up to time T.

To divide the problem of bounding the fluctuation into the problems of bounding the event of single early/late points, we state

Lemma 2 (No log-early/late point implies logarithmic fluctuation)**.** *For Theorem 1 it is sufficient to show that for each γ there is an $a = a(\gamma)$ such that for all r sufficiently large there are l, $m < a \ln r$ such that*

$$\mathbb{P}\big(\mathcal{L}^l(T)\big) + \mathbb{P}\big(\mathcal{E}^m(T)\big) \le r^{-\gamma}.$$

Remark. *Late points quantify the inner error, $r - \inf\{|z| : z \notin A(\pi r^2)\}$, i.e., the largest deviation of $A(\pi r^2)$ from B_r to the inside, and early points target the outer error, $\sup\{|z| : z \in A(\pi r^2)\} - r$. Note that $\mathcal{L}^l(T)$ and $\mathcal{E}^m(T)$ are monotonically decreasing in l and m (since $L_z^l \supset L_z^{l+1}$, $E_z^m \supset E_z^{m+1}$).*

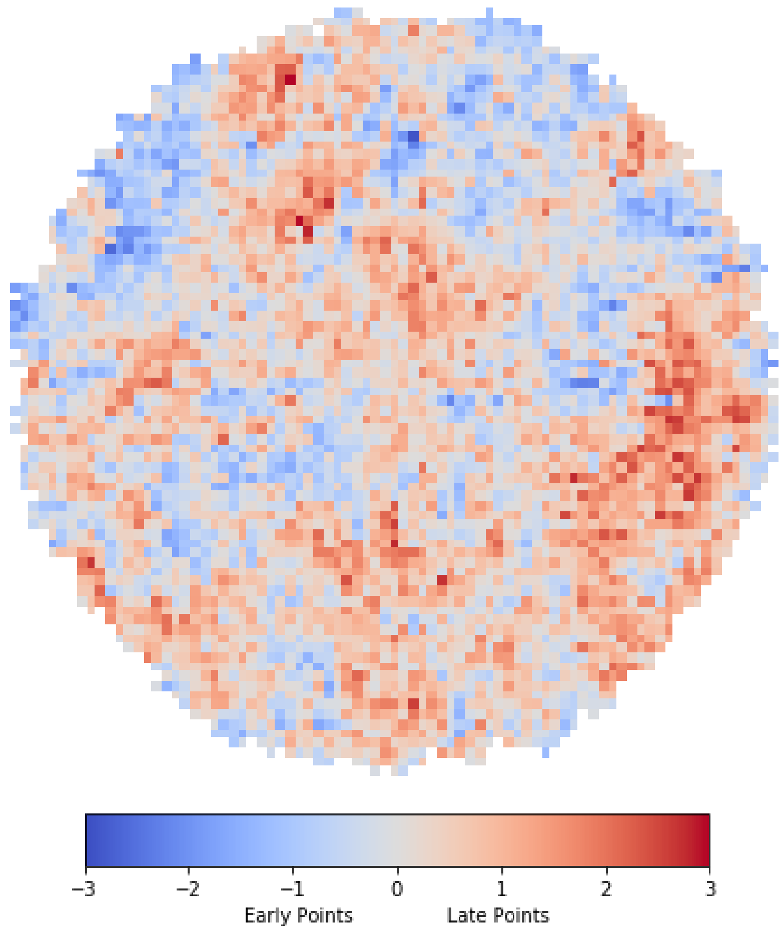

Figure 2: Early (blue) and late (red) points of a sample of an IDLA cluster at time 5000. Points $z \in \mathbb{Z}^2$ joining the IDLA cluster at time about $\pi|z|^2$ are colored in gray. No point is earlier than 3-early or later than 3-late. See Figure 11 in Section 6.1 for a histogram of early and late points.

Proof of Lemma 2. It is

$$\begin{aligned}
\mathcal{E}^m(T) &= \bigcup_{z\in A(T)} \left\{z \in A(\pi(|z|-m)^2)\right\} \\
&= \bigcup_{z\in A(T)} \bigcup_{n\leq\pi(|z|-m)^2} \{z \in A(n)\} \\
&= \bigcup_{n\leq T} \left\{\text{there is a } z : z \in A(n) \;\text{ and }\; z \notin \mathbb{B}_{\sqrt{n/\pi}+m}\right\} \\
&= \bigcup_{n\leq T} \left\{\mathbb{B}_{\sqrt{n/\pi}+m} \not\supset A(n)\right\},
\end{aligned} \tag{1.2}$$

similarly,

$$\mathcal{L}^l(T) = \bigcup_{n\leq T} \left\{\mathbb{B}_{\sqrt{n/\pi}-l} \not\subset A(n)\right\}.$$

Therefore, using the monotonicity of $\mathcal{L}^l(T)$ and $\mathcal{E}^m(T)$ in l and m, we get for $T = \pi r^2$ and l, $m < a\ln r$,

$$\begin{aligned}
&\mathbb{P}\big(\mathbb{B}_{r-a\ln r} \not\subset A(T) \;\text{ or }\; A(T) \not\subset \mathbb{B}_{r+a\ln r}\big) \\
&\quad \leq \mathbb{P}\big(\mathbb{B}_{r-l} \not\subset A(T)\big) + \mathbb{P}\big(A(T) \not\subset \mathbb{B}_{r+m}\big) \\
&\quad \leq \mathbb{P}\big(\mathcal{L}^l(T)\big) + \mathbb{P}\big(\mathcal{E}^m(T)\big).
\end{aligned}$$

□

1.4 Proof Sketch

Lemma 2 implies that in order to prove Theorem 1 it suffices to show that l-late and m-early points are unlikely for l and m of logarithmic order. Here, I will briefly sketch the proof of this statement. For readability in text passages I often refer to probabilities of events as if they were boolean expressions.

We define the **grid** $\mathcal{G}$ by,

$$\mathcal{G} := \{x + iy \,|\, x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\} \subset \mathbb{C}. \tag{1.3}$$

On the grid we define in Section 3.2 a continuous process which we call grid Brownian motion and for integer times behaves like a random walk on $\mathbb{Z}^2$ (as a subset of $\mathcal{G}$); for intermediate times it behaves like a one dimensional

Brownian motion on the edges of $\mathcal{G}$. Furthermore, we will give a definition of harmonicity on the grid (see Section 3.1) and prove that a function of grid Brownian motion is also a martingale if the function is grid harmonic (see Theorem 13). This result for grid-harmonic functions is analogous to the fact that harmonic functions applied to Brownian motions are martingales. For $\zeta \in \mathbb{Z}^2$ we will define a function H^ζ on the grid, see (4.1), which is grid harmonic and approximates the discrete Poisson kernel for the ball $\mathbb{B}_{|\zeta|}$. Moreover is H^ζ close to the continuum Poisson kernel for the ball $B_{|\zeta|}$ (Lemma 17). We will derive some properties of H^ζ such as an approximate mean-value property (see Lemma 16 (f)) from the continuum Poisson kernel.

We will define the grid IDLA (see Section 5.1), whose underlying particles are grid Brownian motions with similar stopping rules as the particles (random walks) of the IDLA. Hence, for integer times the grid IDLA behaves like the IDLA.

Our main tool is the process M^ζ, which is defined by the values of H^ζ on the particles of the grid IDLA (see Section 5.2). Since H^ζ is grid harmonic and since the underlying particles of the grid IDLA are grid Brownian motions (and therefore continuous), we can conclude that M^ζ is a continuous martingale (see Lemma 18). Hence, we can represent M^ζ by a time-changed Brownian motion (using Theorem 5). Applying this outcome and a bound on large deviations of Brownian motions (Lemma 4), we can conclude that the deviation of M^ζ is unlikely to be large while its quadratic variation is small (Lemma 6). Analyzing the behavior of M^ζ and its quadratic variation on the event of early and late points near ζ will be the next aim.

If we choose ζ outside the ball $\mathbb{B}_r$ (from which we want to measure the fluctuations of $A(T)$, for $T = \pi r^2$), then M^ζ is large on the event of an early point near ζ and no late point (see (7.5)). For this implication, we use that close to an early point there are many points part of the IDLA cluster (see Section 7.2) and that H^ζ is large near ζ (Lemma 16). In addition, on the same event the quadratic variation is small (see (7.4)). According to Lemma 6, however, the quadratic variation is large if the martingale is large, i.e., the event —an early point but and no late point— is unlikely to occur; in other words: early points imply late points (see Lemma 21).

Under assistance of the same tools we will prove that late points imply early points (Lemma 22). If we choose ζ inside $\mathbb{B}_r$ and ζ is a late point, then no particle reaches ζ. Hence, and by the mean-value property of H^ζ, the martingale M^ζ is small (large deviation). If in addition there is no early point, then its quadratic variation is large (Lemma 24). Again, the result—late points imply early points—follows since by Lemma 6 the deviation of the martingale M^ζ cannot be large while its quadratic variation is small.

Lemma 21 and 22 are the key ingredients for the final step of the proof of the main result (provided by Section 6), iterating alternatingly the contraposition of Lemma 21 (i.e., no late point implies no early point) and Lemma 22 (i.e., no early point implies no late point). We will recursively define sequences l_i and m_i starting with l_0 being of order $\sqrt{T}$. Then Lemma 20 (an a priori bound on the event of the absence of $\sqrt{T}$-late points) implies that there is no l_0-late point by time T. Choosing m_0 to be l_0 up to a multiplicative constant, Lemma 21 (no l_0-late points imply no m_0-early points) gives us the absence of m_0-early points by time T. Similarly, we can conclude from Lemma 22 (no m_0-early point implies no l_1-late point) that there is no l_1-late points by time T if we choose l_1 to be approximately $\sqrt{m_0}$. By this choice l_1 is smaller than l_0, i.e., even less late points are unlikely. Hence, if we keep on assigning l_i and m_i according to these rules (see also (6.3)), we obtain decreasing sequences for which there are no l_i-late and m_i-early points. The assumptions of Lemma 21 or 22 are fulfilled for m_i or l_i being larger than $\ln T$. Therefore, we stop the iteration when reaching this threshold and we end up with l and m being of order $\ln T$. According to Lemma 2 this is what we needed to show.

2 Preliminaries

2.1 Brownian Motions

Let $\mathcal{B}(t)$, $t \geq 0$ be a standard Brownian motion starting in the origin, and by $\tau_{(a,b)} = \inf\{s \geq 0 \,|\, \mathcal{B}(s) \notin (a,b)\}$ denote its exit time from the interval (a,b). Lemma 3 provides an upper bound for this exit time and Lemma 4 bounds large deviations of $\mathcal{B}$. In Section 2.2 and 7.5 we will use these lemmas to bound martingales, which will be represented by Brownian motions.

Lemma 3 (Exit times of Brownian motions)**.** *Let* $0 < d \leq c$ *and* $\lambda > 0$ *with* $\sqrt{\lambda}(c+d) \leq \frac{3}{\sqrt{2}}$, *then*

$$\mathbb{E}\big(e^{\lambda\tau_{(-d,c)}}\big) \leq 1 + 20\lambda cd.$$

Proof. The first part of the proof follows the idea of [RY99], Ch. II, Prop. 3.7. The estimations of the second part follow Lemma 5 in [JLS12].

Obtain that for a standard Brownian motion $\mathcal{B}(t)$ started in 0,

$$M^s(t) := \exp\Big(is\Big(\mathcal{B}(t) - \frac{c-d}{2}\Big) + \frac{s^2}{2}t\Big)$$

is a (complex) martingale. Hence, the same holds true for

$$N^s(t) := \frac{1}{2}\Big(M^s(t) + M^{-s}(t)\Big) = \exp(t\,s^2/2) \cdot \cos\Big(s\Big(\mathcal{B}(t) - \frac{c-d}{2}\Big)\Big).$$

Since $N^s(t \wedge \tau_{(-d,c)})$ is bounded by $\exp(cs^2/2)$, it is uniformly integrable. Therefore, if $s \in [0, \pi(c+d)^{-1})$, it is by optional sampling theorem ([RY99], Ch. II, Cor. 3.6),

$$\begin{aligned}
&\mathbb{E}\big(\exp\big(\tau_{(-d,c)}\,s^2/2\big)\big) \\
&= \mathbb{E}\Big(\exp\big(\tau_{(-d,c)}\,s^2/2\big) \cdot \cos\Big(s\Big(\mathcal{B}(\tau_{(-d,c)}) - \frac{c-d}{2}\Big)\Big)\Big) \cdot \Big(\cos s\frac{c+d}{2}\Big)^{-1} \\
&= \mathbb{E}\big(N^s(\tau_{(-d,c)})\big) \cdot \Big(\cos s\frac{c+d}{2}\Big)^{-1} \\
&= \mathbb{E}\big(N^s(0)\big) \cdot \Big(\cos s\frac{c+d}{2}\Big)^{-1} \\
&= \Big(\cos s\frac{c-d}{2}\Big) \cdot \Big(\cos s\frac{c+d}{2}\Big)^{-1} \\
&= \cos(sa) \cdot \cos(sb)^{-1},
\end{aligned}$$

where in the last equation we defined $a = \frac{c-d}{2}$, $b = \frac{c+d}{2}$.

Cosine is decreasing and concave on $[0, \frac{\pi}{2}]$; as a result, we have

$$\cos y - \cos x \geq (y - x)\cos' y,$$

for $0 \leq x \leq y \leq \frac{\pi}{2}$. Hence, for $0 \leq sa \leq sb \leq \frac{\pi}{2}$ the estimation

$$\cos(sa) \leq \cos(sb) + s(b-a)\sin(sb)$$

provides

$$\cos(sa) \cdot \cos(sb)^{-1} \leq 1 + s(b-a)\tan(sb).$$

Thus, using $\tan x < 10x$, for $0 < x \leq \frac{3}{2}$,

$$\begin{aligned}\cos(sa) \cdot \cos(sb)^{-1} &\leq 1 + 10s^2 b(b-a) \\ &\leq 1 + 10s^2(b-a)(b+a) \\ &= 1 + 20\lambda cd,\end{aligned}$$

for $0 \leq \sqrt{\lambda}(c-d) \leq \sqrt{\lambda}(c+d) \leq \frac{3}{\sqrt{2}}$. □

Lemma 4 (Exponential inequality)**.** *For $a \geq 0$, it is*

$$\mathbb{P}\left(\sup_{s\in[0,t]} \mathcal{B}(s) \geq at\right) \leq e^{-a^2 t/2}.$$

The proof is an application of Doob's L^1-inequality ([RY99], Ch. II, Theorem 1.7) to the martingale $M^\alpha(t) = \exp\left(\alpha\mathcal{B}(t) - \frac{\alpha^2}{2}t\right)$. The detailed proof can be found in [RY99], Ch. II, Prop. 1.8.

2.2 Martingales

In this section, we collect some properties of martingales useful for this thesis.

In what follows, we let $X(t)$, $t \geq 0$ be a real-valued process; $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $\mathcal{F}_t$ a filtration of $\mathcal{F}$.

If $X(t)$ is adapted to $\mathcal{F}_t$ and

- $\mathbb{E}(|X(t)|) < \infty$ for every $t \geq 0$ and
- $\mathbb{E}(X(t) \,|\, \mathcal{F}_s) = X(s)$ a.s. for all $0 \leq s < t$,

then X is a **martingale** (w.r.t. $\mathcal{F}_t$). In order to define the quadratic variation of X, let Δ be a subdivision of the interval $[0,t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$ and denote the modulus of Δ by $|\Delta| = \sup_i |t_i - t_{i-1}|$. We set

$$T_t^\Delta = \sum_{i=1}^{n} \left(X(t_{i+1}) - X(t_i)\right)^2.$$

We say X is of finite quadratic variation if there exists a process $\langle X, X\rangle$ such that for every sequence Δ_i of subdivisions of $[0,t]$ with $|\Delta_i| \to 0$ as $i \to \infty$,

$$\lim_{i\to\infty} \mathbb{P}(|T_t^{\Delta_i} - \langle X, X\rangle_t| > \epsilon) = 0,$$

for every $\epsilon > 0$. The process $\langle X, X\rangle_t$ (sometimes $\langle X\rangle_t$) is called the **quadratic variation** (QV) of X.

The following lemma enables us to represent any martingale with divergent quadratic variation as a time-changed Brownian motion.

Theorem 5 (Dambis Dubins-Schwarz). *Let $M(\cdot)$ be an a.s. continuous $\mathcal{F}_t$-martingale, which vanishes at time 0 and with $\langle M\rangle_\infty = \infty$. Set*

$$T_t = \inf\{s \geq 0 \,|\, \langle M\rangle_s > t\}.$$

Then $\mathcal{B}(t) := M(T_t)$ is a Brownian motion adapted to $\mathcal{F}_t$ and, vice versa, $M(\cdot)$ can be represented as

$$M(t) = \mathcal{B}(\langle M\rangle_t).$$

Proof. I will only sketch the two key arguments of the proof. For the complete proof I refer to [RY99], Ch. V, Theorem 1.6.

For the time change T_t, $t \geq 0$ and the martingale M the assumptions for [RY99], Ch. V, Prop. 1.5 are fulfilled. It states that the quadratic variation of the time-changed process $M_{T_\bullet}$ equals the time-changed quadratic variation process of M. Hence,

$$\langle \mathcal{B}\rangle_t = \langle M_{T_\bullet}\rangle_t = \langle M\rangle_{T_t} = t.$$

Then, by Lévy's characterization theorem ([RY99], Ch. IV, Theorem 3.6) $\mathcal{B}$ is a $\mathcal{F}_{T_t}$-Brownian motion.

Combining Theorem 5 with Lemma 4 leads to the following large deviation bound for martingales.

Lemma 6 (Small QV implies small martingale). *Let $M(t)$, $t \geq 0$ be a continuous martingale, which fulfills the assumptions of Theorem 5. Then,*

$$\mathbb{P}\big(M(t) \geq l,\ \langle M\rangle_t \leq k\big) \leq e^{-l^2/(2k)}$$

and

$$\mathbb{P}\big(M(t) \le -l,\ \langle M\rangle_t \le k\big) \le e^{-l^2/(2k)},$$

for all $0 < l,\ k$.

In words, if the QV of a martingale remains small, then a large deviation of the martingale itself is unlikely. Or at the level of the time-changed BM $\mathcal{B}$: if the time $\langle M\rangle_t$ elapses slow, then it is unlikely that $\mathcal{B}(\langle M\rangle_t)$ is large.

Proof. By Theorem 5 there is a Brownian motion $\mathcal{B}(t)$ such that $M(t) = \mathcal{B}(\langle M\rangle_t)$. Hence,

$$\begin{aligned}\mathbb{P}(M(t) \ge l,\ \langle M\rangle_t \le k) &= \mathbb{P}(\mathcal{B}(\langle M\rangle_t) \ge l,\ \langle M\rangle_t \le k)\\ &\le \mathbb{P}\Big(\sup_{s\in[0,k]} \mathcal{B}(s) \ge l\Big)\\ &\le e^{-l^2/(2k)}.\end{aligned}$$

The second inequality of the lemma follows from the first with the observation $\langle -M\rangle_t = \langle M\rangle_t$. □

2.3 Discrete Potential Theory

In this section, we will define the potential kernel, which will be used in Section 4 (approximating discrete Poisson kernel) to define a specific harmonic function. In Lemma 7, we give a general estimate for the potential kernel and compute some specific values.

Consider $\mathbb{Z}^2$ as a subset of $\mathbb{C}$ and denote $V = \{1, -1, i, -i\}$. The two dimensional **simple random walk** $S(n)$ starting in 0 can be defined as a Markov chain with state space $\mathbb{Z}^2$, start $S(0) = 0$, and transition probabilities

$$\mathbb{P}\big(S(n+1) = z \,\big|\, S(n) = y\big) = \frac{1}{4}, \quad \text{for } \ z - y \in V.$$

For $A \subset \mathbb{Z}^2$ let

$$\partial_o A = \{z \in \mathbb{Z}^2 \setminus A \,|\, z - y \in V, \ \text{ for some } y \in A\}$$

denote the **outer boundary** of A and $\bar{A} := A \cup \partial_o A$ the **discrete closure** of A. For a function $g : \bar{A} \to \mathbb{R}$ denote the **discrete Laplacian** Δ by

$$\Delta g(z) = \sum_{v \in V} \frac{1}{4} g(z+v) - g(z),$$

for $z \in A$. The function $g : \bar{A} \to \mathbb{R}$ is called **harmonic** in A (with respect to the simple random walk) if

$$\Delta g(z) = 0, \tag{2.1}$$

for all $z \in A$.

Remark. *For such a function g and a random walk $S(n)$ the process $Y(n) = g((n \wedge \tau_A))$ is a martingale (by a direct calculation; see for instance [LL10] Prop 6.1.1), where τ_A denotes the exit time of $S(n)$ from A.*

Define the **potential kernel** $a : \mathbb{Z}^2 \to \mathbb{R}_{\geq 0}$ by

$$a(z) = \sum_{n=0}^{\infty} \big(\mathbb{P}(S(n) = 0) - \mathbb{P}(S(n) = z)\big). \tag{2.2}$$

Remark. *The convergence of $a(z)$ clearly holds in the transient case; in two dimensions [Spi76], Ch. 12, Prop. 1 proves the convergence.*

Moreover, $2a(x-y)$ is equal to the expected number of visits of $x \in \mathbb{Z}^2$ by a random walk started in x before hitting $y \in \mathbb{Z}^2$, which we can obtain from [LL10] Proposition 4.6.3 and the translation invariance of the random walk.

