An Introduction to the Theory of Groups

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Fourth Edition

Problems

Chapter 1

1.13

(i) A permutation $\alpha \in S_n$ is ***regular*** if either α has no fixed points and it is the product of disjoint cycles of the same length, or $\alpha = \mathbb{1}$. Prove that α is regular iff it is a power of an n-cycle β; that is, $\alpha = \beta^m$ for some m. (*Hint*: if $\alpha = (a_1 a_2 \ldots a_k)(b_1 b_2 \ldots b_k) \ldots (z_1 z_2 \ldots z_k)$, where there are m letters a, b, ..., z, then let $\beta = (a_1 b_1 \ldots z_1 a_2 b_2 \ldots z_2 \ldots a_k b_k \ldots z_k)$.)

Solution: β^m takes a_1 through $b_1 \ldots z_1$ to a_2 as desired. $\mathbb{1}$ can be expressed as β^n for $\beta(j) = j+1$ an n-cycle. For a general regular α, disjointess of the sets a_j, b_j, ...z_j guaranteed that the β from the hint is an n-cycle. If there's some n-cycle β with $n = mk$, and we take the mth power, we also get m disjoint, length-k cycles, as desired.

(ii) If α is an n-cycle, then α^k is a product of $\gcd(n,k)$ disjoint cycles, each of length ${}^{n}/_{\gcd(n,k)}$.

Solution: $\alpha^n = \mathbb{1}$. If n is a multiple of k, then $\alpha^n = \left(\alpha^k\right)^{n/k}$. α^k would then be a product of k ${}^{n}/_{k}$-cycles. In the case where n is not a multiple of k, but they have a non-trivial gcd, then starting at α_0, α would take us to α_1. α^k will take us to α_k. It takes α_k to α_{2k}, and so on until we get to $\alpha_{mk} = \alpha_0$. This happens if $m = \frac{n}{\gcd(n,k)}$, but I don't know how to prove that.

(iii) If p is prime, then every power of a p cycle is either a p-cycle or $\mathbb{1}$.

Solution: This is a corollary of the last exercise, noting that $\gcd(p,k) = 1$ if $k \neq p$ and p if $k = p$.

1.17 How many $\alpha \in S_n$ are there with $\alpha^2 = \mathbb{1}$?

Solution: There's $\mathbb{1}$, and there's disjoint unions of transpositions. In terms of single transpositions, there are $\binom{n}{2}$ of them. If I'm going to put together a product of j transpositions, there are $\binom{n}{2}$ ways to choose the first transposition, $\binom{n-2}{2}$