Lemma 7 (Properties of a)**.** *The potential kernel $a(\cdot)$*

(a) is harmonic everywhere except 0,

$$\Delta a(z) = \delta_0(z) = \begin{cases} 1, & z = 0 \\ 0, & z \neq 0, \end{cases}$$

(b) can be precisely estimated by

$$a(z) = \frac{2}{\pi} \ln |z| + \lambda + O(|z|^{-2}),$$

for a constant $\lambda > 0$,

(c) is invariant under the dihedral symmetries, i.e., for all $z \in \mathbb{Z}^2$,

$$a(z) = a(-i\bar{z}) = a(iz) = a(\bar{z}) = a(-\bar{z}) = a(-iz) = a(i\bar{z}) = a(-z).$$

(d) $a(0) = 0, \quad a(1) = 1, \quad$ *and* $\quad a(1+i) = \frac{4}{\pi}$.

(e) is subadditive, i.e., for all $x, y \in \mathbb{Z}^2$ *it is*

$$a(x+y) \leq a(x) + a(y).$$

Proof. (a) Using the notation $p^n(z) := \mathbb{P}(S(n) = z)$, we have

$$\Delta(p^n(0) - p^n(z)) = p^n(z) - p^{n+1}(z),$$

for all $z \in \mathbb{Z}^2$. Hence,

$$\begin{aligned}
\Delta a(z) &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta\big(p^i(0) - p^i(z)\big) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta\big(p^i(z) - p^{i+1}(z)\big) \\
&= \lim_{n\to\infty} p^0(z) - p^{n+1}(z) \\
&= p^0(z) = \delta_0.
\end{aligned}$$

(b) This statement was first proven by Stöhr [Sto50] (for a more comprehensible proof see [KS04]). Both use the following explicit integral representation of $a(\cdot)$, which appears as Prop. 4.4.3 in [LL10],

$$a(x) = (2\pi)^{-2} \int_{[-\pi,\pi]^2} \frac{1 - \cos x \cdot \theta}{1 - \phi(\theta)} d\theta, \tag{2.3}$$

where $\phi(\theta) = (\cos\theta_1 + \cos\theta_2)/2$ denotes the characteristic function of the simple random walk in $\mathbb{Z}^2$. This equality can be used to derive the result.

(c) This follows immediately from the symmetry properties of $\mathbb{P}(S(n) = z)$.

(d) By definition it is $a(0) = 0$. With $z = 0$ in (a) we obtain

$$a(1) = 1.$$

The next calculation follows Chapter 15 of [Spi76]. By (2.3) and the addition formula,

$$\begin{aligned} a(1+i) &= \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{1-\cos(\theta_1+\theta_2)}{1-(\cos\theta_1+\cos\theta_2)/2} d\theta \\ &= \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} \frac{1-\cos(\theta_1+\theta_2)}{1-\left(\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)\right)} d\theta. \end{aligned}$$

With the transformation $x = (\theta_1+\theta_2)/2$, $y = (\theta_1-\theta_2)/2$ and the symmetry properties of the integrand we can conclude

$$\begin{aligned} a(1+i) &= \frac{1}{2\pi^2} \int_{\{|x|+|y|\le\pi\}} \frac{1-\cos 2x}{1-\cos x\cos y} d(x,y) \\ &= \frac{1}{4\pi^2} \int_{[-\pi,\pi]^2} \frac{1-\cos 2x}{1-\cos x\cos y} d(x,y) \\ &= \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{1-\cos 2x}{|\sin x|} dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{(\sin x)^2}{|\sin x|} dx = \frac{4}{\pi}. \end{aligned}$$

(e) Fix $y \in \mathbb{Z}^2$ and define $g(x) = a(x+y) - a(x)$. By (a), g is harmonic everywhere except in 0 and $-y$. By maximum principle g attains its maximum in 0 or $-y$; since $g(-y) = -a(y) \le a(y) = g(0)$, it attains its maximum in 0, i.e., for all $x, y \in \mathbb{Z}^2$,

$$a(y) = g(0) \ge g(x) = a(x+y) + a(x).$$

$\square$

Remark. *By Lemma 7 (c), which describes the symmetries of the potential kernel, it suffices to compute $a(\cdot)$ for all $z \in \mathbb{Z}^2$ with $0 \le Im(z) \le Re(z)$. It is still costly to compute explicit values of $a(\cdot)$. McCrea-Whipple [MW40] were the first who computed a large number of values of $a(\cdot)$.*

2.4 Potential Theory in Continuum

The main goal of this section is to bound the Poisson kernel for the ball (Lemma 11). The Poisson kernel is harmonic in the ball. Hence, we may apply some general statements about harmonic functions which we state first.

2.4.1 Harmonic Analysis

For $U \subset \mathbb{R}^2$ open, define the **Laplacian** Δ of a C^2 function $u : U \to \mathbb{R}$ by

$$\Delta u = \partial_{x_1}\partial_{x_1} u + \partial_{x_2}\partial_{x_2} u.$$

We call u **harmonic** if

$$\Delta u = 0.$$

We use $B(z, r) = z + B_r$ to denote the ball with radius r and center in $z \in \mathbb{R}^2$.

Theorem 8 (Mean-value formula). *Suppose $u \in C^2(U)$ is harmonic, then*

$$u(z) = \frac{1}{\pi r^2} \int_{B(z,r)} u(y) dy,$$

for each $B(z, r) \subset U$.

Theorem 9 (Maximum principle). *If $u \in C^2(U)$ is harmonic and continuous on $\bar{U}$, then*

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

Proofs of these basic properties of harmonic functions can be found for instance in [Eva10] Chapter 2.2.

2.4.2 Poisson Kernel for the Ball

The aim of this section is to introduce a harmonic function F^ζ, which plays a large role in the definition of a discrete harmonic function that we introduce in Section 4. In Lemma 10 we state basic properties of F^ζ, so is F^ζ a shift of the Poisson kernel for the two dimensional Euclidean ball. The estimations of F^ζ in Lemma 11 are of more specific character.

Define $F^\zeta : \mathbb{C} \setminus \{\zeta\} \to \mathbb{R}$ by

$$F^\zeta(z) = \operatorname{Re} \frac{\zeta}{|\zeta|(\zeta - z)}. \tag{2.4}$$

Lemma 10 (Properties of F^ζ). *Let $\zeta \in \mathbb{C} \setminus \{0\}$, then*

(a) denote the Poisson kernel for the ball $B_{|\zeta|}$ with pole in ζ by

$$K^\zeta(z) = \frac{|\zeta|^2 - |z|^2}{2|\zeta||\zeta - z|^2}.$$

We have,

$$F^\zeta(z) - \frac{1}{2|\zeta|} = K^\zeta(z), \quad \text{for all } z \neq \zeta.$$

(b) The level curves of F^ζ are the circles tangent to $B_{|\zeta|}$ at point ζ. The level set $F^\zeta(\cdot) = 0$ is the tangent line to $B_{|\zeta|}$ at point ζ.

The integral curves of the vector field ∇F^ζ are circles through ζ with center on the tangent line to $B_{|\zeta|}$ at ζ (see Figure 5) and the line through 0 and ζ.

For both sets of curves ζ is always excluded.

(c) F^ζ is harmonic in $B_{|\zeta|}$.

(d) For all $z \neq \zeta$, it is

$$|\nabla F^\zeta(z)| = \frac{1}{|z - \zeta|^2}.$$

(e) For $0 < R < |\zeta|$,

$$\sup_{z \in B_R} F^\zeta(z) = F^\zeta\left(\frac{R}{|\zeta|}\zeta\right) = \frac{1}{|\zeta| - R}.$$

(f) For $z \in \mathbb{C}$ and $r > 0$ so that $\zeta \notin \bar{B}(z, r)$, it is

$$\sup_{\omega \in B(z,r)} F^\zeta(\omega) - F^\zeta(z) \leq \frac{1}{\big(|\zeta - z| - r\big)^2}.$$

Proof. (a) Using $\operatorname{Re} z = (z+\bar{z})/2$,

$$\begin{aligned} F^{\zeta}(z) - \frac{1}{2|\zeta|} &= \frac{1}{2|\zeta||\zeta - z|^2}\Big(\zeta\,\overline{(\zeta - z)} + \bar{\zeta}(\zeta - z)\Big) - \frac{1}{2|\zeta|} \\ &= \frac{1}{2|\zeta|}\left(\frac{1}{|\zeta - z|^2}\Big(|\zeta|^2 + |\zeta - z|^2 - |z|^2\Big) - 1\right) \\ &= K^{\zeta}(z). \end{aligned}$$

(b) We first show that the level sets of K^1 are the circles tangent to ∂B_1 at 1. For $z = x + iy \neq 1$ and $R \neq -1$ it is

$$2K^1(z) = \frac{1 - x^2 - y^2}{(1-x)^2 + y^2} = R \tag{2.5}$$

if and only if

$$\Big(\frac{1}{1+R}\Big)^2 = \Big(x - \frac{R}{1+R}\Big)^2 + y^2,$$

i.e., $z \in \partial B(m, r)$, where $m = R/(1+R)$ and $r = 1/(1+R)$. Then the result follows since $r + m = 1$ and $\operatorname{Im}(m) = 0$.

For $R = -1$, Equation (2.5) yields to the equation of the tangent line (or degenerate circle) $x = 1$.

With the $\phi(z) := \lambda e^{\alpha i} z$ for $\alpha \in [0, 2\pi]$, $\lambda \in \mathbb{R} \setminus \{0\}$ we obtain the invariance under rotation-dilation,

$$K^{\phi(1)}\big(\phi(z)\big) = K^1(z). \tag{2.6}$$

Hence our statement (b) holds for all $\zeta = \lambda e^{i\alpha} \neq 0$, and by (a) it holds for F^{ζ} as well.

To verify that the integral curves of ∇F are the circles through ζ with center on the tangent line to $B_{|\zeta|}$ at ζ, we show geometrically that such a circle is orthogonal to all level curves. As illustrated in Figure 3 below we let $\varphi : (0, 2\pi) \to \mathbb{R}^2$ be a curve so that the image of the curve is such a circle. Let φ start and end at ζ, i.e., ζ is not in the image of φ. We let φ be directed as in Figure 3 below.

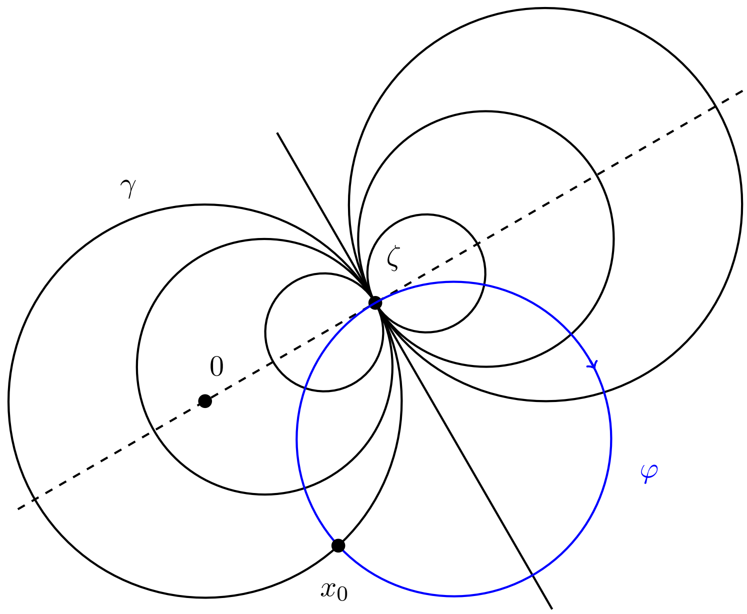

Figure 3: Shown is an integral curve φ of ∇F^{ζ} (blue) and the level sets of F^{ζ} (black). Considering the kite defined by the vertices ζ, x_0, and the centers of φ and γ, orthogonality of φ and γ in x_0 is evident.

For each point x_0 in the image of φ there is a level curve γ such that $\gamma(u_0) = \varphi(s_0) = x_0$ for some s_0, u_0, and because F^{ζ} is constant on the level curves, it is

$$0 = \nabla F^{\zeta}(\gamma(u_0))\gamma'(u_0).$$

Using this and the orthogonality of $\gamma'(u_0)$ and $\varphi'(s_0)$, we get that

$$\varphi'(s) = \lambda(\varphi(s))\nabla F^{\zeta}(\varphi(s)), \tag{2.7}$$

for all s in the domain of φ and some real function λ. Note that $\lambda(\varphi(s)) > 0$ for all s and that λ is continuous (since ∇F^{ζ} and ϕ' are continuous). Equation (2.7) means that φ is an integral curve of the vector field $\lambda\nabla F$. The result follows if we show that the images of the two integral curves ∇F^{ζ} and $\lambda\nabla F^{\zeta}$ are identical. To see this, define the time change $\tau(t)$ by

$$\int_{s_0}^{\tau(t)} \lambda(\varphi(s))ds = t - t_0,$$

and by differentiating we can derive

$$1 = \lambda(\varphi(\tau(t)))\tau'(t).$$

Hence, denoting $\phi(t) := \varphi(\tau(t))$, it is

$$\phi'(t) = \lambda(\phi(t))\,\nabla F^{\zeta}(\phi(t))\,\tau'(t) = \nabla F^{\zeta}(\phi(t)),$$

and $\phi(t_0) = \varphi(\tau(t_0)) = x_0$, which completes this proof.

(c) $F^{\zeta}(z)$ is the real part of $f^{\zeta}(z) := \zeta/(|\zeta|(\zeta - z))$. Since f^{ζ} satisfies the Cauchy-Riemann equations for all $z \neq \zeta$, it is analytic in $B_{|\zeta|}$. Then F^{ζ} is harmonic in $B_{|\zeta|}$, since the real (and the imaginary) part of analytic functions is harmonic.

(d) Let f^{ζ} be defined as in (c). Denote $u(z) = F^{\zeta}(z) = \operatorname{Re} f^{\zeta}$ and $v(z) := \operatorname{Im} f^{\zeta}$. As in (c), f^{ζ} satisfies the Cauchy-Riemann equations. Hence using Cauchy-Riemann equations, for all $z = x + iy \neq \zeta$ it is

$$\begin{aligned}
|\nabla F^{\zeta}(z)|^2 &= \big(\partial_x u(z)\big)^2 + \big(\partial_y u(z)\big)^2 \\
&= \big(\partial_x u(z)\big)^2 + \big(\partial_x v(z)\big)^2 \\
&= \left|\frac{\partial}{\partial_z}\frac{\zeta}{|\zeta|(\zeta - z)}\right|^2 \\
&= \left|\frac{\partial}{\partial_z}\frac{1}{\zeta - z}\right|^2,
\end{aligned}$$

and the result follows from

$$\frac{\partial}{\partial_z}\frac{1}{\zeta - z} = \frac{1}{(\zeta - z)^2}, \quad \text{for all } z \neq \zeta.$$

(e) By maximum principle (Theorem 9) and because F^ζ is harmonic on B_R, it attains its maximum on the boundary ∂B_R. The point $z \in \partial B_R$ with minimum distance to ζ is $z = \frac{R}{|\zeta|}\zeta$. By the representation of F^ζ provided in (a) and since $|z|$ is constant on ∂B_R, F^ζ attains its maximum on ∂B_R in z.

(f) Since $\zeta \notin \bar{B}(z,r)$, F^ζ attains its maximum on $\partial B(z,r)$. Hence, using (d),

$$\begin{aligned}
\sup_{\omega \in B(z,r)} F^\zeta(\omega) - F^\zeta(z) &\leq \max_{\omega \in \partial B(z,r)} \int_{[z,\omega]} |\nabla F^\zeta| \\
&\leq \max_{\omega \in \partial B(z,r)} \frac{1}{|\zeta - \omega|^2} \\
&= \frac{1}{\left|\zeta - r\frac{\zeta - z}{|\zeta - z|} - z\right|^2} \\
&= \frac{1}{\left(|\zeta - z| - r\right)^2},
\end{aligned}$$

where $[z,\omega]$ denotes the line segment from z to ω.

Remark. *Using the notation in the proof of Lemma 10 (b) we obtain that $r \to \infty$ as $R \to -1$. Hence, using (2.6) and $K^1(z) = F^1(z) - 1/2$, we get the asymptotic behavior*

$$\lim_{|z| \to \infty} F^\zeta(z) = 0.$$

Figure 4 below exemplifies the behavior of F^ζ on circles ∂B_r, for $r < |\zeta|$, which will help for later considerations.

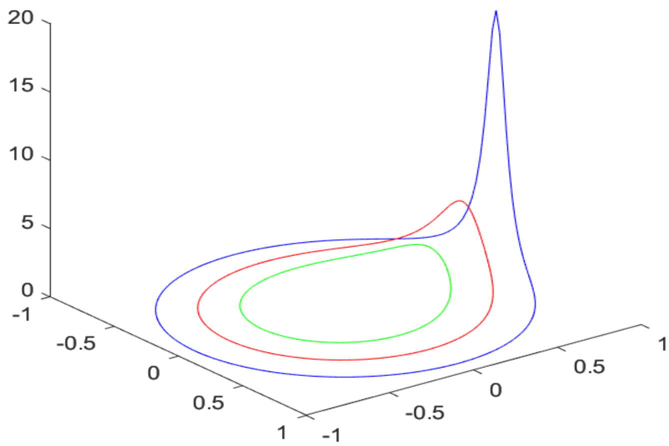

Figure 4: The Poisson kernel K^1 on the circles ∂B_r, for $r = 0.5$ (green), $r = 0.7$ (red), and $r = 0.9$ (blue).

We now state a lemma which bounds F^ζ on circles close to $B_{|\zeta|}$. It will be used to show that the level sets of a later defined function do not deviate a lot from the level sets of F^ζ.

Lemma 11. *For* $|\zeta| > 4C > 24$,

$$F^\zeta(z) - \frac{1}{2|\zeta|} \leq -\frac{C}{|z-\zeta|^2}, \quad \textit{for } z \in \partial B_{|\zeta|+14C},$$
$$F^\zeta(z) - \frac{1}{2|\zeta|} \geq \frac{C}{|z-\zeta|^2}, \quad \textit{for } z \in \partial B_{|\zeta|-14C}.$$

Proof. Figure 5 sketches the integral curves of ∇F^ζ, which by Lemma 10 (b) are circles through ζ with center on the tangent line to $B_{|\zeta|}$ at ζ. Let p_1 and p_2 denote the points where these circles are tangent to $\partial B_{|\zeta|+2C}$, and let R be the short segment of $\partial B_{|\zeta|+2C}$ from p_1 to p_2 (red in Figure 5 below).

Step 1. Show that

$$F^\zeta(z) - \frac{1}{2|\zeta|} \leq -\frac{C}{|z-\zeta|^2}, \quad \text{for all } z \in \partial B_{|\zeta|+2C} \setminus R. \tag{2.8}$$

To verify this, fix a point $z \in \partial B_{|\zeta|+2C} \setminus R$. By $\gamma_z = \gamma : [a, b] \to \mathbb{C}$ denote the section of the integral curve through z starting at z and ending at some point $\tilde{z} \in \partial B_{|\zeta|}$ with $\tilde{z} \neq \zeta$ (as in Figure 5 below).

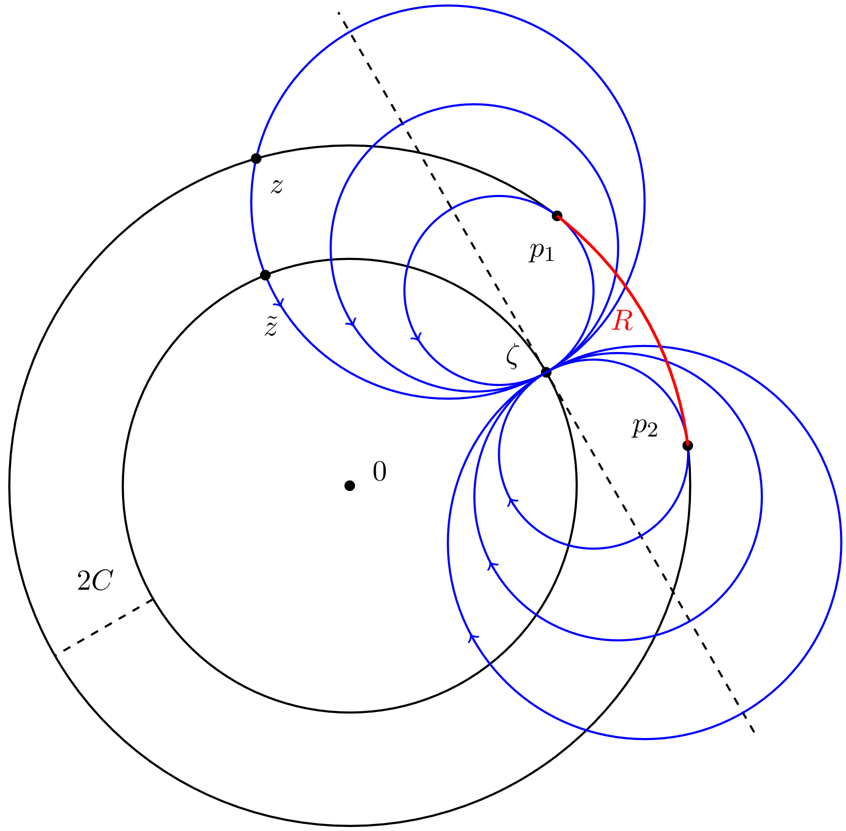

Figure 5: Integral curves of F^ζ (blue) connecting $\partial B_{|\zeta|}$ and $\partial B_{|\zeta|+2C}$. Note the difference in the length of γ_z between $z \in R$ and $z \in \partial B_{|\zeta|+2C} \setminus R$. We defined $R \subset \partial B_{|\zeta|+2C}$ (red) by the arc between p_1 and p_2, in which the integral curves are tangent to $\partial B_{|\zeta|+2C}$

Since $F^\zeta = \frac{1}{2|\zeta|}$ on $\partial B_{|\zeta|} \setminus \{\zeta\}$, it is $F^\zeta(\tilde{z}) = \frac{1}{2|\zeta|}$. Using Lemma 10 (c) we see that

$$\begin{aligned}
\frac{1}{2|\zeta|} - F^{\zeta}(z) &= \int_{\gamma} \nabla F^{\zeta} \\
&= \int_{[a,b]} \nabla F^{\zeta}(\gamma(t)) \cdot \gamma'(t)dt \\
&= \int_{[a,b]} |\nabla F^{\zeta}(\gamma(t)) \cdot \gamma'(t)|dt \\
&\geq \min_{y\in\gamma} \frac{1}{|\zeta - y|^2} \int_{[a,b]} |\gamma'(t)|dt \\
&= \frac{\text{Length}(\gamma)}{\max_{y\in\gamma} |\zeta - y|^2},
\end{aligned}$$

where $y \in \gamma$ means that y is an element of the image of γ. Fix some $z \in \partial B_{|\zeta|+2C} \setminus R$. Then the distance of y to ζ changes just a little for y along $\gamma_z = \gamma$. More precisely,

$$\max_{y\in\gamma} |\zeta - y|^2 \leq 2|\zeta - z|^2. \tag{2.9}$$

This can be seen by noting that γ_z is contained in a square with vertices z and $\tilde{z}$. Hence,

$$\max_{y\in\gamma} |\zeta - y| \leq |\zeta - z| + \sqrt{2|z - \tilde{z}|}.$$

Since the integral curve through p_1 is tangent to $B_{|\zeta|+2C}$ at p_1, we obtain $|z - \tilde{z}| \leq |p_1 - \tilde{p}_1| = |p_1 - \zeta| \leq |z - \zeta|$. Hence, with $C > 6$ we have

$$\max_{y\in\gamma} |\zeta - y| \leq |\zeta - z| + \sqrt{2|z - \zeta|} \leq \sqrt{2}|\zeta - z|,$$

which shows (2.9). By (2.9) and the fact that each curve connecting $\partial B_{|\zeta|+2C}$ and $\partial B_{|\zeta|}$ has length at least $2C$, Equation (2.8) holds.

Step 2. Now we will use (2.8) to complete the first inequality of this lemma. Our strategy is to choose a circle sufficiently larger than $\partial B_{|\zeta|+2C}$ so that for each point z on that circle the level curve $F^{\zeta}(\cdot) = F^{\zeta}(z)$ can be traced to a point where the level curve intersects $\partial B_{|\zeta|+2C} \setminus R$. Then apply (2.8) on the intersection point.

We choose a cone so that R is inside the cone. Fix ζ and C with $|\zeta| > 4C$. Assume $\operatorname{Im}\zeta = 0$. We define the cone with vertex in 0 by its height h (as in Figure 6). Also let s, k, and ω be defined as in Figure 6 below.

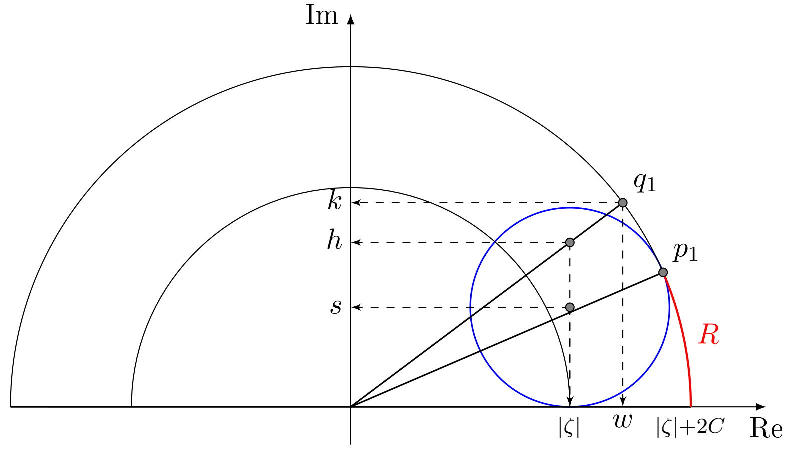

Figure 6: This figure defines s, h, k, $\omega \in \mathbb{R}$ and $q_1 \in \mathbb{C}$. Where s denotes the radius of the integral curve tangent to $B_{|\zeta|+2C}$ (blue). We consider k, ω, and q_1 to be determined by h. Otherwise, the setting is (up to rotation) the same as in Figure 5.

The radius of the integral curve tangent to $\partial B_{|\zeta|+2C}$ (blue in Figure 6), which is defined by s, is smaller than $2C$ (since the center is inside the annulus $B_{|\zeta|+2C} \setminus B_{|\zeta|}$). Thus, we choose $h = 2C$. We will show now that

$$|\zeta| + C < \omega < |\zeta| + 2C, \quad \text{and } k \leq 3C. \tag{2.10}$$

The upper bound of ω is obvious. For the lower bound, note that

$$\frac{\omega}{|\zeta|} = \frac{|\zeta| + 2C}{\sqrt{h^2 + |\zeta|^2}}.$$

Hence, $|\zeta| + C < \omega$ holds if

$$\left(\frac{|\zeta|(|\zeta| + 2C)}{|\zeta| + C}\right)^2 - |\zeta|^2 > h^2 = 4C^2.$$

The left-hand side equals

$$\frac{|\zeta|^2(2C|\zeta|+3C^2)}{(|\zeta|+C)^2} = \frac{2C|\zeta|^2}{|\zeta|+C}\cdot\frac{|\zeta|+3C/2}{|\zeta|+C}$$
$$\geq \frac{2C|\zeta|^2}{|\zeta|+C} \geq \frac{2C|\zeta|^2}{|\zeta|+|\zeta|/4} = \frac{8}{5}C|\zeta| > 4C^2.$$

Again we use intercept theorem and $|\zeta| \geq 4C$ to show

$$k = h\frac{|\zeta|+2C}{|\zeta|} = 2C + \frac{4C^2}{|\zeta|} \leq 3C.$$

Now we will find an upper bound for the radius of the circle, which runs through q_1 and is tangent to $B_{|\zeta|}$ at ζ (see radius r in Figure 7 below).

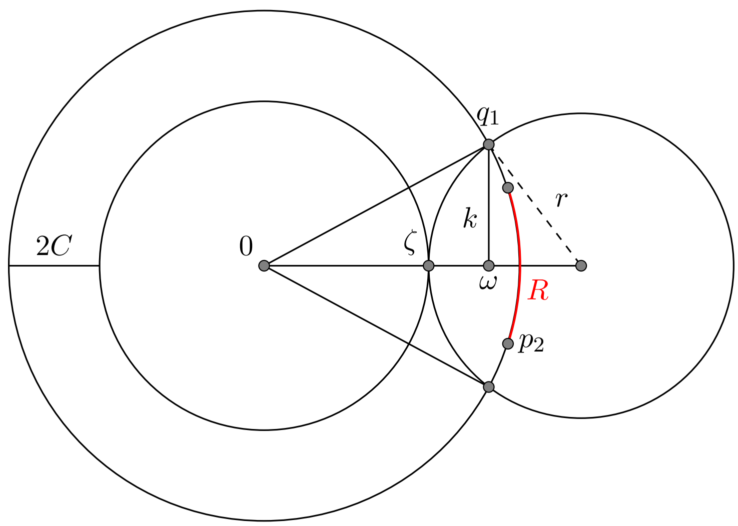

Figure 7: k, ω, R, q_1, and p_2 are as in Figure 6 and 5 above. The radius of the circle tangent to $\partial B_{|\zeta|}$ at ζ and through q_1 is denoted by r. This circle is a level curve of F^ζ and contains the arc R, where points are excluded from the inequalities (2.8).

We can compute the radius r using $r^2 = k^2 + (r-(\omega-|\zeta|))^2$. Hence with (2.10),

$$r = \frac{k^2+(\omega-|\zeta|)^2)^2}{2(\omega-|\zeta|)} < \frac{(3C)^2+(2C)^2}{2C} < 7C.$$

Since the level curves partition $\mathbb{C} \setminus \{\zeta\}$, we can map each point $z \in B_{|\zeta|+2r}$ onto a point $\hat{z} \in B_{|\zeta|+2C} \setminus R$ (by following a level curve). Apply (2.8) to $\hat{z}$ and the result follows since $F^{\zeta}(z) = F^{\zeta}(\hat{z})$.

With the same steps, one can prove the second inequality of Lemma 11.

3 Stochastic Analysis on the Grid

Let $\mathcal{G} \subset \mathbb{C}$ be the $\mathbb{Z}^2$-grid as defined in (1.3). Analogous to discrete functions that are harmonic with relation to the random walk (see (2.1)), we want to give a definition of a harmonic function on the grid and an (a.s.) continuous process on the grid such that similar properties hold. We start with some topological definitions on the grid.

The grid is equipped with the canonical topology induced by $\mathbb{C}$. An **edge** in $\mathcal{G}$ is a closed line segment of length one, which connects two points of $\mathbb{Z}^2$ (which we consider as a subset of $\mathcal{G}$). Denote $V = \{1, -1, i, -i\}$, and for $v \in V$ let e_v be the edge connecting 0 and v. The **0-cross** E denotes the interior of the union of the four edges containing 0. Note that $\partial E = V$ in the topology of $\mathcal{G}$.

3.1 Grid-Harmonic Functions

Using these definitions, we can now define harmonicity for functions which are defined on the grid.

Let U be an open subset of $\mathcal{G}$, then call a continuous function $f : U \to \mathbb{R}$ **grid-harmonic** on U if

- f is linear on each interval I, for I being a subset of $U \cap e$, for some edge e of $\mathcal{G}$, and
- for each lattice point z in U, the sum of the slopes of f on the four line segments starting in z is zero; i.e, if we denote $d_v = \partial_t f(z + tv)$ for

$v \in V$ and such $t > 0$ that $z + tv$ is part of the connected component of z in $U \cap (z + E)$, then we have

$$\sum_{v \in V} d_v = 0.$$

Remark. *If in addition $\partial U \subset \mathbb{Z}^2$, i.e., U is a union of crosses $z + E$ with $z \in \mathbb{Z}^2$, then for each $z \in U \cap \mathbb{Z}^2$ we can write*

$$f|_{z+E}(y) = f(z) + f_z(y - z), \tag{3.1}$$

where f_z is grid-harmonic in E with $f_z(0) = 0$; for $x \in E$ this enables us to write

$$f_z(x) = \sum_{v \in V} \alpha_v |x| \mathbf{1}_{e_v}(x), \tag{3.2}$$

for some $\alpha_v \in \mathbb{R}$ (depending on z) with

$$\sum_{v \in V} \alpha_v = 0.$$

Note, if we extend a discrete harmonic function on $\mathbb{Z}^2$ continuously and linearly along the edges of $\mathcal{G}$, then this function is grid-harmonic.

Lemma 12 (Maximum principle)**.** *Let U be an open subset of $\mathcal{G}$. For a continuous function $f : \bar{U} \to \mathbb{R}$, which is grid-harmonic in U, we have*

$$\max_{\bar{U}} f = \max_{\partial U} f.$$

Proof. Suppose f attains a strict maximum in $x_0 \in U \cap \mathbb{Z}^2$. There is an $\epsilon > 0$ such that $\bar{B}(x_0, \epsilon) \cap \mathcal{G} \subset U$. Then,

$$f(x_0) = \sum_{z \in \partial B(x_0, \epsilon) \cap \mathcal{G}} \frac{1}{4} f(z)$$

leads to a contradiction since $f(z) < f(x_0)$.

Since f is linear along the edges, f can neither achieve a strict maximum on an open edge. $\square$

3.2 Grid Brownian Motion

Construct an a.s. continuous process with state space $\mathcal{G}$, which acts like a Brownian motion on the edges and like a simple random walk on the lattice points of $\mathcal{G}$.

Define the grid Brownian motion $\beta_t : \Omega \to \mathcal{G}$ by specifying a particles movement from one lattice point to the next. Let $\mathcal{B}$ be a one-dimensional standard Brownian motion and

$$\tau_{(-1,1)} = \inf\{t \geq 0 \ \mid \mathcal{B}(t) \notin (-1,1)\}$$

its exit time from the interval $(-1,1)$. By $\tilde{\mathcal{B}}(t) = \mathcal{B}(t \wedge \tau_{(-1,1)})$ denote the stopped Brownian motion. Note that $\tilde{\mathcal{B}}$ a.s. returns infinitely often to the origin before exiting $(-1,1)$. For $t > 0$ let

$$\begin{aligned} a(t) &= \sup\{s \leq t \,|\, \tilde{\mathcal{B}}(s) = 0\}, \\ b(t) &= \inf\{s \geq t \,|\, \tilde{\mathcal{B}}(s) = 0\}, \end{aligned}$$

and call $(a(t), b(t))$ the **interval of excursion** straddling t, with the convention $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. After sampling $\tilde{\mathcal{B}}$, choose for each interval of excursion a direction in V, uniformly and independently of one another. The process $\varphi(t)$, $t \geq 0$ denotes the direction of the interval of excursion straddling t, chosen as described. Since $\{t \geq 0 \,|\, \tilde{\mathcal{B}}(t) = 0 \ \}$ is a null set, the process $\varphi(t)$ is defined almost everywhere. By the sampled Brownian motion and the process φ, define a process $X(t) : \Omega \to \bar{E}$, which describes a particles movement on the 0-cross by:

$$X(t) := \varphi(t) \cdot \tilde{\mathcal{B}}(t).$$

We call X a **cross motion** on E. In particular, the particle's distance to 0 is distributed like $|\tilde{\mathcal{B}}(\cdot)|$. By $\mathcal{F}^X$ denote the natural filtration of X.

Let X^1, X^2, X^3, ... be independent processes defined as above; for each such process let $\tau_E^i = \inf\{t \geq 0 \,|\, X^i(t) \notin E\}$ be the hitting time of ∂E. Then for $t \geq 0$ define

$$T_t = \max\Big\{n \in \mathbb{N} \,\Big|\, \sum_{i=1}^{n} \tau_E^i \leq t\Big\},$$

the largest number of stopping times τ_E^i, which can elapse before time t. Furthermore, for $t \geq 0$ denote the overhanging time

$$t' := t - \sum_{i=1}^{T_t} \tau_E^i \in [0, \tau_E^{n+1}).$$

Now we define the **grid Brownian motion** $\beta(t)$ by the sum of the lattice points adjacent to 0 reached by the first T_t processes $X^1, \ldots, X^{T_t}$ plus the position of the last particle, which has not been stopped by time t', i.e.,

$$\beta(t) = \sum_{i=1}^{T_t} X^i(\tau_E^i) + X^{T_t+1}(t'). \tag{3.3}$$

Remark (on grid Brownian motions). *With*

$$\gamma^0 = 0, \quad \gamma^{n+1} = \inf\big\{t > \gamma^n \,\big|\, \beta(t) \in \mathbb{Z}^2 \setminus \{\beta(\gamma^n)\}\big\},$$

the process $Y(n) = \beta(\gamma^n)$ restricted to $\mathbb{Z}^2$, is distributed like a simple random walk.

It is $\{T_t = k\} \in \sigma(\tau_E^1, \ldots, \tau_E^{k+1})$, but T_t is not a stopping time with respect to the natural filtration of β.

Remark (on cross motions). *Since the directions φ are chosen after sampling the Brownian motion, the elementary Markov property does not hold by definition for X. As a substitute for the Markov property, we can define a process $\hat{X}^y(\cdot)$, for $y \in E$, such that for each Borel set A of E and $s, u \geq 0$, it is*

$$\mathbb{P}\big(X(s+u) \in A \,\big|\, \mathcal{F}_s^X\big) = \mathbb{P}\big(\hat{X}^{X(s)}(u) \in A \,\big|\, X(s)\big). \tag{3.4}$$

Roughly, $\hat{X}^y$ can be constructed by running a one-dimensional Brownian motion started in $|y|$ and stopped on exiting $(-1,1)$. For the first interval of excursion, choose the direction $d(y)$, which for $y \in E$ can be defined by

$$d(y) = \sum_{v \in V} v \cdot \mathbf{1}_{e_v}(y) \in V.$$

For later intervals of excursion, uniformly and independently choose directions from V. In case of $y = 0$, for all intervals of excursion choose the direction uniformly and independently. As for X, define $\hat{X}^y(t)$ by the product of the Brownian motion (started in $|y|$ stopped on exiting $(-1,1)$) at time t and the direction of the interval of excursion straddling t.

3.3 Harmonic Functions of Grid Brownian Motions

The main result of this section is Theorem 13. It states that applying a grid-harmonic function to a grid Brownian motion preserves the martingale property. It may be considered as an analog (or combination) of the fact that a discrete harmonic function applied to a random walk or a harmonic function applied to a Brownian motion is a martingale (the discrete case holds by definition; continuous case can be shown by Itô's formula). Lemma 14 and 15 are the preliminary work for the proof of Theorem 13.

In this section, for a grid BM β and $A \subset \mathcal{G}$ we let $\tau_A = \inf\{s \geq 0 \,|\, \beta(s) \notin A\}$ and the left limit

$$f\big(\beta(t \wedge \tau_A -)\big) = \lim_{u \uparrow \tau_A} f\big(\beta(t \wedge u)\big).$$

By $\mathcal{F}$ denote the natural filtration of β.

Theorem 13. *Let β be a grid Brownian motion and A be an open subset of $\mathcal{G}$. Assume that $f : A \to \mathbb{R}$ is grid-harmonic on A. Then*

$$f\big(\beta(t \wedge \tau_A -)\big),\ t \geq 0$$

is a martingale with relation to $\mathcal{F}_t$.

We will reduce the problem to subsets $A \subset \mathcal{G}$ with $\partial A \subset \mathbb{Z}^2$. For such an A and $z \in A \cap \mathbb{Z}^2$, write

$$f|_{z+E}(x) = f(z) + f_z(x - z),$$

where $f_z : E \to \mathbb{R}$ is as in (3.2); in particular grid-harmonic and $f_z(0) = 0$. By $\hat{f}_z : \bar{E} \to \mathbb{R}$ denote the function f_z that is continuously extended to $\bar{E}$.

Remark. *We have not assumed that $f : A \to \mathbb{R}$ is continuously extendable to the boundary. Hence, (according to Figure 8) for $z_1 + 1 = z_2 + i = \zeta \in \partial A$ with $\zeta \in \mathbb{Z}^2$ it may be*

$$f(z_1) + \hat{f}_{z_1}(1) \neq f(z_2) + \hat{f}_{z_2}(i),$$

where z_1, $z_2 \in A$; i.e., on the event $\beta(\tau_A) = \zeta$ the limit $f(\beta(\tau_A-))$ depends on whether the last lattice point β passed, before hitting ζ, was z_1 or z_2.

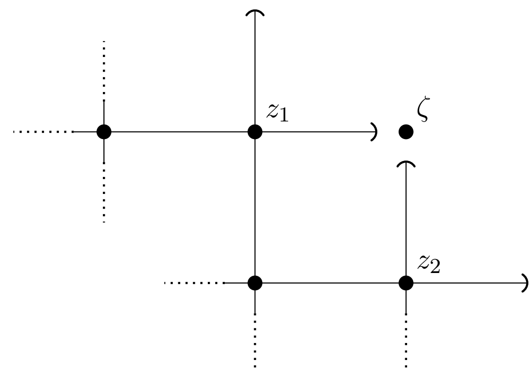

Figure 8: Set $A \subset \mathcal{G}$ with $\zeta \in \partial A$, $\zeta \notin A$, and $\zeta \in \mathbb{Z}^2$.

Therefore, Theorem 13 makes use of the left limit. If we additionally assume that f is continuous on the whole grid $\mathcal{G}$, then the theorem holds without the left limit. Since $H^\zeta : \mathcal{G} \to \mathbb{R}$ (defined in the next section) is continuous on $\mathcal{G}$ (see (2.2); the potential kernel $a(z)$ converges for each $z \in \mathbb{Z}^2$) and grid-harmonic on Ω_ζ, we will not need to consider the left limit of H^ζ.

To prove this theorem, we will reduce the problem to the martingale property of cross motions X, defined in 3.3. In the sequel let $\mathcal{F}^X$ denote its natural filtration and let

$$\tau_B^X = \inf\{t \geq 0 \,|\, X(t) \notin B\}, \quad \text{and} \quad \tau_B^i = \tau_B^{X^i},$$

for some cross motion X^i and a subset B of E. The main ingredients for the proof of Theorem 13 are the following two lemmas.

Lemma 14. *Let X be a cross motion; B be an open subset of the 0-cross E; and $g : B \to \mathbb{R}$ grid-harmonic on B, then*

$$g(X(t \wedge \tau_B^X)),\ t \geq 0$$

is a martingale w.r.t. $\mathcal{F}^X$.

We will now introduce some definitions related to the grid BM β. For A as in Theorem 13 define

$$\nu^0 = 0,$$
$$\nu^{l+1} = \inf\big\{u > \nu^l \,\big|\, \beta(u \wedge \tau_A) \in \mathbb{Z}^2 \setminus \{\beta(\nu^l \wedge \tau_A)\}\big\}.$$

For $t \geq 0$ denote

$$N_t = \max\{k \in \mathbb{N} \,|\, \nu^k \leq t\}$$

and $t^l = t - \nu^l$.

The next lemma states that to prove the martingale property of $f(\beta(t \wedge \tau_A -))$, it suffices to consider only the first cross motion after time ν^{N_s}; that is the time of β's last hit of a lattice point before time $s \wedge \tau_A$.

Lemma 15. *If A is an open subset of $\mathcal{G}$ such that $\partial A \subset \mathbb{Z}^2$, f is grid-harmonic on A, and β a grid Brownian motion, then on the event $\{N_s = j\} \cap \{\beta(\nu^j) = z\}$ for some $j \in \mathbb{N}$, $z \in \mathbb{Z}^2$ it is*

$$\mathbb{E}\big(f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_s\big) = f(z) + \mathbb{E}\big(f_z(X^{j+1}(t^j)) \,\big|\, \mathcal{F}_s\big),$$

for $0 \leq s < t$.

Before proving Theorem 13, Lemma 14, and Lemma 15, we introduce some notation. For independent random variables $L : \Omega_1 \to \mathcal{G}$ and $M : \Omega_2 \to \mathcal{G}$ on the probability spaces $(\Omega_l, \mathcal{A}_l, \mathbb{P}_l)$, $l = 1, 2$, and for $h : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ measurable and bounded, denote

$$\mathbb{E}^L\big(h(L, M)\big) = \int_{\Omega_1} h(L, M)\mathbb{P}_1(d\omega_1).$$

For $L = X^i$ write $\mathbb{E}^i(h(X^i, M)) = \mathbb{E}^{X^i}(h(X^i, M))$.

Proof of Theorem 13. Without a loss of generality, assume $A \subset \mathcal{G}$ is open, connected, and $0 \in A$. Define

$$\tilde{A} = \bigcup_{z \in A \cap \mathbb{Z}^2} (z + E).$$

Then $A \subset \tilde{A}$ and $\partial \tilde{A} \subset \mathbb{Z}^2$. Furthermore, extend f linearly along the edges of $\tilde{A}$; then $f : \tilde{A} \to \mathbb{R}$ is grid-harmonic on $\tilde{A}$. Note that due to optional stopping theorem ([RY99] Ch. II, Theorem 3.3) it suffices to prove the theorem for the extended function f on the enlarged set $\tilde{A}$.

Under these conditions we can apply Lemma 15. Hence, on the event $\{N_s = j\} \cap \{\beta(\nu^j) = z\} \in \mathcal{F}_s$ it is

$$\mathbb{E}\big(f(\beta(t \wedge \tau_A-)) \,\big|\, \mathcal{F}_s\big) = f(z) + \mathbb{E}\big(\hat{f}_z(X^{j+1}(t^j)) \,\big|\, \mathcal{F}_s\big),$$

for some $z \in \mathbb{Z}^2$ and $j \in \mathbb{N}$. Note that $X^{j+1}(s^j) =: Y$ is $\mathcal{F}_s$-measurable. Let $u = t - s$ and for $y \in \bar{E}$ let $\hat{X}^y$ be defined as in (3.4), in particular independent of $\mathcal{F}_s$. Applying Lemma 14 to $\hat{X}^y$, gives

$$\begin{aligned}
\mathbb{E}\big(f_z(X^{j+1}(t^j)) \,\big|\, \mathcal{F}_s\big) &= \mathbb{E}\big(f_z(X^{j+1}(u + s^j)) \,\big|\, \mathcal{F}_s\big) \\
&= \mathbb{E}^{\hat{X}}\big(f_z(\hat{X}^Y(u))\big) \\
&= f_z(\hat{X}^Y(0)) \\
&= f_z(X^{j+1}(s^j)).
\end{aligned}$$

Plugging in, yields

$$\mathbb{E}\big(f(\beta(t \wedge \tau_A-)) \,\big|\, \mathcal{F}_s\big) = f(z) + f_z(X^{j+1}(s^j)) = f(\beta(s \wedge \tau_A-)),$$

on the event $\{N_s = j\} \cap \{\beta(\nu^{N_s}) = z\}$; and the result follows. □

Proof of Lemma 14. Since $X(0) = 0$, we may assume that B is open, connected, and $0 \in B$. We can extend $g : B \to \mathbb{R}$ to a grid-harmonic function on E, then by optional stopping theorem ([RY99] Ch. II, Theorem 3.3) it suffices to prove the lemma for $B = E$. Prove,

$$\mathbb{E}\big(g(X(t)) - g(X(s)) \,\big|\, \mathcal{F}_s^X\big) = 0, \quad \text{for } 0 \leq s < t.$$

We may assume $g(0) = 0$, which enables us to write g as in Equation (3.2). Let

$$\tau^s = \inf\{t \geq s \,|\, X(t) = 0\},$$

the time of X's first return to the origin after s. Split the conditional expectation into the increment of $g(X(\cdot))$ over time s to τ^s, and the increment over τ^s to t:

$$\begin{aligned}
\mathbb{E}\big(g(X(t)) - g(X(s)) \,\big|\, \mathcal{F}_s^X\big) =& \mathbb{E}\big(g(X(t)) - g(X(t \wedge \tau^s)) \,\big|\, \mathcal{F}_s^X\big) \\
&+ \mathbb{E}\big(g(X(t \wedge \tau^s)) - g(X(s)) \,\big|\, \mathcal{F}_s^X\big).
\end{aligned} \tag{3.5}$$

At first, show that the first summand of (3.5) equals zero.

It is $\tau^s \geq s$ and $g(X(\tau^s)) = 0$, hence,

$$\mathbb{E}\big(g(X(t)) - g(X(t \wedge \tau^s)) \,\big|\, \mathcal{F}_s^X\big) = \mathbb{E}\big(\mathbf{1}_{\{t>\tau^s\}}\, \mathbb{E}\big(g(X(t)) \,\big|\, \mathcal{F}_{\tau^s}^X\big) \,\big|\, \mathcal{F}_s^X\big).$$

Consider the event $\{t > \tau^s\}$; here, we can conclude by the definition of $\hat{X}^0 =: \hat{X}$ that

$$\mathbb{P}(\hat{X}(u) \in A) = \mathbb{P}(X(\tau^s + u) \in A \,|\, \mathcal{F}_{\tau^s}^X),$$

for all Borel subsets A of E and $u \geq 0$. In addition, $\hat{X}$ is independent of $\mathcal{F}_{\tau^s}^X$. Hence, by setting $u = t - \tau^s > 0$, we get

$$\begin{aligned}
\mathbb{E}\big(g(X(t)) \,\big|\, \mathcal{F}_{\tau^s}^X\big) &= \mathbb{E}\big(g(\hat{X}(t - \tau^s)) \,\big|\, \mathcal{F}_{\tau^s}^X\big) \\
&= \mathbb{E}^{\hat{X}}\big(g(\hat{X}(u))\big) \\
&= \mathbb{E}^{\hat{X}}\left(\sum_{v\in V} \alpha_v |\hat{X}(u)| \mathbf{1}_{\{\hat{X}(u)\in e_v\}}\right) \\
&= \sum_{v\in V} \alpha_v\, \mathbb{E}^{\hat{X}}\big(|\hat{X}(u)|\big)\, \mathbb{E}^{\hat{X}}\big(\mathbf{1}_{\{\hat{X}(u)\in e_v\}}\big) \\
&= \mathbb{E}^{\hat{X}}\big(|\hat{X}(u)|\big) \frac{1}{4} \sum_{v\in V} \alpha_v \\
&= 0,
\end{aligned} \tag{3.6}$$

where we used the independence of $|\hat{X}(u)|$ and $\{\hat{X}(u) \in e_v\}$. By plugging in, we observe that the first summand of (3.5) equals 0.

Proceed with the second summand of (3.5) and show

$$\mathbb{E}\big(g(X(t \wedge \tau^s)) \,\big|\, \mathcal{F}_s^X\big) = g(X(s)).$$

Note that the particles $X(t \wedge \tau^s)$ and $X(s)$ are on the same edge. Furthermore, the information of the current edge e_v, $v \in V$ on which $X(t)$ is located is in $\mathcal{F}_t^X$. Denoting $\tau_{e_v}^s = \inf\{u \geq s \,|\, X(u) \in \partial e_v\}$, the first time X hits ∂e_v after s, one can obtain

$$\mathbb{E}\big(g(X(t \wedge \tau^s)) \,\big|\, \mathcal{F}_s^X\big) = \sum_{v\in V} \mathbf{1}_{\{X(s)\in e_v\}} \mathbb{E}\big(g(X(t \wedge \tau_{e_v}^s)) \,\big|\, \mathcal{F}_s^X\big). \tag{3.7}$$

We may assume $X(s) \in e_v$. For $y \in e_v$ let $\hat{X}^y$ be defined as in (3.4); let

$$\hat{\tau}^y_{e_v} = \inf\{u \geq 0 \,|\, \hat{X}^y(u) \in \partial e_v\},$$

and let $r = t - s$. Again, we use (3.4), the pseudo Markov property of X, to get

$$\begin{aligned} \mathbb{E}\big(g(X(t \wedge \tau^s_{e_v})) \,\big|\, \mathcal{F}^X_s\big) &= \mathbb{E}\big(g(\hat{X}^{X(s)}((t \wedge \tau^s_{e_v}) - s)) \,\big|\, X(s)\big) \\ &= \mathbb{E}\big(g(\hat{X}^{X(s)}(r \wedge \hat{\tau}^{X(s)}_{e_v})) \,\big|\, X(s)\big). \end{aligned} \tag{3.8}$$

Recall the construction of $\hat{X}$; then with the notation $X(s)(\omega) = y = \lambda v$ for some $y \in e_v$ and $\lambda \in (0,1)$, the modulus $|\hat{X}^y(u \wedge \hat{\tau}^y_{e_v})|$ is distributed as $\mathcal{B}^\lambda(u \wedge T_{(0,1)})$, where $\mathcal{B}^\lambda$ denotes a Brownian motion started in λ and $T_{(0,1)}$ the first exit of $\mathcal{B}^\lambda$ from $(0,1)$. As a consequence, a.s.,

$$\begin{aligned} \mathbb{E}\big(g(\hat{X}^{X(s)}(r \wedge \hat{\tau}^{X(s)}_{e_v})) \,\big|\, X(s)\big)(\omega) &= \mathbb{E}^{\hat{X}}\left(\alpha_v |\hat{X}^y(r \wedge \hat{\tau}^y_{e_v})|\right) \\ &= \alpha_v \, \mathbb{E}\big(\mathcal{B}^\lambda(r \wedge T_{(0,1)})\big) \\ &= \alpha_v \lambda \\ &= g(X(s))(\omega). \end{aligned}$$

Combining this with (3.8) and (3.7) proves the lemma.

Proof of Lemma 15. Fix $0 \leq s < t$. Consider all equations of this proof on the event $\{N_s = j\}$. It is $\mathbb{E}(|N_t|) < \infty$, hence,

$$\mathbb{E}\big(f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_s\big) = \sum_{k=j}^{\infty} \mathbb{E}\big(\mathbf{1}_{\{N_t = k\}} f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_s\big).$$

In the next step we will show that for each summand on the right-hand side where $k \geq j+1$ the following holds

$$\mathbb{E}\big(\mathbf{1}_{\{N_t = k\}} f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_s\big) = \mathbb{E}\big(\mathbf{1}_{\{N_t = k\}} f(\beta(\nu^{j+1})) \,\big|\, \mathcal{F}_s\big). \tag{3.9}$$

This can be proven by an iteration of the tower property. In detail, since

$$\mathcal{F}_s \subset \mathcal{F}_{\nu^{j+1}} \subset \mathcal{F}_{\nu^{j+2}} \subset \ldots \subset \mathcal{F}_{\nu^k},$$

we may project $\mathbf{1}_{\{N_t=k\}} f(\beta(t \wedge \tau_A -))$ on $\mathcal{F}_{\nu^k}$ down to $\mathcal{F}_{\nu^{j+1}}$ before projecting on $\mathcal{F}_s$. To handle $\mathbf{1}_{\{N_t=k\}}$, make use of the equality

$$\{N_t = k\} = \{\nu^k \leq t < \nu^{k+1}\}.$$

Hence, $\mathbf{1}_{\{N_t=k\}} = \mathbf{1}_{\{\nu^k \leq t\}} \mathbf{1}_{\{t<\nu^{k+1}\}}$ and for $k \geq j+1$ we get

$$\mathbb{E}\big(\mathbf{1}_{\{N_t=k\}} f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_{\nu^k}\big) = \mathbf{1}_{\{\nu^k \leq t\}} \mathbb{E}\big(\mathbf{1}_{\{t<\nu^{k+1}\}} f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_{\nu^k}\big).$$

Note that $\beta(\nu^k)$ and ν^k are $\mathcal{F}_{\nu^k}$-measurable. By considering the event $\{y = \beta(\nu^k)\}$, we fix the center of the last cross motion X^{k+1} of β before time $t \wedge \tau_A$. Therefore, on this event we can now get rid of the left limit,

$$\hat{f}_y(\beta(t \wedge \tau_A -) - y) = \hat{f}_y(\beta(t \wedge \tau_A) - y).$$

Hence, on the event $\{\nu^k \leq t\} \cap \{\beta(\nu^k) = y\}$ it is

$$\begin{aligned}
&\mathbb{E}\big(\mathbf{1}_{\{t<\nu^{k+1}\}} f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_{\nu^k}\big) \\
&\quad = \mathbb{E}(\mathbf{1}_{\{t<\nu^{k+1}\}} \,|\, \mathcal{F}_{\nu^k}) f(y) + \mathbb{E}\big(\mathbf{1}_{\{t<\nu^{k+1}\}} \hat{f}_y(\beta(t \wedge \tau_A) - y) \,\big|\, \mathcal{F}_{\nu^k}\big).
\end{aligned}$$

By Lemma 14 the second summand equals zero. More precisely, it is

$$\begin{aligned}
&\mathbb{E}\big(\mathbf{1}_{\{t<\nu^{k+1}\}} \hat{f}_y(\beta(t \wedge \tau_A) - y) \,\big|\, \mathcal{F}_{\nu^k}\big) \\
&\quad = \mathbb{E}^{k+1}\big(\mathbf{1}_{\{t<\nu^{k+1}\}} \hat{f}_y(X^{k+1}(t^k))\big) \\
&\quad = \mathbb{E}^{k+1}\Big(\mathbf{1}_{\{t^k<\tau_E^{k+1}\}} \sum_{v \in V} \alpha_v |X^{k+1}(t^k)| \mathbf{1}_{\{X^{k+1}(t^k) \in e_v\}}\Big) \\
&\quad = 0.
\end{aligned}$$

With the independence of $\{t^k < \tau_E^{k+1}\}$ and $\{X^{k+1}(t^k) \in e_v\}$ the last equation follows by (3.6). Plugging in, gives us

$$\mathbb{E}\big(\mathbf{1}_{\{N_t=k\}} f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_{\nu^k}\big) = \mathbb{E}\big(\mathbf{1}_{\{N_t=k\}} \,\big|\, \mathcal{F}_{\nu^k}\big) f(\beta(\nu^k)).$$

For $k = j+1$, this implies (3.9). With the same arguments for larger k we get by the tower property

$$\begin{aligned}
\mathbb{E}\big(\mathbb{E}\big(\mathbf{1}_{\{N_t=k\}} f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_{\nu^k}\big) \,\big|\, \mathcal{F}_{\nu^{k-1}}\big) &= \mathbb{E}\big(\mathbf{1}_{\{N_t=k\}} f(\beta(\nu^k)) \,\big|\, \mathcal{F}_{\nu^{k-1}}\big) \\
&= \mathbb{E}(\mathbf{1}_{\{N_t=k\}} \,|\, \mathcal{F}_{\nu^{k-1}}) f(\beta(\nu^{k-1})).
\end{aligned}$$

Iterating this argument we get (3.9) for all $k \geq j+1$.

Now reunite the events $\{N_t \geq j+1\}$ and $\{N_t = j\}$. If $N_t \geq j+1$, it is $t^j = \tau_E^{j+1}$. Therefore, on the event $\{N_s = j\} \cap \{\beta(\nu^j) = z\}$ we have

$$\begin{aligned}
&\mathbb{E}\big(f(\beta(t \wedge \tau_A -)) \,\big|\, \mathcal{F}_s\big) \\
&\quad = f(z) + \mathbb{E}\big(\mathbf{1}_{\{N_t = j\}} \hat{f}_z(X^{j+1}(t^j)) \,\big|\, \mathcal{F}_s\big) + \mathbb{E}\big(\mathbf{1}_{\{N_t \geq j+1\}} \hat{f}(X^{j+1}(\tau_E^{j+1})) \,\big|\, \mathcal{F}_s\big) \\
&\quad = f(z) + \mathbb{E}\big(\hat{f}_z(X^{j+1}(t^j)) \,\big|\, \mathcal{F}_s\big).
\end{aligned}$$

$\square$

4 Approximating Discrete Poisson Kernel

In this section we define a function H^ζ on the grid $\mathcal{G}$, which is a discrete analog of F^ζ, and derive some of its properties. We define a ball-shaped set $\Omega_\zeta \subset \mathcal{G}$ in which H^ζ is grid harmonic. The function H^ζ will be used to define the main martingale for the proof of Theorem 1 (see (5.3)).

First, define H^ζ for ζ in the cone $K = \{z \in \mathbb{Z}^2 \,|\, 0 \leq \operatorname{Im} z \leq \operatorname{Re} z\}$ as a sum of potential kernels $a(\cdot)$ (defined in Section 2.3) and then use the symmetries of the potential kernel $a(\cdot)$ to define H^ζ for all $\zeta \in \mathbb{Z}^2$. For $\zeta = x + iy \in K$ define the scalars $\lambda_1 = (x-y)/|\zeta|$, $\lambda_2 = y/|\zeta|$. Let

$$H^\zeta(z) = \frac{\pi}{2}\big(\lambda_1 a(z-(\zeta+1)) + \lambda_2 a(z-(\zeta+1+i)) - (\lambda_1+\lambda_2) a(z-\zeta)\big) \tag{4.1}$$

and for arbitrary $\zeta \in \mathbb{Z}^2$ choose an element ψ in the group of symmetries as in Lemma 7 (c) such that $\psi(\zeta) \in K$, and define

$$H^\zeta(z) := H^{\psi(\zeta)}(\psi(z)). \tag{4.2}$$

Define $H^\zeta(z)$ for all $z \in \mathcal{G}$ by extending it linearly along the edges of $\mathcal{G}$. We let Ω_ζ be the connected component of 0 in the set

$$\Big\{z \in \mathcal{G} \,\Big|\, H^\zeta(z) - \frac{1}{2|\zeta|} > 0\Big\} \setminus \{\zeta\}.$$

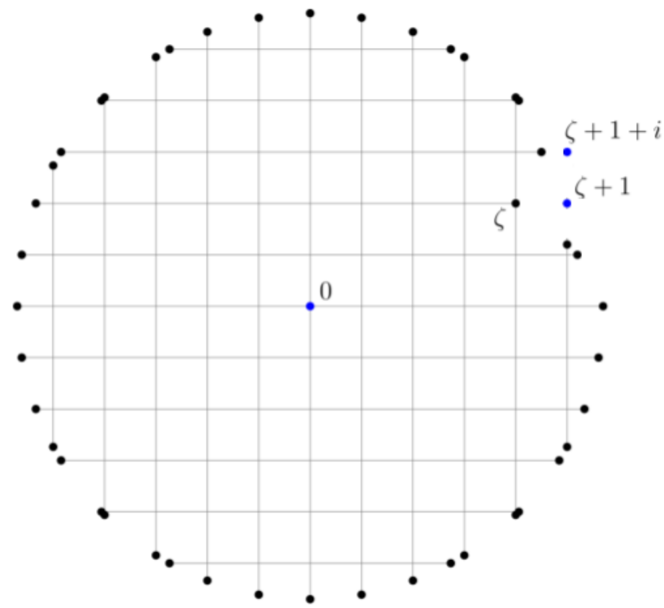

Figure 9: Sketch of Ω_ζ for $\zeta = 4+2i$. In Lemma 16 (d), we will show that Ω_ζ is close to the ball $B_{|\zeta|} \cap \mathcal{G}$. The black dots are the boundary of $\Omega_\zeta \subset \mathcal{G}$. The lattice points ζ, $\zeta+1$, and $\zeta+1+i$ are not in Ω_ζ.

Lemma 16 (Properties of H^ζ)**.** *There is a constant $C_1 < \infty$ such that the following statements hold:*

(a) $1 \le H^\zeta(\zeta)$, $\quad H^\zeta(\zeta+1) < 0$, $\quad H^\zeta(\zeta+1+i) < 0$, $\quad H^\zeta(z) \le 2$, *for* $\zeta \in K$ *and all* $z \in \mathcal{G}$,

(b) $\zeta \in \partial\Omega_\zeta$ *and for* $z \in \partial\Omega_\zeta \setminus \{\zeta\}$ *it is* $H^\zeta(z) = 1/(2|\zeta|)$.

(c) H^ζ *is grid-harmonic in* Ω_ζ.

(d) $\mathbb{B}_{|\zeta|-C_1} \subset \Omega_\zeta \subset B_{|\zeta|+C_1}$

(e) For each $z \in \mathbb{B}_r$ *with* $0 \le r < |\zeta| - C_1$ *we have,*

$$\frac{1}{2|\zeta|} \le H^\zeta(z) \le \frac{1}{|\zeta| - r - C_1}.$$

(f) *For* $0 \le r \le |\zeta|$ *we have the mean-value properties:*

$$\left|\sum_{z\in\mathbb{B}_r}\left(H^\zeta(z)-H^\zeta(0)\right)\right| + \left|\sum_{z\in\Omega_\zeta\cap\mathbb{Z}^2}\left(H^\zeta(z)-H^\zeta(0)\right)\right| \le C_1 \ln|\zeta|.$$

Recall the definition of F^ζ given in (2.4). The next lemma allows us to derive some properties of H^ζ from F^ζ, for instance from mean-value property (Theorem 8). Lemma 17 will be our main factor in the proof of Lemma 16 (d), (e), and (f).

Lemma 17 (H^ζ is close to F^ζ)**.** *There is a constant* $C_0 > 0$ *such that for all* $z \in \mathcal{G}$*,*

$$|H^\zeta(z) - F^\zeta(z)| \le \frac{C_0}{|\zeta - z|^2}$$

Remark. *Note the similarities between* H^ζ *and* F^ζ*:*

- F^ζ *is harmonic in* $B_{|\zeta|}$, H^ζ *is grid-harmonic in* Ω_ζ *(Lemma 10 (c) and Lemma 16 (c));*
- $F^\zeta = 1/(2|\zeta|)$ *on* $(\partial B_{|\zeta|}) \setminus \{\zeta\}$ *and* $H^\zeta = 1/(2|\zeta|)$ *on* $(\partial\Omega_\zeta) \setminus \{\zeta\}$*, see Lemma 10 (a) and Lemma 16 (b);*
- F^ζ *is invariant under rotation,* H^ζ *is invariant under dihedral symmetries, see* (2.6) *and* (4.2)*.*

Remark. H^ζ *approximates the discrete Poisson kernel for the ball* $\mathbb{B}_{|\zeta|}$ *(that is the probability that a random walk started at* $z \in \mathbb{B}_{|\zeta|}$ *exits the ball* $\mathbb{B}_{|\zeta|}$ *in* ζ*). The discrete Poisson kernel can be expressed in terms of potential kernels* $a(\cdot)$*, see [LL10] Lemma 6.3.6 and Proposition 4.6.2.*

We will make use of properties of the continuum Poisson kernel (Lemma 10) and of the potential kernel (Lemma 7) to prove Lemma 16.

Proof of Lemma 16. By the symmetries of H^ζ, we may assume $\zeta \in K$.

(a) Note that in the definition of H^ζ the λ_i are the coefficients of the linear combination of $\zeta/|\zeta|$ with respect to the basis 1, $1+i$. Since $\zeta/|\zeta|$ is in the convex hull of $\{0, 1, 1+i\}$, we have $\lambda_1, \lambda_2 \in [0, 1]$ and

$$\lambda_1 + \lambda_2 \le 1. \tag{4.3}$$

Recall that by Lemma 7 (d), $a(1+i) = 4/\pi$, $a(1) = 1$. Then, since $a(1+i) > a(1)$, $\lambda_i \in [0,1]$, and Lemma 7 (c) (symmetries of the potential kernel) we see $H^\zeta(\zeta+1)$, $H^\zeta(\zeta+1+i) < 0$.
Since $a(\cdot)$ is subadditive (see Lemma 7 (e)) for any $z \in \mathbb{Z}^2$ it is $a(z-\zeta-1) \le a(z-\zeta) + a(1)$ and $a(z-\zeta-1-i) \le a(z-\zeta) + a(1+i)$. Hence, using the exact values of $a(1)$ and $a(1+i)$, for any $z \in \mathbb{Z}^2$,

$$H^\zeta(z) \le \frac{\pi}{2}\big(\lambda_1 a(1) + \lambda_2 a(1+i)\big) \le 2(\lambda_1+\lambda_2) \le 2,$$

where we used (4.3). Thus, by linearity $H^\zeta(z) \le 2$ for all $z \in \mathcal{G}$. On the other hand,

$$H^\zeta(\zeta) = \frac{\pi}{2}\lambda_1 + 2\lambda_2 \ge \frac{\pi}{2}(\lambda_1+\lambda_2) = \frac{\pi}{2}\mathrm{Re}\frac{\zeta}{|\zeta|} \ge \frac{\pi}{2}\mathrm{Re}\frac{1+i}{|1+i|} > 1.$$

(b) By (a) it is $H^\zeta(\zeta) > 1 > 1/(2\zeta)$ and the statement follows from the definition of Ω_ζ.

(c) By Lemma 7 the only critical points are ζ, $\zeta+1$, and $\zeta+1+i$. However, by (a) and the definition of ζ these points are not in Ω_ζ.

(d) For the first inclusion, $\mathbb{B}_{|\zeta|-C_1} \subset \Omega_\zeta$, it suffices to show $H^\zeta(z) > 1/(2|\zeta|)$ for all $z \in \partial B_{|\zeta|-C_1} \cap \mathcal{G}$ for some constant $C_1 > 0$ (by Lemma 12).

By Lemma 17, there is a constant C_0, such that

$$H^\zeta(z) - F^\zeta(z) \ge -\frac{C_0}{|\zeta - z|^2},$$

for all $z \in \mathcal{G}$. Choosing $C_1 := 28C_0$ and $C := 2C_0$ in Lemma 11, leads to

$$F^\zeta(z) - \frac{1}{2|\zeta|} \ge \frac{2C_0}{|\zeta-z|^2},$$

for all $z \in \partial B_{|\zeta|-C_1} \cap \mathcal{G}$. Combining both inequalities,

$$\begin{aligned} H^\zeta(z) - \frac{1}{2|\zeta|} &= H^\zeta(z) - F^\zeta(z) + F^\zeta(z) - \frac{1}{2|\zeta|} \\ &\ge -\frac{C_0}{|\zeta-z|^2} + \frac{2C_0}{|\zeta-z|^2} > 0. \end{aligned}$$

Using maximum principle for grid-harmonic functions, Lemma 11, and Lemma 17 and the same choices of C_1 and C, it is for all $z \in B_{|\zeta|+C_1} \cap \mathcal{G}$,

$$\begin{aligned} H^\zeta(z) - \frac{1}{2|\zeta|} &= H^\zeta(z) - F^\zeta(z) + F^\zeta(z) - \frac{1}{2|\zeta|} \\ &\leq \frac{C_0}{|\zeta - z|^2} - \frac{2C_0}{|\zeta - z|^2} < 0, \end{aligned}$$

which proves the inclusion $\Omega_\zeta \subset B_{|\zeta|+C_1}$.

(e) By (d) it is $\mathbb{B}_r \subset \Omega_\zeta$, hence the lower bound follows by the definition of Ω_ζ.
For the upper bound use Lemma 17 and Lemma 10 (e), to get

$$\begin{aligned} \max_{z \in \mathbb{B}_r} H^\zeta(z) \leq \sup_{z \in B_r} F^\zeta(z) + \sup_{z \in B_r} \frac{C_0}{|\zeta - z|^2} &\leq \frac{1}{|\zeta| - r} + \frac{C_0}{(|\zeta| - r)^2} \\ &\leq \frac{1}{|\zeta| - r - C_1}, \end{aligned}$$

where the last inequality holds by the choice $C_1 = 28C_0$ we made in the proof of (d).

(f) At first assume $0 < r \leq |\zeta| - C_1$. By mean-value property (Theorem 8) we have,

$$\int_{B_r} \big(F^\zeta(z) - F^\zeta(0)\big) = 0, \tag{4.4}$$

since F^ζ is harmonic in B_r. Using Lemma 17, we will approximate

$$\sum_{z \in \mathbb{B}_r} \big(H^\zeta(z) - H^\zeta(0)\big) \tag{4.5}$$

by (4.4), and get an error of order $\ln|\zeta|$. For this approximation we proceed in three steps. At first, we see that by Lemma 17,

$$\begin{aligned} &\left| \sum_{z \in \mathbb{B}_r} \big(H^\zeta(z) - H^\zeta(0)\big) - \sum_{z \in \mathbb{B}_r} \big(F^\zeta(z) - F^\zeta(0)\big) \right| \\ &\leq \sum_{z \in \mathbb{B}_r} \left(\frac{C_0}{|\zeta - z|^2} + \frac{C_0}{|\zeta|^2} \right) \leq 8\pi C_0 \ln|\zeta|. \end{aligned}$$

Second, if $\Box_z$ denotes the unit square with center in $z \in \mathbb{Z}^2$ and

$$B_r^{\Box} := \bigcup_{z \in \mathbb{B}_r} \Box_z,$$

then Lemma 10 (f) implies

$$\sup_{\omega \in \Box_z} F^{\zeta}(\omega) - F^{\zeta}(z) \leq \frac{2}{|\zeta - z|^2}.$$

Hence,

$$\begin{aligned}
\left| \int_{B_r^{\Box}} F^{\zeta}(z) dz - \sum_{z \in \mathbb{B}_r} F^{\zeta}(z) \right| &= \left| \sum_{z \in \mathbb{B}_r} \int_{\Box_z} \big(F^{\zeta}(\omega) - F^{\zeta}(z) \big) d\omega \right| \\
&\leq \sum_{z \in \mathbb{B}_r} \frac{2}{|\zeta - z|^2} \\
&\leq 8\pi \ln |\zeta|.
\end{aligned}$$

Third,

$$\begin{aligned}
\left| \int_{B_r^{\Box}} F^{\zeta}(z)\, dz - \int_{B_r} F^{\zeta}(z)\, dz \right| &= \left| \int_{B_{r+1} \setminus B_{r-1}} F^{\zeta}(z)\, dz \right| \\
&\leq \int_{B_{r+1} \setminus B_{r-1}} \frac{1}{|\zeta - z|}\, dz \\
&= \int_0^{2\pi} \int_{r-1}^{r+1} \frac{1}{|se^{i\theta} - \zeta|} s\, ds\, d\theta \\
&\leq 4 \int_0^{2\pi} \frac{1}{|re^{i\theta} - \zeta|} r\, d\theta \\
&\leq 8 \int_{\alpha}^{\alpha+\pi} \Big(\frac{d}{d\theta} \ln |re^{i\theta} - \zeta| \Big)\, d\theta \\
&\leq 8\pi \ln |\zeta|,
\end{aligned}$$

where α denotes the angle for which $re^{i\alpha}$ has minimal distance to ζ. In addition,

$$\left| \int_{B_r} F^{\zeta}(0)\, dz - \sum_{z \in \mathbb{B}_r} F^{\zeta}(0) \right| = \frac{|\pi r^2 - |\mathbb{B}_r||}{|\zeta|} \leq 9.$$

Using this, the three steps above, and (4.4), the result can be derived.

It remains to prove the approximate mean-value property for $\mathbb{B}_r$, when $|\zeta|-C_1 \le r < |\zeta|$, and for $\Omega_\zeta \cap \mathbb{Z}^2$. By (e) we have $\mathbb{B}_{|\zeta|-C_1} \subset \Omega_\zeta \cap \mathbb{Z}^2 \subset \mathbb{B}_{|\zeta|+C_1}$; therefore, the sum (4.5) differs at most by the lattice points in the annulus $R = \mathbb{B}_{|\zeta|+C_1} \setminus \mathbb{B}_{|\zeta|-C_1}$, i.e., by

$$\begin{aligned}\sum_{z\in R} \left(|H^\zeta(z)| + |H^\zeta(0)|\right) &\le \sum_{z\in R} \min\left(\frac{2}{|\zeta - z|}, 2\right) + |R|\frac{2}{|\zeta|} \\ &\le 32\pi C_1 \ln|\zeta|,\end{aligned}$$

where we used $H^\zeta \le 2$ (see Lemma 16 (a)), Lemma 17, and $|R| \le 16C_1|\zeta|$. This completes the proof.

Proof of Lemma 17. This result is based on the approximation $a(z) \approx \ln|z|$ (see Lemma 7 (b)). Here the choice of the coefficients λ_1, λ_2 comes into play.

By Lemma 7 (b) we can see that

$$\begin{aligned}&H^\zeta(z) - \big(\lambda_1(\ln|z-\zeta-1| - \ln|z-\zeta|) + \lambda_2(\ln|z-\zeta-1-i| - \ln|z-\zeta|)\big) \\ &\quad= O\left(\frac{1}{|\zeta - z|^2}\right).\end{aligned} \tag{4.6}$$

Then if γ denotes the line segment from $z-\zeta$ to $z-\zeta-1$, we get

$$\begin{aligned}\ln|z-\zeta-1| - \ln|z-\zeta| &= \mathrm{Re}\int_\gamma \frac{1}{\omega} \\ &= -\int_0^1 \mathrm{Re}\frac{1}{z-\zeta-t}dt \\ &= \mathrm{Re}\frac{1}{z-\zeta} + O\left(\frac{1}{|z-\zeta|^2}\right),\end{aligned}$$

where for the last equality we used the Taylor series in $t=0$. Plugging this (and the analog for $\ln|z-\zeta-1-i| - \ln|z-\zeta|$) into Equation (4.6) above, we get

$$\begin{aligned}&\lambda_1(\ln|z-\zeta-1| - \ln|z-\zeta|) + \lambda_2(\ln|z-\zeta-1-i| - \ln|z-\zeta|) \\ &= -\lambda_1\mathrm{Re}\frac{1}{z-\zeta} - \lambda_2\mathrm{Re}\frac{1+i}{z-\zeta} + O\left(\frac{1}{|z-\zeta|^2}\right) \\ &= F^\zeta + O\left(\frac{1}{|z-\zeta|^2}\right).\end{aligned}$$

This finishes the proof.

5 Grid IDLA

In order to define a continuous martingale, we introduce a growth model similar to the IDLA, where the underlying particles are grid Brownian Motions. The martingale will be defined by the values of H^ζ on those particles. To ensure the martingale property, the particles are stopped on exiting Ω_ζ, i.e., before reaching a point where harmonicity of H^ζ fails.

5.1 Definition

Define a process $A^\zeta(t)$, $t \geq 0$ on the grid, which for t being an integer, behaves like the IDLA process on $\mathbb{Z}^2$.

We let

$$S := (\Omega_\zeta \cap \mathbb{Z}^2) \cup \partial\Omega_\zeta \tag{5.1}$$

denote the lattice points in Ω_ζ and its boundary in $\mathcal{G}$. Let $\tilde{\beta}^1$, $\tilde{\beta}^2$, $\tilde{\beta}^3$, ... be independent grid Brownian motions and

$$\tau^n = \begin{cases} 0 & , n = 1 \\ \inf\{t \geq 0 \,|\, \tilde{\beta}^n(t) \in (\mathbb{Z}^2 \setminus A^\zeta(n)) \cup \partial\Omega_\zeta\} & , n \geq 2 \end{cases}$$

be the first time the nth grid Brownian motion either reaches the boundary of Ω_ζ or hits a lattice point (in Ω_ζ), which is not already occupied by $A^\zeta(n)$. Define the time change $s \mapsto s'' := \frac{s}{1-s}$, $[0,1) \to \mathbb{R}$. For each n define a time-changed and stopped grid BM by

$$\beta^n(s) = \tilde{\beta}^n(s'' \wedge \tau^n), \tag{5.2}$$

for $s \in [0,1)$. Note that a.s. $\beta^n(1) = \lim_{s \to 1} \beta^n(s) = \tilde{\beta}^n(\tau^n)$. For $t \in [0,1]$ define the **grid IDLA** $A^\zeta(t)$ by

$$A^\zeta(t) = \big(\beta^1(t)\big),$$

and for $t \in (n, n+1]$ define $A^\zeta(t) : \Omega \to S^n \times \bar{\Omega}_\zeta$ by

$$A^\zeta(t) = \big(\beta^1(1), ..., \beta^n(1), \beta^{n+1}(t-n)\big).$$

We will refer to A^ζ in different ways: if it occurs in the context of set operators, as in the definition of τ^n, we refer to A^ζ as a set; if we iterate over A^ζ, we refer to A^ζ as a multiset.

Remark. *The first particles of A^ζ are either lattice points or on the boundary of Ω_ζ; the last particle can be somewhere in $\bar{\Omega}_\zeta$. Hence, although not generally, the set A^ζ is increasing over integer time steps. Moreover, points of A^ζ in Ω_ζ are almost surely of multiplicity at most one; however, at the boundary $\partial\Omega_\zeta$ the grid IDLA may have points of larger multiplicity.*

5.2 Main Martingale

In this section we introduces a process that plays a crucial role in the proof of Theorem 1. Showing that this process is a martingale (Lemma 18), is the main result of this section.

Let $t \in (n, n+1]$. For $i \in \{1, ..., n+1\}$ let π^i be the projection on the ith coordinate. Then $M^\zeta(t) : \Omega \to \mathbb{R}$ can be defined by

$$M^\zeta(t) = \sum_{i=1}^{n+1} f\big(\pi^i(A^\zeta(t))\big) = \sum_{i=1}^{n} f\big(\beta^i(1)\big) + f\big(\beta^{n+1}(t-n)\big), \tag{5.3}$$

where

$$f : \mathcal{G} \to \mathbb{R}, \quad f(z) = H^\zeta(z) - H^\zeta(0).$$

Multiset notation enables us to write

$$M^\zeta(t) = \sum_{z \in A^\zeta(t)} f(z).$$

Lemma 18. *M^ζ is a martingale w.r.t. $\mathcal{F}_t = \sigma(A^\zeta(s) \,|\, 0 \le s \le t)$.*

Lemma 19. *M^ζ can be written as a time change $t \mapsto \langle M^\zeta \rangle_t$ of a standard Brownian motion $\mathcal{B}$, i.e.,*

$$M^\zeta(t) = \mathcal{B}(\langle M^\zeta \rangle_t).$$

Remark. *If* $T_s = \inf\{t \geq 0 \,|\, \langle M^\zeta \rangle_t \geq s\}$ *denotes the time at which* $\langle M^\zeta \rangle$ *reaches* s*, then the time change* $\langle M^\zeta \rangle_t$ *,* $t \geq 0$ *is a family of stopping times with respect to* $(\mathcal{F}_{T_s})_{s \geq 0}$*. Furthermore, the BM* $\mathcal{B}$ *is adapted to the filtration* $\mathcal{F}_{T_s}$*.*

Proof of Lemma 19. Show that M^ζ satisfies the assumptions of Theorem 5; then the result follows.

We have seen that M^ζ is a continuous martingale. M^ζ clearly vanishes in $t = 0$. It remains to show that $\langle M \rangle_\infty = \infty$. Remember that M^ζ is the sum of H^ζ over the particles of A^ζ. The particles act like Brownian motions on the edges of $\mathcal{G}$, and H^ζ is linear on the edges. Since for a BM $\mathcal{B}$,

$$\lim_{t \to \infty} \langle \mathcal{B} \rangle_t = \lim_{t \to \infty} t = \infty,$$

we can conclude $\langle M \rangle_\infty = \infty$.

Proof of Lemma 18. Fix $\zeta \in \mathbb{Z}^2$ and let f, S, and $\mathcal{F}_t$ be defined as above in this section. Let $t \in (n, n+1]$ and regard A^ζ as a process $A^\zeta(t) : (\Omega, \mathcal{F}_t) \to (S^n \times \bar{\Omega}_\zeta,\, \mathcal{P}(S^n) \otimes \mathcal{B}(\bar{\Omega}_\zeta))$.

Since f and the projections π^i are measurable, $M^\zeta(t)$ is adapted to $\mathcal{F}_t$. Moreover, f is bounded on $\bar{\Omega}_\zeta$, and since we sum over $n+1$ many particles in $\bar{\Omega}_\zeta$ at time t, it is $\mathbb{E}(|M^\zeta(t)|) < \infty$.

Now prove the martingale property. Theorem 13 states that grid-harmonic functions of grid Brownian motions are martingales. This will be our key argument.

By an iteration of tower property it suffices to show

$$\mathbb{E}(M^\zeta(t) \,|\, \mathcal{F}_s) = M^\zeta(s), \quad \text{for } s \in [n, t).$$

Moreover, since $M^\zeta(n)$ is $\mathcal{F}_n \subset \mathcal{F}_s$ measurable, it is

$$\begin{aligned}\mathbb{E}(M^\zeta(t) \,|\, \mathcal{F}_s) &= \mathbb{E}\bigg(\sum_{i=1}^{n} f\big(\pi^i(A^\zeta(n))\big) + f\big(\pi^{n+1}(A^\zeta(t))\big) \,\bigg|\, \mathcal{F}_s \bigg) \\ &= M^\zeta(n) + \mathbb{E}\big(f(\beta^{n+1}(t-n)) \,\big|\, \mathcal{F}_s\big).\end{aligned}$$

So it suffices to prove that

$$\mathbb{E}\big(f(\beta^{n+1}(t-n)) \,\big|\, \mathcal{F}_s\big) = f(\beta^{n+1}(s-n)).$$

The notation $\beta^{n+1}(u) = \tilde{\beta}^{n+1}(\frac{u}{1-u} \wedge \tau^{n+1})$ for $u \in [0,1]$ refers to the definition of the grid IDLA (5.2). For simplicity denote $\beta := \beta^{n+1}$, $\tilde{\beta} := \tilde{\beta}^{n+1}$, and $\tau := \tau^{n+1}$. Let $t' := t - n$, $s' := s - n$ be the shifted times in the domain of β, and $t'' := \frac{t'}{1-t'}$, $s'' := \frac{s'}{1-s'}$ the accelerated times used for the definition of the grid Brownian motion. Obtain that the last grid BM $\beta(t')$ only depends on its own history and on $A^\zeta(n)$; the trajectory of $A^\zeta(\cdot)$ up to time n does not have any effect. Hence,

$$\begin{aligned}
&\mathbb{E}\big(f(\beta(t')) \,\big|\, \mathcal{F}_s\big) \\
&\quad = \mathbb{E}\big(f(\beta(t')) \,\big|\, \mathcal{F}_n,\, \beta(u)\ : 0 \le u \le s'\big) \\
&\quad = \mathbb{E}\big(f(\beta(t')) \,\big|\, A^\zeta(n),\, \beta(u)\ : 0 \le u \le s'\big) \\
&\quad = \mathbb{E}\big(f(\tilde{\beta}(t'' \wedge \tau)) \,\big|\, A^\zeta(n),\, \tilde{\beta}(u \wedge \tau)\ : 0 \le u \le s''\big) \\
&\quad = \sum_{A \in S^n} \mathbb{E}\big(f(\tilde{\beta}(t'' \wedge \tau)) \,\big|\, A^\zeta(n) = A,\, \tilde{\beta}(u)\ : 0 \le u \le s''\big) \cdot \mathbf{1}_{\{A^\zeta(n) = A\}}.
\end{aligned}$$

For the last equality note that $|S^n| < \infty$, where the set S is as in (5.1). Next we apply Theorem 13 on each summand of the equation above; for this note that on the event $\{A^\zeta(n) = A\}$ the time τ, which stops $\tilde{\beta}$ on hitting $\partial\Omega_\zeta \cup (\mathbb{Z}^2 \setminus A)$, equals the exit time $\tau_{\tilde{A}} = \inf\{t \ge 0 \,|\, \tilde{\beta}(t) \notin \tilde{A}\}$, where $\tilde{A} \subset \mathcal{G}$ is an open set constructed by the union of the open crosses $z + E$ with centers that are the lattice points of A:

$$\tilde{A} = \Omega_\zeta \cap \bigcup_{z \in A \cap \mathbb{Z}^2} (z + E).$$

Since H^ζ is grid-harmonic in Ω_ζ and $\tilde{A} \subset \Omega_\zeta$, we can conclude that f is grid-harmonic on $\tilde{A}$ and we can apply Theorem 13 (see also the remark of Theorem 13). Hence, on the event $\{A^\zeta(n) = A\}$ we get

$$\begin{aligned}
&\mathbb{E}\big(f(\tilde{\beta}(t'' \wedge \tau)) \,\big|\, A^\zeta(n) = A,\, \tilde{\beta}(u)\ : 0 \le u \le s''\big) \\
&\quad = \mathbb{E}\big(f(\tilde{\beta}(t'' \wedge \tau_{\tilde{A}})) \,\big|\, A^\zeta(n) = A,\, \tilde{\beta}(u)\ : 0 \le u \le s''\big) \\
&\quad = \mathbb{E}\big(f(\tilde{\beta}(t'' \wedge \tau_{\tilde{A}})) \,\big|\, \tilde{\beta}(u)\ : 0 \le u \le s''\big) \\
&\quad = f(\tilde{\beta}(s'' \wedge \tau_{\tilde{A}})) \\
&\quad = f(\beta(s')).
\end{aligned}$$

In total this gives us $\mathbb{E}(f(\beta^{n+1}(t')) \,|\, \mathcal{F}_s) = f(\beta^{n+1}(s'))$. $\square$

6 Logarithmic Fluctuations

The aim of this section is to prove Theorem 1 on a high level, i.e., only using the lemmas 20, 21, and 22 that are about the occurrence of early and late points. While stated here, we will prove these lemmas in Section 7. We begin with an a priori bound on the probability of very late points.

Lemma 20 (No very late point)**.** *There are constants C_2, $c_2 > 0$ such that if $T \geq 100\pi$ and $l \geq \sqrt{\frac{T}{100\pi}}$, then*

$$\mathbb{P}\big(\mathcal{L}^l(T)\big) \leq C_2 e^{-c_2\sqrt{T}}.$$

Fix $\gamma > 1$ and define the constants

$$\begin{aligned} C_3 &= \max\{(\gamma + 5 + \ln C_2)/c_2,\ C_1 1000/b\}, \\ C_4 &= \Big(2700 + \frac{1}{c_2}\Big)(\gamma + 3) + \frac{2\ln C_2}{c_2} + 2C_1 + 2, \end{aligned}$$

where C_1 and b are as defines in Lemma 16 and Lemma 25.

Lemma 21 (Early points imply late points)**.** *Fix $T \geq 19$. Then for all l and m satisfying $m \geq C_3 \ln T$ and $l \leq (b/1000)m$,*

$$\mathbb{P}\big(\mathcal{E}^m(T) \cap \mathcal{L}^l(T)^c\big) \leq T^{-(\gamma+1)}.$$

Lemma 22 (Late points imply early points)**.** *If l, $m \in \mathbb{N}$ such that $l \geq C_4 \ln T$ and $m \leq l^2/(C_4 \ln T)$, then*

$$\mathbb{P}\big(\mathcal{L}^l(T) \cap \mathcal{E}^m(T)^c\big) \leq T^{-(\gamma+1)}.$$

Remark. *In Theorem 1 (see also Lemma 2) l can be chosen much smaller than in Lemma 20, which bounds the probability of an l-late point for l being of order $\sqrt{T}$.*

6.1 Proof of Theorem 1

This section contains the proof of Theorem 1 with the help of the lemmas above. We begin with a brief overview of the proof.

We define sequences of l and m for which l-late and m-early points are unlikely. Lemma 20 states that $\sqrt{T}$-late points are unlikely. Hence, we start with l being of order $\sqrt{T}$ and choose (according to Lemma 21 and 22) iteratively smaller values for l and m; we end up with l and m being of order $\ln T$. While the assumptions $m \geq C_3 \ln T$ and $l \geq C_4 \ln T$ (of Lemma 21 and Lemma 22) give an absolute lower bound on how much l and m may decline, the assumptions $l \leq \frac{b}{1000} m$ and $m \leq l^2/(C_4 \ln T)$ bound how much m and l may decrease with each iteration. The absolute lower bounds will eventually determine the logarithmic fluctuations.

Proof of Theorem 1. Fix $\gamma > 1$. Lemma 2 tells us that it suffices to find a constant a and some $l, m < a \ln r$ such that

$$\mathbb{P}\big(\mathcal{L}^l(T)\big) + \mathbb{P}\big(\mathcal{E}^m(T)\big) \leq r^{-\gamma}. \tag{6.1}$$

To proof this, we use an iteration procedure.

Fix $T > 100\pi$. Start the iteration by choosing l_0 according to Lemma 20, i.e., $l_0 := \sqrt{T/(100\pi)}$. Thus,

$$\mathbb{P}\big(\mathcal{L}^{l_0}(T)\big) \leq Ce^{-c\sqrt{T}} \leq T^{-(\gamma+1)} \tag{6.2}$$

holds. Provided l_i is fixed, we choose m_i according to Lemma 21's assumption $l \leq m\,b/1000$, and provided m_i is fix, we choose l_{i+1} according Lemma 22, i.e., for $i \geq 0$ we define alternating recursively

$$\begin{aligned} m_i &:= \frac{1000}{b} l_i, \\ l_{i+1} &:= \sqrt{m_i C_4 \ln T}, \end{aligned} \tag{6.3}$$

i.e, each element of the sequence l_i (and also m_i) decays like the square root of its predecessor. Since

$$\mathbb{P}\big(\mathcal{E}^{m_i}(T)\big) \leq \mathbb{P}\big(\mathcal{E}^{m_i}(T) \cap \mathcal{L}^{l_i}(T)^c\big) + \mathbb{P}\big(\mathcal{L}^{l_i}(T)\big),$$

Lemma 21 states that the occurrence of an m_i-early point is unlikely if an l_i-late point is unlikely. Hence, using (6.2),

$$\mathbb{P}(\mathcal{E}^{m_0}(T)) \leq 2T^{-(\gamma+1)}. \tag{6.4}$$

Similarly, we use

$$\mathbb{P}\big(\mathcal{L}^{l_{i+1}}(T)\big) \le \mathbb{P}\big(\mathcal{L}^{l_{i+1}}(T) \cap \mathcal{E}^{m_i}(T)^c\big) + \mathbb{P}\big(\mathcal{E}^{m_i}(T)\big)$$

with Lemma 22 and (6.4), to get

$$\begin{aligned}\mathbb{P}\big(\mathcal{L}^{l_1}(T)\big) &\le \mathbb{P}\big(\mathcal{L}^{l_1}(T) \cap \mathcal{E}^{m_0}(T)^c\big) + \mathbb{P}\big(\mathcal{E}^{m_0}(T)\big) \\ &\le T^{-(\gamma+1)} + 2T^{-(\gamma+1)}.\end{aligned} \tag{6.5}$$

By induction we derive directly from definition

$$\begin{aligned}l_i &= \alpha^{1-1/2^i}\, l_0^{1/2^i}, \\ m_i &= \alpha^{1-1/2^i}\, m_0^{1/2^i},\end{aligned} \tag{6.6}$$

where $\alpha := (1000/b)C_4 \ln T$.

Define

$$C_5 = \max\{1, 1000/b\} \cdot \max\{C_3, C_4\}.$$

We claim that for the sequences l_i and m_i and for $a := 3C_5$ there is a $k = k(T)$ such that

- **(i)** for all $n \le k$ it is $l_n \ge C_4 \ln T$ and $m_n \ge C_3 \ln T$,
- **(ii)** $l_k, m_k \le a \ln T$ (i.e., fluctuation is at most logarithmic), and
- **(iii)** $k << \ln T$ (i.e., the upper bounds on $\mathbb{P}\big(\mathcal{L}^{l_k}(T)\big)$ and $\mathbb{P}\big(\mathcal{E}^{l_k}(T)\big)$ are not weakened too much).

The following sketch outlines the constraints (i), (ii), and (iii).

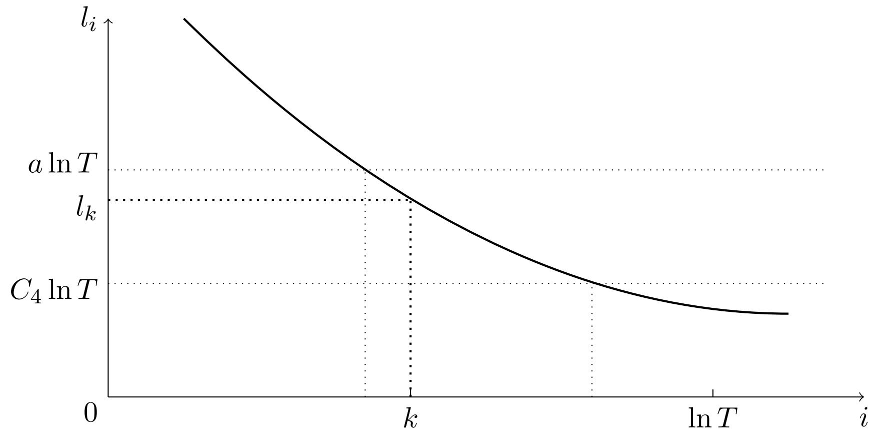

Figure 10: This sketch illustrates the decline of l_i over i, i.e., with each step we prove a smaller fluctuation from circularity to the inside. The dotted vertical lines mark the valid range for k (see (i), (ii), and (iii)).

By (i) the assumptions of Lemma 21 and Lemma 22 are fulfilled for all l_n, m_n with $n < k$. Hence, we can inductively repeat the steps (6.4) and (6.5), which gives us

$$\begin{aligned} &\mathbb{P}\big(\mathcal{L}^{l_k}(T)\big) \leq (2k+1)T^{-(\gamma+1)}, \quad \text{and} \\ &\mathbb{P}(\mathcal{E}^{l_k}(T)) \leq (2k+2)T^{-(\gamma+1)}. \end{aligned}$$

Therefore, using (ii) and (iii),

$$\begin{aligned} \mathbb{P}\big(\mathcal{L}^{a\ln T}(T)\big) &\leq \mathbb{P}\big(\mathcal{L}^{l_k}(T)\big) \\ &\leq (2k+1)T^{-(\gamma+1)} \\ &\leq \frac{1}{2}T^{-\gamma}, \end{aligned}$$

for T sufficiently large. With the same arguments,

$$\mathbb{P}\big(\mathcal{E}^{a\ln T}(T)\big) \leq \frac{1}{2}T^{-\gamma}.$$

We rephrase the fluctuation bounds in terms of the radius r instead of the time $T = T(r) := \pi r^2$. Note that

$$3a \ln r \geq a \ln T,$$

for r sufficiently large, i.e., mapping r to T changes the fluctuation only by a constant factor. Hence, we have

$$\begin{aligned}\mathbb{P}(\mathcal{E}^{3a\ln r}(T)) + \mathbb{P}(\mathcal{L}^{3a\ln r}(T)) &\leq \mathbb{P}(\mathcal{E}^{a\ln T}(T)) + \mathbb{P}(\mathcal{L}^{a\ln T}(T)) \\ &\leq \frac{1}{2}T^{-\gamma} + \frac{1}{2}T^{-\gamma} \\ &\leq r^{-\gamma},\end{aligned}$$

which by (6.1) proves the theorem.

It remains to prove the existence of a $k = k(T)$ such that (i), (ii), and (iii) hold. The non-recursive form of l_i and m_i, see (6.6), and some simple estimations lead to the statement

$$i \geq \log_2 \ln T, \quad \text{implies} \quad l_i,\, m_i \leq a \ln T. \tag{6.7}$$

Choose k to be the smallest integer such that $l_k \leq a \ln T$. By this choice (ii) holds (using that l_i, m_i are monotonically decreasing). Furthermore, using the contraposition of (6.7), it is

$$k \leq \log_2 \ln T + 1,$$

which proves (iii). By definition of k it is

$$\begin{aligned}&l_i > a \ln T \geq C_4 \ln T, \quad \text{and} \\ &m_i > l_i > a \ln T \geq C_3 \ln T,\end{aligned}$$

for all $i \leq k-1$. Hence, it remains to prove (i) for the last index k, where we can use the recursive definition of l_k and m_k,

$$l_k = \sqrt{C_4 (\ln T) m_{k-1}} \geq \sqrt{C_4 (\ln T) 3 C_5 \ln T} \geq C_4 \ln T,$$

similarly,

$$m_k \geq \frac{1000}{b} C_4 \ln T \geq C_3 \ln T,$$

which completes the proof.

Remark. *The next histogram indicates how equally early and late points are distributed. This can be interpreted as a consequence of Lemma 21 and Lemma 22.*

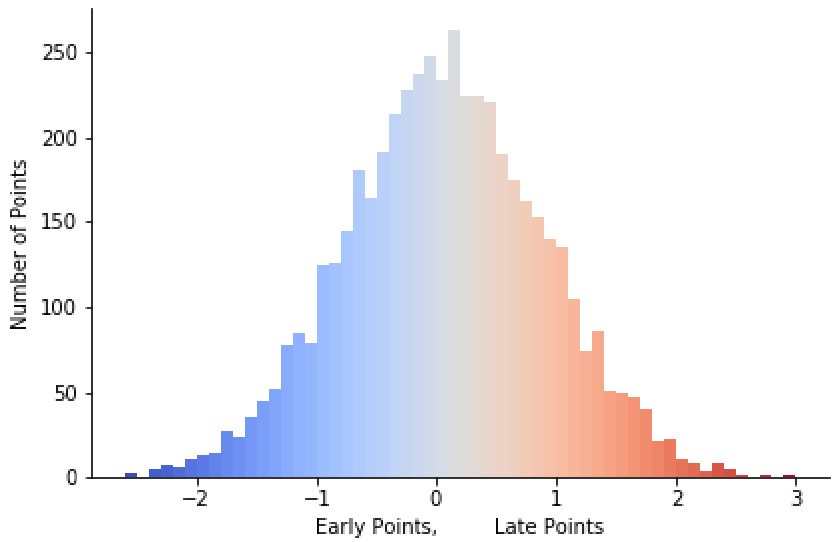

Figure 11: Number of early and late points of an IDLA cluster at time 5000 (same realization and color scale as in Figure 2).

7 Detect Early and Late Points

In this section we prove Lemma 20, 21, and 22, which were the core lemmas of the proof of Theorem 1. In order to detect early and late points, we will analyse the behavior of M^ζ and its quadratic variation on different events.

7.1 Quadratic Variation Bounds

To prove Lemma 21 and 22 we will state two lemmas, which bring the martingale from Section 5.2 into play. Both lemmas show that the absence of early points indicates a small quadratic variation $\langle M^\zeta \rangle$.

For time $t \geq 0$ define the radius

$$r_0 := \sqrt{t/\pi} + 4m + 2C_1, \tag{7.1}$$

where C_1 is defined in Lemma 16. For $|\zeta| > r_0$ and on the event $\mathcal{E}^m(t)^c$ the cluster $A^\zeta(t)$ is inside Ω_ζ. In this case, we get the following bound for the quadratic variation $\langle M^\zeta \rangle_t$.

Lemma 23 (No early point implies small quadratic variation)**.** *For each time $t \in \mathbb{N}$ and $\zeta \in \mathbb{Z}^2$ with $|\zeta| \geq r_0$, it is*

$$\mathbb{P}\big(\mathcal{E}^{m+1}(t)^c \cap \{\langle M^\zeta \rangle_t > s\}\big) \leq t^{80} e^{-s},$$

for $s > 0$.

For ζ closer to the origin particles of $A^\zeta(t)$ may accumulate at $\partial\Omega_\zeta$, which may yield to a larger quadratic variation of $M^\zeta(t)$.

Lemma 24 (No early point implies small quadratic variation 2)**.** *Fix $m \geq 2C_1 + 2$, $l \leq m$, and $\zeta \in \mathbb{Z}^2$ with $|\zeta| \geq l$. Let $t = \pi(|\zeta| + l)^2$, then*

$$\mathbb{P}\big(\mathcal{E}^{m}(t)^c \cap \{\langle M^\zeta \rangle_t > s\}\big) \leq t^{80} e^{1260m} e^{-s},$$

for $s > 0$.

In the proof of Lemma 21 we will choose ζ outside B_{r_0} (and Lemma 23 applies), whereas in Lemma 22 we choose ζ smaller than $\sqrt{t/\pi} - l$ (where Lemma 24 applies). Lemma 23 and 24 will be proven in Section 7.5.

7.2 No Thin Tentacles

We set

$$\mathbb{B}(z, r) = B(z, r) \cap \mathbb{Z}^2.$$

The next lemma states that if $z \in A(n)$, then the IDLA cluster $A(n)$ occupies more then a constant fraction of points in the ball $\mathbb{B}(z, m)$ with high probability. It is used in Lemma 21 to bound the probability of the event that a point $z \in \mathbb{Z}^2$ is m-early, but there are just a few points in $\mathbb{B}(z, m)$ part of the IDLA cluster (see Figure 12 below).

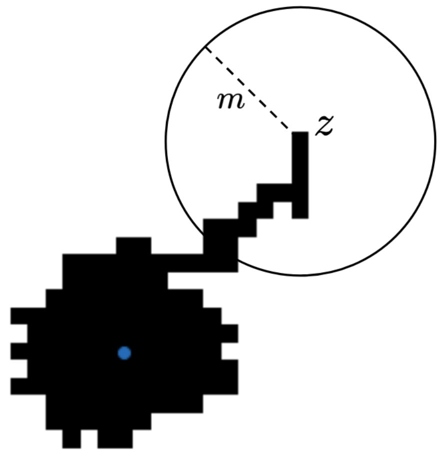

Figure 12: A thin tentacle of an IDLA cluster by time 119.

Lemma 25 (No thin tentacles)**.** *There are constants b, C_2, $c_2 > 0$ such that for all $m > 0$ and $z \in \mathbb{Z}^2$ with $0 \notin \mathbb{B}(z, m)$,*

$$\mathbb{P}\big(z \in A(n),\ |A(n) \cap \mathbb{B}(z, m)| \leq bm^2\big) \leq C_2 e^{-c_2 m}.$$

For the proof, I refer to Jerison, Levine, and Sheffield who proved a similar result for higher dimensions as well (see [JLS12], Lemma A).

7.3 Proof: Early Points Imply Late Points

We will prove Lemma 21 in this section. The main effort will be dedicated to bound $\langle M^\zeta \rangle_t$ and $M^\zeta(t)$ on events which partition $\mathcal{E}^m(t) \cap \mathcal{L}^l(t)^c$.

Proof of Lemma 21. Since $\mathcal{L}^{l+1}(T) \subset \mathcal{L}^l(T)$, it suffices to prove the lemma for $l = bm/1000$. If $m > T - \sqrt{T/\pi}$, then $\mathcal{E}^m(T) = \emptyset$. Hence, we may assume that $m \leq T - \sqrt{T/\pi}$. For each $z \in \mathbb{Z}^2$ and each integer $t \in 1, ..., T$ define the event that z is the first m-early point and joins the IDLA cluster at time t by

$$Q_{z,t} = \{z \in A(t) \setminus A(t-1)\} \cap E_z^m \cap \mathcal{E}^m(t-1)^c.$$

If there is at least one m-early point up to time T, then one point in $\mathbb{B}_T$ needs to be the first m-early point. This enables us to write $\mathcal{E}^m(T)$ as a disjoint union of the events $Q_{z,t}$,

$$\mathcal{E}^m(T) = \bigcup_{t \le T} \bigcup_{z \in \mathbb{B}_T} Q_{z,t}. \tag{7.2}$$

On the event $Q_{z,t}$ it is clear that $A(t) \backslash \{z\} \subset \mathbb{B}_{\sqrt{t/\pi}+m}$ and also that z cannot be $(m+1)$-early (since otherwise a neighbor of z would be m-early at time $t-1$). Hence, and since z is the first m-early point, it is $A(n) \subset \mathbb{B}_{\sqrt{n/\pi}+m+1}$ for all $n \le t$, and by (1.2),

$$Q_{z,t} \subset \mathcal{E}^{m+1}(t)^c \tag{7.3}$$

holds for each $t \in 1, ..., T$ and $z \in \mathbb{B}_T$. Fix the time $t \in 1, ..., T$ and a point $z \in \mathbb{B}_t$. Recall (7.1), the definition of $r_0 = r_0(t, m)$. In order to apply Lemma 23, we choose for each $Q_{z,t}$ a ζ with distance to the origin larger than r_0, but close enough to the fixed z such that we can find a sufficient large lower bound of H^ζ for points close to z. Therefore, consider the unit square with lattice point corners containing $r_0 \frac{z}{|z|}$, and define $\zeta = \zeta(z, t, m)$ to be the corner of the square that is farthest from the origin. Then,

$$r_0 \le |\zeta| \le r_0 + \sqrt{2}.$$

Choose $s = (2\gamma + 100) \ln T$. So Lemma 23 leads to

$$\begin{aligned} \mathbb{P}(Q_{z,t} \cap \{\langle M^\zeta \rangle_t > s\}) &\le \mathbb{P}(\mathcal{E}^{m+1}(t)^c \cap \{\langle M^\zeta \rangle_t > s\}) \\ &\le T^{-(2\gamma+20)}. \end{aligned} \tag{7.4}$$

In words, on the event $Q_{z,t}$ it is unlikely that the quadratic variation of $M^\zeta(t)$ is bigger than $\ln(T)$. On the other hand, on the event $Q_{z,t} \cap \mathcal{L}^l(t)^c$ it is likely that the martingale $M^\zeta(t)$ is large; more precisely, we will show later that

$$\mathbb{P}\big(Q_{z,t} \cap \mathcal{L}^l(t)^c \cap \{M^\zeta(t) < m(b/25)\}\big) < T^{-(\gamma+5)}. \tag{7.5}$$

However, by Corollary 6 it is unlikely that the quadratic variation is small while the martingale itself is large. Hence,

$$\begin{aligned} \mathbb{P}\big(\langle M^\zeta \rangle_t \le s,\ M^\zeta(t) \ge m\, b/25\big) &\le \mathbb{P}\big(\langle M^\zeta \rangle_t \le s,\ M^\zeta(t) \ge s\big) \\ &\le e^{-s/2} < T^{-(\gamma+5)}, \end{aligned} \tag{7.6}$$

where we also used that $m\,b/25 \geq (b/25)C_3 \ln T \geq s$. In conclusion, on the event $Q_{z,t} \cap \mathcal{L}^l(t)^c$ the quadratic variation is likely to be small (see (7.4)), and the martingale itself is likely to be large (see (7.5)). However, by (7.6) it is unlikely that the martingale is large while its quadratic variation is small. In keeping with this idea we get

$$\begin{aligned} \mathbb{P}\big(Q_{z,t} \cap \mathcal{L}^l(t)^c\big) &\leq \mathbb{P}\big(Q_{z,t} \cap \{\langle M^\zeta \rangle_t > s\}\big) \\ &\qquad + \mathbb{P}\big(Q_{z,t} \cap \mathcal{L}^l(t)^c \cap \{M^\zeta(t) < m\,b/25\}\big) \\ &\qquad + \mathbb{P}\big(\langle M^\zeta \rangle_t \leq s,\ M^\zeta(t) \geq m\,b/25\big) \\ &\leq 3T^{-(\gamma+5)}. \end{aligned} \tag{7.7}$$

By summing over all $Q_{z,t}$ (and since $T > 6\pi$), it follows immediately with (7.2) that

$$\begin{aligned} \mathbb{P}(\mathcal{E}^m(t) \cap \mathcal{L}^l(t)^c) &= \sum_{t=1}^{T} \sum_{z \in \mathbb{B}_T} \mathbb{P}\big(Q_{z,t} \cap \mathcal{L}^l(t)^c\big) \\ &\leq T \cdot (2\pi T^2) \cdot 3T^{-(\gamma+5)} \\ &\leq T^{-(\gamma+1)}. \end{aligned}$$

It only remains to prove (7.5). On the event $Q_{z,t}$, it is

$$A(t) \subset \mathbb{B}_{\sqrt{t/\pi}+m+1} \subset \mathbb{B}_{r_0 - C_1} \subset \Omega_\zeta. \tag{7.8}$$

The first inclusion above holds by Equation (7.3), for the second use the definition of r_0 (see (7.1)), and the third we get from Lemma 16 (d) and since $|\zeta|$ is nearly r_0. Consequently, $A^\zeta(t)$ consists of lattice points and no particle (grid Brownian motions) of $A^\zeta(t)$ has been stopped because of reaching the boundary of Ω_ζ, which gives

$$A^\zeta(t) = A(t),$$

on the event $Q_{z,t}$.

In order to bound $M^\zeta(t)$ on the event $Q_{z,t} \cap \mathcal{L}^l(t)^c$, partition $A^\zeta(t)$ into the sets

$$A^1 = A^\zeta(t) \cap \mathbb{B}_{\sqrt{t/\pi}-l}, \quad A^2 = A^\zeta(t) \cap \mathbb{B}(z, m), \quad A^3 = A^\zeta(t) \setminus (A^1 \cup A^2),$$

as illustrated in Figure 13 below.

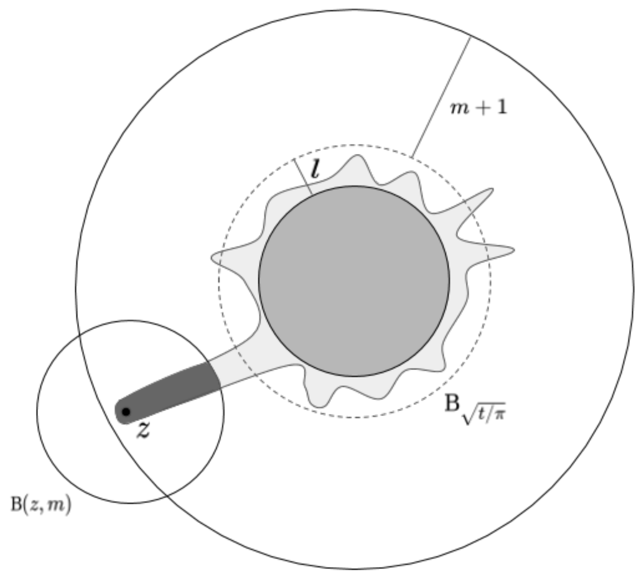

Figure 13: IDLA cluster at time t on the event $Q_{z,t} \cap \mathcal{L}^l(t)^c$. The cluster is partitioned into A^1 (gray), A^2 (dark gray), and A^3 (light gray).

For each such set, bound its contribution to $M^\zeta(t)$ from below. More precisely, consider the event $W := \{|A^2| > bm^2\} \cap Q_{z,t} \cap \mathcal{L}^l(t)^c$ and find a sufficiently large lower bound of $M^\zeta(t)$ on this event. As a consequence, if $M^\zeta(t)$ is lower than that bound, then $|A^2| \leq bm^2$, which is unlikely to occur by Lemma 25 (No thin tentacles). We will give a more thorough explanation in (7.9) and the subsequent inclusions.

Start with **A^3's contribution** to $M^\zeta(t)$, which is small since on the event W the fraction A^3 cannot contain many points. On the event $\mathcal{L}^l(t)^c$ it is $\mathbb{B}_{\sqrt{t/\pi}-l} \subset A^\zeta(t)$, hence

$$A^1 = \mathbb{B}_{\sqrt{t/\pi}-l},$$

i.e., most of the particles of $A^\zeta(t)$ are contained in A^1. Hence, the number of particles in A^2 or A^3 can be bounded by

$$|A^2 \cup A^3| = |A^\zeta(t)| - |A^1| \leq t - \pi(\sqrt{t/\pi} - l - 1)^2 \leq 8\, l\sqrt{t}.$$

By Lemma 17 we have $H^\zeta(0) \leq 2/r_0$. Hence,

$$\sum_{z\in A^3} \big(H^\zeta(z) - H^\zeta(0)\big) \geq -|A^3| \cdot H^\zeta(0) \geq -8\, l\sqrt{t} \cdot \frac{2}{r_0} \geq -30\, l \geq -\frac{b}{30}m,$$

where for the first inequality we have used that by Ω_ζ's definition H^ζ is positive on Ω_ζ.

The **contribution of A^1** is small, since the approximate mean-value property of H^ζ (see Lemma 16 (f)) implies

$$\sum_{z\in A^1} \big(H^\zeta(z) - H^\zeta(0)\big) \geq -C_1 \ln|\zeta| \geq -C_1 \ln T \geq -\frac{b}{1000}m,$$

where we used the assumption $m \geq C_3 \ln T$ and $C_3 b/1000 \geq C_1$, which holds by the definition of C_3.

On the event W the **contribution of A^2** is the largest, since the fraction A^2 contains at least bm^2 points and H^ζ is larger on A^2 (A^2 is closer to ζ than A^1 and A^3). The distance between z and ζ is (up to a constant) $3m$. In addition, z and ζ are (nearly) on the same ray through 0. Therefore, using Lemma 10 (a),

$$F^\zeta(z) = \frac{|\zeta| + |z| + |\zeta - z|}{2|\zeta||\zeta - z|} \approx \frac{1}{3m}$$

Hence, for $\tilde{z} \in \mathbb{B}(z, m)$ using the estimations Lemma 17 and Lemma 10 (f), we get

$$H^\zeta(\tilde{z}) = H^\zeta(\tilde{z}) - F^\zeta(\tilde{z}) + F^\zeta(\tilde{z}) - F^\zeta(z) + F^\zeta(z) = O\Big(\frac{1}{m^2}\Big) + \frac{1}{3m}.$$

By using Lemma 17 once again, it is $H^\zeta(0) = 1/r_0 + O(1/r_0^2) \leq 1/4m + O(1/m^2)$, and we obtain $H^\zeta(\tilde{z}) - H^\zeta(0) \geq 1/12m + O(1/m^2) \geq 1/13m$. Thus,

$$\sum_{\tilde{z}\in A^2} \big(H^\zeta(\tilde{z}) - H^\zeta(0)\big) \geq |A^2|\frac{1}{13m} > bm^2\frac{1}{13m} = \frac{b}{13}m.$$

Summing over $\boldsymbol{A^1}$, $\boldsymbol{A^2}$, and $\boldsymbol{A^3}$ provides

$$M^{\zeta}(t) > \Big(\frac{b}{13} - \frac{b}{30} - \frac{b}{1000}\Big) m > \frac{b}{25} m,$$

on the event W. Hence,

$$\Big\{ M^{\zeta}(t) > \frac{b}{25} m \Big\} \supset \{|A^2| > bm^2\} \cap Q_{z,t} \cap \mathcal{L}^l(t)^c, \tag{7.9}$$

taking the complement,

$$\Big\{ M^{\zeta}(t) \leq \frac{b}{25} m \Big\} \subset \{|A^2| \leq bm^2\} \cup Q_{z,t}^c \cup \mathcal{L}^l(t),$$

and intersecting with $Q_{z,t} \cap \mathcal{L}^l(t)^c$ gives

$$\Big\{ M^{\zeta}(t) \leq \frac{b}{25} m \Big\} \cap Q_{z,t} \cap \mathcal{L}^l(t)^c \subset \{|A^2| \leq bm^2\} \cap Q_{z,t} \cap \mathcal{L}^l(t)^c.$$

However, by Lemma 25 it is unlikely that $|A^2| \leq bm^2$, i.e.,

$$\begin{aligned} \mathbb{P}\big(\big\{ M^{\zeta}(t) \leq (b/25) m \big\} \cap Q_{z,t} \cap \mathcal{L}^l(t)^c\big) &\leq \mathbb{P}\big(\{|A^2| \leq bm^2\} \cap Q_{z,t} \cap \mathcal{L}^l(t)^c\big) \\ &< C_2 e^{-c_2 m} \\ &< T^{-(\gamma+5)}, \end{aligned}$$

which finishes the proof.

7.4 Proof: Late Points Imply Early Points

In this section we prove Lemma 22. It can be considered as the reverse analog of Lemma 21, and the main tools we will need were already used in the proof of Lemma 21. Similar to the proof of Lemma 21 we will partition $\mathcal{L}^l(T) \cap \mathcal{E}^m(T)^c$ and bound M^{ζ} and its quadratic variation on each set in that partition. While for Lemma 21 we chose for certain subevents of $\mathcal{E}^m(T)$ a ζ such that A^{ζ} is contained inside Ω_{ζ} (see (7.8)), our choice of ζ will now imply that some particles of A^{ζ} need to accumulate on $\partial\Omega_{\zeta}$.

Proof of Lemma 22. Start with the observation that, since $\mathcal{E}^m(T)$ is decreasing in m, we may assume without a loss of generality $m = l^2/(C_4 \ln T)$.

Since $l \geq C_4 \ln T$ we have $m \geq l$. Partition $\mathcal{L}^l(T)$ into the events that single points are l-late, which we defined in (1.1),

$$\mathcal{L}^l(T) = \bigcup_{\zeta\,:\,|\zeta|\leq\sqrt{T/\pi}-l} L_\zeta^l.$$

Now fix a point $\zeta \in \mathbb{B}_{\sqrt{T/\pi}-l}$ that is potentially l-late by time T. Define the time $T_\zeta = \pi(|\zeta| + l)^2$ at which ζ is not part of the IDLA cluster on the event L_ζ^l.

To bound the probability of the event $\mathcal{L}^l(T) \cap \mathcal{E}^m(T)^c$ we consider (similar to Equation (7.7) in the proof of Lemma 21) $\mathcal{E}^m(T)^c$ on the event $\langle M^\zeta \rangle_{T_\zeta} > s$ and L_ζ^l on the counter event, i.e.,

$$\begin{aligned} \mathbb{P}(L_\zeta^l \cap \mathcal{E}^m(T)^c) &\leq \mathbb{P}(\mathcal{E}^m(T)^c \cap \{\langle M^\zeta \rangle_{T_\zeta} > s\}) \\ &\quad + \mathbb{P}(L_\zeta^l \cap \{\langle M^\zeta \rangle_{T_\zeta} \leq s\}), \end{aligned} \tag{7.10}$$

for an s, which we will pick in the next step.

Consider the first summand on the right side and note that $T_\zeta \leq T \leq e^m$. Suppose $|\zeta| \geq l$, which enables us to apply Lemma 24; hence,

$$\begin{aligned} \mathbb{P}(\mathcal{E}^m(T)^c \cap \{\langle M^\zeta \rangle_{T_\zeta} > s\}) &\leq T_\zeta^{80} e^{1260m} e^{-s} \leq e^{1340m-s} \\ &\leq e^{-10m} \leq e^{-10(\gamma+3)\ln T} \\ &\leq T^{-(\gamma+3)}, \end{aligned} \tag{7.11}$$

where we set $s = 1350m$.

Next bound the second summand of (7.10). For this purpose, show that on the event L_ζ^l it is $M^\zeta(T_\zeta) \leq -l$. The strategy is to split the sum of $H^\zeta(z) - H^\zeta(0)$ over all $z \in A^\zeta(T_\zeta)$ into

- the particles on $\partial\Omega_\zeta \setminus \{\zeta\}$ (here, $H^\zeta(\cdot) - H^\zeta(0)$ is small and constant)
- and the particles in Ω_ζ (here, the sum of $H^\zeta(\cdot) - H^\zeta(0)$ over all these particles is small by the approximate mean-value property of H^ζ that can be applied if all sites in Ω_ζ are occupied, see Lemma 16 (f)).

On the event L_ζ^l no particle in $A^\zeta(T_\zeta)$ reaches ζ, where H^ζ is much larger than on every other boundary point, where H^ζ equals $\frac{1}{2|\zeta|}$ (see Lemma 16).

In addition, by the choice of ζ and T_ζ some particles of $A^\zeta(T_\zeta)$ need to accumulate at $\partial\Omega_\zeta \setminus \{\zeta\}$. More precisely, since $\Omega_\zeta \subset B_{|\zeta|+C_1}$ (see Lemma 16 (d)), we have

$$T^\zeta - |\Omega_\zeta \cap \mathbb{Z}^2| \geq \pi(|\zeta| + l)^2 - \pi(|\zeta| + C_1 + 1)^2 \geq \pi|\zeta| l,$$

i.e, on the event L^l_ζ at least $\pi|\zeta|l$ particles accumulate at $\partial\Omega_\zeta \setminus \{\zeta\}$. For these particles we have by Lemma 17,

$$H^\zeta(z) - H^\zeta(0) \leq -\frac{1}{2|\zeta|} + O\Big(\frac{1}{|\zeta|^2}\Big).$$

Hence,

$$\sum_{z\in A^\zeta(T_\zeta)\cap\partial\Omega_\zeta} \big(H^\zeta(z) - H^\zeta(0)\big) \leq -\pi|\zeta|l \cdot \frac{1}{2|\zeta|} \leq -\frac{3}{2}l. \tag{7.12}$$

Otherwise, provided the event L^l_ζ, our martingale $M^\zeta(T_\zeta)$ is maximized if all sites in Ω_ζ are occupied by $A^\zeta(T_\zeta)$. This results from the definition of Ω_ζ, according to which H^ζ achieves its minimum over $\bar{\Omega}_\zeta$ in every point on $\partial\Omega_\zeta \setminus \{\zeta\}$. In this case, when all points in $\Omega_\zeta \cap \mathbb{Z}^2$ are occupied by $A^\zeta(T_\zeta)$, we can use Lemma 16 (f), which implies

$$\sum_{z\in\Omega_\zeta\cap\mathbb{Z}^2} (H^\zeta(z) - H^\zeta(0)) \leq C_1 \ln|\zeta| \leq C_1 \ln T \leq \frac{1}{2}l, \tag{7.13}$$

where for the last inequality we used $l \geq C_4 \ln T$ and $C_4 > 2C_1$. The statements (7.12) and (7.13) give us

$$\begin{aligned} M^\zeta(T_\zeta) &\leq \sum_{z\in A^\zeta(T_\zeta)\cap\partial\Omega_\zeta} \big(H^\zeta(z) - H^\zeta(0)\big) + \sum_{z\in\Omega_\zeta\cap\mathbb{Z}^2} \big(H^\zeta(z) - H^\zeta(0)\big) \\ &\leq -l, \end{aligned}$$

on the event L^l_ζ, i.e., $L^l_\zeta \subset \{M^\zeta(T_\zeta) \leq -l\}$; and by Corollary 6 it is unlikely that the modulus of the martingale is large while its quadratic variation is small. Thus,

$$\begin{aligned} \mathbb{P}\big(L^l_\zeta \cap \{\langle M^\zeta\rangle_{T_\zeta} \leq s\}\big) &\leq \mathbb{P}\big(\{M^\zeta(T_\zeta) \leq -l\} \cap \{\langle M^\zeta\rangle_{T_\zeta} \leq s\}\big) \\ &\leq e^{-\frac{l^2}{2700m}} \leq T^{-(\gamma+3)}, \end{aligned} \tag{7.14}$$

where we used the inequality $l^2/(2700m) > (\gamma+3)\ln T$, which holds by the assumption on m and since $C_4 \geq 2700(\gamma+3)$.

Combining (7.10), (7.11), and (7.14), yields

$$\mathbb{P}\big(L^l_\zeta \cap \mathcal{E}^m(T)^c\big) \leq 2T^{-(\gamma+3)},$$

for all ζ with $l \leq |\zeta| \leq \sqrt{T/\pi} - l$.

Using Lemma 20, we can handle all ζ where $|\zeta| < l$ at once,

$$\mathbb{P}\Bigg(\bigcup_{\zeta\,:\,|\zeta|<l} L^l_\zeta\Bigg) = \mathbb{P}\big(\mathcal{L}^l(4\pi l^2)\big) \leq C_2 e^{-c_2 l} < T^{-(\gamma+3)}.$$

In total,

$$\begin{aligned}
\mathbb{P}(\mathcal{L}^l(T) \cap \mathcal{E}^m(T)^c) &\leq \mathbb{P}\Bigg(\bigcup_{|\zeta|<l} L^l_\zeta \cap \mathcal{E}^m(T)^c\Bigg) + \mathbb{P}\Bigg(\bigcup_{l\leq|\zeta|<\sqrt{T/\pi}-l} L^l_\zeta \cap \mathcal{E}^m(T)^c\Bigg)\\
&\leq T^{-(\gamma+3)} + T \cdot 2T^{-(\gamma+3)}\\
&\leq T^{-(\gamma+1)}.
\end{aligned}$$

□

7.5 Proof: Quadratic Variation Bounds

Here we prove Lemma 23 and 24, which bound $\langle M^\zeta\rangle$ on the event of no m-early point. We first state Lemma 26, which bounds increments of $\langle M^\zeta\rangle$ of size one by exit times of Brownian motions. In both proofs of Lemma 23 and 24 we will split $\langle M^\zeta\rangle$ into increments of size one, apply Lemma 26, and then use Lemma 3 to bound the exit times.

For $A \subset \mathcal{G}$ define

$$\tilde{A} = \Omega_\zeta \cap \bigcup_{z\in A\cap\mathbb{Z}^2} (z+E),$$

where $E \subset \mathcal{G}$ denotes the open cross, defined in Section 3. With the notation of Lemma 19, let $\tilde{\mathcal{B}}^n(u) := \mathcal{B}(\langle M^\zeta\rangle_n + u) - \mathcal{B}(\langle M^\zeta\rangle_n)$, which is, since $\langle M^\zeta\rangle_n$ is an $\mathcal{F}_{T_s}$-stopping time, a Brownian motion (see for instance [RY99], Ch. III, Cor. 3.6). Denote the exit times

$$\tau^n_{(-a_n,b_n)} = \inf\{u \geq 0 \,|\, \tilde{\mathcal{B}}^n(u) \notin [-a_n, b_n]\}.$$

Lemma 26. *Fix $\zeta \in \mathbb{Z}^2$. For $n \in \mathbb{N}$ let*

$$\begin{aligned} -a_n &= \min \left\{ H^\zeta(z) - H^\zeta(0) \,\middle|\, z \in \partial \tilde{A}^\zeta(n) \right\}, \\ b_n &= \max \left\{ H^\zeta(z) - H^\zeta(0) \,\middle|\, z \in \partial \tilde{A}^\zeta(n) \right\}, \end{aligned}$$

then we have

$$\langle M^\zeta \rangle_{n+s} - \langle M^\zeta \rangle_n \le \tau^n_{(-a_n, b_n)},$$

for all $s \in [0,1]$.

Proof. H^ζ is grid-harmonic on $\tilde{A}^\zeta(n) \subset \Omega_\zeta$ (see Lemma 16 (c)). By applying maximum principle (Lemma 12) to $H^\zeta(\cdot) - H^\zeta(0)$, we get

$$H^\zeta(z) - H^\zeta(0) \in [-a_n, b_n],$$

for all $z \in \tilde{A}^\zeta(n)$. From the definition of $M^\zeta(t)$ it is

$$M^\zeta(t) - M^\zeta(n) \in [-a_n, b_n],$$

for all $t \in [n, n+s]$. Using Lemma 19, the representation of M^ζ as a Brownian motion, we conclude for $r \in [\langle M^\zeta \rangle_n, \langle M^\zeta \rangle_{n+s}]$ that

$$\mathcal{B}(r) - \mathcal{B}(\langle M^\zeta \rangle_n) \in [-a_n, b_n],$$

which is equivalent to the statement that

$$\tilde{\mathcal{B}}^n(u) = \mathcal{B}(\langle M^\zeta \rangle_n + u) - \mathcal{B}(\langle M^\zeta \rangle_n) \in [-a_n, b_n],$$

for all $u \in [0, \langle M^\zeta \rangle_{n+s} - \langle M^\zeta \rangle_n]$. This result can be rephrased as follows: the first exit of $\tilde{\mathcal{B}}^n(\cdot)$ from the interval $[-a_n, b_n]$ cannot occur before time $\langle M^\zeta \rangle_{n+s} - \langle M^\zeta \rangle_n$, which enables us to write,

$$\langle M^\zeta \rangle_{n+s} - \langle M^\zeta \rangle_n \le \tau^n_{(-a_n, b_n)}.$$

□

Recall that Lemma 23 assumes $|\zeta| > r_0 = r_0(t, m)$ for r_0 defined as in (7.1). In this case, we have good bounds for $-a_n$ and b_n on the event $\mathcal{E}^{m+1}(t)^c$, for $-a_n$, b_n as defined in Lemma 26.

Proof of Lemma 23. Fix time $t > 0$. The event $\mathcal{E}^{m+1}(t)^c$ is equal to the event that

$$A(n) \subseteq \mathbb{B}_{\sqrt{n/\pi}+m+1}, \quad \text{for all } n = 1, ..., t.$$

Lemma 16 (d) gives us $\mathbb{B}_{\sqrt{n/\pi}+m+1} \subset \Omega_\zeta$, which implies that in the event $\mathcal{E}^{m+1}(t)^c$, it is

$$A^\zeta(n) = A(n)$$

We now apply Lemma 16 (e) with the radius $\sqrt{n/\pi} + m + 2 < |\zeta| - C_1$, and get the bounds

$$-a := -\frac{1}{2r_0} \leq H_\zeta(z) - H_\zeta(0) \leq \frac{1}{r_0 - \sqrt{n/\pi} - m - 2 - C_1} =: b_n,$$

for all integers $n \leq t$ and $z \in A^\zeta(n)$ on the event $\mathcal{E}^{m+1}(t)^c$. Hence, on $\mathcal{E}^{m+1}(t)^c$ Lemma 26 implies (with $s = 1$) that

$$\langle M^\zeta \rangle_{n+1} - \langle M^\zeta \rangle_n \leq \tau^n_{(-a,b_n)},$$

where $\tau^n_{(-a,b_n)}$ are independent exit times of standard Brownian motions starting at zero. Summing over all increments of size one,

$$\begin{aligned}
\exp(\langle M^\zeta \rangle_t) \mathbf{1}_{\mathcal{E}^{m+1}(t)^c} &\leq \exp\left(\langle M^\zeta \rangle_t \, \mathbf{1}_{\mathcal{E}^{m+1}(t)^c}\right) \\
&= \exp\Big(\sum_{n=1}^{t} (\langle M^\zeta \rangle_n - \langle M^\zeta \rangle_{n-1})\, \mathbf{1}_{\mathcal{E}^{m+1}(t)^c}\Big) \\
&\leq \exp\Big(\sum_{n=1}^{t} \tau^n_{(-a,b_n)}\Big).
\end{aligned}$$

To bound the exit times $\tau^n_{(-a,b_n)}$ observe that for the interval $(-a, b_n)$ and $\lambda = 1$ the assumptions of Lemma 3 are fulfilled, and since the exit times are independent

$$\begin{aligned}
\mathbb{E}\Big(\exp\Big(\sum_{n=1}^{t} \tau^n_{(-a,b_n)}\Big)\Big) &= \prod_{n=1}^{t} \mathbb{E}\big(\exp(\tau^n_{(-a,b_n)})\big) \leq \prod_{n=1}^{t} (1 + 20ab_n) \\
&= \exp\ \ln \prod_{n=1}^{t} (1 + 20ab_n) = \exp \sum_{n=1}^{t} \ln(1 + 20ab_n) \\
&\leq \exp\Big(20a \sum_{n=1}^{t} b_n\Big).
\end{aligned}$$

To find a more handy bound denote $R = r_0 - m - 3 - C_1$. Using

$$b_n = \frac{1}{R + 1 - \sqrt{n/\pi}} \le \frac{1}{R - \sqrt{(n-1)/\pi}},$$

we can dominate b_n, $n = 1, ..., t$ and get

$$\begin{aligned}
\sum_{n=1}^{t} b_n &= \sum_{n=1}^{t} \frac{1}{R + 1 - \sqrt{n/\pi}} \\
&\le \int_0^t \frac{dx}{R - \sqrt{x/\pi}} \\
&\le 2\pi \int_0^{\sqrt{t/\pi}} \frac{y\,dy}{R - y} \\
&\le 2\pi \int_{R-\sqrt{t/\pi}}^{R} \frac{R - z}{z} dz \\
&= 2\pi R \Big(\ln\left(\frac{R}{R - \sqrt{t/\pi}} \right) - \sqrt{t/\pi} \Big) \\
&\le 2\pi r_0 \ln(r_0/C_1).
\end{aligned}$$

Together with the Markov inequality,

$$\begin{aligned}
\mathbb{P}(\mathcal{E}^{m+1}(t)^c \cap \{\langle M^\zeta \rangle_t > s\}) &\le \frac{1}{e^s} \mathbb{E}\big(\exp(\langle M^\zeta \rangle_t)\, \mathbf{1}_{\mathcal{E}^{m+1}(t)^c} \big) \\
&\le \frac{1}{e^s} \mathbb{E}\Big(\exp \sum_{n=1}^{t} \tau^n_{(-a, b_n)} \Big) \\
&\le \frac{1}{e^s} \exp\Big(20a \sum_{n=1}^{t} b_n \Big) \\
&\le \frac{1}{e^s} \exp(40a\pi r_0 \ln(r_0/C_1)) \\
&\le \frac{1}{e^s} \left(\frac{r_0}{C_1} \right)^{20\pi} \le \frac{1}{e^s} t^{80}.
\end{aligned} \tag{7.15}$$

□

In the last proof, on the event of no m-early point the grid IDLA $A^\zeta(t)$ stayed inside balls, which are subsets of Ω_ζ and keep some distance to Ω_ζ. Hence, Lemma 16 (e) provides a good bound on $H^\zeta(z) - H^\zeta(0)$ for all z in

$A^\zeta(t)$. In Lemma 24 ζ may be chosen much smaller than in Lemma 23. It follows that particles of $A^\zeta(t)$ may accumulate at $\partial\Omega_\zeta$; in particular, they might hit ζ. For these particles, we will need different bounds of $H^\zeta(\cdot) - H^\zeta(0)$, see (7.17). Since this bound is too rough to apply it to all particles of $A^\zeta(t)$, we will pick some time t_0 before t such that $A^\zeta(t_0)$ is inside $\mathbb{B}_{|\zeta|-C_1}$; for the cluster $A^\zeta(t_0)$ we can apply Lemma 23.

Proof of Lemma 24. First consider the case $|\zeta| \geq 5m$. Choose t_0 so that $r_0(t_0)$ is just a little smaller than $|\zeta|$, here, $t_0 = \pi(|\zeta| - 4m - 2C_1 - 1)^2$. By $t_0, ..., t_N$ denote the sequence t_0, $\lceil t_0 \rceil$, $\lceil t_0 \rceil + 1$, ..., $\lfloor t \rfloor$, t. To bound $\mathbb{E}(e^{\langle M^\zeta \rangle_t}\, \mathbf{1}_{\mathcal{E}^m(t)^c})$ consider separately the increment of $\langle M^\zeta \rangle_\bullet$ up to time t_0 and the increments over the intervals $[t_0, t_1], ..., [t_{N-1}, t_N]$, i.e., write

$$\begin{aligned} e^{\langle M^\zeta \rangle_t}\, \mathbf{1}_{\mathcal{E}^m(t)^c} &\leq e^{\langle M^\zeta \rangle_t} \mathbf{1}_{\mathcal{E}^m(t_0)^c} \\ &= e^{\langle M^\zeta \rangle_{t_0}} \mathbf{1}_{\mathcal{E}^m(t_0)^c} \prod_{n=1}^{N} e^{\langle M^\zeta \rangle_{t_n} - \langle M^\zeta \rangle_{t_{n-1}}}. \end{aligned} \tag{7.16}$$

Lemma 16 (a) and (e) provide the estimations,

$$-a := -\frac{1}{2|\zeta|} \leq H^\zeta(z) - H^\zeta(0) \leq 2 =: b, \tag{7.17}$$

which hold on the entire set $\bar{\Omega}_\zeta$. Hence, Lemma 26 implies

$$\langle M^\zeta \rangle_{t_n} - \langle M^\zeta \rangle_{t_{n-1}} \leq \tau^{t_n}_{(-a,b)},$$

for all $n = 1, ..., N$. Moreover, since the Brownian motions $\tilde{\mathcal{B}}^{t_n}(\cdot)$ are independent of each other, of $\mathcal{F}_{t_0}$, and therefore of $\mathcal{E}^m(t_0)^c \in \mathcal{F}_{t_0}$ as well, the same holds for the exit times $\tau^{t_n}_{(-a,b)}$. Applying all this to (7.16), yields

$$\begin{aligned} \mathbb{E}\big(e^{\langle M^\zeta \rangle_t}\, \mathbf{1}_{\mathcal{E}^m(t)^c}\big) &\leq \mathbb{E}\Big(e^{\langle M^\zeta \rangle_{t_0}} \mathbf{1}_{\mathcal{E}^m(t_0)^c} \prod_{n=1}^{N} e^{\tau^{t_n}_{(-a,b)}}\Big) \\ &= \mathbb{E}\big(e^{\langle M^\zeta \rangle_{t_0}} \mathbf{1}_{\mathcal{E}^m(t_0)^c}\big) \prod_{n=1}^{N} \mathbb{E}\big(e^{\tau^{t_n}_{(-a,b)}}\big) \end{aligned} \tag{7.18}$$

For the product of the expected exit times in (7.18), Lemma 3 gives us

$$\begin{aligned}\prod_{n=1}^{N} \mathbb{E}\Big(e^{\tau^{t_n}_{(-a,b)}}\Big) &\leq \exp \ln \prod_{n=1}^{N} \frac{20}{|\zeta|} \\ &\leq \exp \sum_{n=1}^{N} \frac{20}{|\zeta|} \\ &\leq \exp\left((t - t_0 + 2)\frac{20}{|\zeta|}\right) \\ &\leq \exp(960m),\end{aligned} \tag{7.19}$$

where for the last inequality we used that t_0 and $t = \pi(|\zeta| + l)^2$ are not all that far from each other (use also $|\zeta| \geq 5m$ and $m \geq 2C_1 + 2$).

To bound the factor $\mathbb{E}(e^{\langle M^\zeta\rangle_{t_0}} \mathbf{1}_{\mathcal{E}^m(t_0)^c})$ in (7.18), note that since $r(t_0) < |\zeta|$, the assumptions of Lemma 23 are fulfilled, thus, by (7.15),

$$\mathbb{E}\big(e^{\langle M^\zeta\rangle_{t_0}}\, \mathbf{1}_{\mathcal{E}^m(t_0)^c}\big) \leq t_0^{80} \leq t^{80}. \tag{7.20}$$

Plugging (7.19) and (7.20) into (7.18), gives us

$$\mathbb{E}\big(e^{\langle M^\zeta\rangle_{t}}\, \mathbf{1}_{\mathcal{E}^m(t)^c}\big) \leq t^{80} e^{960m}, \tag{7.21}$$

For the case $|\zeta| \leq 5m$, use the equations (7.19) and (7.18) with $t_0 = 0$. Then,

$$\begin{aligned}\mathbb{E}\big(e^{\langle M^\zeta\rangle_{t}}\, \mathbf{1}_{\mathcal{E}^m(t)^c}\big) &\leq \exp\left((t+2)\frac{20}{|\zeta|}\right) \leq \exp\left(80\pi|\zeta| + \frac{40}{|\zeta|}\right) \\ &\leq \exp(1260m)\end{aligned} \tag{7.22}$$

Using Markov inequality, (7.21), and (7.22),

$$\mathbb{P}(\mathcal{E}^m(t)^c \cap \{\langle M^\zeta\rangle_t > s\}) \leq \mathbb{E}\big(e^{\langle M^\zeta\rangle_{t}}\, \mathbf{1}_{\mathcal{E}^m(t)^c}\big)\, e^{-s} \leq t^{80} e^{1260m} e^{-s}.$$

□

7.6 Proof: No Very Late Point

We now present the proof of Lemma 20. It is basically a reformulation of [LBG92] Lemma 6 in terms of late points.

Proof of Lemma 20. Rephrasing the first conclusion of [LBG92] Lemma 6, for each $\epsilon > 0$ and r sufficiently large, it is

$$\mathbb{P}\Big(z \notin A\big(\pi\big(r\sqrt{1+\epsilon}\big)^2\big)\Big) \leq e^{-cr}, \tag{7.23}$$

for each $z \in \mathbb{B}_{r(1-\epsilon)}$ and a constant $c > 0$ that only depends on ϵ. In terms of late points, (7.23) means that we have an exponential bound for l-late points if l is proportional to r, where r is the radius of the ball, which describes the expected shape of the cluster by time $T = \pi r^2$.

Fix $l \geq 1$. It suffices to prove the lemma for $T = 100\pi l^2$, and since

$$\mathcal{L}^l(100\pi l) = \bigcup_{z \in \mathbb{B}_{9l}} \big\{z \notin A\big(\pi(|z|+l)^2\big)\big\}, \tag{7.24}$$

we may assume $|z| \leq 9l$. To apply (7.23) to the event of z being l-late, find ϵ and r such that

$$\begin{aligned} &r(1-\epsilon) > |z|, \quad \text{and} \\ &r\sqrt{1+\epsilon} < |z| + l. \end{aligned} \tag{7.25}$$

The first inequality ensures that $z \in \mathbb{B}_{r(1-\epsilon)}$. Together with (7.23) we obtain from the second inequality

$$\mathbb{P}\big(z \notin A\big(\pi(|z|+l)^2\big)\big) \leq e^{-cr}.$$

Assume that r is of the form $r = a(\epsilon)(|z|+l)$, for some a just depending on ϵ. Such an r, which fulfills the inequalities of (7.25), exists if there is an $a = a(\epsilon)$, which fulfills

$$\frac{9}{10}\frac{1}{1-\epsilon} < a < \frac{1}{\sqrt{1+\epsilon}}.$$

Such an $a(\epsilon)$ exists if we can find an ϵ with

$$\sqrt{1+\epsilon} < \frac{10}{9}(1-\epsilon).$$

This inequality is fulfilled for $\epsilon = 1/20$. Hence,

$$\mathbb{P}\big(z \notin A\big(\pi(|z|+l)^2\big)\big) \leq e^{-c'l},$$

for a constant $c' > 0$. Summing over all $z \in \mathbb{B}_{9l}$ as in (7.24), gives

$$\mathbb{P}\big(\mathcal{L}^l(100\pi l)\big) \leq 200\pi l^2 e^{-c'l}.$$

Choosing c_2, C_2 suitably, such that the right side is at most $C_2 e^{-c_2\sqrt{T}}$, and large enough that we can use the same constants c_2, C_2 as in Lemma 25, completes the proof.

Index of Notation

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